COERCIVE ESTIMATES FOR THE LAPLACE-BELTRAMI OPERATOR

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by
ERIN QUINN

Dr. Dorina Mitrea, Thesis Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

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And hereby certify that in their opinion it is worthy of acceptance.

[redacted]

Professor Dorina Mitrea

[redacted]

Professor Ioan Kosztin

[redacted]

Professor Tanya Christiansen

[redacted]

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Contents

Chapter 1

Introduction

Recall a classical result in Partial Differential Equations (PDEs) corresponding to the entire Euclidean space. If $f \in L^2(\mathbb{R}^n)$, $E(x) = c_n \frac{1}{\|x\|^{n-2}}$ for $n \geq 3$, and $c_n = \frac{1}{n(n-2)\alpha(n)}$, $\alpha(n)$ denoting the volume of the unit ball in $\mathbb{R}^n$, then $u(x) := (E * f)(x)$ is a solution of the Poisson problem for the Laplacian:

$$\Delta u = f \text{ in } \mathbb{R}^n. \tag{1.1}$$

Here, $*$ denotes the convolution operator. Then, using properties of the Fourier transform and Plancherel's Theorem, for $j, k \in \{1, 2, \dots, n\}$ we have that

$$\begin{aligned} \|\partial_j \partial_k u\|_{L^2(\mathbb{R}^n)} &= C \, \|\xi_j \, \xi_k \, \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \\ &= C \left\| \xi_j \, \xi_k \, \hat{E}(\xi) \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)} \\ &= C \left\| \frac{\xi_j}{\|\xi\|} \cdot \frac{\xi_k}{\|\xi\|} \, \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)} \\ &\leq C \|\hat{f}\|_{L^2(\mathbb{R}^n)} \\ &= C \|f\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{1.2}$$

where C denotes constants that depend only on n and $\hat{}$ denotes the Fourier transform.

Hence,

$$\sum_{j,k=1}^{n} ||\partial_j \partial_k u||_{L^2(\mathbb{R}^n)} \leq C ||\Delta u||_{L^2(\mathbb{R}^n)}. \tag{1.3}$$

If we now set $\dot{W}^{2,2}(\mathbb{R}^n)$ to be the homogeneous Sobolev space of functions u for which

$$||u||_{\dot{W}^{2,2}(\mathbb{R}^n)} := \sum_{j,k=1}^{n} ||\partial_j \partial_k u||_{L^2(\mathbb{R}^n)} < \infty, \tag{1.4}$$

then based on estimate (1.3) we observe that any solution u of (1.1) satisfies

$$||u||_{\dot{W}^{2,2}(\mathbb{R}^n)} \approx ||\Delta u||_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

where, for $a, b \in \mathbb{R}$, $a \approx b$ means that there exist constants c_1 and c_2 such that $c_1\, a \leq b \leq c_2\, a$.

The goal of our work is to explore similar estimates on bounded surfaces. Such settings are natural to consider since they arise in real-life problems. For example, we are led to considering PDEs on domains on surfaces when analyzing the flow of water in the ocean, the flow of air on airplane wings, and the flow of heat through a body.

In the case when a surface $\mathcal{S}$ is flat in $\mathbb{R}^{n+1}$, that is, $\mathcal{S}$ is an open, smooth, bounded domain in $\mathbb{R}^n$, the Kadlec-Miranda-Talenti identity shows that for a solution u of $\Delta u = f \in L^2(\mathcal{S})$ in $\mathcal{S}$ with homogeneous Dirichlet boundary condition (i.e. $u = 0$ on $\partial\mathcal{S}$, the boundary of $\mathcal{S}$), that

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} |\partial_j \partial_k u|^2 \, dS = \int_{\mathcal{S}} |\Delta u|^2 \, dS - \int_{\partial\mathcal{S}} (\partial_\nu u)^2 \, \mathcal{G}_{\partial\mathcal{S}} \, ds, \tag{1.6}$$

where ν is the outward unit normal to $\mathcal{S}$ and $\partial_\nu u$ is the directional derivative of u in the direction of ν, also called the normal derivative. See [7], [10], [11] page 340, [8] and the references therein. Here we would also like to mention the work in [4] where it is shown that if Ω is a smooth bounded domain then the problem $\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ has a unique solution $u \in W^{2,p}(\Omega)$ for every $f \in L^p(\Omega)$, $1 < p < \infty$. Coercive estimates are also obtained, however the constants depend on $\partial\Omega$. Two important aspects arise. First, one needs to consider boundary conditions for u. Second, the geometries of the domain and boundary play an important role in this new setting. For example, if $\mathcal{G}_{\partial\mathcal{S}}$ is non-negative, which is the case if the domain $\mathcal{S}$ is convex, then (1.6) implies (1.5) with $\mathbb{R}^n$ replaced by $\mathcal{S}$, where $\mathcal{G}_{\partial\mathcal{S}}$ is as defined in (7.24). It is important also to point out that there exist domains for which the equivalent of (1.5) fails. Here is an example in $\mathbb{R}^2$.

Fix $\theta \in (0, \pi)$ and define $\Omega_\theta := \{z \in \mathbb{C} : |z| \leq 1 \text{ and } -\theta < arg\, z < \theta\}$.

If $z = x + iy \equiv (x, y)$ and we let φ be a smooth, compactly supported function with support contained in the ball centered at 0 of radius $\frac{1}{2}$, and we further define

$$u(z) := Re\left(z^{\frac{\pi}{2\theta}}\right)\varphi(z), \quad z \in \Omega_\theta, \tag{1.7}$$

then

$$u\left(\rho e^{\pm i\theta}\right) = Re\left(\rho^{\frac{\pi}{2\theta}} e^{\pm i\frac{\pi}{2}}\right)\varphi\left(\rho e^{\pm i\theta}\right) = 0 \text{ for } 0 \leq \rho, \tag{1.8}$$

so $u = 0$ on $\partial\Omega_\theta$. Furthermore,

$$(\Delta u)(z) = (\Delta\varphi)(z)\, Re\left(z^{\frac{\pi}{2\theta}}\right) + \left(\Delta Re\left(z^{\frac{\pi}{2\theta}}\right)\right)\cdot\varphi(z) + \left(\nabla Re\left(z^{\frac{\pi}{2\theta}}\right)\right)\nabla\varphi(z). \tag{1.9}$$

Since $z^{\frac{\pi}{2\theta}}$ is holomorphic in $\mathbb{C}\backslash(-\infty, 0]$, $\Delta Re\left(z^{\frac{\pi}{2\theta}}\right) = 0$ in Ω_θ, and from (1.9) we see that

$$\left|\Delta u(z)\right| \sim \left|z^{\frac{\pi}{2\theta}-1}\right| \text{ near } 0. \tag{1.10}$$

In particular, if $z = re^{i\omega}$, then

$$\begin{aligned} \int_{\Omega_\theta} \left|\Delta u(z)\right|^2 dz &\approx \int_0^1 \left(r^{\frac{\pi}{2\theta}-1}\right)^2 r\,dr \\ &\approx \int_0^1 r^{\frac{\pi}{\theta}-1}\,dr < \infty, \end{aligned} \tag{1.11}$$

for every $\theta \in (0, \pi)$.

On the other hand, for each $j, k \in \{1, \dots, n\}$,

$$\begin{aligned} \partial_j\partial_k u(z) = (\partial_j\partial_k\varphi) Re\left(z^{\frac{\pi}{2\theta}}\right) + \left(\partial_j\partial_k Re\left(z^{\frac{\pi}{2\theta}}\right)\right)\varphi(z) \\ + \partial_j Re\left(z^{\frac{\pi}{2\theta}}\right)\partial_k\varphi(z) + \left(\partial_k Re\left(z^{\frac{\pi}{2\theta}}\right)\right)\partial_j\varphi(z), \end{aligned} \tag{1.12}$$

and

$$\left|\partial_j\partial_k u(z)\right| \sim \left|z^{\frac{\pi}{2\theta}-2}\right| \text{ near } 0. \tag{1.13}$$

Thus,

$$\begin{aligned} \int_{\Omega_\theta} |\partial_j\partial_k u(z)|^2 dz &\approx \int_0^1 \left(r^{\frac{\pi}{2\theta}-2}\right)^2 r\,dr \\ &\approx \int_0^1 r^{\frac{\pi}{\theta}-3}\,dr. \end{aligned} \tag{1.14}$$

However, $\int\limits_0^1 r^{\frac{\pi}{\theta}-3}\,dr < \infty$ if and only if $\frac{\pi}{\theta} - 3 > -1$, that is,

$$\frac{\pi}{2} > \theta > 0. \tag{1.15}$$

This means that the estimate

$$\|u\|_{\dot{W}^{2,2}(\Omega_\theta)} \approx \|\Delta u\|_{L^2(\Omega_\theta)} \tag{1.16}$$

holds if and only if $0 < \theta < \frac{\pi}{2}$, which in fact corresponds to the case when Ω_θ is convex. This example clearly shows that one cannot hope to prove coercive estimates for solutions to the Poisson problem considered on arbitrary domains and that the geometry of the underlying domain plays an important role. The case when homogeneous Neumann boundary conditions are imposed was treated in [6] and [1]. In these papers, the authors obtain a formula relating the integral over a smooth bounded domain Ω of a certain quadratic form in the second derivative of a function $u \in W^{2,2}(\Omega)$ with the integral on $\partial\Omega$ of the squared gradient of u weighted by a curvature term.

To state our main result we need to introduce some notation. If $\mathcal{S} \subset \mathbb{R}^n$ is an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν, γ a unit normal to $\partial\mathcal{S}$, and u a real-valued function on $\mathcal{S}$, we denote by $\Delta_{\mathcal{S}} u$ the surface Laplacian of u and by $\nabla_{\mathcal{S}} u$ the surface gradient of u. For more precise definitions of $\nabla_{\mathcal{S}} u$ and $\Delta_{\mathcal{S}} u$, see Definition 7.2. We also let $\mathcal{G} = \text{div}\, \nu$ and $R = (\partial_k \nu_j(x))_{j,k}$ where ν is an extension of the unit normal with the properties proved in Proposition 5.1. For more thorough definitions and properties of R and $\mathcal{G}$, see Proposition 5.5 and Theorem 5.3 respectively. If $f \in C^2(\mathcal{S})$, and if u is such that $\Delta_{\mathcal{S}} u = f$ on $\mathcal{S}$, then we prove in Proposition 7.3 that, if $\mathcal{D}_j = \partial_j - \nu_j \partial_\nu$, $j = 1, \ldots, n$, then

$$\begin{aligned}
\int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 dS = &\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS + \int_{\mathcal{S}} \langle R \nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u \rangle \mathcal{G}\, dS \\
&- 2 \int_{\mathcal{S}} |R \nabla_{\mathcal{S}} u|^2 dS - \sum_{j=1}^{n} \int_{\partial\mathcal{S}} (\mathcal{D}_j u) \partial_\gamma (\mathcal{D}_j u)\, ds \\
&+ \sum_{k=1}^{n} \int_{\partial\mathcal{S}} (\partial_\gamma u)(\mathcal{D}_k \mathcal{D}_k u)\, ds \\
&- \sum_{j,k=1}^{n} \int_{\partial\mathcal{S}} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k] u) \gamma_k\, ds.
\end{aligned} \tag{1.17}$$

This integral identity allows us to obtain coercive estimates for the solution of the Poisson problem for $\Delta_{\mathcal{S}}$ with homogeneous Dirichlet

$$(D) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ u\big|_{\partial\mathcal{S}} = 0 \\ u \in W^{2,2}(\mathcal{S}), \end{cases} \tag{1.18}$$

Neumann

$$(N) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ \partial_\gamma u\big|_{\partial\mathcal{S}} = 0 \\ u \in W^{2,2}(\mathcal{S}), \end{cases} \tag{1.19}$$

and mixed boundary conditions

$$(M) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ u\big|_{\Gamma_1} = 0 \\ \partial_\gamma u = 0 \text{ on } \Gamma_2 \\ u \in W^{2,2}(\mathcal{S}), \end{cases} \tag{1.20}$$

where $W^{2,2}(\mathcal{S}) = \left\{ u : \sum_{|\alpha|\leq 2} \|\mathcal{D}^\alpha u\|_{L^2(\mathcal{S})} < \infty \right\}$ if $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ and Γ_1 and Γ_2 are such that $\Gamma_1 \cup \Gamma_2 = \partial\mathcal{S}$.

As such, we obtain the following results.

If u is a solution of (D) and $\mathcal{G}_{\partial\mathcal{S}} \geq 0$,

or

u is a solution of (N) and $\left(t_j \gamma_k(x)\right)_{j,k}$ is positive, definite for $x \in \partial\mathcal{S}$,

or

u is a solution of (M), $\mathcal{G}_{\partial\mathcal{S}} \geq 0$ on Γ_1 and $\left(t_j \gamma_k(x)\right)_{j,k}$ is positive, definite for $x \in \Gamma_2$,

then

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 \, dS \leq C \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS, \text{ for some } C > 0 \text{ independent of } u. \tag{1.21}$$

Observe that our results generalize the Kadlec-Miranda-Talenti identity, (1.6). Indeed, if $\mathcal{S}$ is flat in $\mathbb{R}^{n+1}$, then $\Delta_{\mathcal{S}} \equiv \Delta$ in $\mathbb{R}^n$, $\sum\limits_{j,k=1}^{n+1} (\mathcal{D}_j\mathcal{D}_k u)^2 = \sum\limits_{j,k=1}^{n} |\partial_j\partial_k u|^2$, $\nabla_{\mathcal{S}} u = \nabla u$, $R \equiv 0$, $\nu = (0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, and $\mathcal{G}_{\partial\mathcal{S}}$ is the Gauss curvature of the boundary of the domain $\mathcal{S}$.

The layout of the thesis is as follows. In Chapter 1 we review some notation and basic definitions which are relevant to our work, and we prove a partition of unity result. Chapter 2 deals with the theory of integration on surfaces. Here we also show how to define a unit normal N to a surface which is locally given by graphs of C^k functions, $k \geq 2$. For such surfaces we also have a local description of the unit normal γ to the boundary of a surface. In Chapter 3, we discuss first-order tangential differential operators. Here we also prove that tangential operators annihilate functions which are constant on a surface.

We begin Chapter 4 by proving the existence of a distinguished extension to a neighborhood of the surface of the unit normal to a surface. We then introduce the Gauss curvature for a surface and prove that it is actually the divergence of our extension of the unit normal. We finish the chapter by proving some useful properties of a particular family of tangential operators and defining the $n \times n$ matrix-valued function R which appears in the identities on surfaces we prove later in the thesis. Chapter 5 focuses on two integration by parts results. In Chapter 6 we prove identity (1.17). This is done using the formalism and properties of a particular family of operators $(\mathcal{D}_j)_{j=1}^n$ defined here as well as the work done in previous chapters. We finish with Chapter 7 in which we prove Sobolev norm estimates on surfaces for solutions to the Poisson problem for the Laplace-Beltrami operator on surfaces with homogeneous Dirichlet, Neumann, and mixed boundary conditions.

Chapter 2

Preliminaries

Let $n \in \mathbb{N}$ and set $\mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, n times. We denote points in $\mathbb{R}^n$ by $x = (x_1, \dots, x_n)$, and for such a point x let $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x'' := (x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$. Throughout we shall use the usual definition of the magnitude $|x|$ of a vector $x \in \mathbb{R}^n$, given by

$$|x| := \sqrt{x_1^2 + \cdots + x_n^2}. \tag{2.1}$$

The scalar product of two vectors $x, y \in \mathbb{R}^n$ is given by

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k. \tag{2.2}$$

If $x \in \mathbb{R}^n$ and $r > 0$, the open ball with center x and radius r is defined by

$$B_r(x) := \{y \in \mathbb{R}^n :\mid y - x \mid< r\}. \tag{2.3}$$

A set $U \subset \mathbb{R}^n$ is said to be open if for every $x \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. A function f defined on a set $\mathcal{O} \subset \mathbb{R}^n$ is said to be of class $C^k(\mathcal{O})$ if f and all of its derivatives up to order k are continuous. The partial derivative of f with respect to $x_i \in \mathbb{R}$ will be denoted by $\partial_{x_i} f$, $\frac{\partial f}{\partial x_i}$, or $\partial_i f$. As is well known, if $f \in C^2(\mathbb{R}^n)$, then

$$\partial_i \partial_j f = \partial_j \partial_i f, \text{ for every } i, j \in \{1, \dots, n\}. \tag{2.4}$$

The gradient of a scalar function f at a point $x \in \mathbb{R}^n$ is denoted by

$$\nabla f := (\partial_1 f, \partial_2 f, \ldots, \partial_n f). \tag{2.5}$$

We will denote by

$$\nabla_\tau f(x) := \langle \tau, \nabla f \rangle \tag{2.6}$$

the directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction of a vector $\tau \in \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$. The directional derivative of a scalar function f in the direction of the normal N is called the normal derivative, and it is given by

$$\partial_N f := \langle N, \nabla f \rangle. \tag{2.7}$$

The tangential derivative of f, denoted $\nabla_{tan} f$, is given by

$$\nabla_{tan} f := \nabla f \Big|_{\mathcal{S}} - (\partial_N f) N. \tag{2.8}$$

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function, $f(x) = (f_1(x), \ldots, f_n(x))$ with $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, then the divergence of f is given by

$$\text{div} f := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n}. \tag{2.9}$$

The Jacobian matrix of such a function f is given by

$$\mathcal{J} f := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \in M_{n \times n}, \tag{2.10}$$

where $M_{n\times n}$ denotes the space of all $n \times n$ matrices.

Let $v_1, v_2, \dots, v_{n-1}$ be vectors in $\mathbb{R}^n$. The vector product $v_1 \times v_2 \times \cdots \times v_{n-1}$ is defined as the vector which is obtained by formally computing the determinant of the matrix V, formed with $v_1, v_2, \dots, v_{n-1}$ as the first $(n-1)$ rows and $(e_1, e_2, \dots, e_n)$ on the n^{th} row, by expanding along the n^{th} row of the matrix.

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \cdots & v_{n-1,n} \\ e_1 & e_2 & \cdots & e_n \end{pmatrix} = \sum_{j=1}^{n} A_{nj} e_j, \qquad (2.11)$$

where A_{nj} is the cofactor of e_j defined by $A_{nj} = (-1)^{n+j} M_{nj}$ and M_{nj} is the minor of matrix V, formed by eliminating row n and column j from V.

Remark 2.1. *If $v_1, \dots, v_{n-1}$ are vectors in $\mathbb{R}^n$, with $v_i = (v_{i,1}, v_{i,2}, \dots, v_{i,n})$ for every $i \in \{1, \dots, n-1\}$, we observe that $v_1 \times v_2 \times \cdots \times v_{n-1}$ is orthogonal on v_j, for every $j \in \{1, \dots, n-1\}$. Indeed, since*

$$\langle v_1 \times v_2 \times \cdots \times v_{n-1}, w \rangle = det \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \cdots & v_{n-1,n} \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}, \qquad (2.12)$$

it follows that $\langle v_1 \times v_2 \times \cdots \times v_{n-1}, w \rangle = 0$ whenever $w = v_j$, for $j = 1, \dots, n-1$.

The infimum of a set $U \subseteq \mathbb{R}$ is denoted by inf(U). We denote by $\overset{o}{U}$ the interior of the set U and $\overline{U}$ the closure of the set U. The set of all points which are members of both the closure of the set U and the closure of the complement of U is known as the boundary of U, and throughout we use the usual notation ∂U. We denote the distance between points $x,\ y \in \mathbb{R}^n$ by dist$(x,\ y) = |x - y|$. The support of f, denoted by supp$f$, is the closure of the set of points where f is nonzero. If a continuous function $f : U \longrightarrow \mathbb{R}$ is compactly supported we write $f \in C_o(U)$.

Theorem 2.2. *(Partition of Unity for compact sets) Let* $K \subset \mathbb{R}^n$ *be compact,* $K \subset \bigcup\limits_{j=1}^{J} U_j$ *where* U_j *is open for* $j \in \{1, \ldots, J\}$*. Then there exists a finite collection of* C^∞ *functions* $\{\varphi_j\}_{j=1}^J$ *such that*

(i) For every $1 \leq j \leq J$, *supp*(φ_j) *is compact and contained in* U_j*;*

(ii) For every $1 \leq j \leq J$, $0 \leq \varphi_j \leq 1$;

(iii) $\sum\limits_{j=1}^{J} \varphi_j(x) = 1,$ *for every* $x \in K$.

Proof of Theorem 2.2. Before proceeding with the proof of Theorem 2.2, we state and prove two lemmas which will be useful in the sequel.

Lemma 2.3. *If C is compact, and U is an open set such that* $C \subset U$*, then there exists a compact set D such that* $C \subset \overset{o}{D} \subset D \subset U$*.*

Proof. Let $V = U^c \cap \overline{B_R(0)}$, where $R > 0$ is large enough so that $\overline{U} \subset B_R(0)$. Then V is compact, so there exists $a > 0$ such that

$$a = \text{dist}(V, C) = \inf_{\substack{x \in C \\ y \in V}} |x - y|. \tag{2.13}$$

Let

$$D = \bigcup_{x \in C} \overline{B_{\frac{a}{4}(x)}}. \tag{2.14}$$

Then D is compact, and $C \subset \overset{o}{D} \subset D \subset U$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *If D is a compact set, and U is an open set such that $D \subset U$, then there exists $\psi \in \mathcal{C}^{\infty}$ such that $\psi > 0$ on D, and $\psi = 0$ outside some open set contained in U.*

Proof. Let

$$f(x) = \begin{cases} e^{-\frac{1}{(x-1)^2}} \cdot e^{-\frac{1}{(x+1)^2}}, & x \in (-1,1) \\ 0, & x \notin (-1,1). \end{cases} \tag{2.15}$$

Then $f \in \mathcal{C}^{\infty}(\mathbb{R})$, and $f > 0$ on $(-1,1)$. We now let $\epsilon > 0$. Also, for every $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, let $g_a(x) := f\left(\frac{x_1 - a_1}{\epsilon}\right) \cdots f\left(\frac{x_n - a_n}{\epsilon}\right)$, where f is as defined in (2.15). Then $g_a \in \mathcal{C}^{\infty}(\mathbb{R}^n)$, and

$$\begin{cases} g_a > 0 \text{ on } (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon) \\ g_a = 0 \text{ elsewhere.} \end{cases} \tag{2.16}$$

Set $\alpha = \operatorname{dist}(D, \overline{U}) > 0$. For every $x \in D$, $B_{\frac{a}{4}}(x) \subset U$. Moreover, there exists $\epsilon' > 0$ such that $O_{x_l} = (x_1 - \epsilon', x_1 + \epsilon') \times \cdots \times (x_n - \epsilon', x_n + \epsilon') \subset U$. Hence, $D \subset \bigcup\limits_{x_l \in D} O_{x_l}$. Since D is compact, we can extract a finite subcover such that $D \subset \bigcup\limits_{l=1}^{M} O_{x_l}$. Let

$$\psi(x) = \sum_{l=1}^{M} g_{x_l}(x). \tag{2.17}$$

Then $\psi \in \mathcal{C}^{\infty}$, $\psi > 0$ on $\bigcup\limits_{l=1}^{M} O_{x_l}$, and $\psi = 0$ outside $\bigcup\limits_{l=1}^{M} O_{x_l}$. This proves Lemma 2.4. □

Let $C_1 := K \setminus \bigcup\limits_{j=2}^{J} U_j$. Then C_1 is compact, and $C_1 \subset U_1$. By Lemma 2.3, there exists a compact set D_1 such that $C_1 \subset \overset{o}{D}_1 \subset D_1 \subset U_1$. Thus $K \subset D_1 \cup \bigcup\limits_{j=2}^{J} U_j$. By induction, we can construct the sets $D_1, D_2, \dots, D_J$ such that $K \subset \bigcup\limits_{j=1}^{J} \overset{o}{D}_j$, where D_j is compact, and $D_j \subset U_j$ for every $1 \leq j \leq J$. Indeed, suppose $K \subset \overset{o}{D}_1 \cup \overset{o}{D}_2 \cup \cdots \cup \overset{o}{D}_k \cup U_{k+1} \cup \cdots \cup U_j$, and let

$$C_{k+1} := K \setminus \left[\bigcup_{j=1}^{k} \overset{o}{D}_j \cup \bigcup_{j=k+2}^{J} U_j \right]. \tag{2.18}$$

Then $C_{k+1} \subset U_{k+1}$. Therefore, by Lemma 2.3, there exists a compact set D_{k+1} such that $C_{k+1} \subset \overset{o}{D}_{k+1} \subset D_{k+1} \subset U_{k+1}$. By Lemma 2.4, there exists $\psi_1, \ldots, \psi_J$ such that $\psi_j \in \mathcal{C}^\infty$, $\psi_j > 0$ on D_j, and $\psi = 0$ outside some open set contained in U_j. Thus,

$$\sum_{j=1}^{J} \psi_j(x) > 0 \text{ for every } x \in \bigcup_{j=1}^{J} \overset{o}{D}_j\ . \tag{2.19}$$

We now define the functions φ_j:

$$\varphi_j(x) := \frac{\psi_j(x)}{\sum_{j=1}^{J} \psi_j(x)}. \tag{2.20}$$

Then $\varphi_j \in \mathcal{C}^\infty$, $\operatorname{supp} \varphi_j \subset U_j$, $0 \leq \varphi_j \leq 1$ for every $j \in \{1, 2, \ldots, J\}$, and if $x \in K$, then $x \in \bigcup\limits_{j=1}^{J} \overset{o}{D}_j$ and $\sum\limits_{j=1}^{J} \varphi_j(x) = 1$. This completes the proof of Theorem 2.2. $\square$

Chapter 3

Analysis on Surfaces

We begin this chapter by recalling the definition of a surface of class C^k.

Definition 3.1. *For $k \in \mathbb{N}$, $\mathcal{S} \subset \mathbb{R}^n$ is called a surface of class C^k if $\mathcal{S}$ is compact and if for every $x \in \mathcal{S}$, there exists $r_0 > 0$, an open set $\mathcal{O}' \subset \mathbb{R}^{n-1}$ and a function $P : \mathcal{O}' \to \mathbb{R}^n$ of class C^k which is one-to-one and such that $P(0) = x_0$, $\partial_1 P(x'), \partial_2 P(x'), \ldots, \partial_{n-1} P(x')$ are linearly independent for every $x' \in \mathcal{O}'$, and $P(\mathcal{O}') = \mathcal{S} \cap B_{r_0}(x_0)$. We call such P a local parametrization for the surface $\mathcal{S}$.*

Lemma 3.2. *If $\mathcal{S} \subset \mathbb{R}^n$ is a surface of class C^k and P is a local parametrization of $\mathcal{S}$ as in Definition 3.1, then the vector*

$$N(P(x')) = \frac{\partial_1 P(x') \times \cdots \times \partial_{n-1} P(x')}{|\partial_1 P(x') \times \cdots \times \partial_{n-1} P(x')|} \tag{3.1}$$

is normal to $\mathcal{S}$.

Proof. This is an immediate application of Remark 2.1. □

Based on Definition 3.1 it follows that for a surface $\mathcal{S}$ of class C^k there exists $J \in \mathbb{N}$,

$r_1, \dots, r_J > 0$, $x_1, \dots, x_J \in \mathcal{S}$, and a family of local parametrizations

$$\begin{aligned} &P_j : \mathcal{O}'_j \to \mathcal{S},\ P_j \in C^k,\ j = 1, \dots, J, \text{ such that} \\ &\mathcal{S} \subset \bigcup_{j=1}^{J} P_j(\mathcal{O}'_j), \text{ and } P(\mathcal{O}'_j) = \mathcal{S} \cap B_{r_j}(x_j) \text{ for } j = 1, \dots, J. \end{aligned} \tag{3.2}$$

Definition 3.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k and $\mathcal{O}'$ be an open set such that $\mathcal{O}' \subset \mathbb{R}^{n-1}$, and consider $P : \mathcal{O}' \to \mathcal{S}$ such that P is a local parametrization for $\mathcal{S}$. If $f : \mathcal{S} \to \mathbb{R}$ is a continuous function such that supp $f \subseteq P(\mathcal{O}')$, then*

$$\int_{\mathcal{S}} f dS := \int_{\mathcal{O}'} f(P(x'))|\partial_1 P(x') \times \partial_2 P(x') \times \cdots \times \partial_{n-1} P(x')|dx'. \tag{3.3}$$

Definition 3.4. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k and let $\{\mathcal{O}'_j\}_{j=1}^J$ and $\{P_j\}_{j=1}^J$ be as in (3.2). If $f : \mathcal{S} \to \mathbb{R}$ is a continuous function, then*

$$\int_{\mathcal{S}} f dS := \sum_{j=1}^{J} \int_{\mathcal{S}} \xi_j f dS, \tag{3.4}$$

where $\{\xi_j\}_{j=1}^J$ is a partition of unity associated to the open cover $\{P(\mathcal{O}'_j)\}_{j=1}^J$ of $\mathcal{S}$ as given by Theorem 2.2.

Definition 3.5. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k. Let $\mathcal{O}'' \subseteq \mathbb{R}^{n-2}$. A mapping $p : \mathcal{O}'' \to \mathbb{R}^n$ is a local parametrization of $\partial\mathcal{S}$ provided $\partial_1 p, \dots, \partial_{n-2} p$ are linearly independent and $p(\mathcal{O}'') \subset \partial\mathcal{S}$.*

Definition 3.6. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k. Let $f : \partial\mathcal{S} \to \mathbb{R}$ be continuous with supp $f \subseteq p(\mathcal{O}'')$, where $p : \mathcal{O}'' \to \mathbb{R}^n$ is a local parametrization of $\partial\mathcal{S}$ (as in Definition 3.5). Then, for $x'' = (x_1, x_2, \dots, x_{n-2}) \in \mathcal{O}''$,*

$$\int_{\partial\mathcal{S}} f ds := \int_{\mathcal{O}''} f(p(x''))|\partial_1 p(x'') \times \cdots \times \partial_{n-2} p(x'') \times N(p(x''))|dx''. \tag{3.5}$$

Definition 3.7. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k. Let $f : \partial\mathcal{S} \to \mathbb{R}$ be a continuous function. Let $\{\xi_j\}_{j=1}^N \subset C_o^\infty(\mathbb{R}^n)$ be a sequence of functions such that $\sum\limits_{j=1}^N \xi_j = 1$ in an open set containing $\partial\mathcal{S}$, and supp $\xi_j \cap \partial\mathcal{S} \subseteq p_j(\mathcal{O}''_j)$, where $p_j : \mathcal{O}''_j \to \mathbb{R}$ is a local parametrization of $\partial\mathcal{S}$ for $j \in \{1, \dots, J\}$ and $\partial\mathcal{S} \subset \bigcup\limits_{j=1}^J p_j(\mathcal{O}''_j)$. Then*

$$\int_{\partial\mathcal{S}} f ds := \sum_{j=1}^J \int_{\partial\mathcal{S}} \xi_j f ds. \tag{3.6}$$

Remark 3.8. *Definitions 3.3, 3.4, 3.6, and 3.7 are independent of the parametrizations and partitions of unity considered.*

In the last part of this chapter we will show how using some given parametrizations for $\mathcal{S}$ and $\partial\mathcal{S}$ we can construct at each point $x \in \partial\mathcal{S}$ vectors N and γ such that N is an outward unit normal to $\mathcal{S}$, γ is a unit normal to $\partial\mathcal{S}$, and $N \perp \gamma$.

Lemma 3.9. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k. Let $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$ and let $\varphi : \mathcal{O}' \to \mathbb{R}$ be such that graph $\varphi \subseteq \mathcal{S}$. Then $P : \mathcal{O}' \to \mathbb{R}^n$, $P(x') = (x', \varphi(x'))$ for every $x' \in \mathcal{O}'$ is a local parametrization of $\mathcal{S}$, and*

$$N(x', \varphi(x')) = \frac{(-\nabla\varphi(x'),\ 1)}{\sqrt{1 + |\nabla\varphi(x')|^2}}, \qquad \text{for every } x' \in \mathcal{O}'. \tag{3.7}$$

Proof.

$$\partial_j P(x') = (0, \dots, 1, 0, \dots, \partial_j\varphi(x')), \ \text{for } j = 1, \dots, n-1; \tag{3.8}$$

thus, by Lemma 3.2,

$$
\begin{aligned}
|\partial_1 P(x') \times \cdots \times \partial_{n-1} P(x')|\, N(x', \varphi(x')) &= \partial_1 P(x') \times \cdots \times \partial_{n-1} P(x') \\
&= det \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_1 \varphi(x') \\ 0 & 1 & \cdots & 0 & \partial_2 \varphi(x') \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-1} \varphi(x') \\ e_1 & e_2 & \cdots & e_{n-1} & e_n \end{pmatrix} \qquad (3.9) \\
&= e_1 (-1)^{n-1} (-1)^{n-2} \partial_1 \varphi(x') + \cdots + e_{n-1} (-1)^{2n-1} \partial_{n-1} \varphi(x') + e_n (-1)^{2n} \\
&= (-\nabla \varphi(x'), 1),
\end{aligned}
$$

and (3.7) now follows. □

Lemma 3.10. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a surface of class C^k. Let $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$ and let $\varphi : \mathcal{O}' \to \mathbb{R}$ be such that graph $\varphi \subseteq \mathcal{S}$, and $\varphi(\partial \mathcal{O}') \subseteq \partial \mathcal{S}$. We consider the function $\psi : \mathcal{O}'' \to \mathbb{R}$, where $\mathcal{O}'' \subseteq \mathbb{R}^{n-2}$ and ψ is such that graph $\psi \subseteq \partial \mathcal{O}'$, and the function $p : \mathcal{O}'' \to \mathbb{R}^n$, $p(x'') = (x'', \psi(x''), \varphi(x'', \psi(x'')))$ for every $x'' \in \mathcal{O}''$ is a local parametrization of $\partial \mathcal{S}$. Then a unit normal to $\partial \mathcal{S}$ that is also perpendicular to N is given by the vector*

$$\gamma(p(x'')) = -\frac{\partial_1 p(x'') \times \cdots \times \partial_{n-2} p(x'') \times N(p(x''))}{|\partial_1 p(x'') \times \cdots \times \partial_{n-2} p(x'') \times N(p(x''))|}, \qquad (3.10)$$

where, as proved in Lemma 3.9,

$$N(p(x'')) = N(x'', \psi(x''), \varphi(x'', \psi(x''))) = \frac{(-\nabla \varphi(x'', \psi(x'')), 1)}{\sqrt{1 + |\nabla \varphi(x'', \psi(x''))|^2}} \qquad (3.11)$$

is the unit normal to the surface $\mathcal{S}$ at $p(x'')$.

Moreover, the components of $\gamma(p(x''))$ are

$$
\begin{aligned}
w_j = \frac{-1}{\sqrt{1 + |\nabla \varphi(x'', \psi(x''))|^2}} \Big[& \partial_j \psi(x'') (1 + |\nabla \varphi(x'', \psi(x''))|^2) \\
& + \partial_j \varphi(x'', \psi(x'')) \partial_{n-1} \varphi(x'', \psi(x'')) \sum_{k=1}^{n-2} \partial_k \varphi(x'', \psi(x'')) \partial_k \psi(x'') \Big]
\end{aligned} \qquad (3.12)
$$

for $1 \le j \le n-2$,

$$\begin{aligned} w_{n-1} = \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big[& 1 + |\nabla\varphi(x'',\psi(x''))|^2 - (\partial_{n-1}\varphi(x'',\psi(x'')))^2 \\ & + \partial_{n-1}\varphi(x'',\psi(x'')) \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')) \Big], \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} w_n = & \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \cdot \\ & \cdot \Big[\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')) \Big]. \end{aligned} \tag{3.14}$$

Proof. Since $\partial_1 p(x''), \dots, \partial_{n-1}p(x'')$ are vectors parallel to ∂S at $p(x'')$, (3.10) is an immediate application of Remark 2.1.

Next, setting $w = (w_1, \dots, w_n) := \partial_1 p \times \cdots \times \partial_{n-2}p \times N$, we have that $\gamma = \frac{w}{|w|}$, and we compute w_j, $j \in \{1, \dots, n\}$. Since for $1 \le j \le n-2$ we have

$$\partial_j p(x'') = \Big(0, \dots, 1, 0, \dots, \partial_j\psi(x''), \partial_j\varphi(x'',\psi(x'')) + \partial_{n-1}\varphi(x'',\psi(x''))\partial_j\psi(x'')\Big), \tag{3.15}$$

we can write

$$-\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}\,(\partial_1 p(x'') \times \cdots \times \partial_{n-2}p(x'') \times N(x'')) =$$

$$= det \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_1\psi & \partial_1\varphi + \partial_{n-1}\varphi\,\partial_1\psi \\ 0 & 1 & \cdots & 0 & \partial_2\psi & \partial_2\varphi + \partial_{n-1}\varphi\,\partial_2\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-2}\psi & \partial_{n-2}\varphi + \partial_{n-1}\varphi\,\partial_{n-2}\psi \\ \partial_1\varphi & \partial_2\varphi & \cdots & \partial_{n-2}\varphi & \partial_{n-1}\varphi & -1 \\ e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} & e_n \end{pmatrix}. \tag{3.16}$$

Next, we multiply the first $(n-2)$ rows by $\partial_j\varphi$ and subtract each from the $(n-1)^{st}$ row to obtain the matrix

$$det \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_1\psi & \partial_1\varphi + \partial_{n-1}\varphi\,\partial_1\psi \\ 0 & 1 & \cdots & 0 & \partial_2\psi & \partial_2\varphi + \partial_{n-1}\varphi\,\partial_2\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-2}\psi & \partial_{n-2}\varphi + \partial_{n-1}\varphi\,\partial_{n-2}\psi \\ 0 & 0 & \cdots & 0 & A & B \\ e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} & e_n \end{pmatrix}, \tag{3.17}$$

where

$$A = \partial_{n-1}\varphi(x'', \psi(x'')) - \sum_{j=1}^{n-2} \partial_j\varphi(x'', \psi(x''))\partial_j\psi(x''), \tag{3.18}$$

and

$$B = -1 - \sum_{j=1}^{n-2} \partial_j\varphi(x'', \psi(x''))[\, \partial_j\varphi(x'', \psi(x'')) + \partial_{n-1}\varphi(x'', \psi(x''))\partial_j\psi(x'')\,]. \tag{3.19}$$

Using formula (2.11) we compute w_1 directly:

$$w_1 = \frac{(-1)^{n+1}}{\sqrt{1 + |\nabla\varphi(x'', \psi(x''))|^2}} \cdot \\ \cdot\, det \begin{pmatrix} 0 & 0 & \cdots & 0 & \partial_1\psi & \partial_1\varphi + \partial_{n-1}\varphi\,\partial_1\psi \\ 1 & 0 & \cdots & 0 & \partial_2\psi & \partial_2\varphi + \partial_{n-1}\varphi\,\partial_2\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-2}\psi & \partial_{n-2}\varphi + \partial_{n-1}\varphi\,\partial_{n-2}\psi \\ 0 & 0 & \cdots & 0 & A & B \end{pmatrix}. \tag{3.20}$$

Interchanging rows 1 and 2, then 2 and 3, and so on, $(n-3)$ times,

$$
\begin{aligned}
w_1 = {} & \frac{(-1)^{n+1}(-1)^{n-3}}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \cdot \\
& det \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_2\psi & \partial_2\varphi + \partial_{n-1}\varphi\, \partial_2\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \partial_{n-2}\psi & \partial_{n-2}\varphi + \partial_{n-1}\varphi\, \partial_{n-2}\psi \\ 0 & 0 & \cdots & 0 & \partial_1\psi & \partial_1\varphi + \partial_{n-1}\varphi\, \partial_1\psi \\ 0 & 0 & \cdots & 0 & A & B \end{pmatrix} \\
= {} & \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} det \begin{pmatrix} \partial_1\psi & \partial_1\varphi + \partial_{n-1}\varphi\partial_1\psi \\ A & B \end{pmatrix} \qquad (3.21) \\
= {} & \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big\{ -\partial_1\psi(x'') - \partial_1\psi(x'') \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x'')) \\
& \Big(\partial_k\varphi(x'',\psi(x'')) + \partial_{n-1}\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big) \\
& - \Big(\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big) \\
& \cdot \Big(\partial_1\varphi(x'',\psi(x'')) + \partial_{n-1}\varphi(x'',\psi(x''))\partial_1\psi(x'')\Big)\Big\},
\end{aligned}
$$

where for the last equality we have used (3.18) and (3.19). Then, by direct computation, (3.21) becomes

$$
\begin{aligned}
& \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Bigg\{ -\partial_1\psi(x'') - \partial_1\psi(x'') \sum_{k=1}^{n-2} \partial_k^2\varphi(x'',\psi(x'')) \\
& - \partial_1\psi(x'')\partial_{n-1}\varphi(x'',\psi(x'')) \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'') \\
& - \partial_1\varphi(x'',\psi(x''))\partial_{n-1}\varphi(x'',\psi(x'')) - \partial_{n-1}^2\varphi(x'',\psi(x''))\partial_1\psi(x'') \qquad (3.22) \\
& + \partial_1\varphi(x'',\psi(x'')) \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'') \\
& + \partial_{n-1}\varphi(x'',\psi(x''))\partial_1\psi(x'') \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'') \Bigg\} \\
= {} & \frac{-1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big[\partial_1\psi(x'')(1+|\nabla\varphi(x'',\psi(x''))|^2) \\
& + \partial_1\varphi(x'',\psi(x''))\Big(\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2} \partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big)\Big]. \qquad (3.23)
\end{aligned}
$$

A similar computation works for determining w_j, $j = 2, 3, \ldots, n-2$, and we conclude that

$$\begin{aligned} w_j = \frac{-1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big[&\partial_j\psi(x'')(1+|\nabla\varphi(x'',\psi(x''))|^2) \\ &+ \partial_j\varphi(x'',\psi(x''))\Big(\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big)\Big] \end{aligned} \tag{3.24}$$

for $1 \le j \le n-2$.

Moreover,

$$\begin{aligned} w_{n-1} &= \frac{(-1)^{n+n-1}}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} B \\ &= \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big[1 + |\nabla\varphi(x'',\psi(x''))|^2 - (\partial_{n-1}\varphi(x'',\psi(x'')))^2 \\ &\quad + \partial_{n-1}\varphi(x'',\psi(x''))\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big]. \end{aligned} \tag{3.25}$$

Finally,

$$\begin{aligned} w_n &= \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} A \\ &= \frac{1}{\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2}} \Big[\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big]. \end{aligned} \tag{3.26}$$

This completes the proof of Lemma 3.10. □

Chapter 4

Tangential Operators

We now define first-order differential operators and some of their properties, and we then focus our attention on a certain group of tangential operators which will play an important role in our subsequent calculations. Let $m, n, r \in \mathbb{N}$ and

$$Lu = \left(\sum_{\beta=1}^{m} \sum_{j=1}^{n} a_j^{\alpha\beta} \partial_j u_\beta + \sum_{\beta=1}^{m} b^{\alpha\beta} u_\beta \right)_{1 \le \alpha \le r} \tag{4.1}$$

be a first-order differential operator acting on the vector-valued function $u = (u_\beta)_{1 \le \beta \le m}$, $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The adjoint of L acting on the vector-valued function $v = (v_\alpha)_{1 \le \alpha \le r}$, $v : \mathbb{R}^n \longrightarrow \mathbb{R}^r$, is by definition

$$L^* v = \left(-\sum_{\alpha=1}^{r} \sum_{j=1}^{n} \partial_j (a_j^{\alpha\beta} v_\alpha) + \sum_{\alpha=1}^{r} b^{\alpha\beta} v_\alpha \right)_{1 \le \beta \le m} . \tag{4.2}$$

The symbol of L is the matrix-valued function

$$\sigma(L, \xi) = \left(\sum_{j=1}^{n} a_j^{\alpha\beta} \xi_j \right)_{\substack{1 \le \alpha \le r \\ 1 \le \beta \le m}} \quad \text{for } \xi = (\xi_j)_j,\ 1 \le j \le n. \tag{4.3}$$

In general if Ω is a bounded domain in $\mathbb{R}^n$, whose boundary $\partial\Omega$ is a surface of class C^k in $\mathbb{R}^n$ with outward unit normal N and $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $v : \mathbb{R}^n \longrightarrow \mathbb{R}^r$, are of class $C^1(\mathbb{R}^n)$, then the following integration by parts formula holds:

$$\int_\Omega \langle Lu(x), v(x)\rangle dx = \int_{\partial\Omega} \langle \sigma(L,N)u(x), v\rangle ds(x) + \int_\Omega \langle u(x), L^* v(x)\rangle dx. \tag{4.4}$$

A particular class of first-order operators that we are interested in studying are tangential operators to a surface $\mathcal{S}$.

Definition 4.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k with unit normal N. The operator L as in (4.1) is said to be tangential if $\sigma(L,N) = 0$ on $\mathcal{S}$.*

Proposition 4.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k. Let τ be a tangent vector to $\mathcal{S}$ at $(x', \varphi(x'))$, and $u : \mathbb{R}^n \to \mathbb{R}$ a C^1 function. If u is constant on $\mathcal{S}$, then $\nabla_\tau u = 0$ on $\mathcal{S}$.*

Proof. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $u = C$ on $\mathcal{S}$ for some constant C. Since $\nabla_\tau u = 0$ on $\mathcal{S}$ is a local phenomenon, without loss of generality we can assume that $\mathcal{S}$ is the graph of a C^k function $\varphi : \mathcal{O}' \to \mathbb{R}$, $\mathcal{O}' \subset \mathbb{R}^{n-1}$. Then $P : \mathcal{O}' \to \mathbb{R}^n$, $P(x') = (x', \varphi(x'))$, $x' \in \mathcal{O}'$ is a parametrization of $\mathcal{S}$ and if $\tau_j(x', \varphi(x')) := \partial_j P(x')$ where $\partial_j P(x') = (0, \dots, 1, 0, \dots, \partial_j \varphi(x'))$, $j = 1, \dots, n-1$, then we have that the set $\{\tau_j(x', \varphi(x')) : j = 1, \dots, n-1\}$ is a basis for the tangent plane to $\mathcal{S}$ at the point $(x', \varphi(x')) \in \mathcal{S}$. Hence, any tangent vector τ to $\mathcal{S}$ at $(x', \varphi(x'))$ is a linear combination of $\tau_j's$, and the conclusion of Proposition 4.2 follows if we prove that $\nabla_{\tau_j} u = 0$ on $\mathcal{S}$ for each $j = 1, \dots, n-1$. Fix such a j. Then

$$\begin{aligned} \nabla_{\tau_j} u(x', \varphi(x')) &= \Big\langle \tau_j(x', \varphi(x')), (\nabla u)(x', \varphi(x')) \Big\rangle \\ &= (\partial_j u)(x', \varphi(x')) + (\partial_n u)(x', \varphi(x')) \partial_j \varphi(x'). \end{aligned} \tag{4.5}$$

Since $u = C$ on $\mathcal{S}$, we have that for t in a neighborhood of 0,

$$u(x' + te_j, \varphi(x' + te_j)) = C. \tag{4.6}$$

Hence,

$$\left[\frac{d}{dt}\left[u(x' + te_j, \varphi(x' + te_j))\right]\right]_{t=0} = 0. \tag{4.7}$$

By taking the derivative in (4.7) we obtain

$$\begin{aligned} 0 &= \left[\frac{d}{dt}\left[u(x' + te_j, \varphi(x' + te_j))\right]\right]_{t=0} \\ &= \Big[(\partial_j u)(x' + te_j, \varphi(x' + te_j)) + (\partial_n u)(x' + te_j, \varphi(x' + te_j))\partial_j\varphi(x' + te_j)\Big]_{t=0} \\ &= (\partial_j u)(x', \varphi(x')) + (\partial_n u)(x', \varphi(x'))\partial_j\varphi(x'). \end{aligned} \tag{4.8}$$

The proof of Proposition 4.2 can be completed by combining (4.8) and (4.5). ☐

In what follows, a particular family of tangential operators will prove to be very useful. More precisely, if $\mathcal{S}$ is a C^k oriented surface with unit normal N, set

$$M_{jk} := N_j\partial_k - N_k\partial_j, \qquad j, k \in \{1, \dots, n\},\ j \neq k. \tag{4.9}$$

Then the following is true.

Lemma 4.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k with unit normal N, and let $u : \mathbb{R}^n \to \mathbb{R}$ be of class C^1 such that $u = C$ on $\mathcal{S}$ for some constant C. Then $M_{jk}u = 0$ on $\mathcal{S}$ for all $j, k \in \{1, \dots, n\}$.*

Proof. Using a localization argument, we can assume that $\mathcal{S}$ is the graph of a C^k function $\varphi : \mathcal{O}' \to \mathbb{R}$, $\mathcal{O}' \subset \mathbb{R}^{n-1}$. Fix $j, k \in \{1, \dots, n\},\ j \neq k$. We distinguish two cases.

Case 1. $1 \leq j, k \leq n-1$.

Making use of (3.7), for $x' \in \mathcal{O}'$ we have

$$\begin{aligned}(M_{jk}u)(x', \varphi(x')) &= \frac{-\partial_j\varphi(x')(\partial_k u)(x', \varphi(x'))}{\sqrt{1+|\nabla\varphi(x')|^2}} + \frac{\partial_k\varphi(x')(\partial_j u)(x', \varphi(x'))}{\sqrt{1+|\nabla\varphi(x')|^2}} \\ &= \frac{-\partial_j\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}\left[\frac{d}{dt}u(x'+te_k, \varphi(x'+te_k))\right]_{t=0} \\ &\quad + \frac{\partial_k\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}\left[\frac{d}{dt}u(x'+te_j, \varphi(x'+te_j))\right]_{t=0} \\ &= 0,\end{aligned} \tag{4.10}$$

with the last two equalities obtained using (4.8).

Case 2. $1 \leq j \leq n-1$, $k = n$.

Again using (3.7), for $x' \in \mathcal{O}'$ we have

$$\begin{aligned}&(M_{jn}u)(x', \varphi(x')) \\ &\quad = \frac{-\partial_j\varphi(x')(\partial_n u)(x', \varphi(x'))}{\sqrt{1+|\nabla\varphi(x')|^2}} - \frac{(\partial_j u)(x', \varphi(x'))}{\sqrt{1+|\nabla\varphi(x')|^2}} \\ &\quad = \frac{\left[\frac{d}{dt}u(x'+te_j, \varphi(x'+te_j))\right]_{t=0}}{\sqrt{1+|\nabla\varphi(x')|^2}} \\ &\quad = 0.\end{aligned} \tag{4.11}$$

This completes the proof of Lemma 4.3 □

Proposition 4.4. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k with unit normal N, $u : \mathbb{R}^n \to \mathbb{R}$ a C^1 function, and L a first-order operator as defined in (4.1). If u is constant on $\mathcal{S}$ and L is tangential with zero lower order terms (i.e. $b^{\alpha\beta} = 0$ for all $1 \leq \alpha \leq r$, $1 \leq \beta \leq m$), then $Lu = 0$ on $\mathcal{S}$.*

Proof. Using a localization argument, we can assume that $\mathcal{S}$ is the graph of a C^k function $\varphi : \mathcal{O}' \to \mathbb{R}$, $\mathcal{O}' \subset \mathbb{R}^{n-1}$. Suppose L is as in (4.1) with $b^{\alpha\beta} = 0$ for all $1 \leq \alpha \leq r$,

$1 \leq \beta \leq m$ and $\sigma(L, N) = 0$ on $\mathcal{S}$. Recall M_{jk} as in (4.9). Then

$$\partial_j = \sum_{k=1}^{n} N_k N_k \partial_j = \sum_{k=1}^{n} N_k M_{jk} + N_j \sum_{k=1}^{n} N_k \partial_k. \tag{4.12}$$

Hence, for $1 \leq \alpha \leq r,\ 1 \leq \beta \leq m$, we use (4.12) and the tangentiality of L to write

$$\begin{aligned} \sum_{j=1}^{n} a_j^{\alpha\beta} \partial_j &= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j^{\alpha\beta} N_k M_{jk} + \sum_{j=1}^{n} a_j^{\alpha\beta} N_j \sum_{k=1}^{n} N_k \partial_k \\ &= \sum_{j,k=1}^{n} a_j^{\alpha\beta} N_k M_{jk}. \end{aligned} \tag{4.13}$$

In particular, since $u = C$ on $\mathcal{S}$ for some constant C, (4.13) and Lemma 4.3 give that $Lu = 0$ on $\mathcal{S}$. ☐

Corollary 4.5. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k with unit normal N, and $u : \mathbb{R}^n \to \mathbb{R}$ a C^1 function. If u is constant on $\mathcal{S}$, then $\nabla_{tan} u = 0$ on $\mathcal{S}$, where $\nabla_{tan} u$ is as defined in (2.8).*

Proof. Let $L = \nabla_{tan}$. Then $\sigma(L, N) = N - N\langle N, N\rangle = 0$ on $\mathcal{S}$. Hence, by Proposition 4.4, $\nabla_{tan} u = 0$ on $\mathcal{S}$. ☐

Chapter 5

A Distinguished Extension of the Unit Normal

In this chapter we extend the unit normal N that is originally defined at each point of a surface $\mathcal{S} \subset \mathbb{R}^n$ to a neighborhood of $\mathcal{S}$. This extension will have certain properties that are important for us in the sequel. The extension result is listed below. The proof follows the work in [2].

Proposition 5.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, and let N be the unit normal to $\mathcal{S}$ given by this orientation. For $x \in \mathbb{R}^n$, define the function*

$$\rho(x) := \begin{cases} \mathit{dist}(x, \mathcal{S}), & x \text{ above } \mathcal{S}, \\ -\mathit{dist}(x, \mathcal{S}), & x \text{ below } \mathcal{S}. \end{cases} \tag{5.1}$$

Then the following hold:

(i) $\partial_N \rho = 1$ *on* $\mathcal{S}$;

(ii) $\nabla \rho \Big|_{\mathcal{S}} = N$;

(iii) $\nabla \left(\frac{\rho}{|\nabla \rho|} \right) = \frac{\nabla \rho}{|\nabla \rho|}$ *on* $\mathcal{S}$;

(iv) There exists $\epsilon_0 > 0$ *such that if we set* $U := \{x + tN(x) \, : \, x \in \mathcal{S}, t \in (-\epsilon_0, \epsilon_0)\}$ *and* $\nu(x) := \frac{\nabla \rho(x)}{|\nabla \rho(x)|}, x \in U$, *then* ν *is a unitary extension of* N *to the neighborhood* U *(i.e.*

$\nu\Big|_{\mathcal{S}} = N,\ |\nu| = 1$) *which satisfies*

(a) $\partial_k \nu_j - \partial_j \nu_k = 0$ *on* $\mathcal{S},\ j,\ k = 1, \dots, n;$

(b) $\partial_\nu \nu_j = 0$ *on* $\mathcal{S},\ j = 1, \dots, n;$

(c) div $\nu\Big|_{\mathcal{S}}$ *is independent of the extension.*

Proof.

(i) Since $\mathcal{S} \subset \mathbb{R}^n$ is a surface of class $C^k,\ k \geq 2$, locally it is given by the graphs of functions of class C^k. Fix $x_0 \in \mathcal{S}$. Then there exists $r > 0$ such that $B_r(x_0) \cap \mathcal{S}$ is given by the graph of $\varphi : \mathcal{O}' \to \mathbb{R}$ for some $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$. Thus $\mathcal{O}' \ni x' \to (x', \varphi(x')) \in \mathcal{S}$ is a local parametrization for $\mathcal{S}$,

$$N(x', \varphi(x')) = \frac{(-\nabla\varphi(x'), 1)}{\sqrt{1 + |\nabla\varphi(x')|^2}}, \text{ for } x' \in \mathcal{O}', \tag{5.2}$$

and $x_0 = (x_0', \varphi(x_0'))$, for some $x_0' \in \mathcal{O}' \subset \mathbb{R}^{n-1}$. Let ρ be defined as in (5.1). Clearly $\rho(x) = 0$ if and only if $x \in \mathcal{S}$. In addition,

$$\rho(x) = \inf\{|x - y| : y \in \mathcal{S}\}, \tag{5.3}$$

and if x is in a small neighborhood of x_0 then

$$\rho(x) = \inf\{|x - (y', \varphi(y'))| : y' \in \mathcal{O}'\}. \tag{5.4}$$

We will prove that there exists $\epsilon_0 > 0$ such that for every $t \in [0, \epsilon_0)$ and $x \in \mathcal{S} \cap B_r(x_0)$, $\rho(x + tN(x)) = t$. To this end, we make the following claim:

$$t^2 = \inf\{|x_0 + tN(x_0) - (y', \varphi(y'))|^2 : y' \in \mathcal{O}'\} \text{ for small } t. \tag{5.5}$$

To prove (5.5), let $F : \mathcal{O}' \to [0, \infty)$, $F(y') = |x_0 + tN(x_0) - (y', \varphi(y'))|^2$. Then

$$F(y') = \sum_{j=1}^{n-1} (x_{0_j} + tN_j(x_0) - y_j)^2 + (\varphi(x_0') + tN_n(x_0) - \varphi(y'))^2, \tag{5.6}$$

and

$$F(x_0') = |x_0 + tN(x_0) - (x_0', \varphi(x_0'))|^2 = |tN(x_0)|^2 = t^2. \tag{5.7}$$

We will show that $F(y') \geq F(x_0')$ for all $y' \in \mathcal{O}'$ if t is small. The Taylor series of F at x_0 is given by

$$F(y') = F(x_0') + \big\langle (y' - x_0'), \nabla F(x_0') \big\rangle + \frac{1}{2}(\text{Hess}F)(\xi)(y' - x_0')^2, \tag{5.8}$$

where $\xi \in [x_0', y']$ and Hess $(F) = (\partial_j \partial_k F)_{j,k}$ is the Hessian of F. Differentiating F with respect to y_k in (5.6), we obtain

$$\partial_{y_k} F(y') = -2(x_{0_k} + tN_k(x_0) - y_k) - 2(\varphi(x_0') + tN_n(x_0) - \varphi(y'))\partial_{y_k}\varphi(y'). \tag{5.9}$$

Substituting $y' = x_0'$, we have

$$(\partial_{y_k} F)(x_0') = -2tN_k(x_0) - 2tN_n(x_0)\partial_{y_k}\varphi(x_0'), \text{ for all } k = 1, \ldots, n-1. \tag{5.10}$$

Now by (5.2), the above expression becomes

$$\frac{2t}{\sqrt{1 + |\nabla\varphi(x')|^2}} [\partial_{y_k}\varphi(x_0') - \partial_{y_k}\varphi(x_0')] = 0, \tag{5.11}$$

hence,

$$\nabla F(x_0') = 0. \tag{5.12}$$

We now let $A :=$ Hess $F(\zeta)$ be the $(n-1) \times (n-1)$ matrix $(A_{jk})_{jk}$ such that $A_{jk}(\zeta) = (\partial_{y_j} \partial_{y_k} F)(\zeta)$. What is important for us is that

$$\langle Az, z \rangle \geq 0 \quad \text{for every } z \in \mathbb{R}^{n-1}. \tag{5.13}$$

To see why (5.13) holds, we differentiate (5.9) with respect to y_j to obtain

$$\partial_{y_j}\partial_{y_k}F(y') = 2\delta_{jk}+2\partial_{y_j}\varphi(y')\partial_{y_k}\varphi(y')-2(\varphi(x_0')+tN_n(x_0)-\varphi(y'))\partial_{y_j}\partial_{y_k}\varphi(y'). \tag{5.14}$$

Then

$$\begin{aligned}\sum_{j,k=1}^{n-1} A_{jk}(\zeta)z_jz_k &= 2\sum_{j,k=1}^{n-1}\delta_{jk}z_jz_k + 2\sum_{j,k=1}^{n-1}\partial_{y_j}\varphi(\zeta)\partial_{y_k}\varphi(\zeta)z_jz_k \\ &\quad - 2\sum_{j,k=1}^{n-1}(\varphi(x_0') + tN_n(x_0) - \varphi(\zeta))(\partial_{y_j}\partial_{y_k}\varphi(\zeta))z_jz_k.\end{aligned} \tag{5.15}$$

The second term in the right-hand side of (5.15) is equal to $2\langle\nabla\varphi(\zeta), z\rangle^2$. Since $\varphi \in C^k$, $k \geq 2$, gives a local parametrization, we can assume that $\overline{\mathcal{O}'}$ is compact and that $||\nabla\varphi||_{L^\infty(\overline{\mathcal{O}'})} \leq C(\varphi)$ and $||\partial_j\partial_k\varphi||_{L^\infty(\overline{\mathcal{O}'})} \leq C(\varphi)$ for $j, k = 1, \ldots, n-1$ and for some constant $C(\varphi) \geq 0$. Then, turning our attention to the third term in the right-hand side of (5.15), we have

$$2\left|\sum_{j,k=1}^{n-1}(\varphi(x_0') - \varphi(\zeta))\partial_j\partial_k\varphi(\zeta)z_jz_k\right| \leq 2(n-1)C^2(\varphi)|x_0' - \zeta||z|^2 \leq \frac{1}{4}|z|^2 \tag{5.16}$$

if $|x_0' - \zeta| < \frac{1}{8(n-1)C^2(\varphi)} =: \epsilon(\varphi)$. Also,

$$\left|\frac{2t}{\sqrt{1+|\nabla\varphi(\zeta)|^2}}\sum_{j,k=1}^{n-1}\partial_j\partial_k\varphi(\zeta)z_jz_k\right| \leq 2tC(\varphi)(n-1)|z|^2 \leq \frac{1}{4}|z|^2 \tag{5.17}$$

if $0 < t_\varphi < \frac{1}{8C(\varphi)(n-1)}$. Hence,

$$\sum_{j,k=1}^{n-1} A_{jk}(\zeta)z_jz_k \geq 2|z|^2 + 2\langle\nabla\varphi(\zeta), z\rangle^2 - \frac{1}{2}|z|^2 \geq 0 \tag{5.18}$$

if $|x_0' - \zeta| < \frac{1}{8(n-1)C^2(\varphi)} = \epsilon(\varphi)$ and $0 < t_\varphi < \frac{1}{8C(\varphi)(n-1)}$. Thus by (5.12) and (5.18), we have that

$$F(y') = F(x_0') + \big\langle(y' - x_0'), \nabla F(x_0')\big\rangle + \frac{1}{2}\text{Hess } F(\xi)(y' - x_0')^2 \geq F(x_0'), \tag{5.19}$$

for $\xi \in [x'_0, y']$ and for all y' such that $|y' - x'_0| < \epsilon(\varphi)$ and $t \in (0, t_\varphi)$.

There exist finitely many points $x'_1, \dots, x'_J \in \mathcal{O}'$ such that $\mathcal{O}' \subset \bigcup_{j=1}^{J} B_{\epsilon(\varphi)}(x'_j)$. Thus, if $x \in \mathcal{S} \cap \varphi(\mathcal{O}')$, then $x = (x', \varphi(x'))$ and there exists $j \in \{1, \dots, J\}$ such that $|x' - x'_j| < \epsilon(\varphi)$. Based on what we proved so far, $\rho(x + tN(x)) = t$ if $t \in (0, t_\varphi)$. Moreover, $\mathcal{S}$ is compact, so there exist $\varphi_1, \dots, \varphi_J$ as in Theorem 2.2. For each such φ_j, we obtain t_{φ_j} that gives $\rho(x + tN(x)) = t$. Let

$$\epsilon_0 = \min\{t_{\varphi_1}, \dots, t_{\varphi_N}\}. \tag{5.20}$$

Then

$$\rho(x + tN(x)) = t \text{ for every } t \in [0, \epsilon_0). \tag{5.21}$$

Taking the derivative of (5.21) we obtain

$$\big\langle \nabla\rho(x + tN(x)), N(x) \big\rangle = 1, \tag{5.22}$$

which yields (i) by setting $t = 0$.

(ii) By (2.8), $\nabla\rho\Big|_{\mathcal{S}} = \nabla_{tan}\rho + (\partial_N\rho)N$. Since $\rho\Big|_{\mathcal{S}} = 0$, by Corollary 4.5 $\nabla_{tan}\rho = 0$, which together with (i) implies that $\nabla\rho\Big|_{\mathcal{S}} = N$.

(iii) $\nabla\left(\frac{\rho}{|\nabla\rho|}\right) = \frac{1}{|\nabla\rho|}\nabla\rho + \rho\,\nabla(\frac{1}{|\nabla\rho|})$. Since $\rho = 0$ on $\mathcal{S}$, we conclude that $\nabla\left(\frac{\rho}{|\nabla\rho|}\right) = \frac{\nabla\rho}{|\nabla\rho|}$ on $\mathcal{S}$.

(iv) a) A direct computation gives

$$\begin{aligned} \partial_k\nu_j - \partial_j\nu_k &= \partial_k\left(\frac{\partial_j\rho}{|\nabla\rho|}\right) - \partial_j\left(\frac{\partial_k\rho}{|\nabla\rho|}\right) \\ &= \frac{1}{|\nabla\rho|}(\partial_k\partial_j\rho - \partial_j\partial_k\rho) + ((\partial_j\rho)\partial_k - (\partial_k\rho)\partial_j)\left(\frac{1}{|\nabla\rho|}\right). \end{aligned} \tag{5.23}$$

$\rho \in \mathcal{C}^2$, so by (2.4), $\partial_k\partial_j\rho - \partial_j\partial_k\rho = 0$. Using the definition of ν_j and by observing that $\frac{1}{|\nabla\rho|} = 1$ on $\mathcal{S}$, we see that on $\mathcal{S}$ $\partial_k\nu_j - \partial_j\nu_k = |\nabla\rho|(N_j\partial_k - N_k\partial_j)(1)$. Thus, by Lemma 4.3, $\partial_k\nu_j - \partial_j\nu_k = 0$ on $\mathcal{S}$. b) Using the definition of the normal derivative and the definition of ν_j we can write

$$\begin{aligned}
\partial_\nu\nu_j = \langle \nu, \nabla\nu_j\rangle &= \left\langle \nu, \nabla\left(\frac{\partial_j\rho}{|\nabla\rho|}\right)\right\rangle = \left\langle \frac{\nabla\rho}{|\nabla\rho|}, \nabla\left(\frac{\partial_j\rho}{|\nabla\rho|}\right)\right\rangle \\
&= \sum_{k=1}^{n} \frac{\partial_k\rho}{|\nabla\rho|}\partial_k\left(\frac{\partial_j\rho}{|\nabla\rho|}\right) \\
&= \sum_{k=1}^{n} \frac{\partial_k\rho}{|\nabla\rho|}\,\frac{\partial_k\partial_j\rho}{|\nabla\rho|} + \sum_{k=1}^{n}\frac{\partial_k\rho}{|\nabla\rho|}\,\partial_j\rho\,\partial_k\left(\frac{1}{|\nabla\rho|}\right).
\end{aligned} \tag{5.24}$$

By substituting ν_j for $\frac{1}{|\nabla\rho|}\partial_j\rho$ in the second term of the right-most expression in (5.24), we obtain

$$\begin{aligned}
&\frac{1}{|\nabla\rho|^2}\cdot\frac{1}{2}\partial_j\sum_{k=1}^{n}(\partial_k\rho)^2 + \sum_{k=1}^{n}\partial_k\rho(\nu_j\partial_k - \nu_k\partial_j)\left(\frac{1}{|\nabla\rho|}\right) \\
&\qquad + \sum_{k=1}^{n}(\partial_k\rho)\nu_k\partial_j\left(\frac{1}{|\nabla\rho|}\right) \\
&= \frac{1}{|\nabla\rho|^2}\cdot\frac{1}{2}\partial_j\left(|\nabla\rho|^2\right) + \sum_{k=1}^{n}\partial_k\rho\left(\frac{\partial_k\rho}{|\nabla\rho|}\right)\partial_j\left(\frac{1}{|\nabla\rho|}\right) \text{ on } \mathcal{S},
\end{aligned} \tag{5.25}$$

using the fact that $|\nabla\rho| = 1$ on $\mathcal{S}$ and applying Lemma 4.3 to $\nu_j\partial_k - \nu_k\partial_j$ on $\mathcal{S}$. By direct computation, (5.25) becomes

$$\begin{aligned}
&\frac{1}{|\nabla\rho|^2}\cdot\frac{1}{2}\partial_j(\partial_1^2\rho + \cdots + \partial_n^2\rho) + \frac{|\nabla\rho|^2}{|\nabla\rho|}\frac{(-1)}{|\nabla\rho|^2}\cdot\frac{1}{2}(|\nabla\rho|^2)^{-\frac{1}{2}}\cdot 2\partial_j\rho \\
&= \frac{1}{|\nabla\rho|^2}\partial_j\rho - \frac{\partial_j\rho}{|\nabla\rho|^2} = 0 \text{ on } \mathcal{S}.
\end{aligned} \tag{5.26}$$

c) Direct computation yields

$$\begin{aligned}
\operatorname{div}\nu = \sum_{j=1}^{n}\partial_j\nu_j &= \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k\nu_k\partial_j\nu_j \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k[\nu_k\partial_j - \nu_j\partial_k]\nu_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\nu_k\nu_j(\partial_k\nu_j).
\end{aligned} \tag{5.27}$$

Remark. Set $U := \{x + tN(x) : x \in \mathcal{S}, t \in (\epsilon_0, \epsilon_0)\}$ where ϵ_0 is as defined in (5.20). ν is extended in a neighborhood U of $\mathcal{S}$ such that $|\nu| = 1$ in U, which implies that

$$\sum_{j=1}^{n} \nu_j \partial_{x_k} \nu_j = \frac{1}{2}\partial_{x_k}\left(\sum_{j=1}^{n} \nu_j^2\right) = \frac{1}{2}\partial_{x_k}|\nu|^2 = \frac{1}{2}\partial_{x_k}(1) = 0, \tag{5.28}$$

for $j, k = 1, \dots, n$. Hence,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \nu_k \nu_j (\partial_{x_k}\nu_j) = 0 \text{ in } U. \tag{5.29}$$

By (5.29), (5.27) becomes

$$\text{div}\, \nu = \sum_{j=1}^{n}\sum_{k=1}^{n} (\nu_k)[\nu_k\partial_j - \nu_j\partial_k](\nu_j), \tag{5.30}$$

which shows that div $\nu\Big|_{\mathcal{S}} = \sum_{j=1}^{n}\sum_{k=1}^{n} N_k M_{kj} N_j$ is independent of the extension. This completes the proof of Proposition 5.1.

□

Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k with unit normal N and recall the first-order tangential differential operators

$$M_{jk} = N_j\partial_k - N_k\partial_j, \qquad 1 \leq j, k \leq n \tag{5.31}$$

as defined in (4.9). If ν is as defined in Proposition 5.1, then each operator M_{jk} extends accordingly by setting $M_{jk} = \nu_j\partial_k - \nu_k\partial_j$. We now define a stronger notion of tangentiality for operators.

Definition 5.2. *Let $\mathcal{S} \in \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν as in Proposition 5.1. The operator L as in (4.1) is said to be strongly tangential if $\sigma(L, \nu) = 0$.*

Next we define the Gauss Curvature for a surface $\mathcal{S}$. Let $\mathcal{S}, U, \nu$ be as in Proposition 5.1. For a point $x \in \mathcal{S}$ there exists $P : \mathcal{O}' \to \mathcal{S}$ local parametrization of $\mathcal{S}$ in a neighborhood of $P(x') = x$. From $\langle \nu(P(x')), \nu(P(x')) \rangle = 1$ we have that for all $j = 1, \dots, n-1$, $\frac{\partial}{\partial x_j} \langle \nu(P(x')), \nu(P(x')) \rangle = 0$. Hence, we observe that $\langle \frac{\partial}{\partial x_j}[\nu(P(x'))], \nu(P(x')) \rangle = 0$, which implies that $\frac{\partial}{\partial x_j}[\nu(P(x'))]$ is tangential to $\mathcal{S}$ at $P(x')$ for all $1 \le j \le n-1$. Thus there exist real numbers b_{jk}, $1 \le j, k, \le n-1$, such that

$$\frac{\partial}{\partial x_j}[\nu(P(x'))] = \sum_{k=1}^{n-1} b_{jk}[\partial_{x_k} P(x')] \text{ for all } 1 \le j \le n-1. \tag{5.32}$$

We now define the matrix

$$B := (b_{jk})_{1 \le j,k \le n-1} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n-1} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n-1,1} & \cdots & \cdots & b_{n-1,n-1} \end{pmatrix} \tag{5.33}$$

and the Gauss Curvature

$$\mathcal{G} := \text{ Trace } B = \sum_{j=1}^{n-1} b_{jj}. \tag{5.34}$$

Theorem 5.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \ge 2$, ν as in Proposition 5.1, and $\mathcal{G}$ as in (5.34). Then div $\nu = \mathcal{G}$ on $\mathcal{S}$.*

Proof. Without loss of generality we can assume that $\mathcal{S}$ is the graph of a C^k function $\varphi : \mathcal{O}' \to \mathbb{R}$, $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$. We start by observing that for $1 \le j \le n-1$ and $x' \in \mathcal{O}'$,

$$b_{jj} = \partial_{x_j} \left[\frac{-\partial_{x_j} \varphi(x')}{\sqrt{1 + |\nabla \varphi(x')|^2}} \right]. \tag{5.35}$$

Indeed the left hand side of (5.32) becomes

$$\partial_{x_j} \nu(P(x')) = \left(\partial_{x_j} \left[\frac{-\partial_{x_1} \varphi(x')}{\sqrt{1 + |\nabla \varphi(x')|^2}} \right], \dots, \partial_{x_j} \left[\frac{1}{\sqrt{1 + |\nabla \varphi(x')|^2}} \right] \right). \tag{5.36}$$

Substituting (3.8) into the right hand side of (5.32), we obtain

$$\begin{aligned}\sum_{k=1}^{n-1} b_{jk}(0,\dots,1,0,\dots,\partial_{x_k}\varphi(x')) &= \sum_{k=1}^{n-1}(0,\dots,b_{jk},0,\dots,b_{jk}\partial_{x_k}\varphi(x')) \\ &= \left(b_{j1},b_{j2},\dots,b_{j\,n-1},\sum_{k=1}^{n-1} b_{jk}\partial_{x_k}\varphi(x')\right).\end{aligned} \tag{5.37}$$

This proves (5.35). For $1 \leq j,k \leq n$, we again recall the tangential operators M_{jk} as defined in (4.9) and extended for ν as defined in Proposition 5.1. Then by (5.30),

$$(\text{div}\ \nu)(P(x')) = \sum_{j=1}^{n}\sum_{k=1}^{n} \nu_k M_{kj}\nu_j(P(x')) \text{ for } x' \in \mathcal{O}'. \tag{5.38}$$

If $j = k = n$, then $\nu_n M_{nn}\nu_n = 0$. Thus,

$$\begin{aligned}(\text{div}\ \nu)(P(x')) &= \sum_{j=1}^{n}\sum_{k=1}^{n} \nu_k M_{kj}\nu_j(P(x')) \\ &= \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \nu_k M_{kj}\nu_j(P(x')) + \sum_{j=1}^{n-1} \nu_n M_{nj}\nu_j(P(x')) \\ &\qquad + \sum_{k=1}^{n-1} \nu_k M_{kn}\nu_n(P(x')) \\ &= \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \nu_k \Bigg[\frac{-\partial_{x_k}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_j}[\nu_j(P(x'))] \\ &\qquad + \frac{\partial_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_k}[\nu_j(P(x'))]\Bigg] \\ &\qquad + \sum_{j=1}^{n-1} \nu_n \frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_j}[\nu_j(P(x'))] \\ &\qquad - \sum_{k=1}^{n-1} \nu_k \frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_k}[\nu_n(P(x'))].\end{aligned} \tag{5.39}$$

By substituting in the appropriate expression for ν, (5.39) becomes

$$\begin{aligned}
&\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_j}\left[\frac{-\partial_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}\right] \\
&\quad - \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_k}\left[\frac{-\partial_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}\right] \\
&\quad + \sum_{j=1}^{n-1} \left(\frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}}\right)^2 \cdot \frac{\partial}{\partial x_j}\left[\frac{-\partial_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}\right] \\
&\quad + \sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}} \cdot \frac{\partial}{\partial x_k}\left(\frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}}\right).
\end{aligned} \tag{5.40}$$

By direct computation, (5.40) becomes

$$\begin{aligned}
&\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{(\partial_{x_k}\varphi(x'))^2}{(\sqrt{1+|\nabla\varphi(x')|^2})^4} \Big[-\partial^2_{x_j}\varphi(x') \cdot \sqrt{1+|\nabla\varphi(x')|^2} \\
&\qquad + \partial_{x_j}\varphi(x')\partial_{x_j}\left(\sqrt{1+|\nabla\varphi(x')|^2}\right)\Big] \\
&\quad - \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x') \cdot \partial_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^4} \Big[-\partial_{x_k}\partial_{x_j}\varphi(x') \cdot \sqrt{1+|\nabla\varphi(x')|^2} \\
&\qquad + \partial_{x_j}\varphi(x')\partial_{x_k}\left(\sqrt{1+|\nabla\varphi(x')|^2}\right)\Big] \\
&\quad + \sum_{j=1}^{n-1} \frac{1}{(\sqrt{1+|\nabla\varphi(x')|^2})^4} \Big[-\partial^2_{x_j}\varphi(x') \cdot \sqrt{1+|\nabla\varphi(x')|^2} \\
&\qquad + \partial_{x_j}\varphi(x')\partial_{x_j}\left(\sqrt{1+|\nabla\varphi(x')|^2}\right)\Big] \\
&\quad - \sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^4} \cdot \partial_{x_k}\left(\sqrt{1+|\nabla\varphi(x')|^2}\right) \\
&= -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{(\partial_{x_k}\varphi(x'))^2}{(\sqrt{1+|\nabla\varphi(x')|^2})^3} \cdot \partial^2_{x_j}\varphi(x') \\
&\quad + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')\partial_{x_j}\varphi(x')\partial_{x_k}\partial_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^3} \\
&\quad - \sum_{j=1}^{n-1} \frac{\partial^2_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}
\end{aligned}$$

$$\begin{aligned} &= -\sum_{j=1}^{n-1} \frac{\partial^2_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}\left[\sum_{k=1}^{n-1}(\partial_{x_k}\varphi(x'))^2+1\right] \\ &\quad + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')\partial_{x_j}\varphi(x')\partial_{x_k}\left(\partial_{x_j}\varphi(x')\right)}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}. \end{aligned} \tag{5.41}$$

Hence,

$$\begin{aligned} (\operatorname{div}\nu)(P(x')) &= -\sum_{j=1}^{n-1} \frac{\partial^2_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}\left(|\varphi(x')|^2+1\right) \\ &\quad + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')\partial_{x_j}\varphi(x')\partial_{x_k}\left(\partial_{x_j}\varphi(x')\right)}{(\sqrt{1+|\nabla\varphi(x')|^2})^3} \\ &= -\sum_{j=1}^{n-1} \frac{\partial^2_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \\ &\quad + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_k}\varphi(x')\partial_{x_j}\varphi(x')\partial_{x_k}\left(\partial_{x_j}\varphi(x')\right)}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}. \end{aligned} \tag{5.42}$$

Next we use the quotient rule in formula (5.35) to obtain

$$\begin{aligned} \sum_{j=1}^{n-1} b_{jj} &= \sum_{j=1}^{n-1} \frac{-\partial^2_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} \\ &\quad + \sum_{j=1}^{n-1} \frac{\partial_{x_j}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^2}\cdot\partial_{x_j}(\sqrt{1+|\nabla\varphi(x')|^2}). \end{aligned} \tag{5.43}$$

To complete the proof we observe that

$$\partial_{x_j}\left[1+\sum_{k=1}^{n-1}(\partial_{x_k}\varphi(x'))^2\right] = 2\sum_{k=1}^{n-1}\partial_{x_k}\varphi(x')\partial_{x_j}\partial_{x_k}\varphi(x'). \tag{5.44}$$

Then, by (5.44), equation(5.43) becomes

$$\sum_{k=1}^{n-1} b_{jj} = \sum_{j=1}^{n-1} \frac{-\partial^2_{x_j}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial_{x_j}\varphi(x')\partial_{x_k}\varphi(x')\partial_{x_j}\partial_{x_k}\varphi(x')}{(\sqrt{1+|\nabla\varphi(x')|^2})^3}. \tag{5.45}$$

By comparing our definition of $\mathcal{G}$ in formula (5.34) to our formula for div ν in equation (5.42), we conclude that div $\nu = \mathcal{G}$. □

Lemma 5.4. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, and let M be as in (5.31), ν as in Proposition 5.1, and $\mathcal{G}$ as in Theorem 5.3. The following formulas hold:*

(i) $M_{jk} = -M_{kj}$, *for all* $1 \leq j,\ k \leq n$;

(ii) $\partial_k = \sum\limits_{j=1}^{n} \nu_j M_{jk} + \nu_k \partial_\nu$, *for all* $1 \leq k \leq n$;

(iii) $\sum\limits_{k=1}^{n} M_{jk}\nu_k = \nu_j \mathcal{G}$, *for all* $1 \leq j \leq n$.

Proof.

(i) $M_{jk} = \nu_j\partial_k - \nu_k\partial_j = -\nu_k\partial_j + \nu_j\partial_k = -M_{kj}$.

(ii)

$$\sum_{j=1}^{n} \nu_j M_{jk} + \nu_k\partial_\nu = \sum_{j=1}^{n}(\nu_j^2\partial_k - \nu_j\nu_k\partial_j) + \nu_k\sum_{j=1}^{n}\nu_j\partial_j = \partial_k\sum_{j=1}^{n}\nu_j^2 = \partial_k.$$

(iii)

$$\begin{aligned}\sum_{k=1}^{n} M_{jk}\nu_k &= \sum_{k=1}^{n}(\nu_j\partial_k - \nu_k\partial_j)\nu_k = \sum_{k=1}^{n}\left[\nu_j(\partial_k\nu_k) - \nu_k(\partial_j\nu_k)\right]\\ &= \nu_j \text{div } \nu - \frac{1}{2}\partial_j(|\nu|)^2 = \nu_j\mathcal{G},\end{aligned}$$

with the last equality obtained using (5.28) and Theorem 5.3. □

Proposition 5.5. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, and U, ν as given in Proposition 5.1. Then for the $n \times n$ matrix-valued function*

$$R(x) := \nabla\nu(x) = (\partial_k\nu_j(x))_{j,k},\ x \in U,\ \textit{for all } j,k = 1,\ldots,n, \tag{5.46}$$

the following hold true:

(i) $R\nu = 0$ *in* U;

(ii) $Tr(R) = \mathcal{G}$ *in U, where* $\mathcal{G}$ *is the Gauss Curvature for* $\mathcal{S}$.

Moreover, when restricted to the surface $\mathcal{S}$, R *has the following additional properties:*

(iii) R *depends only on* $\mathcal{S}$ *and not on the choice of* ν*;*

(iv) $R^T = R$ *on* $\mathcal{S}$*;*

(v) $(Ru)\,|_{\mathcal{S}}$ *is tangential to* $\mathcal{S}$ *for any vector field* $u : \mathcal{S} \to \mathbb{R}^n$.

Proof.

(i) $R\nu = \nabla|\nu|^2 = 0$ in U.

(ii) $\text{Tr}(R) = \text{Tr}(\partial_k \nu_j(x))_{j,k} = \text{div } \nu = \mathcal{G}$ by Theorem 5.3.

(iii) Using (iv) b in Proposition 5.1, $\nabla_\nu \nu\Big|_{\mathcal{S}} = 0$. Also, $\nu\Big|_{\mathcal{S}} = N$ by Proposition 5.1, and $\nabla\nu\big|_{\mathcal{S}} = \nabla_{tan}\nu + (\nabla_\nu \nu)\nu$. Together these conditions imply that $\nabla\nu\Big|_{\mathcal{S}}$ depends only on the surface $\mathcal{S}$ and not on the extension ν. Since $R(x) = \nabla\nu(x)$, (iii) follows.

(iv) By (iv) a of Proposition 5.1, $\partial_k \nu_j - \partial_j \nu_k = 0$ on $\mathcal{S}$ for $j, k = 1, \ldots, n$, which implies that $R = R^T$ on $\mathcal{S}$.

(v) On $\mathcal{S}$, we have that $\langle \nu, Ru \rangle = \langle \nu, (\nabla\nu)u \rangle = \nu \cdot \nabla\nu \cdot u = 0$, since $\nu \cdot \nabla\nu = 0$ on $\mathcal{S}$ by Proposition 5.1. This completes the proof of Proposition 5.5. ☐

Chapter 6

Integration by Parts on Surfaces

A version of the integration by parts formula (4.4) can be obtained when one replaces Ω by a surface $\mathcal{S}$ in $\mathbb{R}^n$. However, the drawback of such a result is the fact that the adjoint appearing in the right-hand side of (4.4) will have to be computed with respect to the surface $\mathcal{S}$ and not $\mathbb{R}^n$. This is a difficult task and it would be desirable to have an integral formula that involves the adjoint of the operator considered when acting in $\mathbb{R}^n$, and not just on the surface. Theorem 6.2 addresses this issue. To prove Theorem 6.2 we need the following lemma.

Lemma 6.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with unit normal N, extended unit normal ν as in Proposition 5.1, and γ the unit normal to $\partial\mathcal{S}$ as defined in (3.10). If $f, g \in C^1(\mathcal{S})$, then for every $1 \leq j < k \leq n$,*

$$\int_{\mathcal{S}} (M_{jk}f) g \, dS = -\int_{\mathcal{S}} f(M_{jk}g) \, dS + \int_{\partial\mathcal{S}} (N_j\gamma_k - N_k\gamma_j) fg \, ds. \tag{6.1}$$

Proof. It is enough to show that

$$\int_{\mathcal{S}} M_{jk}f \, dS = \int_{\partial\mathcal{S}} (N_j\gamma_k - N_k\gamma_j) f \, ds. \tag{6.2}$$

Let $x' = (x_1, \dots, x_{n-1}) = (x'', \psi(x'')) \in \mathcal{O}' \subset \mathbb{R}^{n-1}$, where ψ is as defined in Lemma 3.10. We now distinguish several cases. *Case 1.* $1 \leq j, k \leq n-2$.

We use Definition 3.3, so that the left-hand side of (6.2) becomes

$$\begin{aligned}
&\int_S (\nu_j \partial_k - \nu_k \partial_j) f \, dS \\
&= \int_{\mathcal{O}'} \frac{-1}{\sqrt{1 + |\nabla\varphi(x')|^2}} [\partial_j \varphi(x')(\partial_k f)(x', \varphi(x')) \\
&\qquad - \partial_k \varphi(x')(\partial_j f)(x', \varphi(x'))] \sqrt{1 + |\nabla\varphi(x')|^2} \, dx' \\
&= -\int_{\mathcal{O}'} \Big\{ \partial_j \varphi(x') \Big[(\partial_k f)(x', \varphi(x')) + (\partial_n f)(x', \varphi(x')) \partial_k \varphi(x') \Big] \\
&\qquad - \partial_k \varphi(x') \Big[(\partial_j f)(x', \varphi(x')) + (\partial_n f)(x', \varphi(x')) \partial_j \varphi(x') \Big] \Big\} dx' \\
&= -\int_{\mathcal{O}'} \Big(\partial_{x_k} [\partial_j \varphi(x') f(x', \varphi(x'))] - \partial_{x_j} [\partial_k \varphi(x') f(x', \varphi(x'))] \Big) dx'.
\end{aligned} \tag{6.3}$$

If we let $(N'_1, \dots, N'_{n-1})$ be the unit normal to $\partial\mathcal{O}'$ in $\mathbb{R}^{n-1}$ as defined by the parametrization ψ based on Lemma 3.9, integration by parts yields for (6.3)

$$\begin{aligned}
&-\int_{\partial\mathcal{O}'} [N'_k(x', \varphi(x')) \partial_j \varphi(x') - N'_j(x', \varphi(x')) \partial_k \varphi(x')] f(x', \varphi(x')) ds \\
&= \int_{\mathcal{O}''} \left[\frac{\partial_k \psi(x'')}{\sqrt{1 + |\nabla\psi(x'')|^2}} \partial_j \varphi(x'', \psi(x'')) - \frac{\partial_j \psi(x'')}{\sqrt{1 + |\nabla''\psi(x'')|^2}} \partial_k \varphi(x'', \psi(x'')) \right] \\
&\qquad \cdot f(x'', \psi(x''), \varphi(x'', \psi(x''))) \sqrt{1 + |\nabla\psi(x'')|^2} dx'' \\
&= \int_{\mathcal{O}''} \Big[\partial_k \psi(x'') \partial_j \varphi(x'', \psi(x'')) - \partial_j \psi(x'') \partial_k \varphi(x'', \psi(x'')) \Big] \\
&\qquad \cdot f(x'', \psi(x''), \varphi(x'', \psi(x''))) dx''.
\end{aligned} \tag{6.4}$$

On the right-hand side of (6.2) we apply first Definition 3.6, then we substitute the appropriate expression for γ as given in Lemma 3.10 to obtain

$$\begin{aligned}
&\int_{\partial S}(N_j\gamma_k - N_k\gamma_j)f\,ds \\
&= \int_{\mathcal{O}''}\Big\{\frac{\partial_j\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big[\partial_k\psi(x'')[1+|\nabla\varphi(x'',\psi(x''))|^2] \\
&\quad + \partial_k\varphi(x'',\psi(x''))[\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')]\Big] \\
&\quad - \frac{\partial_k\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big[\partial_j\psi(x'')[1+|\nabla\varphi(x'',\psi(x''))|^2] \\
&\quad + \partial_j\varphi(x'',\psi(x''))[\partial_{n-1}\varphi(x'',\psi(x'')) - \sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')]\Big]\Big\} \\
&\qquad \cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\,dx'' \\
&= \int_{\mathcal{O}''}\Big[\partial_j\varphi(x'',\psi(x''))\partial_k\psi(x'') - \partial_k\varphi(x'',\psi(x''))\partial_j\psi(x'')\Big] \\
&\qquad \cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\,dx''.
\end{aligned} \tag{6.5}$$

Thus (6.2) is proved for $1 \le j, k \le n-2$. *Case 2.* $1 \le j \le n-2,\ k = n-1$.

We again use Definition 3.3 so that the left-hand side of (6.2) becomes

$$\begin{aligned}
&\int_S(\nu_j\partial_{n-1} - \nu_{n-1}\partial_j)f(x',\varphi(x'))\,dS \\
&= -\int_{\mathcal{O}'}\Big[\frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}}(\partial_j\varphi(x')(\partial_{n-1}f)(x',\varphi(x')) \\
&\qquad - \partial_{n-1}\varphi(x')(\partial_j f)(x',\varphi(x'))\Big]\cdot\sqrt{1+|\nabla\varphi(x')|^2}\,dx' \\
&= \int_{\mathcal{O}'}\Big(-\partial_{x_{n-1}}[\partial_j\varphi(x')f(x',\varphi(x'))] + \partial_{x_j}[\partial_{n-1}\varphi(x')f(x',\varphi(x'))]\Big)dx'.
\end{aligned} \tag{6.6}$$

Integrating by parts, (6.6) becomes

$$\begin{aligned}
&\int_{\partial\mathcal{O}'}[-N'_{n-1}\partial_j\varphi(x') + N'_j\partial_{n-1}\varphi(x')]f(x',\varphi(x'))ds \\
&= \int_{\mathcal{O}''}\left[\frac{-1}{\sqrt{1+|\nabla\psi(x'')|^2}}(\partial_j\varphi)(x'',\psi(x'')) - \frac{\partial_j\psi(x'')}{\sqrt{1+|\nabla\psi(x'')|^2}}(\partial_{n-1}\varphi)(x'',\psi(x''))\right] \\
&\qquad \cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\sqrt{1+|\nabla''\psi(x'')|^2}\,dx'' \\
&= -\int_{\mathcal{O}''}\Big[(\partial_j\varphi)(x'',\psi(x'')) + \partial_j\psi(x'')(\partial_{n-1}\varphi)(x'',\psi(x''))\Big]\cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned} \tag{6.7}$$

We again apply both Definition 3.6 and Lemma 3.10 and observe that the right-hand side of (6.2) becomes

$$
\begin{aligned}
&\int_{\partial S}(N_j\gamma_{n-1}-N_{n-1}\gamma_j)f(x',\varphi(x'))ds\\
&=\int_{\mathcal{O}''}\Big\{-\frac{\partial_j\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big[1+|\nabla\varphi(x'',\psi(x''))|^2\\
&\quad-(\partial_{n-1}\varphi(x'',\psi(x'')))^2+\partial_{n-1}\varphi(x'',\psi(x''))\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big]\\
&\quad-\frac{\partial_{n-1}\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big[\partial_j\psi(x'')[1+|\nabla\varphi(x'',\psi(x''))|^2]\\
&\quad+\partial_j\varphi(x'',\psi(x''))[\partial_{n-1}\varphi(x'',\psi(x''))-\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')]\Big]\Big\}\\
&\qquad\cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''\\
&=-\int_{\mathcal{O}''}[\partial_j\varphi(x'',\psi(x''))+\partial_{n-1}\varphi(x'',\psi(x''))\partial_j\psi(x'')]\\
&\qquad\cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned}
\tag{6.8}
$$

Case 3. $1\leq j\leq n-2,\ k=n.$

Using Definition 3.3 and integration by parts, the left-hand side of (6.2) becomes

$$
\begin{aligned}
&\int_S(\nu_j\partial_n-\nu_n\partial_j)f(x',\varphi(x'))dS=\\
&=\int_{\mathcal{O}'}\frac{-1}{\sqrt{1+|\nabla\varphi(x')|^2}}[\partial_j\varphi(x')(\partial_nf)(x',\varphi(x'))\\
&\qquad+(\partial_jf)(x',\varphi(x'))]\cdot\sqrt{1+|\nabla\varphi(x')|^2}\,dx'\\
&=-\int_{\mathcal{O}'}\partial_{x_j}[f(x',\varphi(x'))]dx'=-\int_{\partial\mathcal{O}'}N_j'f(x',\varphi(x'))ds\\
&=\int_{\mathcal{O}''}\frac{\partial_j\psi(x'')}{\sqrt{1+|\nabla\psi(x'')|^2}}\cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\sqrt{1+|\nabla\psi(x'')|^2}\,dx''\\
&=\int_{\mathcal{O}''}\partial_j\psi(x'')f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned}
\tag{6.9}
$$

By Definition 3.6 and Lemma 3.10, the right-hand side of (6.2) becomes

$$\begin{aligned}
&\int_{\partial S}(N_j\gamma_n - N_n\gamma_j)f(x',\varphi(x'))ds\\
&=\int_{O''}\Big\{\frac{-\partial_j\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\cdot\\
&\qquad\cdot\Big[\partial_{n-1}\varphi(x'',\psi(x''))-\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big]\\
&\quad+\frac{1}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big[\partial_j\psi(x'')[1+|\nabla\varphi(x'',\psi(x''))|^2]\\
&\quad+\partial_j\varphi(x'',\psi(x''))(\partial_{n-1}\varphi(x'',\psi(x''))\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x''))\Big]\\
&\qquad\cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\Big\}dx''\\
&=\int_{O''}\partial_j\psi(x'')f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned}\tag{6.10}$$

Case 4. $j = k = n-1$.

$$\int_S(\nu_{n-1}\partial_{n-1}-\nu_{n-1}\partial_{n-1})f(x',\varphi(x'))dS = 0,\tag{6.11}$$

and

$$\int_{\partial S}(N_{n-1}\gamma_{n-1}-N_{n-1}\gamma_{n-1})f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx'' = 0.\tag{6.12}$$

Case 5. $j = n-1,\ k = n$.

Using Definition 3.3 and integrating by parts, the left-hand side of (6.2) becomes

$$
\begin{aligned}
&\int_S (\nu_{n-1}\partial_n - \nu_n\partial_{n-1}) f(x',\varphi(x'))dS \\
&= \int_{\mathcal{O}'} \Big[\frac{-\partial_{n-1}\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}(\partial_n f)(x',\varphi(x')) \\
&\qquad + \frac{-1}{\sqrt{1+|\nabla\varphi(x')|^2}}(\partial_{n-1} f)(x',\varphi(x'))\Big]\sqrt{1+|\nabla\varphi(x')|^2}dx' \\
&= -\int_{\mathcal{O}'} \partial_{x_{n-1}}\big[f(x',\varphi(x'))\big]dx' \qquad\qquad (6.13)\\
&= -\int_{\partial\mathcal{O}'} N'_{n-1} f(x'',\psi(x''),\varphi(x'',\psi(x'')))ds \\
&= \int_{\mathcal{O}''} \frac{1}{\sqrt{1+|\nabla\psi(x'')|^2}} f(x'',\psi(x''),\varphi(x'',\psi(x'')))\sqrt{1+|\nabla\psi(x'')|^2}\,dx'' \\
&= \int_{\mathcal{O}''} f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned}
$$

By Definition 3.6 and Lemma 3.10, the right-hand side of (6.2) becomes

$$
\begin{aligned}
&\int_{\partial S} (N_{n-1}\gamma_n - N_n\gamma_{n-1}) f(x',\varphi(x'))ds \\
&= \int_{\mathcal{O}''} \Big[\frac{-\partial_{n-1}\varphi(x'',\psi(x''))}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big(\partial_{n-1}\varphi(x'',\psi(x'')) \\
&\qquad - \sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x'')\Big) \\
&\quad - \frac{1}{(\sqrt{1+|\nabla\varphi(x'',\psi(x''))|^2})^2}\Big(1+|\nabla\varphi(x'',\psi(x''))|^2 \\
&\qquad - (\partial_{n-1}\varphi(x'',\psi(x'')))^2 + (\partial_{n-1}\varphi(x'',\psi(x'')))\sum_{k=1}^{n-2}\partial_k\varphi(x'',\psi(x''))\partial_k\psi(x''))\Big) \\
&\qquad\quad \cdot f(x'',\psi(x''),\varphi(x'',\psi(x'')))\Big]dx'' \\
&= \int_{\mathcal{O}''} f(x'',\psi(x''),\varphi(x'',\psi(x'')))dx''.
\end{aligned}
\tag{6.14}
$$

Case 6. $j = k = n$.

$$\int_S (\nu_n\partial_n - \nu_n\partial_n) f dS = 0, \tag{6.15}$$

and

$$\int_{\partial S} (N_n\gamma_n - N_n\gamma_n) f ds = 0. \tag{6.16}$$

In each case we have proved equation (6.2). Since the cases exhaust all possibilities, this completes the proof of Lemma 6.1. ☐

Theorem 6.2. *Let $S \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with unit normal N. Let L be as in equation (4.1) with coefficients of class C^1 in $\mathbb{R}^n$. Let ν be an extension of N, as in Proposition 5.1. If $\sigma(L, N) = 0$, then L extends uniquely to an operator (still denoted by L) which acts on C^1 vector-valued functions defined on S such that $(Lu)\Big|_S = L\left(u\Big|_S\right)$ for every $u \in C^1(U)$. Furthermore, similar considerations apply to L^*, and for all $u, v \in C^1(S)$ and $1 \leq j, k \leq n$, $1 \leq \alpha \leq r$, $1 \leq \beta \leq m$,*

$$\int_S \langle Lu, v\rangle \, dS = \int_S \langle u, L^*v\rangle \, dS + \sum_{j,\alpha,\beta} \int_S (\partial_N a_j^{\alpha\beta}) N_j u_\beta v_\alpha \, dS + \int_{\partial S} \langle \sigma(L, \gamma)u, v\rangle \, ds. \tag{6.17}$$

Moreover, if $\sigma(L, \nu) = 0$ in U, then for all $u, v \in C^1(S)$,

$$\int_S \langle Lu, v\rangle \, dS = \int_S \langle u, L^*v\rangle \, dS + \int_{\partial S} \langle \sigma(L, \gamma)u, v\rangle \, ds. \tag{6.18}$$

Proof. Suppose $\sigma(L, N) = 0$. By Lemma 5.4, on S we have

$$\begin{aligned}(Lu)_\alpha &= -\sum_{j=1}^{n}\sum_{\beta} a_j^{\alpha\beta}\left(\sum_{k=1}^{n} N_k M_{jk} + N_j\partial_N\right) u_\beta + \sum_\beta b^{\alpha\beta} u_\beta \\ &= -\sum_{j,\beta,k} a_j^{\alpha\beta} N_k M_{jk} u_\beta - \sum_{j,\beta} a_j^{\alpha\beta} N_j \partial_N u_\beta + \sum_\beta b^{\alpha\beta} u_\beta.\end{aligned} \tag{6.19}$$

The second term of the rightmost expression in (6.19) contains $\sigma(L, N) = 0$, so we have

$$(Lu)_\alpha = -\sum_{j,\beta,k} a_j^{\alpha\beta} N_k M_{jk} u_\beta + \sum_\beta b^{\alpha\beta} u_\beta \qquad \text{on } S. \tag{6.20}$$

We now let ν be as defined in Proposition 5.1. Then

$$\int_{\mathcal{S}} \langle Lu, v\rangle \, dS = -\sum_{j,k}\sum_{\alpha,\beta}\int_{\mathcal{S}} (M_{jk}u_\beta)(a_j^{\alpha\beta}\nu_k v_\alpha)\, dS + \sum_{\alpha,\beta}\int_{\mathcal{S}} b^{\alpha\beta}u_\beta v_\alpha \, dS. \tag{6.21}$$

By Lemma 6.1, the right-hand side of (6.21) becomes

$$\begin{aligned} &\sum_{j,k}\sum_{\alpha,\beta}\int_{\mathcal{S}} u_\beta \left[M_{jk}\left(a_j^{\alpha\beta}\nu_k v_\alpha\right)\right] dS + \sum_{\alpha\beta}\int_{\mathcal{S}} b^{\alpha\beta}u_\beta v_\alpha \, dS \\ &\qquad -\sum_{j,k}\sum_{\alpha,\beta}\int_{\partial\mathcal{S}} (\nu_j\gamma_k - \nu_k\gamma_j)u_\beta a_j^{\alpha\beta}\nu_k v_\alpha \, ds. \end{aligned} \tag{6.22}$$

For the term under the boundary integral we write

$$\begin{aligned} &-\sum_{j,k}\sum_{\alpha,\beta}\left(\nu_j\gamma_k a_j^{\alpha\beta}\nu_k u_\beta v_\alpha - \nu_k\gamma_j a_j^{\alpha\beta}\nu_k u_\beta v_\alpha\right) \\ &\qquad\qquad = -\langle\gamma,\nu\rangle\langle\sigma(L,\nu)u, v\rangle + |\nu|^2\langle\sigma(L,\gamma)u, v\rangle \\ &\qquad\qquad = \langle\sigma(L,\gamma)u, v\rangle, \end{aligned} \tag{6.23}$$

since $\langle\gamma,\nu\rangle = 0$ on $\mathcal{S}$. The term under the first surface integral in (6.22) can be expanded to obtain

$$\begin{aligned} &\sum_{j,k}\sum_{\alpha,\beta} u_\beta\left[M_{jk}\left(a_j^{\alpha\beta}\nu_k v_\alpha\right)\right] \\ &\qquad = \sum_{j,k}\sum_{\alpha,\beta} u_\beta a_j^{\alpha\beta} v_\alpha M_{jk}\nu_k + \sum_{j,k}\sum_{\alpha,\beta} u_\beta \nu_k M_{jk}\left(a_j^{\alpha\beta} v_\alpha\right) \\ &\qquad =: I + II. \end{aligned} \tag{6.24}$$

Then, by Lemma 5.4, we observe that

$$I = \sum_{j,\alpha,\beta} u_\beta a_j^{\alpha\beta} v_\alpha \nu_j \mathcal{G} = \langle\sigma(L,\nu)u, v\rangle\,\mathcal{G} = 0 \text{ on } \mathcal{S}. \tag{6.25}$$

Next, we again apply (ii) in Lemma 5.4 to obtain

$$\begin{aligned}
II + \sum_{\alpha,\beta} b^{\alpha\beta} u_\beta v_\alpha &= - \sum_{j,\alpha,\beta} u_\beta \partial_j \left(a_j^{\alpha\beta} v_\alpha \right) + \sum_{j,\alpha,\beta} u_\beta \nu_j \partial_\nu \left(a_j^{\alpha\beta} v_\alpha \right) + \sum_{\alpha,\beta} b^{\alpha\beta} u_\beta v_\alpha \\
&= \langle u, L^* v \rangle + \sum_{j,\alpha,\beta} \left[u_\beta \nu_j \left(\partial_\nu a_j^{\alpha\beta} \right) v_\alpha + u_\beta \nu_j a_j^{\alpha\beta} \partial_\nu v_\alpha \right] \\
&= \langle u, L^* v \rangle + \sum_{j,\alpha,\beta} \left[u_\beta \partial_\nu \left(\nu_j a_j^{\alpha\beta} \right) v_\alpha - u_\beta \left(\partial_\nu \nu_j \right) a_j^{\alpha\beta} v_\alpha \right] \\
&\quad + \langle \sigma(L,\nu) u, \nabla_\nu v \rangle \\
&= \langle u, L^* v \rangle + \langle [\partial_\nu \sigma(L,\nu)]\, u, v \rangle - \langle \sigma(L, \nabla_\nu \nu) u, v \rangle \\
&\quad + \langle \sigma(L,\nu) u, \nabla_\nu v \rangle .
\end{aligned} \tag{6.26}$$

But on $\mathcal{S}$, $\sigma(L,\nu) = 0$, and

$$\nabla_\nu \nu = \left(\partial_\nu \nu_j \right)_{j=1}^n = 0 \tag{6.27}$$

by Proposition 5.1. Hence,

$$II + \sum_{\alpha,\beta} b^{\alpha\beta} u_\beta v_\alpha = \langle u, L^* v \rangle + \langle [\partial_\nu \sigma(L,\nu)]\, u, v \rangle \quad \text{on } \mathcal{S}. \tag{6.28}$$

Furthermore,

$$\begin{aligned}
\langle [\partial_\nu \sigma(L,\nu)]\, u, v \rangle &= \sum_{j,\alpha,\beta} u_\beta \left(\partial_\nu a_j^{\alpha\beta} \right) \nu_j v_\alpha + \sum_{j,\alpha,\beta} u_\beta a_j^{\alpha\beta} (\partial_\nu \nu_j) v_\alpha \\
&= \sum_{j,\alpha,\beta} u_\beta \left(\partial_N a_j^{\alpha\beta} \right) N_j v_\alpha \quad \text{on } \mathcal{S},
\end{aligned} \tag{6.29}$$

since $\partial_\nu(\nu_j) = 0$ on $\mathcal{S}$. Hence, for every $u, v \in C^1(\mathcal{S})$,

$$\begin{aligned}
&\int_S \langle Lu, v \rangle \, dS \\
&\quad = \int_S \langle u, L^* v \rangle \, dS + \sum_{j,\alpha,\beta} \int_S \left(\partial_N a_j^{\alpha\beta} \right) N_j u_\beta v_\alpha \, dS + \int_{\partial S} \langle \sigma(L,\gamma) u, v \rangle \, ds.
\end{aligned} \tag{6.30}$$

When L is strongly tangential, $\partial_\nu \sigma(L,\nu) = 0$ by definition, and the theorem is proved. □

Chapter 7

Identities on Surfaces

We begin this section by introducing some new operators which will be useful for us in the sequel. Let

$$\mathcal{D}_j := \partial_j - \nu_j \partial_\nu = \partial_j - \sum_{k=1}^{n} \nu_j \nu_k \partial_k, \text{ for } j = 1, 2, \ldots, n. \tag{7.1}$$

Proposition 7.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν as in Proposition 5.1, and γ the unit normal to $\partial\mathcal{S}$ as defined in (3.10). The following formulas hold:*

(i) $\mathcal{D}_j = \sum_{k=1}^{n} \nu_k M_{kj}$ *for all* $1 \leq j \leq n$;

(ii) $M_{jk} = \nu_j \mathcal{D}_k - \nu_k \mathcal{D}_j$ *for all* $1 \leq j, k \leq n$;

(iii) $\sum_{j=1}^{n} \nu_j \mathcal{D}_j = 0$ *and* $\sum_{j=1}^{n} \mathcal{D}_j \nu_j = \mathcal{G}$ *on* $\mathcal{S}$ *for all* $1 \leq j \leq n$;

(iv)

$$[\mathcal{D}_j, \mathcal{D}_k] = \nu_j \langle \nabla \nu_k, \nabla \rangle - \nu_k \langle \nabla \nu_j, \nabla \rangle \text{ on } \mathcal{S}, \text{ for all } 1 \leq j, k \leq n, \tag{7.2}$$

where $[\mathcal{D}_j, \mathcal{D}_k]$ is the commutator of $\mathcal{D}_j$ and $\mathcal{D}_k$, that is,

$$[\mathcal{D}_j, \mathcal{D}_k] = \mathcal{D}_j \mathcal{D}_k - \mathcal{D}_k \mathcal{D}_j; \tag{7.3}$$

(v) For every $f, g \in \mathcal{C}^1(\mathcal{S})$, and for every $1 \leq j \leq n$,

$$\int_{\mathcal{S}} (\mathcal{D}_j f) g \, dS = \int_{\mathcal{S}} (-f(\mathcal{D}_j g) + \nu_j \mathcal{G} f g) \, dS + \int_{\partial \mathcal{S}} \gamma_j f g \, ds. \tag{7.4}$$

Proof.

(i) Let $j \in \{1, \dots, n\}$. Then,

$$\sum_{k=1}^{n} \nu_k M_{kj} = \sum_{k=1}^{n} \nu_k (\nu_k \partial_j - \nu_j \partial_k) = \partial_j - \nu_j \sum_{k=1}^{n} \nu_k \partial_k = \mathcal{D}_j. \tag{7.5}$$

(ii) Fix $1 \leq j, k \leq n$. Then,

$$\begin{aligned} \nu_j \mathcal{D}_k - \nu_k \mathcal{D}_j &= \nu_j (\partial_k - \nu_k \partial_\nu) - \nu_k (\partial_j - \nu_j \partial_\nu) \\ &= \nu_j \partial_k - \nu_j \nu_k \partial_\nu - \nu_k \partial_j + \nu_j \nu_k \partial_\nu = M_{jk}. \end{aligned} \tag{7.6}$$

(iii) Let $j \in \{1, \dots, n\}$. Then,

$$\sum_{j=1}^{n} \nu_j \mathcal{D}_j = \sum_{j=1}^{n} \nu_j (\partial_j - \nu_j \partial_\nu) = \sum_{j=1}^{n} \nu_j \partial_j - \partial_\nu = 0. \tag{7.7}$$

Also, on $\mathcal{S}$,

$$\begin{aligned} \sum_{j=1}^{n} \mathcal{D}_j \nu_j &= \sum_{j=1}^{n} (\partial_j - \nu_j \partial_\nu) \nu_j \\ &= \sum_{j=1}^{n} \partial_j \nu_j - \sum_{j=1}^{n} \nu_j \partial_\nu \nu_j = \sum_{j=1}^{n} \partial_j \nu_j = \mathcal{G}, \end{aligned} \tag{7.8}$$

where for the last two equalities we used (iv)b in Proposition 5.1 and Theorem 5.3, respectively.

(iv) Observe that on $\mathcal{S}$, for each $j, k \in \{1, \dots, n\}$,

$$\begin{aligned}
\mathcal{D}_j\mathcal{D}_k &= (\partial_j - \nu_j\partial_\nu)(\partial_k - \nu_k\partial_\nu) \\
&= \partial_j\partial_k - \partial_j(\nu_k\partial_\nu) - \nu_j\partial_\nu\partial_k + \nu_j\partial_\nu(\nu_k\partial_\nu) \\
&= \partial_j\partial_k - (\partial_j\nu_k)\partial_\nu - \nu_k(\partial_j\partial_\nu) - \nu_j\partial_\nu\partial_k + \nu_j\nu_k\partial_\nu\partial_\nu + \nu_j(\partial_\nu\nu_k)\partial_\nu \\
&= \partial_j\partial_k - (\partial_j\nu_k)\partial_\nu - \sum_{l=1}^{n}[\nu_k\nu_l(\partial_j\partial_l) + \nu_k(\partial_j\nu_l)\partial_l + \nu_j\nu_l\partial_l\partial_k] + \nu_j\nu_k\partial_\nu^2,
\end{aligned} \tag{7.9}$$

where in the last equality we again used (iv)b in Proposition 5.1. We observe that the expression

$$\partial_j\partial_k - (\partial_j\nu_k)\partial_\nu - \sum_{l=1}^{n}[\nu_k\nu_l\partial_j\partial_l + \nu_j\nu_l\partial_l\partial_k] + \nu_j\nu_k\partial_\nu^2$$

is symmetric in j and k. Hence,

$$\begin{aligned}
[\mathcal{D}_j, \mathcal{D}_k] &= -\sum_{l=1}^{n}\nu_k(\partial_j\nu_l)\partial_l + \sum_{l=1}^{n}\nu_j(\partial_k\nu_l)\partial_l \\
&= -\nu_k\sum_{l=1}^{n}(\partial_l\nu_j)\partial_l + \nu_j\sum_{l=1}^{n}(\partial_l\nu_k)\partial_l,
\end{aligned} \tag{7.10}$$

where in the last equality we used (iv)a in Proposition 5.1. Directly computing (7.10) yields (7.2).

(v) $\mathcal{D}_j$ is a first order differential operator defined in a neighborhood U of $\mathcal{S} \subset \mathbb{R}^n$, which has symbol $\sigma(\mathcal{D}_j, \nu) = \nu_j - \nu_j\sum_{i=1}^{n}\nu_i\nu_i = 0$. Hence, by Definition 5.2, $\mathcal{D}_j$ is strongly tangential to $\mathcal{S}$. Also, $\sigma(\mathcal{D}_j, \gamma) = \gamma_j$. In order to apply Theorem 6.2 we need to compute $\mathcal{D}_j^*$. We have

$$
\begin{aligned}
(\mathcal{D}_j)^* &= \left(\partial_j - \nu_j \sum_{k=1}^{n} \nu_k \partial_k\right)^* \\
&= -\partial_j + \sum_{k=1}^{n} [\partial_k(\nu_k \nu_j) + \nu_j \nu_k \partial_k] \\
&= -\partial_j + \sum_{k=1}^{n} [\nu_j \partial_k \nu_k + \nu_k \partial_k \nu_j + \nu_j \nu_k \partial_k] \\
&= -\partial_j + \nu_j \mathcal{G} + \langle \nu, \nabla \nu_j \rangle + \sum_{k=1}^{n} \nu_j \nu_k \partial_k \\
&= -\mathcal{D}_j + \nu_j \mathcal{G} + \partial_\nu \nu_j,
\end{aligned}
\tag{7.11}
$$

and when restricted to $\mathcal{S}$ the last term in (7.11) vanishes. Now (7.4) follows from the above and Theorem 6.2. □

Definition 7.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν as in Proposition 5.1. For any smooth, real-valued function f on $\mathcal{S}$ and any smooth tangential field $u = (u_1, \dots, u_n)$ on $\mathcal{S}$, define*

(i) $div_{\mathcal{S}} u := \sum_{j=1}^{n} \mathcal{D}_j u_j$ for all $1 \leq j \leq n$;

(ii) $\nabla_{\mathcal{S}} f := (\mathcal{D}_1 f, \mathcal{D}_2 f, \dots, \mathcal{D}_n f)$;

(iii) $\nabla^2_{\mathcal{S}} f := (\mathcal{D}_j \mathcal{D}_k f)_{j,k}$, for all $1 \leq j, k \leq n$;

(iv) $\Delta_{\mathcal{S}} f := div_{\mathcal{S}} \nabla_{\mathcal{S}} f = \sum_{j=1}^{n} \mathcal{D}_j^2 f$ for all $1 \leq j \leq n$.

Proposition 7.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν as in Proposition 5.1, and γ the unit normal to $\partial\mathcal{S}$ as defined in (3.10). Let*

$f \in C^2(\mathcal{S})$. If u is such that $\Delta_{\mathcal{S}} u = f$ on $\mathcal{S}$, then

$$\begin{aligned} \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 dS = &\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS + \int_{\mathcal{S}} \langle R \nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u \rangle \mathcal{G}\, dS \\ &- 2 \int_{\mathcal{S}} |R \nabla_{\mathcal{S}} u|^2 dS - \sum_{j=1}^{n} \int_{\partial \mathcal{S}} (\mathcal{D}_j u) \partial_\gamma (\mathcal{D}_j u)\, ds \\ &+ \sum_{k=1}^{n} \int_{\partial \mathcal{S}} (\partial_\gamma u)(\mathcal{D}_k \mathcal{D}_k u)\, ds - \sum_{j,k=1}^{n} \int_{\partial \mathcal{S}} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k] u) \gamma_k\, ds. \end{aligned} \tag{7.12}$$

Proof. To simplify the writing, in what follows we will use the repeated indices summation convention. For example, $\sum_{j=1}^{n} \mathcal{D}_j u \mathcal{D}_j u$ will simply be written as $\mathcal{D}_j u \mathcal{D}_j u$ with the understanding that since j is repeated, we sum over j. By formula (7.4), we have

$$\begin{aligned} \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 dS &= \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_j u)(\mathcal{D}_k \mathcal{D}_k u)\, dS \\ &= - \int_{\mathcal{S}} \mathcal{D}_j u (\mathcal{D}_j \mathcal{D}_k \mathcal{D}_k u)\, dS + \int_{\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_k \mathcal{D}_k u) \nu_j \mathcal{G}\, dS \\ &\quad + \int_{\partial \mathcal{S}} (\gamma_j \mathcal{D}_j u)(\mathcal{D}_k \mathcal{D}_k u)\, ds \\ &= - \int_{\mathcal{S}} \mathcal{D}_j u (\mathcal{D}_j \mathcal{D}_k \mathcal{D}_k u)\, dS + \int_{\partial \mathcal{S}} (\gamma_j \mathcal{D}_j u)(\mathcal{D}_k \mathcal{D}_k u)\, ds \\ &=: I + II, \end{aligned} \tag{7.13}$$

where in the second to last equality we have used (iii) in Proposition 7.1 By formula (7.3), the integral I becomes

$$\begin{aligned} I &= - \int_{\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_j \mathcal{D}_k \mathcal{D}_k u) dS \\ &= - \int_{\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_k \mathcal{D}_j \mathcal{D}_k u) dS - \int_{\mathcal{S}} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k] \mathcal{D}_k u) dS. \end{aligned} \tag{7.14}$$

Integrating the first term of (7.14) by parts and again using formula (7.3), we further obtain

$$\begin{aligned}
I &= \int_{S} (\mathcal{D}_k\mathcal{D}_j u)(\mathcal{D}_j\mathcal{D}_k u)\, dS - \int_{S} (\mathcal{D}_j u)(\mathcal{D}_j\mathcal{D}_k u)\nu_k \mathcal{G}\, dS \\
&\quad - \int_{\partial S} (\mathcal{D}_j u)(\mathcal{D}_j\mathcal{D}_k u)\gamma_k\, ds - \int_{S} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]\mathcal{D}_k u)\, dS \\
&= \int_{S} ([\mathcal{D}_k, \mathcal{D}_j]u)(\mathcal{D}_j\mathcal{D}_k u)\, dS + \int_{S} (\mathcal{D}_j\mathcal{D}_k u)^2\, dS \\
&\quad - \int_{S} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]u)\nu_k\mathcal{G}\, dS - \int_{S} (\mathcal{D}_j u)(\mathcal{D}_k\mathcal{D}_j u)\nu_k\mathcal{G}\, dS \\
&\quad - \int_{\partial S} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]u)\gamma_k\, ds - \int_{\partial S} (\mathcal{D}_j u)(\mathcal{D}_k\mathcal{D}_j u)\gamma_k\, ds \\
&\quad - \int_{S} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]\mathcal{D}_k u)\, dS \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned} \tag{7.15}$$

By (iii) in Proposition 7.1 we have that $I_4 = 0$. Also, using Proposition 7.1 (iv) we can write

$$\begin{aligned}
I_7 &= -\int_{S} (\mathcal{D}_j u)\nu_j\big(\langle\nabla\nu_k, \nabla\rangle\mathcal{D}_k u\big)\, dS + \int_{S} (\mathcal{D}_j u)\nu_k\big(\langle\nabla\nu_j, \nabla\rangle\mathcal{D}_k u\big)\, dS \\
&= \int_{S} (\mathcal{D}_j u)\big(\langle\nabla\nu_j, \nabla\rangle(\nu_k\mathcal{D}_k u)\big)\, dS - \int_{S} (\mathcal{D}_j u)(\mathcal{D}_k u)\langle\nabla\nu_j, \nabla\rangle\nu_k\, dS \\
&= -\int_{S} (\mathcal{D}_j u)(\mathcal{D}_k u)\langle\nabla\nu_j, \nabla\nu_k\rangle\, dS \\
&= -\int_{S} |R\nabla_{\mathcal{S}} u|^2\, dS,
\end{aligned} \tag{7.16}$$

where for the last two equalities again we used (iii) in Proposition 7.1. Next we analyze I_1.

$$\begin{aligned}
I_1 &= \int_{S} (\mathcal{D}_j\mathcal{D}_k u)\nu_k\langle\nabla\nu_j, \nabla u\rangle\, dS - \int_{S} (\mathcal{D}_j\mathcal{D}_k u)\nu_j\langle\nabla\nu_k, \nabla u\rangle\, dS \\
&= \int_{S} ([\mathcal{D}_j, \mathcal{D}_k]u)\nu_k\langle\nabla\nu_j, \nabla u\rangle\, dS + \int_{S} (\mathcal{D}_k\mathcal{D}_j u)\nu_k\langle\nabla\nu_j, \nabla u\rangle\, dS \\
&= \int_{S} \nu_j\langle\nabla\nu_k, \nabla u\rangle\nu_k\langle\nabla\nu_j, \nabla u\rangle\, dS - \int_{S} \nu_k\langle\nabla\nu_j, \nabla u\rangle\nu_k\langle\nabla\nu_j, \nabla u\rangle\, dS \\
&= -\int_{S} |\langle\nabla\nu_j, \nabla u\rangle|^2\, dS \\
&= -\int_{S} |R\nabla_{\mathcal{S}} u|^2 dS.
\end{aligned} \tag{7.17}$$

For the first equality in (7.17) we used (7.2), for the second the fact that $\nu_j \mathcal{D}_j = 0$ on $\mathcal{S}$, for the third we used again (7.2) and that $\nu_k \mathcal{D}_k = 0$ on $\mathcal{S}$, for the fourth equality we used that

$$\begin{aligned} \nu_j \langle \nabla \nu_j, \nabla u \rangle &= \nu_j (\partial_l \nu_j)(\partial_l u) \\ &= \frac{1}{2} \partial_l \left(\sum_{j=1}^{n} \nu_j^2 \right) \partial_l u = 0, \end{aligned} \tag{7.18}$$

and the last equality is immediate. We continue using (7.2) and $\nu_j \mathcal{D}_j = 0$ on $\mathcal{S}$ to see that

$$\begin{aligned} I_3 &= - \int_{\mathcal{S}} (\mathcal{D}_j u) \nu_j \langle \nabla \nu_k, \nabla u \rangle \nu_k \mathcal{G} \, dS + \int_{\mathcal{S}} (\mathcal{D}_j u) \nu_k \langle \nabla \nu_j, \nabla u \rangle \nu_k \mathcal{G} \, dS \\ &= \int_{\mathcal{S}} \langle R \nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u \rangle \mathcal{G} \, dS. \end{aligned} \tag{7.19}$$

Combining (7.15), (7.16), (7.17), and (7.19), (7.12) follows. This completes the proof of Proposition 7.3. ☐

We now define new operators whose properties will be particularly useful in proving three corollaries to Proposition 7.3. For each $j = 1, \ldots, n$, let

$$t_j := \mathcal{D}_j - \gamma_j \partial_\gamma. \tag{7.20}$$

Computing the symbol of t_j with respect to ν and to γ, we see that

$$\sigma(t_j, \nu) = \nu_j - \nu_j \langle \nu, \nu \rangle - \gamma_j \langle \gamma, \nu \rangle = 0, \tag{7.21}$$

and

$$\sigma(t_j, \gamma) = \gamma_j - \nu_j \langle \gamma, \nu \rangle - \gamma_j \langle \gamma \, \gamma \rangle = 0. \tag{7.22}$$

Hence, t_j is a first order differential operator tangent to $\partial \mathcal{S}$ for each $j = 1, \ldots, n$.

Corollary 7.4. *If* $u\mid_{\partial\mathcal{S}} = 0$, *then*

$$\begin{aligned}\int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 dS = \sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS + \int_{\mathcal{S}} \langle R\nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u\rangle \mathcal{G}\, dS \\ - 2\int_{\mathcal{S}} |R\nabla_{\mathcal{S}} u|^2 dS + \int_{\partial\mathcal{S}} (\partial_\gamma u)^2 \mathcal{G}_{\partial\mathcal{S}}\, ds,\end{aligned} \tag{7.23}$$

where

$$\mathcal{G}_{\partial\mathcal{S}} = \sum_{j=1}^{n} t_j \gamma_j. \tag{7.24}$$

Proof. Since

$$\sigma([\mathcal{D}_j, \mathcal{D}_k], \gamma) = [\gamma_j - \nu_j\langle\gamma,\nu\rangle][\gamma_k - \nu_k\langle\gamma,\nu\rangle] - [\gamma_k - \nu_k\langle\gamma,\nu\rangle][\gamma_j - \nu_j\langle\gamma,\nu\rangle] = 0,$$

we observe that $[\mathcal{D}_j, \mathcal{D}_k]$ is a first order operator tangential to $\partial\mathcal{S}$. Hence, we have $[\mathcal{D}_j, \mathcal{D}_k]u = 0$ on $\partial\mathcal{S}$ and

$$-\int_{\partial\mathcal{S}} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]u)\gamma_k\, ds = 0 \tag{7.25}$$

Noting that

$$\gamma_j t_j = \gamma_j \mathcal{D}_j - \partial_\gamma = \gamma_j \partial_j - \gamma_j \nu_j \partial_\nu - \partial_\gamma = 0, \tag{7.26}$$

in light of (7.25) and (7.20) the remaining boundary terms in (7.12) become

$$\begin{aligned}&\int_{\partial\mathcal{S}} (\partial_\gamma u)(\mathcal{D}_j \mathcal{D}_j u) ds - \int_{\partial\mathcal{S}} (\mathcal{D}_j u)\partial_\gamma(\mathcal{D}_j u) ds \\ &= \int_{\partial\mathcal{S}} [(\partial_\gamma u)(t_j + \gamma_j\partial_\gamma)(t_j u + \gamma_j \partial_\gamma u) - \gamma_j(\partial_\gamma u)\partial_\gamma(t_j u + \gamma_j\partial_\gamma u)]\, ds \\ &= \int_{\partial\mathcal{S}} (\partial_\gamma u) t_j(\gamma_j \partial_\gamma u) ds \\ &= \int_{\partial\mathcal{S}} (\partial_\gamma u)[\gamma_j t_j(\partial_\gamma u) + (\partial_\gamma u)(t_j\gamma_j)] ds \\ &= \int_{\partial\mathcal{S}} (\partial_\gamma u)^2 \mathcal{G}_{\partial\mathcal{S}} ds.\end{aligned} \tag{7.27}$$

For the second equality in (7.27) we used the fact that $t_j(t_j u) = 0$ on $\partial\mathcal{S}$, while for the last one we used (7.26). Combining (7.12) and (7.27) we conclude that for u satisfying $\Delta_{\mathcal{S}} u = f$ in $\mathcal{S}$ and $u\big|_{\partial\mathcal{S}} = 0$, the formula (7.23) holds true. □

Corollary 7.5. *If* $\partial_\gamma u = 0$ *on* $\partial\mathcal{S}$*, then*

$$\begin{aligned}\int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 d\mathcal{S} = \sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j\mathcal{D}_k u)^2 d\mathcal{S} + \int_{\mathcal{S}} \langle R\nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u\rangle \mathcal{G}\, d\mathcal{S}\\ - 2\int_{\mathcal{S}} |R\nabla_{\mathcal{S}} u|^2 d\mathcal{S} + \sum_{j,k=1}^{n} \int_{\partial\mathcal{S}} (t_j u)(t_j\gamma_k)(t_k u) ds.\end{aligned} \tag{7.28}$$

Proof. Again we make use of the repeated indices summation convention and we start with (7.12). We only need to analyze the boundary terms in (7.12), the second of which is zero if $\partial_\gamma u = 0$ on $\partial\mathcal{S}$. Hence the remaining terms are

$$\begin{aligned}&-\int_{\partial\mathcal{S}} (\mathcal{D}_j u)\partial_\gamma(\mathcal{D}_j u) ds - \int_{\partial\mathcal{S}} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k]u)\gamma_k\, ds\\ &\qquad = -\int_{\partial\mathcal{S}} (\mathcal{D}_j u)\big(\mathcal{D}_k\mathcal{D}_j u\big)\gamma_k\, ds - \int_{\partial\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_j\mathcal{D}_k u - \mathcal{D}_k\mathcal{D}_j u)\gamma_k ds\\ &\qquad = -\int_{\partial\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_j\mathcal{D}_k u)\gamma_k\, ds\\ &\qquad = -\int_{\partial\mathcal{S}} (\mathcal{D}_j u)\mathcal{D}_j(\partial_\gamma u) ds + \int_{\partial\mathcal{S}} (\mathcal{D}_j u)(\mathcal{D}_j\gamma_k)(\mathcal{D}_k u) ds\\ &\qquad = -\int_{\partial\mathcal{S}} (t_j u + \partial_\gamma u)\big(t_j(\partial_\gamma u) + \gamma_j\partial_\gamma(\partial_\gamma u)\big)\, ds\\ &\qquad\quad + \int_{\partial\mathcal{S}} (t_j u + \partial_\gamma u)[t_j\gamma_k + \gamma_j(\partial_\gamma\gamma_k)](t_k u + \partial_\gamma u)\, ds\\ &\qquad = -\int_{\partial\mathcal{S}} (t_j u)\gamma_j\partial_\gamma(\partial_\gamma u) ds + \int_{\partial\mathcal{S}} (t_j u)[t_j\gamma_k + \gamma_j(\partial_\gamma\gamma_k)](t_k u) ds\\ &\qquad = \int_{\partial\mathcal{S}} (t_j u)(t_j\gamma_k)(t_k u) ds.\end{aligned} \tag{7.29}$$

For the first equality in (7.29) we used formula (7.3) and also the fact that $\gamma_k\mathcal{D}_k = \gamma_k(\partial_k - \nu_k\partial_\nu) = \partial_\gamma - \langle\gamma, \nu\rangle\partial_\nu = \partial_\gamma$. For the third equality we made use of $(\mathcal{D}_j\mathcal{D}_k u)\gamma_k = \mathcal{D}_j(\gamma_k\mathcal{D}_k u) - \mathcal{D}_j(\gamma_k)\mathcal{D}_k u$. Hence, for a function u that satisfies $\Delta_{\mathcal{S}} u = f$ in $\mathcal{S}$ and $\partial_\gamma u = 0$ on $\partial\mathcal{S}$, formula (7.28) holds true. □

Corollary 7.6. *Let* $\partial\mathcal{S} = \Gamma_1 \cup \Gamma_2$ *where* Γ_1 *and* Γ_2 *are connected. If* $u\big|_{\Gamma_1} = 0$ *and* $\partial_\gamma u = 0$ *on* Γ_2*, then*

$$\int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 dS = \sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS + \int_{\mathcal{S}} \langle R\nabla_{\mathcal{S}} u, \nabla_{\mathcal{S}} u\rangle \mathcal{G}\, dS - 2\int_{\mathcal{S}} |R\nabla_{\mathcal{S}} u|^2 dS$$

$$+ \int_{\Gamma_1} (\partial_\gamma u)^2 \mathcal{G}_{\partial\mathcal{S}}\, ds + \sum_{j,k=1}^{n} \int_{\Gamma_2} (t_j u)(t_j \gamma_k)(t_k u) ds. \tag{7.30}$$

Proof. Again we make use of the repeated indices summation convention. Combining the proofs of Corollaries 7.4 and 7.5, we see that if we consider the boundary terms of (7.12), we have

$$-\int_{\Gamma_1} (\mathcal{D}_j u)([\mathcal{D}_j, \mathcal{D}_k u])\gamma_k\, ds = 0. \tag{7.31}$$

and the remaining boundary terms on Γ_1 follow the calculations of (7.27) to become

$$\int_{\Gamma_1} (\partial_\gamma u)^2 \mathcal{G}_{\partial\mathcal{S}}\, ds. \tag{7.32}$$

The boundary integrals over Γ_2 follow (7.29) to become

$$\int_{\Gamma_2} (t_j u)(t)j\gamma_k)(t_k u) ds. \tag{7.33}$$

We combine (7.12), (7.32), and (7.33) to complete the proof of Corollary 7.6. □

We close this section with a simplification of the proof of Theorem 3.1.1.1 in [5]. The lemma below can be used to replace the main step in the proof of the aforementioned theorem in the setting when $\mathcal{S}$ is the C^2 boundary of an open bounded subset of $\mathbb{R}^n$. We would like to point out that the formalism developed so far allows us to work globally on $\mathcal{S}$. This is in contrast with the approach in [5] where the computation is done in local coordinates.

Lemma 7.7. *Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν and such that $\partial\mathcal{S} = \varnothing$. Then for $u, v \in C^1(\mathbb{R}^n)$,*

$$\begin{aligned}&\int_{\mathcal{S}} [\langle u, \nu\rangle\, \mathit{div}\, u - \langle\langle u, \nabla\rangle\, u, \nu\rangle] dS \\ &\quad = \int_{\mathcal{S}} [\langle -2u_{\tan}, \nabla_{\tan}\langle \nu, u\rangle\rangle + \langle R u_{\tan}, u_{\tan}\rangle + \mathcal{G}\,\langle u, \nu\rangle^2] dS\end{aligned} \tag{7.34}$$

where $u_{tan} = u - \langle u, \nu\rangle\nu$.

Proof. Consider the left-hand side of equation (7.34):

$$\langle u, \nu\rangle \operatorname{div} u = \langle u, (\operatorname{div} u)\nu\rangle\,, \tag{7.35}$$

and

$$\langle\langle u, \nabla\rangle\, u, \nu\rangle = \sum_{j,k=1}^{n} u_j(\partial_j u_k)\nu_k = \left\langle u, \left(\sum_{k=1}^{n}(\partial_j u_k)\nu_k\right)_{j=1}^{n}\right\rangle. \tag{7.36}$$

We therefore define the operator L as follows:

$$Lu := (\operatorname{div} u)\nu - \left(\sum_{k=1}^{n}(\partial_j u_k)\nu_k\right)_{j=1}^{n}. \tag{7.37}$$

Then

$$\sigma(L, \nu)u = \langle \nu, u\rangle\, \nu - \langle u, \nu\rangle\, \nu = 0. \tag{7.38}$$

Hence, by Definition 5.2, L is a strongly tangential operator. To determine L^*, the adjoint of L in $\mathbb{R}^n$, let $u, v \in C_o^\infty(\mathbb{R}^n)$. Then integration by parts implies

$$\begin{aligned}\int_{\mathbb{R}^n} \langle Lu, v\rangle &= \int_{\mathbb{R}^n} \left[\langle(\operatorname{div} u)\nu, v\rangle - \sum_{j,k=1}^{n} v_j(\partial_j u_k)\nu_k\right] \\ &= \int_{\mathbb{R}^n} \left[\langle -\nabla\,\langle \nu, v\rangle\,, u\rangle + \sum_{j,k=1}^{n} u_k\partial_j(v_j\nu_k)\right] \\ &= \int_{\mathbb{R}^n} \left[\langle -\nabla\,\langle \nu, v\rangle\,, u\rangle + \left\langle u, \left(\sum_{j=1}^{n}(\partial_j v_j)\nu_k + \sum_{j=1}^{n} v_j\partial_j\nu_k\right)_{k=1}^{n}\right\rangle\right] \\ &= \int_{\mathbb{R}^n} [\langle -\nabla\,\langle \nu, v\rangle\,, u\rangle + \langle u, (\operatorname{div} v)\nu\rangle + \langle u, Rv\rangle]\,.\end{aligned} \tag{7.39}$$

Since $\nabla\langle\nu, v\rangle = \nabla_{\tan}\langle\nu, v\rangle + [\partial_\nu\langle\nu, v\rangle]\nu$, we can write the adjoint of L as

$$L^*v = -\nabla_{\tan}\langle\nu, v\rangle - \partial_\nu\langle\nu, v\rangle\nu + Rv + (\text{div } v)\nu. \tag{7.40}$$

Now, on $\mathcal{S}$,

$$\begin{aligned} \text{div}_{\mathcal{S}} v &= \text{div}_{\mathcal{S}} v_{tan} + \text{div}_{\mathcal{S}}[\langle v, \nu\rangle\nu] \\ &= \text{div}_{\mathcal{S}} v_{tan} + \sum_{j=1}^{n} \nu_j \mathcal{D}_j(\langle v, \nu\rangle) + \langle v, \nu\rangle \sum_{j=1}^{n} \mathcal{D}_j \nu_j \\ &= \text{div}_{\mathcal{S}} v_{tan} + \langle v, \nu\rangle \mathcal{G}, \end{aligned} \tag{7.41}$$

using (iii) in Proposition 7.1 for the last equality. On the other hand, on $\mathcal{S}$, we use (i) in Definition 7.2 and (7.1) to write

$$\begin{aligned} \text{div}_{\mathcal{S}} v = \sum_{j=1}^{n} \mathcal{D}_j v_j &= \sum_{j=1}^{n} (\partial_j v_j)\Big|_{\mathcal{S}} - \sum_{j=1}^{n} \nu_j \partial_\nu v_j \\ &= (\text{div } v)\Big|_{\mathcal{S}} - \langle\nu, \partial_\nu v\rangle. \end{aligned} \tag{7.42}$$

Combining (7.41) and (7.42) we see that

$$(\text{div } v)\ |_{\mathcal{S}} = \text{div}_{\mathcal{S}}(v_{\text{tan}}) + \mathcal{G}\langle v, \nu\rangle + \langle(\partial_\nu v), \nu\rangle, \tag{7.43}$$

and

$$\langle(\partial_\nu v), \nu\rangle = \partial_\nu\langle\nu, v\rangle + (\partial_\nu \nu)v = \partial_\nu\langle\nu, v\rangle \tag{7.44}$$

since $\partial_\nu \nu = 0$ on $\mathcal{S}$ by (6.27). Hence,

$$\begin{aligned} (L^*v)\ |_{\mathcal{S}} = &-\nabla_{\tan}\langle\nu, v\rangle - [\partial_\nu\langle\nu, v\rangle]\nu \\ &+ \text{div }_{\mathcal{S}}(v_{\text{tan}})\nu + \mathcal{G}\langle v, \nu\rangle\nu + [\partial_\nu\langle\nu, v\rangle]\nu + Rv \\ = &-\nabla_{\tan}\langle\nu, v\rangle + \text{div }_{\mathcal{S}}(v_{\text{tan}})\nu + \mathcal{G}\langle v, \nu\rangle\nu + Rv. \end{aligned} \tag{7.45}$$

Thus, since L is strongly tangential to $\mathcal{S}$ and $\partial\mathcal{S} = \varnothing$, if we apply Theorem 6.2 we get that

$$\begin{aligned}\int_{\mathcal{S}} \langle Lu, u\rangle \, dS &= \int_{\mathcal{S}} \langle u, L^{*}u\rangle \, dS \\ &= \int_{\mathcal{S}} [\langle -u_{\tan}, \nabla_{\tan} \langle \nu, u\rangle\rangle + \langle u, Ru\rangle + \mathcal{G}\langle \nu, u\rangle^{2} \\ &\qquad + \langle \mathrm{div}_{\mathcal{S}}(u_{\tan})\nu, u\rangle] dS.\end{aligned} \tag{7.46}$$

We integrate the last term of (7.46) by parts to obtain $-\langle u_{tan}, \nabla_{tan}\langle \nu, u\rangle\rangle$. Moreover, recalling from Proposition 5.5 that $R\nu = 0$ on $\mathcal{S}$ and $R = R^{T}$, (7.46) becomes

$$\int_{\mathcal{S}} \langle u, Lu\rangle \, dS = \int_{\mathcal{S}} [\langle -2u_{\tan}, \nabla_{\tan}\langle \nu, u\rangle\rangle + \langle u_{\tan}, Ru_{\tan}\rangle + \mathcal{G}\langle \nu, u\rangle^{2}] dS. \tag{7.47}$$

This finishes the proof of Lemma 7.7. □

Chapter 8

Coercive Estimates

In this section we are finally able to obtain coercive estimates for the Laplace-Beltrami operator on C^k surfaces in $\mathbb{R}^n$. Define for each $k \in \mathbb{N}$ the Sobolev space

$$W^{k,2}(\mathcal{S}) := \left\{ u : \left(\sum_{|\alpha| \leq k} \int_{\mathcal{S}} |\mathcal{D}^{\alpha} u|^2 ds \right)^{\frac{1}{2}} < \infty \right\}, \tag{8.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex.

Proposition 8.1. *If $\mathcal{S} \subset \mathbb{R}^n$ is an oriented surface of class C^k, $k \geq 2$ with extended unit normal ν and $\mathcal{G}_{\partial\mathcal{S}} \geq 0$ on $\partial\mathcal{S}$, then there exists $C > 0$ such that for any u solution of*

$$(D) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ u\big|_{\partial\mathcal{S}} = 0 \\ u \in W^{2,2}(\mathcal{S}), \end{cases}$$

we have

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS \leq C \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS. \tag{8.2}$$

Proof. Assume $\mathcal{G}_{\partial S} \geq 0$ on ∂S and u is as in the statement of Proposition 8.1. Then

$$\begin{aligned}\sum_{j,k=1}^{n} \int_S (\mathcal{D}_j \mathcal{D}_k u)^2 dS &\leq \sum_{j,k=1}^{n} \int_S (\mathcal{D}_j \mathcal{D}_k u)^2 dS + \int_{\partial S} (\partial_\gamma u)^2 \mathcal{G}_{\partial S}\, ds \\ &\leq \int_S |\Delta_S u|^2\, dS + \int_S |\langle R\nabla_S u, \nabla_S u\rangle| \cdot |\mathcal{G}|\, dS \\ &\quad + 2\int_S |R\nabla_S u|^2\, dS \\ &\leq \int_S |\Delta_S u|^2\, dS + C\int_S |\nabla_S u|^2\, dS.\end{aligned} \tag{8.3}$$

The first inequality in (8.3) is obvious; the second follows from Corollary 7.4; the third is a consequence of the fact that $\|R\|_{L^\infty(S)} \leq C$. To further estimate $\int_S |\nabla_S u|^2\, dS$, we use Poincaré's Inequality which gives for u $\big($recall that $u\big|_{\partial S} = 0\big)$

$$\int_S |u|^2\, dS \leq C \int_S |\nabla_S u|^2\, dS. \tag{8.4}$$

Moreover, integration by parts gives that

$$\int_S u\, \Delta_S u\, dS = \int_S |\nabla_S u|^2\, dS. \tag{8.5}$$

Using Hölder's Inequality in (8.5), we further have that for any $\epsilon > 0$,

$$\begin{aligned}\int_S |\nabla_S u|^2\, dS &\leq \left(\int_S |u|^2\, dS\right)^{\frac{1}{2}} \left(\int_S |\Delta_S u|^2\, dS\right)^{\frac{1}{2}} \\ &\leq \frac{\epsilon^2}{2}\int_S |u|^2\, dS + \frac{1}{2\epsilon^2}\int_S |\Delta_S u|^2\, dS \\ &\leq \frac{\epsilon^2}{2}\int_S |\nabla_S u|^2\, dS + \frac{1}{2\epsilon^2}\int_S |\Delta_S u|^2\, dS,\end{aligned} \tag{8.6}$$

where in the last inequality we used (8.4). Now we choose ϵ small enough so that $\frac{\epsilon^2}{2} < \frac{1}{2}$ and subtract $\frac{\epsilon^2}{2}\int_S |\nabla_S u|^2\, dS$ from the left-hand side of (8.6) to complete the proof of (8.2). □

Proposition 8.2. *If $\mathcal{S} \subset \mathbb{R}^n$ is an oriented surface of class C^k, $k \geq 2$, with extended unit normal ν, γ the unit normal to $\partial\mathcal{S}$, and $\left(t_j\gamma_k(x)\right)_{j,k}$ is a positive definite matrix for $x \in \partial\mathcal{S}$, then there exists $C > 0$ such that for any u solution of*

$$(N) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ \partial_\gamma u\big|_{\partial\mathcal{S}} = 0 \\ u \in W^{2,2}(\mathcal{S}), \end{cases}$$

there holds

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j\mathcal{D}_k u)^2 dS \leq C \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS. \tag{8.7}$$

Proof. Assume u is as in the statement of Proposition 8.2. Using Corollary 7.5 and letting $\left(t_j\gamma_k(x)\right)_{j,k}$ be a positive definite matrix for $x \in \partial\mathcal{S}$, we again obtain

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j\mathcal{D}_k u)^2 dS \leq \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS + C \int_{\mathcal{S}} |\nabla_{\mathcal{S}} u|^2 \, dS. \tag{8.8}$$

We now use a variant of Poincaré's Inequality that gives

$$\int_{\mathcal{S}} |u - u_{\mathcal{S}}|^2 \, dS \leq C \int_{\mathcal{S}} |\nabla_{\mathcal{S}} u|^2 \, dS, \tag{8.9}$$

where $u_{\mathcal{S}} = \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} u \, dS$ is the average of u over $\mathcal{S}$. Also, since uniqueness in (N) is up to constants, we can assume that $u_{\mathcal{S}} = 0$, and we have (8.4). Again (8.5) and (8.6) hold, and the proof of Proposition 8.2 is completed in the same manner as that of Proposition 8.1. $\square$

Proposition 8.3. *Let $\mathcal{S}$ be as in the statement of Proposition 8.2. If $\partial\mathcal{S} = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are connected, $\mathcal{G}_{\partial\mathcal{S}} \geq 0$ on Γ_1 and $\left(t_j\gamma_k(x)\right)_{j,k}$ is a positive definite matrix for $x \in \Gamma_2$, then there exists $C > 0$ such that for any u solution of*

$$(M) \qquad \begin{cases} \Delta_{\mathcal{S}} u = f \in L^2(\mathcal{S}) \text{ on } \mathcal{S} \\ u = 0 \text{ on } \Gamma_1 \\ \partial_\gamma u = 0 \text{ on } \Gamma_2 \\ u \in W^{2,2}(\mathcal{S}), \end{cases}$$

there holds

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS \leq C \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS. \tag{8.10}$$

Proof. Assume $\mathcal{G}_{\partial\mathcal{S}} \geq 0$ on Γ_1, $\left(t_j \gamma_k(x)\right)_{j,k}$ is a positive definite matrix for $x \in \Gamma_2$, and u is as in the statement of Proposition 8.3. Using Corollaries 7.4 and 7.5, the properties of a positive definite matrix, and the fact that $\|R\|_{L^\infty(\mathcal{S})} \leq C$, we again obtain

$$\sum_{j,k=1}^{n} \int_{\mathcal{S}} (\mathcal{D}_j \mathcal{D}_k u)^2 dS \leq \int_{\mathcal{S}} |\Delta_{\mathcal{S}} u|^2 \, dS + C \int_{\mathcal{S}} |\nabla_{\mathcal{S}} u|^2 \, dS. \tag{8.11}$$

Since $u\big|_{\Gamma_1} = 0$, we can again use (8.4). Then (8.5) and (8.6) still hold, and the proof of Proposition 8.3 is completed in the same manner as that of Proposition 8.1. $\square$

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