GLOBAL BIFURCATION OF ANTI-PLANE SHEAR EQUILIBRIA

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

Bifurcation theoretic methods are used to construct families of solutions for two problems arising in non-linear elasticity. These solution curves are shown to exhibit interesting phenomena that are both mathematically challenging and physically relevant. In the first part, we consider an unbounded elastic slab subjected to anti-plane shear deformation and under the influence of a body force. For one class of materials and forces, there is shown to be a loss of ellipticity at the terminal end of the bifurcation curve. More specifically, the ellipticity of the governing equations degenerates as the strain reaches a critical value determined by the material in question. For another class of materials and forces, we prove that broadening occurs; that is, the displacements within our family of equilibria remain uniformly bounded, but their effective supports become arbitrarily large.

In the next part, we investigate anti-plane shear deformations on a semi-infinite slab with a non-linear mixed traction displacement boundary condition. Energy estimates are used to show that broadening *cannot* occur in this setting. Once more we apply global bifurcation theory and deduce extreme behavior at the terminal end of the curve. It is shown that arbitrarily large strains are encountered for a class of idealized materials that do not allow for a loss of ellipticity. We also consider degenerate materials, prove that ellipticity breaks down, and most importantly show that a concurrent blow-up in the second derivative occurs.

Chapter 1

Introduction

1.1 General equations

Consider an elastic material that occupies the set $\mathcal{B} \subset \mathbb{R}^3$ in its reference configuration. A C^1 map of the form $\mathbf{f} : \mathcal{B} \to \mathbb{R}^3$ is called a *deformation*. In general one assumes that $\mathbf{f}$ is invertible and $\det \mathbf{F} > 0$, where $\mathbf{F} = \nabla \mathbf{f}$ is known as the *deformation gradient*, so that $\mathbf{f}$ is injective and orientation preserving. The *displacement* is given by

$$\mathbf{u} = \mathbf{f} - \mathrm{id}.$$

We make the standing assumption that all materials considered are hyperelastic and homogeneous. It is well known that such materials admit an associated strain energy density function of the form $W := W(\mathbf{F})$. The equilibrium equations are

$$\begin{cases} \dfrac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{kl}} u_{k,jl} = -\mathrm{b}_i(\mathbf{u}) & \text{in } \mathcal{B}, \text{ for } i \in \{1,2,3\}, \\ \mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\mathcal{B}_1, \\ \dfrac{\partial W(\mathbf{F})}{\partial F_{ij}} n_j = \tau_i & \text{on } \partial\mathcal{B}_2, \text{ for } i \in \{1,2,3\}, \end{cases} \tag{1.1}$$

where summation convention is adopted, $\partial\mathcal{B}_1 \cup \partial\mathcal{B}_2 = \partial\mathcal{B}$, $\mathbf{b}$ is a body force, $\boldsymbol{\varphi}$ is a prescribed boundary displacement, $\mathbf{n}$ is the outer unit normal, and $\boldsymbol{\tau}$ is a prescribed traction force. The general theory for nonlinear system of partial differential equations, including (1.1), is far from complete.

Additional constraints are often imposed on W to either incorporate reasonable physical assumptions or simply make (1.1) more tractable. For example, material frame indifference expresses the intuitive idea that the behavior of an elastic body does not depend on the frame of reference and is enforced by the requirement

$$W(QA) = W(A) \qquad \text{for all} \qquad Q \in SO(3).$$

An elastic material is said to be *isotropic* if

$$W(AQ) = W(A) \text{ for all } Q \in SO(3),$$

and we interpret this to mean the mechanical properties of the material are uniform throughout the body. The *right Cauchy–Green tensor* is defined by $\mathbf{F}^T\mathbf{F} = \mathbf{C}$. Under the assumptions of material frame indifference and isotropy, it can be shown using the polar decomposition of $\mathbf{F}$ that W is a function of the three principal invariants of $\mathbf{C}$:

$$I_1 = \operatorname{tr}\mathbf{C}, \qquad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\left(\mathbf{C}^2\right)], \qquad I_3 = \det\mathbf{C}.$$

Incompressible elastics are those for which $\det\mathbf{F} = 1$ (note that we also have $I_3 = 1$ in this case).

1.2 Variational considerations

Our investigation will not directly use the calculus of variations, but some of the key concepts will be motivating in the arguments to follow. Problem (1.1) carries a formal variational structure where the energy of a deformation $\mathbf{f} = \mathrm{id} + \mathbf{u}$ is given by

$$E(\mathbf{f}) := \int_{\mathcal{B}} W(\mathbf{F})\, dV - \int_{\mathcal{B}} \mathbf{B}(\mathbf{u})\, dV - \int_{\partial\mathcal{B}_2} \mathbf{T}(\mathbf{u})\, dS,$$

provided that $\mathbf{B}$ and $\mathbf{T}$ are potentials for $\mathbf{b}$ and $\boldsymbol{\tau}$, respectively. The physically motivated hypothesis

$$W(\mathbf{F}) \to \infty \qquad \text{as} \qquad \det(\mathbf{F}) \to 0^+ \tag{1.2}$$

says that an infinite amount of energy is required to compress a portion of $\mathcal{B}$ to a point.

One then seeks to minimize E over some appropriate class of functions; one reasonable choice for compressible elastics would be

$$\mathcal{A} := \{\mathbf{f} \in W^{1,1}(\mathcal{B}, \mathbb{R}^3) \ : \ E(\mathbf{f}) < \infty,\ \mathbf{u}|_{\partial\mathcal{B}_2} = \boldsymbol{\varphi}\},$$

and for compressible materials the determinant constraint can be incorporated into $\mathcal{A}$. This approach leads to several non-trivial difficulties. The most challenging of these complications stems from the need to select an appropriate set of coercivity and growth conditions on W to both facilitate the direct method of the calculus of variations and respect either (1.2) or the incompressibility constraint. See [2] and the references therein for a thorough discussion of these issues and a presentation of some related open problems.

1.3 Anti-plane shear

Let $\Omega \subset \mathbb{R}^2$ and $\mathcal{B} = \Omega \times \mathbb{R}$. Suppose that a hyperelastic, homogeneous, isotropic, and incompressible material occupies the body $\mathcal{B}$ in its reference configuration. A material is in a state of anti-plane shear if it is deformed in one direction only, and the deformation is independent of this direction. If the material under consideration is subjected to anti-plane shear deformation perpendicularly to Ω, then the displacement may be written as

$$\mathbf{u}(x, y, z) = u(x, y)\mathbf{e}_3. \tag{1.3}$$

The relations $I_1 = I_2 = 3 + |\nabla u|^2$ can be checked with a routine calculation, and they justify the following representation:

$$W(I_1, I_2, I_3) =: \overline{W}(I_1, I_2). \tag{1.4}$$

Let us ignore the boundary conditions momentarily and observe that one obtains an over-determined system consisting of three equations and two unknowns upon combining (1.1) and (1.3). It is natural to expect, at least in some cases, that this system will admit a reduction to a single PDE for the scalar u posed on the two dimensional domain Ω; in particular one hopes to solve a problem of the form

$$\begin{cases} \nabla \cdot (\mathcal{W}'(|\nabla u|^2 \nabla u) = \mathrm{b}_3 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega_1, \\ \mathcal{W}'(|\nabla u|^2)\nabla u \cdot \nu = \tau & \text{on } \partial\Omega_2, \end{cases} \tag{1.5}$$

Knowles discovered the necessary conditions in [22]:

$$\mathrm{c}\frac{\partial \overline{W}}{\partial I_1} + (\mathrm{c}-1)\frac{\partial \overline{W}}{\partial I_2} = 0, \qquad \text{for some } \mathrm{c} \in \mathbb{R}. \tag{1.6}$$

Note that a solution of (1.5) is not guaranteed to be compatible with (1.1). Condition (1.6) is easily satisfied whenever W is independent of I_2, and for simplicity we shall include this as a hypothesis. Finally, we introduce the variable $q := |\nabla u|^2$ and make the simple change of variables:

$$\mathcal{W}(q) := \overline{W}(3+q).$$

We will use $\mathcal{W}$ when formulating our governing equations and in the statements of structure conditions.

Chapter 2

Broadening global families of anti-plane shear equilibria

2.1 Introduction

Our primary goal is to rigorously prove the existence of interesting families of equilibria far from the reference configuration in the context of nonlinear elastostatics. Global bifurcation theorems for nonlinear elasticity have been established in, for example, [15], [14] and [12]. Due to the generality of the systems these authors consider, one must often accept that several alternatives may hold along the resulting global continuum. Efforts to develop global theories with complete characterizations have been limited. As one might expect, this requires some restrictions on the material, domain, and deformation. To this end, we focus on materials whose reference configuration is an unbounded cylinder that are in a state of anti-plane shear displacement and subjected to a parameter dependent body force. Anti-plane shear deformations are useful for many pilot problems. For example, it is in this context that Saint-Venant's principle for nonlinear elasticity was first probed (see [19, Section 6]). Our investigation into broadening and ellipticity breakdown are in the same spirit.

As with many global theories, the natural first step is to construct a perturbative family. We begin by further developing the local bifurcation theory established in [5, Section 3]. With this in hand, we employ the analytic global bifurcation theory of [4] to obtain a branch of solutions to the corresponding static equilibria problem that abides by a series of alternatives, which are in the same spirit as the ones mentioned above, that hold for a large class of materials. Sharp results are obtained by imposing reasonable assumptions on the material and body force; one set of conditions ensures broadening, while another leads to a loss of ellipticity. We make extensive use of monotonicity properties of solutions, elliptic type estimates, and a conserved quantity of the system in order to develop this global theory.

Broadening is characterized by the existence of a sequence of solutions to the relevant static equilibria problem, for which each element is uniformly bounded in an appropriate Hölder norm and decays to zero in the unbounded direction of the cylinder, yet does not admit a uniformly convergent subsequence. Such a sequence would fail to be uniformly spatially localized. It is in this sense that the displacements become broad. We also mention that after appropriately translating each element of such a sequence that one obtains a front type solution in the limit; see Figure 2.1. Broadening has been a topic of interest

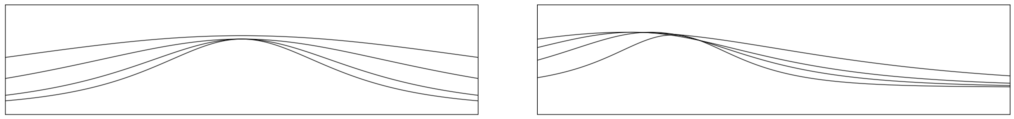

Figure 2.1: Left: Broadening along the center line $y = 0$. Right: Shifted functions converging to a front along the center line $y = 0$.

in the study of interfacial solitary water waves since at least the 1980s. Amick and Turner developed a global bifurcation theory which includes broadening as a possible alternative in [1], while Turner and Vanden-Broeck predicted the phenomena numerically in [34]. Whether broadening does indeed occur for such waves is still an open question. To the best of our knowledge, we are the first to rigorously construct a family of solutions that exhibit broadening in the PDE context.

A loss of ellipticity occurs when the governing equations change type. This is possible for some materials subjected to deformations with sufficiently large gradients. Knowles explores the relationship between ellipticity and crack formation for nonlinear elastics in [23] and again with Sternberg in [21]. We construct families of solutions whose maximum deformation gradient must limit to the critical state where ellipticity breaks down. As far as we know, no previous global construction has been shown to exhibits such behavior.

2.1.1 The problem

Consider a homogeneous, isotropic, incompressible, hyperelastic material occupying the region $\mathcal{D} := \Omega \times \mathbb{R}$, where $\Omega = \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Let u be the unknown displacement as in (1.3). Assume that $u = 0$ on $\partial\Omega$, which may be interpreted physically as a clamped boundary condition. The structure of the static equilibrium equations is largely determined by the strain energy density function W. It is well known that for these materials $W = W(I_1, I_2, I_3)$, where $I_{1,2,3}$ are the principal invariants of the Cauchy–Green tensor, and that $I_1 = I_2 = 3 + |\nabla u|^2$ for anti-plane shear deformations (see [19, Section 4]). Also, note that incompressibility implies $I_3 = 1$. Let us consider generalized neo-Hookean materials, in which case W depends on I_1 alone. We write

$$W(I_1, I_2, I_3) =: \overline{W}(I_1)$$

to simplify the notation. Finally, let

$$\mathcal{W}(q) := \overline{W}(3 + q),$$

where $\frac{1}{2}\mathcal{W}(q)$ is the so called *modulus of shear* at amount of shear q [17].

We also suppose, following for example [15], [12], and [5], that a parameter dependent live load acts on the material. Let $b = b(z, \lambda)$ denote the associated force density. Both $\mathcal{W}$ and b are required to be analytic in their arguments. Near the reference configuration, it is assumed that they have the expansions

$$\mathcal{W}(q) = q + c_1 q^2 + c_2 q^3 + O(|q|^4) \tag{2.1a}$$

$$b(z, \lambda) = (\lambda - 1)z + b_1 z^3 + O(|z|^5) \tag{2.1b}$$

where z is displacement and λ the parameter of the live load. We consider the case in which $b_1 \leq 0$, $c_1 < 0$, and b is odd in z. The more general assumption that b is odd in z and $b_1 + 2c_1 < 0$ is shown to be a requirement for the existence of spatially localized solutions near the reference configuration in [5].

In general, the equations that describe anti-plane shear are an over-determined system consisting of three equations and two unknowns. Knowles gives necessary conditions for non-trivial states of anti-plane shear in the absence of body forces in [22]. As he points out, generalized neo-Hookean materials always satisfy these conditions, and hence the governing equations are reduced to a single scalar PDE:

$$\begin{cases} \nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla u) - b(u, \lambda) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Naturally, the structure of both $\mathcal{W}$ and b will have a great effect on the qualitative features of the equilibria.

The first elastic model we consider, which we refer to as Model I, is equation (2.2) with a corresponding $\mathcal{W}$ satisfying

$$\mathcal{W}'(q) + 2q\mathcal{W}''(q) > \xi_1 > 0, \quad q \geq 0 \tag{2.3a}$$

$$q + c_1 q^2 + c_2 q^3 \leq \mathcal{W}(q), \quad q \geq 0. \tag{2.3b}$$

Note that (3.4) and (2.1a) force the relation $c_1^2 < \frac{5}{3}c_2$ and guarantee the Baker-Ericksen inequality [21, Equation 1.15]. Condition (3.4) ensures that (2.2) is uniformly elliptic regardless of the magnitude of the deformation gradient, and is equivalent to the strict convexity of the function $\mathcal{W}(p_1^2 + p_2^2)$ with respect the the p_i variables. For a discussion of variational approaches to the anti-plane shear problem and the connection to the convexity properties of strain energy density functions see [35].

The combined properties (2.3) are used to establish important a priori estimates. There is no universal choice for the growth conditions of $\mathcal{W}$, but we mention that polynomial models are common in the applied literature. In particular, the reduced polynomial model for incompressible materials has strain energy density given explicitly by:

$$\mathcal{W}(q) = \sum_{i=1}^{n} C_i q^i. \tag{2.4}$$

After normalization so that $C_1 = 1$, we find that for a large class of coefficients (2.4) will satisfy the assumptions of Model I. When $n = 3$, (2.4) becomes the widely used Yeoh model [38]. The parameters of the Yeoh model may be chosen so that $C_2 < 0$, which is one of our assumptions, in order to capture some of the experimental properties of rubber [38]. Our final assumption for Model I is that the the live load b satisfies the following conditions:

$$(\lambda - 1)z + b_1 z^3 \leq b(z, \lambda), \quad \text{for} \quad z \geq 0 \tag{2.5a}$$

$$-b_z(0, \lambda) < 1, \quad \text{for} \quad \lambda > 0. \tag{2.5b}$$

The condition (2.5a) is used to help obtain a priori estimates, and (2.5b) is a requirement of the local theory for homoclinic solutions in [5].

We also consider a second model, Model II, which is again governed by equation (2.2), but where $\mathcal{W}$ satisfies

$$\mathcal{W}'(q) + 2q\mathcal{W}''(q) > 0 \text{ for all } q \in [0, q_1) \tag{2.6a}$$

$$\mathcal{W}'(q) + 2q\mathcal{W}''(q) \to 0 \ \text{ as } \ q \to q_1^- \tag{2.6b}$$

$$q\mathcal{W}'(q) - \mathcal{W}(q) < 0, \text{ for } \ q > 0. \tag{2.6c}$$

Here, (3.5a) and (3.5b) mean simply that (2.2) is elliptic so long as $|\nabla u|^2 < q_1$, and that (2.2) loses ellipticity as $|\nabla u|^2 \to q_1^-$. This loss of ellipticity coincides with the failure of convexity in the function $\mathcal{W}'(p_1^2+p_2^2)$ as $|p|^2 \to q_1$. Furthermore, in this case, we suppose that b is concave in z and satisfies (2.5). Condition (2.6c) along with the concavity of b will be used to help rule out broadening for Model II.

Let us now consider a basic instantiation of Model II. The functions

$$\mathcal{W}(q) = q + c_1 q^2, \ \ b(\lambda, z) = (\lambda - 1)z, \tag{2.7}$$

with $c_1 < 0$, correspond to a "softening" elastic material undergoing simple harmonic forcing. The criteria for a softening material in the present notation is simply $\mathcal{W}''(q) < 0$, for all $q > 0$. See [17] for more details on hardening or softening materials subject to anti-plane shear.

2.1.2 Main results

The primary contribution of this paper is the construction of global families of solutions to (2.2) that either broaden or lose ellipticity.

Theorem 2.1 (Model I)**.** *There is a curve* $\mathcal{C}^I$ *of solutions to* (2.2)*, under the assumptions of Model I, admitting the* C^0 *parameterization*

$$\mathcal{C}^I = \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega}) \times (0, \infty)$$

and $(u(s), \lambda(s)) \to (0, 0)$ *as* $s \to 0^+$*. Moreover, we have that* $\mathcal{C}^I$ *satisfies the following:*

(a) (Symmetry and monotonicity) *Each* $(u(s), \lambda(s)) \in \mathcal{C}^I$ *is monotone in the sense that*

$$\partial_x u(s) < 0 \text{ for } x > 0 \partial_y u(s) < 0 \text{ for } y > 0,$$

and $u(s)$ *is even in both* x *and* y*.*

(b) (Analyticity) *The curve* $\mathcal{C}^I$ *is locally real analytic.*

(c) (Bounds on λ) *There exists some* $0 < \lambda_1^- < \lambda_1^+ < \infty$ *for which* $s \gg 1$ *implies*

$$\lambda_1^- < \lambda(s) < \lambda_1^+.$$

(d) (Bounds on displacement) *There exists* $C = C(c_1, c_2, b_1) > 0$ *for which*

$$\sup_{s \geq 0} |u(s)|_{3+\alpha} \leq C.$$

(e) (Broadening) *There is a sequence* $\{(u_n, \lambda_n)\} \subset \mathcal{C}^I$, *and a sequence* $\{x_n\}$, *with* $x_n \to \infty$, *such that*

$$u_n(\cdot + x_n, \cdot) \xrightarrow{C^3_{\mathrm{loc}}(\overline{\Omega})} \tilde{u} \in C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}),$$

where $\tilde{u}$ *is a solution to* (2.2), $\tilde{u} \not\equiv 0$, *and* $\partial_x \tilde{u} \leq 0$.

We call $\tilde{u}$ in the broadening alternative a front since it has distinct limiting states as $x \to \infty$ and $x \to -\infty$. In fact, we will see that $\tilde{u}$ must decay to 0 as $x \to \infty$ but limit to a non-trivial x-independent solutions of (2.2) as $x \to -\infty$. Fronts are of great interest in, for example, the study of reaction-diffusion systems, hydrodynamics, and mathematical biology. In [6], the authors consider a problem similar to the one posed here. However, they assume that $b_1 + 2c_1 > 0$ and are able to construct a global family of front-type solutions whose displacement grow arbitrarily large. Note that our assumption $b_1, c_1 < 0$ is not quite complementary to the one above; however, when taken together these conditions exhaust the global theory for a wide range the parameters.

Our second main result concerns Model II. As in Theorem 3.1, we establish monotonicity, local analyticity, and an upper bound on λ along the global curve. However, in this case the strictly positive lower bound on λ and the upper bound on the displacement are lost. Furthermore, the loss of ellipticity, which is the distinctive feature of Theorem 3.2, is an impossibility under the assumptions of Model I.

Theorem 2.2 (Model II). *There is a curve $\mathcal{C}^{II}$ of solutions to* (2.2)*, under the assumptions of Model II, with C^0 parameterization*

$$\mathcal{C}^{II} = \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times (0, \infty)$$

and $(u(s), \lambda(s)) \to (0, 0)$ as $s \to 0^+$. Moreover, we have that $\mathcal{C}^{II}$ satisfies the following:

(a) (Symmetry and monotonicity) *Each $(u(s), \lambda(s)) \in \mathcal{C}^I$ is such that $u(s)$ is monotone in the sense that*

$$\partial_x u(s) < 0 \text{ for } x > 0$$

$$\partial_y u(s) < 0 \text{ for } y > 0,$$

and $u(s)$ is even in both x and y.

(b) (Analyticity) *The curve $\mathcal{C}^{II}$ is locally real analytic.*

(c) (Bounds on λ) *There exists some $0 < \lambda_2^+ < \infty$ for which $s \gg 1$ implies*

$$0 < \lambda(s) < \lambda_2^+$$

(d) (Loss of ellipticity) *Following $\mathcal{C}^{II}$ to its extreme, the system loses ellipticity in that*

$$\lim_{s\to\infty} \inf_{\overline{\Omega}} \left(\mathcal{W}'(q) + 2q\mathcal{W}''(q)\right)\Big|_{q=|\nabla u(s)|^2} = 0 \tag{2.8}$$

We note that working under only the assumptions of (2.1), (2.5b), and $b_1, c_1 < 0$, our methods would show the existence of a global curve of solutions that satisfy (a) and (b) as above. Moreover, the condition $0 < \lambda(s)$ is retained. The conditions (2.3b) and (2.5a) are only used once (in Section 2.4). They help ensure global bounds on $|u(s)|_0$ and $|\lambda(s)|$, which will be shown to control $|u(s)|_{3+\alpha}$. Without (2.3b) and (2.5a), we would be left with the alternatives (i) $\sup_{s\geq 0} |u(s)|_0 \to \infty$ or $\sup_{s\geq 0} \lambda(s) \to \infty$; (ii) broadening occurs; or (iii) there is a loss of ellipticity in the limit, which may or may not coincide with either $\lambda(s) \to 0$ or $\lambda(s) \to \infty$. It seems that some restrictions on the growth of $\mathcal{W}$ and b are required for a satisfactory global theory. Perhaps by another set of assumptions on $\mathcal{W}$ and b not utilized here one could force a blow-up in either $|u(s)|_0$ or $\lambda(s)$.

2.1.3 History

Let us briefly recall some of the relevant history. Healey and Simpson obtained global branches of static equilibria for a non-linear elastic mixed boundary value problems in [15]. This general theory includes alternatives such as a loss of ellipticity, failure of compatibility conditions, or a return to the trivial branch of solutions. Healey and Rosakis [14] construct unbounded solution branches, which are sometimes referred to as "solutions in the large." This theory leaves open the possibility that the loading parameter or the norm of the deformation grows arbitrarily large as one follows the global curve. Each of these works are concerned with compressible elastics. Recently, Healey developed global bifurcation results for nonlinear incompressible elastics with conclusions similar to [14]. The displacements and domains in the theories mentioned above are more general than the ones used in this paper. However, note that each of these works are concerned with bounded domains. Because our problem is posed on an infinite cylinder, there are serious additional complications due to the lack of compactness properties for the underlying PDE. This difficulty is overcome with the analytic global bifurcation theory presented in [4]. These authors also recently developed a center manifold reduction to construct small solutions to the anti-plane shear problem in [5], which we use for our local bifurcation theory. We also mention the work of [9, 29], which treat quasilinear elliptic PDE on the whole space using degree theoretic global bifurcation theory. In contrast to [4], these authors impose assumptions that ensure local properness. Because broadening represents a loss of local properness we find the approach of [4] to be more natural in this context.

A word is in order about our choice of domain. Firstly, we are interested in developing a global theory for homoclinic type solutions. Demonstrating broadening behavior is also of great interests to us, which requires the existence of a sequence of spatially localized functions whose effective supports grow without bound. A natural setting for either such analysis is an infinite cylinder. Moreover, there has been a considerable amount of work regarding exponential decay estimates for anti-plane shear on semi-infinite strips of the form $(0,\infty)\times(0,h)$; see [19, Section 6] and the reference therein for a good overview. Thus, interest in anti-plane shear deformations in unbounded domains is well established.

2.1.4 Preliminaries

Let us fix some notation that will be used in the remainder of the paper. First, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$, let

$$C_{\mathrm{b}}^{k+\alpha}(\overline{\Omega}) := \{u \in C^k(\overline{\Omega}) \ : \ |u|_{k+\alpha} < \infty\},$$

where $C^k(\overline{\Omega})$ denotes the space of functions which are k times continuously differentiable on Ω up to the boundary, and $|\cdot|_{k+\alpha}$ is the usual Hölder norm. Much of our analysis will be concerned with solutions whose derivatives decay uniformly to 0, which leads us to consider:

$$C_0^{k+\alpha}(\overline{\Omega}) := \left\{ u \in C_{\mathrm{b}}^{k+\alpha}(\overline{\Omega}) \ : \ \lim_{r\to\infty} \sup_{|x|=r} |\partial_\beta u(x)| = 0, \ \ 0 \le |\beta| \le k \right\}.$$

Next, we define the Banach spaces X and Y by

$$X := \{u \in C_{\mathrm{b,e}}^{3+\alpha}(\overline{\Omega}) \cap C_0^2(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} \tag{2.9}$$

and

$$Y := C_{\mathrm{b,e}}^{1+\alpha}(\overline{\Omega}) \cap C_0^0(\overline{\Omega}) \tag{2.10}$$

where the subscript e denotes evenness in the x and y variables. Equation (2.2) can be written in operator form as

$$\mathcal{F}(u, \lambda) = 0,$$

where $\mathcal{F} : X \times \mathbb{R} \to Y$ is real analytic. We will show that $\mathcal{C}^{I,II} \subset X \times \mathbb{R}$ (here, and in the sequel, $\mathcal{C}^{I,II}$ will be used to indicate that a statement holds for both $\mathcal{C}^{I}$ and $\mathcal{C}^{II}$). A detailed investigation of the linearized operator $\mathcal{F}_u$ along a local curve of solutions to (2.2) is required in order to establish the existence of $\mathcal{C}^{I,II}$. The following spaces are useful for this task:

$$X_{\mathrm{b}} := \{u \in C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} \text{ and } Y_{\mathrm{b}} := C^{1+\alpha}_{\mathrm{b}}(\overline{\Omega}). \tag{2.11}$$

Similarly, let $X_0 := \{u \in C^{3+\alpha}_0(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ and $Y_0 := C^{1+\alpha}_0(\overline{\Omega})$. Finally, we define an exponentially weighted space that plays a role in the local bifurcation theory. The norm for this space is

$$|f|_{C^{k+\alpha}_\mu(\Omega)} := \sum_{\beta \le k} |w_\mu \partial^\beta f|_{C^0} + \sum_{|\beta|=k} |w_\mu| \partial^\beta f|_\alpha|_{C^0}$$

where $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$, and $w_\mu(x) := \operatorname{sech}(\mu x)$. We may then define

$$C^{k+\alpha}_\mu(\overline{\Omega}) := \{f \in C^{k+\alpha}(\overline{\Omega}) \;:\; |f|_{C^{k+\alpha}_\mu(\Omega)} < \infty\}.$$

2.1.5 Outline

In Section 2.2, we recall the existence of a local curve of solutions to (2.2) established in [5, Section 3] and then prove some monotonicity and symmetry properties that will be important for the later analysis. This section ends by showing the linearized operator along the local curve is invertible; a fact which is essential to the global theory. In Section 2.3, we apply the global bifurcation theory of [4] to the local solutions. We also show that the monotonicity properties of the local curve are preserved. Bounds on $|u(s)|_{3+\alpha}$ and λ are derived in Section 2.4 through elliptic estimates, monotonicity, and a conserved quantity. Finally, in Section 2.5 we stitch together the previous work and systematically eliminate alternatives of the global bifurcation theorem to prove both Theorem 3.1 and Theorem 3.2.

2.2 Local bifurcation

Our ultimate aim is to construct non-peturbative solutions, but first this will require us to refine our understanding of the local theory. After establishing the existence of a local curve, we show some monotonicity and symmetry properties, which will be extended to the global curve in Section 2.3.2. The proof of existence for $\mathcal{C}^{I,II}$ will rely upon invertibility properties of the linearized operator $\mathcal{F}_u$ along the local curve; these are investigated at the end of the section.

2.2.1 Existence and uniqueness

In [5, Section 3], the authors establish the existence of a local curve of homoclinic solutions to (2.2) bifurcating from $(u, \lambda) = (0, 0)$ under the assumptions

$$\begin{aligned} b(u,\lambda) =& (\lambda - 1)u + b_1 u^3 \\ \mathcal{W}'(q) =& 1 + 2c_1 q \end{aligned} \tag{2.12}$$

with $b_1 + 2c_1 < 0$. When this inequality is reversed, front-type solutions are instead obtained. The authors extend this argument to deal with more generalized b, including the form (2.1b), in [5, Appendix B.1]. Those arguments can also be used to show the existence of local solutions under the more general assumptions of Model I and Model II. This is the content of the next theorem.

Theorem 2.3. *There exists an* $\epsilon_0 > 0$ *and a local* C^0 *curve*

$$\mathcal{C}_{\text{loc}}^{I,II} = \{(u^\epsilon, \epsilon^2) \ : \ 0 < \epsilon < \epsilon_0\} \subset X_0 \times \mathbb{R}$$

of solutions to (2.2)*, corresponding to Model I or Model II, with the asymptotics*

$$u^\epsilon(x, y) = a_1 \epsilon \operatorname{sech}(\epsilon x) \cos(y) + O(\epsilon^2) \qquad \text{in } \ C_{\text{b}}^3(\overline{\Omega}), \tag{2.13}$$

where $a_1 = \dfrac{2}{\sqrt{3|b_2 + 2c_1|}}$.

Proof. We reparametrize with $\lambda = \epsilon^2$ for convenience. As mentioned above, an existence result was obtained in [5, Section 3] under the conditions (2.12). We will follow closely that proof and focus on the places where deviations are necessary to accommodate the more general form of $\mathcal{W}$ we consider.

Let $L := \mathcal{F}_u(0,0)$ and L' be defined as the restriction of L to x-independent functions (L' is called the transversal linearized operator). The center manifold reduction result in [5] requires that 0 is a simple eigenvalue of L'. The operator L corresponding to (2.12), Model I, or Model II is simply $\Delta + 1$ as seen by the structure of (2.1) and (2.12). Clearly L' satisfies the requirements mentioned above. Now, the center manifold reduction given by [5, Theorem 1.1] shows that solutions of (2.2), that lie in a sufficiently small neighborhood of the origin in $C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R}$ can be expressed as

$$u(x+\tau, y) = v(x)\varphi_0(y) + v'(x)\tau\varphi_0(y) + \Psi(v(x), v'(x), \epsilon)(\tau, y), \tag{2.14}$$

where $v(x) := u(x,0)$, $\varphi_0(y)$ generates the kernel of L', and $\Psi : \mathbb{R}^3 \to C^{3+\alpha}_{\mu}(\overline{\Omega})$ is a C^4 coordinate map. Here $\mu > 0$ is a positive constant depending on the largest non-zero eigenvalue of L'. Moreover, if $(u, \epsilon^2) \in C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R}$ is any sufficiently small solution to (2.2), then, by [4, Theorem 1.1], v solves the second-order ODE

$$v'' = f(v, v', \epsilon^2), \qquad \text{where} \qquad f(A, B, \epsilon^2) := \left.\frac{d}{dx^2}\right|_{x=0} \Psi(A, B, \epsilon)(x, 0). \tag{2.15}$$

Thus, we are left to show that the change in Ψ resulting from the conditions of Model I or Model II does not affect the existence or general form of u^ϵ in (3.20). Let us point out that Ψ inherits the following symmetry properties from the original PDE (2.2):

$$\Psi(-A,-B,\epsilon) = -\Psi(A,B,\epsilon) \qquad \text{and} \qquad \Psi(A,-B,\epsilon)(-x,y) = \Psi(A,B,\epsilon)(x,y). \tag{2.16}$$

From (2.15) it follows that

$$f(-A,-B) = -f(A,B) \qquad \text{and} \qquad f(A,-B) = f(A,B). \tag{2.17}$$

To derive an expression for f, we exploit [5, Theorem 1.2] to conclude that Ψ admits a Taylor expansion of the form

$$\Psi(A,B,\epsilon) = \sum_{\mathcal{J}} \Psi_{ijk} A^i B^j \epsilon^k + \mathcal{R}, \tag{2.18}$$

where the index set

$$\mathcal{J} = \{(i,j,k) \in \mathbb{N} \ : \ i + 2j + k \leq 3, i + j + k \geq 2, i + j \geq 1\},$$

the coefficients $\Psi_{ijk} \in C^{3+\alpha}_{\mu}(\overline{\Omega})$, and the error term $\mathcal{R}$ is of order $O((|A| + |B|^{1/2} + \epsilon)^4)$ in $C^{3+\alpha}_{\mu}(\overline{\Omega})$.

Combining (2.14) and (2.18) yields

$$u(x,y) = (A + Bx)\varphi_0(y) + \sum_{\mathcal{J}} \Psi_{ijk} A^i B^j \epsilon^k + \mathcal{R}, \tag{2.19}$$

where $A = v(0)$ and $B = v'(0)$. For a fixed i, j, k the general theory now allows us to solve for Ψ_{ijk} via a hierarchy of equations of the form

$$\begin{cases} L(\Psi_{ijk}) = F_{ijk} \\ Q(\Psi_{ijk}) = 0, \end{cases} \tag{2.20}$$

where Q is the projection onto $\ker L'$. The F_{ijk} terms are obtained by iteratively feeding truncations of (2.19) into $\mathcal{F}^r - L$ where $\mathcal{F}^r$ is $\mathcal{F}$ precomposed with a certain cutoff function. The key point here is that the Q is unchanged by our modification of $\mathcal{W}$, and the F_{ijk} terms are independent of terms of the order $O(|A| + |B|^{1/2} + \epsilon)^4)$ in $C^{3+\alpha}_{\mu}(\overline{\Omega})$. Our generalized $\mathcal{W}$ introduces, for example, the extra nonlinear term $c_2 \nabla \cdot (|\nabla u|^4 \nabla u)$ into (2.2) near $(u, \lambda) = (0, 0)$. We see that applying this to (2.19) yields only terms of order $O((|A| + |B|^{1/2} + \epsilon)^4)$. Hence, from this point on the argument for existence of solutions to (2.2) carries through without change. In particular, one can solve for Ψ_{ijk} in the exact manner presented in [5, Section 3.1] and [5, Appendix B.1].

Although the rest of the argument now follows verbatim from [5], we continue the sketch because it will help to explain some later reasoning. Having calculated Ψ_{ijk}, we find that f takes the form

$$f(A, B, \epsilon) = \epsilon^2 A + \frac{3(b_1 + 2c_1)}{4} A^3 + r(A, B, \epsilon), \tag{2.21}$$

where $r \in C^3$ is an error term of the order $O(|A|(|A| + |B|^{1/2} + \epsilon)^3 + |B|(|A| + |B|^{1/2} + \epsilon)^2)$. Using the re-scaled variables

$$x =: X/\epsilon, \qquad v(x) =: \epsilon V(x), \qquad v_x(x) =: \epsilon^2 W(x)$$

we may now write (2.15) as the planar system

$$\begin{cases} V_X = W \\ W_X = V - a_1^{-2} V^3 + R(V, V, \epsilon) \end{cases}, \tag{2.22}$$

where the rescaled error $R(V, W, \epsilon) = O(|\epsilon|(|V| + |W|))$. When $\epsilon = 0$ the system has the explicit homoclinic orbit

$$V = a_1 \text{sech}(X) \qquad W = -a_1 \text{sech}(X) \tanh(X). \tag{2.23}$$

This solution crosses the V-axis transversely. Since (2.22) has the reversal symmetries

$$(V(X), W(X)) \mapsto (V(-X), -W(-X)) \qquad \text{and} \qquad (V(X), W(X)) \mapsto (V(X), -W(X)),$$

which it inherits from (2.17) and (2.21), this intersection will persist for small ϵ, so we obtain a family of homoclinic solutions. Undoing the scaling and appealing to [5, Theorem 1.1] shows that the family (3.20) are indeed solutions to (2.2). ∎

We now establish some qualitative properties of small solutions to (2.2).

Theorem 2.4. *Suppose that $(u, \epsilon^2) \in X_0 \times \mathbb{R}$ is a solution to (2.2) under the assumptions of Model I or Model II. There exists δ_0 such that if $|u|_{3+\alpha} + \epsilon^2 < \delta_0$, then $(u, \epsilon^2) \in \mathcal{C}^{I,II}_{\text{loc}}$ after a possible translation in x or reflection about the xy-plane. Moreover, if $(u, \epsilon^2) \in \mathcal{C}^{I,II}_{\text{loc}}$, then u is even in x and y and monotone in that $u_x < 0$ for $x > 0$.*

Proof. First, we show there exists δ_0 small enough to ensure $u > 0$. The Malgrange preparation theorem allows us to write $b(\lambda, z) = zw(\lambda, z)$ for a smooth

w defined in some neighborhood of $(0,0)$, see for example [7, Theorem 7.1]. Then, (2.2) becomes

$$\begin{cases} \mathfrak{a}_1 u_{xx} + \mathfrak{a}_2 u_{yy} + 4\mathcal{W}''(|\nabla u|^2) u_x u_{xy} u_y - uw(\lambda, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

where

$$\mathfrak{a}_1 = \mathcal{W}'(|\nabla u|) + 2\mathcal{W}''(|\nabla u|^2)(u_x)^2 \qquad \text{and} \qquad \mathfrak{a}_2 = \mathcal{W}'(|\nabla u|) + 2\mathcal{W}''(|\nabla u|^2)(u_y)^2.$$

Thus $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are uniformly positive for small enough δ_0 (note that this is only a concern for Model II, since (3.4) ensures such a lower bound holds for Model I). We write (2.24) this way in order to view it as a linear elliptic PDE and apply a comparison principle argument.

Consider the function

$$\Phi^\delta = \Phi^\delta(y) := \log{(2 + \sqrt{\delta} y)} \cos(\sqrt{1-\lambda} y). \tag{2.25}$$

An elementary calculation reveals that

$$\begin{aligned} a_2 \Phi^\delta_{yy} - w(\lambda, u)\Phi^\delta = \Big(&\frac{-\epsilon \cos(\sqrt{1-\lambda} y)}{(2+\sqrt{\epsilon} y)^2} - \frac{2\sqrt{\epsilon(1-\lambda)} \sin(\sqrt{1-\lambda} y)}{2+\sqrt{\epsilon} y} \Big)(1 + O(\epsilon^2)) \\ &+ O(\epsilon^2 \cos(\sqrt{1-\lambda} y)) \;\text{ in }\; C^0(\overline{\Omega}), \end{aligned} \tag{2.26}$$

where we have used the asymptotics of b and $\mathcal{W}$ in (2.1), which hold for small enough ϵ_0. The right hand side of (2.26) is strictly negative whenever $0 \le y \le \frac{\pi}{2}$ and δ is small enough. Moreover, $\Phi^\delta > 0$ in $\overline{\Omega}$. So, if we establish non-negative boundary values for u on the region $(-\infty, \infty) \times (0, \frac{\pi}{2})$, then we may invoke the maximum principle for uniformly elliptic operators with a positive super-solution (see Theorem A.1.(iv)) to conclude that $u > 0$ on $\mathbb{R} \times (0, \frac{\pi}{2})$.

We already know $u = 0$ on $\mathbb{R} \times \{\frac{\pi}{2}\}$, and a phase plane analysis will show $u > 0$ on $\mathbb{R} \times \{0\}$. Indeed, $v := u(x, 0)$ solves the ODE (2.15) by [5, Theorem 1.1]. If we write this as a planar system, which has the same structure as (2.22), then the symmetries $(V, W) \mapsto (-V, -W)$ and $(V, W) \mapsto (V(-X), -W(-X))$ imply that a homoclinic orbit that intersects the positive V axis meets the W axis only at $(0, 0)$. Hence, after a possible reflection $u(x, 0) > 0$, so $u > 0$ for $0 < y < \frac{\pi}{2}$ by the remarks at the end of the previous paragraph. Redoing the above analysis with $\Phi^{\delta}(-y)$ shows $u > 0$ in Ω.

Now that the positivity of u is established, we find from a moving planes argument in [24, Theorem 3.2] that u is even in x about some line $x = x_1$ with $u_x < 0$ for $x > x_1$. The translation $x \mapsto x - x_1$ sends u to a positive solution of (2.2) with the desired monotonicity and evenness properties in x. The phase plane analysis for (2.22) in Theorem 3.4 shows that $u^{\epsilon}_x(0, y) = 0$, where $u^{\epsilon} \in \mathcal{C}^{I,II}_{\mathrm{loc}}$, whence it follows that u^{ϵ} is even about $x = 0$.

Observe that the previous paragraphs established the uniqueness of small solutions to (2.2) up to translations and reflections in x. In particular, the elements of $\mathcal{C}^{I,II}_{\mathrm{loc}}$ are the unique positive and even solutions to (2.2) in a sufficiently small neighborhood of $(0,0)$ in $X_0 \times \mathbb{R}$. Finally, the elements of $\mathcal{C}_{\text{loc}}^{I,II}$ must be even in y since the reflection $y \mapsto -y$ will take an element of $\mathcal{C}_{\text{loc}}^{I,II}$ to another positive solution that is even and monotone in x. ∎

2.2.2 Linearized problem

In this section, we show the linearized operator $\mathcal{F}_u(0,\lambda) : X \to Y$ is invertible for $0 < \lambda \leq 1$. This fact plays an important role in the analysis to follow. In particular, it implies the Fredholmness of $\mathcal{F} : X \to Y$, which will extend to the global curve. A simple calculation yields

$$\mathcal{F}_u(0,0) = \Delta + 1 \tag{2.27}$$

for Model I or Model II. The notion of a limiting operator is needed for the next two lemmas. If

$$L = a_{ij}(x,y)\partial_{x_i}\partial_{x_j} + b_i(x,y)\partial_{x_i} + c(x,y),$$

and as $x \to \pm\infty$ we have

$$a_{ij}(x,y) \to \tilde{a}_{ij}(y), \quad b_i(x,y) \to \tilde{b}_i(y), \quad c(x,y) \to \tilde{c}(y),$$

where each of $\tilde{a_{ij}}, \tilde{b}_i$, and $\tilde{c}$ belongs to $C_{\mathrm{b}}^{\alpha}[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the limiting operator $\tilde{L}$ is defined as

$$\tilde{L} := \tilde{a}_{ij}\partial_{x_i}\partial_{x_j} + \tilde{b}_{x_i}\partial_{x_i} + \tilde{c}.$$

Lemma 2.5 (Invertibility of linearized operator at 0)**.** *For all* $0 < \lambda \leq 1$, $\mathcal{F}_u(0,\lambda) : X \to Y$ *is invertible.*

Proof. Fix $0 < \lambda \leq 1$ and let $\varphi(y) = \cos(\sqrt{1-\lambda}y)$. Since $\varphi > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we may write $u =: v\varphi$, so that $\mathcal{F}_u(0,\lambda)u = 0$ implies

$$\begin{cases} \Delta v + \dfrac{2\varphi_y}{\varphi} v_y = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

Let L be the linear operator associated with (2.28) which acts on v. Note that $L : X_\mathrm{b} \to Y_\mathrm{b}$ has trivial kernel by the strong maximum principle (Theorem A.1.(i)). For $\gamma \in \mathbb{R}$, let $L_\gamma = L - \gamma$ and denote by $\mathcal{B}$ the corresponding bilinear operator:

$$\mathcal{B}[w,w] = \int_\Omega \left(|\nabla w|^2 - \frac{\varphi_y}{\varphi} w w_y + \gamma w^2 \right) dx\, dy = \int_\Omega \left(|\nabla w|^2 + \left(\frac{\partial}{\partial y} \frac{2\varphi_y}{\varphi} \right) w^2 + \gamma w^2 \right) dx\, dy,$$

for $w \in H_0^1$. When γ is large enough, $\mathcal{B}$ is coercive and hence Lax–Milgram implies $L_\gamma : H_0^1 \to L^2$ is invertible.

We will next show that $L_\gamma : X_\mathrm{b} \to Y_\mathrm{b}$ is invertible. The argument is similar to the one found in [36, Appendix A.2]. Let $\rho_\epsilon(x) := \operatorname{sech}(\epsilon x)$. Conjugating by ρ_ϵ the problem $L_\gamma = f$ may be transformed into the equivalent one

$$L_\gamma^\epsilon u_\epsilon = L_\gamma u_\epsilon - \frac{2\partial_x \rho_\epsilon}{\rho_\epsilon} \partial_x u_\epsilon + \left(\frac{\partial_x^2 \rho_\epsilon}{\rho_\epsilon} - \frac{2(\partial_x \rho_\epsilon^2)}{\rho_\epsilon^2} \right) u_\epsilon = f_\epsilon$$

where $u_\epsilon := u\rho_\epsilon$ and $f_\epsilon := f\rho_\epsilon$. If $f \in Y_\mathrm{b}$ then $f_\epsilon \in L^2$, and the equation $L_\gamma(u_\epsilon) = f_\epsilon$ is solvable by the work above. Note that

$$\|L_\gamma^\epsilon - L_\gamma\|_{X_\mathrm{b} \to Y_\mathrm{b}} = \left\| \frac{2\partial_x \rho_\epsilon}{\rho_\epsilon} \partial_x + \left(\frac{\partial_x^2 \rho_\epsilon}{\rho_\epsilon} - \frac{2(\partial_x \rho_\epsilon^2)}{\rho_\epsilon^2} \right) \right\|_{X_\mathrm{b} \to Y_\mathrm{b}} \longrightarrow 0, \quad \text{as} \quad \epsilon \to 0,$$

so for small enough ϵ_0, the pertubation L_γ^ϵ of L_γ remains invertible whenever $0 < \epsilon < \epsilon_0$.

From [11, Theorem 8.8] and [11, Theorem 9.19], we know $u_\epsilon \in C^{3+\alpha}(\overline{\Omega}) \cap C_{\mathrm{b}}^{\alpha}(\overline{\Omega})$. Moreover, by Schauder estimates and injectivity, we have the bound

$$|u_\epsilon|_{2+\alpha} \leq C|f_\epsilon|_\alpha,$$

where $C > 0$ is independent of ϵ. Therefore, we are able to extract a subsequence $\epsilon_n \to 0$ for which $u_{\epsilon_n} \to u$ in $C^2_{\mathrm{loc}}(\overline{\Omega})$ with $u \in C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega})$. Letting $n \to \infty$ in the above equation we find the $L_\gamma u = f$.

Now that the invertibility of $L_\gamma : X_{\mathrm{b}} \to Y_{\mathrm{b}}$ has been established, we will make use of the continuity of the Fredholm index to conclude that $L : X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is invertible. Let $L_{\gamma t} := L - t\gamma$. It is clear that $L_{\gamma t}$ is its own limiting operator for $t \in [0, 1]$, since its coefficients are x-independent. The limiting problem has no non-trivial solutions because $L_{t\gamma}$ satisfies the strong maximum principle (Theorem A.1.(i)). Lemma A.8 of [37] now shows $L_{t\gamma}$ must be semi-Fredholm with index $< \infty$. Thus, the Fredholm index must then be preserved along the family $\{L_{t\gamma}\}_{t\in[0,1]}$. We can now conclude that $L : X_{\mathrm{b}} \to Y_{\mathrm{b}}$ has Fredholm index 0, just as the operator L_γ. Hence, $L : X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is in fact invertible since it also has a trivial kernel. From [37, Lemma A.12], $L : X_0 \to Y_0$ must also have Fredholm index 0, and again the kernel is trivial so that $L : X_0 \to Y_0$ is invertible. Finally, it is not hard to see from the structure of L that data

$f \in Y \subset Y_0$ must have a corresponding solution $u \in X$. For example, if f is even in y, and v is the unique solution to $Lv = f$, then a quick check shows that $Lv(x, -y) = f$ as well. Hence, $L : X \to Y$ is invertible. ∎

Now consider the linearized operator $\mathcal{F}_u(u, \lambda)$ with $(u, \lambda) \in \mathcal{C}^{I,II}_{\mathrm{loc}}$. We know $\mathcal{F}_u(u, \lambda)\partial_x u = 0$ by translation invariance and elliptic regularity. Thus, $\mathcal{F}_u(u, \lambda)$ has nontrivial kernel acting on X_{b}. However, if we instead restrict to X, which by definition imposes even symmetry, then we will have injectivity.

Lemma 2.6 (Trivial kernel)**.** *For all $(u, \lambda) \in \mathcal{C}^{I,II}_{\mathrm{loc}}$, $\mathcal{F}_u(u, \lambda) : X \to Y$ is injective, whenever $(u, \lambda) \in \mathcal{C}^{I,II}_{\mathrm{loc}}$.*

Proof. From [5, Theorem 1.6] and [5, Appendix B.1.], if $\dot{u} \in C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega})$ is a solution of $\mathcal{F}_u(u, \lambda)\dot{u} = 0$, then $\dot{v} := \dot{u}(\cdot, 0)$ solves the linearized reduced ODE

$$\dot{v}'' = r_B \dot{v}' + (\lambda + \frac{9(b_1 + 2c_2)}{4} v^2 + r_A)\dot{v}, \tag{2.29}$$

where $v := u(\cdot, 0)$. As noted above, $\partial_x u$ is in the kernel of $\mathcal{F}_u(u, \lambda)$, so v_x is an odd and bounded solution to (2.29). Suppose that we had another bounded solution $w \in C^2_{\mathrm{b}}(\mathbb{R})$ to (2.29) that is linearly independent of v. From Abel's identity

$$W(x) = W(0) \exp\left(\int_0^x \mathrm{tr}(P(s))\, ds\right),$$

where $W(x)$ is the Wronskian of v and w evaluated at x, and P is the matrix defined by

$$P := \begin{pmatrix} 0 & 1 \\ \lambda + \frac{9(b_1+2c_1)}{4}c_1v^2 + r_A(v,v',\epsilon) & r_B \end{pmatrix}.$$

Since u_x, u_{xx}, w, and w_x are all bounded, and u_x, u_{xx} each decay at infinity, we see that

$$|\text{det}W(x)| \leq (w^2(x) + w_x^2(x)) \cdot (u_x^2(x,0) + u_{xx}(x,0)) \to 0 \quad \text{as} \quad |x| \to \infty.$$

But then we must have

$$\int_0^x r_B(u_x(t,0), u_{xx}(t,0))\,dt \to -\infty \quad \text{and} \quad \int_{-x}^0 r_B(u_x(t,0), u_{xx}(t,0))\,dt \to \infty \quad \text{as} \quad x \to \infty. \tag{2.30}$$

Recalling the symmetry properties of f in (2.17) and the explicit form given in (2.21), it follows that

$$r_B(A,B) = -r_B(A,-B) = r_B(-A,-B).$$

This would imply

$$\begin{aligned}\lim_{x\to\infty} \int_{-x}^0 r_B(u_x(t,0), u_{xx}(t,0))\,dt &= \lim_{x\to\infty} -\int_0^x r_B(u_x(-t,0), u_{xx}(-t,0))\,dt \\ &= \lim_{x\to\infty} \int_0^x r_B(u_x(t,0), u_{xx}(t,0))\,dt,\end{aligned} \tag{2.31}$$

where we used the properties of r_B, oddness of u_x and evenness of u_{xx}. Equations (2.30) and (2.31) together force a contradiction. Hence, there cannot be two linearly independent bounded solutions to (2.29).

At this point we may conclude that v_x generates the solution set of (2.29). Thus, $\mathcal{F}_u : X \to Y$ has trivial kernel, since any non-zero element would necessarily be odd. To see this, suppose, by a slight abuse of notation, that some $w(x,y) \in C^3_{\mathrm{b}}(\overline{\Omega})$ satisfies $\mathcal{F}_u(u,\lambda)w = 0$. Recall that [5, Theorem 1.1] gives the expansion

$$w(x,y) = \varphi_0(x)w(x,0) + \Psi(w(x,0), w_x(x,0), \lambda)(0,y),$$

where $w(x,0)$ is odd in x by the work above. The symmetries in (2.16) imply the additional symmetry

$$\Psi(-A,B,\lambda)(0,y) = -\Psi(A,B,\lambda)(0,y),$$

from which we may conclude that w is odd in x. ∎

Finally, we show that $\mathcal{F}_u$ is invertible along the local curve.

Lemma 2.7 (Invertibility)**.** *For any $(u,\lambda) \in \mathcal{C}^{I,II}_{loc}$, the linearized operator $\mathcal{F}_u(u,\lambda) : X \to Y$ is invertible.*

Proof. We found that $\mathcal{F}_u(u,\lambda) : X \to Y$ has trivial kernel whenever $(u,\lambda) \in \mathcal{C}^{I,II}_{\mathrm{loc}}$ in Lemma 2.6. It therefore suffices to show that this operator is Fredholm index 0. The limiting operator of $\mathcal{F}_u(u,\lambda)$ is simply $\mathcal{F}_u(0,\lambda)$ because u decays as $x \to \pm\infty$. Recall that $\mathcal{F}_u(0,\lambda)$ was shown to be invertible in Lemma 2.5. By [37, Lemma A.13] it follows that the Fredholm indices of $\mathcal{F}_u(u,\lambda)$ and $\mathcal{F}_u(0,\lambda)$ match. Hence, $\mathcal{F}_u(u,\lambda)$ is in fact Fredholm index 0, and the result follows. ∎

2.3 Global bifurcation

2.3.1 Background theory

We begin this section by recalling some of the global bifurcation theory developed in [4, Section 6]. The results stated here are tailored to the problem at hand. Let $\mathcal{I} = (0,1)$ and

$$\mathcal{O} = \bigcup_{\delta>0} \mathcal{O}_\delta \qquad \text{where}$$
$$\mathcal{O}_\delta = X \cap \left\{u \in C^3(\overline{\Omega}) \,:\, \liminf_{(x,y)\in\overline{\Omega}} \left(\mathcal{W}'(q) + 2q\mathcal{W}''(q)\right)\Big|_{q=|\nabla u(x,y)|^2} > \delta\right\}. \tag{2.32}$$

Theorem 2.8. *There is a curve of solutions $\mathcal{C}^{I,II} \subset \mathcal{F}^{-1}(0)$, where $\mathcal{F}$ corresponds to either Model I or Model II, parameterized as $\mathcal{C}^{I,II} := \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset \mathcal{O} \times \mathcal{I}$ with the following properties.*

(a) *One of the following alternatives holds.*

(i) (Blowup) *As $s \to \infty$*

$$N(s) := |u(s)|_{3+\alpha} + \frac{1}{dist(u(s), \partial\mathcal{O})} + \lambda(s) + \frac{1}{dist(\lambda(s), \partial\mathcal{I})} \to \infty \tag{2.33}$$

(ii) (Loss of compactness) *There exists a sequence $s_n \to \infty$ such that $\sup_n N(s_n) < \infty$ but $\{u(s_n)\}$ has no subsequences converging in X.*

(b) *Near each point $(x(s_0), \lambda(s_0)) \in \mathcal{C}$, we can reparametrize $\mathcal{C}$ so that $s \mapsto (x(s), \lambda(s))$ is real analytic.*

(c) *$(x(s), \lambda(s)) \notin \mathcal{C}_{\text{loc}}$ for s sufficiently large.*

Proof. We have shown that the linearized operator is invertible along the local curve and the result follows directly from [4, Theorem 6.1]. ∎

Alternative (i) encapsulates several interesting possibilities. We note that a blow-up in (3.62) can be achieved by a loss of ellipticity, λ returning to 0, or the more obvious unboundedness of λ or $|u(s)|_{3+\alpha}$. Throughout the rest of the paper we investigate alternatives (i) and (ii) for Models I and II. This will ultimately lead us to discover that broadening occurs invariably in Model I and that a loss of ellipticity is ensured for Model II. At times we focus on segments of the curve $\mathcal{C}^{I,II}$ of the form

$$\mathcal{C}_\delta^{I,II} := \mathcal{C}^{I,II} \cap \mathcal{O}_\delta. \tag{2.34}$$

Note that $\mathcal{C}^{II} = \mathcal{C}_{\xi_1}^{II}$ by (3.4).

At this point, it is convenient to recall another result from [4] which helps characterize alternative (ii) of Theorem 3.23.

Theorem 2.9 (Chen, Walsh, Wheeler [4]). *If $\{(u_n, \lambda_n)\}$ is a sequence of solutions to (2.2) that is uniformly bounded in $C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega}) \times \mathbb{R}$, with the additional monotonicity property*

$$u_n(x,y) \ \textit{is even in } x \textit{ and } \ u_x \le 0 \ \textit{ for } \ x \ge 0 \tag{2.35}$$

for each n as well as the asymptotic condition

$$\lim_{|x|\to\infty} u_n(x,y) = U(y) \ \textit{ uniformly in } y \tag{2.36}$$

for some fixed function $U \in C_{\mathrm{b}}^{3+\alpha}([-\frac{\pi}{2}, \frac{\pi}{2}])$*, then either*

(i) *we can extract a subsequence* $\{u_n\}$ *so that* $u_n \to u$ *in* $C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega})$*; or*

(ii) *we can extract a subsequence and find* $x_n \to \infty$ *so that the translated sequence* $\{\tilde{u}_n\}$ *defined by* $\tilde{u}_n = u_n(\cdot + x_n, \cdot)$ *converges in* $C_{\mathrm{loc}}^{3}(\overline{\Omega})$ *to some* $\tilde{u} \in C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega})$ *that solves* (2.2) *and has* $\tilde{u} \not\equiv U$ *with* $\tilde{u}_x \leq 0$*.*

Note that this theorem requires some symmetry and monotonicity properties in u_n. The following subsection demonstrates these properties, and more, for elements of $\mathcal{C}^{I,II}$.

2.3.2 Monotonicity and nodal properties

We show that elements of $\mathcal{C}_{\mathrm{loc}}^{I,II}$ exhibit certain qualitative features by using the asymptotics (3.20) and maximum principle arguments. In fact, we have already established that (2.35) and (2.36) (with $U(y) = 0$) hold along $\mathcal{C}_{\mathrm{loc}}^{I,II}$ in Theorem 2.4. Our goal is to prove that these persist along $\mathcal{C}^{I,II}$. The following sets will be useful for our analysis:

$$\begin{aligned}
\Omega^+ &:= \{(x,y) \in \Omega \ : \ x > 0\} \\
\Omega_+ &:= \{(x,y) \ : \ |x| < R, 0 < y \leq \frac{\pi}{2}\} \\
L &:= \{(0,y) \ : \ -\pi/2 < y < \pi/2\} \\
T &:= \{(x,\pi/2) \ : \ 0 < x < \infty\} \\
B &:= \{(x,-\pi/2) \ : \ 0 < x < \infty\} \\
M &:= \{(x,0) \ : \ 0 \leq x < \infty\}.
\end{aligned} \tag{2.37}$$

The nodal properties we are concerned with are as follows:

$$\begin{gathered}
u_x < 0 \quad \text{on} \quad \Omega^+ \\
u_y < 0 \quad \text{on} \quad \Omega_+ \\
u_{xx} < 0 \quad \text{on} \quad L \\
u_{xy} > 0 \quad \text{on} \quad T \\
u_{xxy} > 0 \quad \text{at} \quad (0, \frac{\pi}{2}) \ \text{and} \ u_{xxy} > 0 \quad \text{at} \quad (0, -\frac{\pi}{2}) \\
u_{yy} < 0 \quad \text{on} \quad M
\end{gathered} \tag{2.38}$$

The reason for such a long list is owed to the style of argument. Roughly speaking, we will split the right half (or upper half) of $\overline{\Omega}$ into a finite rectangle and infinite tail region (or into a finite rectangle and *two* tail regions). The conditions in (3.65) will help gain control on the sign of either u_x or u_y near the boundary. The following result gives a condition which ensures a sign on the x derivative of small solutions to (2.2).

Lemma 2.10 (Asymptotic monotonicity)**.** *There exists $\epsilon_0 > 0$ such that, if $u \in C^3_{\mathrm{b}}(\overline{\Omega})$ and $(u, \lambda) \in \mathcal{O}_\delta \cap \mathcal{F}^{-1}(0)$, for some $\delta > 0$, $\lambda > 0$, $u_x \leq 0$ on $L_{x_0} := \{(x,y) \in \Omega \ : \ x = x_0\}$, and*

$$|u|_2 < \epsilon_0,$$

then $u_x \leq 0$ in $\Omega \cap \{(x,y) \ : \ x \geq x_0\}$.

Proof. Fix λ with $0 < \lambda \leq 1$. Differentiating (2.2) with respect to x gives

$$\begin{cases} \nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla v + 2\mathcal{W}''(|\nabla u|^2)(\nabla u \otimes \nabla u)\nabla v) - b_u(u,\lambda)v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.39}$$

where $v = u_x$. We see that (2.39) is uniformly elliptic by (3.4) in the case of Model I, and the fact that $u \in \mathcal{O}_\delta$ in the case of Model II. Let $v := \varphi z$, where

$$\varphi(y) = \cos(\sqrt{k}y) \tag{2.40}$$

and $1 - \lambda < k < 1$. After plugging (2.40) into (2.39), we find that z satisfies a uniformly elliptic equation with zeroth order term

$$\frac{1}{\varphi}(\partial_y(\mathcal{W}'(|\nabla u|^2)\varphi + 2\mathcal{W}''(|\nabla u|^2)(u_y^2 + u_x u_y)\varphi). \tag{2.41}$$

If ϵ_0 is chosen small enough, then from (2.1a) and (2.1b) it follows that (2.41) admits the $C^0(\overline{\Omega})$ expansion

$$\frac{1}{\varphi}((1 - \lambda - k)\varphi + O(\epsilon_0^2)) < 0. \tag{2.42}$$

Thus, (2.42) implies that z satisfies the strong maximum principle (Theorem A.1.(i)). Note $z = 0$ on $\partial\Omega$, and $z \leq 0$ on L_{x_0}, so it follows from the maximum principle that $z \leq 0$ in $\overline{\Omega} \cap \{(x,y) \,:\, x \geq x_0\}$. Since $\varphi(y) > 0$, we must have $u_x < 0$ in $\overline{\Omega} \cap \{(x,y) \,:\, x \geq x_0\}$ as well.

If $\lambda > 1$, then $v = u_x$ still solves (2.39). For ϵ_0 sufficiently small, $-b_z(u,\lambda) \leq 0$, by (2.1b). As before, the strong maximum principle and boundary conditions now yield the desired conclusion. ∎

Remark 2.11. The above lemma is stated for the half strip $(x_0, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$, for some $x_0 > 0$, but a similar result holds for sets of the form $(x_0, \infty) \times (0, \frac{\pi}{2})$ or $(-\infty, -x_0) \times (0, \frac{\pi}{2})$.

Next, we consider the nodal properties of a monotone solution.

Lemma 2.12 (Nodal properties)**.** *Let* $(u, \lambda) \in \mathcal{O}_\delta \cap \mathcal{F}^{-1}(0)$, *for some* $\delta > 0$. *Suppose that* $u_x < 0$ *in* Ω^+, $u_y < 0$ *in* Ω_+, *and* $u \in X$. *Then* u *satisfies* (3.65).

Proof. Note $u_x = 0$ on $\partial\Omega^+$ from the boundary conditions and evenness in the x variable. In particular, $u_x = u_{xx} = 0$ on T. The Hopf lemma (Theorem A.1.(ii)) shows that $u_{xx} < 0$ on L, $u_{xy} < 0$ on B, and that $u_{xy} > 0$ on T. Moreover, $u_{xy} = u_{xyy} = 0$ on L, since $u_x = 0$ on L. If (s_1, s_2) is a unit outward pointing vector at $(0, \frac{\pi}{2})$ with $s_1 < 0$ and $s_2 > 0$, then Serrin's lemma (Theorem A.1.(iii)) requires $\partial_s^2 u_x < 0$ at $(0, \frac{\pi}{2})$ since $u_{xx} = u_{xy} = 0$ at $(0, \frac{\pi}{2})$. A simple calculation shows $\partial_s^2 u_x = s_1^2 u_{xxx} + 2 s_1 s_2 u_{xxy} + s_2^2 u_{xyy} < 0$ at $(0, \frac{\pi}{2})$. From this we see $u_{xxy} > 0$ at $(0, \frac{\pi}{2})$. A similar argument shows $u_{xxy} < 0$ at $(0, -\frac{\pi}{2})$. We are left only to show that $u_{yy} < 0$ on M. The evenness of u in the y variable implies that $u_y = 0$ along M, and the result follows from the Hopf lemma. ∎

We now show that the collection of (u, λ) satisfying (3.65) is both open and closed in an appropriate relative topology.

Lemma 2.13 (Open property)**.** *Let* $(u, \lambda), (\tilde{u}, \tilde{\lambda}) \in \mathcal{O}_\delta \cap \mathcal{F}^{-1}(0)$, *for some* $\delta > 0$. *Suppose that* $0 < \lambda, \tilde{\lambda}$ *and* $u, \tilde{u} \in C^3_{\mathrm{b,e}}(\overline{\Omega}) \cap C^2_0(\overline{\Omega})$. *If* u *satisfies* (3.65), *then there is some* $\epsilon_0 > 0$ *for which* $|u - \tilde{u}|_3 + |\lambda - \tilde{\lambda}| < \epsilon_0$ *implies* $\tilde{u}$ *also satisfies* (3.65).

Proof. We will establish the sign of either $\tilde{u}_x$ or $\tilde{u}_y$ in several finite regions, and then invoke Lemma 2.10 to determine the signs in a leftover tail region. See Figure 2.3.2 for a sketch of the domains used. Now, because $u \in C^2_0(\overline{\Omega})$, there is an $R > 0$ large enough so that $|u|_2 < \epsilon/2$ for $x > R$, where ϵ is chosen to satisfy Lemma 2.10. If $|u - \tilde{u}|_2 < \epsilon_1 = \epsilon/2$, then $|\tilde{u}(x,y)|_2 < \epsilon$ for $x > R$. Let $\Omega^{+,2R}$ be the rectangle $(0, 2R) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and Ω^+_k the inscribed rectangle with distance $1/k$ from $\Omega^{+,2R}$. Let us define several regions useful for our analysis:

$$T_k := \{(x, \frac{\pi}{2}) \ : \ 1/k < x < 2R - 1/k\}$$
$$B_k ;= \{(x, -\frac{\pi}{2}) \ : \ 1/k < x < 2R - 1/k\}$$
$$L_k := \{(0, y) \ : \ -\frac{\pi}{2} + 1/k < y < \frac{\pi}{2} - 1/k\}.$$

For a given $k > 0$, there is an ϵ_k such that $|u - \tilde{u}|_3 < \epsilon_k$ implies $\tilde{u}_x < 0$ in $\overline{\Omega}_k$, $\tilde{u}_{xx} > 0$ on L_k, $\tilde{u}_{xy} > 0$ on T_k, and $\tilde{w}_{xy} < 0$ on B_k.

Suppose $\epsilon_0 < 1$, and consider the Taylor expansion of $\tilde{u}_x$ at a point $(x_0, \frac{\pi}{2})$ on T_k:

$$\tilde{u}_x(x_0, y) = \tilde{u}_{xy}(x_0, \frac{\pi}{2})(y - \frac{\pi}{2}) + O((y - \frac{\pi}{2})^2) \quad \text{in} \quad C^0(\overline{\Omega}), \tag{2.43}$$

where $\frac{\pi}{2} - 1/k < y < \frac{\pi}{2}$. When k is large enough, the remainder term in (2.43) is dominated by the first term and $\tilde{u}_x(x_0, y) < 0$. Analogous arguments show that for large enough k, $u_x < 0$ in the rectangle $(0, 1/k) \times (-\frac{\pi}{2} + 1/k, \frac{\pi}{2} - 1/k)$, and that $u_x < 0$ in $(1/k, 2R - 1/k) \times (0, 1/k)$.

We still need to deal with the corners. For a given k, consider the quarter circle of radius $\frac{\sqrt{2}}{k}$ in $\Omega^{+,2R}$ centered at $(0, \frac{\pi}{2})$. Because $u_x, u_{xx}, u_{xy}, u_{xxx} = 0$ at $(0, \frac{\pi}{2})$,

$$\tilde{u}_x(x, y) = \tilde{u}_{xxy}(0, \frac{\pi}{2})(x)(y - \frac{\pi}{2}) + O((y - \frac{\pi}{2})^2) \quad \text{in} \quad C^0(\overline{\Omega}).$$

For a given k there exists an ϵ'_k so small that $|u - \tilde{u}|_3 < \epsilon'_k$ implies that $\tilde{u}_{xxy}(0, \frac{\pi}{2}) > 0$. Arguing like before, we see that $\tilde{u}_x(x, y) < 0$ in the quarter circle, whenever k is sufficiently large. A similar argument shows that $u_x < 0$ in quarter circle of radius k centered at $(0, -\frac{\pi}{2})$.

From the work above, we find that if k is taken sufficiently large, and ϵ taken sufficiently small, then $\tilde{u}_x < 0$ in $\Omega^{+,2R}$. In particular, $\tilde{u}_x < 0$ on the line segment with $x = R$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Lemma 2.10 now implies that $\tilde{u}_x < 0$ in Ω^+. If we can show that $\tilde{u}_y < 0$ in Ω_+, then we will be able invoke Lemma 2.12 to get the desired result.

The argument to establish a sign on $\tilde{u}_y$ is similar to the one just given for $\tilde{u}_x$, so we provide only a sketch. Let $\Omega_{+,2R} = \Omega_+ \cap \{(x, y) \ : \ |x| \leq 2R\}$ and $\Omega_{+,k} = \Omega_+ \cap \{(x, y) \ : \ |x| \leq 2R, y > \frac{1}{k}\}$ and $M^{2R} = M \cap \{(x, y) \ : \ |x| \leq 2R\}$.

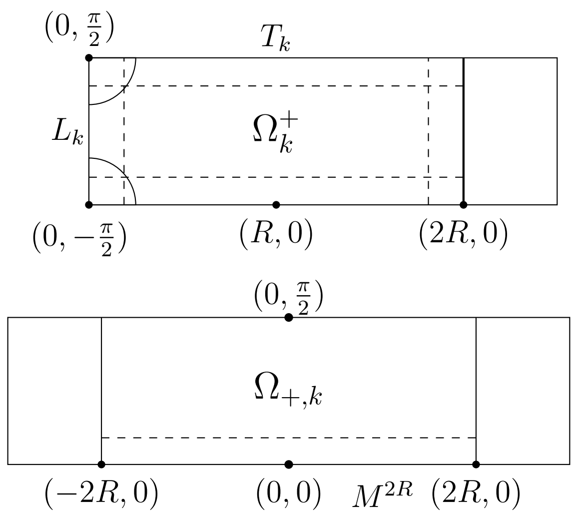

Figure 2.2: Left: Regions use to control the sign of $\tilde{u}_x$. Right: Regions used to control the sign of $\tilde{u}_y$.

We see that (2.39) holds for $v = u_y$ in Ω_+, except the homogeneous Dirichlet condition is lost. Thus, u_y satisfies a uniformly elliptic PDE with a non positive zeroth order coefficient in the tail region $\Omega_+ \cap \{(x, y) \ : \ |x| \geq R\}$, where R is the same constant from the above argument. For small enough ϵ_0 and k we find that $\tilde{u}_y < 0$ on $\Omega_{+,k}$ and $\tilde{u}_{yy} < 0$ on M^{2R}. If k is sufficiently large, then from a Taylor expansion along M^{2R} we find that $\tilde{u}_y < 0$ in $\Omega_{+,2R}$. Thus, a sign condition for $\tilde{u}_y$ is established in the finite region $\Omega_{+,2R}$. To deal the corresponding infinite tails, we just need to establish good boundary values, since we know $\tilde{u}_y$ satisfies the maximum principle whenever $|x| > R$ (see Remark 2.11). From the above argument, we have seen that if ϵ_0 is small enough, then $\tilde{u}_{xy} > 0$ on T. This, along with the decay of $\tilde{u}$ at $x = \infty$, is enough to establish that $\tilde{u}_y < 0$ on all of T. A symmetric argument will show that $u_y < 0$ on the line segment $\{(x, \frac{\pi}{2}) : -\infty < x < 0\}$. Also, $\tilde{u}_y = 0$ on M by evenness in the y variable. Finally, since $\tilde{u}_y \leq 0$ on the line segment with $x = R$ and $0 \leq y \leq \frac{\pi}{2}$ (and on the segment with $x = -R$ and $0 \leq y \leq \frac{\pi}{2}$, by evenness in the x variable), we conclude that $\tilde{u}_y < 0$ on Ω_+. ∎

Lemma 2.14 (Closed property)**.** *Let* $\{(u_n, \lambda_n)\} \subset \mathcal{O}_\delta \cap \mathcal{F}^{-1}(0)$*, for some* $\delta > 0$*. Suppose that* $(u_n, \lambda_n) \to (u, \lambda)$ *in* $C^3_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R}$*. If each* u_n *satisfies* (3.65)*, then so does* u*, unless* $u \equiv 0$*.*

Proof. By continuity we have that $u_x \leq 0$ in Ω^+, $u_x = 0$ on $\partial\Omega$, and $u_y \leq 0$ in Ω_+. So u_x and u_y each satisfy the strong maximum principle (Theorem A.1.(i)) in the relevant domain because $\mathcal{F}_u(u, \lambda)u_x = 0$ and $\mathcal{F}_u(u, \lambda)u_y$. Hence, if u_x is not trivial, then $u_x < 0$ in Ω^+ and $u_y < 0$ in Ω_+. Lemma 2.12 now implies that u satisfies (3.65). ∎

Next, we show that (3.65) holds along $\mathcal{C}^{I,II}_{\mathrm{loc}}$, which in turn shows that they hold on all of $\mathcal{C}^{I,II}$.

Lemma 2.15 (Nodal properties of the local curve)**.** *If* $(u^\epsilon, \epsilon^2) \in \mathcal{C}^{I,II}_{\mathrm{loc}}$ *and* $0 < \epsilon \ll 1$*, then* u^ϵ *exhibits the nodal properties* (3.65)*.*

Proof. In Theorem 2.4, we established that $u^\epsilon_x < 0$ in Ω^+. Since $u^\epsilon_x = 0$ on T, the Hopf lemma (Theorem A.1.(ii)) implies that $u^\epsilon_{xy} > 0$ on T. From (3.20), we know that $u^\epsilon_y(0, \frac{\pi}{2}) < 0$ for small enough ϵ. Combining this with the decay of u^ϵ_y at infinity allows us to conclude that $u^\epsilon_y < 0$ along all of T.

We now proceed as in the proof of Theorem 2.4. We have seen that u^ϵ satisfies equation (2.39). If we consider the corresponding uniformly elliptic operator acting on x-independent functions of the form $v = f(y)\cos(\sqrt{1-\lambda}y)$, then we obtain an expression with the following asymptotics in $C^0(\overline{\Omega})$

$$(1+O(\epsilon^2)(f''\cos(\sqrt{1-\lambda}y) - 2\sqrt{1-\lambda}f'\sin(\sqrt{1-\lambda}y)) \\ +O(\epsilon^2)(f'\cos(\sqrt{1-\lambda}y) - \sqrt{1-\lambda}f\sin(\sqrt{1-\lambda}y)) + O(\epsilon^2 f\cos(\sqrt{1-\lambda}y)). \tag{2.44}$$

Inspecting (2.44) shows that if we choose $f = \Phi_\epsilon$, as in the proof of Theorem 2.4, then for sufficiently small ϵ we can ensure (2.44) is negative for $0 \leq y \leq \frac{\pi}{2}$. The boundary condition on T, evenness in y (which implies $u^\epsilon_y = 0$ on M), maximum principle for uniformly elliptic operators with a positive super-solution (Theorem A.1.(iv)), and decay in x are now enough to conclude that $u^\epsilon_y < 0$ in Ω_+. Now, Lemma 2.12 implies the result. ∎

Theorem 2.16 (Global nodal properties)**.** *Every* $(u,\lambda) \in \mathcal{C}^{I,II}$ *exhibits the nodal properties* (3.65)*.*

Proof. Let $(u,\lambda) \in \mathcal{C}^{I,II}_{\text{loc}}$. From Lemma 2.15 u satisfies (3.65). Since the nodal properties are both open and closed in the relative topology of $\mathcal{C}^{I,II}$ by Lemma 2.13 and Lemma 3.26, we conclude that they hold everywhere on $\mathcal{C}^{I,II}$. ∎

2.4 Uniform regularity and bounds on loading parameter

The main result of this section, which is stated in Proposition 2.24, is that $|u(s)|_{3+\alpha}$ is uniformly bounded along $C_\delta^{I,II}$. This is achieved by first using Schauder theory to estimate $|u(s)|_{3+\alpha}$ in terms of $|\nabla u(s)|_0$ and then estimating $|\nabla u(s)|_0$ in terms of $|u(s)|_0$ and $|\lambda(s)|$ by a maximum principle argument. Upper bounds on $|u(s)|_0$ and $|\lambda(s)|$ are then established for $C_\delta^{I,II}$. Finally, for $s \gg 1$ it is shown that there is a positive uniform lower bound on $\lambda(s)$ along $\mathcal{C}_\delta^{I,II}$.

2.4.1 A conserved quantity and L^p estimates

We derive a conserved quantity of the system that will play a key role establishing uniform bounds on $|u(s)|_0$ and $|\lambda(s)|$ along $\mathcal{C}_\delta^{I,II}$. These results, in tandem with Lemma 2.21, give our desired a priori estimates. The following calculation is valid for any C^2 solution of (2.2). Let

$$\mathcal{L}(z,\xi,\eta,\lambda) := \frac{1}{2}\mathcal{W}(|\xi^2+\eta^2|^2) + B(z,\lambda), \tag{2.45}$$

where

$$B(z,\lambda) := \int_0^z b(t,\lambda)\,dt.$$

The anti-plane elastostatic problem (2.2) is formally the Euler–Lagrange equation given formally by

$$\delta \int_\Omega \mathcal{L}(u, |\nabla u|^2, \lambda)\, dx\, dy = 0.$$

Naturally, the translation invariance in x of our system leads us to expect a corresponding conserved quantity. Consider the functional

$$\begin{aligned}\mathcal{H}(u, \lambda; x) :&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\mathcal{L}(u, |\nabla u|^2, \lambda) - \mathcal{L}_\xi(u, |\nabla u|^2, \lambda) u_x)\, dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1}{2}\mathcal{W}(|\nabla u|^2) - \mathcal{W}'(|\nabla u|^2) u_x^2 + B(u, \lambda))\, dy.\end{aligned} \tag{2.46}$$

If (u, λ) solves (2.2), then $\mathcal{H}(u, \lambda; \cdot)$ is constant in x:

$$\begin{aligned}\partial_x \mathcal{H} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\mathcal{L}_z u_x + \mathcal{L}_\xi u_{xx} + \mathcal{L}_\eta u_{xy} - (\partial_x \mathcal{L}_\xi) u_x - \mathcal{L}_\xi u_{xx})\ dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_x (\mathcal{L}_z - \partial_y \mathcal{L}_\eta - \partial_x \mathcal{L}_\xi)\, dy = 0,\end{aligned}$$

where we used integration by parts and that

$$(\mathcal{L}_z - \partial_y \mathcal{L}_\eta - \partial_x \mathcal{L}_\xi)(y, u, u_x, u_y, \lambda) = \mathcal{F}(u, \lambda) = 0.$$

It is clear that $\mathcal{H}(u(x, y), \lambda; x) \to 0$ as $x \to \infty$, so $\mathcal{H}$ is identically 0. We record this as a lemma. Note that the arguments of $\mathcal{H}$ will often be suppressed in the sequel.

Lemma 2.17 (Conserved quantity)**.** *Let $u \in C^2(\overline{\Omega})$ be a solution to (2.2) for a fixed λ. Then $\mathcal{H}$ is constant in x. In particular, if $u \in X$, then $\mathcal{H} = 0$.*

The conserved quantity and growth conditions of both $\mathcal{W}$ and b are enough to obtain a uniform bound on $|u(s)|_0$ (and on $|\lambda(s)|$, as shown in Subsection 2.4.3).

Lemma 2.18 (L^p bounds)**.** *There exists a constant $C(c_1, c_2, b_1)$ such that if $u \in X$ is a solution to (2.2), corresponding to Model I, with $0 < \lambda$, then*

$$\|u(0,\cdot)\|_2, \;\; \|u_y(0,\cdot)\|_6, \;\; |u(0,\cdot)|_{1/2} \le C, \tag{2.47}$$

where $\|\cdot\|_p$ denotes the L^p norm on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, for any $x_0 \in \mathbb{R}$ we have

$$\|u(x_0,\cdot)\|_2, \;\; |u(x_0,0)|_0 \le C$$

Proof. From (2.46), Lemma 2.17, (2.3b) and (2.5a) we see that when $x = 0$, then

$$\begin{aligned} 0 = 2\mathcal{H} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{L}\,dy \ge & \|u_y(0,\cdot)\|_2^2 + c_1\|u_y(0,\cdot)\|_4^4 + c_2\|u_y(0,\cdot)\|_6^6 \\ & + (\lambda - 1)\|u(0,\cdot)\|_2^2 + \frac{b_1}{2}\|u(0,\cdot)\|_4^4. \end{aligned}$$

For the remainder of the proof, we will suppress arguments of $u(0,\cdot)$ and $u_y(0,\cdot)$ appearing in $L^p[-\frac{\pi}{2}, \frac{\pi}{2}]$ norms. Wirtinger's inequality implies that $\|u_y\|_2^2 - \|u\|_2^2 \ge 0$, and $\|u\|_4 \le \pi\|u_y\|_4$ by Friedrichs's inequality, so that

$$c_1\|u_y\|_4^4 + c_2\|u_y\|_6^6 + \frac{b_1\pi}{2}\|u_y\|_4^4 \le -\lambda\|u\|_2^2 + (\|u\|_2^2 - \|u_y\|_2^2) \le 0. \tag{2.48}$$

Hölder's inequality yields

$$\|u_y\|_4^4 \leq \pi^{\frac{1}{3}} \|u_y\|_6^4.$$

Altogether these give

$$(c_1 + \frac{b_1 \pi}{2}) \pi^{\frac{1}{3}} \|u_y\|_6^4 + c_2 \|u_y\|_6^6 \leq 0 \tag{2.49}$$

so that

$$\|u_y\|_6^2 \leq \frac{|c_1 + \frac{b_1}{2}\pi| \pi^{\frac{1}{3}}}{c_2}.$$

Thus, $\|u_y\|_6$ is uniformly bounded. An application of Hölder's inequality shows that $\|u_y\|_2$ is uniformly bounded too. As mentioned above, $\|u\|_2 \leq \|u_y\|_2$. Hence, we obtain a uniform bound on $|u(0,\cdot)|_{1/2}$ by Sobolev embedding. Because of the monotonicity of u established in (3.65), we see that $\|u(x_0,\cdot)\|_2$ and $|u(x_0,\cdot)|_0$ are maximized at $x_0 = 0$. Thus, the L^2 and L^∞ norms of u are uniformly bounded on any transversal line in Ω. ∎

Remark 2.19. Note that this says nothing about solutions of Model II. If $u \in \mathcal{C}_\delta^{II}$, with $\delta > 0$, then $|u| < C$, where C depends on q_1. This is a direct consequence of (3.5a), (3.5b), and the homogeneous Dirichlet conditions ($|\nabla u|^2 < q_2$ along $\mathcal{C}_\delta^{II}$).

Remark 2.20. More generally, if $\mathcal{H} = M$, then the above argument shows that $\|u_y(0,\cdot)\|_6$ is bounded uniformly by a constant C that depends on c_1, c_2, b_1 and M, so long as one assumes sufficient growth of $\mathcal{W}$ relative to b.

At this stage, we are left to establish control on $|\lambda(s)|$ to complete our desired estimates of $|u(s)|_{3+\alpha}$

2.4.2 Uniform regularity

We begin by using the so called "P-function" technique (see [33]) along with standard elliptic estimates to gain some control on $|u(s)|_{3+\alpha}$.

Lemma 2.21. *Let $(u, \lambda) \in \mathcal{O}_\delta \cap \mathcal{F}^{-1}(0)$, for some $\delta > 0$. If λ and K are positive, and $|u|_0 + \lambda < K$, then there is a constant $C(K, \delta) > 0$ for which $|u|_{3+\alpha} \leq C(K, \delta)$. If (u, λ) is a solution which corresponds to Model I, then the above estimate holds for some $C = C(K)$.*

Proof. We prove this result by using a maximum principle of Payne and Philippin. First, we obtain bounds on $|\nabla u(s)|^2$. Recall, as mentioned in Remark 2.19 that this is trivial for Model II, so let us assume for now the conditions of Model I. By Theorem 1 of [26] the function

$$P(x, y) = \int_0^{|\nabla u(x,y)|^2} (\mathcal{W}'(\xi) + 2\xi\mathcal{W}''(\xi))d\xi - 2\int_0^{u(x,y)} b(\eta, \lambda)\eta \, d\eta,$$

obtains its maximum either on $\partial\Omega$ or at a critical point of u. We should note that in [26] the results are stated for bounded $C^{2+\alpha}$ domains. So, our application includes the additional possibility that the maximum of P occurs in the limit as $x \to \pm\infty$. However, the decay of u precludes this scenario for nontrivial solutions. The homogeneous Dirichlet boundary conditions of (2.2) and monotonicity properties of (3.65) now imply that P is maximized at $(0,0)$, which is the only critical point of u. Thus,

$$(2q\mathcal{W}'(q) - \mathcal{W}(q))\big|_{q=|\nabla u(x,y)|^2} - 2\int_0^{u(x,y)} b(\eta,\lambda)\eta\, d\eta \leq -2\int_0^{u(0,0)} b(\eta,\lambda)\eta\, d\eta.$$

So,

$$2q\mathcal{W}'(q) - \mathcal{W}(q) \leq -2\int_{u(x,y)}^{u(0,0)} b(\eta,\lambda)\eta\ d\eta \leq 2(u(0,0))^2 \max_{(x,y)\in\overline{\Omega}} |b(u(x,y),\lambda)|. \tag{2.50}$$

Since b is analytic in both z and λ, it follows that the right hand side of (2.50) is bounded by $C(K)$. Moreover, the left hand side of (2.50) satisfies

$$2q\mathcal{W}'(q) - \mathcal{W}(q) = \int_0^q \left(\mathcal{W}' + 2q\mathcal{W}''(q)\right)\ dq \geq q\xi_1,$$

whenever $q \geq 0$, by (3.4). Hence,

$$|\nabla u|_0^2 \leq \frac{2(u(0,0))^2}{\xi_1} \max_{(x,y)\in\overline{\Omega}} |b(u(x,y),\lambda)|. \tag{2.51}$$

Standard elliptic theory can now be invoked to upgrade a uniform bound in $|\nabla u|$ into a uniform bound in $C^{3+\alpha}(\overline{\Omega})$. As we have seen in (2.39), $\partial_x u$ solves a divergence form elliptic equation. In particular, from (2.50) it follows that we may view $\partial_x u$ as the solution to a linear PDE with uniformly bounded coefficients. An application of [11, Theorem 8.29] yields that for some $\alpha' \in (0,\alpha]$

$$|u_x|_{C^{\alpha'}(\Omega_M)} \leq C,$$

where $\Omega_M = \Omega \cap \{(x, y) \ : \ M \leq x \leq M+1\}$, and both α' and C depend on K and δ (or only K in the case of Model I). An analogous bound for $|u_y|_{C^{\alpha'}(\Omega_M)}$ is obtained by differentiating in y instead. Now, by viewing (2.2) as a linear equation with coefficients that depend on u_x and u_y, which we have just shown are uniformly bounded in $C^{\alpha'}(\Omega_M)$, we may apply (linear) Schauder theory to obtain a uniform bound on $|u|_{C^{2+\alpha'}(\Omega_M)}$. This gives control over $|u|_{C^{1+\alpha}(\Omega_M)}$, so that by repeating the previous argument we gain control of $|u|_{C^{2+\alpha}(\Omega_M)}$. Now, Schauder estimates applied to the linearized equations for either u_x or u_y provide a uniform bound on $|u|_{C^{3+\alpha}(\Omega_M)}$. ∎

2.4.3 Bounds on loading parameter

Now, we show that as we follow either $\mathcal{C}^I$ or $\mathcal{C}^{II}_\delta$, for some $\delta > 0$, that $\lambda(s)$ cannot return to 0 without the corresponding solutions returning to the reference configuration or the equation undergoing a loss of ellipticity. Moreover, an upper bound on $\lambda(s)$ is derived for either case. These estimates will be used to establish bounds on $|u(s)|_{3+\alpha}$ and (3.62).

Lemma 2.22. *If* $\{(u_n, \lambda_n)\} \subset \mathcal{C}^I$*, or* $\{(u_n, \lambda_n)\} \subset \mathcal{C}^{II}_\delta$*, for some* $\delta > 0$*, is sequence of solutions to* (2.2)*, that are uniformly bounded in* $C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega})$ *and for which* $\lambda_n \to 0$*, then* $u_n \to 0$ *in* X*.*

Proof. Assume that $\{u_n\}$ does not converge to 0 in $C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega})$. By the hypothesis and (3.65), we may then invoke Theorem 2.9. From this, one may conclude that either there is a subsequence converging in $C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega})$ to a solution $(u, 0) \in X \times \mathbb{R}$ of (2.2), or there is a subsequence of translates

$$u_n(\cdot + x_n, \cdot) \xrightarrow{C^2_{\mathrm{b}}(\overline{\Omega})} \tilde{u}(\cdot, \cdot),$$

where $x_n \to \infty$, and $(\tilde{u}, 0) \in C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R}$ solves (2.2). Moreover, $\tilde{u}_x, \tilde{u}_y \to 0$ uniformly in x, y as $x \to \infty$, and $\tilde{u}_x \leq 0$. From (3.65) we also know that $\tilde{u}_y \geq 0$ for $y \in [-\frac{\pi}{2}, 0)$ and $\tilde{u}_y \leq 0$ for $y \in (0, \frac{\pi}{2}]$. An application of the strong maximum principle for signed solutions (see Theorem A.1.(i)) shows that the inequalities for $\tilde{u}_x$ and $\tilde{u}_y$ must in fact be strict in the interior of Ω. Throughout the rest of the proof, the properties of u and $\tilde{u}$ that concern us are the same, namely that they are monotone bounded solutions to (2.2) that decay as $x \to \infty$. For simplicity, let us only write u in the sequel, with the understanding that the argument applies equally well to $\tilde{u}$.

Let $0 < R_1 < R_2$ and $\varphi(y) = \cos(y)$. Multiplying (2.2) by φ and then integrating we find

$$\begin{aligned} 0 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} (\nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla u) - b(u, \lambda))\varphi \, dxdy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{W}'(|\nabla u|^2) u_x \varphi \Big|_{R_1}^{R_2} dy + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} \left(-\mathcal{W}'(|\nabla u|^2) u_y \varphi_y - b(u, \lambda)\varphi\right) dxdy. \end{aligned}$$

Note that $u_y \varphi_y > 0$ by the comments above. For R_1 sufficiently large,

$$\frac{1}{2} < \mathcal{W}'(|\nabla u|^2) < 1 \quad \text{in} \quad (R_1, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$$

because of the decay in u_x and u_y. Recall that b has the form (2.1b) when $|\nabla u|^2$

is sufficiently small. So for large enough R_1 we also have $-b(u,\lambda)-(1-\lambda)u \geq 0$ whenever $x > R_1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Letting $R_2 \to \infty$ we see that

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{R_1}^{\infty} (-\mathcal{W}'(|\nabla u|^2) + 1) u_y \varphi_y + (-b(u,\lambda) - (1-\lambda)u)\varphi \, dxdy$$
$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{W}'(|\nabla u|^2) u_x \varphi(R_1, y)\, dy > 0.$$

This is of course a contradiction. ∎

Lemma 2.23 (Bounds on λ)**.** *Let $(u,\lambda) \in \mathcal{C}^I \setminus \mathcal{C}^I_{\text{loc}}$. Then there exists some positive constants $\lambda_1^\pm = \lambda^\pm(c_1, c_2, b_1)$ for which $0 < \lambda^- < \lambda < \lambda^+ < \infty$. If instead we take $(u,\lambda) \in \mathcal{C}^{II}_\delta \setminus \mathcal{C}_{\text{loc}}$, for some $\delta > 0$, then the result still holds with positive constants $\lambda_2^\pm$, except that λ^- now depends on δ as well.*

Proof. First, let us suppose that there is some sequence $\{(u_n, \lambda_n)\} \subset \mathcal{C}^{I,II}_\delta \setminus \mathcal{C}^{I,II}_{\text{loc}}$ for which $\lambda_n \to 0$. By Lemma 2.21 and Lemma 2.18 we see that $\{u_n\}$ is uniformly bounded in $C^{3+\alpha}_{\text{b}}(\overline{\Omega})$ (in the case of Model II there is dependence on δ). Then, from Lemma 2.22, it follows that $u_n \to 0$ in $C^{3+\alpha}_{\text{b}}(\overline{\Omega})$. However, Theorem 2.4 then implies that $u_n \in \mathcal{C}^{I,II}_{\text{loc}}$ for large enough n, and this contradicts part (c) of Theorem 3.23.

To establish an upper bound on λ, see that (2.5a) and Lemma 2.17 imply

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\mathcal{W}(|\nabla u|^2) + 2B)\big|_{x=0}\, dy \geq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{W}(|\nabla u|^2)\big|_{x=0}\, dy + (\lambda - 1)\|u(0,\cdot)\|_2^2 + \frac{b_1}{2}\|u(0,\cdot)\|_4^4,$$

where we are using $\|\cdot\|_p$ to denote the L^p norm on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $\mathcal{W}(q) > 0$ for $q > 0$, it follows that $(\lambda-1)\|u(0,\cdot)\|_2^2 + \frac{b_1}{2}\|u(0,\cdot)\|_4^4 \leq 0$. Appealing to

Lemma 2.18, We find that

$$\lambda \leq \left(1 + \frac{|b_1|}{2} \sup_{s>0} |u(s)|^2\right) \|u(0,\cdot)\|_2^2 \leq C(c_1, c_2, b_1). \blacksquare$$

We are now ready to state the main a priori estimate.

Proposition 2.24. *If $(u, \lambda) \in \mathcal{C}^I$, then then there exists some $C(c_1, c_2, b_1) > 0$ for which $|u|_{3+\alpha} \leq C$. If instead $(u, \lambda) \in \mathcal{C}_\delta^{II}$, then there exists $C(c_1, c_2, b_1, \delta) > 0$ for which $|u|_{3+\alpha} \leq C$.*

Proof. For $(u, \lambda) \in \mathcal{C}^I$, the result follows from Lemma 2.18 and Lemma 2.23 combined with Lemma 2.21. For $(u, \lambda) \in \mathcal{C}_\delta^{II}$, the estimate can be obtained from Remark 2.19 and Lemma 2.23 in conjunction with Lemma 2.21. ∎

2.5 Proof of the main results

We are now prepared to prove the main results. The existence of a global solution branch, for either model, is shown in Section 2.3. The key difference between the global behavior of $\mathcal{C}^I$ and $\mathcal{C}^{II}$ is related to alternative (i) of Theorem 3.23; for $\mathcal{C}^I$ it is shown to be impossible, thus forcing alternative (ii), whereas for $\mathcal{C}^{II}$ it is shown to hold (note that (ii) is *not* necessarily excluded in this case).

Proof of Theorem 3.1. By Theorem 3.23, there exists a curve of solutions $\mathcal{C}^I$ extending $\mathcal{C}_{\text{loc}}^I$, which is locally real analytic with C^0 parameterization

$$\mathcal{C}^I = \{(u(s), \lambda(s)) \ : \ 0 < s < \infty\} \subset X \times \mathbb{R}.$$

The symmetry and monotonicity properties of Theorem 3.1.(a) are proved throughout in Theorem 2.16. The bounds on $\lambda(s)$ and $\sup_{s\geq 0}|u(s)|_{3+\alpha}$ from Theorem 3.1.(c) and Theorem 3.1.(d) are established in Lemma 2.23 and Proposition 2.24, respectively. This establishes all parts of Theorem 3.1 except for the broadening in (e). From the alternatives in Theorem 3.23, we see that (e) must hold if

$$N(s) = |u(s)|_{3+\alpha} + \frac{1}{\operatorname{dist}(u(s), \partial\mathcal{O})} + \lambda(s) + \frac{1}{\operatorname{dist}(\lambda(s), \partial\mathcal{I})}$$

is bounded uniformly in s. The bound on the first term follows directly from Lemma 2.21 and those on the third and fourth terms follow from the estimates on λ established in Lemma 2.23. Recall that $\mathcal{O}$ is defined by (2.32), so (3.4) implies that the second term remains bounded as well. Hence, we have the desired control over $N(s)$, and the result must hold. ∎

We need to prove one more lemma in preparation for Theorem 3.2.

Lemma 2.25 (Nonexistence of monotone fronts)**.** *Let $\mathcal{F}$ correspond to Model I. Then, $\mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, for $\delta > 0$, is locally pre-compact in X. In particular, alternative* (ii) *of Theorem* 2.9 *cannot hold for a sequence $\{(u_n, \lambda_n)\} \subset \mathcal{C}_\delta^{II}$, $\delta > 0$.*

Proof. From Theorem 2.9 we find that if $\mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$ fails to be locally pre-compact, then Theorem 2.9.(ii) must hold. Suppose that $\{u_n\}$ is a sequence satisfying Theorem 2.9.(ii). Then $\lim_{n\to\infty} u_n(\cdot + x_n, y) =: \tilde{u}(x,y) \in C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega})$ must solve (2.2). Moreover, $U(y) := \lim_{x\to-\infty} \tilde{u}(x,y)$ satisfies

$$((\mathcal{W}'((U_y)^2)U_y)_y - b(U,\lambda) = 0, \tag{2.52}$$

which can be seen by [6, Lemma 2.3]. Multiplying (2.52) by $U(y)$ and integrating by parts yields

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\mathcal{W}'(U_y^2)U_y^2 + b(U,\lambda)U)\, dy.$$

From Lemma 2.17 we know that $\mathcal{H}(u_n,\lambda;x) = 0$, and hence that $\mathcal{H}(U,\lambda;x) = 0$. Written explicitly this becomes

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\mathcal{W}(U_y^2) + B(U,\lambda)\right)\, dy.$$

After combining these equations, we find

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\mathcal{W}'(U_y^2)U_y^2 - \mathcal{W}(U_y^2) + b(U,\lambda)U - 2B(U,\lambda))\, dy. \tag{2.53}$$

A simple calculation will show

$$b(z,\lambda)z - 2B(z,\lambda) < 0 \quad \text{for} \quad z > 0,$$

where the concavity of $b(z,\lambda)$ and the fact that $b(0,\lambda) = 0$ are used. Recall that $q\mathcal{W}'(q) - \mathcal{W} < 0$ by (2.6c). But then the right hand side of (2.53) is negative, which is a contradiction, hence the result holds. ∎

Proof of Theorem 3.2. From Theorem 3.23, we see there is a curve of solutions $\mathcal{C}^{II}$, extending $\mathcal{C}^{II}_{\text{loc}}$, which is locally real analytic with C^0 parameterization

$$\mathcal{C}^{II} = \{(u(s), \lambda(s)) \ : \ 0 < s < \infty\} \subset X \times \mathbb{R}.$$

As in the proof of Theorem 3.1, we wish to understand the alternatives in Theorem 3.23.(a). The quantity

$$\inf_{\overline{\Omega}} \left(\mathcal{W}'(q) + 2q\mathcal{W}''(q)\right)\Big|_{q=|\nabla u(s)|^2}$$

is not bounded below a priori along $\mathcal{C}^{II}$ as it was for $\mathcal{C}^{I}$. This leads us to consider $N(s)$ (see (3.62) or the proof of Theorem 3.1) on a segment $\mathcal{C}^{II}_{\delta}$ of $\mathcal{C}^{II}$, with $\delta > 0$. An estimate of the form $|u(s)|_{3+\alpha} < C(\delta)$ is obtained whenever $(u(s), \lambda(s)) \in \mathcal{C}^{II}_{\delta}$, by Proposition 2.24.

Now, if we assume $\mathcal{C}^{II} = \mathcal{C}^{II}_{\delta^*}$, for some $\delta^* > 0$, then the first term of $N(s)$ is uniformly bounded along $\mathcal{C}^{II}$ by the paragraph above. Of course, this assumption also implies that the second term is uniformly bounded along $\mathcal{C}^{II}$ by definition. Furthermore, Lemma 2.23 implies that the third and fourth terms are also controlled. Thus, there is some $C'(\delta)$ for which $s \gg 1$ implies $|N(s)| \leq C'(\delta)$. Hence, alternative (ii) of Theorem 3.23 must hold. However, this contradicts the impossibility of fronts established in Lemma 2.25. So we must have

$$\lim_{s\to\infty} \inf_{\overline{\Omega}} \left(\mathcal{W}'(q) + 2q\mathcal{W}''(q)\right)\Big|_{q=|\nabla u(s)|^2} = 0.$$

In particular, we must have $|\nabla u(s)|^2 \to q_1$. Note that our estimates on $|u(s)|_{3+\alpha}$ and $\lambda(s)$ breakdown as $\delta \to 0$. This leaves open the possibility that $\lambda(s)$ approaches either 0 or ∞, or that a blow-up in $C^{3+\alpha}(\overline{\Omega})$ (note that $|u(s)|_0$ and $|u_y(s)|_0$ are indeed bounded, but the elliptic estimates depend on δ) occurs concurrently with the loss of ellipticity. This establishes Theorem 3.2.(ii)

The monotonicity and symmetry properties of Theorem 3.2.(a) are prove in Theorem 2.16. Finally, the bound on $\lambda(s)$ of Theorem 3.2.(c) is proved in Lemma 2.23. ∎

Chapter 3

Anti-plane shear equilibria in the large

3.1 Introduction

We consider a class of quasilinear elliptic partial differential equations with a non-linear boundary condition arising in elastostatics. Specifically, they relate to *anti-plane* shear deformations, which have received a great deal of attention in the literature [13, 17, 19, 22, 23, 31, 35]. The governing equations for nonlinear elastostatics in the most general setting constitute a system of quasilinear elliptic PDEs. Complications stemming from compressibility constraints, ellipticity, and mixed boundary conditions pose serious obstacles to a satisfactory existence and regularity theory. Global bifurcation theory combined with a detailed qualitative analysis and sufficient a priori estimates allows us to investigate the complex interplay between these mechanisms. For deformations of anti-plane shear type, they can be reduced to a single scalar equation provided the elastic material satisfies certain physical conditions, see [23, 35] or

Section 1.3. The resulting problem becomes more tractable and provides value insights for the general case. However, there is a relative scarcity of results in this direction. There are a number of works in which deformations localized to a reference state or solving a reduced ODE model are obtained, but these are unable to account for the phenomena previously mentioned. On the other hand, global families of equilibria for the full system have been obtained in a series of important papers [12,14,15]. The terminal behavior of these curves are described by several outcomes including, for example, blow-up in an appropriate Hölder space or the failure of ellipticity. The inherent difficulties presented by systems of non-linear elliptic PDEs leaves the exact outcome uncertain in general. A primary motivation for this paper is to concretely demonstrate blow-up and loss of ellipticity for families of anti-plane shear equilibria.

Our investigation is significantly complicated by the inclusion of degenerate ellipticity and nonlinear boundary conditions. These features are essential to models incorporating material failure and deformation dependent boundary conditions. In this paper, we establish global families of anti-plane shear equilibria on a semi-infinite strip for a large class of materials and traction forces. The corresponding solutions exhibit arbitrarily large strains or kink-type singularities. The observed outcome depends upon the ellipticity of the underlying equations and the structure of the nonlinear boundary condition. A key component of our analysis is a new, more general, gradient maximum principle for a class of degenerate quasi-linear elliptic PDEs. This result is vital to our proof that a loss of regularity must accompany a loss of ellipticity.

The breakdown of ellipticity has been linked to failure mechanics in a series of well-known papers exploring crack propagation [21, 23]. Notably, Knowles points out the similarities between the equations of anti-plane shear and those of a steady, irrotational, ideal gas; in the latter case, one recognizes the loss of ellipticity as marking a switch from subsonic to supersonic flow.

3.1.1 The problem

We are now ready to state the problem considered in this chapter. The assumptions of Section 1.3 are adopted with $\mathcal{B} = \Omega \times \mathbb{R}$ and $\Omega = (0, \infty) \times (-h/2, h/2)$. Suppose that the material in question is in a state of anti-plane shear with displacement perpendicular to Ω. The lateral sides (denoted by T and B) are assumed to be traction free while the near-end (denoted by L) is subjected to a displacement dependent traction force $f = f(\lambda, z)$. The governing equations for static equilibrium are

$$\begin{cases} \nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla u) = 0 & \text{in } \Omega \\ u_y = 0 & \text{on } T \cup B. \\ \mathcal{W}'(|\nabla u|^2) u_x = -f(\lambda, u) & \text{on } L. \end{cases} \tag{3.1}$$

The nonlinear boundary condition in (3.1) expresses a relation between the traction force applied to the left boundary, the displacement, and a quantity λ that serves as both a loading and bifurcation parameter. This is an example of what is referred to as a "live boundary" condition in the literature [12, 27, 32].

These can used to account for complex interactions between an elastic material and its environment which depend on displacement. Several authors have studied a reduced model of elastic contact whose boundary conditions are analogous to ours [20, 31].

3.1.2 Additional assumptions

The structure of $\mathcal{W}$ plays a crucial in our analysis of (3.1). It is assumed that $\mathcal{W}$ may be expanded near $q = 0$ as

$$\mathcal{W}(q) = q + c_1 q^2 + c_2 q^3 + \cdots . \tag{3.2}$$

Note that $\mathcal{W}(0) = 0$ and $\mathcal{W}'(0) = 1$. Given $\mathcal{W}$, we introduce the two important functions

$$\mathcal{E}(q) := \mathcal{W}'(q) + 2q\mathcal{W}''(q), \quad \tau(k) := 2\mathcal{W}'(k^2)k. \tag{3.3}$$

The quantity $\mathcal{E}$ is useful when describing our structural assumptions and τ is known as the *shear response.* The following (strict) ellipticity condition is assumed in parts of this paper:

$$\mathcal{E}(q) > \xi > 0, \quad q \geq 0, \tag{3.4}$$

while at others we impose the following degenerate-ellipticity conditions:

$$\mathcal{E}(q) > 0 \qquad \text{for} \qquad q \in [0, q_{\text{cr}}), \tag{3.5a}$$

$$\mathcal{E}(q) \to 0 \qquad \text{as} \qquad q \to q_{\text{cr}}^-, \tag{3.5b}$$

$$\mathcal{E}'(q_{\rm cr}) = 3\mathcal{W}''(q_{\rm cr}) + 2q_{\rm cr}\mathcal{W}'''(q_{\rm cr}) < 0, \tag{3.5c}$$

where $0 < q_{\rm cr} < \infty$. Note that (3.4) guarantees (3.1) is elliptic for all values of q. This is equivalent to the strict convexity of $\mathcal{W}(k^2)$. Such an assumption is commonly adopted when studying quasilinear PDE. On the other hand, (3.5a) and (3.5b) make (3.1) degenerate elliptic.

Let us fix some key assumptions on f. For the remainder of the paper we suppose that f is odd in z and that

$$f_z(\lambda^*, 0) = \frac{\pi}{h}, \quad f_{\lambda z}(\lambda^*, 0) \neq 0, \quad 8c_1\left(\frac{\pi}{h}\right)^3 - \partial_z^3 f(\lambda^*, 0) \neq 0, \tag{3.6}$$

where $\lambda^* > 0$. The non-vanishing of $f_{\lambda z}$ is related to a so-called transversality condition required in our construction of local bifurcation curves. The oddness of f is a key component in our proof of the monotonicity properties of u. It will be assumed for some of the global theory that f satisfies the following growth conditions:

$$f_z(0,0) = 0, \tag{3.7a}$$

$$f_{\lambda z}(\lambda, 0) > 0 \qquad \text{for} \qquad \lambda \in \mathbb{R}, \tag{3.7b}$$

$$f(\lambda, z) \geq \rho(\lambda)z - bz^3 \qquad \text{for} \qquad z > 0, \tag{3.7c}$$

$$f(\lambda, z) = 0 \qquad \text{implies} \qquad f_z(\lambda, z) \neq 0 \qquad \text{for} \qquad \lambda > 0, \tag{3.7d}$$

where $\rho(0) = 0$, $\rho'(\lambda) > 0$ for $\lambda \in (0, \infty)$, and $\rho(\lambda) \to \infty$ as $\lambda \to \infty$. For the degenerate case, we impose the further restrictions that

$$\mathcal{W}'(q) \leq 1, \tag{3.8a}$$

$$f(\lambda, z) = \lambda z + Bz^3, \tag{3.8b}$$

$$\left(\frac{2\pi}{3h}\right)^3 < 2Bm^2 q_{\mathrm{cr}}, \tag{3.8c}$$

where $B > 0$ and $m = \min\limits_{q\in[0,q_{\mathrm{cr}}]} \mathcal{W}'(q)$. This condition is used to show that when the gradient achieves the critical value, it must have nontrivial tangential component. This is a key hypothesis for our proof that a loss of regularity accompanies the breakdown of ellipticity.

3.1.3 Main results

Our efforts are focused primarily on the construction and investigation of solutions to (3.1). The following general theorem pertains to a wide class of strain-energy functions and traction forces. It will serve as the starting point for a more detailed global theory.

Theorem 3.1. *Suppose that $\mathcal{W}$ satisfies one of* (3.4) *or* (3.5)*. In addition, let f be of the form* (3.6)*. Then there is a curve $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ of solutions to* (3.1) *with $\mathcal{C}^+$ admitting the C^0 parameterization*

$$\mathcal{C}^+ = \{(\lambda(s), u(s)) : 0 \leq s < \infty\} \subset \mathbb{R} \times C^{3,\alpha}_{\mathrm{b}}(\overline{\Omega})$$

and $(\lambda(0), u(0)) = (\lambda^*, 0)$. Moreover, $\mathcal{C}^+$ satisfies the following:

(a) (Symmetry and monotonicity) *Each* $(\lambda(s), u(s)) \in \mathcal{C}^+$ *is monotone in the sense that*

$$\partial_x u(s) < 0 \quad \textit{for} \quad (x, y) \in [0, \infty) \times (0, h/2]$$
$$\partial_y u(s) > 0 \quad \textit{for} \quad (x, y) \in [0, \infty) \times (-h/2, h/2).$$

Furthermore, $u(s)$ *is odd* y *and* $\mathcal{C}^- = \{(\lambda(s), -u(s)) \mid (\lambda(s), u(s)) \in \mathcal{C}^+\}$.

(b) (Analyticity) *The curve* $\mathcal{C}$ *is locally real analytic.*

(c) (Alternatives) *At least one of the following alternatives hold:*

(i) (Blow-up) $\mathcal{C}^+$ *is not bounded in* $\mathbb{R} \times C^{3,\alpha}(\overline{\Omega})$ *in that*

$$\limsup_{s\to\infty} \left(|\lambda(s)| + |u(s)|_{C^{3,\alpha}(\Omega)}\right) \to \infty. \tag{3.9}$$

(ii) (Loss of ellipticity) *Following* $\mathcal{C}^+$ *to its extreme, the system loses ellipticity in that*

$$\liminf_{s\to\infty} \inf_{\overline{\Omega}} \mathcal{E}(q)\big|_{q=|\nabla u(s)|^2} = 0 \tag{3.10}$$

The price of generality in our assumptions on $\mathcal{W}$ and f is paid for in the ambiguity of Theorem 3.1.(c). The following theorem demonstrates how this result can be sharpened by assuming uniform ellipticity and suitable growth conditions on the traction force f.

Theorem 3.2. *Let* $\mathcal{W}$ *and* f *satisfy* (3.4) *and* (3.6)*, respectively. Suppose also that* f *satisfies* (3.7)*. Then, the curve* $\mathcal{C}^+$ *constructed in Theorem* 3.1 *exhibits the following behavior:*

$$\limsup_{s\to\infty} |\nabla u(s)|_{C^0(\Omega)} \to \infty. \tag{3.11}$$

Our proof of gradient blow-up in Theorem 3.2 relies crucially on (3.4). We prove an analogous result under the weaker condition (3.5).

Theorem 3.3. *Suppose that* $\mathcal{W}$ *and* f *satisfy* (3.5), (3.6), *and* (3.7)*. Then, the curve* $\mathcal{C}^+$ *constructed in Theorem* 3.1 *must also satisfy the following:*

(a) (Loss of ellipticity) *Alternative* (c).(ii) *of Theorem* 3.1 *occurs, and*

(b) (Bounds on λ) *There exists* $\lambda^+ = \lambda^+(\mathcal{W}, b, h)$ *for which*

$$0 \le \lambda(s) \le \lambda^+,$$

(c) (Blow-up) *If* (3.8) *is satisfied as well, then*

$$\limsup_{s\to\infty} |D^2 u(s)|_0 \to \infty \tag{3.12}$$

The proof of Theorem 3.3.(c) is accomplished using a novel gradient maximum principle for degenerate elliptic PDEs that is of independent interest.

3.1.4 Notation

Our analysis will take place primarily within Hölder spaces. For $k \in \mathbb{N}$, let

$$V_k \coloneqq \{v \in C^k(\overline{\Omega}) \,|\, \partial_y^n v(\cdot, \pm h/2) = 0,\ n \in \mathbb{N},\ \beta\ \text{ odd},\ \text{and}\ n \le k\}.$$

Note that the homogeneous Neumann boundary conditions on T and B are incorporated into the definition of V_k. The vanishing higher order derivatives are needed in several parts of the paper to ensure the existence of sufficiently regular periodic extensions. The spaces relevant to our functional analytic framework are

$$X \coloneqq C^{3,\alpha}_{\mathrm{b,o,0}}(\overline{\Omega}) \cap V_3$$

$$Y \coloneqq Y_1 \times Y_2 = C^{0,\alpha}_{\mathrm{b,o,0}}(\overline{\Omega}) \cap V_1 \times C^{1,\alpha}_{\mathrm{o}}(L) \cap \{v \in C^1(L) \mid v_y(0, \pm h/2) = 0\}.$$

Here, the subscript o denotes oddness about the line $y = 0$, and the subscript 0 is used to indicate uniform decay to 0 as $x \to \infty$ in $C^3(\Omega)$ (for X) and in $C^1(\Omega)$ (for Y).

Let us now define the nonlinear operator $\mathcal{F}$ by

$$\mathcal{F} \coloneqq (\mathcal{F}_1, \mathcal{F}_2) : \mathbb{R} \times X \to Y, \tag{3.13}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$\mathcal{F}_1(\lambda, u) = \nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla u) \qquad \text{and} \qquad \mathcal{F}_2(\lambda, u) = \mathcal{W}'(|\nabla u|^2)\partial_x u + f(\lambda, u). \tag{3.14}$$

Equation (3.1) may now be recast as the abstract operator equation

$$\mathcal{F}(\lambda, u) = 0. \tag{3.15}$$

Given $\mathcal{W}$ satisfying (3.4) or (3.5), the following functional acting on $u \in X$ gives a quantitative measure of ellipticity for (3.1) at u :

$$\mathcal{E}^*(u) \coloneqq \inf_{\Omega} \mathcal{E}(|\nabla u|^2). \tag{3.16}$$

For $\delta > 0$, let

$$\mathcal{O}_\delta \coloneqq (-1/\delta, 1/\delta) \times \{u \in X \ : \ \mathcal{E}^*(u) > \delta\}, \qquad \text{and} \qquad \mathcal{O} = \bigcup_{\delta > 0} \mathcal{O}_\delta.$$

Lastly, we introduce some geometric notation. For $l > 0$, we define the following truncated subdomains of Ω:

$$\Omega_l \coloneqq \Omega \cap \{(x, y) \ : \ 0 < x < l\}$$
$$\Omega_l^{l+1} \coloneqq \Omega_{l+1} \setminus \overline{\Omega_l} \qquad \text{for} \qquad l > 0.$$

We also adopt the nonstandard notation

$$\mathbb{R}^2_+ \coloneqq \{(x, y) \in \mathbb{R}^2 \ : \ x > 0\}.$$

3.2 Local bifurcation

Observe that the anti-plane shear system (3.1) admits the trivial solutions $(\lambda, 0)$ for any λ. The main result of this section is that there exists an exceptional parameter value λ^* at which a curve of solutions bifurcates from the the trivial family.

We begin by studying the linearized problem at $(\lambda^*, 0)$:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v_y = 0 & \text{on } T, B. \\ v_x = -f_z(\lambda^*, 0)v & \text{on } L. \end{cases} \tag{3.17}$$

It was assumed in (3.6) that $f_z(\lambda^*, 0) = \frac{\pi}{h}$. It is easy to check that

$$\Phi_0(x, y) := e^{-\lambda_0 x} \sin(\lambda_0 y) \tag{3.18}$$

belongs to X and solves (3.17). This suggests that equilibria near $(\lambda^*, 0)$ should look like scaled pertubations of Φ_0, which is indeed the case. Let

$$\Phi_k := e^{-\lambda_k x} \sin(\lambda_k y) \qquad \text{and} \qquad \lambda_k := \frac{(2k+1)\pi}{h} \tag{3.19}$$

and note that Φ_k solves an analogous problem when $f_z(\lambda^*, 0) = \lambda_k$.

Theorem 3.4 (Local bifurcation)**.** *Suppose that f satisfies* (3.6)*. Then there exists an $\epsilon_0 > 0$ and a local C^0 curve parametrized as*

$$\mathcal{C}_{\text{loc}} := \{(\lambda(s), u(s)) \ : \ |s| < \epsilon_0\} \subset \mathbb{R} \times X,$$

where $\lambda(0) = \lambda^*$ and each $(\lambda(s), u(s))$ solves (3.1)*. Each $u(s) = u^s(x, y)$ obeys the asymptotic*

$$u^s(x, y) = \Psi(s, s\Phi_0) = s\Phi_0(x, y) + o(s) \qquad \text{in} \qquad C_{\text{b}}^{3,\alpha}(\overline{\Omega}), \tag{3.20}$$

where $\Psi : \mathbb{R} \times X \to Y$ is analytic. Moreover, there exists an open set V, with $(\lambda^*, 0) \in V \subset \mathbb{R} \times X$, such that any solution $(\lambda, v) \in V$ necessarily lies in $\mathcal{C}_{\text{loc}}$.

Remark 3.5. Theorem 3.4 is proved using a version of the Crandall–Rabinowitz theorem suited for analytic operators. Analogous results remain valid under less stringent regularity assumptions.

Our goal is to show that the linearization of (3.13) at $(\lambda^*, 0)$, which is given by

$$\mathcal{F}_u(\lambda^*, 0) = (\Delta, \partial_x + \lambda_0), \tag{3.21}$$

satisfies the hypothesis of the Crandall–Rabinowitz theorem Theorem B.1. The most demanding part of this strategy is verifying $\mathcal{F}_u(\lambda^*, 0) : X \to Y$ is a Fredholm operator of index 0. As an intermediate step, let us consider the family of operators defined by

$$L_t := (\Delta - \gamma t, \partial_x + \lambda_0 - \gamma t), \tag{3.22}$$

where $t \in [0, 1]$ and $\gamma > 0$ is a constant yet to be specified. Each L_t will be shown to be semi-Fredholm with index $< \infty$ (this amounts to showing L_t has a closed range and finite dimensional kernel). Homotopy invariance may then be used to deduce information about $L_0 = \mathcal{F}_u(\lambda^*, 0)$ from the invertibility of L_1. The unboundedness of Ω and lack of smoothness in $\partial\Omega$ at the corner points causes some technical difficulties in the following analysis.

Lemma 3.6. *The operator $L_t : X \to Y$ is semi-Fredholm with index $< \infty$ for all $t \in [0, 1]$.*

Proof. It is well-known that L_t is semi-Fredholm with index $< \infty$ if the operator is locally proper. In our setting, L_t will be locally proper if any bounded sequence of the form $\{u_j\} \subset X$, such that $L_t u_j =: (v_j, w_j) \to (v, w) \in Y$, admits a subsequence converging in X. Schauder estimates may be used directly when dealing with smooth bounded domains, but modifications are needed for our problem.

The Arzelà–Ascoli theorem ensures the existence of a subsequence $\{u_k\}$ converging in $C^3_{\text{loc}}(\Omega)$ to some u for which $L_t u = (v, w)$. Moreover, u is odd about the line $y = 0$, and $u_y = 0$ on T and B. We must show the convergence takes place in $C^{3,\alpha}(\overline{\Omega})$. For each k, we may differentiate in the interior equation in x to see $\partial_x u_k$ solves the Dirichlet problem

$$(\Delta - \gamma t)\partial_x u_k = \partial_x v_k \text{ in } \Omega \qquad \text{and} \qquad \partial_x u_k = w_k - (\lambda_0 - \gamma t)u_k \text{ on } L.$$

Note that u_k can be extended to a $2h$-periodic function belonging to $C^{3,\alpha}(\mathbb{R}^2_+)$ via even reflection about the lines T and B (recall that elements of the space X have vanishing first and third y derivatives on T and B). Likewise, v_k and w_k admit extensions in $C^{1,\alpha}(\mathbb{R}^2_+)$ and $C^{2,\alpha}(\mathbb{R})$, respectively. This allows us to ignore any complications caused by the irregular boundary points $(0, \pm h/2)$. Hence, Schauder estimates [11, Theorem 6.6] can be invoked to obtain

$$|\partial_x u_k|_{2,\alpha} \leq C(|\partial_x u_k|_0 + |w_k - (\lambda_0 - \gamma t)u_k|_{2,\alpha;L} + |\partial_x v_k|_{0,\alpha}), \tag{3.23}$$

where the constant C is independent of k. Differentiating the interior equation and left boundary condition in y yields a linear elliptic PDE with a uniformly oblique boundary condition satisfied by $\partial_y u_k$. Mimicking the argument above, using instead now [11, Theorem 6.29], produces an inequality analogous to (3.23) that is valid for $\partial_y u_k$. Finally, applying [11, Theorem 6.6, Theorem 6.29] to the undifferentiated equation for u_k, and combining the result with those above, reveals that

$$|u_k|_{3,\alpha} \le C(|u_k|_0 + |v_k|_{1,\alpha} + |w_k|_{2,\alpha;L}). \tag{3.24}$$

Hence, $u_k \to u$ in $C^{3,\alpha}_{\mathrm{b}}(\overline{\Omega})$ provided $u_k \to u$ in $C^0_{\mathrm{b}}(\overline{\Omega})$.

We are left only to show that u_k converges pointwise uniformly to u. Suppose that this is not the case. Define the sequence ρ_j by $\rho_j := u_j - u$. Then, there must exist $\{(x_j, y_j)\} \subset \overline{\Omega}$ with $x_j \to \infty$ and $y_j \to y_0 \in [-h/2, h/2]$ for which some subsequence $\{\rho_j\}$ satisfies

$$|\rho_j(x_j, y_j)| > c > 0. \tag{3.25}$$

Let us define $\theta_n(x, y) := \rho_n(x + x_n, y)$, which is well-defined on the shifted domain $(-x_n, \infty) \times (-h/2, h/2)$. After extracting a subsequence, we find that there exists a twice continuously differentiable function θ for which $\theta_n \to \theta$ in $C^2_{\mathrm{loc}}((-\infty, \infty) \times [-h/2, h/2])$, and

$$\begin{cases} \Delta\theta - \gamma t\theta = 0 & \text{in } (-\infty,\infty)\times(-h/2,h/2) \\ \theta_y(x, \pm h/2) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{3.26}$$

Note that θ is bounded as $|x| \to \infty$, since $|u_j|_{2,\alpha}$ is uniformly bounded. The maximum principle and Hopf lemma imply that $|\theta|$ cannot achieve its maximum value at any point belonging to either the interior or the lateral boundaries of $\mathbb{R} \times \{\pm h/2\}$. Hence, θ is maximized as $|x| \to \infty$. For any $x_1 < x_2$, we may integrate by parts using (3.26) to obtain

$$0 < \gamma t \int_{-h/2}^{h/2} \int_{x_1}^{x_2} \theta^2 \, dx \, dy + \int_{-h/2}^{h/2} \int_{x_1}^{x_2} |\nabla \theta|^2 \, dx \, dy = \int_{-h/2}^{h/2} \theta_x \theta \Big|_{x=x_1}^{x=x_2} dx \, dy < \infty. \tag{3.27}$$

Moreover, θ is odd about $y = 0$, so Wirtinger's inequality and the boundedness of (3.27) imply

$$\lim_{x \to \infty} \int_{-h/2}^{h/2} \int_{x}^{x+1} \theta^2 + |\nabla \theta|^2 \, dx \, dy \leq \left(1 + \frac{h^2}{\pi^2}\right) \lim_{x \to \infty} \int_{-h/2}^{h/2} \int_{x}^{x+1} |\nabla \theta|^2 \, dx \, dy = 0. \tag{3.28}$$

Finally, (3.28) and Morrey's inequality imply the existence of a constant C for which

$$\limsup_{x \to \infty} |\theta(x, \cdot)|_0^2 \leq C \left(1 + \frac{h^2}{\pi^2}\right) |\nabla \theta|_0^2 \int_{-h/2}^{h/2} \int_{x}^{x+1} |\nabla \theta|^2 \, dx \, dy = 0.$$

One can argue by similar means to find that θ vanishes as $x \to -\infty$. Hence, $\theta = 0$. However, this is a contradiction, because equation (3.25) implies that $|\theta(0, y_0)| > 0$. ∎

The next lemma shows invertibility for $t = 1$ and sufficiently large γ.

Lemma 3.7 (L_1 is invertible)**.** *Let $\gamma > \lambda_0$ in* (3.22)*. Then, the operator $L_1 : X \to Y$ is invertible.*

We defer the proof of Lemma 3.7 to Section 3.6. The Fredholm index of $\mathcal{F}(\lambda^*, 0)$ is now deduced from our analysis of the operators L_t.

Lemma 3.8. $\mathcal{F}_u(\lambda^*, 0) : X \to Y$ *is Fredholm with index* 0.

Proof. Recall that $\mathcal{F}_u(\lambda^*, 0) = L_0$. We have shown that $L_1 : X \to Y$ is invertible, hence it has Fredholm index 0. Moreover, in Lemma 3.6 it was shown that L_t if semi-Fredholm with index $< \infty$, for $t \in [0, 1]$. Thus, L_0 must have Fredholm index 0 by the continuity of the index. ∎

It is now shown that Φ_0, which was defined in (3.18), generates the kernel of $\mathcal{F}_u(\lambda^*, 0)$ and that the so-called transversality condition holds.

Lemma 3.9. *The kernel of* $\mathcal{F}_u(\lambda^*, 0) : X \to Y$ *is* 1*-dimensional and generated by* $\{\Phi_0\}$*. Moreover, the condition*

$$\mathcal{F}_{\lambda u}(\lambda^*, 0)\Phi_0 \notin \operatorname{ran}(\mathcal{F}_u(\lambda_0, 0)).$$

Proof. For any $w \in X$ we have the half-range Fourier expansion

$$w(x, y) = \sum_{k=0}^{\infty} \varphi_k(x) \sin(\lambda_k y), \qquad \text{where} \qquad \varphi_k(x) = \frac{2}{h} \int_L w(x, y) \sin(\lambda_k y)\, dy.$$

If $w \in \ker(\mathcal{F}_u(\lambda^*, 0))$, then each φ_k must satisfy the ODE

$$\begin{cases} \varphi_k''(x) - \lambda_k^2 \varphi_k(x) = 0 & \text{in } [0, \infty) \\ \varphi_k'(x) + \lambda_k \varphi_k(x) = 0 & \text{for } x = 0, \end{cases}$$

supplemented with the condition that φ_k decays to 0 as $x \to \infty$. Thus, we obtain a solution only when $k = 0$, and it must be $e^{-\lambda_0 x}$.

Finally, we must verify that

$$\mathcal{F}_{\lambda u}(\lambda^*, 0)\Phi_0 = (f_{\lambda z}(\lambda^*, 0)\Phi_0, 0) \notin \operatorname{ran}(\mathcal{F}_u(\lambda^*, 0)).$$

Let $u \in X$. Then, multiplying Δu by Φ_0 and integrating by parts twice shows

$$\int_\Omega (\Delta u)\Phi_0 \, dx \, dy = -\int_L \Phi_0(\partial_x u + \lambda u) \, dy. \tag{3.29}$$

Hence, if there exists some $u \in X$ for which $\mathcal{F}(\lambda^*, 0)u = \mathcal{F}_{\lambda u}(\lambda^*, 0)\Phi_0$, then

$$0 = \int_L f_{\lambda z}(\lambda^*, 0)\Phi_0^2 \, dy. \tag{3.30}$$

Thus, the transversality condition holds whenever $f_{\lambda z}(\lambda^*, 0) \neq 0$, and this condition is assumed by (3.6). ∎

We now have all of the ingredients required for the proof of Theorem 3.4.

Proof of Theorem 3.4. This theorem follows from an application of the Crandall–Rabinowitz theorem (for analytic operators) as found in [3] (see Theorem B.1 for a precise statement). The key hypothesis are verified in Lemma 3.8 and Lemma 3.9. ∎

Finally, we establish the invertibility of $\mathcal{F}_u$ along $\mathcal{C}_{\text{loc}}$. The argument will also reveal qualitative information about the bifurcation diagram.

Lemma 3.10. *The operator* $\mathcal{F}_u(\lambda, u) : X \to Y$ *is invertible for* $(\lambda, u) \in \mathcal{C}_{\text{loc}} \setminus \{(\lambda^*, 0)\}$.

Proof. Let $(\lambda, u) \in \mathcal{C}_{\text{loc}} \setminus \{(\lambda^*, 0)\}$. Consider the family of operators

$$F_t \coloneqq \mathcal{F}_u(\lambda^*, 0) - t(\mathcal{F}_u(\lambda^*, 0) - \mathcal{F}_u(\lambda, u)).$$

Note that F_t is locally proper for $t \in [0,1]$ by an argument similar to the one given in Lemma 3.6. Thus, the Fredholm index is constant in t along $\{F_t\}_{t\in[0,1]}$. If we can now show that $\mathcal{F}_u(\lambda, u)$ is injective, then $\mathcal{F}_u(\lambda, u)$ will necessarily be invertible. Moreover, the kernel of $\mathcal{F}_u(\lambda, u)$ will be trivial provided $\dot{\lambda}(s) \neq 0$ by Proposition 8.3.4 found in [3].

The structures of $\mathcal{W}$ and f show that (3.1) is invariant with respect to the reflection $(\lambda, u) \mapsto (\lambda, -u)$. This implies $\dot{\lambda}(0) = 0$. However, if we can show that $\ddot{\lambda}(0) \neq 0$, then $\dot{\lambda}(s) \neq 0$ must hold for $s > 0$ sufficiently small. Recall that $u^s = s\Phi_0 + o(s)$ in $C^{3,\alpha}_{\text{b}}(\overline{\Omega})$. If we take into consideration the odd reflection symmetry of $\mathcal{C}_{\text{loc}}$, then we are led to the more precise expansion

$$u^s = s\Phi_0 + O(s^3) \qquad \text{in} \qquad C^{3,\alpha}_{\text{b}}(\overline{\Omega}). \tag{3.31}$$

Multiplying (3.1) by Φ_0, integrating by parts, and making use of the boundary condition produces

$$\int_\Omega \mathcal{W}'(|\nabla u(s)|^2)\nabla u(s) \cdot \nabla \Phi_0 \, dx\, dy = \int_L f(\lambda(s), u(s))\Phi_0 \, dy. \tag{3.32}$$

On the other hand, Φ_0 is harmonic and a similar calculation reveals

$$\int_\Omega \nabla u(s) \cdot \nabla \Phi_0 \, dx\, dy = \lambda_0 \int_L u(s)\Phi_0 \, dy. \tag{3.33}$$

After subtracting (3.33) from (3.32), and utilizing the expansions of f, $\mathcal{W}$, $\lambda(s)$, and u^s, we find

$$2c_1 s^3 \int_\Omega |\nabla\Phi_0|^4\,dx\,dy = s^3 \frac{\ddot{\lambda}(0)}{2} f_{\lambda z}(\lambda^*, 0) \int_L \Phi_0^2\,dy + \frac{f_{zzz}(\lambda^*, 0)}{6} \int_L \Phi_0^4\,dy + O(s^4). \tag{3.34}$$

This calculation takes into account (3.2), (3.6), that $\Phi_0 \in H^1(\Omega)$, and the symmetry relations

$$\partial_\lambda^n f(\lambda, 0) = \partial_z^{2n} f(\lambda, 0) = 0, \qquad \text{for} \qquad n \in \mathbb{N}.$$

The explicit form of Φ_0 may now be used to calculate the integrals in (3.34), and so by cancelling a factor of s^2 and letting $s \to 0$ one sees

$$\ddot{\lambda}(0) = \frac{8c_1\lambda_0^3 - f_{zzz}(\lambda^*, 0)}{8 f_{\lambda z}(\lambda^*, 0)},$$

which is nonzero by (3.6). ∎

3.3 A priori estimates

The curve $\mathcal{C}_{\mathrm{loc}}$ constructed in the previous section will be extended using tools from analytic global bifurcation theory. In this section, we obtain suitable a priori estimates that are crucial to our analysis of the global curve.

3.3.1 Conserved quantity and L^2 estimates

The strain energy density functions considered in this paper are independent of the spatial variables. As a result, our system is invariant under positive translations in the x-variable. Naturally, one expects to find an associated conserved quantity. As proved in [6],

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{W}(|\nabla u|^2) - 2\mathcal{W}'(|\nabla u|^2)u_x^2\right)\big|_{x=x_0}\, dy = 0 \qquad \text{for } x_0 \geq 0, \tag{3.35}$$

which may be checked by differentiating (3.35) in x, integrating by parts, and making use of the condition $|\nabla u|^2 \to 0$ as $x \to \infty$. An immediate consequence of (3.35) is that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(2\mathcal{W}'(|\nabla u|^2)|\nabla u|^2 - \mathcal{W}(|\nabla u|^2)\right)|_{x=x_0}\, dy = \int_{-\frac{h}{2}}^{\frac{h}{2}} 2\mathcal{W}'(|\nabla u|^2)u_{y^2}|_{x=x_0}\, dy, \tag{3.36}$$

for $x_0 > 0$. We note that the integrand on the left hand side of (3.36) is related to the Legendre transform of $\mathcal{W}$ in a quadratic variable. Recalling (3.3), one sees that $\mathcal{W}(k^2)$ transforms as

$$\mathcal{W}^*(p) = kp - \mathcal{W}(k^2), \qquad \text{where} \qquad \tau(k) = p. \tag{3.37}$$

Equation (3.37) is well-defined for $k \in [0, \infty)$ under condition (3.4) and for $k \in [0, \sqrt{q_{\text{cr}}})$ when instead (3.5) holds. Replacing p in (3.37) allows us to write

$$\mathcal{W}^*(\tau(k)) = 2\mathcal{W}'(k^2)k^2 - \mathcal{W}(k^2)$$

for appropriate values of k. It follows that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{W}^*(\tau(|\nabla u|)) - 2\mathcal{W}'(|\nabla u|^2)u_y^2\right)|_{x=x_0}\, dy = 0, \qquad x_0 \geq 0. \tag{3.38}$$

Hence, (3.38) provides a conserved quantity relating $\mathcal{W}'(|\nabla u|^2)u_y^2$ and $\mathcal{W}^*$, whereas (3.35) gives one relating instead $\mathcal{W}'(|\nabla u|^2)u_x^2$ and $\mathcal{W}$. Combining

(3.35), the nonlinear boundary condition of (3.1), and Hölder's Inequality yields

$$\int_{-h/2}^{h/2} \mathcal{W}(|\nabla u|^2)\big|_{x=0}\, dy = -2\int_{-h/2}^{h/2} f(\lambda,u)u_x\big|_{x=0}\, dy \leq 2\|f(\lambda,u(0,\cdot))\|_2\|u_x(0,\cdot)\|_2. \tag{3.39}$$

The relation in (3.35) be used to establish upper bounds on both $\|u(0,\cdot)\|_2$ and $|u(0,\cdot)|_{0,1/2}$. We must prove a simple structural lemma concerning $\mathcal{W}$ before showing this.

Lemma 3.11. *Suppose that $\mathcal{W}$ satisfies* (3.4) *or* (3.5). *Then, there is a constant $m > 0$ for which*

$$\mathcal{W}'(q) > m \qquad \textit{whenever} \qquad q \in [0, q^*),$$

where q^* is taken to be ∞ when (3.4) *holds and q_{cr} for* (3.5).

Proof. Recall that (3.4) states

$$\mathcal{W}'(q) + 2q\mathcal{W}''(q) \geq \xi. \tag{3.40}$$

This inequality can be integrated once in q to obtain

$$2\mathcal{W}'(q) \geq \xi + q^{-1}\mathcal{W}(q), \tag{3.41}$$

and then a second time to show

$$\mathcal{W}(q) \geq \frac{\xi q}{2} + \frac{1}{2}\int_0^q \frac{\mathcal{W}(q)}{q}\, dq. \tag{3.42}$$

The first claim follows from (3.41) by noticing (3.42) implies $\mathcal{W} > 0$ (recall that $\mathcal{W}'(0) = 1$ by (3.2)). In particular, one can take $m = \xi/2$. The second claim follows by taking $\xi = 0$ in the work above because each of the analogous inequalities are then valid for $q \in [0, q_{\mathrm{cr}}]$. Moreover, we can use (3.41) and (3.42) to give m explicitly by

$$m = \inf_{q \in (0, q_{\mathrm{cr}})} \frac{\mathcal{W}(q)}{2q} > 0. \blacksquare \tag{3.43}$$

Note that m obtained in Lemma 3.11 depends only on the structure of $\mathcal{W}$.

Lemma 3.12. *Suppose that* $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$. *Let* $\mathcal{W}$ *satisfy either* (3.4) *or* (3.5). *If* $|f(\lambda, z)| \leq K$, *then*

$$|u|_{C^{0,1/2}(L)} \leq \frac{K}{m}(2\sqrt{h} + h),$$

where $m > 0$ *is the constant from Lemma* 3.11.

Proof. It follows from Lemma 3.11 that $m|\nabla u|^2 \leq \mathcal{W}'(|\nabla u|^2)$. The inequalities of (3.39) now imply

$$m\|\nabla u(0, \cdot)\|_2^2 \leq \int_{-h/2}^{h/2} \mathcal{W}(|\nabla u|^2)\big|_{x=0} \, dy \leq 2K\sqrt{h}\|u_x(0, \cdot)\|_2.$$

Hence, $\|\nabla u(0, \cdot)\|_2 \leq 2K\sqrt{h}m^{-1}$, and an application of Morrey's inequality in one dimension yields

$$|u(0, \cdot)|_{C^{0,1/2}(L)} \leq (1 + \sqrt{h}/2)\|u_y(0, \cdot)\|_2.$$

We conclude that

$$|u|_{C^{0,1/2}(L)} \leq \frac{K}{m}(2\sqrt{h} + h). \blacksquare$$

3.3.2 Uniform regularity

This sub-section provides several quantitative estimates related to $\mathcal{F}$ and $\mathcal{F}_u$. Most of the estimates will depend upon the ellipticity assumptions imposed on $\mathcal{W}$. A quick calculation will show the components of $\mathcal{F}_u(\lambda, u)$ are given by

$$\mathcal{F}_{1,u}(\lambda,u)\Theta = \nabla\cdot((\mathcal{W}'(|\nabla u|^2)I + (\nabla u\otimes\nabla u)\mathcal{W}''(|\nabla u|^2))\nabla\Theta)$$

$$\mathcal{F}_{2,u}(\lambda,u)\Theta = (\mathcal{W}'(|\nabla u|^2) + 2u_x^2\mathcal{W}''(|\nabla u|^2))\partial_x\Theta + 2u_xu_y\mathcal{W}''(|\nabla u|^2)\partial_y\Theta - f_z(\lambda,u)\Theta \tag{3.44}$$

Suppose that $(\lambda, u) \in \mathcal{O}_\delta$, for some $\delta > 0$. Then (3.44) constitutes a uniformly elliptic differential operator in Ω with a uniformly oblique boundary operator on L.

Lemma 3.13. *Suppose that $|\lambda| + |\nabla u|^2 \leq K$ and $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, for some $\delta > 0$. Then there exists a constant $C = C(K, \delta, h, f, \mathcal{W})$ for which $|u|_{3,\alpha} \leq C$.*

Proof. We let C denote a constant $C = C(K, \delta, h, f, \mathcal{W})$ throughout the course of this argument. Note that $|\nabla u|^2 \leq K$ implies that $|u|_0 \leq \frac{h\sqrt{K}}{2}$, since u is odd and vanishes along $y = 0$. We claim that [25, Theorem 3] may be applied in our setting to show

$$|u|_{C^{2,\alpha'}(\Omega)} \leq C \tag{3.45}$$

for some $\alpha' \in (0, \alpha]$. The arguments of the quoted theorem are carried out locally near a $C^{2,\alpha}$ boundary. If we first extend u periodically about T and B, as done in Lemma 3.6, then the non-smoothness of $\partial\Omega$ at $(0, \pm h/2)$ can be ignored. The remaining hypothesis of [25, Theorem 3] are satisfied because $(\lambda, u) \in \mathcal{O}_\delta$, and it follows that there is some $\alpha' \in (0, \alpha)$ and $C > 0$ for which

$$|D^2 u|_{C^{0,\alpha'}(\Omega \setminus \Omega_2)} \leq C. \tag{3.46}$$

The bounds on $|u|$ and (3.46) can be interpolated to yield (3.45).

Next, we differentiate (3.1) in the y-variable, including the left boundary condition, and find that u_y solves the equations given by $\mathcal{F}_u(\lambda, u)$. Recall that $\mathcal{F}_u(\lambda, u)$ is given explicitly by (3.44), so u_y solves a uniformly elliptic PDE in Ω with a uniformly oblique boundary condition on L. Moreover, we could have first extended u periodically, and then differentiated in y, so without loss of generality we may assume that $u_y \in C^{2,\alpha}(\mathbb{R}^2_+)$ and solves the PDE obtained by extending the coefficients of $\mathcal{F}_u(\lambda, u)$ periodically in the y-direction. The inequality (3.45) bounds the coefficients of $\mathcal{F}_{1,u}(\lambda, u)$ in $C^{0,\alpha'}(\overline{\Omega})$ and those of $\mathcal{F}_{2,u}$ in $C^{1,\alpha'}(L)$. Linear Schauder theory, for example Theorems 6.4 and 6.29 of [11], may be applied to show

$$|u_y|_{2,\alpha'} \leq C.$$

If instead the interior equation of (3.1) is differentiated in the x-variable, then upon solving for u_{xxx} (which is possible since the coefficient of u_{xxx} is bounded below by δ) it becomes apparent that $|u_{xxx}|_{0,\alpha'} \leq C$. Hence, $|u|_{3,\alpha'} \leq C$. In particular, $|u|_{2,\alpha} \leq C$, and the argument above can be repeated to provide the desired inequality $|u|_{3,\alpha} \leq C$. ∎

The estimate provided by Lemma 3.13 requires ellipticity, obliqueness, gradient control, and bounds on the parameter λ. The first two of these conditions are dependent on the structure of $\mathcal{W}$ (and also on the size of the gradient in the degenerate case). Estimates on λ are obtained when f satisfies some mild growth conditions.

Lemma 3.14 (Bound on λ). *Let $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, for some $\delta > 0$, and suppose $\partial_x u < 0$ on $\Omega \cap \{(x,y) \in \mathbb{R}^2 \ : \ y > 0\}$ and $\partial_y u > 0$ on Ω. If f satisfies* (3.7) *and $\rho(\lambda) > 0$, then*

$$\rho(\lambda) \leq b|u|_0^2 + \lambda_0 |\mathcal{W}'(|\nabla u|^2)|_0.$$

If f satisfies (3.8) *and $\lambda > 0$, then we obtain the refined estimate*

$$\lambda \leq \lambda_0 - 2B|u|_0^2.$$

Proof. Recall that the harmonic function Φ_0, which is defined in (3.18), solves (3.17). Multiplying (3.1) by Φ_0 and integrating by parts yields

$$0 < \int_{-\frac{h}{2}}^{\frac{h}{2}} f(\lambda, u)\Phi_0 \, dy = \int_\Omega \mathcal{W}'(|\nabla u|^2)\nabla u \cdot \nabla \Phi_0 \, dx \, dy,$$

whereas multiplying (3.17) by u and integrating gives

$$0 < \lambda_0 \int_{-\frac{h}{2}}^{\frac{h}{2}} u\Phi_0 \, dy = \int_\Omega \nabla u \cdot \nabla \Phi_0 \, dx \, dy.$$

Combining these equations with (3.7) and the sign conditions imposed on u_x and u_y shows

$$0 < \rho(\lambda) \int_L u\Phi_0 \; dy - b \int_{-h/2}^{h/2} u^3 \Phi_0 \, dy \leq \lambda_0 |\mathcal{W}'(|\nabla u|^2)|_0 \int_{-h/2}^{h/2} u\Phi_0 \; dy. \quad (3.47)$$

The first claimed inequality follows from (3.47) after observing

$$\rho(\lambda) \int_L u\Phi_0 \, dy \le b|u|_0^2 \int_{-h/2}^{h/2} u\Phi_0 \, dy + \lambda_0 |\mathcal{W}'(|\nabla u|^2)|_0 \int_{-h/2}^{h/2} u\Phi_0 \, dy$$

and then dividing by the remaining (non-zero) integral. If f satisfies (3.8), then (3.47) implies

$$\lambda \int_L u\Phi_0 \, dy + B \int_{-h/2}^{h/2} u^3 \Phi_0 \, dy \le \lambda_0 \int_{-h/2}^{h/2} u\Phi_0 \, dy \tag{3.48}$$

An application of Hölder's inequality shows that

$$\left(\int_{-h/2}^{h/2} u\Phi_0 \, dy \right)^3 \le \frac{4h^2}{\pi^2} \int_{-h/2}^{h/2} u^3 \Phi_0 \, dy \tag{3.49}$$

which can be combined with (3.48) to show

$$\lambda + \frac{B\pi^2}{4h^2} \left(\int_{-h/2}^{h/2} u\Phi_0 \, dy \right)^2 \le \lambda_0. \tag{3.50}$$

Notice that $|u|_0 = u(0, h/2)$ (the maximum principle implies u is maximized on L, and $u_y \ge 0$). A simple integration by parts shows that

$$\int_{-h/2}^{h/2} u\Phi_0 \, dy > \int_{-h/2}^{h/2} u \sin\left(\frac{\lambda_0}{2} y\right) \, dy = \frac{2}{\lambda_0} \int_{-h/2}^{h/2} u_y \cos\left(\frac{\lambda_0}{2} y\right) \, dy > \frac{2\sqrt{2}}{\lambda_0} |u|_0$$

Applying this estimate to (3.50) gives the second claimed inequality. ∎

This result also supplies an estimate on $|f(\lambda(s), u(s))|$ when (3.8) holds (assuming that $\lambda(s) \ge 0$). It is immediate that $|u|_0 \le \sqrt{\frac{\lambda(s)-\lambda_0}{2B}}$. Notice that $f_z(\lambda, z) > 0$, so after plugging in the upper bound for $|u|_0$ into explicit form of f we find

$$|f(\lambda(s), u(s))| \le \sqrt{\frac{\lambda_0 - \lambda(s)}{2B}} \left(\frac{\lambda_0 + \lambda(s)}{2} \right).$$

Optimizing this inequality for $\lambda(s) \in [0, \lambda_0]$ yields

$$|f(\lambda(s), u(s))| \leq \frac{1}{\sqrt{2B}} \left(\frac{2\pi}{3h}\right)^{3/2} < m\sqrt{q_{\mathrm{cr}}}, \tag{3.51}$$

where we have used the definition of λ_0 and (3.8).

3.3.3 Compactness

The unboundedness of Ω suggests that bounded subsets of $\mathcal{F}^{-1}(0)$ may fail to be locally pre-compact. Indeed, it was shown in [16] that broadening occurs for a large class of materials and body forces subjected to anti-plane shear in an infinite strip. We derive energy-type estimates in the spirit of [18] and use them to establish exponential decay. This approach, when combined with Schauder estimates, establishes compactness results needed elsewhere. We provide a proof applicable to our setting for convenience. For $u \in C^2(\overline{\Omega})$ we define the following energy functional:

$$E(u; x_0) := \int_{x_0}^{\infty} \int_{-h/2}^{h/2} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \, dy \, dx = \int_{\Omega_{x_0}} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \, dx \, dy, \tag{3.52}$$

We will frequently omit the dependence on u in (3.52) and write $E(x)$.

Theorem 3.15. *Suppose that $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$ for some $\delta > 0$. Then, $|u|_0 \leq M$ implies the following exponential decay estimate:*

$$E(u; x) \leq E(u; 0)e^{-kx}, \qquad \text{for } x > 0, \tag{3.53}$$

where $k > 0$. Moreover, k depends only on $M, \mathcal{W}$, and the constant $m > 0$ from Lemma 3.11.

Proof. First, let us notice that

$$E(z) = \int_z^\infty \int_{-h/2}^{h/2} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \, dy \, dx = -\int_{-h/2}^{h/2} \mathcal{W}'(|\nabla u|^2) u_x u \Big|_{x=z} dy, \tag{3.54}$$

which can be seen from integration by parts and (3.1). Also,

$$E'(z) = -\int_{-h/2}^{h/2} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \Big|_{x=z} dy.$$

Our goal now is to find $k > 0$ for which $kE(z) + E'(z) \leq 0$. To this end, let $L(z) = \{(x, y) \in \Omega \, : \, x = z\}$ and estimate (3.54) by

$$E(z) \leq \frac{|u|_0}{l} \int\limits_{L(z)\cap\{|\nabla u|>l\}} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \, dy + \left(\int\limits_{L(z)} \mathcal{W}'(|\nabla u|^2) u_x^2 \, dy \int\limits_{L(z)\cap\{|\nabla u|<l\}} \mathcal{W}'(|\nabla u|^2) u^2 \, dy \right)^{1/2}$$

where $l > 0$ and Hölder's inequality was used to obtain the last term. Now, let $M(l) = \max\limits_{q\in[0,l^2]} \mathcal{W}'(q)$ and continue the estimate above by

$$E(z) \leq \left(\frac{|u|_0}{l} + \frac{h\sqrt{M(l)}}{\pi\sqrt{m}} \right) \int\limits_{L(z)} \mathcal{W}'(|\nabla u|^2)|\nabla u|^2 \, dy,$$

where the Wirtinger inequality was used to estimate $\|u(z, \cdot)\|_2$. Let $k(l)$ be defined by

$$k(l) = \frac{l\sqrt{m}\pi}{\sqrt{m}\pi|u|_0 + lh\sqrt{M(l)}}, \tag{3.55}$$

so that $k(l)E(z) + E'(z) \leq 0$ whenever $l > 0$. Integrating this inequality in z yields

$$E(z) \leq E(0)e^{-k(l)z}, \qquad \text{for} \qquad l > 0. \tag{3.56}$$

The choice $l = |u|_0$ gives $k(l)$ with the desired properties. ∎

Remark 3.16. Note the decay constant k is independent of δ.

Remark 3.17. This theorem can be generalized to hold for the higher dimensional setting. The decay constant would then depend on the Poincare constant for a cross section of the cylinder.

Remark 3.18. Note that the PDE in (3.1) is simply the Laplacian if $\mathcal{W}'(q) = 1$. In this case, we see that $k(l) \to \frac{\pi}{h}$ as $l \to \infty$; this is known to be optimal.

Next, we will upgrade the energy estimate into pointwise bounds.

Lemma 3.19 (Exponential decay of u)**.** *Suppose that* $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$ *for some* $\delta > 0$*. If* $|\nabla u|_0^2 \leq K$*, then there exists* $C(K, h, m)$ *for which*

$$|u(x, \cdot)|_{0,1/2} \leq C E(0)^{1/4} e^{-kx/4},$$

where the constants m *and* k *are given by Lemma* 3.11 *and Theorem* 3.15*, respectively.*

Proof. Consider the truncated energy $\tilde{E}(u; x) := E(u; x) - E(u; x+1)$ where $x \geq 1$. It follows that

$$m \int_{-h/2}^{h/2} \int_{x_0}^{x_0+1} |\nabla u|^4 \, dx \, dy \leq K \int_{-h/2}^{h/2} \int_{x_0}^{x_0+1} \mathcal{W}'(|\nabla u|^2) |\nabla u|^2 \, dx \, dy = \tilde{E}(u; x_0).$$

Moreover, $\tilde{E}(u; x_0) \leq E(u; x_0)$ and we see that

$$\int_{-h/2}^{h/2} \int_{x_0}^{x_0+1} |\nabla u|^4 \, dx \, dy \leq \frac{K}{m} E(u; 0) e^{-kx_0}.$$

Finally, the Sobolev embedding theorem yields the desired estimate. ∎

The following result establishes pointwise decay in the full norm of X.

Lemma 3.20. *Let* $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$*, for some* $\delta > 0$*, and suppose that* $|\lambda| + |\nabla u|_0^2 \leq K$*. Then, there exists* $C = C(h, \delta, K) > 0$ *for which*

$$|u|_{C^{3,\alpha}(\Omega_x^{x+1})} \leq Ce^{-kx/4},$$

where k *is taken according to Theorem* 3.15 *and* $x \geq 2$*.*

Proof. Throughout the proof C will be used for a constant $C = C(h, \delta, K) > 0$, and it may change from line to line. The result will follow by a combination of interior Schauder estimates and the exponential decay of u. Indeed, it follows from Lemma 3.13 that there exists some C for which $|u|_{3,\alpha} < C$. We may assume, as justified in Lemma 3.13, that u is $2h$-periodic, $u \in C_{\mathrm{b}}^{3,\alpha}(\mathbb{R}_+^2)$, and $\mathcal{F}(\lambda, u) = 0$ on $\mathbb{R}_+^2$. We have assumed that $x > 2$, so

$$\Omega_x^{x+1} \subset (x-1, x+2) \times [-h, h] =: R(x) \subset \mathbb{R}_+^2.$$

Theorem 6.2 of [11] now implies that

$$|u|_{C^{2,\alpha}(\Omega_x^{x+1})} \leq C|u|_{C^0(R(x))}. \tag{3.57}$$

Recall that u_y solves the uniformly elliptic PDE and left boundary condition given by $\mathcal{F}_u(\lambda, u)$. Moreover, $u_y = 0$ on $T \cup B$. So, we may combine [11, Theorem 6.4] and (3.57) to show

$$|u_y|_{C^{2,\alpha}(\Omega_x^{x+1})} \leq C|u_y|_{C^0(\Omega_x^{x+1})} \leq C|u|_{C^0(R(x))}. \tag{3.58}$$

On the other hand, $\mathcal{F}_{1,u}(\lambda, u)[u_x] = 0$. Solving for u_{xxx}, which is possible since $(\lambda, u) \in \mathcal{O}_\delta$, yields

$$|u|_{C^{3,\alpha}(\Omega_x^{x+1})} \leq C|u|_{C^0(R(x))}. \tag{3.59}$$

Finally, we may appeal to (3.59) and Lemma 3.19 to conclude

$$|u|_{C^{3,\alpha}(\Omega_x^{x+1})} \leq Ce^{-kx/4} \blacksquare \tag{3.60}$$

We are now prepared to prove a local pre-compactness result.

Lemma 3.21. *Let* $\{(\lambda_n, u_n)\} \subset \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, *for some* $\delta > 0$. *Suppose further that there exists* $K > 0$ *for which* $|\nabla u|^2 \leq K$. *Then, there exists a subsequence of* $\{(\lambda_n, u_n)\}$ *converging in* $\mathbb{R} \times C^{3,\alpha}(\overline{\Omega})$ *to* $(\lambda, u) \in \mathcal{F}^{-1}(0)$.

Proof. The hypothesis of Lemma 3.13 are met and it follows that $|u_n| \leq C$, for some constant $C > 0$ which is independent of n. The Arzelà–Ascoli theorem and a diagonalization argument imply the existence of a subsequence of $\{(\lambda_n, u_n)\}$ converging in $C^{2,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ to some $(\lambda, u) \in \mathcal{F}^{-1}(0)$. Moreover, the elements u_n can be taken to converge to u in $C^0_{\mathrm{b}}(\overline{\Omega})$ by Lemma 3.20. We now show that $v_n := u_n - u$ solves a uniformly elliptic elliptic PDE and invoke Schauder estimates to obtain the desired estimate. Let us temporarily adopt a slight abuse of notation and rewrite $\mathcal{F}(\lambda, u)$, as defined in (3.14), in the form

$$\mathcal{F}(z, p, r, \lambda) : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}.$$

In this format, we have $\mathcal{F}(u, Du, D^2u, \lambda) = 0$. A quick calculation will show that

$$
\begin{cases} a_n^{ij}\partial_{ij}v_n + b_n^i\partial_i v_n = 0 & \text{in } \Omega \\ \beta^i\partial_i v_n + \mu_n v_n = 0 & \text{on } L, \\ \partial_y v_n = 0 & \text{on } T \cup B, \end{cases} \tag{3.61}
$$

where

$$
\begin{aligned}
a_n^{ij} &:= \int_0^1 \mathcal{F}_{1,r^{ij}}(Du_n^{(t)}, D^2u_n^{(t)})\,dt, & b_n^i &:= \int_0^1 \mathcal{F}_{1,p^i}(Du_n^{(t)}, D^2u_n^{(t)})\,dt, \\
\beta_n^i &:= \int_0^1 \mathcal{F}_{2,p^i}(u_n^{(t)}, Du_n^{(t)}, \lambda^{(t)})\,dt, & \mu_n &:= \int_0^1 \mathcal{F}_{2,z}(u_n^{(t)}, Du_n^{(t)}, \lambda^{(t)})\,dt,
\end{aligned}
$$

and $u_n^{(t)}$ and $\lambda^{(t)}$ are given by

$$
u_n^{(t)} := tu_n + (1-t)u \qquad \lambda_n^{(t)} := t\lambda_n + (1-t)\lambda.
$$

If $\mathcal{W}$ obeys (3.4), then the ellipticity and obliqueness of (3.61) is clear. If instead (3.5) holds, then these properties follow from the observation $|\nabla u_n^{(t)}| \leq t|\nabla u_n| + (1-t)|\nabla u| < q_{\mathrm{cr}}$. Arguing as in Lemma 3.13 leads us to conclude

$$
|v_n|_{3,\alpha} \leq C,
$$

where C is independent of n. Classical Schauder estimates now supply the estimate $|v_n|_{3,\alpha} \leq \tilde{C}|v_n|_0$, where $\tilde{C}$ is once more independent of n. As remarked above, $u_n \to u$ in $C_{\mathrm{b}}^0(\overline{\Omega})$, and we obtain the desired convergence. ∎

3.4 Global theory

In this section, we combine the local theory and a priori estimates from sections 3.2 and 3.3 with some results from abstract global bifurcation theory to extend the local curve $\mathcal{C}_{\text{loc}}$ to the non-peturbative regime. Alternatively, one could opt for lesser regularity requirements on f and $\mathcal{W}$, carry out the arguments of Section 3.2 with minor modifications, and then extend $\mathcal{C}_{\text{loc}}$ using degree-theoretic techniques. A number of qualitative properties are also shown to hold for any element of this extended solution family; these will prove to be important later as we characterize the limiting behavior.

3.4.1 Existence of a global curve

We are almost prepared to state a preparatory global bifurcation theorem. First, let us prove a preliminary result concerning the Fredholm properties of $\mathcal{F}_u$.

Lemma 3.22. *Let $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, for some $\delta > 0$. Suppose that u is non-trivial. Then, $\mathcal{F}_u(\lambda, u) : X \to Y$ is Fredholm with index* 0.

Proof. Recall that $L_0 = \mathcal{F}_u(\lambda^*, 0) : X \to Y$ is Fredholm with index 0. Therefore, it suffices to show $\mathcal{F}_u(\lambda, u)$ is semi-Fredholm with index $< \infty$ by the continuity of the Fredholm index (consider the homotopy $tL_0+(1-t)\mathcal{F}_u(\lambda, u)$).

The argument follows by making appropriate modifications to the one found in Lemma 3.6. To this end, suppose that $\{\Theta_n\} \in X$ is such that $|\Theta_n|_{3,\alpha} \leq K$ and $\mathcal{F}_u(\lambda, u)[\Theta_n] =: (v_n, w_n) \to (v, w)$ in Y. It follows from the Arzelà–Ascoli theorem and a diagonalization argument that there exists $\Theta \in C^{3,\alpha}(\overline{\Omega})$ such that $\Theta_n \to \Theta$ in $C^3_{\text{loc}}(\overline{\Omega})$ and $\mathcal{F}_u(\lambda, u)[\Theta] = (v, w)$. The estimate of Lemma 3.20

shows that

$$|u|_{C^{3,\alpha}(\Omega_{x_0}^{x_0+1})} \le Ce^{-kx_0/4} \qquad \text{for} \qquad x_0 > 0,$$

and therefore the elements of matrix valued function $\mathcal{W}'(|\nabla u|^2)I + (\nabla u \otimes \nabla u)\mathcal{W}''(|\nabla u|^2)$ from (3.44) converge uniformly to the identity matrix as $x \to \infty$. The proof of the claim can now be completed as in Lemma 3.6. ∎

The following theorem shows that $\mathcal{C}_{\text{loc}}$ can be extended into a locally real analytic curve contained in $\mathcal{F}^{-1}(0)$.

Theorem 3.23. *There is a curve $\mathcal{C} \subset \mathcal{F}^{-1}(0) \cap \mathcal{O}$ parameterized as $\mathcal{C} := \{(\lambda(s), u(s)) : -\infty < s < \infty\}$ with the following properties:*

(a) (No closed loop) *Let $\mathcal{C}^{\pm} := \{(\lambda(\pm s), u(\pm s)) \ : \ 0 \le s < \infty\}$. Then $\mathcal{C}^+ \cap \mathcal{C}^- = \{(\lambda^*, 0)\}$.*

(b) (Blow-up) *As $s \to \infty$*

$$N(s) := |u(s)|_{3,\alpha} + \frac{1}{\text{dist}((\lambda(s), u(s)), \partial\mathcal{O})} + |\lambda(s)| \to \infty. \tag{3.62}$$

(c) *$\mathcal{C}^+$ can be reparametrized near any point $(\lambda(s_0), u(s_0)) \in \mathcal{C}^+ \setminus \{(\lambda^*, 0)\}$ so that $s \mapsto (\lambda(s), u(s))$ is locally real analytic.*

This result establishes the fundamental existence result underpinning Theorem 3.1, Theorem 3.2, and Theorem 3.3. The proof of Theorem 3.23 will require some additional qualitative analysis of $\mathcal{C}$ and is delayed until the end of this section. The next lemma provides a weaker result to set the global theory in motion.

Lemma 3.24. *There is a curve* $\mathcal{C} \subset \mathcal{F}^{-1}(0) \cap \mathcal{O}$ *parameterized as* $\mathcal{C} := \{(\lambda(s), u(s)) : -\infty < s < \infty\}$ *with the following properties:*

(a) *One of the following alternatives holds:*

- (i) (Blow-up) *As* $s \to \infty$

$$N(s) := |u(s)|_{3,\alpha} + \frac{1}{\operatorname{dist}((\lambda(s), u(s)), \partial\mathcal{O})} + |\lambda(s)| \to \infty \quad (3.63)$$

- (ii) (Closed Loop) *There exists* $0 < T < \infty$ *such that* $(\lambda(T), u(T)) = (\lambda^*, 0)$.

(b) $\mathcal{C}^+ := \{(\lambda(s), u(s)) \ : \ s \geq 0\}$ *can be reparametrized near any point* $(\lambda(s_0), u(s_0)) \in \mathcal{C}^+ \setminus \{(\lambda^*, 0)\}$ *so that* $s \mapsto (\lambda(s), u(s))$ *is locally real analytic.*

Proof. We have shown that that the hypothesis of [3, Theorem 9.1.1] are met (in their notation we take $U = \mathcal{O}$ and $\mathcal{S} = \mathcal{O} \cap \mathcal{F}^{-1}$, see Appendix B for more details). In particular, closed bounded subsets of $\mathcal{F}^{-1}(0) \cap \mathcal{O}$ are shown to be locally-precompact in Lemma 3.21, $\mathcal{F}_u(\lambda, u) : X \to Y$ is a Fredholm operator of index 0 whenever $(\lambda, u) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}$ by Lemma 3.22, and $\dot{\lambda} \not\equiv 0$ by the proof of Lemma 3.10. The conclusion of Theorem 9.1.1 supplies a curve $\mathcal{C}$ that satisfies each of the claimed properties above. ∎

3.4.2 Symmetry and monotonicity

We now show that elements of $\mathcal{C}^+$ exhibit certain symmetry and monotonicty properties. The following sets will be useful for our analysis:

$$\begin{aligned}
\Omega^+ &:= \{(x,y) \in \overline{\Omega} \ : \ y > 0\} \\
L &:= \{(0,y) \ : \ -h/2 < y < h/2\} \\
L^+ &:= \{(0,y) \ : \ y \in L, y > 0\} \\
T &:= \{(x,h/2) \ : \ 0 < x < \infty\} \\
B &:= \{(x,-h/2) \ : \ 0 < x < \infty\}.
\end{aligned} \tag{3.64}$$

The nodal properties we are concerned with are as follows:

$$\begin{aligned}
&u_x < 0 \quad \text{on} \ \ \Omega^+ \\
&u_y > 0 \quad \text{on} \ \ \Omega \\
&u_{yy}(0,h/2) < 0 \\
&f_z(\lambda,0) > 0.
\end{aligned} \tag{3.65}$$

Note that the oddness of u about the line $y=0$ allows to extend the claimed properties of u_x and u_{yy} in an obvious manner. We show first that (3.65) holds along $\mathcal{C}^+_{\text{loc}}$. Then, we show that (3.65) is both an open and closed property in the relative topology of $\mathcal{C}^+ \cap X$.

Lemma 3.25 (Nodal properties along $\mathcal{C}^+_{\text{loc}}$)**.** *If* $(\lambda,u) \in \mathcal{C}^+_{\text{loc}}$*, and* $0 < |\lambda^* - \lambda| < \epsilon_0$ *for some* ϵ_0 *sufficiently small, then* (λ,u) *satisfies* (3.65).

Proof. It was assumed in (3.6) that $f_z(\lambda^*, 0) = \lambda_0 > 0$. The regularity of f implies immediately that $f_z(\lambda, 0) > 0$ if ϵ_0 is sufficiently small.

Recall that $\mathcal{F}_u(\lambda, u)[u_y] = 0$. Hence, u_y solves a uniformly elliptic equation no zeroth order term in Ω. It follows that u_y satisfies the strong maximum principle. Therefore, $u_y > 0$ on L would imply $u_y > 0$ in Ω, since $u_y = 0$ on T, B and $u_y \to 0$ as $x \to \infty$. It follows from (3.20) that u is of the form

$$u(x, y) = \epsilon \sin(\lambda_0 y) e^{-\lambda_0 x} + o(\epsilon) \text{ in } C^{3,\alpha}(\overline{\Omega}),$$

for some $0 < \epsilon < \epsilon_0$. Let $0 < c < \lambda_0^2$. Then, for small enough ϵ_0 we can ensure that $u_{yy}(0, \frac{h}{2}) < \epsilon(c - \lambda_0^2)$ and $u_{yy}(0, -\frac{h}{2}) > \epsilon(\lambda_0^2 - c)$. If $0 < d < h/2$, then (3.20) also implies $u_y > 0$ on $[-\frac{h}{2} + d, \frac{h}{2} - d]$, and finally $u_y > 0$ for $\frac{h}{2} - d \leq |y| \leq \frac{h}{2}$ by an application of Taylor's theorem.

Let us now verify the properties associated with u_x. The oddness of u implies that $u(x, 0) = 0$ and consequently $u_x(x, \frac{h}{2}) = 0$ for $x \geq 0$. If we can show that $u > 0$ on L^+, then it will follow that $u_x > 0$ on Ω^+. To see this, recall that $u_{xy} = 0$ on T, so the Hopf lemma prevents u_x from obtaining a maximum or minimum on T. Moreover, $0 < f(\lambda, z)$ when $z > 0$ and $|z|$ is sufficiently small, since $f_z(\lambda, 0) > 0$. Thus, the claim follows from (3.2) and the boundary condition

$$\mathcal{W}'(|\nabla u|^2) u_x = -f(\lambda, u).$$

Finally, it was established that u is increasing in y and the argument is concluded by observing once more that $u(0,0) = 0$. ∎

(3.65) is now shown to hold for limits of appropriate solutions.

Lemma 3.26 (Closed property). *Let* $\{(\lambda_n, u_n)\} \subset \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$, *for some* $\delta > 0$. *Suppose that* f *satisfies* (3.7), $(\lambda_n, u_n) \to (\lambda, u)$ *in* $\mathbb{R} \times C^3_{\mathrm{b}}(\overline{\Omega})$. *If each* u_n *satisfies* (3.65), *then so does* u, *unless* $u \equiv 0$.

Proof. Let us assume that u is not trivial. By continuity it follows that $u_y \geq 0$ in $\partial\Omega$. Furthermore, u_y satisfies the maximum principle since u_y solves and elliptic PDE with no zeroth order term. Hence, $u_y \geq 0$ in Ω. A similar argument shows that $u_x \leq 0$ in Ω^+.

It follows from continuity that $f_z(\lambda, 0) \geq 0$. Suppose for contradiction that $f_z(\lambda, 0) = 0$. Differentiating the boundary condition on L in y and evaluating at $(0,0)$ reveals that $u_{xy} = 0$. Evaluating (3.1) at $(0,x)$, for $x > 0$, shows that $u_{yy}(0,x) = 0$ (and more importantly that $u_{xyy}(0,0) = 0$). Moreover, $u_x = u_{xx} = u_{xxx} = 0$ at $(0,0)$. The Serrin edge point lemma can now be applied to u_x in the region $\Omega \setminus \Omega^+$ to show that $u_{xxy}(0,0) > 0$. However, $u_{xy}(0,x) < 0$ for $x > 0$ by the Hopf Lemma. Hence, $u_{xxy}(0,0) \leq 0$ and we arrive at a contradiction. Now, suppose again for contradiction that $u_y(0,y_0) = 0$ for some $(0,y_0)$ with $|y_0| < \frac{h}{2}$ (if this were not the case, then the maximum principle would imply that $u_y > 0$ in Ω as desired). Then $u_{yy}(0,y_0) = 0$.

Differentiating the left boundary condition in y gives

$$(\mathcal{W}'(u_x^2) + 2u_x^2\mathcal{W}''(u_x^2))u_{xy}(0, y_0) = \mathcal{E}(u_x^2)u_{xy}|_{(0,y_0)} = 0.$$

However, this would force $u_{xy}(0, y_0) = 0$ and contradict the Hopf lemma.

We have already shown that $u_y > 0$ on Ω and that $f_z(\lambda, 0) > 0$. Note that $f_z(\lambda, 0) > 0$ implies $\lambda > 0$ and consequently $f(\lambda, z) > 0$ for $z \in (0, \epsilon)$, for some $\epsilon < 0$. Moreover, $u > 0$ on L^+ because $u(0, 0) = 0$. The left boundary condition implies that $u_x < 0$ when $0 < u(0, y) < \epsilon$. Suppose that $u_x(0, y_0) = 0$ for some $y_0 \in (0, h/2]$. We claim that $y_0 = h/2$. It follows from (3.7) that $f_z(\lambda, u(0, y_0)) < 0$. However, differentiating the left boundary condition in the y-variable and evaluating at y_0 shows $f_z(\lambda, u(0, y_0))u_y(0, y_0) = 0$. Hence, $u_y(0, y_0) = 0$ and the sign condition on u_y forces $y_0 = h/2$. Evaluating the PDE of (3.1) at $(0, h/2)$ shows $u_{xx}(0, h/2) + u_{yy}(0, h/2) = 0$. Recall that $|u|_0 = u(0, h/2)$ and observe that $U \coloneqq |u|_0 - u$ solves the uniformly elliptic PDE in (3.1). Hence, U satisfies the hypothesis of the Serrin edge point lemma from which we can conclude $u_{xx} + u_{yy} > 0$. This is a contradiction and we find $u_x > 0$ on L^+.

We are left to show that $u_{yy}(0, h/2) < 0$. By continuity $u_{yy}(0, h/2) \leq 0$, so let us assume for contradiction that $u_{yy}(0, h/2) = 0$. The boundary condition on T implies immediately that $u_{xy}(0, \frac{h}{2}) = 0$, and (3.1) evaluated at $(0, \frac{h}{2})$

gives

$$(\mathcal{W}'(u_x^2) + 2u_x^2\mathcal{W}''(u_x^2))u_{xx} + \mathcal{W}'(u_x^2)u_{yy} = 0, \tag{3.66}$$

hence $u_{xx}(0, h/2) = 0$. Differentiating the left boundary condition twice in y, evaluating at $(0, h/2)$, and making use of $u_y = u_{xy} = u_{yy}$ at $(0, h/2)$ yields

$$(\mathcal{W}'(u_x^2) + 2u_x^2\mathcal{W}''(u_x^2))u_{xyy}(0, h/2) = 0,$$

and in particular that $u_{xyy}(0, h/2) = 0$. Now, differentiating (3.1) in y and evaluating all known quantities shows that $u_{yyy}(0, h/2)$ (note that $u_{yxx}(0, h/2) = 0$). This contradicts the Serrin edge point lemma. ∎

Lemma 3.27 (Open property). *Let $(\lambda_1, u), (\lambda_2, w) \in \mathcal{F}^{-1}(0) \cap \mathcal{O}_\delta$. If u satisfies* (3.65)*, then for some ϵ_0 we have $|\lambda_1 - \lambda_2| + |u - w|_{3,\alpha} \leq \epsilon_0$ implies that w satisfies* (3.65).

Proof. As we saw in Lemma 3.25, the signs of w_x, w_y on L^+, L can be used to establish (3.65). For any $0 < d < \frac{h}{2}$ there is some $\epsilon_0 > 0$ such that $|u - w|_{3,\alpha} < \epsilon_0$ implies that $w_y(0, y) > 0$ for $|y| \leq \frac{h}{2} - d$ and $w_{yy}(0, h/2) < 0$. Now, it is clear from the Taylor expansion of w_y in the y-variable at $(0, \frac{h}{2})$ that $w_y > 0$ on L (recall that $w_y(0, \frac{h}{2}) = 0$).

Observe that $u_{xx}(0, 0) = u_{xxx}(0, 0) = u_{xyy}(0, 0) = 0$ and consequently $u_{xy}(0, 0) < 0$ by another application of the Serrin edge point lemma. Hence, if $0 < \epsilon < |u_{xy}(0, 0)/2| = \epsilon_0$, then $w_{xy}(0, 0) < u_{xy}(0, 0) + \epsilon < 0$. Hence, there is some $d' > 0$ for which $w_x(0, y) < 0$ on $(0, d')$ whenever $0 < \epsilon < \epsilon_0$. Therefore, we may assume that ϵ is chosen to ensure $w_x < 0$ on $[d'/2, h/2]$. Arguing as in Lemma 3.25 we find that $w_x < 0$ in Ω^+ (note that $f(\lambda_1, u(0,y)) > 0$ for $y \in (0, h/2]$ and $f_z(\lambda_1, 0) > 0$) ∎

We are finally ready to prove the primary theorem of this section.

Proof of Theorem 3.23. Lemma 3.24 establishes the existence and local analyticity of $\mathcal{C}$. It remains to show $\mathcal{C}$ does not form a closed loop. Suppose for contradiction that this were the case. Then, there exists some $0 < T < \infty$ for which $(\lambda(T), u(T)) = (\lambda^*, 0)$.

Let $s^* > 0$ and set $\mathcal{C}^+(s^*) := \{(\lambda(s), u(s)) \ : \ 0 < s < s^*\}$. It now follows from Lemma 3.25, Lemma 3.27, and Lemma 3.26 that (3.65) must remain valid for $\mathcal{C}^+(s^*)$ provided $u(s) \not\equiv 0$ for all $s \in (0, s^*)$. We see from Theorem 3.4 that $\mathcal{F}^{-1}(0) \cap V \subset \mathcal{C}^+_{\text{loc}} \cup \mathcal{C}^-_{\text{loc}}$. An easy calculation shows that $(\lambda, -u) \in \mathcal{F}^{-1}(0)$ for any $(\lambda, u) \in C^+$. Hence, $\mathcal{C}^-_{\text{loc}} = \{(\lambda(s), -u(s)) \ : \ (\lambda(-s), u(-s)) \in \mathcal{C}^+_{\text{loc}}\}$ and (3.65) cannot hold along $\mathcal{C}^-_{\text{loc}} \setminus \{(\lambda^*, 0)\}$. Consequently, there exists some $(\lambda, 0) \neq (\lambda^*, 0)$ such that $(\lambda, 0) \in \mathcal{C}^+$ and $(\lambda, 0)$ must be a point of bifurcation. Let $s_0 \in (0, T)$ be the least value for which $u(s_0) \equiv 0$. Then (3.65) holds along $\mathcal{C}^+(s_0)$. The local analysis carried out in Section 3.2 can be redone at $(\lambda(s_0), 0)$, and we conclude that $f_z(\lambda(s_0), 0) = \lambda_0$. To see this, note that $f_z(\lambda(s_0), 0)$ must equal λ_k for some $k \in \mathbb{N}$, otherwise $\ker(\mathcal{F}_u(\lambda(s_0), 0))$ is trivial and $(\lambda(s_0), 0)$ cannot be a bifurcation point. Also, the only Φ_k that meets the conditions of

(3.65) is Φ_0, and it follows that $f_z(\lambda(s_0), 0) = \lambda_0$ so that $\mathcal{F}_u(\lambda(s_0), 0)[\Phi_0] = 0$. However, this is impossible since $f_{\lambda z}(\lambda, 0) > 0$ and $f_z(\lambda^*, 0) = \lambda_0$. ∎

3.5 Proofs of the main theorems

The main theorems are now proved using results of the previous sections. Most of the required work is already completed, but the arguments must be pieced together.

Proof of Theorem 3.1. The existence and local real analyticity of $\mathcal{C}$ are shown in Theorem 3.23. Moreover, it was shown in the proof of Theorem 3.23 that if $(\lambda, 0) \in \mathcal{C}$, then $\lambda = \lambda^*$. Hence, the symmetry and monotonicity properties claimed in Theorem 3.1.(a) follow from Lemma 3.25, Lemma 3.27, and Lemma 3.26. In particular, these properties hold on $\mathcal{C}^+_{\text{loc}} \setminus \{(\lambda^*, 0)\}$, are both open and closed in the relative topology, and therefore must be satisfied on all of $\mathcal{C}^+ \setminus \{(\lambda^*, 0)\}$. Finally, the alternatives of Theorem 3.1.(c) follow directly Theorem 3.23.(b) (note that the loss of ellipticity corresponds to elements of $\mathcal{C}^+$ approaching $\partial\mathcal{O}$). ∎

Theorem 3.1 lays the foundation for both Theorem 3.2 and Theorem 3.3.

Proof of Theorem 3.2. Theorem 3.1 establishes the existence of $\mathcal{C}$ along with the claimed symmetry, monotonicity, and regularity properties. It remains to show that the blow-up of $u(s)$ in $C^{3,\alpha}$ can be refined to the claimed gradient blow-up in (3.11). Condition (3.4) implies that $\mathcal{E}^*(u(s)) > \xi$ for all s (so (3.1) is strictly elliptic for each $u(s)$).

Suppose for contradiction that $|\nabla u(s)|^2 < K$ for all s and some $K < \infty$. Then $|u|_0^2 \leq Kh^2/4$ and

$$\rho(\lambda(s)) \leq bKh^2/4 + \lambda_0 \max_{q \in [0,K]} \mathcal{W}'(q)$$

by Lemma 3.14, and $\lambda(s)$ is seen to be bounded above (recall that $\rho(\lambda) \to \infty$ as $\lambda \to \infty$). It was shown in the proof of Theorem 3.1 that $\mathcal{C}^+$ satisfies (3.65), so $f_z(\lambda(s), 0) > 0$. Recall that $f_z(0,0) = 0$ and $\lambda^* > 0$, so it follows that $\lambda(s) > 0$. Hence, there exists $\delta > 0$ for which $(\lambda(s), u(s)) \in \mathcal{O}_\delta$. Lemma 3.13 implies the existence of $C > 0$ such that $|u(s)|_{3,\alpha} \leq C$. However, this would force $\sup\limits_{s>0} N(s) < \infty$ and therefore contradict Theorem 3.23. ∎

The argument for blow-up in the degenerate case is much more subtle. We show that a loss of ellipticity is inevitable in this setting. The failure of classical elliptic estimates and maximum principles make it challenging to deduce information about the higher regularity of $u(s)$.

Proof of Theorem 3.3. Note that $\mathcal{C}^+ \subset \mathcal{O}$ by the construction in Theorem 3.1. This implies that $|\nabla u(s)|_0^2 < q_{\text{cr}}$. The oddness of each $u(s)$ and the mean value theorem imply that $|u(s)|_{C^1(\Omega)} \leq C$, where $C > 0$ depends only on h and q_{cr}. The upper bound on $\lambda(s)$ obtained in Lemma 3.14 implies $\lambda(s) \leq \lambda^+$ for some $\lambda^+ > 0$ that is independent of $\mathcal{E}^*(u(s))$. It was shown in the proof of

Theorem 3.1 that $\mathcal{C}^+$ satisfies (3.65), so $f_z(\lambda(s), 0) > 0$. Hence, $|\lambda(s)| \leq \lambda^+$. Now, let us show that

$$\liminf_{s \to \infty} \mathcal{E}^*(u(s)) \to 0^+.$$

If this were not the case, then there would exist some $\delta > 0$ for which $\mathcal{C}^+ \subset \mathcal{O}_\delta$ and $|\nabla u(s)|_0^2 \leq q_{\text{cr}} - \delta$. However, we could then argue as in the proof of Theorem 3.2 to show $\sup\limits_{s > \infty} N(s) < \infty$ and reach a contradiction.

It remains to show $|D^2 u(s)|_0 \to \infty$ as $s \to \infty$ whenever (3.8) holds. Suppose that this were not the case and in particular that $|u(s)|_2 < K < \infty$. Observe first that $|f(\lambda, u)| \leq m\sqrt{q_{\text{cr}}}$ by (3.51) (this equation is valid since $\lambda > 0$). Moreover, there exists $u \in C_{\text{b}}^{1,2/3}(\overline{\Omega})$ and a sequence $\{s_n\}$ for which $(\lambda_n, u_n) := (\lambda(s_n), u(s_n)) \to (\lambda, u)$ in $\mathbb{R} \times C_{\text{loc}}^{1,2/3}(\overline{\Omega})$ and $\mathcal{E}^*(u(s)) \to 0^+$ as $s_n \to \infty$. If $S_{\text{cr}} = \{(x, y) \in \Omega \;:\; |\nabla u|^2(x, y) = q_{\text{cr}}\}$, then u_n can be taken so that $u_n \to u$ in $C_{\text{loc}}^{3,\alpha}(\Omega \backslash S_{\text{cr}})$; this follows from interior Schauder estimates and a diagonlization argument applied to sets of the form $S_N := \{(x, y) \;:\; |\nabla u|^2 < q_{\text{cr}} - \frac{1}{N},\; x < N\}$. Moreover, u solves the interior equation given by (3.1) on $\Omega \setminus S_{\text{cr}}$.

Theorem 3.15 implies $E(u_n; x) \leq E(u_n; 0)e^{-kx}$ for each n, where k is independent of n (recall that k depends on h, $|u_n|_0$, and the constant m given in Lemma 3.11). Hence, there is some $\epsilon_0, x^* > 0$ such that $|\nabla u_n(x, y)|^2 < q_{\text{cr}} - 2\epsilon_0$ whenever $x > x^*$. Thus, $|\nabla u(x, y)|^2 < q_{\text{cr}} - \epsilon_0$ whenever $x > x^*$. We claim that $|\nabla u|^2 < q_{\text{cr}}$ in Ω. Our proof is patterned after a similar result found in [10]. Suppose that the claim is false. Then, there must exist some

$(x_0, y_0) \in (0, x^*) \times [-h/2, h/2]$ at which $|\nabla u(x_0, y_0)|^2 = q_{\text{cr}}$. Recall that $\partial_x u$ vanishes along $y = 0$, so y_0 cannot be 0 by Theorem 3.29. Without loss of generality we may then take $y_0 \in (0, h/2]$. We claim that $u_x^2 = q_{\text{cr}}$ somewhere along T. Suppose for contradiction that this is not the case. Let us assume first that $x_0 > h/2$. We note that $u_y(x_0, h/2) = 0$, and so there must be some $r > 0$ for which $|\nabla u|^2 < q_{\text{cr}}$ in $B_r(x_0, h/2) \cap \overline{\Omega}$. The regularity of u implies that there is a first r', with $r < r' < h/2$, which satisfies $|\nabla u|^2 < q_{\text{cr}}$ inside $B_{r'}(x_0, h/2) \cap \overline{\Omega}$ and $|\nabla u|^2(p) = q_{\text{cr}}$ for some $p \in \partial B_{r'}(x_0, h/2) \cap \Omega_{x_0}$. Let $\nu(p) = (\nu_1, \nu_2)$ be the outer unit normal at p. Note that $\nu_1, \nu_2 \leq 0$. It follows from Theorem 3.29 and the non-characteristic degeneracy condition (3.78) that

$$-2\mathcal{W}''(|\nabla u|^2)(u_x \nu_2 - u_y \nu_1)^2 = 0, \tag{3.67}$$

which is possible only if ν_1 or ν_2 vanishes, since $u_x \leq 0$ and $u_y \geq 0$ in this region (we need only rule out the case in which ν_1 vanishes). If the x-coordinate of p is x_0, then (3.67) implies $u_y^2(p) = q_{\text{cr}}$ (since $u_x(p)$ must vanish). However, $|\nabla u|^2 < q_{\text{cr}}$ in $B_{r'}(x_0, h/2) \cap \Omega$ and (3.78) must then hold in the y direction as well. In particular, this implies $u_y^2(p) < q_{\text{cr}}$, and we arrive at a contradiction. Finally, when $0 < x_0 < h/2$ we may repeat the argument above with a family of ellipses centered at $(x_0, h/2)$ and once more conclude that u_x^2 achieves the critical value somewhere along T. It follows from (3.1) and (3.65) that $\partial_x^2 u_n \geq 0$ along T for each n. But then u_x^2 is non-increasing along T and consequently

$u_x^2(0,h/2) = q_{\mathrm{cr}}$. Hence, $f(\lambda, u(0,h/2)) = m\sqrt{q_{\mathrm{cr}}}$, and this contradicts (3.8) (this follows since (3.8) implies (3.51)). Hence, $|\nabla u|^2 < q_{\mathrm{cr}}$ in Ω.

By the reasoning above, we may take $u \in C^3(\Omega)$ with $|u|_{C^2(\Omega)} \leq K$. Condition (3.8) and equation (3.51) prevent u_x^2 from obtaining the critical value q_{cr} on L, so $|\nabla u(0,y_0)|^2 = q_{\mathrm{cr}}$ for some $y_0 \in (-h/2, h/2)$. The hypothesis of Theorem 3.29 are met and we conclude that u does not belong to $C^{1,2/3}(\overline{\Omega})$, a contradiction. ∎

3.6 Invertibility of L_1

Let us begin by fixing some notation. If $U \subset \mathbb{R}^n$, then we use $W^{k,p}(U)$ to denote the Sobolev space of k-times weakly differentiable functions whose partial derivatives of order k or less belong to $L^p(U)$; the definition of $W^{k,p}_{\mathrm{loc}}(U)$ is analogous to that of $W^{k,p}(U)$ with $L^p(U)$ replaced by $L^p_{\mathrm{loc}}(U)$. Also, the spaces $W^{k,2}(U)$ will be labeled by $H^k(U)$. Additionally, we introduce the extension domain $\Omega_{l,\mathrm{ext}} := (0,l) \times \mathbb{R}$.

Proof of Lemma 3.7. The injectivity of $L_1 : X \to Y$ follows from the maximum principle and Hopf lemma. We are left to show surjectivity. A two step approximation procedure will be used to obtain weak solutions, which will then be used to solve $L_1 u = (v,w) \in Y$. First, we conjugate by $e^{-\epsilon x}$ and obtain the equivalent problem

$$L_1^\epsilon u^\epsilon = (v^\epsilon, w), \qquad L_1^\epsilon := (\Delta + 2\epsilon\partial_x + \epsilon^2 - \gamma, \partial_x + \epsilon - a), \tag{3.68}$$

where $a = \gamma - \lambda_0$ and $v^\epsilon = e^{-\epsilon x} v$. Second, we consider this modified operator on the truncated domain Ω_l with a homogeneous Dirichlet condition imposed at the far end. Let us introduce the spaces

$$H_1(\Omega_l) := \{v \in H^1(\Omega_l) \ : \ u|_{x=l} = 0 \ \text{ in the trace sense}\}$$

and

$$H_2(\Omega_l) := L^2(\Omega_l) \times H^{-1/2}(-h/2, h/2).$$

The set $H_1(\Omega_l)$ becomes a Hilbert space when equipped with the $H^1(\Omega_l)$ norm and obvious inner product. Given $(v^\epsilon, w) \in H_2(\Omega_l)$, we call u^ϵ a weak solution of the equation $L_1^\epsilon u^\epsilon = (v^\epsilon, w)$ if

$$\begin{aligned} \mathcal{B}(u^\epsilon, \varphi) := \int_{\Omega_l} \nabla u^\epsilon \cdot \nabla \varphi - 2\epsilon u_x^\epsilon \varphi + (\gamma - \epsilon^2) u^\epsilon \varphi \, dx\, dy + (a - \epsilon) \int_L u^\epsilon \varphi \, dy \\ = - \int_{\Omega_l} v^\epsilon \varphi \, dx\, dy - \int_L w \varphi \, dy \end{aligned} \tag{3.69}$$

for all $\varphi \in H_1(\Omega_l)$. The bilinear form $\mathcal{B} : H_1(\Omega_l) \times H_1(\Omega_l) \to \mathbb{R}$ is coercive whenever $\gamma > \lambda_0$, $0 < \epsilon < \epsilon_0 < a$, and ϵ_0 is taken small enough. To see this, note that

$$\begin{aligned} \mathcal{B}(u^\epsilon, u^\epsilon) = \int_{\Omega_l} |\nabla u^\epsilon|^2 - 2\epsilon u_x^\epsilon u^\epsilon + (\gamma - \epsilon^2)(u^\epsilon)^2 \, dx\, dy + (a - \epsilon) \int_L (u^\epsilon)^2 \, dy \\ \geq \min\{1 - \epsilon, \gamma - \epsilon - \epsilon^2\} \|u^\epsilon\|^2_{H^1(\Omega_l)}. \end{aligned} \tag{3.70}$$

Furthermore, Ω_l is Lipschitz, so by the trace theorem

$$\|u(0, \cdot)\|_2 \leq C(\Omega_2) \|u\|_{H^1(\Omega_2)} \leq C(\Omega_2) \|u\|_{H^1(\Omega_l)}, \tag{3.71}$$

for all $l > 2$ and $u \in H^1(\Omega_l)$. Applying (3.71) and Hölder's inequality to (3.69) yields

$$B(u^\epsilon, \varphi) \leq C\|u\|_{H^1(\Omega_l)}\|\varphi\|_{H^1(\Omega_l)},$$

where $C = C(\Omega_2, \gamma, \epsilon_0)$ and $l > 2$. Hence, $\mathcal{B}$ is also continuous, and the Lax-Milgram theorem ensures the existence of a weak solution $u^\epsilon \in H_1(\Omega_l)$ of (3.68) given any $(v, w) \in H_2(\Omega_l)$. In particular, we obtain such a u^ϵ for a pair (v^ϵ, w) with $(v, w) \in Y$.

The regularity of u^ϵ must now be considered. We claim that $u^\epsilon \in C^{3,\alpha}(\overline{\Omega})$. Appealing to Theorems 8.8 and 9.19 of [11] reveals $u^\epsilon \in C^{3,\alpha}(\Omega_l)$. The functions v^ϵ and w can be extended, via even reflection as done in Lemma 3.6, to a $2h$-periodic functions $(\tilde{v}^\epsilon, \tilde{w}) \in C^{1,\alpha}(\Omega_{l,\text{ext}}) \times C^{2,\alpha}(\mathbb{R})$. A direct calculation shows that the analogous extension of u satisfies $\tilde{u}_\epsilon \in W^{1,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$ (so that $\tilde{u}^\epsilon$ and its first order weak derivatives are well-defined and have finite square integrals on compact subsets of $\overline{\Omega}_{l,\text{ext}}$) and solves the same interior equation as u^ϵ (with the extension of v^ϵ). Using [11, Theorem 8.8] and [11, Theorem 9.19] once more shows $\tilde{u}^\epsilon \in C^{3,\alpha}(\Omega_{l,\text{ext}})$. It remains to estimate $\tilde{u}^\epsilon$ near $\partial\Omega_{l,\text{ext}}$. Recall that $\tilde{u}^\epsilon$ satisfies $\tilde{u}^\epsilon(l, \cdot) = 0$ in the trace sense, so Theorem 8.36 of [11] may be used to show $\tilde{u}^\epsilon$ is $C^{1,\alpha}$ near the line $x = l$. Estimates at the left boundary are now obtained by reducing (3.68) to Poisson's equations with a homogeneous Neumann boundary condition. Let $\Theta = e^{(\epsilon-\alpha)x}\tilde{u}^\epsilon + e^{-x}\tilde{w}(y)$.

Then, $\Theta \in W^{1,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$ and

$$\begin{cases} \Delta\Theta = F & \text{in} \quad \Omega_{l,\text{ext}} \\ \Theta_x = 0 & \text{on} \quad \{(0,y) \mid y \in \mathbb{R}\}, \end{cases} \tag{3.72}$$

where

$$F = e^{(\epsilon-a)x}(\tilde{v}^\epsilon - 2a\tilde{u}^\epsilon_x + (\gamma - 2\epsilon\alpha)\tilde{u}^\epsilon) + e^{-x}(\tilde{w} + \partial_y^2\tilde{w})),$$

and (3.72) is understood in the weak sense. It can be shown that for any $0 < r < R < l$ and $y_0 \in \mathbb{R}$

$$\int_{B_r(0,y_0)\cap\mathbb{R}^2_+} |\nabla\Theta_y|^2 \leq C\left(\frac{1}{(R-r)^2}\int_{B_R\cap\mathbb{R}^2_+}\Theta_y^2 + \int_{B_R\cap\mathbb{R}^2_+} F^2\right),$$

where C is a dimensional constant. Hence, Θ_y, and consequently $\tilde{u}_y$, belong to $W^{1,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$. The interior equation of L^ϵ_1 may now be solved for u^ϵ_{xx}, and it follows that $\tilde{u}^\epsilon_{xx} \in W^{1,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$. This implies $F_x \in W^{1,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$ and that Θ_x solves

$$\begin{cases} \Delta\Theta_x = F_x & \text{in} \quad \Omega_{l,\text{ext}} \\ \Theta_x = 0 & \text{on} \quad \{(0,y) \mid y \in \mathbb{R}\}. \end{cases}$$

Theorem 9.12 of [11] can now be invoked to conclude

$$\|D^2\Theta_x\|_{L^2(B^+_R(0,y_0))} \leq C\|F_x\|_{L^2(B^+_R(0,y_0))},$$

for any $0 < R < l$ and $y_0 \in \mathbb{R}$. Thus, Θ_x, and consequently $\tilde{u}^\epsilon_x$, belong to $W^{2,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$. It may now be seen that $\tilde{u}^\epsilon \in W^{3,2}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$ by differentiating the interior equation in y and solving for $\tilde{u}^\epsilon_{yyy}$. The Sobolev embedding theorem implies firstly that $\tilde{u}^\epsilon \in W^{2,6}_{\text{loc}}(\overline{\Omega}_{l,\text{ext}})$, and secondly that $\tilde{u}^\epsilon \in C^{1,2/3}(\overline{\Omega}_{l,\text{ext}})$.

Hence, $\Delta\tilde{u}^\epsilon \in C^\beta(\overline{\Omega}_{l,\text{ext}})$ with $\beta = \min\{\alpha, 2/3\}$, and $\tilde{u}^\epsilon \in C^{2,\beta}(\overline{\Omega}_{l,\text{ext}})$ by Theorems 6.18 and 6.27 of [11]. Finally, an argument similar to the one found in Lemma 3.6 can be use to bootstrap our estimates and yields the Schauder estimate

$$|u^\epsilon|_{C^{3,\alpha}(\Omega_l)} \leq C_1 \left(|u^\epsilon|_{C^0(\Omega_l)} + |v^\epsilon|_{C^{1,\alpha}(\Omega_l)} + |w|_{C^{1,\alpha}(L)}\right) \tag{3.73}$$

for some constant C_1 depending only on a, h, γ and ϵ_0 (provided $l > 2$).

The last step of our argument is to obtain a subsequence $\{u^\epsilon\}$ converging to some $u \in X$ as $\epsilon \to 0$ and $l \to \infty$. This requires u^ϵ satisfy the symmetry and derivative conditions of X. Note that $L_1^\epsilon[u^\epsilon(x, y) + u^\epsilon(x, -y)] = 0$, so the oddness of u^ϵ follows from the maximum principle and Hopf lemma. Moreover, one finds that $\partial_y^3 u^\epsilon(x, \pm h/2) = 0$, where $0 \leq x < l$, by differentiating the PDE in y and solving for $\partial_y^3 u^\epsilon$. Consider now $\tilde{u}^\epsilon$ restricted to $(0, l) \times (-h, h)$, so that $\tilde{u}^\epsilon$ vanishes on three of the four boundary segments. Theorem 3.7 of [11] implies

$$\sup_{\Omega_l} |u^\epsilon| \leq \sup_L |u^\epsilon| + (e^{2h} - 1)|v^\epsilon|_0. \tag{3.74}$$

If u^ϵ achieves a positive maximum value at some point on L, then $u_x^\epsilon \leq 0$ there, and then by the boundary condition $u^\epsilon \leq \frac{|w|_0}{a-\epsilon}$. An analogous bound holds at a negative minimum obtained on L. Combining this observation with (3.73) and (3.74) gives

$$|u^\epsilon|_{C^{3,\alpha}(\Omega_l)} \leq C_2(|v^\epsilon|_{C^{1,\alpha}(\Omega_l)} + |w|_{C^{1,\alpha}(L)}), \tag{3.75}$$

for a constant $C_2 = C_2(a, h, \epsilon_0, \gamma)$. Denote by u_n^ϵ the solution obtained to the truncated problem with $l = n \in \mathbb{N}$. Thus, $\{u_n^\epsilon\}$ is seen to admit a subsequence converging in $C^3_{\text{loc}}(\Omega)$ to some $u_\infty^\epsilon \in C_{\text{b}}^{3,\alpha}(\Omega)$ by a diagonalization argument. Note that $L^1_\epsilon u^\epsilon_\infty = (v^\epsilon, w)$. The Schauder estimate (3.75) is also uniform in ϵ, so a second diagonlization argument produces a solution of $L_1 u = (v, w)$ with $u \in C^{3,\alpha}_{\text{b,o}}(\Omega)$. The argument is completed once u is shown to decay uniformly to zero. This follows from the proof of Lemma 3.6. ∎

3.7 Examples

The framework developed above allows us to construct global solution curves for a wide class of elastic materials and tractions. In this section, we investigate some concrete examples covered by Theorem 3.2 and Theorem 3.3. Consider the functions

$$\mathcal{W}(q) = q + c_1 q^2 \qquad f(\lambda, z) = \lambda z - b z^3. \tag{3.76}$$

Suppose first that $c_1 > 0$. In this case, Theorem 3.2 is applicable and we find that $\mathcal{C}$ contains a sequence of elements $(\lambda(s_n), u(s_n))$ for which $|\nabla u(s_n)|^2 \to \infty$. Moreover, the left boundary condition of (3.1) can be rewritten for $y > 0$ to read

$$\tau(|\nabla u|) = 2 f(\lambda, u)\sqrt{1 + \frac{u_y^2}{u_x^2}}.$$

Note that $\frac{d}{dk}\tau(k) = 2\mathcal{E}(k^2) \geq \xi$. For any s_n, the maximum of $|\nabla u(s_n)|^2$ is achieved on L and we conclude that $|u_y(s_n)| \to \infty$.

If instead $c_1 < 0$, then Theorem 3.3 applies and we obtain global family $\mathcal{C}$ containing a subsequence $(\lambda(s_n), u(s_n))$ for which $\mathcal{E}^*(u(s_n)) \to 0$ (note that the construction of $\mathcal{C}$ and proof of ellipticity loss do no depend on (3.8)). If b, c_1 and h are chosen appropriately, then we conclude also that $|D^2 u(s_n)|_0 \to \infty$. A quick calculation shows that $m = \frac{2}{3}$ and $q_{\mathrm{cr}} = \frac{1}{6|c_1|}$, so (3.8) is satisfied if $b < 0$ and

$$\left(\frac{\pi}{h}\right)^3 < \frac{|b|}{2|c_1|}.$$

3.8 A general gradient maximum principle

Several gradient maximum principles for solutions of degenerate elliptic PDEs are established in this section. Structural conditions relevant to the investigations of this paper are imposed, but the results are stated under more general hypotheses. Consider the problem

$$\nabla \cdot (\mathcal{W}'(|\nabla u|^2)\nabla u) = 0 \qquad \text{in} \qquad U, \tag{3.77}$$

where $U \subset \mathbb{R}^n$ is a bounded C^2 domain, $\mathcal{W}$ is C^3, and (3.5) is satisfied.

Suppose that a second order elliptic differential operator has the symmetric coefficient matrix $A = (a_{ij})$, with $a_{ij} = a_{ij}(x)$. If A is positive semi-definite on U, then there is said to be a *non-characteristic degeneracy* at $p \in \partial U$ if

$$\ker(A(p)) \neq 0 \qquad \text{and} \qquad A(p)\nu \neq 0, \tag{3.78}$$

where $\nu(p)$ is the outward unit normal at p. Condition (3.78) expresses a relationship between a differential operator and the geometry of U that can be exploited to prove Hopf-type lemmas.

We wish to investigate the properties of solutions exhibiting a loss of ellipticity. To this end, let us consider a neighborhood V of U where $q_{\rm cr} - \epsilon_{\rm cr} < |\nabla u|^2 \leq q_{\rm cr}$. Moreover, it follows from (3.5) that $\epsilon_{\rm cr}$ may be selected so that $\mathcal{E}'(|\nabla u|^2) < 0$ on V. Let us also point out that the strict sign condition $\mathcal{W}''(q_{\rm cr}) < 0$ must hold. This follows from (3.5b) and Lemma 3.11. Hence, by taking $\epsilon_{\rm cr}$ smaller if needed, we may suppose also that $\mathcal{W}'' < 0$ on V. The following notation will be convenient:

$$a_{ij}(x) := \delta_{ij} + 2\frac{\mathcal{W}''(q)u_{,i}u_{,j}}{\mathcal{W}'(q)}\bigg|_{q=|\nabla u(x)|^2}, \tag{3.79}$$

where δ_{ij} is the Kronecker delta function. The explicit dependence of q on both u and x will be omitted for the remainder of this section. Note that the matrix (a_{ij}) is positive semi-definite by the ellipticity of (3.77); in particular,

$$a_{ij}\xi_i\xi_j = |\xi|^2 + 2\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}u_{,i}u_{,j}\xi_i\xi_j \geq 0 \qquad \text{on} \qquad V. \tag{3.80}$$

The Bernstein technique can be used to obtain a differential equation satisfied by $|\nabla u|^2$; by differentiating (3.77) in the $k^{\rm th}$-variable, multiplying the resulting equation by $u_{,k}$, and then summing over each such equation we obtain

$$a_{ij}|\nabla u|^2_{,ij} = 2(u^2_{,ij}) + 2\left(\left(\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\right)^2 - \frac{\mathcal{W}'''(q)}{\mathcal{W}'(q)}\right)u_{,i}u_{,j}|\nabla u|^2_{,i}|\nabla u|^2_{,j} - \frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}|\nabla u|^2_{,i}|\nabla u|^2_{,i}. \tag{3.81}$$

Taking $\xi = \nabla u_{,k}$ in (3.80) yields

$$\nabla u_{,k} \cdot \nabla u_{,k} \geq -2\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)} u_{,i}u_{,j}u_{,ik}u_{,jk} = -\frac{\mathcal{W}''(q)}{2\mathcal{W}'(q)}|\nabla u|^2_{,k}|\nabla u|^2_{,k}. \tag{3.82}$$

Using $\sum_{i,j=1}^{n} u_{,ij}^2 = \nabla u_{,k} \cdot \nabla u_{,k}$ and (3.82) gives

$$2\sum_{i,j=1}^{n} u_{,ij}^2 \geq -\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}|\nabla u|^2_{,k}|\nabla u|^2_{,k}. \tag{3.83}$$

Using (3.80) once more, this time with $\xi = \nabla|\nabla u|^2$, gives

$$|\nabla u|^2_{,k}|\nabla u|^2_{,k} \geq -2\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\left(|\nabla u|^2_{,k}u_{,k}\right)^2. \tag{3.84}$$

Restricting to V and appealing to (3.81), (3.83), and (3.84) shows that

$$a_{ij}|\nabla u|^2_{,ij} \geq 2\left(3\left(\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\right)^2 - \frac{\mathcal{W}'''(q)}{\mathcal{W}'(q)}\right)\left(|\nabla u|^2_{,k}u_{,k}\right)^2 \tag{3.85}$$

The definition of $\mathcal{E}$ and (3.5b) may be used to show

$$3\left(\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\right)^2 - \frac{\mathcal{W}'''(q)}{\mathcal{W}'(q)} = 3\left(\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\right)^2 + \frac{3\mathcal{W}''(q)}{2q\mathcal{W}'(q)} - \frac{\mathcal{E}'(q)}{2q\mathcal{W}'(q)} \to -\frac{\mathcal{E}'(q_{\mathrm{cr}})}{2q_{\mathrm{cr}}\mathcal{W}'(q_{\mathrm{cr}})} > 0 \quad \text{as} \quad q \to q_{\mathrm{cr}},$$

so by shrinking V if needed we conclude

$$a_{ij}|\nabla u|^2_{,ij} \geq 0 \qquad \text{in} \qquad V. \tag{3.86}$$

Moreover, the differential operator

$$Q := a_{ij}\partial^2_{ij} \tag{3.87}$$

is degenerate-elliptic throughout U by (3.5) and (3.86) shows that $Q|\nabla u|^2 \geq 0$ on V. We are now ready to state a Hopf lemma for $|\nabla u|^2$.

Theorem 3.28. *Let $u \in C^2(\overline{U}) \cap C^3(U)$ be a solution of* (3.77) *for which $|\nabla u|^2 < q_{\text{cr}}$ in U. Assume that for some $p \in \partial\Omega$ we have*

$$\sup_U |\nabla u|^2 = |\nabla u|^2(p).$$

If in addition (3.78) *holds at p, then*

$$\frac{\partial|\nabla u|^2}{\partial\nu}\Big|_{x=p>0}$$

Proof. The proof involves the following exponential barrier function

$$w(x) = d(e^{-k|x-y|^2} - e^{-kr^2}), \tag{3.88}$$

where $y = p - r\nu$, r is small enough that $B_r(y) \subset U$, (3.86) holds, and k, d are positive constants yet to be chosen. Let $x \in B_\epsilon(p) \cap B_r(y)$ for some ϵ such that $0 < \epsilon < r$. We find that

$$\begin{aligned} Qw &= 2ke^{-k|x-y|^2}\left[2k(x_i - y_i)(x_j - y_j)a_{ij} - \left(n + 2q\frac{\mathcal{W}''(q)}{\mathcal{W}'(q)}\right)\right] \\ &\geq 2ke^{-k|x-y|^2}\left(\frac{2kC_1(r-\epsilon)^2}{2} - C_2\right) > 0, \end{aligned}$$

where $C_1 > 0$ is determined by the non-characteristic condition (3.78) to satisfy $a_{ij}(x_i - y_i)(x_j - y_j) \geq C_1|x-y|^2$, $C_2 = C_2(n, \mathcal{W}) > 0$, and k is chosen sufficiently large according to C_1 and C_2. Furthermore, if $d > 0$ is taken small enough, then $|\nabla u|^2 - q_{\text{cr}} + dw(x) < 0$ on $\partial\left(B_\epsilon(p) \cap B_r(y)\right)$ and $Q\left[|\nabla u|^2 + dw(x)\right] \geq 0$, so the strong maximum principle implies

$$q_{\text{cr}} - |\nabla u|^2(p - t\nu) > dw(p - t\nu) \quad \text{for } \ 0 < t < \epsilon.$$

It now follows that $\partial_\nu |\nabla u|^2 > 2rkde^{-kr^2}$ and the claim is proved. ∎

The so-called function “P-function” is defined for $u \in C^1(\overline{U})$ by

$$P(u;x) := \int_0^{|\nabla u|^2(x)} (\mathcal{W}'(\zeta) + 2\zeta\mathcal{W}''(\zeta))\, d\zeta = (2q\mathcal{W}'(q) - \mathcal{W}(q))\,\big|_{q=|\nabla u|^2(x)}.$$

We will often adopt a slight abuse of notation and write $P(x)$ in place of $P(u;x)$. Maximum principles for P-functions can be found in [26,33]. However, a strict ellipticity condition of the form (3.4) is assumed throughout these works, and a new argument is required for our setting.

Theorem 3.29. *Let $\mathcal{W}$ be as in* (3.5) *and U a C^2 domain. Suppose $u \in C^1(\overline{U}) \cap C^3(U)$ solves* (3.1)*. Assume also that $|\nabla u|^2 < q_{\mathrm{cr}}$ in the interior of U, that $|u|_{C^2(U)} \leq K$, and $|\nabla u|^2(p) = q_{\mathrm{cr}}$ for some $p \in \partial U$. Then, $u \notin C^{1,\alpha}(\overline{U})$ for any $\alpha \in (1/2, 1)$, provided* (3.78) *holds at p.*

Proof. Let us point out that (3.77) is invariant under translations and rotations. Hence, we may assume that $p = 0$ and that $\nu(p) = -e_n$. Condition (3.78) and the regularity u imply the existence of constants $r_1, C_1 > 0$ for which the inequality $a_{nn} > 2C_1$ and (3.86) are valid within $B_{r_1}(0) \cap U$. Moreover, U satisfies an interior sphere condition and we may assume that $B_{r_2}(x_0) \subset U$, where $x_0 = -r\nu(p)$ and $r_1 < r_2$. A calculation shows

$$Q[P] = \mathcal{E}'(q)a_{ij}|\nabla u|^2_{,i}|\nabla u|^2_{,j} + \mathcal{E}(q)a_{ij}|\nabla u|^2_{ij} \geq \mathcal{E}'(q)|\nabla u|^2_{,i}|\nabla u|^2_{,i} + 2\frac{\mathcal{E}'(q)\mathcal{W}''(q)}{\mathcal{W}'(q)}\left(|\nabla u|^2_{,i}u_i\right)^2. \tag{3.89}$$

The sign conditions $\mathcal{W}''(q_{\rm cr}), \mathcal{E}'(q_{\rm cr}) < 0$ obtained from (3.85) and (3.89) imply

$$Q[P] \geq \mathcal{E}'(q)|\nabla u|_{,i}^2 |\nabla u|_{,i}^2 \qquad \text{in} \qquad B_{r_1}(0) \cap U. \tag{3.90}$$

Let $\Phi : B_{r_1}(0) \cap B_{r_2}(x_0)$ straighten the boundary of $B_{r_2}(x_0)$ near 0; more specifically, take

$$\Phi(x) \coloneqq \left(x_1, ..., x_{n-1}, x_n - r_2 + \sqrt{r_2^2 - x_1^2 - \cdots - x_{n-1}^2} \right).$$

Hence, Φ is of class C^2 near 0 and for some $0 < \kappa < 1$ we have

$$\kappa|\Phi(x_1) - \Phi(x_2)| \leq |x_1 - x_2| \leq \frac{1}{\kappa}|\Phi(x_1) - \Phi(x_2)| \qquad \text{for} \qquad x_1, x_2 \in B_{r_1}(0) \cap B_{r_2}(x_0). \tag{3.91}$$

Consider the transformation $Y = \Phi(x)$. In these coordinates, inequality (3.90) becomes

$$\tilde{a}_{ij}\partial_{ij}^2 \tilde{P} + \tilde{b}_i \partial_i \tilde{P} \geq \mathcal{E}'(|\nabla u|^2)|\nabla u|_{,i}^2 |\nabla u|_{,i}^2 \qquad \text{in} \qquad \Phi(B_{r_1}(0) \cap B_{r_2}(x_0)), \tag{3.92}$$

where

$$\tilde{P}(Y) \coloneqq P(|\nabla u|^2)\big|_{\Phi^{-1}(Y)}, \quad \tilde{a}_{ij} \coloneqq \Phi_{i,l}\Phi_{j,m} a_{lm}\big|_{\Phi^{-1}(Y)} \quad \text{and} \quad \tilde{b}_i \coloneqq \Phi_{i,lm} a_{lm}\big|_{\Phi^{-1}(Y)}.$$

We see from (3.92) and the observation $|\tilde{b}_i \partial_i \tilde{P}| \leq C(\mathcal{W}, K, |\Phi|_1)$ that

$$\tilde{Q}[\tilde{P}] \coloneqq \tilde{a}_{ij}\partial_{ij}^2 \tilde{P} \geq -C_2 \qquad \text{in} \qquad \Phi(B_{r_1}(0) \cap B_{r_2}(x_0)) \tag{3.93}$$

for some $C_2 = C_2(\mathcal{W}, K, |\Phi|_1)$. Note that $(\tilde{a}_{ij}) = D\Phi^T(a_{ij})D\Phi$. Hence, the matrix $(\tilde{a}_{ij})$ is positive semi-definite near 0. Furthermore, we can assume r_1 is selected so that

$$\tilde{a}_{nn} = \Phi_{n,l}\Phi_{n,m}a_{lm} > C_1 > 0 \qquad \text{in} \qquad B_{r_1}(0) \cap U, \tag{3.94}$$

which is possible since $\nabla\Phi_n(0) = -\nu(0)$ and $a_{nn}(0) > 2C_1$.

Let $C_{R,l}$ be the cylinder defined by

$$C_{R,l} = \{Y \in \mathbb{R}^n \ : \ Y_1^2 + Y_2^2 + \cdots + Y_{n-1}^2 < R^2 \ \text{ and } \ 0 \le Y_n \le l\}.$$

Suppose R and l are such that

$$\frac{\kappa r_1}{4\sqrt{n-1}} < R < \frac{\sqrt{3}\kappa r_1}{4\sqrt{n-1}} \quad \text{and} \quad 0 < l < \frac{\kappa r_1}{4}, \tag{3.95}$$

where κ is taken according to (3.91). It follows that $\Phi^{-1}(C_{R,l}) \subset B_{r_1/2}(0) \cap B_{r_2}(x_0)$. Let $\sigma \in (0,1)$ and $\epsilon_0 \in (0, \pi/2)$ a constant yet to be chosen. Also let

$$l = (\pi/2 - \epsilon_0)\sqrt{\frac{C_1 m \sin(\epsilon_0)}{4C_2}} \qquad \text{and} \qquad \beta = \sqrt{\frac{4C_2}{C_1 m \sin(\epsilon_0)}}, \tag{3.96}$$

where $m = \min_{B_{r_1}(0)\cap U} P > 0$. The constant ϵ_0 may be selected so that (3.95) holds as well. If $G := \tilde{P}(1 + \sin(\epsilon_0 + \beta Y_n))^{-1}$, then for appropriately chosen ϵ_0 the inequality

$$(1 + \sin(\epsilon_0 + \beta l))\tilde{Q}[G] + 2\beta\cos(\epsilon_0 + \beta Y_n)\tilde{a}_{in}\partial_i G > \frac{C_1\beta^2 m \sin(\epsilon_0)}{2} - C_2 > 0 \tag{3.97}$$

holds in $C_{R,l}$ by (3.96). Moreover, for $\epsilon > 0$ the inequality

$$\left(\tilde{Q} + \frac{2\beta\cos(\epsilon_0 + \beta Y_n)}{1 + \sin(\epsilon_0 + \beta Y_n)}\tilde{a}_{in}\partial_i\right)[G - G(0) + \epsilon Y_n^{1+\sigma}] > \epsilon(1+\sigma)\beta\cos(\epsilon_0 + \beta Y_n)Y_n^{\sigma} \ge 0 \tag{3.98}$$

holds in $C_{R,l}$. Let $S := \partial C_{R,l} \setminus \{Y \in \mathbb{R}^n \ : \ Y_n = 0 \ \text{ or } \ Y_n = l\}$. It follows that

$$m_R := \sup_S \left(\tilde{P}(Y) - \tilde{P}(0)\right) = \sup_{\Phi^{-1}(S)} \left(P(x) - P(q_{\mathrm{cr}})\right) < 0.$$

Note that (3.91) and (3.95) imply $\operatorname{dist}(\Phi^{-1}(S), 0) > \frac{\kappa^2 r_1}{4\sqrt{n-1}}$. Thus, there is constant $m_{r_1} < 0$, which is independent of l, such that $m_R < m_{r_1}$. Moreover, the proof of Theorem 3.28 can be adapted to the domain $B_{r_1}(0) \cap B_{r_2}(x_0)$ to supply the estimate

$$\sup_{\Phi^{-1}(\partial C_{R,l} \cap \{Y_n = l\})} \left(|\nabla u|^2 - q_{\mathrm{cr}}\right) < -C_3 l, \tag{3.99}$$

where C_3 is independent of R and l (in this case, the normal derivative of $|\nabla u|^2$ may not exist, but the construction of exponential comparison function is not effected). Recall that $\mathcal{E}'(q) < 0$ in $B_{r_1}(0)$, so (3.99) implies

$$m_l := \sup_{\partial C_{R,l} \cap \{Y_n = l\}} \left(\tilde{P}(Y) - \tilde{P}(0)\right) = \sup_{\Phi^{-1}(\partial C_{R,l} \cap \{Y_n = l\})} \left(P(|\nabla u|^2) - P(q_{\mathrm{cr}})\right) < -C_4 l^2,$$

where C_4 is again independent of R and l. Since $G < \tilde{P}$ in $C_{R,l}$, it follows that

$$G - G(0) + \epsilon Y_n^{1+\sigma} < 0 \qquad \text{on} \qquad \partial C_{R,l} \setminus \{Y_n = 0\} \tag{3.100}$$

if l is sufficiently small and ϵ is taken as

$$\epsilon = \frac{C_4}{2} \left((\pi/2 - \epsilon_0) \sqrt{\frac{C_1 m \sin(\epsilon_0)}{4 C_2}} \right)^{1-\sigma}. \tag{3.101}$$

The weak maximum principle may now be invoked using (3.98) and (3.100) to show

$$\tilde{P}(Y) - \tilde{P}(0) \leq -\epsilon Y_n^{1+\sigma} \qquad \text{in} \quad C_{R,l}. \tag{3.102}$$

It follows from (3.91) and (3.102) that

$$P(|\nabla u|^2(-s\nu)) - P(q_{\text{cr}}) \le -C_5 s^{1+\sigma} \tag{3.103}$$

for $s > 0$ small enough and some $C_5 > 0$ which is independent of s. However, it follows from the regularity of P in the q-variable that

$$P(-s\nu) - P(0) \ge \frac{\mathcal{E}'(q_{\text{cr}})}{4} \left(|\nabla u|^2(-s\nu) - q_{\text{cr}}\right)^2 \tag{3.104}$$

for s sufficiently small. The argument is completed by observing (3.103) and (3.104) prevent $u \in C^{1,\alpha}(\overline{U})$, where $\alpha \in (\frac{1+\sigma}{2}, 1)$. ∎

Appendix A

Maximum principles

We recall some variants of the maximum principle which are used repeatedly throughout the paper. In particular, we state the strong maximum principle, including the version which a sign condition on the zeroth order term is replaced with a sign condition on the solution ([30, Lemma 1]), the Hopf boundary lemma, Serrin edge point lemma (see [30]), and finally a variant which holds when a positive supersolution is known to exist (see [28, Section 5 Theorem 10]).

Theorem A.1. *Let Ω be a connected, open set (possibly unbounded), and consider the second order operator L given by*

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x) \tag{A.1}$$

where the coefficients are of class $C_0^0(\overline{\Omega})$. Suppose that L is uniformly elliptic in the sense that there exists $\lambda > 0$ such that

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \textit{for all } \xi \in \mathbb{R}^n,\ x \in \overline{\Omega}, \tag{A.2}$$

and that (a_{ij}) *is symmetric. Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *be a solution of* $Lu = 0$ *in* Ω.

(i) (Strong maximum principle) *Suppose that* u *attains a maximum value on* $\overline{\Omega}$ *at a point in the interior of* Ω. *If* $c \leq 0$ *in* Ω, *or if* $\sup_{\overline{\Omega}} u \leq 0$, *then* u *is a constant function.*

(ii) (Hopf boundary lemma) *Suppose that* u *attains it maximum value on* $\overline{\Omega}$ *at a point* $x_0 \in \partial\Omega$ *for which there exists an open ball* $B \subset \Omega$ *with* $\overline{B} \cap \partial\Omega = \{x_0\}$. *Assume that either* $c \leq 0$ *or else* $\sup_B u = 0$. *Then* u *is a constant function or*

$$\nu \cdot \nabla u(x_0) > 0,$$

where ν *is the outward unit normal to* Ω *at* x_0.

(iii) (Serrin edge point lemma) *Let* $x_0 \in \partial\Omega$ *be an edge point in the sense that near* x_0 *the boundary* $\partial\Omega$ *consists of two transversally intersection* C^2 *hypersurfaces* $\{\sigma(x) = 0\}$ *and* $\{\gamma(x) = 0\}$. *Suppose that* $\sigma, \gamma < 0$ *in* Ω, $u \in C^2(\overline{\Omega})$, $u > 0$ *in* Ω *and* $u(x_0) = 0$. *Assume further that* $a_{ij} \in C^2$ *in a neighborhood of* x_0,

$$B(x_0) = 0, \qquad \text{and} \qquad \partial_\tau B(x_0) = 0,$$

for every differential operator ∂_τ *tangential to* $\{\sigma = 0\} \cap \{\gamma = 0\}$ *at* x_0.

Then for any unit vector s outward from Ω at x_0, either

$$\partial_s u(x_0) < 0, \quad \textit{and } \partial_s^2 u(x_0) < 0.$$

(iv) (When a positive supersolution exists) *Suppose that there exists $v \in C^2(\overline{\Omega})$ such that $v > 0$ and $Lv \leq 0$ in $\overline{\Omega}$. Then either $\frac{u}{v}$ is constant, or $\frac{u}{v}$ cannot achieve a nonnegative maximum in Ω.*

Remark A.2. If, in the context of (iv) above we suppose that that $u \geq$ on Ω, then the existence of a positive superolution implies that $u > 0$ in $\overline{\Omega}$.

Appendix B

Bifurcation theory

In this section, we record some useful results from bifurcation theory. First, we quote a version of the well known Crandall–Rabinowitz theorem as stated in [3, Theorem 8.3.1]. See [8] for the original paper.

Theorem B.1. *Suppose that X and Y are Banach spaces, that $F : \mathbb{R}\times X \to Y$ is of class C^k, $k \geq 2$, and that $F(\lambda, 0) = 0 \in Y$ for all $\lambda \in \mathbb{R}$. Suppose also that*

$$
\begin{aligned}
&L = \partial_x F[(\lambda_0, 0)] \ \text{ is a Fredholm operator of index zero;}\\
&\ker(L) \ \text{ is one dimensional;}\\
&\ker(L) = \{\xi \in X \ : \ \xi = s\xi_0 \ \text{ for some } \ s \in \mathbb{R}\}\\
&\text{the transversality condition holds:}\\
&\qquad \partial^2_{\lambda x} F[(\lambda_0, 0)](1, \xi_0) \notin \mathit{range}(L).
\end{aligned}
$$

Then $(\lambda_0, 0)$ is a bifurcation point. More precisely, there exists $\epsilon > 0$ and a *branch of solutions*

$$\{(\lambda, x) = (\Lambda(s), s\xi(s)) \ : \ s \in \mathbb{R}, |s| < \epsilon\} \subset \mathbb{R} \times X,$$

such that $\Lambda(0) = \lambda_0$*;* $\xi(0) = \xi_0$*;*

$F(\Lambda(s), s\xi(s)) = 0$ *for all* s *with* $|s| < \epsilon$;

Λ *and* $s \mapsto s\xi(s)$ *are of class* C^{k-1}, *and* ξ *is of class* C^{k-2}, *on* $(-\epsilon, \epsilon)$;

there exists an open set $U_0 \subset \mathbb{R} \times X$ *such that* $(\lambda_0, 0) \in U_0$ *and*

$\{(\lambda, x) \in U_0 \ : \ F(\lambda, x) = 0, x \neq 0\} = \{(\Lambda(s), s\xi(s)) \ : \ 0 < |s| < \epsilon\}$;

if F *is analytic,* ξ *and* Λ *are analytic functions on* $(-\epsilon, \epsilon)$.

This result is used to verify the existence of local solution branches. The next theorem allows us to extend such a family and provides information regarding the terminal behavior of the curve. Our presentation is taken from [3, Theorem 9.1.1]. Let us state some relevant hypothesis and notation. Let $U \subset \mathbb{R} \times X$ and $F : U \to Y$ be $\mathbb{R}$-analytic. Assume also that

$$(\lambda, 0) \in U \text{ and } F(\lambda, 0) = 0 \text{ for all } \lambda \in \mathbb{R}. \tag{B.1}$$

$$\partial_x F[(\lambda, x)] \text{ is a Fredholm operator of index zero when } F(\lambda, x) = 0, \ (\lambda, x) \in U. \tag{B.2}$$

$$\text{For some } \lambda_0 \ \ker(L) \text{ is one dimensional and the transversality condition holds.} \tag{B.3}$$

Let $\kappa(s) = s\xi(s)$, where ξ is taken from Theorem B.1, and define the sets

$$\mathcal{R}^+ = \{(\Lambda(s), \kappa(s)) \ : \ s \in (0, \epsilon)\},$$

$$\mathcal{S} = \{(\lambda, x) \in U \; : \; F(\lambda, x) = 0\},$$

$$\mathcal{T} = \{(\lambda, x) \in \mathcal{S} \; : \; x \neq 0\}.$$

Make the addition assumption that ϵ is taken small enough so that $\kappa'(s) \neq 0$ for $|s| < \epsilon$ and $\mathcal{R}^+ \subset \mathcal{T}$.

Theorem B.2. *Suppose that* (B.1) *holds,* $\Lambda' \not\equiv 0$ *on* $(-\epsilon, \epsilon)$ *and that in* $\mathbb{R} \times X$ *all bounded closed subsets of* $\mathcal{S}$ *are compact. Then there exists a continuous curve* $\mathfrak{R}$ *which extends* $\mathcal{R}$ *as follows:*

(a) $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) \; : s \in [0, \infty)\} \subset U$ *where* $(\Lambda, \kappa) : [0, \infty) \to \mathbb{R} \times X$ *is continuous.*

(b) $\mathcal{R}^+ \subset \mathfrak{R} \subset \mathcal{S}$.

(c) The set $\{s \geq 0 \; : \; \ker(\partial_x F[(\lambda(s), \kappa(s))]) \neq \{0\}$ *has no accumulation points.*

(d) At each point, $\mathfrak{R}$ *has an analytic re-parameterization.*

(e) One of the following occurs.

(i) $\|(\Lambda(s), \kappa(s))\| \to \infty$ *as* $s \to \infty$.

(ii) $(\Lambda(s), \kappa(s))$ *approaches the boundary of* U *as* $s \to \infty$.

(iii) $\mathfrak{R}$ *is a closed loop. In other words, for some* $T > 0$,

$$\mathfrak{R} = \{(\Lambda(s), \kappa(s)) \; : \; 0 \leq s \leq T\}$$

and $(\Lambda(T), \kappa(T)) = (\lambda_0, 0)$.

Bibliography

[1] C. J. Amick and R. E. L. Turner, *A global theory of internal solitary waves in two-fluid systems*, Trans. Amer. Math. Soc., 298 (1986), pp. 431–484.

[2] J. M. Ball, *Some Open Problems in Elasticity*, Springer New York, New York, NY, 2002, pp. 3–59.

[3] B. Buffoni and J. Toland, *Analytic Theory of Global Bifurcation*, Princeton University Press, 2003.

[4] R. M. Chen, S. Walsh, and M. H. Wheeler, *Existence and qualitative theory for stratified solitary water waves*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 517–576.

[5] R. M. Chen, S. Walsh, and M. H. Wheeler, *Center manifolds without a phase space for quasilinear problems in elasticity, biology, and hydrodynamics*, arXiv preprint arXiv:1907.04370, (2019).

[6] R. M. Chen, S. Walsh, and M. H. Wheeler, *Global bifurcation for monotone fronts of elliptic equations*, arXiv preprint arXiv:2005.00651, (2020).

[7] S. N. Chow and J. K. Hale, *Methods of bifurcation theory*, vol. 251 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], Springer-Verlag, New York-Berlin, 1982.

[8] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, Journal of Functional Analysis, 8 (1971), pp. 321–340.

[9] H. G. Gebran and C. A. Stuart, *Global continuation for quasilinear elliptic systems on* $\mathbb{R}^N$ *and the equations of elastostatics*, Adv. Nonlinear Stud., 9 (2009), pp. 727–762.

[10] D. GILBARG and M. SHIFFMAN, *On bodies achieving extreme values of the critical mach number, i*, Journal of Rational Mechanics and Analysis, 3 (1954), pp. 209–230.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[12] T. J. Healey, *Classical injective solutions in the large in incompressible nonlinear elasticity*, Arch. Ration. Mech. Anal., 232 (2019), pp. 1207–1225.

[13] T. J. Healey and U. Miller, *Two-phase equilibria in the anti-plane shear of an elastic solid with interfacial effects via global bifurcation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 463 (2007), pp. 1117–1134.

[14] T. J. Healey and P. Rosakis, *Unbounded branches of classical injective solutions to the forced displacement problem in nonlinear elastostatics*, Journal of Elasticity, 49 (1997), pp. 65–78.

[15] T. J. Healey and H. C. Simpson, *Global continuation in nonlinear elasticity*, Arch. Rational Mech. Anal., 143 (1998), pp. 1–28.

[16] T. Hogancamp, *Broadening global families of anti-plane shear equilibria*, SIAM Journal on Mathematical Analysis, 53 (2021), pp. 5853–5879.

[17] C. Horgan and J. Knowles, *The effect of nonlinearity on a principle of saint-venant type*, Journal of Elasticity, 11 (1981), pp. 271–291.

[18] C. Horgan and L. Payne, *Decay estimates for second-order quasilinear partial differential equations*, Advances in Applied Mathematics, 5 (1984), pp. 309–332.

[19] C. O. Horgan, *Anti-plane shear deformations in linear and nonlinear solid mechanics*, SIAM Rev., 37 (1995), pp. 53–81.

[20] S. Jiménez-Alfaro, V. Villalba, and V. Mantič, *Singular elastic solutions in corners with spring boundary conditions under anti-plane shear*, International Journal of Fracture, 223 (2020), pp. 197–220.

[21] J. Knowles and E. Sternberg, *Anti-plane shear fields with discontinuous deformation gradients near the tip of a crack in finite elastostatics*, Journal of Elasticity, 11 (1981), pp. 129–164.

[22] J. K. Knowles, *On finite anti-plane shear for incompressible elastic materials*, J. Austral. Math. Soc. Ser. B, 19 (1975/76), pp. 400–415.

[23] ——, *The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids*, International Journal of Fracture, 13 (1977), pp. 611–639.

[24] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains*, Comm. Partial Differential Equations, 16 (1991), pp. 585–615.

[25] G. M. Lieberman, *Two-dimensional nonlinear boundary value problems for elliptic equations*, Trans. Amer. Math. Soc., 300 (1987), pp. 287–295.

[26] L. E. Payne and G. A. Philippin, *Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature*, Nonlinear Anal., 3 (1979), pp. 193–211.

[27] P. Podio-Guidugli, G. Vergara-Caffarelli, and E. G. Virga, *The role of ellipticity and normality assumptions in formulating live-boundary conditions in elasticity*, Quarterly of applied mathematics, 44 (1987), pp. 659–664.

[28] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Springer Science & Business Media, 2012.

[29] P. J. Rabier and C. A. Stuart, *Global bifurcation for quasilinear elliptic equations on $\mathbb{R}^N$*, Mathematische Zeitschrift, 237 (2001), pp. 85–124.

[30] J. Serrin, *A symmetry problem in potential theory*, Archive for Rational Mechanics and Analysis, 43 (1971), pp. 304–318.

[31] M. Sofonea and A. Matei, *Variational inequalities with applications: a study of antiplane frictional contact problems*, vol. 18, Springer Science & Business Media, 2009.

[32] S. J. Spector, *On uniqueness in finite elasticity with general loading*, Journal of Elasticity, 10 (1980), pp. 145–161.

[33] R. P. Sperb, *Maximum principles and their applications*, vol. 157 of Mathematics in Science and Engineering, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[34] R. Turner and J.-M. Vanden-Broeck, *Broadening of interfacial solitary waves*, The Physics of fluids, 31 (1988), pp. 2486–2490.

[35] J. Voss, H. Baaser, R. J. Martin, and P. Neff, *More on anti-plane shear*, Journal of Optimization Theory and Applications, 184 (2020), pp. 226–249.

[36] M. H. Wheeler, *Large-amplitude solitary water waves with vorticity*, SIAM J. Math. Anal., 45 (2013), pp. 2937–2994.

[37] ——, *Solitary water waves of large amplitude generated by surface pressure*, Arch. Ration. Mech. Anal., 218 (2015), pp. 1131–1187.

[38] O. H. Yeoh, *Some forms of the strain energy function for rubber*, Rubber Chemistry and technology, 66 (1993), pp. 754–771.

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