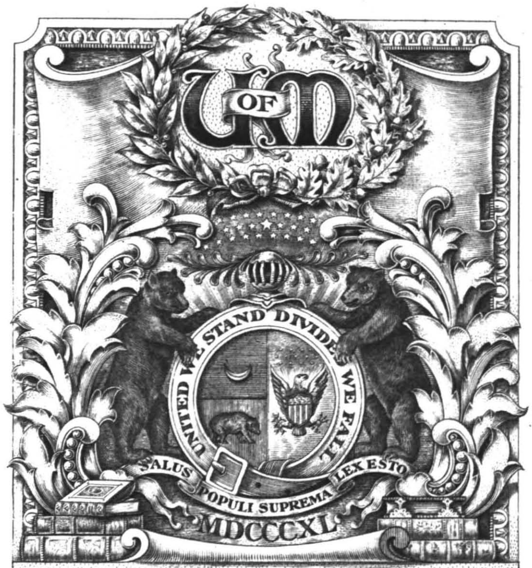
PERRY — Differential Equations THESIS 1903



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SOLUTIONS OF DIFFERENTIAL EQUATIONS NOT OBTAINED BY GIVING PARTICULAR VALUES TO THE CONSTANT OF INTEGRATION IN THE GENERAL SOLUTION.

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In considering the solution of Differential Equations, let the equation be taken in the form $f(x,y,p)=0$, in which p denotes $\frac{dy}{dx}$, and f is a rational, integral, and algebraic function of x, y, and p, of degree n in p. It has been shown that, in general, this equation must have a solution in the form $F(x,y,c)=0$, F will always be a function of x, y, and a variable parameter, c. F will also be of degree n in c , but may not be, in all cases, a rational, integral, and algebraic function in x and y. We can assume f an indecomposable function. Then F will also be indecomposable. For if F could be factored, then to each of these factors would correspond a factor of f.

There are, in some cases, solutions which can not be obtained by assigning particular values to the constant of integration in the general solution. Such a solution of a Differential Equation is called a Singular Solution. The present theory of singular solutions was expounded by Cayley in 1873. His method is, in general, as follows.-

Let $f(x,y,p)=0$, of degree n in p, have a general solution $F(x,y,c)=0$ such that F is a rational, integral, and algebraic function of x, y, and c , of degree n in c. If these functions be represented geometrically so that x and y are the coordinates of a point in a plane, then $f(x,y,p)=0$ defines a system of curves $F(x,y,c)=0$. If values be assigned to x and y

in $f(x,y,p)=0$, n values of p will be determined, corresponding to the directions of n curves through the point (x,y). If values be assigned to x and y in $F(x,y,c)=0$ n values of c will be determined, corresponding to which there will be n curves. The curve $F(x,y,c)=0$ may have a certain number of nodes and cusps. If it be possible, choose $c=c_1$ so that the curve $F(x,y,c_1)=0$ will have an additional node at some point, P., called a "level point". At this point there are n values of c, consisting of c_1 and n-1 other values. Hence through the level point pass a nodal curve and n-1 other curves. At this point, therefore, there are 2+n-1 or n+1 directions of the tangent determined by $f=0$. But since f is of degree n in p, there can not be n+1 values of p, unless f is identically equal to 0 for all values of p.

The c-discriminant, or the condition for equal values of c in $F=0$ is an invariant of the function, and its relation to the system of curves $F=0$ must be explained in connection with the theory of singular solutions. Likewise the p-discriminant of $f=0$ is an invariant of the function f, and its relation to the system of curves defined by $f=0$ must be explained.

If there be any ordinary nodes of $F=0$, they will form a locus called the nodal locus. Since there can be only n directions of the curve at at a point, there can be only n-2 curves besides the nodal curve, which gives two directions. But since there are n values of c, the nodal curve must correspond to equal values of c. Therefore the

nodal locus represents a locus of points for which $F(x,y,c)=0$ gives equal values of c. Hence the nodal locus must be contained in the c-discriminant. Since at a nodal point the n values of p will, in general, be distinct, there will be no equal values of p for points on the nodal locus; and hence the nodal locus will not be contained in the p-discriminant.

There may also be a cuspidal locus, at every point of which there are at least two equal values of p corresponding to the tangent at the cusp counted twice, and $n-2$ other values. This gives $n-1$ curves through the point. But since there are n values of c for each point, the cuspidal curve must be counted twice, and corresponds to equal values of c. From this it is evident that the cuspidal locus must be found in both the c-discriminant and in the p-discriminant.

There may also be a locus called the tac locus, such that there are, at every point, two equal values of p and no equal values of c. This will be a locus of points at which two different curves of the system are tangent. This locus will therefore be found in the p-discriminant but not in the c-discriminant.

In general none of these loci will be solutions of $f=0$, because none of these loci have, at each point, the direction of some curve of the system at that point. This brings us to consider that, if there be in the plane any curve $U=0$ tangent at every point to some curve of the

of the system $F(x,y,c)=c$, then $U=0$ will be a solution of $f(x,y,p)=0$. This is evident from the fact that, at each point of $U=0$, U has the same coordinates (x,y) and the same direction as some curve of the system which is a solution of the differential equation. Hence the values of x, y, and p at each point of $U=0$ will satisfy $f(x,y,p)=0$, and any envelope of the system of curves is a solution of the differential equation. It is to the envelope that Cayley applies the term "Singular Solution".

Through any point on this curve will pass the tangent curve counted twice and n-2 other curves. Since the two-fold value of c at any point on the envelope corresponds to two coincident curves, so the value of p must be two-fold corresponding to two coincident tangents. Therefore the envelope, if there be one, must be found in both the c- and the p-discriminant.

It can be shown that if the nodal, cuspidal, tac, and envelope loci be denoted by $N=0$, $C=0$, $T=0$, and $U=0$, respectively, then,- the c-discriminant is represented by $N^2C^3U=0$, and the p-discriminant is represented by $CTU=0$

The c-discriminant is found by eliminating c between $F(x,y,c)=0$ and $\frac{\partial F}{\partial c}=0$. The p-discriminant is found by eliminating p between $f(x,y,p)=0$ and $\frac{\partial f}{\partial p}=0$.

The question of the existence of a singular solution is not discussed by Cayley except for the particular case where n=[illegible]. In such a case $f(x,y,p)=0$ may be

represented by $(L,M,N \between p,1)^2=0$. Let the general solution be denoted by $(P,Q,R \between c,1)^2=0$. If there be a singular solution it will be $LN-M^2=0$ or a factor of $LN-M^2=0$. But in general $LN-M^2$ is not decomposable and not a singular solution. Hence in general a differential equation of the second degree does not have a singular solution. But, according to Cayley, every algebraic equation, rational and integral in x and y, and depending in any way on a variable parameter c, has an envelope, and the differential equation obtained by eliminating c between $F=0$ and the derived equation has a singular solution. Therefore a differential equation will not, in general, have an algebraic solution which is a rational and integral function of x and y.

In Vol. VIII (of his Mathematical Papers), page 420, Cayley discusses a differential equation which does not seem to agree with his theorem that every system of algebraic curves as described above, ~~always~~ has an envelope. The problem is as follows,-

The general solution of $\left(\frac{dy}{dx}\right)^2 - \frac{9}{4}x = 0$ is, $(y+c)^2 - x^3 = 0$. The p- and c-discriminants are $x=0$ and $x^3=0$ respectively. But this is a cusp locus and not an envelope. Cayley therefore concludes that $(y+c)^2 - x^3=0$ has no envelope. According to Darboux, the above problem comes under the special case in which $\frac{\partial f}{\partial y}=0$, and, in order to get a singular solution, it is necessary to revolve the axes of coordinates through some angle. If in this case the axes be revolved through an angle of 90°, the differential equation becomes $\left(\frac{dy}{dx}\right)^2 + \frac{4}{9y}=0$, and the p-discriminant

becomes $y = -\infty$. Changeing back to the original axes, the singular solution-if it be considered such - becomes $x = \infty$.

Clebsch offers the following explanation in regard to the existence of envelopes and singular solutions. Being given the integral equation $F(x,y,c) = 0$, the differential equation obtained by eliminating c between $F=0$ and the derived equation has not all its generality even though $F=0$ be perfectly general. Hence if we make the differential equation perfectly general, c will not enter in $F=0$ in an arbitrary manner.

Darboux seeks to explain this inconsistency in a different way. Thus,- If $F(x,y,c) = 0$ be a rational, integral, and algebraic function, it will have an envelope, and the differential equation of which it is a solution will have a singular solution. But, as he states, some differential equations have general solutions which are not rational, integral, and algebraic. Hence he concludes that all differential equations which have no singular solution are of the kind which have no rational, integral and algebraic solution. His theory does not hold in the above problem unless $x = \infty$ can be considered a singular solution. The following is, in general, Darboux's method of investigating the existence of singular solutions.

There has existed an error in the theory of singular solutions, which has led to some confusion and has presented a seeming inconsistency. The commonly accepted theory was as follows,- Every differential equation has a solution which is a function of x , y, and a variable parameter ,c. Every function of x,y, and c has an envelope. Therefore every differential equation has a singular solution. But this is not the case. The inconsistency is explained if it is noticed that not every dfferential equation has a solution which is a rational, integral, and algebraic function; and that ,therefore, the integral may not have an envelope. For example,- $p^2-(1-y^2)=0$, is a rational,.integral, and algebraic function of x, y, and p, but the general solution is a transcendental function, $y=\sin(x+c)$

In general we may say that ,if $f(x,y,p)=0$ be of the n'th degree in p , n curves will pass through any point (x',y') , but all that is given to determine these is a developement of y in positive powers of x. The coefficients in these developements are rational functions of x' and y' and one of the roots of $f(x',y',p)=0$, and are ,therefore, algebraic functions of a single parameter. But an envelope can not be obtained by varying this parameter, for at the points of the discriminant of p on the envelope ,if there be one, the developement generally becomes divergent and ceases to represent a solution of the equation. Darboux was probably the first to

give an exhaustive discussion of the existence of singular solutions.

If a differential equation has, in general, a solution which is a finite and continuous function of x, y, and c, then a singular solution is the general rule, and the theory of singular solutions is nothing more than the theory of envelopes. But since this is not the case, the theory of singular solutions should be separated from that of envelopes and should be developed independently, without the assumption of an integral solution.

Thus let $f(x,y,p)=0$ define a system of curves. If there is a singular solution,-that is, an envelope of the system,-then at each point of the envelope $\frac{\partial f}{\partial p}=0$, since at each point there must be equal values of p. Moreover since the envelope is tangent at every point to a curve of the system, then $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0$.
Hence for every point of the envelope, the three following equations must hold.- (1) $f(x\,y\,p)=0$ (2) $\frac{\partial f}{\partial p}=0$ and (3) $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0$

If we eliminate p between (1) and (2), we get an equation in x, and y represented by $A=0$. If we eliminate p between (1) and (3), we get a different function of x and y represented by $B=0$. Then if there be a singular solution, $A=0$ and $B=0$, representing two loci, must have some part in common denoted by $S(x,y)=0$ and which satisfies (1), (2) and (3).

Taking a point on this locus $S(x,y)=0$, a value $p=m$ can be determined satisfying (1), (2) and (3) such that

$f(x,y,m)=0$, $\frac{\partial f(x,y,m)}{\partial m}=0$ and $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}m=0$

In order to have the tangent to the envelope pass through the point, derive (1) which gives, $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p+\frac{\partial f}{\partial m}\frac{dm}{dx}=0.$ Since $\frac{\partial f}{\partial m}=0$, this reduces to $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0$, By comparing this with $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}m=0$, we get $\frac{\partial f}{\partial y}(m-p)=0$. Unless $\frac{\partial f}{\partial}=0$ $m=p$. Assuming that $\frac{\partial f}{\partial y}$ is not equal to zero, and substituting $m=p$ in $f(x,y,m)=0$ we get $f(x,y,p)=0$. Hence the locus $S(x,y)=0$ is a solution of $f(x,y,p)=0$, since at all points the values of x, y, and p satisfy the given equation. But if $\frac{\partial f}{\partial y}=0$ then by (3), $\frac{\partial f}{\partial x}=0$.

This will explain the preceding problem discussed by Cayley. The equation as stated was,- $p^2-4x=0$.

$\frac{\partial f}{\partial y}=0$ unless the axes be revolved through some angle, such that $\frac{\partial f}{\partial y}$ is no longer equal to zero.

Another geometric interpretation may be given if, in the solution $F(x,y,c)=0$, we consider c as a third coordinate, z. $F(x,y,z)=0$ then represents a surface which, for any values of x and y, gives n values of z, and for any value $z=c$ gives the projection on the xy plane of the intersection of $z=c$ with $F(x,y,z)=0$. The differential equation in this case defines a surface such that, for every point on the intersection of $z=c$ with the surface, $f(x,y,p)=0$

The different loci given by the p- and c-discriminants may then be interpreted as follows.—

The nodal locus represents the projection on the xy plane of the intersection of two branches of the surface, and corresponds to equal values of c but not of p.

The cuspidal locus represents the projection on the

xy plane of a cuspidal edge of the surface, and corresponds to equal values of both z and p .

The tac locus is the projection on the xy plane of those points (x',y',z_1) and (x',y',z_2) on the surface at which tangent planes are parallel. This is evident from the fact that if the surfaces at the points (x',y',z_1) and (x',y',z_2) are parallel, then the two intersections of $z = z_1$ and $z = z_2$ with the surface will be parallel, and the projections will be tangent. The tac locus therefore corresponds to equal values of p but unequal values of z.

The envelope locus represents the projection on the xy plane of those points on the surface at which a tangent plane is parallel to the z axis. This corresponds to equal values of both p and z. Hence the envelope locus will be found in both the z- and the p-discriminant. Considering the differential equation in the form $f(x,y,p)=0$ then $\frac{\partial f}{\partial p}=0$ and since the plane parallel to the z axis is tangent to the surface at the point,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0$$