Green's function estimates and elliptic measures for some linear elliptic equations with singular drifts.

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

The purpose of this thesis is to study pointwise estimates on Green's functions and elliptic measures for linear elliptic partial differential equations that have a drift term that diverges at the boundary in several different ways. These questions are of fundamental importance in its own right in partial differential equations, with future applications in other settings such as the time-independent Schroedinger equation, as well as being important for questions of solvability of rough Dirichlet and Neumann problems for a large class of elliptic operators with singular lower order terms.

Chapter 1

Introduction.

We are generally interested in Green's functions corresponding to linear elliptic operators of the form

$$L = \operatorname{div}(A\nabla u) + \mathcal{B} \cdot \nabla u, \tag{1.1}$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ is a uniformly elliptic matrix with variable coefficients $a_{ij} : \Omega \to \mathbb{R}$ satisfying,

$$||A_{ij}||_{L^\infty(\Omega)} \leqslant \Lambda, \text{ and } A(x)\xi \cdot \xi \geqslant \lambda|\xi|^2,$$

$$\text{for almost every } x \in \Omega \text{ and every } \eta \in \mathbb{R}^n, \tag{1.2}$$

with a drift $\mathcal{B} \in L^\infty_{loc}(\Omega; \mathbb{R}^n)$ that diverges like the inverse distance to the boundary:

$$|\operatorname{dist}(\cdot, \Omega^c)^{1-\beta}\mathcal{B}| \leqslant M, \tag{1.3}$$

for some uniform constant M, and for some $0 \leqslant \beta < 1$. Henceforth, we use the notation $\delta_\Omega(x) := \operatorname{dist}(x, \Omega^c)$.

Our first main result, in Chapter 2 develops a new technique and applies it to obtain with pointwise upper estimates for the Green's function with poles in the interior, specifically for the operator,

$$L_1 = \Delta u + \mathcal{B} \cdot \nabla u, \tag{1.4}$$

in the unit ball. Here, the drift diverges at the boundary, as earlier, like $M/(\delta(x))^{1-\beta}$ for some finite constant M and for some $0 < \beta < 1$. Further, we require that the drift is locally in $C^{1,\alpha}$. We prove pointwise upper and lower estimates on the Green function with poles in the interior.

In Chapter 3, we then construct a specific example with the operator of Eq. (1.4) in the unit ball, for the case where the drift is bounded like $M/\delta(x)$, to show that the upper Green function estimate fails and solutions do not exist.

In Chapter 4, we then discuss the case of a drift that diverges on average on every Whitney cube like the inverse distance to the boundary, in a general chord-arc domain, with a sufficiently small prefactor.

More precisely, we have the conditions,

$$(1) : \int_{U_Q} |\mathcal{B}|^2 \delta(X) dV \leqslant \beta^2 l(Q)^{n-1}, \tag{1.5}$$

Here we also require,

$$(2) : \left(\sum_{i=0}^{n} \delta(X)|\mathcal{B}_i(X)| + \sum_{i,j=1}^{n} |\mathcal{A}_{ij}(X)| \right) < M < \infty, \ \ \sum_{i,j=1}^{n} \mathcal{A}_{ij}(X)\xi_i\xi_j \geqslant \lambda|\xi|^2. \tag{1.6}$$

We prove a ‘Bourgain’ type estimate on the elliptic measure as well as expected pointwise bounds on the Green’s function to establish doubling of the elliptic measure. We explain this in greater detail in the Preliminaries section of Chapter 4. In this chapter, we also give a new proof of the Hardy inequality for domains with an Ahlfors regular codimension one boundary, using a stopping time method, which has some novelty of its own.

Every chapter starts with some preliminary details, setting the context for the theorems that follow. We state and prove the theorems in each of these chapters.

In all of these cases, while the lower pointwise bounds on the Green's function is somewhat routine to establish, following the arguments going back at least to [GrWi], the pointwise upper bounds on the Green's function need a lot more work in both Chapters 2 and 4.

In future work, we will aim to consider more general bounded Lipschitz domains for which the methods developed in Chapter 2 can be applied. The existence of solutions for the operator Eq. (1.1) with the condition Eq. (4.4), is guaranteed by the result of [Ha24], and one expects to prove the pointwise estimates for the Green's functions in such a broader class of bounded Lipschitz domains and a large class of lower order terms in the linear elliptic operator, using the methods of Chapter 2.

We also intend to study the question of the upper pointwise estimates, where the pole of the Green's function is arbitrarily close to the boundary. The method introduced here is expected to work for drifts that are in L^p_{loc} for some appropriate $p > 1$, or some drifts that diverge in the interior, and where one expects to obtain pointwise estimates on the Green's function with uniform constants.

We also aim to use the method of this thesis, for the time independent Schroedinger equation (where we naturally have a principal Laplacian term) and study newer classes of potentials for which one is able to obtain pointwise bounds on the Green's function. Specifically, one anticipates to first get a condition of the form in Eq. (2.29) and continue the argument with appropriate modifications. In that case, one would require a similar level of regularity on the potential term as we required for the drift term here. We will also aim in the future to study the question by possibly removing the $C^{1,\alpha}$ regularity on the drift term, as well as to introduce nonlinear potential terms. We will also consider the Green's function corresponding to the parabolic operator with such drifts and potentials.

Chapter 2

Pointwise estimates for Green's functions with drifts diverging at the boundary.

2.1 Preliminaries.

Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^n$ with $n \geqslant 3$. We denote, $\delta(X) = \mathrm{dist}(X, \partial\Omega)$. The coordinates in $\mathbb{R}^n$ are written as $(x_1, \ldots, x_n)$. In this chapter, we consider in $\mathbb{R}^n$ the linear second order operator with a singular drift term, given for some $0 < \beta < 1$,

$$Lu = -\Delta u + \mathcal{B} \cdot \nabla u = 0, \quad (\delta(X))^{1-\beta} |\mathcal{B}(X)| \leqslant M. \tag{2.1}$$

We write, $D = \mathrm{diam}(\Omega)$. Here $0 < M < \infty$.

We find pointwise upper and lower estimates for the Green's function for such an operator when the Green's function is locally in C^3, in compactly supported subdomains of Ω.

We require the pointwise condition that,

$$-\frac{M}{\delta(x)^{2-\beta}} \leqslant \nabla \cdot \mathcal{B} \leqslant 0, \tag{2.2}$$

In particular this also means that $\nabla \cdot \mathcal{B}$ is negative in the distributional sense, that is,

$$\int_\Omega (\mathcal{B} \cdot \nabla v) \geqslant 0, \ \forall v \in C_0^\infty(\Omega) \text{ where } v \geqslant 0, \tag{2.3}$$

Note that when the drift satisfies the pointwise estimate of Eq. (2.3), then according to Theorem 1.2 of [Ha24], solutions to the adjoint equation $L_T u = -\nabla \cdot (\nabla u + \mathcal{B} u) = 0$ exists, and one considers the Green's functions $\mathcal{G}(x, y), \mathcal{G}_T(x, y)$ corresponding to L and L_T respectively, both exist. Then, we also have $\mathcal{G}(x, y) = \mathcal{G}_T(y, x)$.

We have stated the bounds as above for simplicity, but in general it is enough to consider $|\mathcal{B}|^2$ and $|\nabla \cdot \mathcal{B}|$ to belong respectively to the more general Morrey spaces $M^{\frac{n}{2-2\beta}}$ and $M^{\frac{n}{2-\beta}}$ as considered in [Ha24].

In [SSSZ12], operators are considered that have the Laplacian term along with certain divergence free drift terms with similar scale-invariant properties, along with the parabolic counterparts:

$$\partial_t u + \mathcal{B} \cdot \nabla u - \Delta u = 0 \tag{2.4}$$

For the corresponding parabolic operator, with divergence free measurable drift terms $(\mathrm{div}(b) = 0)$, an old result of Nash [Na58] shows the pointwise upper bound:

$$|\mathcal{G}(x, t; y, s)| \leqslant \frac{C}{(t-s)^{\frac{n}{2}}}. \tag{2.5}$$

For the case where $\mathrm{div}(b) = 0$ in Eq. (2.4), in [SSSZ12] several regularity results are presented when writing $b = \nabla \cdot\ d$ for an anti-symmetric matrix d and considering the anti-symmetric part of the matrix $I + d$ to belong a BMO space with the symmetric part of $I + d$ being comparable to the identity matrix I up to a positive constant .(1) In case the $\nabla \cdot (b) = 0$ condition is relaxed, one is also referred to the regularity results in [Naz12] where the condition is relaxed to $\nabla \cdot (b) \leqslant 0$. More recently, one is referred

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(1) One is also referred to the work in [HLMP22] for the question of unique solvability of the rough Dirichlet problem in the upper half plane for a divergence form elliptic operator that has a BMO anti-symmetric part as above.

to [KS19] for pointwise upper bounds on Green's function in bounded domains for the operator with a more general elliptic principal term and where the lower-order terms belong to certain L^p spaces for $p \geqslant n$, and then further work in the setting of not necessarily bounded domains, of Sakellaris [Sak1] for lower-order terms in certain weak $L^{n,q}$ spaces, and of Mourgoglou [Mour23] for the lower order terms in Stummel-Kato spaces and variants of it. For Green's function corrsponding to drifts diverging in points within the domain, one is referred to [MM22].

It is seen that in general bounded domains, our singular drift term does not belong to either the weak L^p space considered in [Sak1], nor the Stummel-Kato spaces considered in Section 6 of [Mour23]. We note that the definition of Morrey spaces considered in [Ha24] is more general than the definition considered in [Mour23], where one requires the Morrey type estimate to hold in balls centered on the boundary as well, and our drift does satisfy the definition of [Ha24].

In this chapter, the drift term is singular and behaves like an exponent of the inverse of the distance to the boundary, the exponent being strictly less than 1. While we work specifically for the case of a drift that is pointwise bounded by $\Big(1/\delta(x)\Big)^{1-\beta}$, the method works as long as we have a drift that is integrable at the boundary for the case of the unit ball considered here. Thus, while the principal term in Eq. (2.1) is the Laplacian, locally the drift is merely integrable near the boundary, and not required to be in L^n or $L^{n,p}$ spaces.

The boundary regularity of the solutions to the Dirichlet problem, goes back to the work of Wiener for the Laplacian [Wi24]. This required pointwise estimates on the Green's function.

More generally, we can define the operator:

$$L_1 u = -\nabla \cdot (\mathcal{A}\nabla u) + \mathcal{B} \cdot \nabla u = 0, \tag{2.6}$$

where we have, some constants $\infty > \Lambda \geqslant \lambda > 0$, and M as before, so that,

$$\mathcal{A}_{ij}(X)\eta_i\eta_j \geqslant \lambda|\eta|^2, |\mathcal{A}_{ij}(X)\eta_i\psi_j| \leqslant \Lambda|\eta||\psi|, \quad \text{for all } \eta, \psi \in \mathbb{R}^n \setminus \{0\},$$
$$\text{and } \delta(X)^{-\beta}|\mathcal{B}(X)| \leqslant M. \tag{2.7}$$

As noted earlier, we require the Green's function $G(\cdot, Y)$ for the operator Eq. (2.1) to be locally in C^3, in order to derive the pointwise upper bounds for the operator L. By the Schauder estimates, it is enough to consider the drift term $|\mathcal{B}|$ to be locally in $C^{1,\alpha}$ for an arbitrary $\alpha > 0$. See for example, Theorem 6.10 of [GT77].

Here we state the three main results of this chapter;

Theorem 2.1. *For the elliptic operator in Eq.* (2.6) *with the coefficients satisfying Eq.* (2.7)*, in* $\mathbb{R}^n$ *with* $n \geqslant 3$*, we have the bound for the Green's function for the operator in Eq.* (2.6)*: for any* $z, y \in \Omega$ *with* $|z - y| \leqslant \frac{1}{2}\delta(y) := \frac{1}{2}dist(y, \partial\Omega)$ *we have*

$$\mathcal{G}(y, z) \geqslant K(M, \lambda)\frac{1}{|z - y|^{n-2}}. \tag{2.8}$$

Theorem 2.2. *Consider the operator of Eq.* (2.1) *and the Dirichlet Green's function corresponding to this operator, in* $\mathbb{R}^n$ *with* $n \geqslant 3$*. Further suppose that* $\mathcal{B} \in C^{1,\alpha}(\Omega')$ *for some* $\alpha > 0$*, for any compactly supported subdomain in* Ω*, and that* $\mathcal{B}$ *satisfies Eq.* (2.3)*. For any* $y \in B(0, r)$*, for any* $x \in \Omega$ *with* $|x| \leqslant \frac{1}{2}$*, we have for some constant* K' *dependent on* M, λ *only,*

$$\mathcal{G}(x, 0) \leqslant K'(M, \lambda)\frac{1}{|x|^{n-2}}. \tag{2.9}$$

Here, the pole of Green's function has been considered to be only at the origin.

In case the solution to the adjoint equation also exists, we can use Theorem 4.13 and some standard arguments to show that,

Theorem 2.3. *Consider the operator of Eq.* (2.1) *and the Dirichlet Green's function corresponding to this operator, in* $\mathbb{R}^n$ *with* $n \geqslant 3$. *Further suppose that* $\mathcal{B} \in C^{1,\alpha}(\Omega')$ *for some* $\alpha > 0$, *for any compactly supported subdomain in* Ω, *and that* $\mathcal{B}$ *satisfies Eq.* (2.3). *Then for any* $y \in B(0,r)$ *with* $r < 1$, *and for any* $x \in \Omega$ *with* $|x-y| \leqslant \frac{1}{2}\delta(y)$, *we have for some constant* K' *dependent on* M, λ *only,*

$$\mathcal{G}(x,y) \leqslant K'(M,\lambda,r)\frac{1}{|x-y|^{n-2}}. \tag{2.10}$$

Here the bound $K'(M,\lambda,r)$ *depends on* r *as well.*

We prove Theorem 4.11 in Section 3, and Theorems 2.3 and 4.13 in Section 4.

Remark 2.4. We can consider potential terms with coefficients $\mathcal{D}$ that are bounded in magnitude by $M/(\delta(X))^2$, and in that case, we would need the kind of negativity condition:

$$\int_\Omega (\mathcal{D}v - \mathcal{B}\cdot\nabla v) \leqslant 0,\ \forall v \in C_0^\infty(\Omega),$$

(such as equation 8.8 of [GT77] or equation 1.5 of [Mour23]) as in Section 4 of this chapter, in particular for the additional term that would appear in Eq. (2.23). Further, in presence of the $\mathcal{D}$ term, we need to modify the integrating factor for the inequality corresponding to Eq. (2.64). We leave the question of such potential terms for future work. One is referred to [An86, She94, She95, She99], and also more recently to [Pog24]

Sections 3 and 4, contain the main results of this chapter. The result of Section 3 is short and similar to the proof in [GrWi] for the case of $|\mathcal{B}| = 0$. We also note that for the case of $\beta = 0$, we have a counterexample to the existence of solutions, and pointwise upper bounds on the Green's function fail to exist [Pat24]. There also, the pole of the Green's function is at the origin.

We will also note that in Lemmas 2.10 and 2.11, we see the distinction in the behavior of the Green's function when the gradient is small, in comparison to the case where the gradient is large. In this instance, it is actually easier to estimate the decay of the Green's function in case the gradient is small, as is done in the argument after the completion of Lemma 2.11. In other contexts, this dichotomy of the behavior of solutions to elliptic equations where the divergence is large, versus where it is small, has appeared in the literature. See, for example, [Moo15], and the references therein.

The basic novelties of the proof for the upper bound in Section 4 are in introducing a modified Lorentz norm when restricting to values of the Green's function within a ball containing the pole and separated from the boundary, and then to finally reduce the problem to looking at the level sets of the Green's function, using the polar form of the Laplacian and the points at minimal and maximal distance of these level sets from the pole.

While we have worked specifically for the drift of Eq. (2.1), the method introduced here will be used in several directions in the future, as outlined in Section 5.

2.2 Properties of the Green's function.

We adopt some of the standard arguments of Chapter 8 of [GT77] .

Following the notation of [GT77], when u is only weakly differentiable, then in a weak or generalized sense, u is said to satisfy $L_1 u = 0(\geqslant 0, \leqslant 0)$ in Ω if

$$\mathcal{L}_1(u,v) = \int_\Omega \Big((\mathcal{A}_{ij}\partial_j u)(\partial_i v) - \mathcal{B}_i(\partial_i u)v\Big)dx = 0(\leqslant 0, \geqslant 0) \quad (2.11)$$

for all non-negative functions $v \in C_0^1(\Omega)$ which is the space of compactly supported functions whose first derivatives are continuous in Ω.

Consider the $f, g^i,\ i = 1, 2, \ldots, n$, locally integrable functions in Ω. For our purposes of constructing the Green's function, it is enough to consider compactly supported f, g^i, $i = 1, 2, \ldots, n$. Then a weakly differentiable function u is called a weak or generalized solution of the inhomogeneous equation

$$L_1 u = f - \partial_i g^i, \quad (2.12)$$

in Ω if

$$\mathcal{L}_1(u,v) = \int_\Omega (g^i \partial_i v + f v)dx = F(v), \quad \forall v \in C_0^1(\Omega). \quad (2.13)$$

We say that a function u is a solution of the generalized Dirichlet problem: $L_1 u = f + \partial_i g^i, u = \phi$ on $\partial\Omega$ if u is a generalized solution of Eq. (2.12), $\phi \in W^{1,2}(\Omega)$ and $u - \phi \in W_0^{1,2}(\Omega)$.

With a modification of the argument of Theorem 8.1 of [GT77], we first outline a routine proof of the maximum principle, which will be used in the course of our proof.

Theorem 2.5 (Maximum principle)**.** *Consider the operator* L_1 *in Eq.* (2.6)*, and consider* $u \in W^{1,2}(\Omega)$ *so that* $L_1 u \geqslant 0$ *(*$L_1 u \leqslant 0$*). Then,*

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+, (\inf_\Omega u \geqslant \inf_{\partial\Omega} u^+) \quad (2.14)$$

Outline of the proof: The proof follows almost exactly as in Theorem 8.1 of [GT77]; when $\mathcal{B} = 0$ the proof is the same as the corresponding case in [GT77]. In the case that $\mathcal{B} \neq 0$, note that it remains to consider the case where $\sup_{\Omega} u$ is only attained in the interior of the domain Ω, say at a point x_0 and thus considering $l = \sup_{\partial\Omega} u^+$ and in the case that $l = \sup_{\partial\Omega} u^+ < \sup_{\Omega} u$, we consider a $l \leqslant k < \sup_{\Omega} u$ with $k \uparrow \sup_{\Omega} u$. In that case the proof is completed by contradiction as in [GT77], by considering the compactly supported functions $(u-k)^+ \in C_0^1(\Omega)$, with a uniform upper bound on the magnitude of the $\mathcal{B}$ term in some ball $B(x_0, r) \subset \Omega$, where without loss of generality $r \leqslant \delta(x_0)/2$. We omit the details. ∎

We immediately get from Theorem 2.5, the following corollary, as Corollary 8.2 of [GT77].

Corollary 2.6. *Let $u \in W_0^{1,2}(\Omega)$ satisfy $L_1 u = 0$ in Ω. Then $u = 0$ in Ω.*

We note that Eq. (2.2) , guarantees, with the use of the argument of Theorem 8.1 of [GT77] as above, that the solution to the adjoint equation when it exists, satisfies the maximum principle as well. This will be used in the proof of Theorem 2.3 as well.

As in Section 4 of Chapter III of [HL01], we assume that the Dirichlet problem is solvable for the operator L_1, and that the Green's function is also well defined when taking limiting functions for the Dirac delta distribution at the pole of the Green's function. We would conclude in the limit that the Green's function is positive.

For further consideration, for any point $y \in \Omega$, call $\Omega_\rho = \Omega \cap B_\rho(y)$ where $B_\rho(y)$ is the ball of radius ρ centered at y. Also define,

$$f_\rho(x, y) = |B_\rho(y)|^{-1} \mathbf{1}_{\Omega_\rho}(x), \ x \in \Omega. \tag{2.15}$$

2.3 Lower bounds on the Green's function.

The lower bound follows by an argument similar to that used in [GrWi, Pat24]. (2) We prove this here.

Proof of Theorem 4.11. The proof essentially follows by extending the argument of the proof of Eq.(1.9) of [GrWi]. Take $r := |z-y|$. Consider a smooth cut-off function η which is 1 on $B_r(y) \setminus B_{r/2}(y)$ and zero outside $B_{3r/2}(y) \setminus B_{r/4}(y)$, and further $0 \leqslant \eta \leqslant 1$ and $|\nabla \eta| \leqslant \frac{K}{r}$.

Henceforth, we use the Einstein summation convention, where the summation sign is implied.

Given the domain Ω, for any admissible test function ϕ, the Green's function satisfies the following adjoint equation;

$$\int_{\Omega} \Big((\nabla \phi) \cdot \nabla \mathcal{G}(y,x) - \mathcal{G}(y,x) \mathcal{B} \cdot (\nabla \phi) \Big) dx = \phi(y). \tag{2.16}$$

Here $\mathcal{A}^T_{ij}$ denotes the transpose of the matrix $\mathcal{A}_{ij}$.

We note that for Ω, with $D = \operatorname{diam}(\Omega) < \infty$, we have for the fixed $\beta > 0$, that,

$$\Big(\frac{1}{\delta(x)} \Big)^{1-\beta} \leqslant \frac{D^{\beta}}{\delta(x)}. \tag{2.17}$$

Consider the test function $\phi = G(\cdot, y)\eta$, the bound on the drift term $\mathcal{B}$, the Cauchy inequality with ϵ's, and the bounds on the cut-off function η introduced above, that,

$$\begin{aligned} \int\limits_{r/2<|x-y|<r} |\nabla \mathcal{G}(y,x)|^2 dx &\leqslant \Big(K_1 \frac{1}{r^2} \cdot \int\limits_{r/4<|x-y|<3r/2} \mathcal{G}(y,x)^2 dx \Big) \\ &+ \Big(K_2 \frac{1}{r\delta(y)} \cdot \int\limits_{r/4<|x-y|<3r/2} \mathcal{G}(y,x)^2 dx \Big) + \Big(\frac{K_2}{\delta(y)} \cdot \int\limits_{r/4<|x-y|<3r/2} \mathcal{G}(y,x) |\nabla \mathcal{G}(y,x)| dx \Big). \end{aligned} \tag{2.18}$$

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(2) This was also outlined in the argument of Lemma 4.3 in Section III in [HL01].

Noting that we have $r \leqslant \frac{1}{2}\delta(y)$, using the Cauchy inequality with ϵ's again for the last term on the right, and finally adjusting the terms, we get

$$\begin{aligned}\int\limits_{r/2<|x-y|<r} |\nabla \mathcal{G}(y,x)|^2 dx &\leqslant \tilde{K}\frac{1}{r^2}\cdot\Big(\int\limits_{r/4<|x-y|<3r/2} \mathcal{G}(y,x)^2 dx\Big)\\ &\leqslant \tilde{K} r^{n-2}\big(\sup_{r/4\leqslant|x-y|\leqslant 3r/2} \mathcal{G}(y,x)^2\big). \qquad (2.19)\end{aligned}$$

Again as in [GrWi], choose a similar cut-off function ϕ that is 1 on $B_{r/2}(y)$ and zero outside $B_r(y)$, and using it as the test function we get,

$$\begin{aligned}1 &= \int\limits_{r/2\leqslant|x-y|\leqslant r} (\mathcal{A}_{ij}\partial_i\mathcal{G}(y,x)\partial_j\phi + \mathcal{G}(y,x)\mathcal{B}_i\partial_i\phi)\Big)dx)\\ &\leqslant M\frac{K}{r}\int\limits_{r/2\leqslant|x-y|\leqslant r} |\nabla\mathcal{G}(y,x)(\cdot,y)|dx + \Big(\frac{MK}{r\delta(y)}\cdot\int\limits_{r/2\leqslant|x-y|\leqslant r} |\mathcal{G}(y,x)|dx) \qquad (2.20)\end{aligned}$$

Using the identity of Eq. (4.69), and Cauchy's inequality for the first term on the right, along with a trivial volume bound, and finally Harnack's inequality,

$$1 \leqslant K r^{n-2} \sup_{r/4\leqslant|x-y|\leqslant 3r/2} |\mathcal{G}(y,x)| \leqslant K|z-y|^{n-2}|\mathcal{G}(y,z)|. \qquad (2.21)$$

∎

2.4 Upper bounds on the Green's function.

Here we prove Theorem 4.13.

We will prove this upper bound on the Green's function $\mathcal{G}(x,0)$ for any $x \in \Omega$ and with $|x| \leqslant \frac{1}{2}$. In doing so, we initially adopt the argument of Proposition 5.10 of [KS19] and introduce a modified Lorentz norm for our purpose. The argument will be broken up into several different parts. In this argument for obtaining the upper bound, we will also a-priori use the lower bound of Theorem 4.11.

Proof of Theorem 4.13. For any ρ, consider as in [GrWi] and in the case of Proposition 5.10 of [KS19] the test function given by

$$\phi(x) = \Big(\frac{1}{s} - \frac{1}{\mathcal{G}_\rho(x,0)}\Big)^+, \quad \text{and} \quad \Omega_s := \{x \in \Omega : \mathcal{G}_\rho(x,0) \geqslant s\}. \tag{2.22}$$

Here we use the standard notation that $h(x)^+ := \max(h(x), 0)$.

Using ϕ as a test function, we get, with only the drift term, that,

$$\int_{\Omega_s} \nabla \mathcal{G}_\rho(x,0) \cdot \nabla \phi dx \leqslant \frac{1}{s} - \int_{\Omega_s} \mathcal{B} \cdot (\nabla \mathcal{G}_\rho(x,0))\phi dx. \tag{2.23}$$

We have that $\nabla \phi = \nabla \mathcal{G}_\rho(\cdot,0)/\mathcal{G}_\rho(\cdot,0)^2$ where the derivative is understood to be with respect to the x-variable.

Using Eq. (2.11), using a standard integral by parts, we get,

$$\int_{\Omega_s} \nabla \mathcal{G}_\rho(x,0) \cdot \frac{\nabla \mathcal{G}_\rho(x,0)}{\mathcal{G}_\rho(x,0)^2} dx \leqslant \frac{1}{s} + \int_{\Omega_s} \mathcal{B} \cdot \Big(\frac{\nabla \mathcal{G}_\rho(x,0)}{\mathcal{G}_\rho(x,0)}\Big) dx. \tag{2.24}$$

For the given fixed y, set $w_{s,\rho}(x) = (\ln(\mathcal{G}_\rho(x,0)/s))^+$. In this case, $\nabla w_{s,\rho}(x) = \nabla \mathcal{G}_\rho(x,y)/\mathcal{G}_\rho(x,y)$ in the set Ω_s. Then using the ellipticity condition on $\mathcal{A}$ and the triangle inequality and Cauchy inequality, we have

$$\lambda \int_{\Omega_s} |\nabla w_{s,\rho}|^2 dx \leqslant \frac{1}{s} + \int_{\Omega_s} |\mathcal{B}||\nabla w_{s,\rho}| dx \leqslant \frac{1}{s} + \frac{\lambda}{2} \int_{\Omega_s} |\nabla w_{s,\rho}|^2 + \frac{1}{2\lambda} \int_{\Omega_s} |\mathcal{B}|^2 dx, \tag{2.25}$$

and thus we have

$$\frac{\lambda}{2} \int_{\Omega_s} |\nabla w_{s,\rho}|^2 dx \leqslant \frac{1}{s} + \frac{1}{2\lambda} \int_{\Omega_s} |\mathcal{B}|^2 dx. \tag{2.26}$$

Next we use the Sobolev inequality, and Holder's inequality on the last term on the right, and noting $\Omega_{2s,y} \subset \Omega_{s,y}$ we have,

$$\frac{\lambda}{2} (\ln 2)^2 |\Omega_{2s}|^{\frac{n-2}{n}} \leqslant \frac{\lambda}{2} \Big(\int_{\Omega_s} |w_{s,\rho}|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}} \leqslant C_n^2 \frac{\lambda}{2} \Big(\int_{\Omega_s} |\nabla w_{s,\rho}|^2 dx \Big)$$

$$\leqslant \frac{C_n^2}{s} + \frac{C_n^2}{2\lambda}\Big(\int_{\Omega_s} |\mathcal{B}|^n dx\Big)^{\frac{2}{n}} |\Omega_s|^{\frac{n-2}{n}}. \quad (2.27)$$

Write $f(s) = |\Omega_{s,y_0}|^{\frac{n-2}{n}}$. Thus, we have constants $K_1, K_2 > 0$ so that

$$f(2s) \leqslant \frac{K_1}{s} + K_2\Big(\int_{\Omega_s} |\mathcal{B}|^n dx\Big)^{\frac{2}{n}} f(s). \quad (2.28)$$

We make a digression to state that all the bounds we obtain in the end of this section, will be independent of ρ, and without loss of generality, we can henceforth work with the Green's function $\mathcal{G}(\cdot, 0)$ in place of $\mathcal{G}_\rho(\cdot, 0)$.

Now consider the following two cases:

(i) There exists a large enough absolute constant C so that for any configuration of the drift in the ball $B(0,1)$, we have $f(2s) \geqslant \frac{1}{C} f(s)$ for all $s \geqslant \widetilde{s} := \max\limits_{x \in S(0,\frac{1}{2})} \mathcal{G}(x, 0)$.

(ii) The above fails, and so for all positive integers N large enough, we have some drift configuration $|\mathcal{B}_N|$ for which we have $f(s_N) > N f(2s_N)$ for some $s_N \geqslant \widetilde{s_N}$. Here we define, $\widetilde{s_N} := \max\limits_{x \in S(0,\frac{1}{2})} \mathcal{G}_N(x, 0)$, where $\mathcal{G}_N(x, 0)$ is the Green's function corresponding to the operator with the drift $|\mathcal{B}_N|$. We deal with this case later, in a manner similar to the argument used in the later part of the argument for Case (i).

Case (i). We deal with Case(i) above. Consider a fixed positive real number $K \leqslant 1/(2C)$. Now consider a real number $L \geqslant 2$ large enough so that upon defining $s^* := \max\limits_{x \in S(0,1/L)} \mathcal{G}(x, 0)$, for $s \geqslant s^*$, noting the bound $|\mathcal{B}| \lesssim \frac{M}{\delta_\Omega(y)}$ in Ω_s, and using the maximum principle and noting that $\Omega_s \subset \overline{B_{0,L}} := \overline{B(0, 1/L)}$ for $s \geqslant s^*$, using a trivial volume bound on the integral on the right of Eq. (2.28), we get for all $s \geqslant s^*$,

$$
\begin{aligned}
f(2s) &\leqslant \frac{K_1}{s} + K_2\Big(\int_{B(0,1/L)} |\mathcal{B}|^n dx\Big)^{\frac{2}{n}} f_{y_0}(s) \\
&\leqslant \frac{K_1}{s} + K_2\Big(\int_{B(0,1/L)} \Big|\frac{M}{\delta_\Omega(y)}\Big|^n dx\Big)^{\frac{2}{n}} f_{y_0}(s) \leqslant \frac{K_1}{s} + K f_{y_0}(s). \quad (2.29)
\end{aligned}
$$

We have required that L be large enough so that $K_2 c(\frac{M}{L})^2 \leqslant K$ with the constant volume prefactor of c, so that the last inequality in Eq. (2.29) holds.

Thus we have for all $s \geqslant s^*$,

$$
f(2s) - KCf(2s) \leqslant f(2s) - Kf(s) \leqslant \frac{K_1}{s}.
$$

Since we have $KC \leqslant 1/2$, we get,

$$
\frac{1}{2} f(2s) \leqslant (1 - KC) f(2s) \leqslant \frac{K_1}{s}, \tag{2.30}
$$

and so for all $\lambda \geqslant 2s^*$, we have

$$
f(\lambda) = |\{x : \mathcal{G}(x) \geqslant \lambda\}|^{\frac{n-2}{n}} \leqslant \frac{K_3}{\lambda}. \tag{2.31}
$$

Consider any $p > 1^{(3)}$. We define a modified Lorentz-norm type functional for the Green's function, given by

$$
||\mathcal{G}(\cdot, 0)||^*_{p,\infty} := \sup_{\lambda \geqslant 2s^*} \lambda |\Omega_\lambda|^{\frac{1}{p}} = \sup_{\lambda \geqslant 2s^*} \lambda |\{x : \mathcal{G}(x) \geqslant \lambda\}|^{\frac{1}{p}}. \tag{2.32}
$$

Consider any $p > \alpha > 1$. Modifying a standard argument found for example in the proof of Theorem 1.2.10 of [Gra23], for any subset $E \subset \Omega_t$ with $t \geqslant 2s^*$, we get,

$$
\int_E \mathcal{G}(x,0)^\alpha dx = \int_0^\infty |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\} \cap E| d\lambda
$$

$^{(3)}$Later we will choose $p = \frac{n}{n-2}$.

$$\leqslant \int_0^\infty \min\Big(|E|, \{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\}\Big) d\lambda \tag{2.33}$$

Consider the value λ_E, defined as the unique value for which $|E| = |\{x : \mathcal{G}(x,0) \geqslant \lambda_E^{\frac{1}{\alpha}}\}|$. In the last integral on the right of Eq. (2.33), we have

$$\begin{aligned} &\int_0^\infty \min\Big(|E|, |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\}|\Big) d\lambda \\ &\qquad = \int_0^{\lambda_E} |E| d\lambda + \int_{\lambda_E}^\infty |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\}| d\lambda \end{aligned} \tag{2.34}$$

We consider the set $S_{B_{0,L}}$ of all measurable subsets $E \subset B(0,1/L)$ so that $\lambda_E \geqslant (2s^*)^\alpha$ and $0 \notin E$. From the above consideration and the monotonic behavior of this distribution function of $\mathcal{G}(x,0)$, the set $S_{B_{0,L}}$ consists of all subsets $E \subset B(0,1/L)$ so that $|E| = |\{x : \mathcal{G}(x,0) \geqslant \lambda_E^{\frac{1}{\alpha}}\}| \leqslant |\{x : \mathcal{G}(x,0) \geqslant 2s^*\}|$:

$$S_{B_{0,L}} = \{E \subset B(0,1/L), 0 \notin E : |E| \leqslant |\{x : \mathcal{G}(x,0) \geqslant 2s^*\}|\}. \tag{2.35}$$

In this case, for any $E \subset S_{B_{0,L}}$, for which $\lambda_E^{\frac{1}{\alpha}} \geqslant (2s^*)$, we have

$$|E| = |\{x : \mathcal{G}(x,0) \geqslant \lambda_E^{\frac{1}{\alpha}}\}| \leqslant \frac{(||\mathcal{G}(\cdot,0)||^*_{p,\infty})^p}{(\lambda_E)^{\frac{p}{\alpha}}}, \tag{2.36}$$

by the definition of Eq. (2.32).

Since the function $\lambda \to (||\mathcal{G}(\cdot,0)||^*_{p,\infty})^p/\lambda^{p/\alpha}$ is decreasing, by Eq. (2.36), the value $\lambda_E^{'}$ that satisfies

$$|E| = \frac{(||\mathcal{G}(\cdot,0)||^*_{p,\infty})^p}{(\lambda_E^{'})^{\frac{p}{\alpha}}} \tag{2.37}$$

is such that $(\lambda_E^{'})^{\frac{1}{\alpha}} \geqslant (\lambda_E)^{\frac{1}{\alpha}} \geqslant (2s^*)$.

Note that

$$(\lambda_E^{'}) = \frac{||\mathcal{G}(\cdot,0)||^*_{p,\infty})^\alpha}{|E|^{\frac{\alpha}{p}}}. \tag{2.38}$$

Thus for each $E \subset S_{B_{0,L}}$ we can write

$$\int_0^\infty \min\Big(|E|, |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\}|\Big) d\lambda \leqslant \int_0^\infty \min\Big(|E|, \frac{(||\mathcal{G}(\cdot,0)||^*_{p,\infty})^p}{\lambda^{\frac{p}{\alpha}}}\Big) d\lambda$$
$$= \int_0^{\lambda'_E} |E| d\lambda + \int_{\lambda'_E}^\infty \frac{(||\mathcal{G}(\cdot,0)||^*_{p,\infty})^p}{\lambda^{\frac{p}{\alpha}}} d\lambda. \quad (2.39)$$

Now from the value of λ'_E given in Eq. (2.37), we get upon performing the integrations above [4],

$$\int\limits_E \mathcal{G}(x,0)^\alpha dx = \int_0^\infty |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\} \cap E| d\lambda$$
$$\leqslant \int\limits_0^\infty \min\Big(|E|, |\{x : \mathcal{G}(x,0) \geqslant \lambda^{\frac{1}{\alpha}}\}|\Big) d\lambda \leqslant \frac{p}{p-\alpha} |E|^{(1-\frac{\alpha}{p})} (||\mathcal{G}(\cdot,0)||^*_{p,\infty})^\alpha. \quad (2.40)$$

From Eqs. (2.31) and (2.32), we have

$$||\mathcal{G}(\cdot,0)||^*_{\frac{n}{n-2},\infty} \leqslant K_3. \quad (2.41)$$

We thus have for all $E \subset S_{B_{0,L}}$, with $1 < \alpha < p = n/(n-2)$,

$$\Big(\frac{1}{|E|} \int\limits_E \mathcal{G}(x,0)^\alpha dx\Big)^{\frac{1}{\alpha}} \leqslant \frac{n}{n-\alpha(n-2)} |E|^{\frac{2-n}{n}} K_3. \quad (2.42)$$

For the special case of $E = B(y,R)$ being a ball of some radius R, Moser's inequality, (see for example Theorem 8.17 of [GT77], we get that [5]

$$\sup_{x \in E/2} \mathcal{G}(,0) \leqslant C' \Big(\frac{1}{|E|} \int\limits_E \mathcal{G}(x,0)^\alpha dx\Big)^{\frac{1}{\alpha}} \leqslant \frac{C'n}{n-\alpha(n-2)} |E|^{\frac{2-n}{n}} K_3, \quad (2.43)$$

for some constant $C'(n, \lambda, M, R)$ and in our domains of the form $B(0, 1/L)$ in consideration, along with the drift term, R is bounded from above. Here $\frac{E}{2}$ denotes the ball with half the radius of E but with the same center.

---

[4] Such an expression appears in the proof of Theorem 1.2.10 of [Gra23]

[5] It is to use this inequality with the drift term that we need an exponent on $\mathcal{G}$ of $\alpha > 1$.

Now in particular in $S_{B_{0,L}}$, we choose a ball E with center $y \in B_{0,L}$, and with radius $r_y := |y|/4$. Then from Eq. (2.43) we get in particular,

$$\mathcal{G}(y,0) \leqslant \frac{C_1}{|y|^{n-2}}. \tag{2.44}$$

Define the subset of balls of the form $S'_{B_{y_0},L} = \{B(y,|y|/4) \subset S_{B_0,L}\}$. Each ball in $S'_{B_0,L}$ has volume $C_2(|y|)^n$ for some uniform constant C_2.

We note from Eq. (2.35) that the volume of sets in $S_{B_{0,L}}$ are upper bounded by $V(0) := |\{x : \mathcal{G}(x,0) \geqslant 2s^*\}|$. Thus, there is an upper bound on the distance from the origin to the centers of the balls in $S'_{B_0,L}$, depending on the value of $V(0)$. In particular, there is a value r_0 dependent on $V(0)$ so that for all $z \in \mathbb{B}_0 := B(0,r_0)$, the balls $B(z,r_z) = B(z,|z|/4) \subset S'_{B_0,L}$ and also $B(z,r_z) \subset B(0,1/L)$. Here we have defined $r_z = |z|/4$.

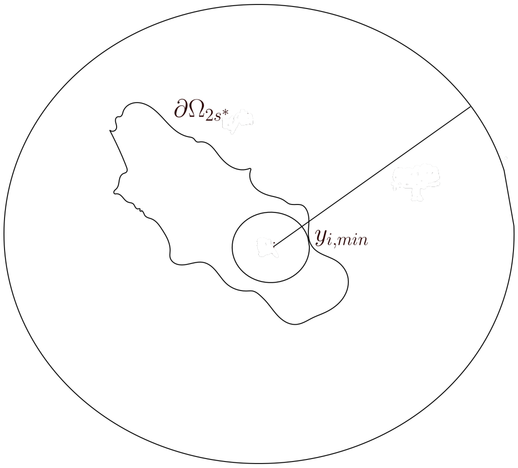

Figure 2.1: The boundary of the set Ω_{2s^*} , the sphere of radius $1/L$ centered at 0, the point on the level set $\partial\Omega_{2s^*}$ at the closest distance from 0, are shown in the figure.

We next show that there exists some constant $\eta > 0$ so that for any configuration of the drift, we have

$$\frac{V(0)}{|B(0,1/L)|} = \frac{|\{x : \mathcal{G}(x,0) \geqslant 2s^*\}|}{|B(0,1/L)|} \geqslant \eta. \tag{2.45}$$

This clearly means that for any configuration of the drift, we have

$$\frac{\mathbb{B}_0}{|B(0,1/L)|} \geqslant \eta. \tag{2.46}$$

Thus the estimate of Eq. (2.44) clearly holds for all $z \in \mathbb{B}_0$, and with a routine Harnack inequality, one extends the same inequality with a modified constant to the entire ball $B(0,1/L)$.

The proof of Eq. (4.23) and the proof of Case(ii) are similar in nature. In particular, in proving Eq. (4.23) and thus Case(i), henceforth we no longer have to use the fact that there is a constant C so that $f(2s) \geqslant \frac{1}{C} f(s)$ for all $s \geqslant \tilde{s}$,and the arguments henceforth can also be employed for Case(ii). We argue by contradiction; that if there does not exist the claimed $\eta > 0$, then there exists a sequence of drifts $|\mathcal{B}_i|$ satisfying Eq. (2.1) and Eq. (2.2) for which the ratio of Eq. (4.23) goes to zero. In particular, if we write the Green's function corresponding to the drift $|\mathcal{B}_i|$ as $\mathcal{G}_i(\cdot,0)$, and $s_i^* := \max\limits_{x\in S(0,1/L)} \mathcal{G}_i(x,0)$, we have as $i \to \infty$,

$$\frac{|\{x : \mathcal{G}_i(x,0) \geqslant 2s_i^*\}|}{|B(0,1/L)|} \to 0. \tag{2.47}$$

In particular, the point $y_{i,\min}$ at the minimum distance from the origin to the level set of $|\{x : \mathcal{G}_i(x,0) = 2s_i^*\}|$ is so that $y_{i,\min} \to 0$ as $i \to \infty$.
We will use the result of Theorem 4.11 for our argument as well.

We first establish the following lemma which will be used later on. Hereafter we write $\mathcal{G}(x,0) = \mathcal{G}(x)$.

Lemma 2.7. *There is a uniform constant* M_0 *so that for any configuration of the drift that satisfies Eq.* (2.1) *and Eq.* (2.2), *we have a sequence* $b_k|_{k=1}^{\infty}$ *of maximum* *points, and a sequence* $a_k|_{k=1}^{\infty}$ *of minimum points, so that,*

$$\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{b_k} b_k^{n-1} < M_0 \Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_k} a_k^{n-1}, \tag{2.48}$$

and $a_k \to 0$*, and* $b_k \to 0$ *as* $k \to \infty$*. In the above equation, it is understood that the pole of the Green's function is at* 0*.*

We need to be careful below about defining the sequence of $a_k|_{k=1}^{\infty}$, since we do not have a-priori information about the structure of the level sets of the Green's function.

Proof of Lemma 5. Assume to the contrary, that for any arbitrarily large $M(\geqslant 1)$, there exists some $i(M)$ and drift $|\mathcal{B}_i|$ satisfying Eq. (2.1) and Eq. (2.2) so that, we have, eventually for any sequence of maximum points $b_{i,k}|_{k=1}^{\infty}$ and any sequence of minimum points $a_{i,k}|_{k=1}^{\infty}$, each sequence converging to 0,

$$I = \inf_k \Big(\Big|\frac{\partial \mathcal{G}_i}{\partial r}\Big|_{b_{i,k}} b_{i,k}^{n-1} \Big) \geqslant M \sup_k \Big(\Big|\frac{\partial \mathcal{G}_i}{\partial r}\Big|_{a_{i,k}} a_{i,k}^{n-1} \Big) := M \cdot S, \tag{2.49}$$

Fix some radius $r_0 > 0$. Without loss of generality, choose the maximum point $b_{i,k_0} = r_0$ and the minimum point a_{i,k_0} to belong to the level set $\mathcal{G}(b_{i,k_0})$, for some $k_0 \geqslant 1$. We suppress the dependence of $i(M)$ on M and the dependence of $\mathcal{G}_i$ on i, below.

Initially we choose the increment h_0 so that,

$$h_0 \cdot \sup_{\mathcal{B}(0,b_{i,k_0}) \backslash B(0,\frac{1}{2}a_{i,k_0})} |\mathcal{G}_i''(x)| \ll \frac{S}{(b_{i,k_0})^{n-1}} \leqslant \frac{I}{(b_{i,k_0})^{n-1}},\ h_0 \ll \frac{1}{2} a_{i,k_0}. \tag{2.50}$$

In this case, define the two sequences in the following way: first consider the point $a_{i,(k_0+1)}$ which is the minimum point of the level set $\{x : \mathcal{G}(x) = \mathcal{G}(a_{i,k_0} - h_0)\}$. By definition, we have, $|(a_{i,k_0}-h_0)| \geqslant |a_{i,k_0+1}|$. We continue this process, till we reach some $t_1 \geqslant 1$ so that $a_{i,k_0+t_1-1} \geqslant \frac{1}{2}a_{i,k_0}$ with possibly $h_0' = |a_{i,k_0+t_1-1} - \frac{1}{2}a_{i,k_0}| \leqslant h_0$, while at this last stage we define a_{i,k_0+t_1} as the minimum point of the level set $\mathcal{G}(a_{i,k_0+t_1-1} - h_0') = \mathcal{G}(\frac{1}{2}a_{i,k_0+t_1})$.

We define the sequence $a_{i,k_0+t_m}|_{m=1}^{\infty}$, for each $m \geqslant 1$ in the same manner as above.

We take,

$$h_m \cdot \sup_{B(0,b_{i,k_0})\backslash B(0,\frac{1}{2}a_{i,k_0+t_m})} |\mathcal{G}_i''(x)| \ll \frac{S}{(b_{i,k_0})^{n-1}} \leqslant \frac{I}{(b_{i,k_0})^{n-1}},\ h_m \ll \frac{1}{2}a_{i,k_0+t_m}. \tag{2.51}$$

Here the implied constants are allowed to depend on M, S as well. Here t_{m+1}, for $m \geqslant 1$ is the greatest integer so that $a_{i,k_0+t_{m+1}-1} \geqslant \frac{1}{2}a_{i,k_0+t_m}$. For all the values $\{a_{i,k_0+t_m}, \dots, a_{i,k_0+t_{m+1}-1}\}$, the decrement at each step is taken as h_m, with a_{i,k_0+t_m+j}, for $1 \leqslant j \leqslant t_{m+1} - t_m$, being the minimum point of the level set of $\mathcal{G}(a_{i,k_0+t_m+j-1} - h_m)$.

It is clear that by this process, the sequence $a_{i,k}|_{k=1}^{\infty}$ so defined converges to y_i. We continue this iteration till we reach some value $t_{m_0} = N$, with $a_{i,k_0+N} = \epsilon$ so that $\mathcal{G}(a_{i,k_0+N}) = \mathcal{G}(\epsilon) \geqslant K(M,\lambda)\left(\frac{1}{\epsilon}\right)^{n-2} \gg \mathcal{G}(b_{i,k_0}) = \mathcal{G}(a_{i,k_0})$, where we have invoked the result of Theorem 1, proved in Section 3.

Starting with b_{i,k_0}, we define $b_{i,k+t}$ as the point on the sphere of radius $|b_{i,k+t-1} - h_{m_0})|$ where the maximum value of the Green's function is attained. By definition, we have that $\mathcal{G}(b_{i,k+t}) \geqslant \mathcal{G}(b_{i,k+t-1} - h_{m_0})$.

Then, we have, for each $k \geqslant 1$ in case of the sequence of $b_{i,k}$'s, and for each $1 \leqslant j \leqslant t_{m+1} - t_m - 1$, for the case of the sequence of the $a_{i,k}$'s, that,

$$\mathcal{G}(b_{i,k} - h_{m_0}) - \mathcal{G}(b_{i,k}) = \left|\frac{\partial \mathcal{G}}{\partial r}\right|_{b_{i,k}} \cdot h_{m_0} + O_{m_0}(h_{m_0}^2),$$

$$\mathcal{G}(a_{i,k_0+t_m+j-1} - h_m) - \mathcal{G}(a_{i,k_0+t_m+j-1}) = \left|\frac{\partial \mathcal{G}}{\partial r}\right|_{a_{i,k_0+t_m+j-1}} \cdot h_m + O_m(h_m^2), \tag{2.52}$$

Here the implied constants depend on the bounds on the second derivative of $\mathcal{G}$ in $\Omega \setminus B(y_i, a_{i,k_0+t_m+j-1})$ for each j considered.

So we have,

$$\Big(\sum_{q=0}^{N} \frac{h_q S}{(a_{i,(k_0+q)})^{n-1}} + O_q(h_q^2)\Big) \geqslant \mathcal{G}(a_{i,k_0+N}) - \mathcal{G}(a_{i,k_0}) \tag{2.53}$$

$$= \mathcal{G}(\epsilon) - \mathcal{G}(a_{i,k_0}) \approx \mathcal{G}(\epsilon) \gg \mathcal{G}(a_{i,k_0}). \tag{2.54}$$

Here we have $h_q = h_m$ whenever $k_0 + t_m \leqslant q \leqslant k_0 + t_{m+1}$.

Using Eqs. (2.50) and (2.51), we find a slightly altered constant S', so that we have,

$$\Big(\sum_{q=0}^{N} \frac{h_q S'}{(a_{i,(k_0+q)})^{n-1}}\Big) \geqslant \mathcal{G}(\epsilon) \gg \mathcal{G}(a_{i,k_0}). \tag{2.55}$$

In this case, we add up the equality in Eq. (2.52) above. First we get,

$$\begin{aligned}\mathcal{G}(b_{i,k_0+N_1}) - \mathcal{G}(b_{i,k_0}) &\geqslant \sum_k \Big(\mathcal{G}(b_{i,k} - h_{m_0}) - \mathcal{G}(b_{i,k})\Big) \\ &\geqslant c\Big(\sum_{q=1}^{N_1} \frac{I h_{m_0}}{(b_{i,(k_0+q)})^{n-1}} + O_{m_0}(h_{m_0}^2)\Big),\end{aligned} \tag{2.56}$$

where b_{i,k_0+N_1} lies in the domain $\Omega \setminus B(y_i, \epsilon)$, with the property that,

$$|b_{i,k_0+N_1} - \epsilon| \leqslant h_{m_0}. \tag{2.57}$$

We terminate the sequence of $b_{i,j}$'s at $j = N_1$. Clearly $N_1 \geqslant N$.

In this case, from Eq. (2.51), we choose the implied constants so that h_{m_0} is small enough, so at each step we have,

$$\frac{I}{(b_{i,k_0+q})^{n-1}} \gg h_{m_0}, \tag{2.58}$$

for all $b_{i,k_0+q} \in \Omega \setminus B(y_i, \epsilon)$, and adding this up N_1 many times, we get with a slightly altered constant that,

$$\mathcal{G}(b_{i,k_0+N_1}) \geqslant \mathcal{G}(b_{i,k_0+N_1}) - \mathcal{G}(b_{i,k_0}) \geqslant cI'\Big(\sum_{m=0}^{N_1} \frac{h_{m_0}}{(b_{i,(k_0+m)})^{n-1}}\Big)$$

$$\geqslant cMS'\Big(\sum_{m=0}^{N_1} \frac{h_{m_0}}{(b_{i,(k_0+m)})^{n-1}}\Big), \quad (2.59)$$

for a slightly altered constant I'.(6)

Further we also have,

$$\mathcal{G}(a_{i,k_0+N}) - \mathcal{G}(a_{i,k_0}) \leqslant c\Big(\sum_{q=0}^{N} \frac{Sh_q}{(a_{i,(k_0+q)})^{n-1}} + O_q(h_q^2)\Big), \quad (2.60)$$

where as before, we have $h_q = h_m$ whenever $k_0 + t_m \leqslant q \leqslant k_0 + t_{m+1}$

Again, we have, by a slightly altered constant S', that,

$$\mathcal{G}(a_{i,k_0+N}) \approx \mathcal{G}(a_{i,k_0+N}) - \mathcal{G}(a_{i,k_0}) \leqslant cS'\Big(\sum_{q=0}^{N} \frac{h_q}{(a_{i,(k_0+q)})^{n-1}}\Big), \quad (2.61)$$

and without loss of generality, we can take $I' \geqslant MS'$ as above.

Thus for M large enough, from Eqs. (2.53), (2.59) and (2.61) using a basic integral test, comparing the values of $\mathcal{G}(b_{i,k_0+N_1}), \mathcal{G}(a_{i,k_0+N})$, we have a violation of Harnack's inequality, on an annular region of width h centered on the sphere $S(b_{i,k_0+N_1})$. This is because of Eq. (2.53), and the right hand side of Eq. (2.59) is then arbitrarily large compared to the right side of Eq. (2.61).

Thus Eq. (2.141) is established. ∎

We continue with the proof of Theorem 4.13, by contradiction. The sets $\Omega_{2s^*} = \{x : \mathcal{G}(x,0) \geqslant 2s^*\} \subset B(0, 1/L)$, with

$$\frac{|\{x : \mathcal{G}_i(x,0) \geqslant 2s_i^*\}|}{|B(0,1/L)|} \to 0, \quad (2.62)$$

---

(6) In fact, one can also work with this same h_{m_0} for the sequences of $a_{i,k}|_{k=k_0}^{\infty}$ in going from a_{i,k_0} to $a_{i,k_0+N} = \epsilon$, which would slightly alter the argument presented above.

as $i \to \infty$.

Consider i large enough to be determined later, and the point $y_{i,\min}$ on the boundary $\partial\Omega_{2s_i^*}$ that is the minimum distance $\delta_{\min,y_i}$ from the origin. As a consequence of Eq. (2.62), for the uniform L, we have $\delta_{\min,y_i} \to 0$ as $i \to \infty$.

In this case, the ball $B(0, \delta_{\min,y_i}) \subset \Omega_{2s_i^*}$. By the Harnack inequality, there is a constant $\beta \gg 1$ independent of r so that for all points on the sphere $S(0,r)$, the values of the Green's function with the pole at origin are comparable and we will have

$$\sup_{S(0,r)} \mathcal{G}_i(\cdot, 0) \leqslant \beta \inf_{S(0,r)} \mathcal{G}_i(\cdot, 0). \tag{2.63}$$

We thus have by definition, $\mathcal{G}_i(y_{i,\min}, 0) \approx 2s^*_{y_i}$.

Consider the sphere $S(0, y_{i,\min})$ of radius $\delta_{\min}$. In this case, we have $S(0, y_{i,\min}) \subset \Omega$ with a point of tangency of this sphere $S(0, y_{i,\min})$ with $\partial\Omega_{2s_i^*}$ at the point $y_{i,\min}$. At $y_{i,\min}$ the gradient $\nabla\mathcal{G}_i(\cdot, 0)$ is inward normal to both $S(0, y_{i,\min}), \partial\Omega_{2s_i^*}$[7]. Further, we must have $\frac{\partial^2\mathcal{G}_i(\cdot,0)}{\partial\theta_k^2}|_{y_{i,\min}} \geqslant 0$ for any of the $(n-1)$ angular coordinates θ_k , otherwise up to second order, noting from above that $\frac{\partial\mathcal{G}_i(\cdot,0)}{\partial\theta_k}|_{y_{i,\min}} = 0$ for all angular coordinates θ_i, and that $\mathcal{G}_i(\cdot, 0)$ is locally in C^3, we will find points on the interior sphere $x \in S(0, y_{i,\min})$ with $\mathcal{G}_i(x, 0) < \mathcal{G}_i(y_{i,\min}, 0)$ that again contradicts the maximum principle.

Again, in this context, we suppress the dependence of $\mathcal{G}_i$ on i below.

From the expression of the Laplacian locally in polar coordinates, up to a rotation

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[7] If there is any nonzero component of the gradient in the tangential direction, then to first order, one would find points $x_1, x_2 \in S(y_i, y_{i,\min})$ on the interior sphere so that $\mathcal{G}(x_1, y_i) < \mathcal{G}(y_{i,\min}, y_i) < \mathcal{G}(x_2, y_i)$ which contradicts the fact that for all $x \in \Omega_{2s^*_{y_i}, y_i}$, by the maximum principle $\mathcal{G}(y_{i,\min}, y_i) < \mathcal{G}(x, y_i)$.

of the coordinate axes, and noting that all the angular first derivatives are zero at the point $y_{i,\min}$, and all the second derivatives with respect to the angular coordinates are positive, we will have

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\big(r^{n-1}\frac{\partial\mathcal{G}(\cdot,0)}{\partial r}\big)+\mathcal{B}\cdot\hat{r}\frac{\partial\mathcal{G}(,0)}{\partial r}$$
$$=\frac{\partial^2\mathcal{G}(\cdot,0)}{\partial r^2}+\frac{(n-1)}{r}\frac{\partial\mathcal{G}(\cdot,0)}{\partial r}+\mathcal{B}\cdot\hat{r}\frac{\partial\mathcal{G}(,0)}{\partial r}\leqslant 0 \quad (2.64)$$

On the sphere $S(0,y_{i,\min})$, the above inequality is valid at the point $y_{i,\min}$.

For some $b_{i,k}$ to be determined in a later step, we begin the iteration with $r_0=a_{i,k}$ which is the minimum distance to the level set $\mathcal{G}_i(\cdot,0)=M_{b_{i,k}}$ where $M_{b_{i,k}}:=\sup\limits_{S(0,b_{i,k})}\mathcal{G}_i(\cdot,0)$.

We call $K_{i,k}:=\overline{B(0,1/L)}\setminus B(0,a_{i,k})$.When we write $\mathcal{G}(r)$, it will be understood from context that we mean $\mathcal{G}_i(r,0)$. In particular, from the statement of Lemma 2.7 we note that,

$$\Big|\frac{\partial\mathcal{G}_i}{\partial r}\Big|_{a_{i,k}}a_{i,k}^{n-1}=G_{i,k}>0. \quad (2.65)$$

Further, we write,

$$\sup_{K_{i,k}}G_i''=H_{i,k}. \quad (2.66)$$

Now we choose the increment h small enough so that,

$$\Big|\frac{\partial\mathcal{G}_i}{\partial r}\Big|_{a_{i,k}}\leqslant\Big|\frac{\partial\mathcal{G}_i}{\partial r}\Big|_{a_{i,k}}a_{i,k}^{n-1}L^{n-1}=G_{i,k}L^{n-1}\gg h\sup_{K_{i,k}}G_i''=hH_{i,k}. \quad (2.67)$$

First we use the Taylor theorem to second order to get, for any $j\geqslant 0$,

$$\mathcal{G}(r_j+h)=\mathcal{G}(r_j)+h\mathcal{G}'(r_j)+\frac{h^2}{2}\mathcal{G}''(r_j+\theta_j h), \quad (2.68)$$

for some $\theta_j \leqslant 1$. We have that for each $k^{(8)}$, using a version of the Hopf lemma, the radial derivative $\frac{\partial \mathcal{G}}{\partial r} \leqslant 0$ at the point of tangency of each level set $\partial\Omega_k$ with this interior sphere. This holds true for each $j \geqslant 0$ with $a_{i,k} \leqslant r_j \leqslant 1/L$.

At the first step for $j = 0$, using Eq. (2.67), we get from Eq. (2.68), that,

$$\mathcal{G}(r_0 + h) \leqslant G(r_0). \tag{2.69}$$

We then define the length $h_0 \leqslant h$ and the point $\overrightarrow{r_1}$ which is at the minimum distance from the origin to the level set of $\mathcal{G} = \mathcal{G}(r_0+h)$ so that $\mathcal{G}(r_0+h)-\mathcal{G}(r_0) = \mathcal{G}(r_1) - \mathcal{G}(r_1 - h_0)$. where $r_1 - h_0$ is on the radial line joining the origin to r_1. Note that $\mathcal{G}(r_0 + h) = \mathcal{G}(r_1)$ by definition and thus $\mathcal{G}(r_0) = \mathcal{G}(r_1 - h_0)$.

At the j'th step ($j \geqslant 1$), we consider $(r_j + h)\hat{r_j}$ where $\hat{r_j}$ is the unit vector in the direction r_j. Consider $\mathcal{G}((r_j + h)\hat{r})$ and next define the point $\overrightarrow{r_{j+1}}$ which is at the minimum distance from the origin to the level set of $\mathcal{G} = \mathcal{G}((r_j + h)\hat{r_j})$. Henceforth, by $\mathcal{G}(r_j), \mathcal{G}(r_j + h)$, we mean the values of the Green's function at the point $r_j\hat{r_j}$ and $(r_j + h)\hat{r_j}$ respectively , and we omit the unit vector from the notation.

Note that using Eq. (2.67), and the subsequent argument till Eq. (2.95), first for the case $j = 0$, that at each subsequent step we have, $\mathcal{G}(r_j + h) \leqslant \mathcal{G}(r_j)$ for each $j \geqslant 1$, and thus we have for each $j \geqslant 1$ that enables us to define $h_j \leqslant h$, $h_j \leqslant h$ so that $\mathcal{G}(r_j + h) - \mathcal{G}(r_j) = \mathcal{G}(r_{j+1}) - \mathcal{G}(r_{j+1} - h_j)$. where $r_{j+1} - h_j$ is on the radial line joining the origin to r_{j+1}. Note that $\mathcal{G}(r_j + h) = \mathcal{G}(r_{j+1})$ by definition and thus $\mathcal{G}(r_j) = \mathcal{G}(r_{j+1} - h_j)$.

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(8) We only need to consider $k \geqslant s^*_{y_i}$.

We have,

$$\mathcal{G}(r_j) = \mathcal{G}(r_j + h) - h\mathcal{G}^{'}(r_j + h) + \frac{h^2}{2}\mathcal{G}^{''}(r_j + h\theta), \tag{2.70}$$

for some $\theta < 1$.

This can be written, for some other constant $\theta_1 < 1$,

$$\frac{\mathcal{G}(r_j + h) - \mathcal{G}(r_j)}{h} = \mathcal{G}^{'}(r_j + h) - \frac{h}{2}\mathcal{G}^{''}(r_j) - \frac{h^2}{2}\theta\mathcal{G}^{'''}(r_j + \theta_1 h). \tag{2.71}$$

Note that, $\mathcal{G}(r_j+h)-\mathcal{G}(r_j) = \mathcal{G}(r_{j+1})-\mathcal{G}(r_{j+1}-h_j)$, with $0 \leqslant h_j \leqslant (r_{j+1}-r_j) \leqslant h$ by definition. Further, we have,

$$\mathcal{G}(r_{j+1} - h_j) = \mathcal{G}(r_{j+1}) - h_j\mathcal{G}^{'}(r_{j+1}) + \frac{h_j^2}{2}\mathcal{G}^{''}(r_{j+1}) - \frac{h_j^3}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h).$$

Using Eq. (2.64), we further get the inequality,

$$\begin{aligned}\mathcal{G}(r_{j+1} - h_j) - \mathcal{G}(r_{j+1}) &\leqslant -h_j\mathcal{G}^{'}(r_{j+1}) \\ &\quad + \frac{h_j^2}{2}\Big(- \Big(\frac{n-1}{r_{j+1}}\Big) - \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}}\Big)\mathcal{G}^{'}(r_{j+1})\frac{h_j^3}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h).\end{aligned} \tag{2.72}$$

This gives us,

$$\begin{aligned}&\mathcal{G}(r_{j+1} - h_j) - \mathcal{G}(r_{j+1}) \\ &\quad \leqslant h_j\Big(- \mathcal{G}^{'}(r_{j+1})\Big)\Big(1 + \frac{n-1}{2r_{j+1}}h_j + \frac{1}{2}\mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}} h_j\Big) - \frac{h_j^3}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h_j).\end{aligned} \tag{2.73}$$

This simplifies to,

$$\begin{aligned}\mathcal{G}^{'}(r_{j+1}) \leqslant \frac{(-1)}{\big(1 + h_j(\frac{n-1}{2r_{j+1}} + \frac{1}{2}\mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\big)}\Big(&\frac{\mathcal{G}(r_{j+1} - h_j) - \mathcal{G}(r_{j+1})}{h_j} \\ &+ \frac{h_j^2}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h_j)\Big).\end{aligned} \tag{2.74}$$

Up to a first order approximation, this is written assuming that h is arbitrarily small compared with $a_{i,k}$, that,

$$\mathcal{G}^{'}(r_{j+1}) \leqslant \big(1 - h_j(\frac{n-1}{2r_{j+1}} + \frac{1}{2}\mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\big)\Big(\frac{\mathcal{G}(r_{j+1}) - \mathcal{G}(r_{j+1} - h_j)}{h_j}\Big) - \frac{h_j^2}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h_j). \quad (2.75)$$

Further note that,

$$\mathcal{G}^{'}(r_j + h) = \mathcal{G}^{'}(r_j) + h\mathcal{G}^{''}(r_j) + \frac{h^2}{2!}\mathcal{G}^{'''}(r_j + \theta_2 h), \quad (2.76)$$

for some $\theta_2 < 1$. Using the fact that $\mathcal{G}(r_j+h)-\mathcal{G}(r_j) = \mathcal{G}(r_{j+1})-\mathcal{G}(r_{j+1}-h_j) < 0$, and $0 \leqslant h_j \leqslant (r_{j+1} - r_j) \leqslant h$, and h_j sufficiently small, we get from Eq. (2.75), that,

$$\mathcal{G}^{'}(r_{j+1}) \leqslant \big(1 - h_j(\frac{n-1}{2r_{j+1}} + \frac{1}{2}\mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\big)\Big(\frac{\mathcal{G}(r_j + h) - \mathcal{G}(r_j)}{h}\Big) + \frac{h_j^2}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h_j). \quad (2.77)$$

Further using Eqs. (2.71) and (2.76), we get,

$$\mathcal{G}^{'}(r_{j+1}) \leqslant \big(1-h_j(\frac{n-1}{2r_{j+1}}+\frac{1}{2}\mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})\big)\Big(\mathcal{G}^{'}(r_j)+\frac{h}{2}\mathcal{G}^{''}(r_j)+\frac{h^2}{2!}\mathcal{G}^{'''}(r_j+\theta_2 h) - \frac{h^2}{2}\mathcal{G}^{'''}(r_j + \theta_1 h)\Big) + \frac{h_j^2}{3!}\mathcal{G}^{'''}(r_{j+1} - \theta h_j). \quad (2.78)$$

Thus we get,

$$\mathcal{G}^{'}(r_{j+1}) \leqslant \mathcal{G}^{'}(r_j) + \frac{h}{2}\mathcal{G}''(r_j) - h_j(\frac{n-1}{2r_{j+1}} + \frac{1}{2}\mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\mathcal{G}'(r_j) + O\Big(\frac{h^2}{2}\mathcal{G}'''(r_j + \theta_1 h) + \frac{h_j^2}{2}\mathcal{G}'''(r_{j+1} + \theta_1 h_j) + \frac{h^2}{2}\mathcal{G}''(r_j)(\frac{n-1}{2r_{j+1}} + \frac{1}{2}\mathcal{B} \cdot \hat{r})\Big), \quad (2.79)$$

for some constants θ_1, θ_2, and so we have, noting that $\mathcal{G}'(r_j) \leqslant 0$, $h_j \leqslant h$, using Eq. (2.64),

$$\begin{aligned} \mathcal{G}^{'}(r_{j+1}) \leqslant \mathcal{G}^{'}(r_j) - \frac{h}{2}\Big(\frac{n-1}{r_j} + \mathcal{B}(r_j)\cdot\hat{r_j}\Big)\mathcal{G}^{'}(r_j) - \frac{h}{2}\Big(\frac{n-1}{r_{j+1}} + \mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}}\Big)\mathcal{G}'(r_j) \\ + O(\frac{h^2}{2}\mathcal{G}'''(r_j + \theta_1 h) + \frac{h_j^2}{2}\mathcal{G}'''(r_{j+1} + \theta_1 h_j) + \frac{h^2}{r_{j+1}}\mathcal{G}''(r_j)). \quad (2.80) \end{aligned}$$

This gives us,

$$\begin{aligned} \mathcal{G}'(r_{j+1}) \leqslant \mathcal{G}'(r_j) - h\Big(\frac{n-1}{2}\Big(\frac{1}{r_j} + \frac{1}{r_{j+1}}\Big) + \frac{1}{2}(\mathcal{B}(r_j)\cdot\hat{r_j} + \mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})\Big)\mathcal{G}'(r_j) \\ + O(\frac{h^2}{2}\mathcal{G}'''(r_j + \theta_1 h) + \frac{h_j^2}{2}\mathcal{G}'''(r_{j+1} + \theta_1 h_j) + \frac{h^2}{r_{j+1}}\mathcal{G}''(r_j)). \quad (2.81) \end{aligned}$$

We note that $r_{j+1} - r_j \leqslant h$, and thus get,

$$\begin{aligned} \mathcal{G}'(r_{j+1}) \leqslant \mathcal{G}'(r_j) - h\Big(\frac{n-1}{r_j} + \frac{1}{2}(\mathcal{B}(r_j)\cdot\hat{r_j} + \mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})\Big)\mathcal{G}'(r_j) \\ + O\Big(\frac{h^2}{2}\mathcal{G}'(r_j) + \frac{h^2}{2}\mathcal{G}'''(r_j + \theta_1 h) + \frac{h_j^2}{2}\mathcal{G}'''(r_{j+1} + \theta_1 h_j) + \frac{h^2}{r_{j+1}}\mathcal{G}''(r_j)\Big). \\ (2.82) \end{aligned}$$

Here we note that the points r_j, r_{j+1} by construction could be far apart depending on the shape of the level sets of the Green's function. Thus we have that,

Now also note by construction that, $0 \leqslant h_j \leqslant r_{j+1} - r_j \leqslant h$, and we have,

$$\begin{aligned} \mathcal{G}(r_{j+1} - h_j) - \mathcal{G}(r_{j+1}) = -\mathcal{G}'(r_{j+1})h_j + \frac{h_j^2}{2}\mathcal{G}''(r_{j+1} + \theta_1 h_j)), \\ \mathcal{G}(r_j) - \mathcal{G}(r_j + h) = -\mathcal{G}'(r_j)h + \frac{h^2}{2}\mathcal{G}''(r_j + \theta_2 h)), \quad (2.83) \end{aligned}$$

Thus we have,

$$\mathcal{G}'(r_{j+1})h_j = \mathcal{G}'(r_j)h - \frac{h_j^2}{2}\mathcal{G}''(r_{j+1} + \theta_1 h_j) + \frac{h^2}{2}\mathcal{G}''(r_j + \theta_2 h), \quad (2.84)$$

and so we have,

$$|\mathcal{G}'(r_{j+1})| = \frac{h}{h_j}\big(|\mathcal{G}'(r_j)| + \frac{h}{2}\mathcal{G}''(r_j + \theta_2 h)\big) + \frac{h_j}{2}\mathcal{G}''(r_{j+1} + \theta_1 h_j)), \quad (2.85)$$

We consider two cases. First, we consider,

$$(a) : |\mathcal{G}'(r_j)|(r_{j+1} - r_j) \geqslant |\mathcal{G}'(r_j)|h_j \geqslant h|\mathcal{G}'(r_j)| - 2h^2 \sup_{K_k} |\mathcal{G}''(r)|. \qquad (2.86)$$

Then, we have

$$\begin{aligned}&(r_{j+1} - r_j)\Big(\frac{n-1}{r_j} + \frac{1}{2}(\mathcal{B}(r_j) \cdot \hat{r_j} + \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\Big)|\mathcal{G}'(r_j)| \\ &\geqslant h\Big(\frac{n-1}{r_j} + \frac{1}{2}(\mathcal{B}(r_j) \cdot \hat{r_j} + \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\Big)|\mathcal{G}'(r_j)| \\ &\qquad - O(h^2 \sup_{K_k} |\mathcal{G}''(r)|(\mathcal{B}(r_j) \cdot \hat{r_j} + \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\Big). \qquad (2.87)\end{aligned}$$

We have uniform bounds on the drift term in the domain K_k, so we can choose h small enough so that the second term on the right is negligible at each step of the iteration in K_k, and thus we get from Eqs. (2.84) and (2.87), noting that by construction, for each j, we have $\mathcal{G}'(r_j) < 0$, that,

$$\begin{aligned}&\mathcal{G}'(r_{j+1}) \leqslant \mathcal{G}'(r_j) - (r_{j+1} - r_j)\Big(\frac{n-1}{r_j} + \frac{1}{2}(\mathcal{B}(r_j) \cdot \hat{r_j} + \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\Big)\mathcal{G}'(r_j) \\ &+O(h^2) = \mathcal{G}'(r_j) - (r_{j+1} - r_j)\Big(\frac{n-1}{r_j} + \frac{1}{2}(\mathcal{B}(r_j) \cdot \hat{r_j} + \mathcal{B}(r_{j+1}) \cdot \widehat{r_{j+1}})\Big)\mathcal{G}'(r_j) + O(h^2). \\ &\qquad (2.88)\end{aligned}$$

On the other hand, when we have the inequality opposite of Eq. (2.86), we get,

$$\begin{aligned}&(b) : |\mathcal{G}'(r_j)|h_j < h|\mathcal{G}'(r_j)| - 2h^2 \sup_{K_k} |\mathcal{G}''(r)| \\ &\qquad \Leftrightarrow h|\mathcal{G}'(r_j)| > |\mathcal{G}'(r_j)|h_j + 2h^2 \sup_{K_k} |\mathcal{G}''(r)|, \qquad (2.89)\end{aligned}$$

with h small enough. We use the binomial approximation to first order to get from Eq. (2.85),

$$|\mathcal{G}'(r_{j+1})| \geqslant |\mathcal{G}'(r_j)| + \frac{2h^2}{h_j} \sup_{K_k} |\mathcal{G}''(r)|$$

$$+\frac{h^2}{2h_j}\mathcal{G}''(r_j+\theta_2 h)\big)+\frac{h_j}{2}\mathcal{G}''(r_{j+1}+\theta_1 h_j)). \tag{2.90}$$

From here we easily see, for h small enough, that,

$$\mathcal{G}'(r_{j+1}) \leqslant \mathcal{G}'(r_j). \tag{2.91}$$

Further, the decay of the Green's function in either case is given for each j by,

$$\mathcal{G}(r_{j+1}) = \mathcal{G}(r_j+h) = \mathcal{G}(r_j) + h\mathcal{G}^{'}(r_j) + \frac{h^2}{2}\mathcal{G}^{''}(r_j+\theta h). \tag{2.92}$$

At each step, we choose h small enough, with $(r_{j+1}-r_j) \leqslant h$, so that we can write the right hand side of Eq. (4.42) ,

$$\begin{aligned}
&\mathcal{G}'(r_{j+1})\\
&\leqslant \mathcal{G}'(r_j)\Big(1-\frac{(n-1)(r_{j+1}-r_j)}{r_j}\Big)\Big(1-(r_{j+1}-r_j)\frac{1}{2}(\mathcal{B}(r_j)\cdot\hat{r_j}+\mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})\Big)\Big)\\
&+O((r_{j+1}-r_j))^2)\\
&\leqslant \mathcal{G}'(r_j)\Big(1-\frac{(n-1)(r_{j+1}-r_j)}{r_j}\Big)\Big(e^{-(r_{j+1}-r_j)\frac{1}{2}(\mathcal{B}(r_j)\cdot\hat{r_j}+\mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})}\Big)+O(h^2).
\end{aligned} \tag{2.93}$$

Iterating this inequality k times, we get an error term $kO(h^2)$ for $\mathcal{G}'(r_{j+k})$.

We choose N large enough so that $Nh \leqslant \Big(1/L - y_{i,\min}\Big) \approx 1/L$. By construction, the equality case for the left hand inequality here is achieved when we have for each j, that $r_{j+1}-r_j = h$.

In this case, the error up to the order of $O(h^2)$ in any of the derivatives considered in either of Eqs. (2.91) and (2.93) at the j'th step with $k \leqslant N$ is $kO(h^2)$. Thus, further, the error in the quantity $|\mathcal{G}'(r_t)|h \lesssim kO(h^3)$, for any t. Thus, the total error in the change of the Green's function is $\sum_{k=1}^{N} kO(h^3) \approx N^2O(h^3) \approx h\cdot(1/L)^2 \to 0$ as $h \to 0$.

Thus the solution to the inequality

$$\mathcal{G}'(r_{j+1}) \leqslant \mathcal{G}'(r_j)\Big(1 - \frac{(n-1)(r_{j+1}-r_j)}{r_j}\Big)\Big(e^{-(r_{j+1}-r_j)\frac{1}{2}(\mathcal{B}(r_j)\cdot\hat{r_j}+\mathcal{B}(r_{j+1})\cdot\widehat{r_{j+1}})}\Big), \tag{2.94}$$

as $h \to 0$, is given by the following differential inequality,

$$\frac{\partial^2 \mathcal{G}(\cdot,0)}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial \mathcal{G}(\cdot,0)}{\partial r} + \mathcal{B}\cdot\hat{r}\frac{\partial \mathcal{G}(,0)}{\partial r} \leqslant 0. \tag{2.95}$$

(9) Using the cases where the above differential inequality Eq. (2.94) holds, along with the cases where we have Eq. (2.91) holding, we now have,

Lemma 2.8. *For any $a_{i,k_0} \leqslant r_1 \leqslant r_2$, noting also that for each r , $\frac{\partial \mathcal{G}}{\partial r} < 0$, we get that*

$$\mathcal{G}(r_1,0) - \mathcal{G}(r_2,0) \geqslant 2C\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_{i,k}} a_{i,k}^{n-1}\Big(\frac{1}{r_1^{n-2}} - \frac{1}{r_2^{n-2}}\Big). \tag{2.97}$$

Specifically we choose $a_{i,k} < r_1 = y_{i,\min} \leqslant r_2 = \frac{\delta_\Omega(y_i)}{L}$, and get,

$$2s_i^* - \frac{1}{C_1}s_i^* \geqslant \frac{1}{C_1}\mathcal{G}(y_{i,\min}) - \mathcal{G}(\frac{\delta_\Omega(y_i)}{L}\widehat{s_{min}}) \geqslant C\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_{i,k}} a_{i,k}^{n-1}\Big(\frac{1}{y_{i,\min}^{n-2}} - L^{n-2}\Big). \tag{2.98}$$

Here, C_1 is a uniform constant arising from the Harnack inequality. In case we have that $y_{i,\min} \to 0$ as $i \to \infty$, then choosing i a posteriori large enough, we get from above that,

$$s_i^* \gtrsim C'\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_{i,k}} a_{i,k}^{n-1} \cdot \frac{1}{y_{i,\min}^{n-2}} \tag{2.99}$$

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(9) If we integrate this, to get for $r \geqslant a_{i,k_0}$, that,

$$\frac{\partial \mathcal{G}}{\partial r}\Big|_r \leqslant C\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_{i,k}}\Big(\frac{a_{i,k}^{n-1}}{r^{n-1}}\Big), \tag{2.96}$$

where the constant C incorporates the exponential factor contribution from the drift term.

for some uniform constant C'.

Now instead of looking the points of minimum of the level sets of the Green's function, we look at the corresponding points of maximum of the level sets. For these points of maximum, we have,

$$\frac{\partial^2 \mathcal{G}_i(\cdot,0)}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial \mathcal{G}_i(\cdot,0)}{\partial r} + \mathcal{B}\cdot\hat{r}\frac{\partial \mathcal{G}_i(\cdot,0)}{\partial r} \geqslant 0, \tag{2.100}$$

We first carry through a general calculation analogous to Eq. (2.70) through Eq. (2.83), for the maximum points. In this case, consider any maximum point $s_i\hat{s_i}$, where $\hat{s_i} = \frac{\vec{s_i}}{|s_i|}$, and the level set $\mathcal{G}(\vec{s_i})$. For any small enough t to be determined later, consider the point $(s_i - t)\hat{s_i}$. and consider the level set of $\mathcal{G}((s_i - t)\hat{s_i})$ and the maximum point $\vec{s_{i+1}}$ of this level set. As before, we remove the reference to the radial unit vector from now on, for simplicity in notation. We enumerate domains $\Omega_m := \{x \in \Omega : \mathcal{G}(x) \leqslant 1/m\}$, $m \geqslant 1$, and take the sequence $m \to \infty$ in the final step of the argument. We also define, for each m, $\Omega'_m := \Omega_m \setminus B(0, b_{i,k_0})$, where $b_{i,k_0} \leqslant y_{i,\min}$ is a maximum point chosen from Lemma 2.7, with $a_{i,k_0} \leqslant y_{i,\min}$ being the corresponding minimum point chosen from the lemma.

First we assume that,

$$\mathcal{G}(s_i - t) = \mathcal{G}(s_i) - t\mathcal{G}'(s_i) + \frac{t^2}{2}\mathcal{G}''(s_i - \theta t), \tag{2.101}$$

for some $\theta \leqslant 1$. In this case we choose $\beta_m := 2\sup_{\Omega'_m} |\mathcal{G}''(r)|$. We first consider the case, where $|\mathcal{G}'(s_i)| > K_m t$, for some $K_m \gg \beta_m$.$^{(10)}$ In that case, it is clear that

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(10) Note that this condition also appears prior to Lemma 2.10 later on.

we must have,

$$\mathcal{G}(s_i - t) \geqslant \mathcal{G}(s_i). \tag{2.102}$$

We then define the length t_i so that, $\mathcal{G}(s_{i+1} + t_i) = \mathcal{G}(s_i)$. By construction, we have, $0 \leqslant t_i \leqslant s_i - s_{i+1} \leqslant t$. Further, we have, $\mathcal{G}(s_i - t) = \mathcal{G}(s_{i+1})$. We can carry through calculations exactly analogous to Eq. (2.70) through Eq. (2.83) here. The case where we have $|\mathcal{G}'(s_i)| \leqslant K_m t$ is considered in Lemmas 2.10 and 2.11 and the argument following Lemma 2.11.

Lemma 2.9. *Given a large positive integer* m*, for any level set* $\{x : \mathcal{G}(x) = \mathcal{G}(\vec{s_i})\}$ *with the value* $\mathcal{G}(\vec{s_i}) \geqslant 1/m$*, with* $|\mathcal{G}'(s_i)| > K_m t$*, where* $\vec{s_i}$ *is the maximum point of this level set, with* $|\vec{s_i}| \geqslant b_{i,k_0}$ *, for* s_{i+1} *defined above, with* t *arbitrarily small in comparison to* b_{i,k_0} *as well as arbitrarily small in comparison to* $\inf\{\mathit{dist}(x, \partial\Omega) : \mathcal{G}(x) = 1/m\}$*, we have,*

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \mathcal{G}^{'}(s_i) + t\Big(\frac{n-1}{s_i} + \frac{1}{2}\mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}}) + \frac{1}{2}\mathcal{B}(s_i) \cdot \widehat{s_i})\Big)\mathcal{G}^{'}(s_i) + O_m(t^2). \tag{2.103}$$

The condition that t be arbitrarily small in comparison to b_{i,k_0} and $\inf\{\mathrm{dist}(x, \partial\Omega) : \mathcal{G}(x) = 1/m\}$ ensures that t/s_i and $t\mathcal{B}(s_i) \cdot \widehat{s_{i+1}}$ are arbitrarily small.

Proof. We have,

$$\mathcal{G}(s_i) = \mathcal{G}(s_i - t) + t\mathcal{G}'(s_i - t) + \frac{t^2}{2}\mathcal{G}''(s_i - t\theta'). \tag{2.104}$$

for some $\theta' < 1$. We write this, with some other constant θ_1, that,

$$\frac{\mathcal{G}(s_i) - \mathcal{G}(s_i - t)}{t} = \mathcal{G}'(s_i - t) + \frac{t}{2}\mathcal{G}''(s_i) - \frac{t^2}{2}\theta_1'\mathcal{G}'''(s_i - \theta_1 t). \tag{2.105}$$

Further, we get,

$$\mathcal{G}(s_{i+1}+t_i)=\mathcal{G}(s_{i+1})+t_i\mathcal{G}^{'}(s_{i+1})+\frac{t_i^2}{2}\mathcal{G}^{''}(s_{i+1})+\frac{t_i^3}{3!}\mathcal{G}^{'''}(s_{i+1}+\theta t). \tag{2.106}$$

Using Eq. (2.100), we get from above that,

$$\mathcal{G}(s_{i+1}+t_i)-\mathcal{G}(s_{i+1})\geqslant t_i\mathcal{G}^{'}(s_{i+1})+\frac{t_i^2}{2}\Big(-\Big(\frac{n-1}{s_{i+1}}\Big)-\mathcal{B}(s_{i+1})\cdot\widehat{s_{i+1}}\Big)\mathcal{G}^{'}(s_{i+1}) + \frac{t_i^3}{3!}\mathcal{G}^{'''}(s_{i+1}-\theta t)\Big). \tag{2.107}$$

This gives us,

$$\mathcal{G}(s_{i+1}+t_i)-\mathcal{G}(s_{i+1})\geqslant t_i\Big(\mathcal{G}^{'}(s_{i+1})\Big)\Big(1-\frac{n-1}{2s_{i+1}}t_i-\frac{1}{2}\mathcal{B}(s_{i+1})\cdot\widehat{s_{i+1}}t_i\Big) - \frac{t_i^3}{3!}\mathcal{G}^{'''}(s_{i+1}-\theta t_i). \tag{2.108}$$

This simplifies, noting that, $\mathcal{G}'(s_{i+1})\leqslant 0$, and that $t_i\leqslant t$ has been chosen arbitrarily small in comparison to $\inf\limits_{\Omega'_m}\operatorname{dist}(x,\partial\Omega)$,

$$\mathcal{G}^{'}(s_{i+1})\leqslant\frac{1}{\big(1-t_i(\frac{n-1}{2s_{i+1}}+\frac{1}{2}\mathcal{B}(s_{i+1})\cdot\widehat{s_{i+1}})\big)}\Big(\frac{\mathcal{G}(s_{i+1}+t_i)-\mathcal{G}(s_{i+1})}{t_i} + \frac{t_i^2}{3!}\mathcal{G}^{'''}(s_{i+1}+\theta t_i)\Big). \tag{2.109}$$

Using the binomial approximation, we have,

$$\mathcal{G}^{'}(s_{i+1})\leqslant\big(1+t_i(\frac{n-1}{2s_{i+1}}+\frac{1}{2}\mathcal{B}(s_{i+1})\cdot\widehat{s_{i+1}})\big)\Big(\frac{\mathcal{G}(s_{i+1}+t_i)-\mathcal{G}(s_{i+1})}{t_i}+\frac{t_i^2}{3!}\mathcal{G}^{'''}(s_{i+1}+\theta t_i)\Big). \tag{2.110}$$

Further note,

$$\mathcal{G}^{'}(s_i-t)=\mathcal{G}^{'}(s_i)-t\mathcal{G}^{''}(s_i)+\frac{t^2}{2!}\mathcal{G}^{'''}(s_i-\theta_2 t), \tag{2.111}$$

for some $\theta_2 < 1$. Using the fact that, $\mathcal{G}(s_i - t) - \mathcal{G}(s_i) = \mathcal{G}(s_{i+1}) - \mathcal{G}(s_{i+1} + t_i) > 0$, and that $0 \leqslant t_i \leqslant s_i - s_{i+1} \leqslant t$, and $t_i \leqslant t$ sufficiently small, we get from Eq. (2.110), that,

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \big(1 + t_i(\frac{n-1}{2s_{i+1}} + \frac{1}{2}\mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}})\big)\Big(\frac{\mathcal{G}(s_i) - \mathcal{G}(s_i - t)}{t} + \frac{t_i^2}{3!}\mathcal{G}^{'''}(s_{i+1} + \theta t_i)\Big). \tag{2.112}$$

Using Eq. (2.105), we get,

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \big(1 + t_i(\frac{n-1}{2s_{i+1}} + \frac{1}{2}\mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}})\big)\Big(\mathcal{G}'(s_i - t) + \frac{t}{2}\mathcal{G}''(s_i) - \frac{t^2}{2}\theta_1'\mathcal{G}'''(s_i - \theta_1 t) + \frac{t_i^2}{3!}\mathcal{G}^{'''}(s_{i+1} + \theta t_i)\Big). \tag{2.113}$$

Now using Eq. (2.111), we get,

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \big(1 + t_i(\frac{n-1}{2s_{i+1}} + \frac{1}{2}\mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}})\big)\Big(\mathcal{G}^{'}(s_i) - \frac{t}{2}\mathcal{G}''(s_i) + \frac{t^2}{2!}\mathcal{G}^{'''}(s_i - \theta_2 t) - \frac{t^2}{2}\theta_1'\mathcal{G}'''(s_i - \theta_1 t) + \frac{t_i^2}{3!}\mathcal{G}^{'''}(s_{i+1} + \theta t_i)\Big). \tag{2.114}$$

So we get, using also Eq. (2.100), that,

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \mathcal{G}^{'}(s_i) + \frac{t_i}{2}\Big(\frac{n-1}{s_{i+1}} + \mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}}\Big)\mathcal{G}^{'}(s_i) + \frac{t}{2}\Big(\frac{n-1}{s_i} + \mathcal{B}(s_i) \cdot \widehat{s_i}\Big)\mathcal{G}^{'}(s_i) + O(t^2), \tag{2.115}$$

where the implied constant for the $O(\cdot)$ term depends on the domain $x : \mathcal{G}(x) \leqslant \mathcal{G}(s_i)$. Noting that $t_i \leqslant t$, and that $s_{i+1} = s_i - O(t)$, we get,

$$\mathcal{G}^{'}(s_{i+1}) \leqslant \mathcal{G}^{'}(s_i) + t\Big(\frac{n-1}{s_i} + \frac{1}{2}\mathcal{B}(s_{i+1}) \cdot \widehat{s_{i+1}}) + \frac{1}{2}\mathcal{B}(s_i) \cdot \widehat{s_i}\Big)\mathcal{G}^{'}(s_i) + O(t^2). \tag{2.116}$$

∎

For a fixed domain Ω'_m, we enumerate the radial distances from y_i to the points of maximum as $\{u_j\}_{j=1}^{\infty}$, with $u_1 = b_{i,k_0}$ and the radial increments as t, where t is arbitrarily small in comparison to b_{i,k_0} as well as arbitrarily small in comparison to $d_m := \min\{\mathrm{dist}(x, \partial\Omega) : x \in \Omega'_m\}$.

Let u_{j+1} be the point of maximum of the level set $\mathcal{G}(u_j+t)$. In case that $\mathcal{G}(u_j+t) < 1/m$, we terminate with a value of $t^* \leqslant t$ and repeat the same argument.

We then start the iteration from u_{j+1}, and iterate with decrements of length t, using Lemma 2.9. We write $v_{j,1} = u_{j+1}$, and then $v_{j,2}$ being the point of maximum of the level set $\mathcal{G}(v_{j,1} - t)$. In this case, we have $\mathcal{G}(v_{j,2} + t_1) = \mathcal{G}(v_{j,1})$ and by definition $t_1 \leqslant t$. For all $n \geqslant 2$, we define the $v_{j,n}$ and t_n analogously. By definition, we have, $0 \leqslant t_n \leqslant (v_{j,n} - v_{j,n+1}) \leqslant t$.

In this case, we get, by an argument similar to Eq. (2.84), that,

$$|\mathcal{G}'(v_{j,n+1})| = \frac{t}{t_n}\Big(|\mathcal{G}'(v_{j,n})| + \frac{t}{2}\mathcal{G}''(v_{j,n} + \theta_2 t)\Big) - \frac{t_n}{2}\mathcal{G}''(v_{j,n+1} + \theta_1 t'_j). \tag{2.117}$$

We consider the two separate cases; first when

$$(a') : |\mathcal{G}'(v_{j,n})|t_n \leqslant |\mathcal{G}'(v_{j,n})|t - \Big(\frac{(n-1)}{b_{i,k_0}} + \frac{M}{d_m^{1-\beta}}\Big)|\mathcal{G}'(v_{j,n})| + 2\sup_{\Omega'_m}|\mathcal{G}''(r)|\Big)t^2. \tag{2.118}$$

In this case, we clearly have from Eq. (2.117), using that $t_n \leqslant t$, that,

$$\begin{aligned}|\mathcal{G}'(v_{j,n+1})| \geqslant{}& |\mathcal{G}'(v_{j,n})|\Big(1 + \Big(\frac{(n-1)}{b_{i,k_0}}\Big)t + \frac{M}{d_m^{1-\beta}}t\Big) + 2\sup_{\Omega'_m}|\mathcal{G}''(r)|\frac{t^2}{t_n} \\ &+ \frac{t^2}{t_n}\mathcal{G}''(v_{j,n} + \theta_2 t) - \frac{t_n}{2}\mathcal{G}''(v_{j,n+1} + \theta_1 t'_j) \\ &\geqslant |\mathcal{G}'(v_{j,n})|\Big(1 + \frac{(n-1)}{b_{i,k_0}}t + \frac{M}{d_m^{1-\beta}}t\Big).\end{aligned} \tag{2.119}$$

On the other hand, when we have the inequality opposite to Eq. (2.118),

$$(b') : \Big(\Big(\frac{(n-1)}{b_{i,k_0}} + \frac{M}{d_m^{1-\beta}}\Big)|\mathcal{G}'(v_{j,n})| + 2\sup_{\Omega'_m}|\mathcal{G}''(r)|\Big)t^2 + |\mathcal{G}'(v_{j,n})|t_n > |\mathcal{G}'(v_{j,n})|t$$

$$\Leftrightarrow -\Big(\Big(\frac{(n-1)}{b_{i,k_0}} + \frac{M}{d_m^{1-\beta}}\Big)|\mathcal{G}'(v_{j,n})| + 2\sup_{\Omega'_m}|\mathcal{G}''(r)|\Big)t^2 + \mathcal{G}'(v_{j,n})t_n < \mathcal{G}'(v_{j,n})t \tag{2.120}$$

then for the sequence $v_{j,n}|_{n\geqslant 1}$ for this fixed j, noting that $t_n \leqslant (v_{j,n} - v_{j,n+1}) \leqslant t$ arbitrarily small, we see first using Lemma 2.9 , that,

$$\mathcal{G}^{'}(v_{j,n+1}) \leqslant \mathcal{G}^{'}(v_{j,n}) + t\Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1})\cdot\widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n})\cdot\widehat{v_{j,n}}\Big)\mathcal{G}^{'}(v_{j,n}) + O_m(t^2). \tag{2.121}$$

Here we have two separate cases, once when,

$$(*) : \Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1})\cdot\widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n})\cdot\widehat{v_{j,n}}\Big) < 0,$$

when, using Eq. (2.121), and substituting the expression for $t\mathcal{G}'(v_{j,n})$ from Eq. (2.120), that,

$$\mathcal{G}^{'}(v_{j,n+1}) \leqslant \mathcal{G}^{'}(v_{j,n}) + t_n\Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1})\cdot\widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n}\cdot\widehat{v_{j,n}})\Big)\mathcal{G}^{'}(v_{j,n}) + O_m(t^2) \leqslant \mathcal{G}^{'}(v_{j,n}) + (v_{j,n} - v_{j,n+1})\Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1})\cdot\widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n})\cdot\widehat{v_{j,n}}\Big)\mathcal{G}^{'}(v_{j,n}) + O_m(t^2). \tag{2.122}$$

On the other hand, as opposed to (*) when we have,

$$(**) : \Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1})\cdot\widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n})\cdot\widehat{v_{j,n}}\Big) \geqslant 0,$$

then we get directly from Eq. (2.121), using $-t \leqslant -(v_{j,n} - v_{j,n+1})$,

$$\mathcal{G}^{'}(v_{j,n+1}) \leqslant \mathcal{G}^{'}(v_{j,n}) + (v_{j,n} - v_{j,n+1})\Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1}) \cdot \widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n}) \cdot \widehat{v_{j,n}}\Big)\mathcal{G}^{'}(v_{j,n}) + O_m(t^2). \quad (2.123)$$

Momentarily ignoring the second order term above, we get from Eq. (2.123) the reversed differential inequality, when $t \to 0$, that

$$\frac{\partial^2 \mathcal{G}_i(\cdot, 0)}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial \mathcal{G}_i(\cdot, 0)}{\partial r} + \mathcal{B} \cdot \hat{r}\frac{\partial \mathcal{G}_i(, 0)}{\partial r} \geqslant 0. \quad (2.124)$$

This calculation was performed for a fixed m as defined earlier.

Define $\alpha_m := \left(\frac{(n-1)}{b_{i,k_0}} + \frac{M}{d_m^{1-\beta}}\right)$, and $\beta_m := 2 \sup\limits_{\Omega'_m} |\mathcal{G}''(r)|$, then, Eq. (2.120) can also be written as,

$$(b'') : |\mathcal{G}'(v_{j,n})| t_n \geqslant |\mathcal{G}'(v_{j,n})| t - \alpha_m |\mathcal{G}'(v_{j,n})| t^2 - \beta_m t^2. \quad (2.125)$$

In the case that $|\mathcal{G}'(u_{j+1})| = |\mathcal{G}'(v_{j,1})| \geqslant K_m t$, for some K_m, dependent on m, to be chosen sufficiently large in comparison to β_m, we get for $r = 1$, from the condition (b") of Eq. (2.125), that,

$$t_1 \geqslant t\big(1 - \alpha_m t - \frac{\beta_m}{|\mathcal{G}'(v_{j,1})|} t\big) \geqslant t\big(1 - \frac{\beta_m}{K_m} - \alpha_m t)\big). \quad (2.126)$$

On the other hand, when, $|\mathcal{G}'(u_{j+1})| = |\mathcal{G}'(v_{j,1})| \leqslant K_m t$, then $|\mathcal{G}'(u_{j+1})| t \leqslant K_m t^2$. For any $n \geqslant 1$, in the situation of Eq. (2.125), we get as deduced in Eq. (2.123), that,

$$|\mathcal{G}^{'}(v_{j,n+1})| \geqslant |\mathcal{G}^{'}(v_{j,n})| + (v_{j,n} - v_{j,n+1})\Big(\frac{n-1}{v_{j,n}} + \frac{1}{2}\mathcal{B}(v_{j,n+1}) \cdot \widehat{v_{j,n+1}} + \frac{1}{2}\mathcal{B}(v_{j,n}) \cdot \widehat{v_{j,n}}\Big)|\mathcal{G}^{'}(v_{j,n})| + O_m(t^2). \quad (2.127)$$

On the other hand, note that in the corresponding case (a') of Eq. (2.119), there is no $O_m(t^2)$ term on the right side of the inequality.

We define the Riemann sum, $\Delta_t(u_j, u_{j+1}) = \sup\limits_{w(t,u_j,u_{j+1})} \sum\limits_{p=0}^{N(t,u_j,u_{j+1})} t|\mathcal{B}(w_p)|$, where the supremum is taken over all sequences $w(t, u_j, u_{j+1})$, where $w_0 = u_j$ and $w_{N(t,u_j,u_{j+1})} = u_{j+1}$, and for each $p \geqslant 1$, we have $|w_p| = |w_{p-1}| + t$.

Then define $\Delta(u_j, u_{j+1}) = \limsup\limits_{t\to 0} \Delta_t(u_j, u_{j+1})$. More generally, one defines for any $a, b \in \Omega$ with $|a| < |b|$, the term $\Delta(a, b) = \limsup\limits_{t\to 0} \Delta_t(a, b)$ in an identical manner.

Now, we show,

Lemma 2.10. *For all t arbitrarily small in comparison to $\alpha_m^{-1}, \beta_m^{-1}$, when $|\mathcal{G}'(v_{j,1})| = |\mathcal{G}'(u_{j+1})| > K_m t$, for any $j \geqslant 1$, with $K_m \gg \beta_m$, there exist constants $c, \Delta(\Omega)$, so that, $K'_m = e^{-\Delta(\Omega)} K_m$, so that for all $n \geqslant 2$, we have $|\mathcal{G}'(v_{j,n})| > K'_m t$.*

We have to ensure that through successive iterations of Eq. (2.127) , the $O_m(t^2)$ term does not dominate the main term on the right of Eq. (2.127).

Proof of Lemma 2.10. Note that for any $n \geqslant 1$, whenever we are in the regime of case (a') of Eq. (2.118), then from Eq. (2.119), we clearly get $|\mathcal{G}'(v_{j,n+1})| > |\mathcal{G}'(v_{j,n})|\Big(1 + \frac{(n-1)}{b_{i,k_0}} t + \frac{M}{d_m^{1-\beta}} t\Big)$. In the case we are in the regime of (b') and thus Eq. (2.127), first note that, $|v_{j,r}| - |v_{j,r+1}| \geqslant t_r$, for any $r \geqslant 1$. For any $1 \leqslant r < n$, using induction, we note that in the regime of (b"), the condition of Eq. (2.126) holds with K_m replaced with K'_m, i.e., $t \geqslant t_r \geqslant t\big(1 - \beta_m/K'_m - t\alpha_m\big)$. Thus t_r is approximable by t when $\beta_m/K'_m \ll 1$, and the number of times the $O_m(t^2)$ term is applied for the j'th iteration, is upper bounded by $\lesssim N_{j,n} < (|u_{j+1}| - |v_{j,n}|)/t_r \approx (|u_{j+1}| - |v_{j,n}|)/t$ where one ignores corrections of the order of $O(t^3)$ or higher.

Thus, the total contribution of the error term is bounded by $N_{j,n} O_m(t^2) \lesssim$

(diamΩ)t. The main contribution now comes from either Eq. (2.119) or the main term of the right of Eq. (2.127), and one gets, when t is small enough, by using the same argument as in Eq. (2.93)(with the reverse inequality), noting that $|v_{j,1}| > |v_{j,n}|$,

$$\begin{aligned} |\mathcal{G}'(v_{j,n})| &\geqslant |\mathcal{G}'(v_{j,1})| \frac{|v_{j,1}|^{n-1}}{|v_{j,n}|^{n-1}} e^{-c\Delta(v_{j,n},v_{j,1})} - C(\mathrm{diam}\Omega)t + O_m(t^2) \\ &\geqslant K_m t e^{-c\Delta(v_{j,n},v_{j,1})} - C(\mathrm{diam}\Omega)t + O_m(t^2) > \frac{1}{2} K_m t e^{-c\Delta(v_{j,n},v_{j,1})}. \end{aligned} \tag{2.128}$$

We note that the drift is integrable, and so we have, $\Delta(a,b) < \Delta$ for any $a,b \in \Omega$ and for some uniform $\Delta \geqslant 0$, and it is enough to choose $K'_m \approx K_m e^{-c\Delta}$. ∎

We can show that,

Lemma 2.11. *When* $|\mathcal{G}'(v_{j,1})| = |\mathcal{G}'(u_{j+1})| \geqslant K_m t$, *we get,*

$$\Big|\mathcal{G}'(u_j)\Big| \geqslant \Big|\mathcal{G}'(u_{j+1})\Big| \Big(\frac{|u_{j+1}|^{n-1}}{|u_j|^{n-1}}\Big) e^{-c\Delta(u_j,u_{j+1})}. \tag{2.129}$$

Proof. This result will follow from repeatedly using Eqs. (2.119) and (2.127). Recall that, $v_{j,1} = u_{j+1}$, and for each $r \geqslant 2$, $v_{j,r+1}$ is the point of maximum of the level set $\mathcal{G}(v_{j,r} - t)$. Hypothetically one of the two following cases happen:

- The values $|v_{j,n}|$ converge, as $n \to \infty$, to some value v_j with $|v_j| > |u_j|$, or
- For some n_0, we have that $|v_{j,n_0}| - |u_{j+1}| = t^{**} \leqslant t$.

In the case where the sequence $|v_{j,n}|$ converges, as $n \to \infty$, and thus also a subsequence $v_{j,n_k} \to v_j$ for some value v_j with $|v_j| > |u_j|$, then we are in the situation of Eq. (2.119) for all large enough k, and thus either $\mathcal{G}'(v_{j,n_k}) \to \infty$ as $k \to \infty$, which is a contradiction to the fact that $\mathcal{G}$ is locally in $C^{3,\alpha}$ outside the pole of the Green's function, or $\mathcal{G}'(u_{j+1}) = 0$ and we would be done as well.

Otherwise in the second case, by using Eq. (2.119) or the main term on the right hand side of Eq. (2.127), whichever is applicable in each step depending on whether condition (a) or (b) is satisfied, noting by Lemma 2.10 that the total contribution of the $N_j O_m(t^2) \to 0$ as $t \to 0$, where $N_j \lesssim (|u_{j+1}| - |u_j|)/t$, we get the claimed result. ∎

Note that for each $j \geqslant 1$, we have $|u_{j+1}| \geqslant |u_j| + t$, by construction. Also, note that as m gets large and thus $d_m \to 0$, the implied constant in the $O_m(t^2)$ term gets large, and for any fixed m, we choose the increment t small enough.

We start the iteration from the maximum point on the sphere $S(0, 1/L)$, and consider the first $j_0 \geqslant 1$ so that $|\mathcal{G}'(u_{j_0})| \leqslant K'_m t$. Then by the arguments in the proof of Lemmas 2.10 and 2.11, we get that for all subsequent $j \geqslant j_0$, $|\mathcal{G}'(u_j)| \leqslant K_m t e^{\Delta}$. In this case, as $t \to 0$, $\mathcal{G}(u_j + t) - \mathcal{G}(u_j) \lesssim K'_m e^{\Delta} t^2 = K_m t^2$ keeping the contribution up to the second order in t. Starting on the sphere $S(0, 1/L)$, till the level set $\mathcal{G}(\cdot) = 1/m$, we have N_m steps in the iteration where by a crude bound, $N_m t \leqslant \mathrm{diam}\Omega$, and thus, the total change in the Green's function is bounded by $N_m e^{c\Delta} K_m t^2 \leqslant (\mathrm{diam}\Omega) K_m t \to 0$ as $t \to 0$. For all $j \leqslant j_0$, the result of Lemma 2.11 holds.

The total change in the Green's function is given up to first order by,

$$\sum_{j=1}^{J} \mathcal{G}(u_j) - \mathcal{G}(u_j + t) \approx |\mathcal{G}'(u_j)| t, \tag{2.130}$$

where for each j, we have $\mathcal{G}(u_j + t) = \mathcal{G}(u_{j+1})$, $|u_1| = 1/L$, and $\mathcal{G}(u_J) = 1/m$. This, in the limit as $t \to 0$, is bounded from above by,

$$\leqslant C' \Big| \frac{\partial \mathcal{G}}{\partial r} \Big|_{b_{i,k}} b_{i,k}^{n-1} \Big(L^{n-2} - \frac{1}{r_{y_i,m}^{n-2}} \Big), \tag{2.131}$$

where $r_{i,m}$ is the point which is the maximum distance from the origin on the level set $\mathcal{G}(\cdot) = 1/m$, and by definition, we have $\mathcal{G}(r_{i,m}) = 1/m$.

After this step, we can take $m \to \infty$. Then in the limit of $m \to \infty$, we have that,

$$\begin{aligned} s_i^* = \mathcal{G}\Big(\frac{1}{L}\widehat{s_{max}}, 0\Big) = \mathcal{G}\Big(\frac{1}{L}\widehat{s_{max}}, 0\Big) - \mathcal{G}(r_i, 0) \leqslant C'\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{b_{i,k}} b_{i,k}^{n-1}\Big(L^{n-2} - \frac{1}{r_{i,m}^{n-2}}\Big), \\ \leqslant C'\Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{b_{i,k}} b_{i,k}^{n-1} L^{n-2},. \quad (2.132) \end{aligned}$$

where r_i is the point which is the maximum distance from y_i to the boundary, and by definition, we have $\mathcal{G}(r_i, 0) = 0$.

Now, we will compare Eq. (2.132) and Eq. (2.99). For this, we will use Lemma 2.7. Given any i, we choose k large enough so that we have two points $a_{i,k}, b_{i,k} \leqslant y_{i,\min}$ satisfying Lemma 2.7. Now comparing the right hand sides of Eqs. (2.99) and (2.132), we conclude that $y_{i,\min}$ can't be arbitrarily small, thus establishing Eq. (4.23), and Case (i) is established.

Case(ii). The proof for Case (ii) follows with the same essential argument as that for Case(i) outlined above. In this case, for each large enough integer N, we have $f(s_N) > Nf(2s_N)$ for some $s_N \geqslant \widetilde{s_N}$. Then the preceding condition implies that there exist boundary points $z_N \in \partial\Omega_N, z_{2N} \in \partial\Omega_{2N}$ with z_{2N} being the point at the minimum distance to $\partial\Omega_{2N}$ from the origin, and z_N being the point at the maximum distance to $\partial\Omega_N$, and the ratio

$$|z_N|/|z_{2N}| \to \infty \text{ as } N \to \infty \tag{2.133}$$

In this case, consider the sphere $S(0, |z_N|)$ of radius $|z_N|$, which by definition is contained inside the sphere $S(0, 1/2)$ since we have $s_N \geqslant \widetilde{s_N} = \max_{S(0,1/2)} \mathcal{G}(x, 0)$.

Then analogous to the earlier case, consider the point $s^* = \max_{x \in S(0,|z_N|)} \mathcal{G}(x,0) = \mathcal{G}(z_N, 0)$, by definition of the point z_N being the point at the maximum distance to $\partial\Omega_N$ from the origin.

In this case, one argues identically as in Case(i), looking at the points of maximum and minimum of the level sets of the Green's function for the proof of Eq. (4.23), using essentially again Eqs. (2.99) and (2.132), using a sequence of points for each fixed N, labelled $a_{N,k}, b_{N,k}|_{k=1}^{\infty}$ converging to 0 as $k \to \infty$ satisfying the condition of Lemma 2.7

This would give us, similar to Eq. (2.132), that,

$$s_N^* \leqslant C' \Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{b_{N,k}} b_{N,k}^{n-1} \frac{1}{|z_N|^{n-2}}. \tag{2.134}$$

Further, anaogous to Eq. (2.99), we would also have for some $C_1 > 1$ that,

$$2s_N^* - \frac{1}{C_1} s_N^* \gtrsim C \Big|\frac{\partial \mathcal{G}}{\partial r}\Big|_{a_{N,k}} a_{N,k}^{n-1} \Big(\frac{1}{|z_{2N}|^{n-2}} - \frac{1}{|z_N|^{n-2}}\Big) \tag{2.135}$$

Using these two equations above, as well as Lemma 2.7, using Eq. (2.133), we get a contradiction as $N \to \infty$.

This concludes the proof of Theorem 4.13. ∎

We now prove Theorem 2.3.

Proof of Theorem 2.3. We use the pointwise upper bound on $\mathcal{G}(x,0)$, the Harnack principle for the solutions for L and L_T, and a standard argument using the Green's functions for balls contained in B(0,1) and the maximum principle. We also use the standard fact mentioned in the introduction that $\mathcal{G}(x,y) = \mathcal{G}_T(y,x)$ where G_T is the Green's function corresponding to the adjoint operator.

Consider without loss of generality any point $w \in S(0, r)$. Call $\delta(w) = d_w$. Then consider the integer m so that $2^m d_w \approx 1$. Then one can choose a sequence of $2m + 1$ many points $\{0 = w_0, w_1, \ldots, w_{2m} = w\}$, where for each $0 \leqslant i \leqslant 2m$, the point w_i lies on the line joining 0 and w, and for each m, we have $|w_{2i} - w_{2i-1}| \approx |w_{2i+1} - w_{2i}| \approx 2^{m-i} d_w$ where the implied constant is independent of m.

By repeated use of Harnack's inequality, we have,

$\mathcal{G}(w_1, 0) = \mathcal{G}(w_1, w_0) = \mathcal{G}_T(w_0, w_1) = K\mathcal{G}_T(w_2, w_1) = K\mathcal{G}(w_1, w_2) = K^2\mathcal{G}(w_3, w_2) = K^4\mathcal{G}(w_5, w_4) = .. = K^{2t}\mathcal{G}(w_{2t+1}, w_{2t}) = \ldots.$

Note that from Theorem 4.13, we have bounds for $\mathcal{G}(w_1, w_0)$. Thus, by another use of the Harnack inequality, we get, for any $x \in S(w, \frac{1}{2}\delta(w))$, with the constant $K(k)$ dependent on k, that

$$\mathcal{G}(x, w) \leqslant K(m). \tag{2.136}$$

Now, consider the Dirichlet Green's function $\mathcal{G}_{ball}(\cdot, w)$ for the ball $B(w, \frac{1}{2}\delta(w))$.

Then we have uniform estimates on $|\mathcal{B}| \leqslant M'(m)$ in this ball $B(w, \frac{1}{2}\delta(w))$. Then by arguments such as in [KS19, Mour23] we have interior estimates for $\mathcal{G}_{ball}$,

$$\mathcal{G}_{ball}(x, w) \leqslant \frac{K'(M'(m))}{|x - w|^{n-2}} \text{ when } x \in B(w, \frac{1}{4}\delta(w)). \tag{2.137}$$

Then by the maximum principle, we have,

$$\mathcal{G}(x, w) - \mathcal{G}_{ball}(x, w) \leqslant K(k), \quad \text{when } \; x \in B(w, \frac{1}{2}\delta(w)). \tag{2.138}$$

In particular, for $x \in B(w, \frac{1}{4}\delta(w))$, we have,

$$\mathcal{G}(x, w) \leqslant K(m) + \frac{K'(M'(m))}{|x - w|^{n-2}}. \tag{2.139}$$

Now consider the length r_w(dependent on m) so that $r_w^{n-2} = \frac{K'(M'(m))}{K(m)}$ In that case, we see that,

$$\mathcal{G}(x,w) \leqslant \frac{2K(m)}{|x-w|^{n-2}}, \quad \text{when } x \in B(w, r_w). \tag{2.140}$$

If we have $r_w \leqslant \frac{1}{4}\delta(w)$, then by using Harnack inequality, we extend the inequality to the entire ball $B(w, \frac{1}{2}\delta(w))$.

In case $r_w > \frac{1}{4}\delta(w)$, then for any $x \leqslant \frac{1}{4}\delta(w)$, we already have the result in the ball $B(w, \frac{1}{4}\delta(w))$ which we also extend to $B(w, \frac{1}{2}\delta(w))$.

For points in $w \in B(0,r)$, we essentially repeat the same argument with a fewer number of steps and better constants. This concludes the proof. ∎

Remark 2.12. We note that the geometry of the unit ball is important in the proof of the result, in Lemma 8 .One expects the same result to hold in more general bounded Lipschitz domains, and in future we expect to work on this question in more general domains.(11)

Remark 2.13. Note that, by arguments similar to Lemma 2.7, we can also prove the following:

Lemma 2.14. *There is a uniform constant M_1 so that for any configuration of the drift that satisfies Eq.* (2.1) *and Eq.* (2.2), *we have a sequence $a_k|_{k=1}^{\infty}$ of minimum points, so that,*

$$\left|\frac{\partial \mathcal{G}}{\partial r}\right|_{a_k} a_k^{n-1} \geqslant M_1. \tag{2.141}$$

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(11) Note that the result of [Ha24] proves the existence of solutions for the case of bounded Lipschitz domains.

This follows by the use of the result of Theorem 4.11; that the Green's function has to grow sufficiently fast as one moves closer to the origin. A similar argument is also used in [Pat24], for the specific case of a drift whose magnitude is radially symmetric in the ball. Using Lemma 2.14, we can give another alternate argument for Lemmas 2.10 and 2.11 and the argument after Lemma 2.11 as well.

Remark 2.15. Note that since the pole of the Green's function in our case is at the origin of the ball, the integrating factor is forced to have a finite value in Lemma 2.10. For points away from the origin, one would need a finer study of the points of maximum of the level sets of the Green's function in $\Omega'_m \setminus B(0, 1/L)$, and we will study this in future.

The argument presented here for the drift term does not immediately generalize to this broader set of operators with elliptic terms in divergence or non-divergence form. Specifically, a version of Eq. (2.64) does not hold for a general elliptic term in place of the Laplacian.

Chapter 3

A counterexample diverging like inverse distance to boundary for solutions to elliptic equations with drifts.

3.1 Preliminaries.

Consider the unit ball of radius 1 with center at origin, $\Omega = B(0,1) \subset \mathbb{R}^n$,$n \geqslant 3$, and the sequence of operators, for $m \geqslant 100^{(1)}$,

$$L_m u = -\nabla \cdot (\nabla u) + \mathcal{B}_m \cdot \nabla u \tag{3.1}$$

where for each m, $\mathcal{B}_m$ is the drift which is of the form,

$$\vec{\mathcal{B}_m} = \begin{cases} -\frac{C}{1-|r|}\hat{r} & |r| \leqslant 1 - \frac{1}{m} \\ -Cm\hat{r} & 1 - \frac{1}{m} < |r| < 1, \end{cases} \tag{3.2}$$

for any arbitrary constant $C \geqslant 1$. Here, we use the notation that $\hat{r}$ is the unit vector at the point $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, with $|\vec{r}|$ the magnitude of the vector $\vec{r}$.

As $m \to \infty$, this approximates a drift that behaves like the inverse distance to the

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(1) The number 100 is chosen large enough for convenience so that the argument of Theorem 4.11 remains unaltered for the special case of $\mathcal{G}_m(x,0)$ in the ball $B(0,1)$, and the test functions considered in that proof are disjoint from the region $|r| > 1 - \frac{1}{m}$, for all $m \geqslant 100$.

boundary, that has been considered in more general domains Ω, in [HL01, KP01];

$$\delta(X)|\mathcal{B}(X)| < M. \tag{3.3}$$

Here, $\delta(X)$ denotes the distance of the point X to the boundary.

One is referred to recent works on the existence of the Green's function in the distributional sense, pointwise bounds on Dirichlet Green's functions in certain settings, and some scale invariant regularity estimates of solutions in the setting of Lorenz spaces and the Kato spaces and for the non-divergence form elliptic operator, in [KS19, Sak1, Sak2, Mour23, HK20].

The following existence and uniqueness (up to a set of measure zero) result for the Green's function is well known, where the drift is bounded, in bounded domains. See for example, Section 5 of [KS19]. or more generally Theorem 6.1 of [Mour23].

Theorem 3.1. *For any bounded domain $\Omega \subset \mathbb{R}^n$, there exists a unique non-negative function $\mathcal{G}_m : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, called the Green's function associated with L_m, such that the following holds:*

1. $G_m(\cdot, y) \in \mathcal{C}(\Omega \setminus B(y,s)) \cap W^{1,2}(\Omega \setminus B(s,y)) \cap W^{1,1}(\Omega) \quad \forall y \in \Omega, \forall s > 0.$
2. $\int_\Omega \Big(\nabla_x \mathcal{G}_m(x,y) \cdot \nabla\phi(x) + \mathcal{B}_m \cdot (\nabla_x \mathcal{G}_m(x,y))\phi(x)\Big)dx = \phi(y)$ *for all* $\phi \in C_c^\infty(\Omega)$.

For the case of an operator with only the elliptic principal term and no lower order terms in a domain Ω, pointwise upper and lower estimates for the Green's function with the pole at $x \in \Omega$ in the interior region $B(x, \delta(x)/2) = \{y \in \Omega | |x-y| \leqslant \delta(x)/2\}$ were obtained in [GrWi]. Precisely, they showed the existence of constants C_1, C_2 uniform on the domain, so that for any $y \in B(x, \delta(x)/2)$, we have

$$\frac{C_1}{|x-y|^{n-2}} \leqslant \mathcal{G}(x,y) \leqslant \frac{C_2}{|x-y|^{n-2}}. \tag{3.4}$$

For the case of a drift that is bounded from above by $\frac{\epsilon}{\delta(X)}$ where ϵ is sufficiently small, we also show pointwise upper and lower bounds on the Green's function in the [H24]. In fact, the argument for the lower bound in this chapter is essentially the same as in [H24]. It will be apparent in Section 3 of this chapter, that when we choose C in Eq. (3.2) small enough, the counterexample fails.

We show the following in Section 3:

Theorem 3.2. *For the operator considered in Eq.* (3.2)*, for any* $m \geqslant 100$*, with the drift considered in Eq.* (3.2)*, with* $C = 1$*, the Dirichlet Green's function, defined in the distributional sense,* $\mathcal{G}_m(x, 0)$*, when evaluated at the point* $(\frac{1}{2}, 0, \ldots, 0)$ *diverges to infinity as* $m \to \infty$ *.*

We note that the claimed proof in Theorem 4.3(a) in Chapter III of [HL01] on the upper bound on the Green's function for the elliptic operator in the half plane in $\mathbb{R}^n$, with $n \geqslant 3$, with a singular drift term bounded by the inverse distance to the boundary, does not work as intended, as the purported elliptic version of Lemma 2.10 in Chapter III of [HL01] can't be proved, since in turn it relies on the upper bound on the Green's function for the parabolic equation with the drift term which is proved in Chapter I of [HL01] and which does not generalize to the elliptic case.

We first show that lower bounds can be obtained for the Green's function, uniform in m, for the Green's function in the domain $B(0, \frac{1}{2})$.

The Green's function $\mathcal{G}_m(x, 0)$ in this special case is dependent only on the radial variable, and thus $\mathcal{G}_m(x, 0) = \mathcal{G}_m(|x|, 0)$.

Theorem 3.3. *The Green's function as defined,* $\mathcal{G}(x, 0)$ *is dependent only on the radial* *variable, and thus we have,* $\mathcal{G}(x,0) = \mathcal{G}(|x|,0)$ *for any* $x \in B(0,1) \setminus \{0\}$.

Proof. We have the equation

$$L_m u_{m,\rho} = \frac{1}{|B_\rho(0)|} 1_{B_\rho(0)}, \tag{3.5}$$

for any $\rho < \frac{1}{2}$, where we write $x = (r, \theta_1, \ldots, \theta_{n-1})$. For convenience, we suppress the dependence on m in the notation. Fix any $\alpha \in \mathbb{S}^{n-1}$, and the co-dimension one hyperplane H_α perpendicular to α and passing through the origin. One can orient the axes to define the variable θ_{n-1} as the azimuthal angle on H_α.

We write, $x'(x) = (r', \theta'_1, \ldots, \theta'_{n-1}) = (r, \theta_1, \ldots, \theta_{n-1} + \beta)$.

Then consider the function $u_{\rho,\beta}(x) = u_\rho(x') = u_\rho(r, \theta_1, \ldots, \theta_{n-1} + \beta)$, for some fixed $0 \leqslant \beta < 2\pi$. For now, we suppress the dependence of $u_{\rho,\beta}$ on α for notational convenience. We note that this is a well defined function in the unit ball, and we show that this also satisfies Eq. (3.5).

We have,

$$\frac{\partial}{\partial r}(u_{\rho,\beta}(x)) = \frac{\partial}{\partial r}(u_\rho(x')) = \frac{\partial u_\rho}{\partial r'}(x')\frac{\partial r'}{\partial r} + \sum_{i=1}^{n-1} \frac{\partial u_\rho}{\partial \theta'_i}(x')\frac{\partial \theta'_i}{\partial r} = \frac{\partial}{\partial r'}(u_\rho(x')). \tag{3.6}$$

and we have $\partial r'/\partial r = 1$ and for each $i \in \{1, \ldots, n-1\}$, that $\partial \theta'_i/\partial r = 0$.

By a similar argument, we also have for any $1 \leqslant i \leqslant (n-1)$,

$$\frac{\partial}{\partial \theta_i}(u_{\rho,\beta}(x)) = \frac{\partial}{\partial \theta'_i}(u_\rho(x')). \tag{3.7}$$

Using the structure of the Laplacian we get, noting also that $\mathcal{B}_m(x), \mathcal{B}_m(x')$ both have only a radially inward pointing component at each of these points, of equal magnitude,

$$\begin{aligned}\nabla \cdot (\nabla u_{\rho,\beta}(x)) + \mathcal{B}_m(x) \cdot \nabla u_{\rho,\beta}(x) &= \nabla' \cdot (\nabla' u_\rho(x') + \mathcal{B}_m(x') \cdot \nabla' u_\rho(x') \\ &= \frac{1}{|B_\rho(0)|} 1_{B_\rho(0)}(x') = \frac{1}{|B_\rho(0)|} 1_{B_\rho(0)}(x), \quad (3.8)\end{aligned}$$

where the last equality follows from the radial symmetry of the function, and we write, $\nabla' := \left(\frac{\partial}{\partial r'}, \frac{\partial}{\partial \theta_1'}, \ldots, \frac{\partial}{\partial \theta_{n-1}'}\right)$.

This gives us a family of solutions $u_{\rho,\beta}$ for $0 \leqslant \beta < 2\pi$. By uniqueness of the solutions, we conclude that all these functions are identical. In particular this forces that for any $r \leqslant 1$, the value of $u_{\rho,\beta}$ on $S(0,r) \cap H_\alpha$, where $S(0,r)$ is the sphere of radius r centered at the origin, is constant.

Now, consider the family of unit vectors $\alpha \in \mathbb{S}^{n-1}$, and repeat this argument for any arbitrary α, rotating the coordinates to have θ_{n-1} to be the azimuthal angle on H_α.

We thus have a family of solutions, all of which coincide., and by repeating the previous argument, this forces $u_\rho(x) = u_\rho(|x|)$.

Now by a standard limiting argument, see, for example Section 6 of [Mour23], we eventually get pointwise convergence almost everywhere of $u_{m,\rho_i} \to u_m$ in $B(0,1) \setminus \{0\}$ along a subsequence $\rho_i \to 0$. We are thus forced to conclude that $u_m(x) = u_m(|x|)$. ∎

3.2 Lower bounds on the Green's function.

The lower bound follows by an argument similar to that used in [GrWi].(2)

While we prove this below for the domain $B(0,1)$, below more generally in an arbitrary not necessarily bounded domain $\Omega \subset \mathbb{R}^n$. We consider the drift more generally considered in Eq. (4.4), and the argument remains the same in the special case of the Green's function $\mathcal{G}(x,0)$ for the operator in Eq. (3.1), for the drift considered in

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(2) This was also outlined in the argument of Lemma 4.3 in Section III in [HL01].

Eq. (3.2). In particular in this case, the constant K in Theorem 4.11 below is independent of the level of truncation m in the drifts Eq. (3.2).

We remark that the Harnack inequality, used routinely in the proof of Theorem 4.11 below, can be used with constants independent of the truncation level m, and can also be used for the limiting drift of Eq. (4.4), as long as the balls considered stay uniformly bounded away from the boundary. That is the case in the setting of Theorem 4.11. One notes the form of the Harnack constant in [GT77] on page 199, for example, which shows that for any ball $B(x, \delta(x))\eta)$, for uniform η, the Harnack constant is bounded from above by C_0^η. As we get arbitrarily close to the boundary, as long as we consider balls bounded away from the boundary in this manner, the Harnack inequality can thus be used routinely.

Theorem 3.4. *For the elliptic operator with the coefficients satisfying Eq.* (3.2) *in* $B(0,1) \subset \mathbb{R}^n$, $n \geqslant 3$, *we have the lower bound for the Green's function for the operator in Eq.* (3.1)*: for any* $z, y \in \Omega$ *with* $|z-y| \leqslant \frac{1}{2}\delta(y) := \frac{1}{2}dist(y, \partial\Omega)$ *we have*

$$\mathcal{G}(y,z) \geqslant c_0 \frac{1}{|z-y|^{n-2}}. \tag{3.9}$$

The proof essentially follows by extending the argument of the proof of Eq.(1.9) of [GrWi]. This argument appears almost identically in Chapter 4 for the more general case of an elliptic principal term. For the sake of completeness and condition Eq. (3.2) that appears in this section, we formulate the proof here, as well as in Chapter 4.

Proof of Theorem 4.11. Take $r := |z-y|$. Consider a smooth cut-off function η which is 1 on $B_r(y) \setminus B_{r/2}(y)$ and zero outside $B_{3r/2}(y) \setminus B_{r/4}(y)$, and further $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant \frac{K}{r}$.

Henceforth, we use the Einstein summation convention, where the summation sign is implied.

Given the domain $B(0,1)$, for any admissible test function ϕ, the Green's function satisfies the following adjoint equation (Theorem 4.11,(2.));

$$\int_{B(0,1)} \Big((\nabla\phi)\cdot\nabla\mathcal{G}(y,x)+\mathcal{G}(y,x)\mathcal{B}\cdot\nabla\phi\Big)dx=\phi(y) \tag{3.10}$$

We consider the test function $\phi(x)=G(y,x)\eta^2(x)$, and first get,

$$\int_{B(0,1)} |\nabla\mathcal{G}(y,x)|^2\eta^2 dx=-\int_{B(0,1)} 2\eta\mathcal{G}(y,x)\nabla\mathcal{G}(y,x)\cdot\nabla\eta+\mathcal{G}(y,x)\eta^2\mathcal{B}\cdot\nabla\mathcal{G}(y,x) \\ +2\mathcal{G}^2(y,x)\eta\mathcal{B}\cdot\nabla\eta)dx. \tag{3.11}$$

Now by using the bound on the drift term $\mathcal{B}$, the Cauchy inequality with ϵ's, with small enough ϵ, for the first two terms on the right,

$$\int_{B(0,1)} \mathcal{G}(y,x)\eta\nabla\mathcal{G}(y,x)\cdot\nabla\eta dx\leqslant\epsilon\int_{B(0,1)}\eta^2|\nabla\mathcal{G}(x,y)|^2dx \\ +\frac{K^2}{\epsilon r^2}\int_{r/4\leqslant|x-y|\leqslant 3r/2} G(y,x)^2dx, \\ \int_{B(0,1)} \mathcal{G}(y,x)\eta^2\nabla\mathcal{G}(y,x)\cdot\mathcal{B}dx\leqslant\epsilon\int_{B(0,1)}\eta^2|\nabla\mathcal{G}(x,y)|^2dx \\ +\frac{K}{\epsilon\delta(y)^2}\int_{r/4\leqslant|x-y|<3r/2}\mathcal{G}(y,x)^2dx. \tag{3.12}$$

Using the bounds on the cut-off function η introduced above, and hiding the term with the square of the gradient of $\mathcal{G}(y,x)$, we get,

$$\int\limits_{r/2<|x-y|<r} |\nabla\mathcal{G}(y,x)|^2dx\leqslant\Big(K_1\frac{1}{r^2}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big) \\ +\Big(K_2\frac{1}{r\delta(y)}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big)+\Big(K_3\frac{1}{\delta(y)^2}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big). \tag{3.13}$$

Noting that $r \leqslant \frac{1}{2}\delta(y)$, we get

$$\int\limits_{r/2<|x-y|<r} |\nabla \mathcal{G}(y,x)|^2 dx \leqslant \tilde{K}\frac{1}{r^2} \cdot \Big(\int\limits_{r/4<|x-y|<3r/2} \mathcal{G}(y,x)^2 dx \Big)$$

$$\leqslant \tilde{K} r^{n-2} \big(\sup_{r/4 \leqslant |x-y| \leqslant 3r/2} \mathcal{G}(y,x)^2 \big). \quad (3.14)$$

Again as in [GrWi], choose a similar cut-off function ϕ that is 1 on $B_{r/2}(y)$ and zero outside $B_r(y)$, and using it as the test function we get,

$$1 = \int\limits_{r/2 \leqslant |x-y| \leqslant r} (\partial_i \mathcal{G}(y,x)\partial_i \phi + \mathcal{G}(y,x)\mathcal{B}_i \partial_i \phi \Big) dx)$$

$$\leqslant M\frac{K}{r} \int\limits_{r/2 \leqslant |x-y| \leqslant r} |\nabla \mathcal{G}(y,x)| dx + \frac{M}{r\delta(y)} \int\limits_{r/2 \leqslant |x-y| \leqslant r} \mathcal{G}(y,x) dx. \quad (3.15)$$

Using the identity of Eq. (4.69), and Cauchy's inequality for the first term on the right, along with a trivial volume bound, and finally Harnack's inequality,

$$1 \leqslant K r^{n-2} \sup_{r/4 \leqslant |x-y| \leqslant 3r/2} \mathcal{G}(y,x) \leqslant K|z-y|^{n-2}\mathcal{G}(y,z). \quad (3.16)$$

∎

Using Harnack inequality, and Theorem 4.11, we immediately get,

Corollary 3.5. *For the elliptic operator with the coefficients satisfying Eq.* (3.2) *in* $B(0,1) \subset \mathbb{R}^n$*, we have the lower bound for the Green's function for the operator in Eq.* (3.1)*: for any* $z, y \in \Omega$ *with* $|y-z| \leqslant \frac{1}{2}\delta(z) := \frac{1}{2} dist(z, \partial\Omega)$ *we have*

$$\mathcal{G}(y,z) \geqslant c_0 \frac{1}{|z-y|^{n-2}}. \quad (3.17)$$

In other words, we can interchange the order of the arguments for the Green's function, in this result.

3.3 Counterexample for uniform upper bounds.

Proof of Theorem 2. In this case, we consider the domain $B(0,1)$ and for all $m \geqslant 3$, the operator L_m with the drift $\mathcal{B}_m$ as considered in Eq. (3.2) in the introduction.

For the points on the x-axis, henceforth we simply write a in place of the point $(a,0,\dots,0)$. We consider the above ODE of Eq. (3.18) along the x-axis.

Consider the pole of the Green's function to be at the origin. Since the Green's function $\mathcal{G}(x)=\mathcal{G}(x,0)$ in this case is dependent only on the radial coordinate, after writing the expression of Eq. (3.1) in polar coordinates, noting that the generalized solution of Theorem 4.11 is in this case also a classical solution to $L_m u = 0$ in $B(0,1)\setminus\{0\}$, we will actually solve in $(0,1)\setminus\{0\}$ the equation,

$$\frac{d\mathcal{G}_m^2}{dr^2}+\frac{n-1}{r}\frac{d\mathcal{G}_m}{dr}+\mathcal{B}_m\cdot\hat{r}\frac{d\mathcal{G}_m}{dr}=0. \tag{3.18}$$

The level sets of the Green's function are the spheres with center at the origin, and $\frac{d\mathcal{G}}{dr}<0$ for all $0<r<1$. By definition, $\mathcal{G}(t)=0$ when $|t|=1$. In this construction, we will later use the fact that the lower bound from Theorem 4.11 exists with point-wise bounds independent of m, in the subdomain $B(0,\frac{1}{4})$.

By using a standard integrating factor, we get from Eq. (3.18) that,

$$(r^{n-1}\mathcal{G}_m' e^{\int_1^r \mathcal{B}_m dt})'=0, \tag{3.19}$$

where $(\cdot)'$ denotes the derivative with respect to r.

$$r^{n-1}\mathcal{G}_m'(r)e^{\int_1^r \mathcal{B}_m dt}=a^{n-1}\mathcal{G}_m'(a)e^{\int_1^a \mathcal{B}_m dt}, \tag{3.20}$$

and thus,

$$\mathcal{G}_m'(r)=\frac{1}{r^{n-1}}a^{n-1}\mathcal{G}_m'(a)e^{-\int_a^r \mathcal{B}_m dt} \tag{3.21}$$

For a fixed m, we distinguish two cases,

- $0 < a < r \leqslant 1 - \frac{1}{m}$.

  In this case, the integral on the exponential gives us,

$$-\int_a^r \mathcal{B}_m dt = \int_a^r \frac{C}{1-t} dt = C \log(\frac{1-a}{1-r}). \tag{3.22}$$

  Thus, we have,

$$\mathcal{G}_m'(r) = \frac{1}{r^{n-1}} a^{n-1} \mathcal{G}_m'(a) \Big(\frac{1-a}{1-r}\Big)^C. \tag{3.23}$$

- $0 < a < 1 - \frac{1}{m} < r < 1$. In this case, the integral on the exponential gives,

$$-\int_a^r \mathcal{B}_m dt = \int_a^{1-\frac{1}{m}} \frac{C}{1-t} dt + \int_{1-\frac{1}{m}}^r \frac{C}{1-(1-\frac{1}{m})} dt = \int_a^{1-\frac{1}{m}} \frac{C}{1-t} dt$$
$$+ (r - 1 + \frac{1}{m}) Cm = C \log\big(m(1-a)\big) + Cm(r - 1 + \frac{1}{m}) \tag{3.24}$$

Note that $\mathcal{G}_m(1) = 0$.

We note, using the maximum principle, that the Green's function is radially non-increasing, and this the radial derivative is non positive. We now note the existence of some positive constant C_0, independent of m, so that there exists a decreasing sequence(3) $a_{k,m} \to 0, k \geqslant 0$, and $a_{0,m} < 1/4$, so that for each $k \geqslant 0$

$$\Big|\frac{d\mathcal{G}_m}{dr}\Big|_{a_{k,m}}\Big| \geqslant \frac{C_0}{a_{k,m}^{n-1}}. \tag{3.25}$$

(Here, as usual, by $a_{k,m}$ we mean the point $(a_{k,m}, 0, \ldots, 0)$.) If this is not true, and for any $\theta > 0$, for all sufficiently small $a \leqslant a_{0,m}$, we have

$$\Big|\frac{d\mathcal{G}_m}{dr}\Big|_a < \frac{\theta}{a^{n-1}}, \tag{3.26}$$

---

(3) Instead of a sequence, it is enough to just find a single point with the stated property.

then we will get a contradiction to the lower bound coming from Theorem 4.11, since in that case by integrating Eq. (3.26), we will get, noting again that $\frac{\partial \mathcal{G}}{\partial r} < 0$,

$$\mathcal{G}_m(p,0) - \mathcal{G}_m(a_{i,m},0) \leqslant \frac{\theta}{n-2}\Big(\frac{1}{p^{n-2}} - \frac{1}{a_i^{n-2}}\Big). \tag{3.27}$$

Thus, when θ is sufficiently small compared to K we get a contradiction to Eq. (4.72) , as Eq. (3.27) shows that the increase of the values of the Green's function is slow enough as $p \to 0$,

$$\mathcal{G}_m(p,0) \leqslant \mathcal{G}_m(a_{i,m},0) + \frac{\theta}{n-2}\Big(\frac{1}{p^{n-2}} - \frac{1}{a_i^{n-2}}\Big). \tag{3.28}$$

so there would have to exist some point $p_i \leqslant a_{0,m}$ so that $\mathcal{G}_m(p_i, y_i) < \frac{K(M)}{p_i^{n-2}}$, which is a contradiction to Theorem 4.11.

For simplicity, consider the special case of $C = 1$. For a fixed m, choosing any $a_{k,m}$, with the above property, we get, $|(a_{k,m})^{n-1}\mathcal{G}'_m(a_{k,m})| > C_0$, independent of m. Now integrating Eq. (4.40), from 1/2 to 1, we get, for any m, using Eq. (3.24),

$$\begin{aligned}\mathcal{G}_m(1) - \mathcal{G}_m(\frac{1}{2}) = -\mathcal{G}_m(\frac{1}{2}) = (a_{k,m}^{n-1}\mathcal{G}'_m(a_{k,m}))\Bigg(\int_{\frac{1}{2}}^{1-\frac{1}{m}} \frac{1}{r^{n-1}}\Big(\frac{1-a_{k,m}}{1-r}\Big)dr \\ + \int_{1-\frac{1}{m}}^{1} \frac{1}{r^{n-1}} e^{-\int_a^r \mathcal{B}_m dt} dr\Bigg)\end{aligned} \tag{3.29}$$

So, we have, noting again that $\mathcal{G}'_m(a_{k,m}) < 0$,

$$\begin{aligned}\mathcal{G}_m(\frac{1}{2}) &\geqslant C_0\Bigg(\int_{\frac{1}{2}}^{1-\frac{1}{m}} \frac{1}{r^{n-1}}\Big(\frac{1-a_{k,m}}{1-r}\Big)dr + \int_{1-\frac{1}{m}}^{1} \frac{1}{r^{n-1}} e^{-\int_a^r \mathcal{B}_m dt} dr\Bigg) \\ &\geqslant C_0\Bigg(\int_{\frac{1}{2}}^{1-\frac{1}{m}} \frac{1}{2r^{n-1}}\Big(\frac{1}{1-r}\Big)dr\Bigg).\end{aligned} \tag{3.30}$$

The above inequality holds with the bound C_0 independent of m. As $m \to \infty$, clearly the right hand side goes to infinity.

Thus, we can't get pointwise upper bounds for $\mathcal{G}_m$, independent of m. ∎

Remark 3.6. We note that if we took the constant $C < 1$ in Eq. (3.2), then for $G_m(x,0)$, one does get a finite value for $G_m(\frac{1}{2})$. More generally, in [**?**], it is shown that for any not necessarily bounded domain Ω, there exists a sufficiently small $\epsilon(\Omega)$ so that when the constant of the limiting drift of Eq. (4.4), is taken as $M = \epsilon(\Omega)$, one gets pointwise upper bounds as expected, in balls bounded away from the pole of the Green's function, and the constants are uniform over the domain. (4)

Remark 3.7. This also immediately gives us that the solution for the Poisson-Dirichlet problem is not well defined in the limit as $m \to \infty$; consider the data $f = 1_{B(\frac{1}{2},\epsilon)}$, a ball of radius ϵ for a small enough ϵ. Consider the problem $L_m u_m = f$ in $B(0,1)$, $u = 0$ on ∂B. In this case, using Harnack's inequality and Theorem 2, we have $u(0) = \int G_m(0,y)f(y)dy \to \infty$ as $m \to \infty$. Further, using Harnack's inequality, we also get that $u_m(t) \to \infty$ as $m \to \infty$ for any $t \in B(0,\frac{1}{4})$, and thus one doesn't get any solution for the limiting drift, in $W^{1,2}(B(0,1))$.

Remark 3.8. From the symmetry of the problem, and the definition of the elliptic measure with pole at the origin, it is clear that for any fixed m, the elliptic measure is equivalent to the Lebesgue measure on the boundary, and thus still doubling. In other words, identifying the boundary with $\mathbb{S}^{n-1}$, if we consider any surface ball $\Delta(p,r) = B(p,r) \cap \mathbb{S}^{n-1}$, for some $p \in \mathbb{S}^{\ltimes - \nVdash}$, and consider the boundary function

$$f(x) = \begin{cases} 1, & x \in B \\ 0, & x \in S^n \setminus B, \end{cases} \tag{3.31}$$

then for each m, the elliptic measure $\omega^0_{L_m}(\Delta(p,r))$ corresponding to the operator L_m with pole at the origin, gives us the relative surface measure of $\Delta(p,r)$ in $\mathbb{S}^{n-1}$.

---

(4) In fact, we show more generally the same result in Chapter 4 of this thesis, when the drift is 'small' on average in Whitney balls.

In Chapter III, Section 4 of [HL01], there is a proposed argument, which is expanded in [H24], giving doubling of the elliptic measure in uniform domains (which contain Harnack chains) when we have both pointwise lower and upper bounds on the Green's function in the form of Eq. (3.4).

Chapter 4

Doubling of elliptic measure for drifts diverging at the boundary with an average smallness condition.

4.1 Preliminaries.

Consider a 1-sided chord arc domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 3$. Consider the linear operator

$$Lu = -\nabla \cdot (\mathcal{A}\nabla u) + \mathcal{B} \cdot \nabla u, \tag{4.1}$$

We assume the drift term is small in an averaged sense over each Whitney region, which is to say that there exists a β small enough so that for each Whitney cube U_Q in the Whitney cube decomposition of Ω we have,

$$(1) : \int_{U_Q} |\mathcal{B}|^2 \delta(X) dV \leqslant \beta^2 l(Q)^{n-1}, \tag{4.2}$$

Here $|\mathcal{B}|$ denotes the magnitude of the drift term. This is a generalization of the case considered in Chapter I of [HL01], with the pointwise smallness estimate of the form

$$|\mathcal{B}(x)| \leqslant \frac{\beta}{\delta(x)}, \tag{4.3}$$

for some small enough β.

Here we also require

$$(2):\left(\sum_{i=0}^{n}\delta(X)|\mathcal{B}_i(X)|+\sum_{i,j=1}^{n}|\mathcal{A}_{ij}(X)|\right)<M<\infty,\ \sum_{i,j=1}^{n}\mathcal{A}_{ij}(X)\xi_i\xi_j\geqslant\lambda|\xi|^2. \quad (4.4)$$

This enables the use of the Harnack inequality in our arguments, and also is important in establishing the Bourgain type estimate for the elliptic measure for this operator.

Also note that the adjoint equation takes the form:

$$L^*u=-\nabla\cdot(\mathcal{A}^*\nabla(u)+\mathcal{B}\cdot(u)), \quad (4.5)$$

We note that solutions for the adjoint equation in general are not a-priori known to exist, whereas the Lax-Milgram lemma can be used in a routine way to show existence of solutions for $Lu=f$ for locally integrable f , once one shows coercivity of the corresponding bilinear form with argument analogous to those used in proving Theorem 4.13.

One motivation for this work comes from perturbative results such as [H24] that establishes L^p Dirichlet solvability for the operator L when the drift satisfies a smallness assumption in a Carleson measure sense as in Eq. (4.7), when we know that the L^p Dirichlet problem for the operator

$$L_0u=-\nabla\cdot(\mathcal{A}\nabla u) \quad (4.6)$$

is solvable. In turn, such a question with the drifts is motivated by the question of the rough Dirichlet solvability for the elliptic operator in non-divergence form, since the non-divergence form operator can be written as a divergence form expression along with a drift.

This means that for every $Q \in \mathbb{D}(\partial\Omega)$, we have,

$$\sup_{x\in\partial\Omega, 0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \int\limits_{\Omega\cap B(x,r)} \sup_{B(t,\delta(t)/2)} \Big(|\mathcal{B}|^2(y)\delta(y)\Big) dt < \infty, \tag{4.7}$$

and further we also have the pointwise bound of the form $|\mathcal{B}(x)| \leqslant M/\delta(x)$ coming from Eq. (4.4). The Carleson measure condition of Eq. (4.7) implies with a standard pigeonholing argument that, for a given small $\epsilon > 0$, for all but a finite number of Whitney cubes contained in $B(x,r)$,

$$\int\limits_{U_Q} \sup_{B(t,\delta(t)/2)} \Big(|\mathcal{B}|^2(y)\delta(y)\Big) dt < \epsilon l(Q)^{n-1}. \tag{4.8}$$

Further, the supremum condition in the integrand forces a pointwise smallness estimate in each Whitney cube of this type. Such a condition on the drift has been considered first in [KP01]. There, the authors also proved a result for Dirichlet solvability for operators with only the principal elliptic term, where the elliptic matrices satisfied the 'DKP' type condition:

$$\sup_{x\in\partial\Omega, 0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \int\limits_{\Omega\cap B(x,r)} \sup_{B(t,\delta(t)/2)} \Big(|\nabla\mathcal{A}|^2(y)\delta(y)\Big) dt < \infty. \tag{4.9}$$

In other results, one establishes pointwise estimates on the Green function, found most recently in [H24] where one has the pointwise smallness condition of Eq. (4.3) on the drift. Further, the so called 'Bourgain' property', for the operator with the Eq. (4.3) assumption, is also found in Chapter 1 of [HL01].

These two ingredients combine to establish the doubling of the elliptic measure, by an argument found in [Ai06], as mentioned in Theorem 4.6 of this chapter. We note that these two ingredients separately are of independent interest. For example, in [HL18], the Bourgain property for the elliptic measure corresponding to the elliptic operator without any lower order terms, is essentially used to prove the fact that an a-priori assumption of BMO-solvability implies L^p Dirichlet solvability. The pointwise estimates on the Green function with lower order terms is itself of interest in other areas of partial differential equations[Mour23, Sak1, KS19]. Although the lower pointwise estimates are proved by adopting the techniques of [GrWi], the upper pointwise estimates require more work, and crucially needs a Calderón Zygmund type decomposition of the Whitney cubes. The Bourgain property in this instance also needs a new argument, in controlling an upper bound on terms of the form $\int |\mathcal{B}|^2 u^2 \psi^2 dV$ for local solutions to $Lu = 0$ and ψ some radially symmetric test function centered on the boundary. In presence of pointwise estimates of Eq. (4.3), as in [HL01], this is somewhat routine. However, with the hypothesis of only Eq. (4.2) on the drift, the arguments become more delicate, in the general setting of the $1-$sided chord arc domains.

In [HMT17], it is pointed out that Dirchlet solvability holds by weakening the hypothesis of Eq. (4.9) in (ii) below, along with the local Lipschitz condition on $\mathcal{A}$ as well as a pointwise estimate on the gradient of $|\nabla \mathcal{A}|$:

$$
\begin{gathered}
(i)\mathcal{A} \text{ in } \mathrm{Lip}_{loc}(\Omega), \quad |\nabla \mathcal{A}(x)| \leqslant \frac{M}{\delta(x)} \\
(ii) \sup_{x\in\partial\Omega, 0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \int\limits_{\Omega\cap B(x,r)} \Big(|\nabla \mathcal{A}|^2(y)\delta(y) \Big) dt < \infty. \qquad (4.10)
\end{gathered}
$$

Thus it is natural to ask whether the results of [H24] can be proved for more general operators with a Carleson measure condition on the drift, where the supremum condition within the integrand is removed,

$$\sup_{x\in\partial\Omega,0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \int\limits_{\Omega\cap B(x,r)} \Big(|\mathcal{B}|^2(y)\delta(y)\Big)dt < \infty. \tag{4.11}$$

This means that for Whitney cubes that typically satisfy the smallness assumption, instead of Eq. (4.8) and thus the pointwise smallness estimate, we only have the average smallness assumption of Eq. (4.2) considered in this chapter, along with Eq. (4.4). Because of the Carleson measure condition, by an elementary counting argument, depending on the Carleson measure constant and β, one sees that all but a finite number of Whitney cubes contained in any $\Omega \cap B(x,r)$ have to satisfy the average smallness condition of Eq. (4.4).

This thesis establishes doubling of the elliptic measure for this more general situation of drifts that have a smallness condition Eq. (4.2) on average on every Whitney cube in a chord arc domain, instead of having a pointwise smallness assumption, along with a background pointwise bound with the large constant as in Eq. (4.4). This result is new even for the half space.

We also note that the Carleson measure involves a square of the drift term. If one worked with a Carleson measure condition of the form

$$\sup_{x\in\partial\Omega,0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \int\limits_{\Omega\cap B(x,r)} |\mathcal{B}|(y)dt < \infty, \tag{4.12}$$

that has a linear drift term in the integrand, the point-wise estimate on the drift in Eq. (4.4) along with Eq. (4.12) immediately gives the Eq. (4.11) condition.

In the theory of linear elliptic operators with lower order terms, one typically deals with some integrability assumption in some L^p or weak L^p space on the drift term, in order to establish the pointwise estimates on the Green function. This has been considered in recent work [KS19, Sak1, Mour23]. When we assume Eq. (4.3), even with a small constant, in general bounded domains, the drift is not integrable. Our situation is much more general in that we assume only an average smallness condition in every Whitney domain.

This work can be seen as the first step in extending the work of [H24], but where the supremum condition is removed from the integrand in the hypothesis of Eq. (4.7) . In [H24] we need to use the doubling property in sawtooth domains where we have the pointwise estimate Eq. (4.3) on the drift.

Even so, there are other adaptations to be made to the methods of [H24] for establishing Dirichlet solvability with this more general Carleson measure condition on the drift. This will be studied in future work.

We remark that, in establishing existence almost everywhere with respect to the elliptic measure, of finite non-tangential limits for solutions to the divergence form elliptic operators, doubling of the elliptic measure is crucial. See the early papers of [HW68, HW70, CFMS81].

We note that our result is proved in the general setting of $1-$sided Chord arc domains. Below, we borrow the terminology from other recent works involving the elliptic measure and chord-arc domains(1) [HMMTZ21].

We prove a local Hardy inequality in Theorem 4.7 in Section 3, and state a global version in Lemma 4.8. This global version is well known from earlier results going back at least to [Lew88]. In [H24] we prove the local version of Theorem 4.7 as a consequence of this global version. However, a direct proof of this local (and thus also

---

(1) In other terminology, these are also called 1-sided non tangentially accessible domains.

the global) Hardy inequality using a stopping time argument here is interesting in it's own right, and to the best knowledge of the author hasn't directly appeared in the literature, except a version of this stopping time argument in [KLT11] for a pointwise Hardy inequality. (2)

Definition 4.1. Corkscrew domain: [see Definition 2.3 of [HMMTZ21]] An open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the interior corkscrew condition if for some uniform constant c wirh $0 < c < 1$, and for every surface ball $\Delta := \Delta(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\partial\Omega$, there is a ball $B(X_\Delta, cr) \subset \Omega \cap B(x, r)$. The point $X_\Delta \subset \Omega$ is called an interior corkscrew point relative to Δ.

Definition 4.2. Harnack Chains: An open connected set $\Omega \subset \mathbb{R}^n$ is said to satisfy the Harnack Chain condition with constants $M, C_1 > 1$ if for every pair of points $x, y \in \Omega$, there is a chain of balls $B_1, B_2, \ldots, B_k \subset \Omega$ with $k \leqslant M(2 + \log_2^+ \Pi)$ that connects x to y, where

$$\Pi := \frac{|x-y|}{\min(\delta(x), \delta(y))}. \tag{4.13}$$

We have $x \in B_1$, $y \in B_k$, $B_j \cap B_{j+1} \neq \phi$ for every $j \in \{1, \ldots, k-1\}$ and for every $1 \leqslant j \leqslant k$, we have,

$$C_1^{-1}\operatorname{diam}(B_j) \leqslant \operatorname{dist}(B_j, \partial\Omega) \leqslant C_1 \operatorname{diam}(B_j). \tag{4.14}$$

Definition 4.3. Ahlfors-David regular(ADR) boundary: We say that a set $E \subset \mathbb{R}^n$ is Ahlford-David regular with constant $C_{AR} > 1$ if for any $q \in E$, and any $0 < r \leqslant \operatorname{diam}(E)$, we have,

$$C_{AR}^{-1} r^{n-1} \leqslant \mathcal{H}^{n-1}(B(q, r) \cap E) \leqslant C_{AR} r^{n-1}. \tag{4.15}$$

---

(2) The author is grateful to Juha Lehrbach for the references.

For such a situation, we have the following well known dyadic decomposition of the boundary.

Definition 4.4. 1-sided Chord arc domain: A domain $\Omega \subset \mathbb{R}^n$ that satisfies the Corkscrew condition and the Harnack Chain condition, along with an Ahlfors-David boundary, is called a 1-sided Chord arc domains.

Lemma 4.5. *(Existence and properties of the "dyadic grid") [DS1, DS2], [Ch90]. Suppose that $E \subset \mathbb{R}^n$ is closed $n-1$-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$ and $C_* < \infty$, depending only on dimension and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(i) $E = \cup_j Q_j^k$ *for each* $k \in \mathbb{Z}$.

(ii) If $m \geqslant k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(iii) For each (j,k) *and each* $m < k$, *there is a unique* i *such that* $Q_j^k \subset Q_i^m$.

(iv) $diam\big(Q_j^k\big) \leqslant C_* 2^{-k}$.

(v) Each Q_j^k *contains some "surface ball"* $\Delta\big(x_j^k, a_0 2^{-k}\big) := B\big(x_j^k, a_0 2^{-k}\big) \cap E$.

(vi) $H^n\big(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leqslant \varrho\, 2^{-k}\}\big) \leqslant C_*\, \varrho^\gamma\, H^n\big(Q_j^k\big)$, *for all* k, j *and for all* $\varrho \in (0, a_0)$.

Recall the Whitney cube decomposition $\mathcal{W} = \mathcal{W}(\Omega)$. See, for example, Chapter 6 of[St70] (the reader is referred to Section 6 of [HMMTZ21] or Section 3 of [HM14], for example, for the notation and a more detailed exposition of the following standard constructions).

Here we briefly introduce the main objects.

For each cube $Q \in \mathbb{D}(\partial\Omega)$, we have the subfamily $\mathcal{W}_Q \subset \mathcal{W}$, and we define

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*. \tag{4.16}$$

Here, I^* is a fattening of the Whitney cube I, and we also write $X(I)$ for the center of each cube I.

These satisfy the following properties: there exists a point $X_Q \in U_Q$, and there are uniform constants k^*, K_0 so that,

$$\begin{aligned} k(Q) - k^* \leqslant k_I \leqslant k(Q) + k^*, \ \forall I \in \mathcal{W}_Q^*, \\ X(I) \to_{U_Q} X_Q \forall I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I, Q) \leqslant K_0 2^{-k(Q)} \forall I \in \mathcal{W}_Q^*. \end{aligned}$$

Here, $X(I) \to_{U_Q} X_Q$ means that the interior of U_Q contains all the balls in a Harnack chain in Ω connecting $X(I)$ and X_Q, and for any point inside any of these balls in the Harnack chain, we have $\operatorname{dist}(Z, \partial\Omega) \approx \operatorname{dist}(Z, \Omega \setminus U_Q)$ with control of the constants uniform on the choice of U_Q.

For any given $Q \in \mathbb{D}(\partial\Omega)$, the Carleson box relative to Q is defined by,

$$T_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \right). \tag{4.17}$$

Here, $\mathbb{D}_Q$ is the subset of the dyadic grid, restricted to Q.

We note that we have not a-priori shown the existence of solutions for this equation $Lu = f + \operatorname{div}\vec{g}$ or of the adjoint equation $L_T u = f + \operatorname{div}\vec{g}$, with the drift satisfying the conditions of Eqs. (4.2) and (4.4), in Chapter 4. One would need to show boundedness and coercivity of the associated bilinear form and invoke the Lax-Milgram lemma, along with the use of Fredholm alternate, as in Section 8.2 of [GT77]. While boundedness of the associated bilinear form for either L or L_T follows readily from Eq. (4.4), the argument for coercivity is involves hiding the contribution from the integral involving the drift term, and is effectively the same as the argument given for the upper bound of the Green's function. Once solutions are established for both L and it's adjoint L_T, we can use some standard approximating arguments to show the existence of the Green's function in the distributional sense, in a manner similar to Section 6 of [Mour23], as an example.

4.1.1 Main result.

The main result of the chapter is the doubling of the elliptic measure corresponding to the operator in Eq. (4.1).

Theorem 4.6. *For the operator Eqs.* (4.1) *and* (4.5) *with the corresponding elliptic measure; for any* $x \in \partial\Omega, r \leqslant \text{diam}\, \partial\Omega$, *with* $A(x,r)$ *a corkscrew point relative to the surface ball* $\Delta(x,r) := B(x,r) \cap \partial\Omega$, *then for any* $y \in \Omega \setminus B(x,2r)$, *we have*

$$c^{-1} r^{n-2} G(y, A(x,r)) \leqslant \omega(y, \Delta(x,r)) \leqslant \omega(y, \Delta(x,2r)) \leqslant c^2 r^{n-2} G(y, A(x,r)). \tag{4.18}$$

The Boundary Hölder regularity and subsequently the Bourgain property is stated and proved in Section 4, and the pointwise upper and lower estimates on the Green function are proved in Section 5.

4.2 Notation.

We use the symbol $\sum\limits_{i,Q_i\subset Q}$ to mean a sum over dyadic sub-cubes contained in the cube Q. The symbol $\sum\limits_{i,Q_i\supset Q}$ is used to mean a sum over the dyadic sub-cubes that contain Q. We use the notation $|\cdot|$ for the volume of a certain set. We also use $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ to denote the distance to the boundary of a point $x \in \Omega$ to the boundary $\partial\Omega$.

4.3 Local boundary Hardy inequality.

Theorem 4.7. *Suppose $\Omega \subset \mathbb{R}^n$ a not necessarily bounded open set, and $Q_0 \in \mathbb{D}(\partial\Omega)$ and $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, and $u|_{\partial\Omega} = 0$. Suppose that the boundary $\partial\Omega$ is $(n-1)-$ dimensional Ahlfors David regular (or simply ADR). In that case, for any $Q_0 \subset \mathcal{D}_k$, there exists a $\overline{Q_0} \subset \mathcal{D}_{l(k)}$, with $l(k) \geqslant k$, $Q_0 \subset \overline{Q_0}$, and where $\frac{l(k)}{k} = O(1)$ with the implied constant independent of the cube Q_0.*

$$\int_{T_{Q_0}} \Big(\frac{u(x)}{\delta(x)}\Big)^2 dx \lesssim \Big(\int_{T_{\overline{Q_0}}} |\nabla u|^2 dx\Big).$$

Proof of Theorem 4.7. The proof follows by the use of the Whitney decomposition of Ω, the Poincare inequality, along with a dyadic version of Hardy's inequality along with the continuity of u at the boundary.

We note that $u = 0$ on the boundary. In this case, for any point $x \in \Omega$, we consider a Harnack path that connects x to $\hat{x}$ which is a point nearest to x on the boundary with $|x - \hat{x}| = \delta(x)$. We enumerate the fattened Whitney cubes.

Up to an implied constant, it will be enough to prove that,

$$\int_{T_{Q_0}} \Big(\frac{u(x)}{\delta(x)}\Big)^2 dV(x) \lesssim \sum_{\substack{U_Q^*,\\ Q\in\mathbb{D}(Q_0)}} \int_{U_Q^*} \Big(\frac{u(x)}{\delta(x)}\Big)^2 dV(x). \tag{4.19}$$

We write,

$$(u_Q) := \frac{1}{|U_Q|}\int_{U_Q} u(x)dV(x), \quad (u_Q^*) := \frac{1}{|U_Q^*|}\int_{U_Q^*} u(x)dV(x). \tag{4.20}$$

We write,

$$\sum_{U_Q}\int_{U_Q}\Big(\frac{u(x)}{\delta(x)}\Big)^2 dV = \sum_{U_Q}\int_{U_Q}\Big(\frac{u(x)-(u_Q)+(u_Q)}{\delta(x)}\Big)^2 dV(x). \tag{4.21}$$

We have,

$$\int_{U_Q}\Big(\frac{u(x)-(u_Q)+(u_Q)}{\delta(x)}\Big)^2 dV \lesssim 2\int_{U_Q}\left(\frac{(u(x)-(u_Q)^2}{\delta(x)^2}+\frac{(u_Q)^2}{\delta(x)^2}\right)dV(x). \tag{4.22}$$

For the first term on the right, one can use Poincare inequality over the fattened Whitney cube U_Q, to get,

$$\lesssim 2\int_{U_Q}\left(|\nabla u(x)|^2+\frac{(u_Q)^2}{\delta(x)^2}\right)dV(x). \tag{4.23}$$

Next, we note that if Q_1 is any dyadic child of Q, then

$$|(u_Q)-(u_{Q_1})| \leqslant |(u_Q)-(u_Q^*)| + |(u_Q^*)-(u_{Q_1})|. \tag{4.24}$$

Here we have the fattened Whitney cubes U_Q^* which contain both of the cubes U_Q, U_{Q_1}. We see by using a crude bound, then Poincare inequality over the region U_Q^{**}, and finally Hölder inequality and another crude bound, that,

$$\begin{aligned}(|(u_Q)-(u_Q^*)|)^2 \leqslant \Big(\frac{1}{|U_Q|}\int_{U_Q^*}|u-(u_Q^*)|dV\Big)^2 &\lesssim \left(\frac{l(Q)}{|U_Q^*|}\int_{U_Q^*}|\nabla u(x)|dV(x)\right)^2\\ &\lesssim \frac{l(Q)^2}{|U_Q|}\int_{U_Q^*}|\nabla u(x)|^2 dV(x).\end{aligned}$$

By an identical argument one also gets with a slightly different implied constant, that,

$$(|u_Q^*-u_{Q_1}|)^2 \lesssim \frac{l(Q_1)^2}{|U_Q|}\int_{U_Q^*}|\nabla u(x)|^2 dV(x).$$

Adding up the two contributions from the above, to get,

$$|(u_Q) - (u_{Q_1})| \lesssim \frac{l(Q_0)^2}{|U_Q|} \int_{U_Q^*} |\nabla u(x)|^2 dV(x).$$

This same holds for any pair of cubes Q_i, Q_{i+1} with $Q_{i+1} \subset Q_i$, being a dyadic child of Q_i, and we get, with the implied constant independent of i, that,

$$|(u_{Q_i}) - (u_{Q_{i+1}})| \lesssim \frac{l(Q_i)^2}{|U_{Q_i}|} \int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x). \tag{4.25}$$

When we sum all the contributions over all the Whitney cubes in the first term on the right in Eq. (4.23), we get up to an implied constant factor, $\int_\Omega |\nabla u(x)|^2 dV(x)$.

We deal with the second term on the right of Eq. (4.23). Take an average over the cube Q,

$$2 \fint_Q \left(\int_{U_Q^*} \frac{(u_Q)^2}{\delta(x)^2} dV(x) \right) d\sigma(y) \approx 2 \frac{1}{\sigma(Q)} \int_Q \left(\frac{(u_Q)^2}{l(Q)^2} |U_Q| \right) d\sigma(y). \tag{4.26}$$

Note that for any $Q' \subset Q$ with both $Q', Q \in \mathbb{D}(Q_0)$, we have

$$l(Q')\sigma(Q') \approx |U_{Q'}| \approx |U_{Q'}^*|, \tag{4.27}$$

where the implied constants are uniform over the cubes in $\mathbb{D}$.

In this case, for any $k \geqslant 1$, and $Q_k \subset Q$ a k-th generation subset of Q with $Q', Q \in \mathbb{D}(Q_0)$, there is a unique sequence of subset $Q = Q_0 \supset Q_1 \supset Q_2 \cdots \supset Q_k$ where for each $1 \leqslant j \leqslant k$, Q_j is a j'th generation subset of Q.

Fix any ϵ, which is to be chosen later. Decompose Q into a countable union of stopping time regimes,

$$Q = \bigcup_{l=1}^{\infty} Q_{n_l},$$

with Q_{n_l} the maximal stopping time cube with the property that $(u_{Q_{n_l}}) \leqslant \epsilon$.

Then, using Eq. (4.27), for a given such sequence, we consider the contribution to the right-hand side of Eq. (4.26), which given by,

$$\begin{aligned}&\frac{1}{\sigma(Q)}\int_{Q_{n_l}}\left(\frac{(((u_{Q_0})-(u_{Q_1}))+\dots((u_{Q_{n_l-1}}-(u_{Q_{n_l}}))+(u_{Q_{n_l}}))^2}{l(Q)^2}|U_Q|\right)d\sigma(y)\\ &\qquad=\frac{\sigma(Q_{n_l})}{\sigma(Q)}|U_Q|\left(\frac{\big(((u_{Q_0})-(u_{Q_1}))+\dots((u_{Q_{n_l-1}}-(u_{Q_{n_l}}))+(u_{Q_{n_l}})\big)^2}{l(Q)^2}\right).\end{aligned}\tag{4.28}$$

Using Eq. (4.27), the above becomes,

$$\approx\frac{l(Q)}{l(Q_{n_l})}\frac{|U_{Q_{n_l}}|}{l(Q)^2}\Big(\sum_{i=0}^{n_l-1}\big((u_{Q_i})-(u_{Q_{i+1}})\big)+(u_{Q_{n_l}})\Big)^2.\tag{4.29}$$

Using the expression Eq. (4.25) which holds for each of the cubes in this sequence, using the Hölder inequality, and writing $l(Q)/l(Q_{n_l})=(l(Q_i)/l(Q_{n_l}))(l(Q)/l(Q_i))$, noting that for each Whitney region we have, $|U^*_Q|=\kappa(l(Q))^n$ for any $Q\in\mathbb{D}(Q_0)$, the main term becomes

$$\begin{aligned}&\lesssim\Big(\sum_{i=0}^{n_l-1}\sqrt{\frac{l(Q_i)}{l(Q)}}\sqrt{\frac{|U_{Q_{n_l}}|}{|U_{Q_i}|}}\sqrt{\frac{l(Q_i)}{l(Q_{n_l})}}\Big(\int_{U^*_{Q_i}}|\nabla u(x)|^2dV(x)\Big)^{\frac{1}{2}})\Big)^2\\ &\qquad\approx\Big(\sum_{i=0}^{n_l-1}\sqrt{\frac{l(Q_i)}{l(Q)}}\Big(\frac{l(Q_{n_l})}{l(Q_i)}\Big)^{\frac{n-1}{2}}\Big(\int_{U^*_{Q_i}}|\nabla u(x)|^2dV(x)\Big)^{\frac{1}{2}})\Big)^2.\end{aligned}\tag{4.30}$$

Note that for each i, the fattened Whitney regions U_{Q_i} and $U_{Q_{i+1}}$ have bounded overlaps. In particular, this implies that the sum of the integrals of the square of the norm of the gradient over the regions U_{Q_i} is a constant multiple of the sum of the integrals of the square of the norm of the gradient over the regions U_{Q_i}, which is the integral over the entire domain. Now use Cauchy Schwarz inequality to upper bound the above by,

$$\begin{aligned}&\lesssim\left(\sum_{i=0}^{n_l-1}\sqrt{\frac{l(Q_i)}{l(Q)}}\right)\left(\sum_{i=0}^{n_l-1}\sqrt{\frac{l(Q_i)}{l(Q)}}\Big(\frac{l(Q_{n_l})}{l(Q_i)}\Big)^{n-1}\Big(\int_{U^*_{Q_i}}|\nabla u(x)|^2dV(x)\Big)\right)+\epsilon'_Q\\ &\qquad=C\left(\sum_{i=0}^{n_l-1}\sqrt{\frac{l(Q_i)}{l(Q)}}\frac{\sigma(Q_{n_l})}{\sigma(Q_i)}\Big(\int_{U^*_{Q_i}}|\nabla u(x)|^2dV(x)\Big)\right)+\epsilon'_Q,\end{aligned}\tag{4.31}$$

where ϵ'_Q is a constant dependent on Q, which we determine below. We note that the first term on the left can be bounded by an infinite geometric progression since we have the inclusion of the dyadic cubes, $Q \supset Q_0 \supset Q_1 \cdots \supset Q_i \cdots \supset Q_{n_l}$ for all $i \leqslant n_l$, and thus can be bounded by a constant K as written above.

We thus get the entire contribution to the right-hand side of Eq. (4.26), using the continuity of u and that it vanishes to 0 to the boundary, and truncating each chain at a suitable stage so that the average of u over the final cube of the chain is as small as chosen. In particular, for this fixed Q, the error ϵ'_Q is bounded by

$$\epsilon'_Q \leqslant \frac{\sigma(Q_{n_l})}{l(Q)}(u_{Q_l})^2 = \frac{l(Q_{n_l})}{l(Q)} l(Q_{n_l})^{n-2}(u_{Q_{n_l}})^2. \tag{4.32}$$

Without loss of generality, we can extend the chains constructed so that the stopping time cubes Q_{n_l} are such that $l(Q_{n_l}) \leqslant 1$, and also obviously $l(Q_{n_l}) \leqslant l(Q)$ by construction. Since $n \geqslant 3$, and we chose $(u_{Q_{n_l}}) \leqslant \epsilon$, we can bound the above term.

Thus for this fixed Q, along a fixed subsequence of decreasing cubes as chosen above, using the continuity of u at the boundary, we can terminate at some Q_l so that $\frac{\sigma(Q_{n_l})}{l(Q)}(u_{Q_l})^2$ is arbitrarily small.

In this process restricted to the cube Q, the number of stopping time regimes are countable. Thus, choosing a countable sequence of (ϵ') 's in a geometric progression, and arguing as above, we only have to deal with a countable sequence of chains contained within each cube, to exhaust the integral over the entire cube Q, up to a total error ϵ_Q which we control as above.

Repeat this process for each of the chains of cubes contained in Q, starting with the expression of Eq. (4.28) in each case.

Enumerate this countable set of chains of cubes contained in Q , which we get in this process, as $\mathcal{C}(\mathcal{Q})_k|_{k=1}^{\infty}$ For a given cube $Q_i \subset Q$, consider the subset of chains from the above in which Q_i appears, and call it $\mathcal{C}(Q, Q_i)$. Each chain in $\mathcal{C}(Q, Q_i)$ terminates in some cube $Q_l \subset Q_i$. Term the length of any chain $c \in \mathcal{C}(Q)$ as $l(c)$ and the maximal stopping time cube for the chain c as Q_c . Also note that the disjoint union of the maximal stopping time cubes of the chains $\mathcal{C}(Q, Q_i)$ is the cube Q_i, which we use in Eq. (4.34).

Now consider the sum in Eq. (4.19), and repeat this above argument for each such Whitney cube U_Q^*. Rearranging the sum in the second step below, the total contribution can be bounded by,

$$\begin{aligned} &K \sum_{Q \in (\mathbb{D}(Q_0)} \left(\sum_{c \in \mathcal{C}(Q)} \sum_{i=0}^{l(c)-1} \sqrt{\frac{l(Q_i)}{l(Q)}} \frac{\sigma(Q_c)}{\sigma(Q_i)} \Big(\int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x) \Big) \right) \\ &\quad = K \sum_{Q \in (\mathbb{D}(Q_0)} \left(\sum_{Q_i \subset Q} \sum_{c \in \mathcal{C}(Q, Q_i)} \sqrt{\frac{l(Q_i)}{l(Q)}} \frac{\sigma(Q_c)}{\sigma(Q_i)} \Big(\int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x) \Big) \right). \end{aligned} \tag{4.33}$$

Here, the sum $\sum_{i}^{l(c)-1}$ is over all the cubes $Q_i \in \mathbb{D}(\partial\Omega)$ that constitute the chain $c \in \mathcal{C}$, with $Q = Q_0 \supset Q_1 \supset \ldots Q_i \supset \cdots \supset Q_l$.

Interchange the two sums in the end, for fixed $Q_i \subset Q$, and sum over the countable stopping time cubes Q_c contained in Q_i, to get that,

$$\begin{aligned} &\sum_{c \in \mathcal{C}(Q, Q_i)} \sqrt{\frac{l(Q_i)}{l(Q)}} \frac{\sigma(Q_c)}{\sigma(Q_i)} \Big(\int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x) \Big) \\ &= \sqrt{\frac{l(Q_i)}{l(Q)}} \left(\int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x) \right) \sum_{c \in \mathcal{C}(Q, Q_i)} \frac{\sigma(Q_c)}{\sigma(Q_i)} = \sqrt{\frac{l(Q_i)}{l(Q)}} \Big(\int_{U_{Q_i}^*} |\nabla u(x)|^2 dV(x) \Big) \Bigg). \end{aligned} \tag{4.34}$$

Returning now to the estimate from Eq. (4.33), and using Eq. (4.34), we have, with a rearrangement of the summation in the second step below,

$$
\begin{aligned}
&K\sum_{Q\in\mathbb{D}(Q_0)}\ \sum_{i,Q_i\subset Q}\left(\sqrt{\frac{l(Q_i)}{l(Q)}}\Big(\int_{U^*_{Q_i}}|\nabla u(x)|^2dV(x)\Big)\right)\\
&=K\sum_{Q\in\mathbb{D}(Q_0)}\ \sum_{k,Q_k\supset Q}\left(\sqrt{\frac{l(Q)}{l(Q_k)}}\Big(\int_{U^*_{Q}}|\nabla u(x)|^2dV(x)\Big)\right)\\
&\qquad\qquad\leqslant K'\sum_{Q\in\mathbb{D}(Q_0)}\left(\int_{U^*_{Q}}|\nabla u(x)|^2dV(x)\right)\leqslant K''\int_{T_{Q_0}}|\nabla u|^2dV(x),\quad (4.35)
\end{aligned}
$$

where the sum over k,

$$
\sum_{k,Q_k\supset Q}\sqrt{\frac{l(Q)}{l(Q_k)}}
$$

is bounded since for a fixed Q the sum runs over the chain of dyadic cubes that contain Q, and are contained in Q_0. Using the bounded overlap of the sets U^*_Q in the last step, we have the required upper bound in terms of $\int_\Omega|\nabla u|^2dV(x)$.

We can further bound the error contribution from each Whitney region U_Q by an arbitrarily small number ϵ_Q. Then combining with the fact that $\int_\Omega|\nabla u(x)|^2dV(x)<\infty$, and a limiting argument, we complete the proof. ∎

We also state the 'global' version of the Hardy inequality that will be used later on in Section 4 as well as Section 5.

Lemma 4.8. *$u\in W^{1,2}(\Omega)\cap C(\overline{\Omega})$, with $u|_{\partial\Omega}=0$, we have* $\big(\int_\Omega(\frac{u}{\delta})^2\big)\lesssim\Big(\int_\Omega|\nabla u|^2\Big)$.

The proof follows from Theorem 2 of [Lew88] with $p=2$, by noting that co-dimension one ADR boundaries satisfy the uniform fatness condition of [Lew88].It can also be proved effectively by the same argument as in the local version Theorem 4.7.

4.4 Boundary Hölder regularity and Bourgain property.

We first show boundary Hölder continuity holds with our singular drift term. This is an adaptation of the proof of Lemma 3.14 in Chapter I of [HL01] for chord-arc domains.

We have the average bounds from Eq. (4.2), for every Whitney region U_Q.

This also implies, up to a constant c_1, that,

$$\int_{U_Q} |\mathcal{B}|^2 dV \leqslant c_1 \beta^2 l(Q)^{n-2}, \tag{4.36}$$

The exact value of β is to be determined later in Step 1 as outlined below.

We get the following theorem,

Theorem 4.9. *Consider the surface ball $\Delta(y, 2r)$ with $y \in \partial\Omega$ and with $r \leqslant diam(\Omega)$. Consider the solution u to Eq.* (4.1) *which vanishes continuously on $\partial\Omega \cap \Delta(y, 2r)$, and suppose that the drift term satisfies the averaged smallness condition of Eq.* (4.2) *and the pointwise condition of Eq.* (4.4). *Then there exist constants $c = c(\gamma_1, M, n)$ and $\alpha = \alpha(\gamma_1, M, n)$, $0 < \alpha < 1 \leqslant c < \infty$, such that*

$$u(z) \leqslant c\big(\frac{|z-y|}{r}\big)^{\alpha} \max_{B(y,2r)\cap\Omega} u, \tag{4.37}$$

whenever $z \in \Omega \cap B(y, r)$.

Proof. We initially follow the proof of Lemma 3.9 in Chapter I of [HL01]. Extend u continuously to 0 outside Ω. Consider a radially symmetric test function $\psi \geqslant 0, \psi \in C_0^\infty(B(y, 2\kappa r))$ with $\psi \equiv 1$ on $B(y, \kappa r)$.

Also, we will choose some constant $c > 0$ with,

$$r||\nabla\psi||_{L^\infty} \lesssim c. \tag{4.38}$$

We use $u\psi^2$ as a test function, to get,

$$\int_{\mathbb{R}^n} (\nabla u\psi^2) \cdot \mathcal{A}\nabla u + (\mathcal{B} \cdot \nabla u)u\psi^2 dV = 0 \tag{4.39}$$

From here we get,

$$\begin{aligned} I_1 = c^{-1}\int_{\mathbb{R}^n} |\nabla u|^2\psi^2 dV \leqslant \int_{\mathbb{R}^n} (\mathcal{A}\nabla u \cdot \nabla u)\psi^2 dV \\ \leqslant c\int_{\mathbb{R}^n} |u||\nabla u|\psi|\nabla\psi|dV + c\int_{\mathbb{R}^n} |u||\mathcal{B}||\nabla u|\psi^2 := I_2 + I_3 \end{aligned}$$

Using Cauchy's inequality with ϵ's , we get that,

$$I_2 \leqslant \frac{1}{2}I_1 + c\int_{\mathbb{R}^n} |\nabla\psi|^2 u^2 dV. \tag{4.40}$$

To deal with I_3, we use the Cauchy inequality with ϵ's, noting that $\psi \in C_0^\infty(B(y, 2\kappa r))$, and Harnack's inequality in the Whitney cubes as above, to get, ⁽³⁾

$$\begin{aligned} \int_\Omega u|\mathcal{B}||\nabla u|\psi^2 dV &\leqslant \epsilon\int_\Omega |\nabla u|^2\psi^2 dV + \frac{1}{\epsilon}\int_\Omega |\mathcal{B}|^2u^2\psi^2 dV \\ &= \epsilon\int_\Omega |\nabla u|^2\psi^2 dV + \frac{1}{\epsilon}\sum_{Q\subset\mathbb{D}(\partial\Omega)}\int_{U_Q} |\mathcal{B}|^2u^2\psi^2 dV. \end{aligned} \tag{4.41}$$

Consider a uniform constant $0 < \theta < 1$ to be fixed later. It is enough to split the set of Whitney cubes in the two following categories:

- Consider $|\text{supp}(\psi) \cap U_Q| > \theta|U_Q|$.

  We can write the second term on the right as,

  $$\int_{U_Q} |\mathcal{B}|^2u^2\psi^2 dV \leqslant (\sup_{\overline{U_Q}})^2(\sup_{\overline{U_Q}}\psi)^2\int_{U_Q} |\mathcal{B}|^2 dV. \tag{4.42}$$

⁽³⁾ Alternately one can also use Hölder's inequality here.

We claim that there exist uniform constants $\eta_1 \geqslant 1, 0 < \eta_2 < 1$ so that, for the Whitney cube U_Q we have on an ample subset $U'_Q \subset (\text{supp}(\psi) \cap U_Q)$ with the property $|U'_Q|/|\text{supp}(\psi) \cap U_Q| \geqslant \eta_2$ and thus $|U'_Q|/|U_Q| \geqslant \eta_2\theta$, that $(\sup_{\overline{U'_Q}} \psi)/(\inf_{\overline{U'_Q}} \psi) < \eta_1$ and $\sup_{\overline{U'_Q}} \psi = (\sup_{\overline{U'_Q}} \psi)$. [4]
This follows by considering that $\psi \subset B(y, 2\kappa r)$ and is radially symmetric within this ball. For any Whitney ball U_Q as described above, consider the point $y_q \in \overline{U_Q}$ with $y_q \in \partial U_Q$, which is closest to the point y, which is where $(\sup_{\overline{U_Q}} \psi)$ is attained. Note that $||\nabla\psi||_{L^\infty} \leqslant \frac{c}{r}$ and that ψ is smooth, and consider the straight line joining y to y_q, and the intersection of this line with U_Q. It is then seen from the structure of ψ that one gets the uniform η_1 and η_2 as described above, with $\sup_{\overline{U'_Q}} \psi = (\sup_{\overline{U'_Q}} \psi) = \psi(y_q)$. Fixing the implied constant dependent on θ, if $l(Q) \ll_\theta \psi(y_q)$, the above claim follows, given any value of $\psi(y_q)$, by looking at the intersection of the ball $B(y, 2\kappa r)$ with U_Q . If $l(Q) \approx_\theta \psi(y_q)$, and in any case we have $l(Q) \lesssim r$, then noting that $||\nabla\psi||_{L^\infty} \leqslant \frac{c}{r} \lesssim \frac{c}{l(Q)}$, and thus that in this case on U_Q, we have $||\nabla\psi||_{L^\infty} \lesssim \frac{c}{\psi(y_q)}$, we also get the result. In case we have $l(Q) \gg_\theta \psi(y_q)$, then we will be in the regime of the second case considered below. Thus for the Whitney cubes that satisfy $|\text{supp}(\psi) \cap U_Q| > \theta|U_Q|$, we get from Eq. (4.42) that,

$$\begin{aligned}\int_{U_Q} |\mathcal{B}|^2 u^2\psi^2 dV &\leqslant (\sup_{\overline{U_Q}} u)^2 (\sup_{\overline{U_Q}} \psi)^2 \int_{U_Q} |\mathcal{B}|^2 dV \leqslant \beta^2 \frac{l(Q)^n}{l(Q)^2} (\sup_{\overline{U_Q}} u)^2 (\sup_{\overline{U_Q}} \psi)^2 \\ &\leqslant \frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 \int_{U'_Q} \frac{(u\psi)^2}{\delta(x)^2} dV \leqslant \frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 \int_{U'_Q} \frac{(u\psi)^2}{\delta(x)^2} dV \\ &\quad + \frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 \int_{U_Q\setminus U'_Q} \frac{(u\psi)^2}{\delta(x)^2} dV \leqslant \frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 \int_{U_Q} \frac{(u\psi)^2}{\delta(x)^2} dV, \quad (4.43)\end{aligned}$$

---

(4)Note that the set $(\text{supp}(\psi) \cap U_Q)$ is a connected set, the intersection of the ball centered at y, with the cube U_Q which has sides parallel to the axes.

where C is a constant due to the Harnack inequality.

- Consider $|\text{supp}(\phi) \cap U_Q| \leqslant \theta |U_Q|$. Consider any point $x_Q \in \text{supp}(\phi) \cap U_Q$ and consider the straight line $l_{x_Q,y}$ joining x_Q with y. Consider the point $y' \in \partial\Omega$ with $y' \in l_{x_Q,y}$ which is the point on the line $l_{x_Q,y}$ at the minimum distance from x_Q .

  Note that y' need not equal y. Then it is not hard to see that there exists a Whitney cube $U_{Q_{ad}}$, intersecting the line segment joining x_Q, y', and not necessarily sharing a side with U_Q so that $U_{Q_{ad}} \subset \text{supp}(\psi)$, and $\inf_{\overline{U_{Q_{ad}}}}(\psi) > \sup_{\overline{U_Q}}(\psi)$. Moreover, we choose the Whitney cube $U_{Q_{ad}}$ in this manner to be that at the minimum distance from U_Q along this line, with the property that $U_{Q_{ad}} \subset \text{supp}(\psi)$, and $\inf_{\overline{U_{Q_{ad}}}}(\psi) > \sup_{\overline{U_Q}}(\psi)$.
  $\text{dist}(U_Q, U_{Q_{ad}}) \approx \text{diam}(U_{Q_{ad}}) \approx \text{diam}(U_Q) \approx \text{dist}(U_Q, \partial\Omega) \approx \text{dist}(U_{Q_{ad}}, \partial\Omega)$, with implied constants uniform over Q. Thus the Whitney cube $U_{Q_{ad}}$ belongs to the set of cubes of the first case above. In fact, for this cube we have, $|\text{supp}(\psi) \cap U_{Q_{ad}}| = |U_{Q_{ad}}|$ Further, on this line, ψ increases from $\psi(x_Q)$ to 1 as one moves from x_Q to y.

  From here, we get using the pointwise bound on the drift term, that,

$$\int_{U_Q} |\mathcal{B}|^2 u^2 \psi^2 dV = \int_{\text{supp}(\psi) \cap U_Q} |\mathcal{B}|^2 u^2 \psi^2 dV \leqslant M^2 \int_{\text{supp}(\psi) \cap U_Q} \frac{u^2 \psi^2}{\delta(x)^2} dV$$
$$= CM^2 \int_{U'_{Q_{ad}}} \frac{u^2 \psi^2}{\delta(x)^2} dV. \quad (4.44)$$

  Here we have chosen a subset $U'_{Q_{ad}} \subset U_{Q_{ad}}$, with, $|U'_{Q_{ad}}| \approx |\text{supp}(\psi) \cap U_Q|$. Here we have used the Harnack inequality for u and incorporated a constant C. Finally, we get that,

$$\int_{U_Q} |\mathcal{B}|^2 u^2 \psi^2 dV = CM^2 \int_{U'_{Q_{ad}}} \frac{u^2\psi^2}{\delta(x)^2} dV \leqslant C'M^2\theta \int_{U_{Q_{ad}}} \frac{u^2\psi^2}{\delta(x)^2} dV. \quad (4.45)$$

Thus, it is enough to consider below,

$$\Big(\int_{U_Q} |\mathcal{B}|^2 u^2\psi^2 dV + \int_{U_{Q_{ad}}} |\mathcal{B}|^2 u^2 \psi^2 dV\Big) \leqslant (\frac{1}{\eta_2\theta} C^2 \eta_1^2 \beta^2 + C'M^2\theta) \int_{U_{Q_{ad}}} \frac{u^2\psi^2}{\delta(x)^2} dV$$
$$\leqslant (\frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 + C'M^2\theta) \Bigg(\int_{U_{Q_{ad}}} \frac{u^2\psi^2}{\delta(x)^2} dV + \int_{U_Q} \frac{u^2\psi^2}{\delta(x)^2} dV \Bigg). \quad (4.46)$$

Here we have used the fact that $U_{Q_{ad}}$ belong to the set of cubes of the first case above, for which the estimate of the first case holds.

Given any Whitney cube U_Q in the second category, we have a unique element $U_{Q_{ad}}$ of the first category constructed above. From the construction above, $U_{Q_{ad}}$ intersects the straight line $l_{x_Q,y}$ and it's distance from U_Q is bounded with a uniform constant times the diameter of U_Q. Thus, there exists a uniform constant n_0 so that any Whitney cube $U_{Q_{ad}}$ of the first category can be used for at most n_0 many Whitney cubes of the second category, in this construction.

Thus we have,

$$\sum_{Q\in\mathbb{D}(\partial\Omega)} \int_{U_Q} |\mathcal{B}|^2 u^2\psi^2 dV \leqslant n_0 (\frac{1}{\eta_2\theta} C^2\eta_1^2\beta^2 + C'M^2\theta) \sum_{Q\in\mathbb{D}(\partial\Omega)} \int_{U_Q} \frac{u^2\psi^2}{\delta(x)^2} dV. \quad (4.47)$$

Thus choosing θ small enough and then β small enough, we get,

$$\sum_{Q\subset\mathbb{D}(\partial\Omega)} \int_{U_Q} |\mathcal{B}|^2 u^2\psi^2 dV \leqslant \sum_{Q\subset\mathbb{D}(\partial\Omega)} \beta'^2 \int_{U_Q} \frac{u^2\psi^2}{\delta(x)^2} dV = \beta'^2 \int_{\Omega} \frac{u^2\psi^2}{\delta(x)^2} dV. \quad (4.48)$$

Here, β' can be a-priori chosen and thus β, θ can be chosen dependent on β'.

Then using the global Hardy inequality, we get that,

$$\beta'^2 \int_\Omega \frac{u^2\psi^2}{\delta(x)^2} dV \leqslant C_1\beta'^2 \int_\Omega |\nabla(u\psi)|^2 dV \leqslant C_2\beta'^2 \int_\Omega |\nabla u|^2\psi^2 dV + C_2\beta'^2 \int_\Omega |\nabla\psi|^2 u^2 dV. \tag{4.49}$$

Combining with Eq. (4.40), hiding the appropriate term, we will get the left hand inequality of the following result, noting that $||\nabla\psi|| \lesssim \frac{1}{r}$,

$$\int_{T_{Q'}} |\nabla u|^2 dV \leqslant cr^{-2} \int_{T_Q} u^2 dV \leqslant c^2 \int_{T_{\overline{Q}}} |\nabla u|^2 dV. \tag{4.50}$$

Here, $\overline{Q} \supset Q$ is the cube considered as in Theorem 4.7, so that, when $Q \in \mathcal{D}(k)$, we have $\overline{Q} \subset \mathcal{D}(l(k))$, with $l(k)/k = O(1)$ with the implied constant the same as in the statement of Theorem 4.7. since we have,

$$r^{-2} \int_{T_Q} u^2 dV \lesssim \int_{T_Q} \frac{u^2(x,t)}{\delta(t)^2} dV(t) \lesssim \int_{T_{\overline{Q}}} |\nabla u|^2 dV. \tag{4.51}$$

Here the second inequality on the right follows by the use of Theorem 4.7.

Now consider the weak solution u_0 so that to $\nabla\cdot(\mathcal{A}\nabla u_0) = 0$, as well as the boundary condition of $u_0 = u$ on the boundary of the Carleson box, which we denote ∂T_Q. The boundary Hölder continuity is known for u_0 , and we also have $u = u_0$ on the boundary ∂T_Q.

In this case, we use $w = u - u_0$ as a test function to get,

$$\int_{T_Q} (\nabla w) \cdot \mathcal{A}\nabla u_0 = 0, \int_{T_Q} (\nabla w) \cdot \mathcal{A}\nabla u + \mathcal{B} \cdot (\nabla u) w = 0. \tag{4.52}$$

Subtracting the first equation from the second, and taking absolute values and then the ellipticity condition on $\mathcal{A}$, noting that $w = u - u_0$, we get that,

$$\int_{T_Q} |\nabla w|^2 dV \leqslant \int_{T_Q} |\mathcal{B}||\nabla u||w| dV. \tag{4.53}$$

One notes using the Cauchy-Schwarz inequality twice on the right,

$$\begin{aligned}\int_{T_Q} |\nabla w|^2 dV \leqslant \int_{T_Q} |\mathcal{B}||\nabla u| w dV &= \sum_{Q'\subset \mathbb{D}(Q)} \int_{U_{Q'}} |\mathcal{B}||\nabla u| w dV \\ &\leqslant \Big(\sum_{Q'\in \mathbb{D}(Q)} (\sup_{U_{Q'}} w)^2 (\int_{U'_Q} \mathcal{B}^2 dV\Big)^{\frac{1}{2}} \Big(\int_{T_Q} |\nabla u|^2 dV\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.54}$$

Thus, using Harnack's inequality, we get up to an implied constant,

$$\begin{aligned}\int_{T_Q} |\nabla w|^2 dV &\leqslant \beta K \Big(\int_{T_Q} \Big(\frac{w}{\delta(X)}\Big)^2 dV\Big)^{\frac{1}{2}} \Big(\int_{T_Q} |\nabla u|^2 dV\Big)^{\frac{1}{2}} \\ &\leqslant \beta K \Big(\int_{T_Q} \Big(\frac{w}{\delta_{T_Q}(X)}\Big)^2 dV\Big)^{\frac{1}{2}} \Big(\int_{T_Q} |\nabla u|^2 dV\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.55}$$

Here, δ_{T_Q} denotes the distance to the boundary of T_Q, and we obviously have using the global dyadic Hardy inequality for the function w in the domain T_Q, that,

$$\int_{T_Q} |\nabla w|^2 dV \leqslant \beta K \big(\int_{T_Q} |\nabla w|^2 dV\big)^{\frac{1}{2}} (\int_{T_Q} |\nabla u|^2 dV)^{\frac{1}{2}}, \tag{4.56}$$

Thus we get,

$$\int_{T_Q} |\nabla w|^2 dV \leqslant \beta^2 K' (\int_{T_Q} |\nabla u|^2 dV). \tag{4.57}$$

For any $Q' \ni y$, we put,

$$\Phi(f, Q) = \frac{1}{l(Q)^{n-2}} \int_{T_Q} |\nabla f|^2 dV. \tag{4.58}$$

For the solution u_0, we have the boundary Hölder regularity Eq. (4.37), which follows by the same argument as that in Lemma 4.1 of [JK82] under the assumption that Ω is a chord-arc domain. Using this estimate, and both sides of the inequality of Eq. (4.74), we get for any $Q_2 \subset Q_1 \in \mathbb{D}$,

$$\Phi(u_0, Q_2) \leqslant c \Big(\frac{l(Q_2)}{l(Q_1)}\Big)^{2\alpha} \Phi(u_0, \overline{Q_1}). \tag{4.59}$$

Here, $\overline{Q_1} \supset Q_1$ is the cube considered as in Theorem 4.7, so that when $Q_1 \in \mathcal{D}(k)$, we have $\overline{Q_1} \subset \mathcal{D}(l(k))$, with $l(k)/k = O(1)$ with the implied constant the same as in the statement of Theorem 4.7.

This follows by noting again that u_0 satisfies boundary Hölder regularity, Eq. (4.37) . So for $Q_2 \subset Q_1 \in \mathbb{D}$, we have,

$$\begin{aligned}\Phi(u_0, Q_2) \leqslant c\frac{1}{l(Q_2)^n}\int_{T_{Q_2}} u_0^2 dV \leqslant c'\Big(\frac{l(Q_2)}{l(Q_1)}\Big)^{2\alpha}(\max_{T_{Q_1}} u_0)^2 \\ \leqslant c''\Big(\frac{l(Q_2)}{l(Q_1)}\Big)^{2\alpha}\frac{1}{l(Q_1)^n}\int_{T_{Q_1}} u_0^2 dV. \quad (4.60)\end{aligned}$$

The last inequality on the right again follows by noting that since u_0 satisfies Eq. (4.37), the supremum attained by u_0 over T_{Q_1} is attained in some Whitney cube U_{Q_j} with $l(Q_j)/l(Q_1) = O(1)$ with the implied constant here independent of Q. Using this fact, and by changing the implied constant c'' in the right-hand expression of Eq. (4.60) , we will get that,

$$\int_{T_{Q_1}} u_0^2 dV \geqslant \int_{U_{Q_j}} u_0^2 dV \geqslant c'''(\max_{T_{Q_1}} u_0)^2 l(Q_j)^n \geqslant c''''(\max_{T_{Q_1}} u_0)^2 l(Q_1)^n.$$

Thus Eq. (4.59) follows from this above, and using the right-hand side of Eq. (4.51).

Now we a-priori modify the pair of cubes (Q_2, Q_1) with $Q_2 \subset Q_1$ to the pair $(\tilde{Q_2}, \tilde{Q_1})$ where, $\tilde{Q_2} \subset Q_2, \tilde{Q_1} \subset Q_1$, with $\tilde{Q_2} \subset Q_2 \subset \tilde{Q_1} \subset Q_1$, so that, $\overline{\tilde{Q_1}} = Q_1$ where $\overline{\tilde{Q_1}}$ was defined in the context of Theorem 4.7.

$$\begin{aligned}&\Phi(u, \tilde{Q_2}) \leqslant 4(\Phi(u_0, \tilde{Q_2}) + \Phi(w, \tilde{Q_2})) \\ &\leqslant c\Big(\frac{l(Q_2)}{l(Q_1)}\Big)^{2\alpha}\Phi(u_0, Q_1) + \Big(\frac{l(Q_1)}{l(Q_2)}\Big)^{n-2}\Phi(w, Q_1) \\ &\qquad \leqslant c'\Big[\Big(\frac{l(Q_2)}{l(Q_1)}\Big)^{2\alpha} + \epsilon'\Big(\frac{l(Q_1)}{l(Q_2)}\Big)^{n-2}\Big]\Phi(u, Q_1). \quad (4.61)\end{aligned}$$

In the last step, we used the fact that $u = u_0$ on ∂T_Q, Harnack and Moser's inequality, on the first term within the brackets, and Eq. (4.57) . Set $l(Q_2)/l(Q_1) = \theta$. Then, using the above inequality, we can choose ϵ' small enough so that,

$$\Phi(u, Q_2) \leqslant \frac{1}{2}\Phi(u, Q_1'). \tag{4.62}$$

With this fixed θ, starting with any cube $Q_1 \ni y$, we can iterate this inequality for cubes of successively smaller radius contained in Q_1 (5), to get the Boundary Hölder continuity result, using the right-hand inequality of Eq. (4.51), Moser's inequality, a crude bound for the term $\Phi(u, Q_2)$, and finally using the left hand inequality of Eq. (4.51) for the term $\Phi(u, Q_1)$. ∎

We note that the above theorem also works in the case of an elliptic function vanishing continuously on a surface ball, even though this has been written for the case of the Dirichlet Green function vanishing on the entire boundary.

Once the boundary Hölder continuity is established (locally) as above, we will get the Bourgain type estimate on the elliptic measure.

Corollary 4.10 (Bourgain estimate)**.** *Given any $x \in \Omega$, and $\hat{x}$ a nearest point to x on the boundary, with $|x - \hat{x}| = r$, there exists a uniform constant $c > 0$, we have, with $\Delta(x) := \Delta(\hat{x}, 10r)$, that $\omega^x(\Delta(x)) \geqslant c$.*

Proof. The function $u(x) := \omega^x(\Delta(x))$ is the unique elliptic continuation with boundary data f with $f = 1$ on Δ and $f = 0$ elsewhere with the condition that $u \to 0$ at ∞. Then consider the function, $v = 1 - u$ which then vanishes continuously on the surface ball $\Delta(x)$, and for which the Boundary Hölder continuity holds for the function v. For points

---

(5) and not necessarily containing y

$x \in B(y, \kappa r) \cap \Omega$ with $\kappa < 1$ sufficiently small, the boundary Hölder continuity gives us that $v(x) \leqslant C\kappa^{\alpha} \max_{T_Q} v$ with $Q \in \mathbb{D}$ with $l(Q) \approx r$, with constants uniform in r, y. This gives us the Bourgain-type estimate for the function $u(x) = 1 - v(x) \geqslant 1 - C\kappa^{\alpha} \max_{T_Q} v$, where by the maximum principle we also have $\max_{T_Q} v \leqslant 1$ and so we have a uniform lower bound on $u(x)$ whenever $x \in B(y, \kappa r) \cap \Omega$. By choosing k appropriately, we get the statement of the result. ∎

4.5 Green's function estimates

In this step we show that when we have the bound of Eq. (4.2)

$$\int_{U_Q} |\mathcal{B}|^2 dV \leqslant \beta^2 l(Q)^{n-2}, \tag{4.63}$$

for every Whitney cube in our domain, with a small parameter β to be determined later, then we have pointwise upper and lower bounds on the Dirichlet Green's function.

We show that the Green function $G(x, y)$ satisfies both the pointwise upper and lower bounds in the ball $B(x, \delta(x)/2)$. The pointwise lower bound on the Green function actually holds more generally with just the background assumption of the pointwise bound on the drift term, $|\mathcal{B}| \leqslant M/\delta(x)$.

The lower bound follows by an argument similar to that used in [GrWi], as well as the corresponding arguments in Chapters 2 and 3 of this thesis. This argument appears almost identically in [Pat24], specifically for the case where we have the Laplacian as the principal term. For the sake of completeness, we include the proof here.

Theorem 4.11. *For the elliptic operator in Eq.* (4.1)*, with* $\geqslant 3$*, with the coefficients satisfying Eq.* (4.4)*, we have the bound: for any* $z, y \in \Omega$ *with* $|y - x| \leqslant \frac{1}{2}\delta(x)$ *we have*

$\mathcal{G}(x,y) > 0$, *and*

$$\mathcal{G}(y,x) \geqslant K(M,\lambda)\frac{1}{|x-y|^{n-2}}. \tag{4.64}$$

Proof of Theorem 4.11. The proof essentially follows by extending the argument of the proof of Eq.(1.9) of [GrWi]. Take $r := |z-y|$. Consider a smooth cut-off function η which is 1 on $B_r(y) \setminus B_{r/2}(y)$ and zero outside $B_{3r/2}(y) \setminus B_{r/4}(y)$, and further $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant \frac{K}{r}$.

Henceforth, we use the Einstein summation convention, where the summation sign is implied.

Given the domain Ω, for any admissible test function ϕ, the Green function satisfies the following adjoint equation;

$$\int_\Omega \Big((\nabla\phi)_i \mathcal{A}^T_{ij}(\nabla\mathcal{G}(y,x))_j - \mathcal{G}(y,x)\mathcal{B}\cdot(\nabla\phi)\Big)dx = \phi(y). \tag{4.65}$$

Here $\mathcal{A}^T_{ij}$ denotes the transpose of the matrix $\mathcal{A}_{ij}$.

We consider the test function $\phi(x) = G(y,x)\eta^2(x)$, and first get,

$$\begin{aligned}\int_{B(0,1)} |\nabla\mathcal{G}(y,x)|^2\eta^2 dx = -\int_{B(0,1)} & 2\eta\mathcal{G}(y,x)\nabla\mathcal{G}(y,x)\cdot\nabla\eta \\ & + \mathcal{G}(y,x)\eta^2\mathcal{B}\cdot\nabla\mathcal{G}(y,x) + 2\mathcal{G}^2(y,x)\eta\mathcal{B}\cdot\nabla\eta)dx. \end{aligned} \tag{4.66}$$

Now by using the bound on the drift term $\mathcal{B}$, the Cauchy inequality with ϵ's, with small enough ϵ, for the first two terms on the right,

$$\begin{aligned}&\int_{B(0,1)} \mathcal{G}(y,x)\eta\nabla\mathcal{G}(y,x)\cdot\nabla\eta dx \leqslant \epsilon\int_{B(0,1)} \eta^2|\nabla\mathcal{G}(x,y)|^2dx \\ &+ \frac{K^2}{\epsilon r^2}\int_{r/4\leqslant|x-y|\leqslant 3r/2} G(y,x)^2dx, \\ &\int_{B(0,1)} \mathcal{G}(y,x)\eta^2\nabla\mathcal{G}(y,x)\cdot\mathcal{B}dx \leqslant \epsilon\int_{B(0,1)} \eta^2|\nabla\mathcal{G}(x,y)|^2dx\end{aligned}$$

$$+\frac{K}{\epsilon\delta(y)^2}\int_{r/4\leqslant|x-y|<3r/2}\mathcal{G}(y,x)^2dx. \quad (4.67)$$

Using the bounds on the cut-off function η introduced above, and hiding the term with the square of the gradient of $\mathcal{G}(y,x)$, we get,

$$\begin{aligned}\int\limits_{r/2<|x-y|<r}|\nabla\mathcal{G}(y,x)|^2dx &\leqslant \Big(K_1\frac{1}{r^2}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big)\\ &+\Big(K_2\frac{1}{r\delta(y)}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big)+\Big(K_3\frac{1}{\delta(y)^2}\cdot\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big).\end{aligned} \quad (4.68)$$

Noting that $r\leqslant\frac{1}{2}\delta(y)$, we get

$$\begin{aligned}\int\limits_{r/2<|x-y|<r}|\nabla\mathcal{G}(y,x)|^2dx &\leqslant \tilde{K}\frac{1}{r^2}\cdot\Big(\int\limits_{r/4<|x-y|<3r/2}\mathcal{G}(y,x)^2dx\Big)\\ &\leqslant \tilde{K}r^{n-2}\big(\sup_{r/4\leqslant|x-y|\leqslant3r/2}\mathcal{G}(y,x)^2\big).\end{aligned} \quad (4.69)$$

Again as in [GrWi], choose a similar cut-off function ϕ that is 1 on $B_{r/2}(y)$ and zero outside $B_r(y)$, and using it as the test function we get,

$$\begin{aligned}1 &= \int\limits_{r/2\leqslant|x-y|\leqslant r}(\partial_i\mathcal{G}(y,x)\partial_i\phi+\mathcal{G}(y,x)\mathcal{B}_i\partial_i\phi\Big)dx)\\ &\leqslant M\frac{K}{r}\int\limits_{r/2\leqslant|x-y|\leqslant r}|\nabla\mathcal{G}(y,x)|dx+\frac{M}{r\delta(y)}\int\limits_{r/2\leqslant|x-y|\leqslant r}\mathcal{G}(y,x)dx.\end{aligned} \quad (4.70)$$

Using the identity of Eq. (4.69), and Cauchy's inequality for the first term on the right, along with a trivial volume bound, and finally Harnack's inequality,

$$1\leqslant Kr^{n-2}\sup_{r/4\leqslant|x-y|\leqslant3r/2}\mathcal{G}(y,x)\leqslant K|z-y|^{n-2}\mathcal{G}(y,z). \quad (4.71)$$

∎

Using Harnack inequality, and Theorem 4.11, we immediately get,

Corollary 4.12. *For the elliptic operator with the coefficients satisfying Eq.* (4.4) *in* Ω*, we have the lower bound for the Green function for the operator in Eq.* (4.1)*: for any* $z, y \in \Omega$ *with* $|y - z| \leqslant \frac{1}{2}\delta(z) := \frac{1}{2}dist(z, \partial\Omega)$ *we have*

$$\mathcal{G}(y, z) \geqslant c_0 \frac{1}{|z - y|^{n-2}}. \tag{4.72}$$

We now give an argument for the pointwise upper bounds in such domains bounded away from the boundary.

Theorem 4.13. *Consider the operator of Eq.* (4.1)*, for* $n \geqslant 3$*, and the Dirichlet Green's function corresponding to this operator, where the drift satisfies the average smallness on Whitney cubes of Eq.* (4.2)*, for some* β *small enough, along with the pointwise bound of Eq.* (4.4)*. For any* $y, x \in \Omega$ *with* $|y - x| \approx \frac{1}{2}\delta(x)$*, we have for some constant* K' *dependent on* M, λ *only,*

$$\mathcal{G}(y, x) \leqslant K'(M, \lambda) \frac{1}{|x - y|^{n-2}}. \tag{4.73}$$

Proof. Consider the operator L considered in Eq. (4.1), and the Dirichlet Green's function as the kernel of the operator L^{-1} that gives the solution u for the data f which is locally in $L^{2_*}_{loc}$, and so $Lu = f$.

Now consider the pole of the Green function x. Without loss of generality, we subsequently only consider any point on the boundary of the ball, $y \in \partial B(x, \delta_\Omega(x)/2)$. Thus we also have, $|x - y| = \frac{1}{2}\delta(x) = c_1\delta(y)$.

Without loss of generality, it is also enough to consider the case where $x = x_{Q_0}$ of the Whitney ball U_{Q_0}. See Remark 4.14.

We find an absolute constant $\kappa < 1$ so that there exist balls B_x, B_y each of radius $\kappa|x - y|$ around the points x, y respectively, that are disjoint, i.e.,

$$B_x = \{z : |z-x| < \kappa|x-y|\}, \quad B_y = \{z : |z-y| < \kappa|x-y|\}, \tag{4.74}$$

and so that we have the point-wise bounds,

$$|\mathcal{B}(x)| \leqslant \frac{\beta'}{\delta(x)}. \tag{4.75}$$

in an ample portion of B_x that is quantified below, with an altered β' , upon performing a Calderón-Zygmund decomposition (as outlined, for example, in Section 3.4 of Chapter 1 of [St70]).

Choose $\theta \ll 1$, an arbitrarily small positive number and some $\theta_1 = K\theta$ with $K \geqslant 1$ to be chosen later. We choose a β' dependent on β , with β'/β sufficiently large so that considering the Calderón-Zygmund decomposition of the function $|\mathcal{B}|^2 1_{U_{Q_0}}$, we find a 'bad set' $\Omega_B \subset U_{Q_0}$ which is a union of bad cubes with disjoint interiors, $\Omega_B = \cup_k P_k$ so that,

$$|\Omega_B| = \sum_i |P_i| \leqslant \frac{C}{\left(\frac{\beta'}{l(Q_0)}\right)^2}\beta^2 l(Q_0)^{n-2} = C\Big(\frac{\beta}{\beta'}\Big)^2\Big(l(Q_0)^n\Big) = C'\Big(\frac{\beta}{\beta'}\Big)^2 |U_{Q_0}|, \tag{4.76}$$

Further, there exists a constant K', so that for every 'bad cube' P_i,

$$\int_{P_i} |\mathcal{B}|^2 dV \leqslant K'\Big(\frac{\beta'}{l(Q_0)}\Big)^2 |P_i|, \tag{4.77}$$

The set Ω_B is the union of cubes $\Omega_B = \cup_{i=} P_i$. Now consider the set $\Omega_B' := \cup_i 5P_i$, where aP_i is the cube with side length $a \cdot l(P_i)$ and with the same center.

Consider the function f:

$$f = \begin{cases} 1 & \text{on } B_x \setminus \Omega_B', \\ 0 & \text{elsewhere,} \end{cases} \tag{4.78}$$

which is the indicator function of the set $B_x \setminus \Omega_B'$, and thus

$$\int_{B_x} f = c(1-\theta'')|x-y|^n, \tag{4.79}$$

with θ'' is a slightly altered constant.

Note that, we have the point-wise estimate of Eq. (4.75) in $(U_{Q_0} \setminus \Omega'_B) \subset (U_{Q_0} \setminus \Omega_B)$. Also, we have $f = 0$ in all the Whitney balls U_Q other than the unique Whitney ball U_{Q_0} so that $B_x \subset U_{Q_0}$.

We have the formula, using the Dirichlet Green's function,

$$u(y) = \int_{B_x} G(y, x) f(x) dx. \tag{4.80}$$

Using u itself as a test function with $u = 0$ on $\partial\Omega$, we get with a standard integral by parts, that,

$$\int_{\Omega} (\mathcal{A}^* \nabla u \cdot \nabla u - \mathcal{B} u \nabla u) = \int_{\Omega} f u = \int_{B_x} f u. \tag{4.81}$$

Using the ellipticity condition on the matrix of coefficients $\mathcal{A}$, we get

$$\Big(\int_{\Omega} |\nabla u|^2 \Big) \leqslant \Big(\int_{\Omega} \mathcal{A}^* \nabla u \cdot \nabla u \Big). \tag{4.82}$$

Here, we use the notation,

$$2_* = \frac{2n}{n+2}, 2^* = \frac{2n}{n-2}. \tag{4.83}$$

We use Eq. (4.81), Eq. (4.75), and Hölder's inequality . We combine this with the pointwise estimates within the balls B_x, B_y, using Cauchy-Schwarz inequality on the first term on the right, in the manner of Eq. (4.54), by first splitting up the first integral on the right of Eq. (4.81) in four separate parts, one over the union of the Whitney regions other than U_{Q_0} where by construction $f = 0$, one over $U_{Q_0} \setminus B_x$ which is a region where again $f = 0$, one integral over $B_x \setminus \Omega'_B$ where $f \neq 0$ and finally one over Ω'_B where again $f = 0$ but where we have by construction the bad cubes for the Calderón-Zygmund decomposition:

$$
\begin{aligned}
|\int_{\Omega}\mathcal{B}u\nabla u| &= |\sum_{Q\neq Q_0}\int_{U_Q}\mathcal{B}u\nabla u+\int_{(U_{Q_0}\setminus B_x)\setminus\Omega_B}\mathcal{B}u\nabla u+\int_{(B_x\setminus\Omega_B)}\mathcal{B}u\nabla u|+\int_{\Omega_B}\mathcal{B}u\nabla u| \\
&\leqslant \sum_{Q\neq Q_0}\int_{U_Q}u|\mathcal{B}||\nabla u|+\int_{(U_{Q_0}\setminus B_x)\setminus\Omega_B}\frac{\beta'}{\delta(x)}u|\nabla u|+\int_{(B_x\setminus\Omega_B)}\frac{\beta'}{\delta(x)}u|\nabla u|+\int_{\Omega_B}|\mathcal{B}||u\nabla u|.
\end{aligned}
\tag{4.84}
$$

Note that, for the ball B_x, we would only be able to use Harnack's inequality with bounds depending on the $||f||_q$ norm of f with some $q > n/2$, and we avoid the necessity to use the Harnack inequality such as in Eq. (4.54), by extracting an ample set $B_x \setminus \Omega_B \supset B_x \setminus \Omega'_B$ where we actually have the point-wise bounds on the $\mathcal{B}$ term.

The first term on the right of Eq. (4.84) is a sum over the Whitney regions that do not intersect B_x, and in each of these Whitney regions we have the estimate of the form Eq. (4.63) and thus an argument identical to Eq. (4.54) works here, with the modification that the constant for the Harnack inequality is worse for the Whitney cubes that are adjacent to U_{Q_0}. Here we note that κ can be a-priori chosen small enough so that the sphere $S_x = \{z : |z-y| = \kappa|x-y|\}$, is an ample distance away from the boundary of U_{Q_0}, so that one can use the Harnack inequality

This is because the term $f = 0$ in $U_{Q_0} \setminus B_x$, but $f \neq 0$ in $B_x \setminus \Omega'_B$. Thus, we do not have the solution $Lu = 0$ in the fixed dilate κU_{Q_1}. Thus we would employ Harnack's inequality in each of the smaller subsets V_j^1, where $U_{Q_1} = \cup_{j=1}^{N} V_j^{(1)}$, with a uniform constant κ' where $(\mathrm{diam}(V_j^{(1)})/\mathrm{diam}U_{Q_1}) \geqslant \kappa'$, leading to a worse Harnack constant but which is still uniform across the domain.

Thus from this first term on the right of Eq. (4.84), we get,

$$
\begin{aligned}
\sum_{Q\neq Q_0}\int_{U_Q}u|\mathcal{B}||\nabla u|dV &\leqslant \Big(\sum_{Q\neq Q_0}(\sup_{U_Q}u)^2\int_{U_Q}\mathcal{B}^2dV\Big)^{\frac{1}{2}}\Big(\sum_{Q\neq Q_0}\int_{U_Q}|\nabla u|^2dV\Big)^{\frac{1}{2}} \\
&\leqslant \beta'\Big(\sum_{Q\neq Q_0}\int_{U_Q}\Big(\frac{u}{\delta}\Big)^2dV\Big)^{\frac{1}{2}}\Big(\sum_{Q\neq Q_0}\int_{U_Q}|\nabla u|^2dV\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\leqslant \frac{\beta'}{2}\Big(\sum_{Q\neq Q_0} \int_{U_Q} \Big(\frac{u}{\delta}\Big)^2 dV + \sum_{Q\neq Q_0} \int_{U_Q} |\nabla u|^2 dV\Big). \quad (4.85)$$

Note that the Harnack inequality is essential to get from the first factor of Eq. (4.85) in the second step, to the first factor in the third step above.

For the second and third terms on the right of Eq. (4.84) we use the point-wise estimate of Eq. (4.75), and Cauchy-Schwarz inequality to get,

$$\int_{(U_{Q_0}\backslash B_x)\backslash\Omega_B} \frac{\beta'}{\delta(x)} u|\nabla u| + \int_{(B_x\backslash\Omega_B)} \frac{\beta'}{\delta(x)} u|\nabla u| \leqslant \beta'\Big(\int_{(U_{Q_0}\backslash B_x)\backslash\Omega_B} \Big(\frac{u}{\delta}\Big)^2 dV$$
$$+ \int_{(U_{Q_0}\backslash B_x)\backslash\Omega_B} |\nabla u|^2 dV + \int_{B_x\backslash\Omega_B} \Big(\frac{u}{\delta}\Big)^2 dV + \int_{B_x\backslash\Omega_B} |\nabla u|^2 dV\Big). \quad (4.86)$$

Finally, for the last term on the right of Eq. (4.84), we get using Eq. (4.77), and an argument similar to Eq. (4.85), using Cauchy Schwarz inequality in the last step:

$$\int_{\Omega_B} |\mathcal{B}|u|\nabla u| dV = \sum_i \int_{P_i} u|\mathcal{B}||\nabla u| dV \leqslant \Big(\sum_i (\sup_{P_i} u)^2 \int_{P_i} |\mathcal{B}|^2 dV\Big)^{\frac{1}{2}} \Big(\sum_i \int_{P_i} |\nabla u|^2 dV\Big)^{\frac{1}{2}}$$
$$\leqslant \beta''\Big(\sum_i (\sup_{P_i} u)^2 \frac{|P_i|}{l(Q_0)^2}\Big)^{\frac{1}{2}} \Big(\sum_i \int_{P_i} |\nabla u|^2 dV\Big)^{\frac{1}{2}}$$
$$\leqslant \frac{\beta''}{2}\Big(\int_{P_i} \Big(\frac{u}{\delta(x)}\Big)^2 dV + \sum_i \int_{P_i} |\nabla u|^2 dV\Big), \quad (4.87)$$

and we have used that $f = 0$ on $\cup_i 5P_i$[6], and that we can use Harnack's inequality.

Using Eqs. (4.85) to (4.87), we finally get, relabelling the small constant as β', that,

$$\int_\Omega \mathcal{A}^* \nabla u \cdot \nabla u = \int_\Omega |\mathcal{B}| u \nabla u + \int_\Omega f u \leqslant \beta'\Big(\int_\Omega |\nabla u|^2 dV + \int_\Omega \Big(\frac{u}{\delta}\Big)^2 dV\Big)$$
$$+ ||f||_{B_2,2_*} ||u||_{B_2,2^*}. \quad (4.88)$$

Here we use the notation, $||\phi||_{B_2,r} := (\int_{B_2} \phi^r)^{1/r}$.

---

[6] We note that the factor of 5 can be altered to any uniform $\kappa > 1$ while worsening the Harnack constant.

We now use the global version of the Hardy inequality given in Lemma 4.8. When β' is small enough, we hide the first term in Eq. (4.88) and using Sobolev's inequality and Harnack inequality, we have,

$$||u||^2_{\Omega,2^*} = \Big(\int_\Omega (u)^{2^*}\Big)^{\frac{2}{2^*}} \lesssim \Big(\int_\Omega |\nabla u|^2\Big) \leqslant C||f||_{B_x,2_*}||u||_{B_x,2^*}. \tag{4.89}$$

This implies,

$$||u||_{B_y,2^*} \leqslant ||u||_{\Omega,2^*} \leqslant \frac{||u||^2_{\Omega,2^*}}{||u||_{B_x,2^*}} \leqslant C||f||_{B_x,2_*}. \tag{4.90}$$

This gives us,

$$\left(\int_{B_y}\left(\int_{B_x} G(y,x)f(x)dx\right)^{2^*} dy\right)^{\frac{1}{2^*}} \leqslant \Big(\int_{B_x} f^{2_*}dx\Big)^{\frac{1}{2_*}}, \tag{4.91}$$

From here, we get, with $r = |x-y|$,

$$\inf_{x\in B_x, y\in B_y} G(y,x)r^n r^{\frac{n}{2^*}} \leqslant Cr^{\frac{n}{2_*}}. \tag{4.92}$$

Recall that the radii of B_x, B_y are comparable to $|x-y|$, with uniform constants, and that f is the indicator of the ball B_x Recall the formula Eq. (4.80), and using Harnack's inequality for the Green function $G(x,y)$ in the ball B_y centered on y, we have from Eq. (4.90),

$$\inf_{y\in B_y, x\in B_x} G(x,y) \leqslant \frac{C}{|x-y|^{(n-2)}}. \tag{4.93}$$

The Harnack inequality now establishes the upper bound with a slightly different constant.∎

Remark 4.14. Note that in general when $x \neq x_Q$, we would have to perform the Calderón Zygmund decomposition for a set of Whitney cubes adjacent to U_Q, and the balls B_x, B_y would belong to a union of these cubes and the argument would proceed similarly as the one presented with minor modifications.

Remark 4.15. We only considered $y \in \Omega$ with $|x - y| \approx \frac{1}{2}\delta(x)$ in Theorem 4.13. This is enough to employ the doubling argument of [Ai06] (See (3.3) of [Ai06]). If we have in particular the Eq. (4.3) assumption, we can prove this result more generally for any $y \in B(x, \delta(x)/2)$, and we do not need a Calderón Zygmund decomposition as in the proof here. However, with the assumption of Eq. (4.2) in the hypothesis of Theorem 4.13, we do not prove this more general result for all $y \in B(x, \delta(x/2))$. In particular, in this general case if we choose the balls B_x, B_y with radii proportional to $|x - y|$ as $|x - y| \to 0$, then in general we can't have control on the lower bound in the integral in Eq. (4.79) upon doing the Calderón Zygmund decomposition, and then further in Eq. (4.91).

4.6 Doubling of the elliptic measure.

We have obtained two sided bounds on the Green function, from Corollary 4.12 and Theorem 4.13. We also have the Bourgain estimate in this case in Corollary 4.10. This immediately give us the following doubling for the elliptic measure.

Proof of Theorem 4.6. The argument follows immediately from the argument for the more general John domains, given in Lemma 3.5 and Lemma 3.6 of [Ai06].∎

We note that an alternate argument for the argument of [Ai06] is also presented in

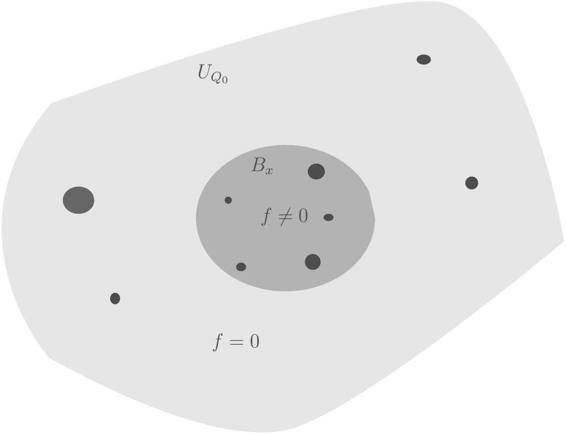

Figure 4.1: The Whitney region U_{Q_0} is shown, along with the ball B_x with center $x = x_{Q_0}$ which without loss of generality we have also taken to be the center of U_{Q_0}. The union of the dark regions is the set $\Omega'_B = \cup_i 5P_i$, and $f = 1$ on the set $B_x \setminus \Omega'_B$ and 0 elsewhere. We have the point-wise bound on the drift term, in $U_{Q_0} \setminus \Omega_B$ with $\Omega_B = \cup P_i$. Here the P_i are the set of bad cubes contained in U_{Q_0}, obtained from the Calderón Zygmund decomposition for the function $|\mathcal{B}|^2 1_{U_{Q_0}}$.

[H24].

Bibliography

[Ai06] Aikawa, Hiroaki, and Kentaro Hirata. "Doubling conditions for harmonic measure in John domains." In Annales de l'institut Fourier, vol. 58, no. 2, (2008): 429-445.

[An86] Ancona, Alano. "On strong barriers and an inequality of Hardy for domains in Rn." Journal of the London Mathematical Society 2, no. 2 (1986): 274-290.

[BL04] Bennewitz, Björn, and John L. Lewis. "On weak reverse Hölder inequalities for nondoubling harmonic measures." Complex Variables, Theory and Application: An International Journal 49, no. 7-9 (2004): 571-582.

[Ch90] Christ, Michael. "AT (b) theorem with remarks on analytic capacity and the Cauchy integral." In Colloquium Mathematicae, vol. 60, no. 2, (1990): 601-628.

[Ch91] Christ, Michael. "On the equation in weighted L 2 norms in C1 equation in weighted L 2 norms in C1." The Journal of Geometric Analysis 1, no. 3 (1991): 193-230.

[CFMS81] Caffarelli, Luis, Eugene Fabes, Stefano Mortola, and Sandro Salsa. "Boundary behavior of nonnegative solutions of elliptic operators in divergence form." Indiana University Mathematics Journal 30, no. 4 (1981): 621-640.

[DJK84] Dahlberg, Björn EJ, David S. Jerison, and Carlos E. Kenig. “Area integral estimates for elliptic differential operators with non-smooth coefficients.” Arkiv för Matematik 22, no. 1-2 (1984): 97-108.

[DKP11] Dindos, Martin, Carlos Kenig, and Jill Pipher. “BMO solvability and the A_∞ condition for elliptic operators.” Journal of Geometric Analysis 21 (2011): 78-95.

[DS1] David, Guy, and Stephen Semmes. “Au-delà des graphes lipschitziens.” Astérisque 193 (1991).

[DS2] David, Guy, and Stephen Semmes. “Analysis of and on uniformly rectifiable sets.” Vol. 38. American Mathematical Soc., 1993.

[Ev22] Evans, Lawrence C. “Partial differential equations”. Vol. 19. American Mathematical Society, 2022.

[FKP91] Fefferman, Robert A., Carlos E. Kenig, and Jill Pipher. “The theory of weights and the Dirichlet problem for elliptic equations.” Annals of Mathematics 134, no. 1 (1991): 65-124.

[Gra23] Grafakos, Loukas. “Fundamentals of Fourier Analysis.” Springer, 2024.

[GT77] Gilbarg, David, Neil S. Trudinger, David Gilbarg, and N. S. Trudinger. “Elliptic partial differential equations of second order.” Vol. 224, no. 2. Berlin: springer, 1977.

[GrWi] Grüter, Michael, and Kjell-Ove Widman. “The Green function for uniformly elliptic equations.” Manuscripta Mathematica 37, no. 3 (1982): 303-342.

[H24] Hofmann, Steve. “The Dirichlet Problem for elliptic equations with singular drift terms” arxiv.org/abs/2502.03665.

[Ha24] Hara, Takanobu. “Global Hö lder solvability of second order elliptic equations with locally integrable lower-order coefficients.” arXiv preprint arXiv:2403.01104 (2024).

[HF11] Han, Qing, and Fanghua Lin. “Elliptic partial differential equations.” Vol. 1. American Mathematical Soc., 2011.

[HK20] Hwang, Sukjung, and Kim, Seick. “Green’s function for second order elliptic equations in non-divergence form.” Potential Analysis 52 (2020): 27-39.

[HL01] Hofmann, Steve., and Lewis, John. L. (2001). “The Dirichlet problem for parabolic operators with singular drift terms (Vol. 719).” American Mathematical Society.

[HL18] Hofmann, Steve, and Phi Le. “BMO solvability and absolute continuity of harmonic measure.” The Journal of Geometric Analysis 28, no. 4 (2018): 3278-3299.

[HLMP22] Hofmann, Steve, Linhan Li, Svitlana Mayboroda, and Jill Pipher. “The Dirichlet problem for elliptic operators having a BMO anti-symmetric part.” *Mathematische Annalen* 382, no. 1-2 (2022): 103-168.

[HM12] Hofmann, Steve, and José María Martell. “$A\infty$ estimates via extrapolation of Carleson measures and applications to divergence form elliptic operators.” *Transactions of the American Mathematical Society* 364, no. 1 (2012): 65-101.

[HM14] Hofmann, Steve, and Martell, José M. "Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in \$ Lp̂\$." In *Annales scientifiques de l'École Normale Supérieure*, vol. 47, no. 3, pp. 577-654. 2014.

[HM17] Hofmann, Steve., Martell, José M. "A sufficient geometric criterion for quantitative absolute continuity of harmonic measure." *preprint* (2017).

[HMT17] Hofmann, Steve, José María Martell, and Tatiana Toro. "A∞ implies NTA for a class of variable coefficient elliptic operators." Journal of Differential Equations 263, no. 10 (2017): 6147-6188.

[HMMTZ21] Hofmann, Steve., Martell, Jose.M., Mayboroda, Svitlana., Toro, Tatiana. and Zhao, Zihui., 2021. "Uniform rectifiability and elliptic operators satisfying a Carleson measure condition." Geometric and Functional Analysis, 31(2): 325-401.

[HW68] Hunt, Richard A., and Wheeden, Richard L. "On the boundary values of harmonic functions." Transactions of the American Mathematical Society 132, no. 2 (1968): 307-322.

[HW70] Hunt, Richard A., and Wheeden Richard L. "Positive harmonic functions on Lipschitz domains." Transactions of the American Mathematical Society 147, no. 2 (1970): 507-527.

[JK82] Jerison, David S., and Kenig, Carlos E. "Boundary behavior of harmonic functions in non-tangentially accessible domains." Advances in Mathematics 46, no. 1 (1982): 80-147.

[Ken94] Kenig, Carlos E. Harmonic analysis techniques for second order elliptic boundary value problems. Vol. 83. American Mathematical Soc., 1994.

[KLT11] Korte, Riikka., Lehrbäck Juha. , and Tuominen, Heli. “The equivalence between pointwise Hardy inequalities and uniform fatness.” Mathematische Annalen 351 (2011): 711-731.

[KP01] Kenig, Carlos E., and Jill Pipher. “The Dirichlet problem for elliptic equations with drift terms.” Publicacions Matemàtiques (2001): 199-217.

[KS19] Kim, Seick., and Sakellaris, Georgios. (2019). “Green’s function for second order elliptic equations with singular lower order coefficients.” Communications in Partial Differential Equations, 44(3), 228-270.

[Lew88] Lewis, John L. “Uniformly fat sets.” Transactions of the American Mathematical Society 308, no. 1 (1988): 177-196.

[MM22] Maz’ya, Vladimir, and Robert McOwen. ”The Fundamental Solution of an Elliptic Equation with Singular Drift.” arXiv preprint arXiv:2209.00058 (2022).

[Moo15] Mooney, Connor. “Harnack inequality for degenerate and singular elliptic equations with unbounded drift.” Journal of Differential Equations 258, no. 5 (2015): 1577-1591.

[Mour23] Mourgoglou, Mihalis. Regularity theory and Green’s function for elliptic equations with lower order terms in unbounded domains. Calc. Var. **62**, 266 (2023).

[Na58] Nash, John. ”Continuity of solutions of parabolic and elliptic equations.” American Journal of Mathematics 80, no. 4 (1958): 931-954.

[Naz12] Nazarov, A. “The Harnack inequality and related properties for solutions of elliptic and parabolic equations with divergence-free lower-order coefficients.” St. Petersburg Mathematical Journal 23, no. 1 (2012): 93-115.

[Pat24] Pathak, Aritro. “A counterexample for pointwise upper bounds on Green’s function with a singular drift at boundary.” arXiv:2405.13313.

[Pog24] Poggi, Bruno. “Applications of the landscape function for Schrödinger operators with singular potentials and irregular magnetic fields.” Advances in Mathematics 445 (2024): 109665.

[Sak1] Sakellaris, Georgios. “On scale-invariant bounds for the Green’s function for second-order elliptic equations with lower-order coefficients and applications.” Analysis & PDE 14, no. 1 (2021): 251-299.

[Sak2] Sakellaris, Georgios. ”Scale invariant regularity estimates for second order elliptic equations with lower order coefficients in optimal spaces.” Journal de Mathématiques Pures et Appliquées 156 (2021): 179-214.

[She94] Shen, Zhongwei. ”On the Neumann problem for Schrödinger operators in Lipschitz domains.” Indiana University Mathematics Journal (1994): 143-176.

[She95] Shen, Zhongwei. “L^p estimates for Schrödinger operators with certain potentials.” In Annales de l’Institut Fourier, vol. 45, no. 2, pp. 513-546. 1995.

[She99] Shen, Zhongwei. ”On fundamental solutions of generalized Schrödinger operators.” Journal of Functional Analysis 167, no. 2 (1999): 521-564.

[SSSZ12] Seregin, Gregory, Luis Silvestre, Vladimír Šverák, and Andrej Zlatoš. "On divergence-free drifts." Journal of Differential Equations 252, no. 1 (2012): 505-540.

[St70] Stein, Elias M. "Singular integrals and differentiability properties of functions." Princeton university press, 1970.

[Wi24] Wiener, Norbert. "The Dirichlet Problem". Journal of Mathematics and Physics, 3 (1924), no. 3, 127–146.

[Z18] Zhao, Zihui. "BMO Solvability and A_∞ Condition of the Elliptic Measures in Uniform Domains." The Journal of Geometric Analysis 28 (2018): 866-908.

VITA

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