A STUDY OF HOMOLOGICAL INVARIANTS

MODULO AN EXACT ZERO DIVISOR

A DISSERTATION IN
Mathematics
and
Computer Science

Presented to the Faculty of the University
of Missouri–Kansas City in partial fulfillment of
the requirements for the degree

DOCTOR OF PHILOSOPHY

by
DEEPAK SIREESHAN

B. E., Anna University, 2015
M.S., University of Missouri-Kansas City, 2020

Kansas City, Missouri
2024

A STUDY OF HOMOLOGICAL INVARIANTS

MODULO AN EXACT ZERO DIVISOR

Deepak Sireeshan, Candidate for the Doctor of Philosophy Degree

University of Missouri–Kansas City, 2024

ABSTRACT

Let Q be a commutative local ring and R a quotient ring of Q by an ideal generated by an exact zero divisor. We use differential graded structures to describe a construction that produces a minimal free resolution of an R-module from the free resolution of the same module, considered as a Q-module, and the free resolution of R as a Q-module. We provide an explicit algorithm for this construction, written for the computer algebra system Macaulay2. Given two R-modules, we then use a mapping cone construction to relate homology of the two modules over Q to homology over R, and we give applications of this construction to the study of several homological invariants, such as complexity, curvature, and generalized Poincaré series.

APPROVAL PAGE

The faculty listed below, appointed by the Dean of the School of Science and Engineering, have examined a dissertation titled "A study of homological invariants modulo an exact zero divisor," presented by Deepak Sireeshan, candidate for the Doctor of Philosophy degree, and certify that in their opinion it is worthy of acceptance.

Supervisory Committee

Liana Şega, Ph.D., Committee Chair
Department of Mathematics and Statistics

Yusuf Uddin, Ph.D.
Department of Computer Science

Eric Hall, Ph.D.
Department of Mathematics and Statistics

Yijie Han, Ph.D.
Department of Computer Science

Noah Rhee, Ph.D.
Department of Mathematics and Statistics

CONTENTS

Appendix

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to Professor Liana M. Şega, my advisor, for her continuous patience, support, and generosity throughout my time at the University of Missouri, Kansas City. Without her guidance, this thesis would not have been possible, so I sincerely thank her. Coming from an engineering background, I could not have asked for a better advisor for my research endeavors.

I would also like to thank the other members of my supervisory committee: Professors Eric Hall, Noah Rhee, Yusuf Uddin, and Yijie Han for the time and effort they invested in my work and for all the valuable input they provided. I would like to thank my parents, Sireeshan and Syamala, as well as my sister and brother-in-law, Divya and Ashok. Without their support, I would never have been able to pursue my dream across continents. I also would like to express my sincere gratitude to my Aunt Nirmala and my grandparents Krishnan and Padmavathi for taking care of me selflessly all these years.

Last but not least, I wish to thank my wife, Shivani. Her continuous support, both mentally and financially, allowed me to pursue my dreams. Thank you for everything.

CHAPTER 1

INTRODUCTION

The importance of solving systems of polynomial equations in several variables cannot be overemphasized in this modern technological era. The theoretical understanding of such polynomial equations comes primarily from the field of *Commutative algebra*, which deals with the study of *rings*. Rings generalize the notion of field (e.g. the real numbers) by no longer requiring nonzero elements to have a multiplicative inverse. When solving a system of linear equations over a field, it is known that the set of all solutions to the system can be spanned by finitely many linearly independent elements from that set. For a linear system defined over an arbitrary ring, the spanning set need not be finitely generated nor be linearly independent. The finiteness can be taken care of by considering a special group of rings called *Noetherian rings*. Most rings that are of interest (e.g. quotients of polynomial rings) are Noetherian. However, the linear independence of a minimal spanning set is not guaranteed. Without such independence, one needs to look at a new system of equations that describes the linear dependence relations between the elements of the spanning set and solve it. We repeat this (quite often, infinite) process and encode that information in an algebraic structure called a *free resolution*. We are concerned with the study of free resolutions of finitely generated *modules*, and homological invariants that are defined in terms of such resolutions.

Let Q and R be two rings such that $\varphi : Q \to R$ is a map that preserves the ring structure (a ring homomorphism). An area of Commutative Algebra that has seen substantial development in recent years consists of understanding how algebraic/homological properties of R-modules compare to those of Q-modules, see [24, 21, 28, 8] for examples of such work. This line of research and the problems involved are generally referred to as *change of ring* problems, which is the common theme of the results in this thesis. We concentrate on a situation where the ring R is a quotient of Q by the ideal generated by one element $f \neq 0$ of Q. The case when f is a nonzero divisor, meaning that multiplication by f is one-to-one on Q, has been much studied, and we will recall the results in this case throughout. We concentrate on the case when f is a zero divisor, and more precisely the case when f has the property that there exists a nonzero element g in Q such that

$$\{h \in Q\colon hf = 0\} = \{gh\colon h \in Q\} \qquad \text{and} \qquad \{h \in Q\colon hg = 0\} = \{fh\colon h \in Q\}\,.$$

Such an element f is called an *exact zero divisor*; the pair (f, g) is also called an exact pair of zero divisors. Such elements bear similarities to the more studied nonzero divisors, in that we have some understanding of their homological behavior, due to the simple structure of the minimal free resolutions of the ideals they generate. Exact zero divisors have been studied in recent literature, see [10, 25, 27], and the ideals they generate are part of the larger class of quasi-complete intersection ideals studied in [4]. A key ingredient of our work comes from the existence of differential graded (dg) structures on resolutions. Such structures provide an effective and elegant tool in commutative algebra, with particularly strong applications in the study of homological behavior under a change of ring, see [5] for the basics of the theory, which will be summarized in Chapter 3.

We now proceed to describe the main results of this work. Since the statements of the major theorems require somewhat specialized terminology, readers who are unfamiliar with or require a refresher on the basic concepts in commutative/homological algebra may find useful to read first Chapter 2, where most of the needed preliminaries are defined and discussed, before returning to the remainder of this introduction.

Let $\varphi\colon Q \to R$ be a ring homomorphism and M an R-module. One would like to be able to describe a free resolution of M over R, given a free resolution A of R over Q and a free resolution U of M over Q. Iyengar [28] provides such a construction when the resolutions A and U have appropriate dg structures and discusses the existence of such structures. When Q is local (meaning also commutative noetherian) with maximal ideal $\mathfrak{m}$ and φ is surjective, we are particularly interested in minimal free resolutions. In this case, if A is a resolution of R over Q admitting a dg algebra structure and U is a minimal free resolution of M over Q admitting a semi-free dg

A-module structure, it is known that $U \otimes_A R$ is a minimal free resolution of M over R, see Lemma 3.1.3.

When $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ and $R = Q/(f)$, a construction of Shamash can be used to build a semi-free dg module structure of U over the Koszul complex on f, see [5, Proposition 2.2.2]. Consequently, this leads to a description of a minimal free resolution of M over R when f is, in addition, a non-zero divisor, cf. [5, Theorem 2.2.3]. In Chapter 3 we extend the Shamash construction when f is a zero divisor as follows:

Theorem 1 (Theorem 3.2.1)**.** *Let $(Q, \mathfrak{m})$ be a Noetherian local ring and let $f, g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ such that $fg = 0$. Set $R = Q/(f)$ and A be the Tate resolution of R over Q described in 3.1.5. If M is a finitely generated R-module and U is a minimal free resolution of M over Q, then:*

(a) *U has a structure of semi-free dg module over A.*

(b) *Furthermore, if (f, g) is an exact pair of zero divisors, then a minimal free resolution of M over R can be constructed using U and A.*

The question of whether a minimal free resolution U of M over Q admits a dg A-module structure was originally asked by Buchsbaum and Eisenbud [12], in the case when A is a Koszul complex. Subsequently, this question was studied in the work of several authors, including Avramov, Kustin, Iyengar, Miller, and Srinivasan, see [7],

[28], [33], [31], [41], [42], with both positive and negative answers. In all previously known results in which existence is established, either A is a Koszul complex, or the resolution U has a finite length. Our result provides a first positive answer (to our knowledge) in a case when both A and U are infinite.

In Appendix A., we present a formal algorithm using the proof of Theorem 3.2.1 to produce multiplication maps that define the dg module structure on the minimal free resolution of an R-module. Using this algorithm, we can define functions in any computer algebra system that produces these maps and further use it to create a minimal free resolution of an R-module M from the minimal free resolution of M over Q. This has been implemented as a package in the computer algebra system Macaulay2 [23] named `TateAlgebra`.

In Chapter 4 and Chapter 5 we proceed to investigate homological properties that encode data that comes from two R-modules M, N. Such properties are often formulated in terms of the homology modules $\operatorname{Tor}^Q(M,N)$ and $\operatorname{Tor}^R(M,N)$, whose definition is recalled in Chapter 2.

If (F,∂) is a complex (see 2.2.1 for definition) of finitely generated R-modules, then a classical construction due to Eisenbud [21] produces degree -2 maps $\widetilde{\tau}\colon \widetilde{F} \to \widetilde{F}$, where $(\widetilde{F},\widetilde{\partial})$ is a lifting of F to Q, and thus $\widetilde{F} \otimes_Q R = F$. With $S = R/\operatorname{ann}_Q(f)R$, the map

$$\tau := \widetilde{\tau} \otimes_Q S\colon F \otimes_R S \to F \otimes_R S$$

is then a degree -2 *chain map* (see 2.2.2 for definition) that we refer to as an *Eisenbud operator*. When f is a non-zero divisor element (and thus $S = R$), $\tau\colon F \to F$ is also called a *CI operator*, and these maps have been essential in the study of free resolutions and (co)homology over complete intersection rings, see [3]. In particular, when f is a non-zero divisor and M, N are finitely generated R-modules, τ induces the maps γ_i in an *exact sequence* (see 2.2.1 for definition).

$$\cdots \to \operatorname{Tor}_{i+1}^{Q}(M,N) \xrightarrow{\alpha_{i+1}} \operatorname{Tor}_{i+1}^{R}(M,N) \xrightarrow{\gamma_i} \operatorname{Tor}_{i-1}^{R}(M,N) \to \operatorname{Tor}_{i}^{Q}(M,N) \to \cdots \tag{1.0.1}$$

in which α_{i+1} is induced by the canonical map $Q \to R$. This sequence provides a tool for investigating the change in various homological invariants (e.g. betti numbers, complexity defined in 2.2.5 and 2.3.1) via the change of ring $Q \to R$, see for example [8, Section 3].

We start our investigation in Chapter 4 by observing that when f is a non-zero divisor, the exact sequence (1.0.1) can be understood as coming from the *mapping cone* W of the Eisenbud operator τ (see 4.2.1 for definition) as seen in Proposition 4.2.4. The *characteristic* of a ring Q denoted by $\operatorname{char}(Q)$ is the smallest positive integer such that $1 + \cdots + 1(n \text{ times}) = 0$; if no such n exists, then $\operatorname{char}(Q) = 0$. In the case when f is an exact zero divisor and the characteristic of the residue field is zero, we are also able to describe the mapping cone of the Eisenbud operator and two exact sequences are needed to describe the change of ring, as described below.

Theorem 2 (Theorem 4.2.6). *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring such that $\operatorname{char}(\mathsf{k}) = 0$. Let (f, g) be an exact pair of zero divisors in Q and set $R = Q/(f)$ and $S = Q/(f, g)$. Let M be an R-module and let (W, ∂^W) denote the mapping cone of the Eisenbud operator τ associated with f.*

For any R-module N with $gN = 0$, there are two long exact sequences

$$\cdots \to \operatorname{Tor}^R_{n-1}(M, N) \to \operatorname{H}_n(W \otimes_S N) \xrightarrow{\psi^N_n} \operatorname{Tor}^R_n(M, N) \xrightarrow{\delta^N_n} \operatorname{Tor}^R_{n-2}(M, N) \to \cdots$$

$$\cdots \to \operatorname{H}_{n+1}(W \otimes_S N) \xrightarrow{\mu^N_{n+1}} \operatorname{Tor}^Q_{n-2}(M, N) \to \operatorname{Tor}^Q_n(M, N) \xrightarrow{\phi^N_n} \operatorname{H}_n(W \otimes_S N) \to \cdots$$

where δ^N is induced by τ and $\psi^N \phi^N$ is equal to the map $\operatorname{Tor}^Q(M, N) \to \operatorname{Tor}^R(M, N)$ induced by the canonical projection $Q \to R$.

We then apply Theorem 2 to deduce results about the vanishing of homology and relate homological invariants via the change of ring $Q \to R$. Such results have been previously investigated by Bergh, Celikbas, and Jorgensen [10] using methods that rely on a change of rings spectral sequence. While we recover many of the results of [10] (in a weaker form, because of our additional assumption on characteristic), the assumption that $\operatorname{char}(\mathsf{k}) = 0$ leads to new results, and in particular we answer one of the questions raised in [10], as described in Theorem 3.

In general, for any $f \in Q$ (not necessarily an exact zero divisor) and M, N finitely generated modules over $R = Q/(f)$ such that $M \otimes_Q N$ has finite length, we are concerned with understanding the asymptotic behavior of the size of the homology modules $\operatorname{Tor}_i^Q(M,N)$, and how this behavior changes under the change of ring $Q \to R$. We use $\ell(-)$ to denote length and we define a generalized Poincaré series

$$\mathrm{P}^Q_{M,N}(t) = \sum_{i=0}^{\infty} \ell(\operatorname{Tor}_i^Q(M,N))t^i \,.$$

The classical Poincaré series of a module M, which is the generating function of the sequence of Betti numbers of M, is $\mathrm{P}^Q_M(t) = \mathrm{P}^Q_{M,\mathsf{k}}(t)$. We use the invariants *length complexity* and *length curvature*, denoted $\ell\,\mathrm{cx}_Q(M,N)$, respectively $\ell\,\mathrm{curv}_Q(M,N)$ (see Section 5.1 for definitions) to measure asymptotic behavior of the coefficients of this series. When $N = \mathsf{k}$, these invariants coincide with the more classical definitions of complexity $\mathrm{cx}_Q\, M$ and curvature $\mathrm{curv}_Q\, M$ that are discussed in [8] and defined in 2.3.1. The behavior of these invariants under the change of ring $Q \to R$ is well understood when f is a non-zero divisor. More precisely, the exact sequence (1.0.1) gives coefficient-wise inequalities

$$\mathrm{P}^Q_{M,N}(t) \preccurlyeq \mathrm{P}^R_{M,N}(t)\cdot(1+t) \qquad \text{and} \qquad \mathrm{P}^R_{M,N}(t) \preccurlyeq \mathrm{P}^Q_{M,N}(t)\cdot(1-t^2)^{-1}$$

from which one obtains the (in)equalities below, see [8, Proposition 4.2.5]:

$$\ell\,\mathrm{cx}_Q(M,N) \le \ell\,\mathrm{cx}_R(M,N) \le \ell\,\mathrm{cx}_Q(M,N) + 1\,,$$

$$\ell\,\mathrm{curv}_R(M,N) = \ell\,\mathrm{curv}_Q(M,N) \text{ when } \ell\,\mathrm{curv}_Q(M,N) \ge 1\,.$$

In Chapter 5 we provide a version of such results for the case when f is an exact zero divisor. We recover two of the results of [10], namely the inequalities providing upper bounds for $\ell\,\mathrm{cx}_Q(M,N)$ and $\mathrm{P}^Q_{M,N}(t)$ stated below. We complete the picture with inequalities that provide upper bounds for $\mathrm{P}^R_{M,N}(t)$ and $\ell\,\mathrm{cx}_R(M,N)$, and we also address the curvature invariant.

Theorem 3 (Theorem 5.1.3, Corollary 5.1.6)**.** *Let $(Q,\mathfrak{m},\mathsf{k})$ be a local ring with $\mathrm{char}\,(\mathsf{k}) = 0$ and (f,g) an exact pair of zero divisors in Q. We set $R = Q/(f)$ and $S = Q/(f,g)$. Let M, N be finitely generated R-modules such that $\ell(M \otimes_Q N) < \infty$ and $gN = 0$.*

The following coefficient-wise inequalities then hold:

$$\mathrm{P}^R_{M,N}(t) \preccurlyeq \frac{(1-t+t^2)\,\mathrm{P}^Q_{M,N}(t)}{(1-t)} \quad \text{and} \quad \mathrm{P}^Q_{M,N}(t) \preccurlyeq \frac{\mathrm{P}^R_{M,N}(t)}{(1-t)}\,.$$

Consequently,

$$\ell\,\mathrm{cx}_R(M,N) - 1 \le \ell\,\mathrm{cx}_Q(M,N) \le \ell\,\mathrm{cx}_R(M,N) + 1 \qquad \text{and}$$

$$\ell\,\mathrm{curv}_Q(M,N) = \ell\,\mathrm{curv}_R(M,N) \quad \text{when} \quad \ell\,\mathrm{curv}_Q(M,N) \ge 1 \text{ and } \ell\,\mathrm{curv}_R(M,N) \ge 1\,.$$

The authors mention in [10] that they do not know whether there exists an R-module M with $\mathrm{cx}_Q(M) < \infty$ and $\mathrm{cx}_R(M) = \infty$ (in the context of the theorem, with $N = \mathsf{k}$), and ask whether an inequality such as $\mathrm{cx}_R(M) \le \mathrm{cx}_Q(M)$ holds. The inequality $\mathrm{cx}_R(M) - 1 \le \mathrm{cx}_Q(M)$ resulting from our theorem provides thus a partial answer. We do not know whether the stronger inequality $\mathrm{cx}_R(M) \le \mathrm{cx}_Q(M)$ holds or if the characteristic assumption can be removed from our result.

While the Poincaré series $\mathrm{P}^Q_M(t)$ are relatively well studied, with several results that imply such series are rational under appropriate assumptions, there are only a few results (see [24], [36]) showing that the generalized Poincaré series $\mathrm{P}^Q_{M,N}(t)$ are rational, or provide formulas for these series. In Chapter 5 we make further contributions in this direction. We start by exploring the question of when equality holds in the second Poincaré inequality of Theorem 3. From [4] it is known that equality holds when $f \notin \mathfrak{m}^2$ and $\mathfrak{m}N = 0$. In Proposition 5.2.4 we consider the case when $f \notin \mathfrak{m}^2$ and $\mathfrak{m}^2 N = 0$, and then we prove the following result, in which $\mathsf{H}_M(t)$ denotes the Hilbert series of a Q-module M.

Theorem 4 (Theorem 5.3.2)**.** *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with $\mathfrak{m}^3 = 0$. Assume* $\operatorname{char}(\mathsf{k}) = 0$*, (f, g) is an exact pair of zero divisors and set $R = Q/(f)$. Let M, N be finitely generated R-modules such that $\mathfrak{m}(M \otimes_R N) = 0$ and $gN = 0$. Then*

$$\mathrm{P}^Q_{M,N}(t) = \frac{\mathsf{H}_M(-t)\,\mathsf{H}_N(-t)}{\mathsf{H}_Q(-t)}\,.$$

Local rings $(Q, \mathfrak{m}, \mathsf{k})$ with $\mathfrak{m}^3 = 0$ that admit exact zero divisors previously appeared in the work of Conca [15] and Avramov, Iyengar and Şega [5]. When $\mathfrak{m}^3 = 0$ and (f, f) is an exact pair of zero divisors, Lemma 5.3.1 shows that the element f is a *Conca generator* (see 5.3 for definition), in the terminology of [5]. The result of the theorem can be viewed as an extension of the fact proved in [5] that when f is a Conca generator, any Q-module with $fM = 0$ is a *Koszul module* (see 5.2.5 for definition). Finally, we should note that the results presented here are contained in our papers [18] and [19].

CHAPTER 2

BACKGROUND

The objective of this chapter is to provide definitions and examples for some of the mainstream concepts involved in our results, in order to make them reasonably accessible to readers with limited algebra background. For more details, please refer to an introductory commutative/homological algebra text such as [34] and [8]. A reader familiar with these topics is advised to skip this chapter.

2.1 Rings, Ideals and Modules

A (commutative) ring is a set with two operations, addition, and multiplication, with zero and an identity element, where addition is closed (meaning for every x in the ring, there exists an additive inverse $-x$ in the ring) satisfying associativity, distributivity, and commutativity. Basic examples are the set of integers $\mathbb{Z}$ or the set of integers modulo an integer $n \geq 2$, denoted $\mathbb{Z}_n$. When working with rings, an important concept is that of an *ideal*: the ideal generated by some elements $f_1, \ldots, f_n$ of the ring Q is denoted $(f_1, \ldots, f_n)$ and consists of all linear combinations of these elements, with coefficients in Q.

2.1.1. *Quotient rings.* A major class of commutative rings is that of polynomial rings in several variables. The notation $k[x_1, \ldots, x_n]$ identifies the polynomial ring in variables $x_1, \ldots, x_n$ (usually considered to have degree 1) over a field k. While variables in a polynomial ring are considered to be algebraically independent, meaning that a polynomial is equal to zero only when its coefficients are all zero, one often needs to describe situations when this is not the case. This can be done by using a construction called *quotient ring*, in which the elements of the ring are replaced with equivalence classes that identify elements whose difference lies in a given ideal. For example, the field $\mathbb{C}$ of complex numbers can be viewed as the polynomial ring $\mathbb{R}[x]$ in which the relation $x^2 = -1$ is introduced by creating the quotient ring $\mathbb{R}[x]/(x^2+1)$. The complex number i can be defined as the equivalence class of the variable x in this new ring.

2.1.2. *Graded Rings.* An important tool in the study of polynomials is the notion of degree. Each polynomial ring Q can be decomposed into its *homogeneous* parts Q_i; the span of monomials of degree i. Take for example, $Q = \mathbb{R}[x]$, then any $r \in Q$ can be uniquely written as $r = r_0 + r_1 + r_2 + \cdots$ with finitely many non-zero $r_i \in Q_i$ where r_i is a degree i monomial. Moreover, multiplication of homogeneous elements respects degree, meaning $r_i r_j \in Q_{i+j}$ for any $i, j \in \mathbb{N}$ when $r_i r_j \neq 0$. A ring with such a structure is called a *graded* ring. For this graded structure to behave well when passing to a quotient ring, the quotient ideal needs to be homogeneous; this means that, for each generator of the ideal, all non-zero monomial terms have the same degree. Such quotient rings $R = Q/I$, where I is a homogeneous ideal of a polynomial ring Q over a field k where each variable is of degree 1 are called *standard graded* k*-algebras*.

2.1.3. *Local rings.* The class of rings we focus on in this thesis are *local rings*. An ideal $\mathfrak{m}$ of a commutative ring Q is a *maximal ideal* of Q when the quotient ring $R = Q/\mathfrak{m}$ is a field. A ring Q is a local ring denoted $(Q, \mathfrak{m}, \mathsf{k})$ if it has a unique maximal ideal $\mathfrak{m} \subsetneq Q$ where $\mathsf{k} = Q/\mathfrak{m}$ is called the residue field of Q. Another characterization of a local ring is for any non-units x and y, $x + y$ is also a non-unit (x is a unit in a ring Q if there exist some $y \in Q$ such that $xy = 1$, the identity element).

The motivation to study local rings comes from studying the local behaviors of functions. Take, for example, the space of real-valued continuous functions X defined on some open interval U around a point p denoted by X_p. We define an equivalence relation on functions $f, g \in X_p$ by setting $f \sim g$ when $f\big|_V = g\big|_V$ for some open interval $V \subseteq U$ around p. The equivalence classes

$$[f] = \{g \in X_p \mid f \sim g\}$$

are commonly referred to as *germs* of X_p and the set of all such equivalence classes form a ring, denoted by $\mathcal{O}_p$. We say that a germ f (abusing notation f to denote the equivalence class $[f]$) is a unit in $\mathcal{O}_p$ if $f(p) \neq 0$; that is there exists another function g with $g(p) \neq 0$ such that $f(p)g(p) = 1$. Note that when you take two non-units in $\mathcal{O}_p$, say a and b, the sum of these germs $a + b$ is also non-unit based on the definition.

Thus, $\mathcal{O}_p$ is a local ring with the maximal ideal corresponding to the set of all the germs f with $f(p) = 0$.

Example 2.1.4. A formal power series in variable x over a ring, say $\mathbb{C}$, is an expression of the form

$$p(x) = \sum_{i=0}^{\infty} p_i x^i$$

with $p_i \in \mathbb{C}$ for all i. The set of all such formal series in x with coefficients in $\mathbb{C}$ form a *ring of formal power series* denoted by $\mathbb{C}[[x]]$. If $p_0 \neq 0$ (constant term is invertible), the inverse of $p(x)$, say $q(x)$, can be iteratively produced by setting

$$q_0 = \frac{1}{p_0} \quad \text{and} \quad q_n = -\frac{1}{p_0} \sum_{i=0}^{n-1} p_{n-i} q_i \, .$$

Moreover, the set of all formal series with zero constant terms (non-invertible constant term) consists of all non-units of $\mathbb{C}[[x]]$ and forms the unique maximal ideal of $\mathbb{C}[[x]]$ which makes it a local ring. In general, a ring of formal power series can be defined over any ring and infinitely many indeterminates.

Example 2.1.5. Consider a polynomial ring $Q = \mathbb{R}[x_1, \cdots, x_n]$ with standard grading. Notice that there are infinitely many maximal ideals of the form:

$$(x_1 - r_1, \cdots, x_n - r_n) \qquad r_i \in \mathbb{R}$$

Clearly, Q is not a local ring.

However, if we only consider the maximal ideals generated by homogeneous elements of Q, there is a unique maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. A graded ring with a unique homogeneous maximal graded ideal is a *graded local ring* (or) **local ring*. The standard facts about graded local rings, which can be viewed as a graded counterpart of the local rings, are proven in [11, Section 1.5]. More importantly, in most situations, results proved on a local ring can be translated to polynomial rings with standard grading.

2.1.6. *Noetherian ring and Artinian local rings.* A ring Q is Noetherian when every ideal I of Q is finitely generated. A local ring $(Q, \mathfrak{m})$ is Artinian when it is Noetherian and satisfies $\mathfrak{m}^n = 0$ for some $n \in \mathbb{N}$.

2.1.7. *Regular sequence.* A non-zero divisor x of a ring Q is also called a regular element of Q. A sequence of elements $(x_1, \cdots, x_n)$ in Q is called Q-regular when x_i is a non-zero divisor in the quotient ring $Q/(x_1, \ldots, x_{i-1})$ for $1 \leq i \leq n$ with $(x_1, \ldots, x_n) \neq Q$.

Example 2.1.8. Consider the quotient ring $Q = \mathsf{k}[x, y, z]/(xz)$. The sequence $(x - 1, xy)$ is Q-regular since $x - 1$ is a non-zero divisor in Q and $xy = y$ (as equivalence classes) is a non-zero divisor in $Q/(x - 1, xz) \cong \mathsf{k}[y]$. In general, the order of the sequence is important. In this example, $(xy, x - 1)$ is not a regular sequence since $xy(z) = 0$ meaning xy is a zero divisor in Q. However, when working over local rings, any permutation of a Q-regular sequence is also Q-regular.

Below, we define two types of rings that hold significance as the origins of change of ring problems upon which numerous results are derived.

2.1.9. *Regular local rings and Complete intersection rings* A Noetherian local ring $(Q, \mathfrak{m})$ whose maximal ideal is minimally generated by Q-regular sequence is a *regular* local ring. A ring Q whose *completion* (see [34, Chapter 3] for definition) is a regular local ring modulo an ideal generated by a Q-regular sequence is called a *complete intersection* ring.

2.1.10. *Modules.* A *module* over a ring Q is a structure defined in the same manner as a vector space over a field; it is a group (with respect to addition) with scalar multiplication satisfying distributivity and associativity. In particular, an ideal of Q is a Q-module. We work with modules M that are *finitely generated*. This means that there exist elements $m_1, \ldots, m_n \in M$ forming a *generating set*, that is, every element $m \in M$ can be written as a linear combination of $m_1, \ldots, m_n$, with coefficients in Q. A submodule of a module M is a subset of M with the module structure. However, unlike subspaces, a submodule of a finitely generated module need not be finitely generated. For example, the ring of polynomials in infinitely many variables over $\mathbb{C}$ is generated by the identity element when regarded as a module over itself, but the ideal generated by the variables is not finitely generated. An important characterization of Noetherian rings is that a submodule of a finitely generated module is also finitely generated. This is one of the main reasons why Noetherian rings are desirable to work with.

A generating set $m_1, \ldots, m_n$ of a Q-module M is said to be *minimal* if no element m_i can be eliminated to still get a generating set. In general, two distinct minimal generating sets of a module M need not have the same cardinality.

Example 2.1.11. Take $\mathbb{Z}$, the ring of integers. Considering $\mathbb{Z}$ as a module over itself, it is easy to see that the sets $\{1\}$ and $\{2, 3\}$ both minimally generate $\mathbb{Z}$ but have different cardinality.

In the case of interest (e.g. when working with local/graded local rings), it is known that all minimal generating sets have the same cardinality, see [34, Theorem 2.3]. This number is called the *minimal number of generators* of M and is a measure of the size of M. Indeed, when working over a vector space, a minimal generating set is also linearly independent, and thus a basis. However, this property is not true in general, for modules.

Example 2.1.12. Considering $\mathbb{Z}$ as a module over itself, the minimal generating set $\{2, 3\}$ is not a basis of $\mathbb{Z}$ since $2(3) + 3(-2) = 0$ (linear dependence).

2.1.13. *Free modules.* A *free* module is a module that has a basis. While the basis can be infinite, we primarily work with finitely generated free modules that have a finite basis. The rank of a Q-free module F is the number of elements in a basis, in which case F isomorphic to Q^n, with the notation in 2.1.15.

We present below some important constructions within the class of modules over a ring.

2.1.14. *Quotient modules.* Given a module M over a ring Q and a sub-module N of M, we define the quotient module of M by N, denoted M/N, by the equivalence relation

$$m_1 \sim m_2 \iff m_1 - m_2 \in N$$

for $m_1, m_2 \in M$. The elements of M/N are the equivalence classes $[m] = \{m + n \mid n \in N\}$.

2.1.15. *Direct sum of modules.* Let M and N be modules over a ring Q. The *direct sum* $M \oplus N$ is the set of all ordered pairs (m, n) where $m \in M, n \in N$ equipped with addition and scalar multiplication:

$$(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2) \quad \text{and} \quad r(m, n) = (rm, rn)\,.$$

In general, for a family of Q-modules $\{M_i\}_{i \in \mathbb{R}}$, we define $\bigoplus_{i \in \Lambda} M_i$ similarly; the elements of this direct sum are of the form $(m_i)_{i \in \Lambda}$ with only finitely many non-zero m_i. We denote by M^n, the direct sum $\bigoplus_{i=1}^{n} M$.

2.1.16. *Tensor products.* Let M, N, P be modules over a ring Q. A map $f : M \oplus N \to P$ is said to be a bilinear map if it is linear (a Q-module homomorphism) in both M and N. For example, when $M = N = \mathbb{R}^2$, the dot product $f(u, v) = u \cdot v$ defined on the vectors of $\mathbb{R}^2$ is a bilinear map.

The tensor product $M \otimes_Q N$ of M and N is a Q-module equipped with a map $M \oplus N \xrightarrow{\otimes} M \otimes_Q N$ that is bilinear. The defining property (up to isomorphism) of this tensor product is that for any Q-module P and bilinear map $f\colon M \oplus N \to P$, there exists a unique homomorphism $\varphi\colon M \otimes_Q N \to P$ such that $f = \varphi \circ \otimes$; see [34, Appendix A] for the construction and properties.

Example 2.1.17. The vector space $\mathsf{M}_2(\mathbb{R})$ of square matrices of size n with entries in $\mathbb{R}$ can be regarded as the tensor product $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Indeed, the map $\otimes$ can be defined to be the bilinear map $\otimes : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathsf{M}_2(\mathbb{R})$ given by

$$\otimes(u, v) = uv^T \quad u, v \in \mathbb{R}^2 .$$

If $f\colon \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ is the dot product map as above, the corresponding map $\varphi\colon \mathsf{M}_2(\mathbb{R}) \to \mathbb{R}$ is given by $\varphi(M) = \operatorname{tr}(M)$, where $\operatorname{tr}(M)$ denotes the trace of the matrix M. Indeed, if $u = (a, b)^T$ and $v = (c, d)^T$, then

$$\operatorname{tr}(uv^T) = \operatorname{tr}\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = ac + bd = u \cdot v$$

The image of $(u, v) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ under $\otimes$ is denoted $u \otimes v$. If e_1, e_2 denotes the standard basis of $\mathbb{R}^2$, then $\mathsf{M}_2(\mathbb{R})$ has a basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.

2.2 Complexes and associated constructions

In this section, we give essential constructions in homological algebra that are used throughout this thesis. The notion of free resolution of a module is discussed at the end of the section.

When $f : U \to V$ is a linear map between vector spaces U and V, the null space of f denoted $\ker(f)$, and the image of f denoted $\operatorname{Im}(f)$ minimally describe U, in the sense that $U \cong \ker(f) \oplus \operatorname{Im}(f)$. This is convenient as knowing the dimension of any two vector spaces involved, for example, will help deduce the dimension of the other. When working over modules, this is rarely true. The following construction helps in this direction in the sense that, certain properties of modules involved are connected to one another.

2.2.1. *Complexes and Homology.* We say that the sequence of Q-modules with Q-linear maps

$$\cdots \to A_{n+1} \xrightarrow{\partial_{n-1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

is a *complex*, denoted by (A, ∂), when $\operatorname{Im}(\partial_k) \subseteq \ker(\partial_{k-1})$ for all k. Equivalently, we have $\partial_i \partial_{i-1} = 0$, for all i. Because of this property, we can define for each n a Q-module called the n^{th} homology of A by setting $\mathrm{H}_n(A) = \ker(\partial_n)/\operatorname{Im}(\partial_{n+1})$. When $\mathrm{H}_n(A) = 0$ for all n, then we say that A is an *exact sequence.* When working over vector spaces as in the paragraph above, we always have an exact sequence:

$$0 \to \ker(f) \to U \xrightarrow{f} \operatorname{Im}(f) \to 0\,.$$

Exact sequences of this form are called *short* exact sequences of modules. It is clear from the rank nullity theorem that dimension is additive in an exact sequence of vector spaces. Meaning, we can deduce that $\dim(U) = \dim(\ker(f)) + \dim(\operatorname{Im}(f))$ from the exact sequence above. A *length* of a module, denoted $\ell(-)$, is a generalization of the dimension to vector spaces. An important property of length is that it is additive over an exact sequence of modules. In general, if there is an exact sequence of the form

$$0 \to A_n \xrightarrow{d_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{d_1} A_0 \to 0\,,$$

we have

$$\sum_{i=0}^{n} (-1)^i \ell(A_i) = 0\,.$$

2.2.2. *Chain maps.* Let (A, d) and (A', d') be complexes of Q-modules. A collection of Q-linear maps $\sigma_n : A_n \to A'_{n+t}$, collectively denoted σ, is called a *chain map of degree* t when $\sigma_{n-1}d_n = d'_{n+t}\sigma_n$ (commutativity of each square in the diagram below) for all n and visualized as

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & A_{n-2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \sigma_{n+1}} & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_{n-2}} & & \\
\cdots & \longrightarrow & A'_{n+t+1} & \xrightarrow{d'_{n+t+1}} & A'_{n+t} & \xrightarrow{d'_{n+t}} & A'_{n+t-1} & \xrightarrow{d'_{n+t-1}} & A'_{n+t-2} & \longrightarrow & \cdots
\end{array}$$

In this thesis, we use the convention that a chain map without any degree associated with it is assumed to have degree 0.

Below, we describe a complex of great importance that is used throughout this thesis.

2.2.3. *Koszul complex* Consider a ring Q with elements $x_1, \ldots, x_n$. We define a complex (K, d) as follows: Set $\mathsf{K}_0 = Q$ and $\mathsf{K}_p = \oplus Q e_{i_i \ldots i_p}$ be a Q-free module with basis

$$\{e_{i_i\dots i_p} \mid 1 \le i_1 < i_2 < \cdots < i_p \le n\}\,.$$

In other words, $\mathsf{K}_p \cong Q^{\binom{n}{p}}$. The Q-linear maps $d_p : \mathsf{K}_p \to \mathsf{K}_{p-1}$ are given by setting

$$d(e_{i_i\dots i_p}) = \sum_{j=1}^{p}(-1)^{j-1}x_{i_j}e_{i_i\dots\hat{i}_j\dots i_p}$$

where the notation $\hat{i}_j$ means the term i_j is deleted, hence $e_{i_i\dots\hat{i}_j\dots i_p}$ is an element in the basis of K_{p-1}. We can verify it is a complex by checking $d_{i-1}d_i = 0$ for all $1 \le i \le n$. This complex is called the Koszul complex on $x_1, \dots, x_n$. It is known that it is an exact sequence when $x_1, \dots, x_n$ is a regular sequence. One can define a multiplicative structure on this complex as shown in this example.

Example 2.2.4. Let $x \in Q$. The Koszul complex on x takes the form:

$$0 \to Qe \xrightarrow[d_1]{.x} Q \to 0\,.$$

Here, the differential d_1 is the map that sends the basis e of K_1 to $x \in \mathsf{K}_0 = Q$. The multiplication on K can be defined by setting $e^2 = 0$ and extending by linearity. With this multiplication, this complex is a non-trivial example of a dg algebra, a concept that will be discussed in more detail in 3.1.1.

2.2.5. *Free resolutions and Betti numbers.* Let Q be a Noetherian ring. If a non-free Q-module M is finitely generated by $m_1, \dots, m_{\beta_0} \in M$, then there exists elements of the form $(r_1, \dots, r_{\beta_0}) \in Q^{\beta_0}$ such that

$$r_1m_1 + r_2m_2 + \cdots + r_{\beta_0}m_{\beta_0} = 0\,.$$

The set of all such elements forms a submodule (finitely generated because of the Noetherian condition) of Q^{β_0}. This module is called the first *syzygy* of M. Essentially, it is the kernel of the map d_0 which sends basis of $F_0 \cong Q^{\beta_0}$ to β_0-generators of M. We have the exact sequence:

$$0 \to \ker(d_0) \to F_0 \xrightarrow{d_0} M \to 0\,.$$

If $\ker(d_0)$ is a free module, the above sequence provides a way to understand a non-free module in terms of free modules. However, if $\ker(d_0)$ is not a free module then we again find the first syzygy of $\ker(d_0)$ defined as the second syzygy of M. We continue this process and encode the maps successively to get an exact(possibly infinite) sequence:

$$(F,d): \qquad \cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0\,. \tag{2.2.1}$$

For each i, the map d_i is a Q-linear map that can be understood as a matrix whose columns are a generating set of the ith syzygy modules. The exact sequence (F,d) (or simply F) is called a *free resolution* of M. The total number (counting from 0) of free modules in the sequence is called the *length* of the resolution. If the generators at each step are chosen minimal, then F is a minimal free resolution of M. When $(Q,\mathfrak{m})$ is a local ring, the minimality of the free resolution is equivalent to the fact that $d_i(F_i) \subset \mathfrak{m}F_{i-1}$, for all i. In this case, the sequence $\{\beta_i(M) = \operatorname{rank}_Q(F_i)\}$ is called the *betti sequence* of M. The betti sequence is an invariant of M, since minimal free resolutions are unique up to the isomorphism of complexes.

Before talking about the structure of free resolution in a graded setting, we need to define below the notion of graded modules and graded homomorphism.

2.2.6. *Graded modules and homomorphisms.* Let Q be a graded ring, meaning $Q = \oplus_{i=0}^{\infty} Q_i$ where Q_i is the i-th homogeneous part of Q. A *graded module* M of Q is Q-module with a family of subgroups $\{M_n\}_{n\geq 0}$ such that

$$M = \bigoplus_{i=0}^{\infty} M_n \quad \text{and} \quad Q_i M_j \subseteq M_{i+j}\,.$$

To put it informally, the scalar multiplication respects degree. The jth *shift* of a graded module M, denoted $M(j)$, is the graded module with $M(j)_i = M_{i+j}$ for all i. For example, the free Q-module of rank 1 generated by an element of degree j is $Q(-j)$.

If M and N are graded Q-modules, a graded homomorphism f is a Q-linear map such that $f(M_i) \subseteq N_i$ and thus respects degree and preserves graded module structure.

When working over graded *local rings, we also have the uniqueness of minimal free resolutions for graded modules. In this case, the maps d_i in (2.2.1) need to be homomorphisms of graded modules; this is illustrated in the example below.

Example 2.2.7. Let $Q = \mathbb{R}[x, y, z]$ be a polynomial ring with the homogeneous maximal ideal $\mathfrak{m} = (x, y, z)$. The minimal free resolution of $\mathbb{R}$ over Q is the Koszul complex on x, y, z:

$$0 \to Q(-3) \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} Q(-2)^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} Q(-1)^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} Q\,.$$

This free resolution is a *linear* free resolution, meaning that all entries of the differential are linear forms. In other words, for each i, the ith free module in the resolution is a sum of finitely many copies of $Q(-i)$.

2.2.8. *Tor.* When (F, d) is a free resolution of a Q-module M as in 2.2.1, then for another Q-module N, we define $F \otimes_Q N$ as the complex

$$\cdots \to F_n \otimes_Q N \xrightarrow{d_n \otimes \mathrm{id}_N} F_{n-1} \otimes_Q N \to \cdots \to F_1 \otimes_Q N \xrightarrow{d_1 \otimes_Q \mathrm{id}_N} F_0 \otimes_Q N\,.$$

where id_N denotes the identity map on the module N. For each n we set

$$\mathrm{Tor}_n^Q(M, N) = \mathrm{H}_n(F \otimes_Q N)\,.$$

This definition does not depend on the choice of the free resolution F, and the same (up to isomorphism) homology module is obtained if using a resolution of N instead.

Over a local ring $(Q, \mathfrak{m}, \mathsf{k})$, $F \otimes_Q \mathsf{k}$ is an exact sequence if and only if M is a Q-free module. Additionally, the betti numbers of M can be described as follows:

$$\beta_n^Q(M) = \dim_{\mathsf{k}}(\mathrm{Tor}_n^Q(M, \mathsf{k}))\,.$$

2.3 Power series and related invariants

Over a regular local ring (e.g. a power series ring over a field), all minimal free resolutions are finite. However, if the ring is not regular (e.g. quotient rings of regular rings), minimal free resolutions are usually infinite. While computer algebra systems such as Macaulay2 [23] can compute the beginning of a minimal free resolution, they cannot be used to analyze asymptotic behavior. A standard technique to describe a possibly infinite sequence is to encode it in a power series and find possible relations between the elements in the sequence. In this section, we look at two different power series that record different invariants of a module.

Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local Noetherian ring.

2.3.1. *Poincaré series.* The Poincaré series of a Q-module M, denoted $\mathrm{P}_M^Q(t)$, is the power series that encodes the Betti sequence $\{\beta_i(M)\}$ as follows:

$$\mathrm{P}_M^Q(t) = \sum_{i=0} \beta_i(M) t^i \,.$$

The Poincaré series of $\mathbb{R}$ in Example 2.2.7 can then be written as

$$\mathrm{P}_M^Q(t) = 1 + 3t + 3t^2 + t^3 \,.$$

In general, we are interested in deriving formulas for Poincaré series of modules without needing to write the free resolution. One such example is given below.

Example 2.3.2. Let $S = \mathbb{R}[x, y, z]$ and $I = (x^2, y^2, z^2)$ be an ideal. Set $Q = S/I$. Consider $N = Q/(x, y, z)$ as a Q-module. It can be shown that this ring satisfies the hypothesis in [25, Theorem 3.3] and hence the Poincaré series of N over the ring Q is given by the formula:

$$\mathrm{P}_N^Q(t) = \frac{1}{(1-t)^3} = \sum_{i=0}^{n} \binom{2+i}{2} t^i \,.$$

While the examples above have rational Poincaré series, it is not always the case. J. P. Serre conjectured that $\mathrm{P}_{\mathsf{k}}^Q(t)$ is rational for all classes of local rings $(Q, \mathfrak{m}, \mathsf{k})$ but Anick[1] provided the first counter-example with a transcendental Poincaré series in 1982. Moreover, Jacobsson [30] showed that the rationality of $\mathrm{P}_{\mathsf{k}}^Q(t)$ does not guarantee the rationality of $\mathrm{P}_M^Q(t)$, for finite Q-modules M. So, naturally, we want to see if there are classes of rings where all Q-modules have rational Poincaré series and if so, whether they have the same denominator. One such class of ring where all finite modules have rational Poincaré series with a common denominator is the class of complete intersection rings. For more examples, please see [38, 36, 32].

2.3.3. *Complexity and Curvature.* For any finitely generated Q-module M, we define a function $b_M : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $b_M(n) = \beta_n(M)$, and we analyze the asymptotic behavior of this function to answer questions about the growth rate of the Betti sequence of M. We define *complexity* of M over Q, denoted by $\mathrm{cx}_Q(M)$, to be the smallest non-negative integer d such that there exists a polynomial $f(x)$ of degree $d-1$ with $\beta_n(M) \leq f(n)$, $\forall n \geq 0$. In other words, b_M is $\mathcal{O}(x^{d-1})$.

However, if such a polynomial does not exist, we say that the complexity is ∞.

In such cases, the notion of *curvature* is better suited. The curvature of a module M over Q, denoted by $\operatorname{curv}_Q(M)$ is defined as the inverse of the radius of convergence of $\mathrm{P}_M^Q(t)$ and is proven to be finite for all modules over any ring (see [8, Corollary 4.1.5] for a proof).

Example 2.3.4. From Example 2.2.7, it is easy to see that $\beta_n(\mathbb{R}) \leq 12/n+1$ which implies $b_{\mathbb{R}}$ is $\mathcal{O}(x^{-1})$. Therefore, we have

$$\operatorname{cx}_Q(\mathbb{R}) = 0 \quad \text{and} \quad \operatorname{curv}_Q(\mathbb{R}) = 0\,.$$

In fact, $\operatorname{cx}_Q(M) = 0 \iff \operatorname{curv}_Q(M) = 0$ for any finitely generated Q-module M. From Example 2.3.2, since $b_N = \dfrac{x(x+1)}{2}$, b_N is $\mathcal{O}(x^2)$. Thus,

$$\operatorname{cx}_Q(N) = 3 \quad \text{and} \quad \operatorname{curv}_Q(N) = 1\,.$$

It is clear that for a finite Q-module M, finite complexity implies $\operatorname{curv}_Q(M) \leq 1$. When Q is a complete intersection, the residue field k of Q has finite complexity meaning the betti numbers are polynomially bounded and so are all finitely generated Q-modules, see [8, Theorem 8.1.2] for a proof. Because of [8, Corollary 8.2.2], the converse holds for complete intersection rings.

2.3.5. *Hilbert Series.* A polynomial ring over a field in several variables regarded as a vector space can be decomposed into its homogeneous parts as discussed in 2.1.2 which are in turn vector spaces. When we want to encode the dimensions of these homogeneous parts, we make use of *Hilbert series*. When Q is a graded vector space and Q_i are the i-th homogeneous part of Q, then the Hilbert series of Q, denoted by $\mathsf{H}_Q(t)$, is given by

$$\mathsf{H}_Q(t) = \sum_{i=0} \dim_{\mathsf{k}}(Q_i) t^i \,.$$

Example 2.3.6. Let $Q = \mathbb{R}[x, y, z]$. We want to understand how many monomials for a given degree i exist in Q since they are a basis of Q_i. It is easy to see that there are $\binom{2+i}{2}$ unique monomials of degree i for each $i \in \mathbb{N}$. The *Hilbert series* of Q is then written as

$$\mathsf{H}_Q(t) = \frac{1}{(1-t)^3} = \sum_{i=0}^{n} \binom{2+i}{2} t^i \,.$$

When $(Q, \mathfrak{m}, \mathsf{k})$ is a local ring and M is a finite Q-module, the Hilbert series of a module M is defined as:

$$\mathsf{H}_M(t) = \sum_{i=0} \dim_{\mathsf{k}} \left(\frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}} \right) t^i \,.$$

When working over Artinian local rings, the Hilbert series is clearly a polynomial since $\mathfrak{m}^n = 0$ for some n.

2.3.7. *Koszul modules (in graded setting).* Let Q be a graded ring. A finitely generated graded module M over Q, generated in degree 0, is said to be *Koszul* if it has a free resolution that is linear. For an example of linear resolution, see Example 2.2.7. Even though Poincaré series and Hilbert series are two different power series that encode different information, they come together for Koszul modules as an important characterization of a Koszul module M is that

$$\mathrm{P}_Q^M(t) = \frac{\mathsf{H}_M(-t)}{\mathsf{H}_Q(-t)}\,.$$

CHAPTER 3

DG MODULE STRUCTURES AND MINIMAL FREE RESOLUTIONS MODULO AN EXACT ZERO DIVISOR

Differential graded(dg) algebra/module structures offer a valuable framework for addressing a variety of challenges within the fields of commutative and homological algebra. They provide a versatile toolset capable of proving statements that may not initially appear to involve them, see [17, Theorem 1.2] for an example. They are also used to extend certain results originally proved for specific rings to a much broader group, see [14] and [37, Theorem A]. Tate's [43] work initially highlighted the importance of dg structures in commutative algebra by describing a process of constructing a free resolution with dg algebra structure and further using this additional structure to better understand the resolution. This process is essential for our approach and will be recalled below. For more details, a comprehensive survey of applications of dg structures to free resolutions can be found in [8, 17].

Let $\varphi\colon Q \to R$ be a ring homomorphism and M an R-module. As pointed out in the introduction, the existence of appropriate dg algebra structures on a free resolution A of R over Q and a free resolution U of M over Q allows one to describe a resolution of M over R. When $(Q, \mathfrak{m}, \mathsf{k})$ is local, appropriate dg structures allow to describe minimal free resolutions, see Lemma 3.1.3. In view of such uses of dg structures, Buchsbaum and Eisenbud [12] posed the question of whether a minimal free resolution U of M over Q admits a dg A-module structure, in the case when A is a Koszul complex. Subsequently, this question was studied in the work of several authors, see [7], [28], [33], [31], [41], [42], with both positive and negative answers.

When $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ and $R = Q/(f)$, a construction of Shamash can be used to build a semi-free dg module structure of U over the Koszul complex on f, see [5, Proposition 2.2.2], leading to a description of a minimal free resolution of M over R when f is, in addition, a non-zero divisor, cf. [5, Theorem 2.2.3]. In this chapter, our goal is to extend the Shamash construction when f is a zero divisor and to use it to describe minimal free resolutions when f is an exact zero divisor.

Throughout this chapter, $(Q, \mathfrak{m}, \mathsf{k})$ denotes a Noetherian local ring. Section 3.1 supplies definitions of dg algebra/module structures and the construction of the Tate resolution of R over Q, where (f, g) is an exact pair of zero divisors in Q. In Section 3.2 we show that the minimal free resolution of M over Q has a semi-free dg module structure over the Tate resolution and further use this information to produce a minimal free resolution of M over $R = Q/(f)$ when $f, g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. The main result is Theorem 3.2.1. The construction in this chapter is formally written as an algorithm in Appendix A. and has also been written as a Macaulay2 [23] package.

3.1 Dg algebras and construction of Tate resolutions

The primary objective of this section is to describe a construction of a minimal free resolution of R over Q endowed with a dg algebra structure when $R = Q/(f)$ and $f \in Q$ is an exact zero divisor. We do this in 3.1.5 by adapting a procedure of adjoining variables to a dg algebra to create new dg algebras, originally used by Cartan and further adapted by Tate.

We start the section by formally defining dg algebras/modules.

3.1.1. *Dg algebras and dg modules.* Let Q denote a commutative ring. If W is a complex of Q-modules, we denote by $W^{\#}$ the underlying graded Q-module and we use $|w|$ for the degree of a homogeneous element w of W.

A *dg algebra* over Q is a complex $A = (A, \partial)$, concentrated in non-negative homological degrees, where $A^{\#}$ is a Q-algebra that is (graded) commutative, meaning that

$$ab = (-1)^{|a||b|}ba \quad \text{for } a, b \in A \quad \text{and} \quad a^2 = 0 \text{ when } |a| \text{ is odd}\,,$$

and such that the Leibniz formula holds:

$$\partial(ab) = (\partial a)b + (-1)^{|a|}a(\partial b) \quad \text{for } a, b \in A\,.$$

If A and B are dg Q-algebras, then $A \otimes_Q B$ is also a dg-algebra, with multiplication given by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}aa' \otimes bb' \tag{3.1.1}$$

and the usual differential defined on a tensor product of complexes. We usually identify A and B with their images in $A \otimes_Q B$ and we write ab instead of $a \otimes b$.

A *dg module* U over A is a complex (U, ∂) with $U^{\#}$ an $A^{\#}$-module such that

$$\partial(au) = (\partial a)u + (-1)^{|a|}a(\partial u) \quad \text{for } a \in A \text{ and } u \in U.$$

A bounded below dg A-module U is said to be *semi-free* if $U^{\#}$ is free over $A^{\#}$.

Existence of semi-free dg module structures is particularly useful, as they give an easy construction of free resolutions over R from those over Q. For a more general construction, that does not require the semi-free assumption, see [28]. The benefit of the setting in Lemma 3.1.3 below is that, in the local case, one obtains a minimal resolution when starting with a minimal one.

Construction 3.1.2. Let $Q \to R$ be a surjective homomorphism of commutative rings. Let A be a dg Q-algebra algebra with $H_0(A) = R$ and let $U = (U, \partial)$ be a dg semi-free A-module. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ denote a basis of $U^{\#}$ over $A^{\#}$. For each integer n, let V_n denote the free Q-summand of U_n with basis $\{e_\lambda \mid \lambda \in \Lambda, |e_\lambda| = n\}$ and let $\pi_{n-1} \colon U_{n-1} \to V_{n-1}$ denote the projection map. Then $U \otimes_A R$ can be identified with the complex $U' = (U', \partial')$, with

$$U'_n = V_n \otimes_Q R \quad \text{and} \quad \partial'_n = \pi_{n-1}\partial_n\big|_{V_n} \otimes_Q 1_R\,.$$

Indeed, assume $R = Q/I$ for an ideal I, and $Q \to R$ is the natural projection. Let $\varepsilon \colon A \to R$ denote the augmentation map and set $J = \operatorname{Ker}(\varepsilon)$. We have then

$R = A/J$, and thus $U \otimes_A R \cong U/JU$. Let $n \geq 0$. The definition of V_n gives $U_n = V_n \oplus W_n$, where $W_n = (A_{\geq 1}U)_n$. We have thus:

$$(U \otimes_A R)_n \cong \left(\frac{U}{JU}\right)_n = \frac{U_n}{IU_n + W_n} \cong \frac{U_n}{W_n} \otimes_Q R \cong V_n \otimes_Q R\,.$$

The differential of U' is induced by the one on $U \otimes_A R$, which is in turn induced by the differential of U, and this yields directly the claimed expression for ∂'.

The following is a standard argument that appears, for instance, in the proof of [8, Theorem 2.2.3].

Lemma 3.1.3. *Let $Q \to R$ be a surjective homomorphism of commutative rings. Let A be a free resolution of R over Q that has a structure of dg algebra, and let $U = (U, \partial)$ be a free resolution of M over Q, that has a structure of semi-free dg-module over A. Then $U \otimes_A R$ is a free resolution of M over R. In particular, if Q is local and U is minimal, then $U \otimes_A R$ is a minimal free resolution of M over R.*

Proof. Since A is a free resolution of R over Q, the augmentation map $\varepsilon : A \to R$ is a quasi-isomorphism (that is, it induces an isomorphism in homology). By [8, Proposition 1.3.2], $\varepsilon\colon A \to R$ induces a quasi-isomorphism $U \to U \otimes_A R$. This shows that $U \otimes_A R$ is a free resolution of M over R. The last statement, regarding minimality, follows directly from Construction 3.1.2. □

3.1.4. *Exterior and divided powers variables.* Let (D, ∂) be a dg algebra over Q and let z be a cycle in D. Following [8, Construction 2.1.7, 6.1.1], we describe a construction that extends D to a dg algebra $D\langle y \mid \partial(y) = z\rangle$ by adjoining a variable y with $\partial(y) = z$. We also use the shortened notation $D\langle y\rangle$.

Case 1: $|z|$ is even. We define $Q\langle y\rangle$ to be a graded free Q-module with basis $\{1, y\}$ such that $|y| = |z| + 1$. We endow $Q\langle y\rangle$ with a Q-algebra structure by setting $y^2 = 0$, and extending by linearity. The dg algebra $D\langle y\rangle$ is defined as $D\langle y\rangle = D \otimes_Q Q\langle y\rangle$, with differential

$$\partial(d_0 + d_1 y) = \partial(d_0) + \partial(d_1)y + (-1)^{|d_1|} d_1 z\,.$$

In this case, y is called an exterior variable.

Case 2: $|z|$ is odd. We define $Q\langle y\rangle$ to be the graded free Q-module with basis $y^{(i)}$, where $y^{(0)} = 1$, $|y| = |z| + 1$ and $|y^{(i)}| = |y|i$ for all $i \geq 0$. We write $y^{(1)} = y$ and we make the convection that $y^{(i)} = 0$ when $i < 0$. We endow $Q\langle y\rangle$ with a Q-algebra structure by setting $y^{(i)}y^{(j)} = \binom{i+j}{i} y^{(i+j)}$, and extending by linearity. The dg algebra $D\langle y\rangle$ is defined as $D\langle y\rangle = D \otimes_Q Q\langle y\rangle$, with differential

$$\partial\left(\sum_i d_i y^{(i)}\right) = \sum_i \partial(d_i) y^{(i)} + \sum_i (-1)^{|d_i|} d_i z y^{(i-1)}\,.$$

In this case, y is called a divided powers variable.

The main purpose of presenting this construction is to describe, following [8, 6.3.1], a free resolution over Q of a quotient $R = Q/I$ that has a dg-algebra structure (in other words, a dg algebra resolution), known as a *Tate resolution* of R over Q. Namely, one successively adjoins finitely many variables in each degree such that the resulting dg algebra, that we shall denote A, is a free resolution of R over Q. This resolution can be chosen to be minimal when $R = \mathsf{k}$, but, in general, it may not be minimal, and it requires adjunction of infinitely many variables. We discuss below a situation when the Tate resolution is minimal and requires adjunction of only two variables.

3.1.5. *Tate resolution in the exact zero divisor case.* Assume (f, g) is an exact pair of zero divisors and set $R = Q/(f)$. Then a Tate resolution A of R over Q is obtained by adjoining one exterior variable y in degree 1 and one divided power variable t in degree 2, namely

$$A = Q\langle y, t \mid \partial(y) = f, \partial(t) = gy\rangle \,. \tag{3.1.2}$$

For notation uniformity, we set $y_{2i} = t^{(i)}$ and $y_{2i+1} = t^{(i)}y$ for all i. With this notation, A is a free Q-algebra with basis $\{y_i\}_{i\geq 0}$, with $|y_i| = i$ for all $i \geq 0$, its differential is described by

$$\partial(y_{2n+1}) = fy_{2n} \quad \text{and} \quad \partial(y_{2n}) = gy_{2n-1}$$

for all $n \geq 0$, and the multiplication is given by

$$y_{2i+1}y_{2j+1} = 0 \quad \text{and} \quad y_{2i}y_{2j+\varepsilon} = \binom{i+j}{i} y_{2(i+j)+\varepsilon} = y_{2j+\varepsilon}y_{2i} \tag{3.1.3}$$

for all $i \geq 0$, $j \geq 0$, where $\varepsilon \in \{0, 1\}$. Using these relations, define $\alpha_{i,j} \in Q$ such that

$$y_iy_j = \alpha_{i,j}y_{i+j} \quad \text{for all } i \geq 0,\, j \geq 0\,.$$

More precisely, we have $\alpha_{i,j} = 0$ when both i and j are odd, $\alpha_{i,j} = \binom{\frac{1}{2}(i+j)}{\frac{i}{2}}$ if both i, j are even, $\alpha_{i,j} = \binom{\frac{1}{2}(i+j-1)}{\frac{i}{2}}$ if i is even and j is odd, and $\alpha_{i,j} = \binom{\frac{1}{2}(i+j-1)}{\frac{i-1}{2}}$ if i is odd and j is even.

For all $i, j \geq 0$, note that

$$\alpha_{i,j} = \alpha_{j,i}\,.$$

Further, one can easily verify that

$$\alpha_{i,j}\alpha_{i+j,k} = \alpha_{j,k}\alpha_{i,j+k} \qquad \text{for all } i,j,k \geq 0 \tag{3.1.4}$$

$$\alpha_{i-1,j}f_i + (-1)^i\alpha_{i,j-1}f_j = \alpha_{i,j}f_{i+j} \qquad \text{for all } i,j \geq 1\,. \tag{3.1.5}$$

In fact, (3.1.4) is equivalent to the associativity of A, namely $(y_iy_j)y_k = y_i(y_jy_k)$, and (3.1.5) is equivalent to the Leibniz rule for A, namely:

$$\partial(y_iy_j) = \partial(y_i)y_j + (-1)^iy_i\partial(y_j)\,.$$

Finally, observe that the Tate construction A is a minimal free resolution of $Q/(f)$ over Q:

$$A\colon \qquad \cdots \to Qy_{2n} \xrightarrow{g} Qy_{2n-1} \xrightarrow{f} Qy_{2n-2} \to \cdots \to Qy_2 \xrightarrow{g} Qy_1 \xrightarrow{f} Qy_0 \to 0\,.$$

Remark 3.1.6. Let A be as in 3.1.5, and set $R = Q/(f)$. Then, with the assumptions and notation introduced in Construction 3.1.2, we have an isomorphism of free Q-modules

$$U_n = V_n \oplus y_1V_{n-1} \oplus y_2V_{n-2} \oplus \cdots \oplus y_nV_0 \cong \bigoplus_{i=0}^{n} V_n\,. \tag{3.1.6}$$

3.2 Semi-free dg module structure

The theorem below is an extension of a construction of Shamash [39], along the lines of the proof in [5, Proposition 2.2.2, Theorem 2.2.3]. A corollary about betti numbers is given at the end of the section.

Theorem 3.2.1. *Let $(Q, \mathfrak{m})$ be a local ring and let $f, g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ such that $fg = 0$. Set $R = Q/(f)$ and $A = Q\langle y, t \mid \partial(y) = f, \partial(t) = gy\rangle$. If M is a finitely generated R-module and U is a minimal free resolution of M over Q, then:*

(a) *U has a structure of semi-free dg module over A.*

(b) *Furthermore, if $\operatorname{ann}(f) = (g)$ and $\operatorname{ann}(g) = (f)$, then $U \otimes_A R$ is a minimal free resolution of M over R.*

Proof. (a) For all integers $i \geq 0$ and n we will construct homomorphisms

$$\sigma_{i,n} \colon U_n \to U_{i+n}$$

and free Q-modules V_n with $V_n \subseteq U_n$ such that:

(1) $\sigma_{0,n} = \mathrm{id}_{U_n}$ for all n.

(2) For all n and $i \geq 1$ the following condition holds:

$$\mathcal{A}(i, n): \qquad \partial_{i+n}\sigma_{i,n} + (-1)^{i+1}\sigma_{i,n-1}\partial_n = f_i\sigma_{i-1,n}\,. \tag{3.2.1}$$

(3) For all n and all i, j with $i \geq 0$ and $j \geq 0$ the following condition holds:

$$\mathcal{B}(i,j,n): \qquad \sigma_{i,n}\sigma_{j,n-j} = \alpha_{i,j}\sigma_{i+j,n-j}\,.$$

(4) For all n the following condition holds:

$$\mathcal{C}(n): \quad \text{The map} \quad \Phi_n\colon V_0 \oplus V_1 \oplus \cdots \oplus V_n \to U_{n+1} \quad \text{given by}$$

$$\Phi_n(x_0, x_1, \ldots, x_n) = \sum_{i=1}^{n+1} \sigma_{i,n+1-i}(x_{n+1-i})$$

is a split injection.

(5) For all n the following holds:

$$\mathcal{D}(n): \quad U_n = \sum_{k=0}^{n} \sigma_{k,n-k}(U_{n-k}) = \bigoplus_{k=0}^{n} \sigma_{k,n-k}(V_{n-k})\,,$$

where the direct sum indicates an *internal* direct sum, that is, every element of U_n can be written uniquely as a sum of elements in $\sigma_{k,n-k}(V_{n-k})$.

Once these conditions are established, we define

$$y_i u = \sigma_{i,n}(u) \quad \text{for all } n,\ i \geq 0 \text{ and } u \in U_n$$

and then extend linearly to a multiplication $A \times U \to U$. Conditions (1)-(3) imply that this rule defines a dg A-module structure on U. Item (5) shows that U is a semi-free dg A-module. Indeed, condition $\mathcal{D}(n)$ can be re-written as $U_n = \bigoplus_{k=0}^{n} y_k V_{n-k}$, implying that a basis of $U^{\#}$ over $A^{\#}$ is $\cup_{i\geq 0} B_i$, where B_i denotes a Q-basis of V_i.

Before proceeding with the bulk of the proof, we show:

Claim. Let n be an integer. Assume $\mathcal{C}(m)$, $\mathcal{D}(m)$ and $\mathcal{B}(i,j,m)$ hold for all m with $0 \le m \le n-1$ and all $i,j \ge 0$. Then

$$\sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k}) = \bigoplus_{k=j}^{n} \sigma_{k,n-k}(V_{n-k}) \subseteq U_n \quad \text{for all } j \ge 1\,. \tag{3.2.2}$$

Proof of Claim. Let $j \ge 1$. We have:

$$\begin{aligned}
\sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k}) &= \sum_{k=j}^{n} \sum_{k'=0}^{n-k} \sigma_{k,n-k}\sigma_{k',n-k-k'}(V_{n-k-k'}) \\
&\subseteq \sum_{k=j}^{n} \sum_{k'=0}^{n-k} \sigma_{k+k',n-(k+k')}(V_{n-(k+k')}) \\
&\subseteq \sum_{l=j}^{n} \sigma_{l,n-l}(V_{n-l}) \\
&\subseteq \sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k})\,.
\end{aligned}$$

We used $\mathcal{D}(n-k)$ in the first line, $\mathcal{B}(k,k',n-k)$ in the second, and $V_i \subseteq U_i$ in the fourth. (Note that $k \ge j \ge 1$ and thus $n-k \le n-1$.) It follows that all inclusions above are in fact equalities. Further, the sum $\sum_{k=j}^{n} \sigma_{k,n-k}(V_{n-k})$ is an internal direct sum, as can be seen from $\mathcal{C}(n-1)$. This finishes the proof of the claim.

We now proceed to construct the maps $\sigma_{i,n}$ and the modules V_n. We start by setting $\sigma_{*,n} = 0$ and $V_n = 0$ whenever $n < 0$. With this definition, note that (1)-(5) hold when $n < 0$. We proceed by induction.

Consider the following statement, depending on an integer k.

Induction statement I. The maps $\sigma_{i,k}\colon U_k \to U_{i+k}$ are defined for all $i \ge 0$ and the free Q-module V_k with $V_k \subseteq U_k$ are defined, and satisfy the properties:

- $\sigma_{0,k} = \mathrm{id}_{U_k}$
- $\mathcal{A}(i,k)$ holds for all $i \geq 1$
- $\mathcal{B}(i,j,k)$ holds for all $i \geq 0$, $j \geq 0$
- $\mathcal{C}(k)$ and $\mathcal{D}(k)$ hold.

As noted above, this statement holds when $k < 0$. Let $n \geq 0$. Assume that Induction statement I holds for all $k \leq n-1$. To complete the induction, we define next $\sigma_{*,n}$ and V_n, and we show that all four items in the Induction statement I hold when $k = n$.

We start by defining $\sigma_{0,n}$ to be the identity map on U_n. Then, we define the free module V_n as follows: Condition $\mathcal{C}(n-1)$ gives that $\mathrm{Im}\Phi_{n-1}$ is a direct summand of U_n. We define V_n to be the complementary direct summand, so that

$$U_n = \mathrm{Im}\Phi_{n-1} \oplus V_n = \mathrm{Im}\Phi_{n-1} \oplus \sigma_{0,n}(V_n)\,.$$

Observe that $\mathcal{D}(n)$ must then hold, as it is a direct consequence of this definition and of $\mathcal{C}(n-1)$. We need to define now $\sigma_{i,n}$ for $i \geq 1$. We proceed by induction.

With n fixed as above, consider the following statement, depending on an integer $l \geq 0$:

Induction statement II. The map $\sigma_{l,n} \colon U_n \to U_{l+n}$ is defined such that

- $\mathcal{A}(l,n)$ holds

- $\mathcal{B}(l, j, n)$ holds for all $j \geq 0$.

Note that these two conditions hold trivially when $l = 0$. Let $i \geq 1$ and assume that Induction statement II holds for all l with $l \leq i - 1$. We now define $\sigma_{i,n}$ and we show that the two items in Induction Statement II also hold with $l = i$.

In view of the direct sum decomposition of U_n given by condition $\mathcal{D}(n)$ (which we know holds, see above), in order to define $\sigma_{i,n}$ it suffices to define the restriction functions $\sigma_{i,n}\big|_{\sigma_{k,n-k}(V_{n-k})}$ for all k with $0 \leq k \leq n$.

Assume first $k > 0$. Note that $\mathcal{C}(n - k)$ implies that the restricted function

$$\sigma_{k,n-k}\big|_{V_{n-k}} : V_{n-k} \to \sigma_{k,n-k}(V_{n-k})$$

is bijective. We define then

$$\sigma_{i,n}\big|_{\sigma_{k,n-k}(V_{n-k})} := \alpha_{i,k}\sigma_{i+k,n-k}\big|_{V_{n-k}} (\sigma_{k,n-k}\big|_{V_{n-k}})^{-1} .$$

In other words, this definition ensures that

$$\sigma_{i,n}\sigma_{k,n-k}\big|_{V_{n-k}} = \alpha_{i,k}\sigma_{i+k,n-k}\big|_{V_{n-k}} \quad \text{when } k > 0 . \tag{3.2.3}$$

We need to define now $\sigma_{i,n}\big|_{V_n}$. To do so, we first claim that

$$\big(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n\big)\,(U_n) \subseteq \partial_{i+n}(U_{i+n}) . \tag{3.2.4}$$

Assuming (3.2.4) holds and recalling that V_n is a free Q-module, we can define $\sigma_{i,n}\big|_{V_n} : V_n \to U_{i+n}$ so that

$$\partial_{i+n}\sigma_{i,n}\big|_{V_n} = f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n\big|_{V_n} . \tag{3.2.5}$$

Indeed, fix a basis of V_n and define $\sigma_{i,n}\big|_{V_n}(e)$ for a basis element e to be the preimage of $\left(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n\right)(e)$ under ∂_{i+n}, and then extend by linearity. Note that this definition depends on the choice of preimage, and thus is not unique.

To prove (3.2.4), we compute:

$$\begin{aligned}
&\partial_{i+n-1}(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n) \\
&\qquad = f_i\partial_{i+n-1}\sigma_{i-1,n} + (-1)^i\partial_{i+n-1}\sigma_{i,n-1}\partial_n \\
&\qquad = f_i\left(f_{i-1}\sigma_{i-2,n} - (-1)^i\sigma_{i-1,n-1}\partial_n\right) + \\
&\qquad\qquad + (-1)^i\left(f_i\sigma_{i-1,n-1}\partial_n - (-1)^{i+1}\sigma_{i-1,n-2}\partial_{n-1}\partial_n\right) \\
&\qquad = 0\,,
\end{aligned}$$

where for the second equality we used the induction hypothesis that $\mathcal{A}(i-1,n)$ and $\mathcal{A}(i,n-1)$ hold, and for the third equality we used $f_if_{i-1} = fg = 0$ and $\partial^2 = 0$. The computation above shows

$$\left(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n\right)(U_n) \subseteq \operatorname{Ker}(\partial_{i+n-1})\,.$$

Since $i \geq 1$ and $n \geq 0$, we have $i+n-1 \geq 0$. When $i+n-1 > 0$, we use the fact that U is acyclic, and hence $\operatorname{Ker}(\partial_{i+n-1}) = \operatorname{Im}(\partial_{i+n})$ and thus (3.2.4) holds. When $i = 1$ and $n = 0$ we have

$$f_1\sigma_{0,0} - (-1)^2\sigma_{1,-1}\partial_0 = f_1\mathrm{id}_{U_0}\,.$$

Since U is a minimal free resolution of M, we have $M = U_0/\partial_1(U_1)$. Since $fM = 0$, we have $f_1 U_0 \subseteq \partial_1(U_1)$. This finishes the proof of (3.2.4) and the definition $\sigma_{i,n}$.

Let $j \geq 0$. We want to prove that $\mathcal{B}(i,j,n)$ holds. Since $\sigma_{0,n}$ is the identity map, $\mathcal{B}(i,0,n)$ holds.

Assume now $j > 0$. In view of the direct sum decomposition provided by $\mathcal{D}(n-j)$, in order to prove $\mathcal{B}(i,j,n)$, it suffices to show

$$\sigma_{i,n}\sigma_{j,n-j}\big|_{\sigma_{k,n-j-k}(V_{n-j-k})} = \alpha_{i,j}\sigma_{i+j,n-j}\big|_{\sigma_{k,n-j-k}(V_{n-j-k})}$$

for all k with $0 \leq k \leq n-j$. We need to check that

$$\sigma_{i,n}\sigma_{j,n-j}\sigma_{k,n-j-k}\big|_{V_{n-j-k}} = \alpha_{i,j}\sigma_{i+j,n-j}\sigma_{k,n-j-k}\big|_{V_{n-j-k}}.$$

Indeed, we have:

$$\begin{aligned}
\sigma_{i,n}\sigma_{j,n-j}\sigma_{k,n-j-k}\big|_{V_{n-j-k}} &= \alpha_{j,k}\sigma_{i,n}\sigma_{j+k,n-(j+k)}\big|_{V_{n-(j+k)}} \\
&= \alpha_{j,k}\alpha_{i,j+k}\sigma_{i+j+k,n-(j+k)}\big|_{V_{n-(j+k)}} \\
&= \alpha_{i,j}\alpha_{i+j,k}\sigma_{i+j+k,n-(j+k)}\big|_{V_{n-(j+k)}} \\
&= \alpha_{i,j}\sigma_{i+j,n-j}\sigma_{k,n-j-k}\big|_{V_{n-j-k}}.
\end{aligned}$$

Here, we used $\mathcal{B}(j,k,n-j)$ in the first equality, (3.2.3) in the second, (3.1.4) in the third, and $\mathcal{B}(i+j,k,n-j)$ in the last. This finishes the proof of $\mathcal{B}(i,j,n)$.

Next, we want to prove that $\mathcal{A}(i,n)$ holds. Given the definition of $\sigma_{i,n}\big|_{V_n}$, we know that the relation holds when we restrict the functions to V_n. We need to also check this relation when the functions are restricted to $\sigma_{k,n-k}(V_{n-k})$ with $k > 0$. To this extent, it suffices to show

$$(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n)\sigma_{k,n-k} = \partial_{i+n}\sigma_{i,n}\sigma_{k,n-k} \quad \text{for all } k > 0\,.$$

Let $k > 0$. The computation below achieves the desired conclusion:

$$\begin{aligned}
&(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n)\sigma_{k,n-k} \\
&\quad = f_i\sigma_{i-1,n}\sigma_{k,n-k} - (-1)^{i+1}\sigma_{i,n-1}\partial_n\sigma_{k,n-k} \\
&\quad = \alpha_{i-1,k}f_i\sigma_{i-1+k,n-k} - (-1)^{i+1}\sigma_{i,n-1}(f_k\sigma_{k-1,n-k} - (-1)^{k+1}\sigma_{k,n-k-1}\partial_{n-k}) \\
&\quad = \alpha_{i-1,k}f_i\sigma_{i-1+k,n-k} + (-1)^{i}\alpha_{i,k-1}f_k\sigma_{i+k-1,n-k} + (-1)^{i+k}\alpha_{i,k}\sigma_{i+k,n-k-1}\partial_{n-k} \\
&\quad = \alpha_{i,k}f_{i+k}\sigma_{i+k-1,n-k} - \alpha_{i,k}(-1)^{i+k+1}\sigma_{i+k,n-k-1}\partial_{n-k} \\
&\quad = \alpha_{i,k}\partial_{i+n}\sigma_{i+k,n-k} \\
&\quad = \partial_{i+n}\sigma_{i,n}\sigma_{k,n-k}\,.
\end{aligned}$$

We used $\mathcal{B}(i-1,k,n)$ and $\mathcal{A}(k,n-k)$ for the second equality, $\mathcal{B}(i,k-1,n-1)$ and $\mathcal{B}(i,k,n-1)$ for the third, (3.1.5) for the fourth, $\mathcal{A}(i+k,n-k)$ for the fifth and $\mathcal{B}(i,k,n)$ (which was proved earlier) for the last equality.

At this point, we verified Induction statement II for $l = i$, hence we have a definition of $\sigma_{i,n}$ for all i, such that $\mathcal{A}(i,n)$ holds and $\mathcal{B}(i,j,n)$ holds for all $i,j \geq 0$. We need to finalize the proof of the Induction statement I for $k = n$. The only remaining item is to show $\mathcal{C}(n)$, namely that the map Φ_n is a split injective. We will show that if $\Phi_n(x) \in \mathfrak{m}U_{n+1}$ for

$$x = (x_0, x_1, \ldots, x_n) \in V_0 \oplus V_1 \oplus \cdots \oplus V_n\,,$$

then $x_i \in \mathfrak{m}V_i$ for all i.

Assume $\Phi_n(x) \in \mathfrak{m}U_{n+1}$. Then $\partial_{n+1}\Phi_n(x) \in \mathfrak{m}^2U_n$. For the purposes of the next computation, we set $x_{n+1} = 0$. We have then:

$$\begin{aligned}
\partial_{n+1}\Phi_n(x) &= \sum_{i=1}^{n+1} \partial_{n+1}\sigma_{i,n+1-i}(x_{n+1-i}) \\
&= \sum_{i=1}^{n+1} f_i\sigma_{i-1,n+1-i}(x_{n+1-i}) - (-1)^{i+1}\sigma_{i,n-i}\partial_{n+1-i}(x_{n+1-i}) \\
&= \sum_{j=0}^{n} f_{j+1}\sigma_{j,n-j}(x_{n-j}) + \sum_{j=0}^{n}(-1)^j\sigma_{j,n-j}\partial_{n+1-j}(x_{n+1-j}) \\
&= \sum_{j=0}^{n} \sigma_{j,n-j}\left(f_{j+1}x_{n-j} - \partial_{n-j+1}(x_{n-j+1})\right)\,.
\end{aligned} \tag{3.2.6}$$

We used $\mathcal{A}(i, n+1-i)$ for the second equality. Next, we set

$$w_j = \sigma_{j,n-j}\left(f_{j+1}x_{n-j} - \partial_{n-j+1}(x_{n-j+1})\right)\,,$$

so that (3.2.6) becomes

$$\partial_{n+1}\Phi_n(x) = \sum_{j=0}^{n} w_j\,.$$

We prove by induction on i with $0 \le i \le n+1$ that $x_{n+1-i} \in \mathfrak{m}V_{n+1-i}$. This is true when $i = 0$, as we defined $x_{n+1} = 0$. Let k be so that $0 \le k \le n$ and assume that $x_{n+1-i} \in \mathfrak{m}V_{n+1-i}$ for all i with $0 \le i \le k$. The fact that that $f_i \in \mathfrak{m}$ for all i and the minimality of U imply

$$w_j \in \mathfrak{m}^2V_{n-j} \quad \text{for all} \quad j \le k-1\,.$$

To complete the induction argument, we show that $x_{n-k} \in \mathfrak{m}V_{n-k}$. We have

$$\begin{aligned} f_{k+1}\sigma_{k,n-k}(x_{n-k}) + \sum_{j=k+1}^{n} w_j &= w_k + \sigma_{k,n-k}\partial_{n-k+1}(x_{n-k+1}) + \sum_{j=k+1}^{n} w_j \\ &= \sigma_{k,n-k}\partial_{n-k+1}(x_{n-k+1}) + \partial_{n+1}\Phi_n(x) - \sum_{j=0}^{k-1} w_j \\ &\in \mathfrak{m}^2 U_n\,. \end{aligned}$$

Recall the direct sum decomposition of U_n given by property $\mathcal{D}(n)$, and notice that $f_{k+1}\sigma_{k,n-k}(x_{n-k})$ and $\sum_{j=k+1}^{n} w_j$ belong to distinct summands in this decomposition. Indeed, we have $f_{k+1}\sigma_{k,n-k}(x_{n-k}) \in \sigma_{k,n-k}(V_{n-k})$, while (3.2.2) implies

$$\sum_{j=k+1}^{n} w_j \in \sum_{j=k+1}^{n} \sigma_{j,n-j}(U_{n-j}) = \bigoplus_{j=k+1}^{n} \sigma_{j,n-j}(V_{n-j})\,.$$

We conclude $f_{k+1}\sigma_{k,n-k}(x_{n-k}) \in \mathfrak{m}^2U_n$, and hence $\sigma_{k,n-k}(x_{n-k}) \in \mathfrak{m}U_n$, since $f_{k+1} \notin \mathfrak{m}^2$. Since Φ_{n-1} is a split injection by $\mathcal{C}(n-1)$, it follows $x_{n-k} \in \mathfrak{m}V_{n-k}$.

We showed thus that $\mathcal{C}(n)$ holds. This finishes the proof of Induction Statement I and concludes the proof of (a).

(b) The fact that (f, g) is an exact pair of zero divisors implies that A is a minimal free resolution of R over Q, as noted in 3.1.5. The conclusion follows then from Lemma 3.1.3, making use of (a). □

Remark 3.2.2. The proof of the Theorem 3.2.1 is inspired by the construction of Shamash described in [5, Proposition 2.2]. It is proved there that if $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, then, with notation as in our theorem, U has a structure of semi-free dg module over the

Koszul complex $K = Q\langle y \mid \partial(y) = f\rangle$. As noted in [5, Remark 2.2.1], since $f\mathrm{id}^U$ and 0^U both induce the zero map on M, they are homotopic, say $f\mathrm{id}^U = \partial\sigma + \sigma\partial$, and the existence of the desired dg module structure is equivalent with the existence of a homotopy σ with $\sigma^2 = 0$. The map $\sigma_{1,*}$ constructed in our proof above is exactly the homotopy σ constructed in the proof of [5, Proposition 2.2]. In fact, all maps $\sigma_{i,*}$ with $i \geq 1$ are homotopies. Indeed, since $f_{i-1}f_i = 0$, we see from condition $\mathcal{A}(i,n)$ in (3.1.2) that $f_{i-1}\sigma_{i,*}$ is a chain map of degree i on U, for all $i \geq 1$. Consequently, $\mathcal{A}(i,n)$ shows that $\sigma_{i,*}$ is a homotopy between $f_i\sigma_{i-1,*}$ and the zero map on U.

Remark 3.2.3. Assume that the hypothesis of the Theorem 3.2.1 holds. In the proof of the theorem, we saw that a basis for $U^\#$ over $A^\#$ can be taken to consist of the union of the bases of the free Q-modules $V_0, V_1, V_2, \ldots$. With this choice of basis, the modules V_n defined in the proof of the theorem coincide with the modules V_n introduced in Construction 3.1.2, and thus the differential of the complex $U \otimes_A R$ can be understood as described there. Further, Construction 3.1.2 and (3.1.6) give

$$U_n \cong \bigoplus_{i=0}^{n} V_i \qquad \text{and} \qquad (U \otimes_A R)_n \cong V_n \otimes_Q R\,. \tag{3.2.7}$$

When R is a local ring and M is a finitely generated R-module, we let $\beta_n^R(M)$ denote the nth *Betti number* of M over R, that is, the rank of the nth free R-module in a minimal free resolution of M, and we let

$$\mathrm{P}_M^R(t) = \sum_{i=0}^{\infty} \beta_i^R(M) t^i$$

denote the *Poincaré series* of M over R.

Corollary 3.2.4 below recovers the Poincaré series formula in [25, Theorem 1.7], and shows that the hypothesis on f, g in part (b) of Theorem 3.2.1 is necessary.

Corollary 3.2.4. *Let $(Q, \mathfrak{m})$ be a local ring and $f, g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ with $fg = 0$. Set $R = Q/(f)$ and $A = Q\langle y, t \mid \partial(y) = f, \partial(t) = gy\rangle$. The following are equivalent:*

1. *(f, g) is an exact pair of zero divisors;*
2. *For all finitely generated R-modules M, if U is a minimal free resolution of M over Q, then $U \otimes_A R$ is a minimal free resolution of M over R;*
3. *For all finitely generated R-modules M, $\mathrm{P}^R_M(t) = (1-t)\,\mathrm{P}^Q_M(t)$.*

Proof. The implication (1) $\Longrightarrow$ (2) is established in Theorem 3.2.1(b).

(2) $\Longrightarrow$ (3): Let $n \geq 0$. Under the hypothesis of (2), we have $\beta^R_n(M) = \mathrm{rank}_R(U \otimes_A R)_n$ and $\beta^Q_n(M) = \mathrm{rank}_Q(U_n)$. Then (3.2.7) gives

$$\beta^Q_n(M) = \sum_{i=0}^{n} \beta^R_i(M)$$

and the Poincaré series formula follows from here.

(3) $\Longrightarrow$ (1): If (3) holds, take $M = R$ in the Poincaré series formula to conclude $P^Q_R(t) = (1-t)^{-1}$, and hence $\beta^Q_n(R) = 1$ for all $n \geq 0$. Since $\mathrm{ann}(f)$ is a second syzygy in a minimal free resolution of $R = Q/(f)$ over Q, we conclude that $\mathrm{ann}(f)$ is principal. If $\mathrm{ann}(f) = (h)$ for some $h \in \mathfrak{m}$, then, since $g \in (h)$ and $g \notin \mathfrak{m}^2$,

we must have $g = uh$ for some unit u, and hence $\operatorname{ann}(f) = (h) = (g)$. Then $\operatorname{ann}(g)$ is a third syzygy in a minimal free resolution of R over Q, and we conclude it must be principal as well. Similarly, we obtain $\operatorname{ann}(g) = (f)$. $\square$

CHAPTER 4

THE MAPPING CONE OF AN EISENBUD OPERATOR

Let $\varphi\colon Q \to R = Q/(f)$ be a surjective ring homomorphism. If (F, ∂) is a complex of finitely generated R-modules, then a classical construction due to Eisenbud [21] produces a degree -2 map τ on F, that we refer to as an *Eisenbud operator*. When f is a non-zero divisor, $\tau\colon F \to F$ is a degree -2 chain map and is also called a *CI operator*. Building on a result of Gulliksen in [24], these maps have been successfully used in the study of free resolutions and (co)homology over complete intersection rings, see [3]. Other equivalent constructions of this operator can be found in [35, 3].

Windle [44] constructed similar operators in the case when f is an exact zero divisor. In this chapter, we adopt a somewhat different construction that better suits our purposes. Mainly, our goal is to use a mapping cone construction in order to describe the change in homology over Q versus homology over R by establishing short exact sequences that resemble the sequence (1.0.1) recalled in the introduction.

The mapping cone is a construction that produces new complexes from a chain map. One of the prominent examples of its application in building free resolution can be seen in [22]. Even the Koszul complex (which is a minimal free resolution if the elements involved form a regular sequence) can be iteratively built using mapping cone constructions. For more examples of its application in building minimal free resolutions, see [20, 40, 29].

Throughout this chapter, $(Q, \mathfrak{m}, \mathsf{k})$ denotes a local ring with $\mathfrak{m}$ as its unique maximal ideal and $\mathsf{k} = Q/\mathfrak{m}$. In Section 4.1 we describe the construction of the operator in the case of an exact zero divisor, and we describe it using differential graded structures. In Section 4.2 we apply the mapping cone construction, and we use it to obtain one of the exact sequences of Theorem 4.2.6. The second sequence in this theorem comes from further uses of dg structures.

4.1 The Eisenbud operator construction

In this section, we extend the classical construction of an Eisenbud operator associated to a regular element $f \in Q$ in order to allow for f to be a zero divisor. We then give a concrete computation of the operator using the language of dg algebras.

4.1.1. *Eisenbud operators.* Let $f \in Q$ and $R = Q/(f)$, and let (F, ∂) be a complex of R-free modules. A *lifting* of the complex F to Q is a sequence $(\widetilde{F}, \widetilde{\partial})$ with

$$\cdots \to \widetilde{F}_{n+1} \xrightarrow{\widetilde{\partial}_{n+1}} \widetilde{F}_n \xrightarrow{\widetilde{\partial}_n} \widetilde{F}_{n-1} \to \cdots$$

such that $\widetilde{F}_n$ is a free Q-module for all n and

$$\widetilde{F} \otimes_Q R = F \quad \text{and} \quad \widetilde{\partial} \otimes_Q R = \partial\,.$$

Note that $(\widetilde{F}, \widetilde{\partial})$ may not be a complex since $\widetilde{\partial}^2$ is not necessarily 0. However, since

$\partial^2 = 0$, we get that $\widetilde{\partial}^2 = f\widetilde{\tau}$ for a map

$$\widetilde{\tau} : \Sigma^{-1}\widetilde{F} \to \Sigma\widetilde{F}.$$

We set

$$S = \frac{R}{(\operatorname{ann}_Q(f))R} = \frac{Q}{(f) + \operatorname{ann}_Q(f)}$$

and identify $\widetilde{F} \otimes_Q S = F \otimes_R S$. We define

$$\tau = \widetilde{\tau} \otimes_Q S \colon \Sigma^{-1}F \otimes_R S \to \Sigma F \otimes_R S\,. \tag{4.1.1}$$

When $f \in Q$ is a regular element, $\operatorname{ann}_Q(f) = 0$ and hence $S = R$. In this case, Eisenbud [21, Proposition 1.1-1.3] proves that $\tau\colon \Sigma^{-1}F \to \Sigma F$ is a chain map, is independent up to homotopy of the choice of the lifting $(\widetilde{F}, \widetilde{\partial})$, and is natural, in the sense of Lemma 4.1.2(b) below. With our extended definition above, we prove in Lemma 4.1.2 that these properties hold without the assumption that f is regular.

Extending the terminology classically used in the case when f is regular, we say that the map τ in (4.1.1) is *the Eisenbud operator* associated to the data $(f, F, \widetilde{F})$. A (different) extension of the notion of the Eisenbud operator to the case of exact zero divisors was explored by Windle in [44].

Lemma 4.1.2. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring, $f \in Q$ and set $R = Q/(f)$. Let (F, ∂) be a complex of free R-modules and let $(\widetilde{F}, \widetilde{\partial})$ be a lifting of F to Q. If τ is the Eisenbud operator associated to the data $(f, F, \widetilde{F})$, then the following hold:*

(a) τ is a chain map;

(b) Let (F', ∂') be another complex of free R-modules and $(\widetilde{F'}, \widetilde{\partial}')$ be a lifting of F' to Q. If $h : F \to F'$ is a chain map of degree 0, and τ' is the Eisenbud operator associated to the data $(f, F', \widetilde{F'})$, then $\tau'(h \otimes_Q S)$ is homotopic to $(h \otimes_Q S)\tau$;

(c) The operator τ is independent of the choice of the lifting up to homotopy.

Proof. (a) We need to show $\tau(\partial \otimes_Q S) = (\partial \otimes_Q S)\tau$, or, in other words

$$\left(\widetilde{\tau}\widetilde{\partial} - \widetilde{\partial}\widetilde{\tau}\right) \otimes_Q S = 0.$$

To establish this conclusion, we show $\operatorname{Im}\left(\widetilde{\tau}\widetilde{\partial} - \widetilde{\partial}\widetilde{\tau}\right) \subseteq \operatorname{ann}_Q(f)\widetilde{F}$.

Indeed, since $\widetilde{\partial}^2 = f\widetilde{\tau}$, we have

$$f\left(\widetilde{\tau}\widetilde{\partial} - \widetilde{\partial}\widetilde{\tau}\right) = (f\widetilde{\tau})\widetilde{\partial} - \widetilde{\partial}(f\widetilde{\tau}) = \widetilde{\partial}^2\widetilde{\partial} - \widetilde{\partial}\widetilde{\partial}^2 = \widetilde{\partial}^3 - \widetilde{\partial}^3 = 0\,.$$

(b) Let $h : F \to F'$ be a chain map. We set $\widetilde{h} : \widetilde{F} \to \widetilde{F'}$ such that $\widetilde{h} \otimes_Q R = h$ and thus we have

$$\widetilde{\partial}'\widetilde{h} - \widetilde{h}\widetilde{\partial} = f\alpha\,, \tag{4.1.2}$$

for some $\alpha : \widetilde{F} \to \Sigma\widetilde{F}'$. To show that $\tau'(h \otimes_R S)$ is homotopic to $(h \otimes_R S)\tau$, we show

$$\tau'(h \otimes_R S) - (h \otimes_R S)\tau = (\partial' \otimes_R S)(\alpha \otimes_R S) + (\alpha \otimes_R S)(\partial \otimes_R S)\,.$$

We reach this conclusion by showing

$$f\left(\widetilde{h}\widetilde{\tau} - \widetilde{\tau}'\widetilde{h} + \widetilde{\partial}'\alpha + \alpha\widetilde{\partial}\right) = 0\,. \tag{4.1.3}$$

Indeed, by definition of τ and τ' we have $f\widetilde{\tau} = \widetilde{\partial}^2$ and $f\widetilde{\tau'} = \widetilde{\partial'}^2$, hence

$$\begin{aligned}
f\widetilde{h}\widetilde{\tau} = \widetilde{h}(f\widetilde{\tau}) = \widetilde{h}\widetilde{\partial}^2 &= (\widetilde{h}\widetilde{\partial})\widetilde{\partial} \\
&= \left(\widetilde{\partial'}\widetilde{h} - f\alpha\right)\widetilde{\partial} && \text{from (4.1.2)} \\
&= \widetilde{\partial'}\widetilde{h}\widetilde{\partial} - f\alpha\widetilde{\partial} \\
&= \widetilde{\partial'}\left(\widetilde{\partial'}\widetilde{h} - f\alpha\right) - f\alpha\widetilde{\partial} && \text{from (4.1.2)} \\
&= \widetilde{\partial'}^2\widetilde{h} - \widetilde{\partial'}f\alpha - f\alpha\widetilde{\partial} \\
&= f\widetilde{\tau'}\widetilde{h} - f\widetilde{\partial'}\alpha - f\alpha\widetilde{\partial}\,.
\end{aligned}$$

This establishes (4.1.3).

(c) The conclusion follows by taking $F' = F$ and $h = \mathrm{id}_F$ in (b). □

4.1.3. *Choice of lifting in the construction of the Eisenbud operator.* Let $f \in Q$ and $R = Q/(f)$, and let A denote a Tate resolution of R over Q obtained by adjoining variables as in 3.1.4. We let $\{y_\lambda\}_{\lambda\in\Lambda}$ denote a basis of $A^{\#}$ over Q, and set

$$\Lambda_i = \{\lambda \in \Lambda\colon |y_\lambda| = i\}$$

for each $i \geq 0$. Since (f) is principal, we may assume that only one variable in degree 1 is adjoined and we let y denote the variable with $|y| = 1$ and $\partial(y) = f$. Further, we see that

$$\partial(y_\lambda) \in \mathrm{ann}_Q(f)y \qquad \text{for all } y_\lambda \in \Lambda_2\,. \tag{4.1.4}$$

When f is a non-zero divisor, we take $\Lambda_i = \emptyset$ for $i > 1$, and when f is an exact zero divisor, recall from 3.1.5 that we may take $\Lambda_i = \{i\}$ for all $i \geq 0$.

Let M be a finite Q-module with $fM = 0$. By [8, Proposition 2.2.7], there exists a free resolution (U, ∂) of M over Q such that U has a semi-free dg module structure over A. Let $\{e_\delta\}_{\delta\in\Delta}$ denote a basis of $U^\#$ over $A^\#$, and for each $n \in \mathbb{Z}$ let V_n denote the free Q-module with basis $\{e_\delta \ \mid \delta \in \Delta, |e_\delta| = n\}$. We write then

$$U_n = V_n \oplus yV_{n-1} \oplus L_n \qquad \text{where} \qquad L_n = \bigoplus_{i=2}^{n} \bigoplus_{\lambda\in\Lambda_i} y_\lambda V_{n-i}\,. \tag{4.1.5}$$

Using the Leibnitz rule and (4.1.4), observe:

$$\partial_n(L_n) \subseteq \operatorname{ann}_Q(f) y V_{n-2} + L_{n-1}\,. \tag{4.1.6}$$

In particular, it follows that

$$(\partial_n \otimes_Q S)(L_n \otimes_Q S) \subseteq L_{n-1} \otimes_Q S \subseteq U_{n-1} \otimes_Q S$$

and we denote $(L \otimes_Q S, \partial \otimes_Q S)$ the subcomplex of $U \otimes_Q S$ defined in this way.

By Lemma 3.1.3, the complex (U', ∂'), with

$$U'_n = V_n \otimes_Q R \quad \text{and} \quad \partial'_n = \pi_{n-1}\partial_n\big|_{V_n} \otimes_Q R$$

is a free resolution of M over R, where $\pi_{n-1} : U_{n-1} \to V_{n-1}$ is the canonical projection map. We denote by (V, ∂^V) the sequence of free Q-modules with V_n as defined above and

$$\partial^V_n = \pi_{n-1}\partial_n\big|_{V_n} : V_n \to V_{n-1}\,.$$

Then (V, ∂^V) is a lifting of U' to Q. From now on, we will assume that the Eisenbud operator defined in (4.1.1) is associated to the data (f, U', V), and thus we have maps

$$\widetilde{\tau}\colon \Sigma^{-1}V \to \Sigma V \quad \text{and} \quad \tau\colon \Sigma^{-1}V \otimes_Q S \to \Sigma V \otimes_Q S$$

where $V \otimes_Q S = U' \otimes_R S$.

Lemma 4.1.4. *Let $n \in \mathbb{Z}$ and $x \in V_{n+1}$, and use the decomposition (4.1.5) to write*

$$\partial_{n+1}(x) = x_n + yx_{n-1} + \overline{x} \qquad \textit{with} \quad x_i \in V_i,\ \overline{x} \in L_n\,.$$

Then the Eisenbud operator $\tau_n\colon V_{n+1} \otimes_Q S \to V_{n-1} \otimes_Q S$ satisfies

$$\tau_n(x \otimes 1_S) = -x_{n-1} \otimes 1_S\,.$$

Proof. Starting with

$$0 = \partial^2(x) = \partial(x_n + yx_{n-1} + \overline{x})\,,$$

we compute $\partial(x_n)$ as follows:

$$\partial(x_n) = -fx_{n-1} + y\partial(x_{n-1}) - \partial(\overline{x})\,.$$

Using (4.1.6) for the second equation, observe

$$-fx_{n-1} \in V_{n-1} \qquad \text{and} \qquad y\partial(x_{n-1}) - \partial(\overline{x}) \in yV_{n-2} + L_{n-1}\,.$$

The definition of ∂^V gives $\partial_n^V(x_n) = -fx_{n-1}$. Then, using the definitions of $\widetilde{\tau}$ and τ, we have

$$\widetilde{\tau}_n(x) = -x_{n-1} \quad \text{and} \quad \tau_n(x \otimes 1_S) = -x_{n-1} \otimes 1_S\,. \qquad \square$$

4.2 The Mapping cone of the Eisenbud Operator

Let (A, d) and (A', d') be complexes of Q-modules and $\sigma : (A, d) \to (A', d')$ be a chain map. The *mapping cone* of σ is a complex (W, ∂) where

$$W_n = A'_n \oplus A_{n-1} \quad \text{and} \quad \partial_n = \begin{pmatrix} d'_n & -f_{n-1} \\ 0 & -d_{n-1} \end{pmatrix} .$$

In general, the mapping cone of f has zero homology (is acyclic) if and only if f is a quasi-isomorphism. So, a mapping cone of f can be regarded as a measure of obstruction of f to be a quasi-isomorphism.

In this section, we retain the notation of the previous section and we construct the mapping cone of the Eisenbud operator $\tau \colon \Sigma^{-1}V \otimes_Q S \to \Sigma V \otimes_Q S$ and describe it using the language of dg algebras. When f is a non-zero divisor, we explain that the long exact sequence arising from the mapping cone is exactly the sequence (1.0.1) in the introduction. When f is an exact zero divisor and the k has characteristic zero, we describe in Theorem 4.2.6 two exact complexes that connect homology over Q to homology over R.

4.2.1. *The mapping cone of τ.* Consider the operator τ as described in 4.1.3. The *mapping cone of* τ is the complex (W, ∂^W) with

$$W_n = (V_{n-1} \oplus V_n) \otimes_Q S = (V_{n-1} \oplus_Q S) \oplus (V_n \otimes_Q S)$$

$$\partial^W_n : (V_{n-1} \oplus V_n) \otimes_Q S \to (V_{n-2} \oplus V_{n-1}) \otimes_Q S$$

$$\partial^W_n = \begin{pmatrix} -\partial^V_{n-1} & -\tau_{n-1} \\ 0 & \partial^V_n \end{pmatrix} \otimes_Q S\,.$$

Let $n \in \mathbb{Z}$, $x \in V_{n-1}$ and $a \in V_n$. As in Lemma 4.1.4, we write

$$\partial_{n-1}(x) = x_{n-2} + yx_{n-3} + \overline{x}$$

$$\partial_n(a) = a_{n-1} + ya_{n-2} + \overline{a}$$

where $x_i, a_i \in V_i$, $\overline{x} \in L_{n-2}$ and $\overline{a} \in L_{n-1}$. Therefore, using Lemma 4.1.4 for τ_{n-1}, we can describe the differential ∂^W of the mapping cone by

$$\partial^W_n((x, a) \otimes 1_S) = (a_{n-2} - x_{n-2}, a_{n-1}) \otimes 1_S \in (V_{n-2} \oplus V_{n-1}) \otimes_Q S.$$

As usual, the mapping cone construction gives rise to a short exact sequence

$$0 \to \Sigma V \otimes_Q S \xrightarrow{\zeta} W \xrightarrow{\gamma} V \otimes_Q S \to 0 \tag{4.2.1}$$

of complexes of free S-modules.

Let N be an S-module. We have an isomorphism of complexes of S-modules

$$(V \otimes_Q S) \otimes_S N \cong (V \otimes_Q R) \otimes_R N = U' \otimes_R N$$

and we identify $\mathrm{H}_n(U' \otimes_R N) = \mathrm{Tor}^R_n(M, N)$. Tensoring the short exact sequence above with N yields a short exact sequence that induces the long exact sequence in homology

$$\cdots \to \operatorname{Tor}^R_{n-1}(M,N) \to \operatorname{H}_n(W \otimes_S N) \to \operatorname{Tor}^R_n(M,N) \xrightarrow{\delta_n} \operatorname{Tor}^R_{n-2}(M,N) \to \dots \tag{4.2.2}$$

where δ_n is induced by τ_n.

We now proceed to closer investigate the complex W.

4.2.2. *The map ω.* Recall from (4.1.5) that $U_n = V_n \oplus yV_{n-1} + L_n$. For each n, we define an S-module homomorphism $\omega_n \colon U_n \otimes_Q S \to W_n$ by setting

$$\omega_n(u \otimes 1_S) = (x,a) \otimes 1_S,$$

where $u \in U_n$ has the following decomposition, per (4.1.5):

$$u = a + yx + l \qquad \text{with } a \in V_n\,, x \in V_{n-1},\ \text{and } l \in L_n\,. \tag{4.2.3}$$

Lemma 4.2.3. *The map $\omega \colon U \otimes_Q S \to W$ defined in 4.2.2 is a homomorphism of complexes of S-modules, and* $\operatorname{Ker}(\omega) = L \otimes_Q S$. *Therefore,*

$$W \cong \frac{U \otimes_Q S}{L \otimes_Q S}\,.$$

If f is a non-zero divisor, then ω is an isomorphism and $W \cong U \otimes_Q S$.

Proof. Let $n \geq 0$. We decompose $u \in U_n$ as in (4.2.3). Recall that $\omega_n(u \otimes 1_S) = (x,a) \otimes 1_S$.

To prove that ω is a chain map, we need to show

$$\omega_{n-1}(\partial_n \otimes_Q S) - \partial^W_n \omega_n = 0$$

for all n. Indeed, with the notation introduced in 4.2.1, we have

$$\partial_n^W \omega_n(u \otimes 1_S) = \partial_n^W((x,a) \otimes 1_S) = (a_{n-2} - x_{n-2}, a_{n-1}) \otimes 1_S\,.$$

Moreover, we use (4.1.6) to write $l = l_{n-2} + \bar{l}$, where $l_{n-2} \in \operatorname{ann}_Q(f) y V_{n-2}$ and $\bar{l} \in L_{n-1}$, and we use the Leibnitz rule towards the computation below:

$$\begin{aligned}
&\omega_{n-1}(\partial_n \otimes_Q S)(u \otimes 1_S) \\
&\qquad = \omega_{n-1}\big((a_{n-1} + y a_{n-2} + \overline{a} + fx - y(x_{n-2} + y x_{n-3} + \overline{x}) + l_{n-2} + \bar{l}) \otimes 1_S\big) \\
&\qquad = \omega_{n-1}\big((a_{n-1} + y(a_{n-2} - x_{n-2}) + \overline{a} - y\overline{x} + \bar{l}) \otimes 1_S\big) \\
&\qquad = (a_{n-2} - x_{n-2}, a_{n-1}) \otimes 1_S\,.
\end{aligned}$$

We showed thus ω is a chain map. It is clearly surjective, and $\operatorname{Ker}(\omega_n) = L_n \otimes_Q S$ for all n. This gives the desired isomorphism of complexes.

When $f \in Q$ is a non-zero divisor, from the discussion in 4.1.3 we have $L_n = 0$ for all n, giving that ω is an isomorphism. □

The following proposition provides an alternate proof for [8, Proposition 3.2.2], towards explaining the long exact sequence (1.0.1) mentioned in the introduction.

Proposition 4.2.4. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring, $f \in Q$ a non-zero divisor and set $R = Q/(f)$. If M, N are finitely generated R-modules, then there is a long exact sequence*

$$\cdots \to \operatorname{Tor}_{n-1}^R(M,N) \to \operatorname{Tor}_n^Q(M,N) \xrightarrow{\varphi_n} \operatorname{Tor}_n^R(M,N) \xrightarrow{\delta_n} \operatorname{Tor}_{n-2}^R(M,N) \to \cdots$$

where δ is induced by the Eisenbud operator associated with f, and φ is induced by the canonical projection $Q \to R$.

Proof. Under the given hypothesis on f, note that the ring S that we worked with so far is equal to R. The exactness of the sequence in the statement is a direct consequence of the isomorphism $W \cong U \otimes_Q S$ of Lemma 4.2.3 and the exact sequence (4.2.2), using the identification $\operatorname{Tor}_n^Q(M,N) = \operatorname{H}_n(U \otimes_Q N)$.

The map φ_n in the statement is a composition

$$\operatorname{H}_n(U \otimes_Q N) \to \operatorname{H}_n(W \otimes_Q N) \to \operatorname{H}_n(V \otimes_Q N)$$

where the first map is induced by the isomorphism ω and the second by the projection γ in (4.2.1), and hence this composition can be viewed as being induced by the canonical projection $\pi \otimes_Q R \colon U \otimes_Q R \to V \otimes_Q R = U'$. Observe that $\pi \otimes_Q R$ is a homomorphism of complexes that lifts the identity map on M, and thus, upon tensoring with N, it describes the canonical map induced in homology by the projection $Q \to R$. □

We now turn our attention to the situation when f is an exact zero divisor.

Lemma 4.2.5. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with $\operatorname{char}(\mathsf{k}) = 0$, (f, g) an exact pair of zero divisors, and $S = Q/(f, g)$. With the notation introduced in 4.2.1,*

$$0 \longrightarrow \Sigma^2 U \otimes_Q S \xrightarrow{\cdot y_2} U \otimes_Q S \xrightarrow{\omega} W \longrightarrow 0 \qquad (4.2.4)$$

is a short exact sequence of complexes of S-modules.

Proof. Let $\alpha\colon \Sigma^2 U \to U$ denote the map given by multiplication by y_2.

Let $n \geq 0$. If $x \in U_n$, then

$$\partial(y_2 x) - y_2\partial(x) = gyx + y_2\partial(x) - y_2\partial(x) = gyx\,.$$

This shows that, upon tensoring with S, the map α becomes a chain map. To show that α is injective, recall that, when f is an exact zero-divisor, we have

$$U_n = \bigoplus_{i=0}^{n} y_i V_{n-i} \qquad \text{and} \qquad y_2 y_{2j+\epsilon} = (1+j)y_{2+2j+\epsilon}$$

for all $n \geq 0$, $j \geq 0$ and $\epsilon \in \{0,1\}$. When $\operatorname{char}(\mathsf{k}) = 0$ we see that $1+j$ is a unit for all $j \geq 0$, and thus $y_2 y_i = 0$ implies $y_i = 0$, for all i. It follows from here that $\alpha_n\colon U_{n-2} \to U_n$ is injective. Another consequence is that $y_2 y_i V_{n-2-i} = y_{i+2} V_{n-2-i}$ and hence

$$\operatorname{Im}(\alpha_n) = y_2 \bigoplus_{i=0}^{n-2} y_i V_{n-2-i} = \bigoplus_{i=2}^{n} y_i V_{n-i} = L_n\,.$$

Since L_n is a direct summand of U_n, the map α_n remains injective when tensored with S. Using Lemma 4.2.3, we further have

$$\operatorname{Im}(\alpha \otimes_Q S) = L \otimes_Q S = \operatorname{Ker}(\omega).$$

Since ω is surjective, this finishes the proof of the exactness of (4.2.4). □

Theorem 4.2.6. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring such that* $\operatorname{char}(\mathsf{k}) = 0$. *Let (f, g) be an exact pair of zero divisors in Q and set $R = Q/(f)$ and $S = Q/(f, g)$. Let M be* *an R-module and let (W, ∂^W) denote the mapping cone of the Eisenbud operator τ associated with f.*

For any R-module N with $gN = 0$, there are have two long exact sequences

$$\cdots \to \operatorname{Tor}^R_{n-1}(M,N) \to \operatorname{H}_n(W \otimes_S N) \xrightarrow{\psi^N_n} \operatorname{Tor}^R_n(M,N) \xrightarrow{\delta^N_n} \operatorname{Tor}^R_{n-2}(M,N) \to \cdots$$

$$\cdots \to \operatorname{H}_{n+1}(W \otimes_S N) \xrightarrow{\mu^N_{n+1}} \operatorname{Tor}^Q_{n-2}(M,N) \to \operatorname{Tor}^Q_n(M,N) \xrightarrow{\phi^N_n} \operatorname{H}_n(W \otimes_S N) \to \cdots$$

where δ^N is induced by τ and $\psi^N\phi^N$ is equal to the map $\operatorname{Tor}^Q(M,N) \to \operatorname{Tor}^R(M,N)$ induced by the canonical projection $Q \to R$.

Proof. Throughout this proof, we use the notation in 4.2.1. The first exact sequence has already been established in (4.2.2). It remains thus to establish the second one.

Tensoring the short exact sequence (4.2.4) of free S-modules with N, we have the short exact sequence

$$0 \to \Sigma^2 U \otimes_Q N \to U \otimes_Q N \xrightarrow{\omega \otimes_R N} W \otimes_Q N \to 0\,. \tag{4.2.5}$$

The induced long exact sequence in homology is exactly the second sequence in the statement, with $\operatorname{H}_n(U \otimes_Q N) = \operatorname{Tor}^Q_n(M,N)$. The map $\psi_n\phi_n$ is the composition

$$\operatorname{H}_n(U \otimes_Q N) \to \operatorname{H}_n(W \otimes_S N) \to \operatorname{H}_n(V \otimes_Q N)$$

where the first map is induced by ω and the second one is induced by the map γ in (4.2.1). We conclude that the composition is induced by the canonical projection $\pi \otimes_Q R \colon U \otimes_Q R \to V \otimes_Q R = U'$. The homomorphism $\pi \otimes_Q R$ lifts the identity map on M, and thus, upon tensoring with N, it describes the canonical map induced in homology by the projection $Q \to R$. $\square$

CHAPTER 5

APPLICATIONS

In this chapter, we use the results of Chapter 4 to investigate the change in asymptotic behavior in homology over Q versus over $R = Q/(f)$ when f is an exact zero divisor. This investigation is motivated by the fact that, when f is a nonzero divisor, and more generally when R is a quotient of Q by an ideal generated by a regular sequence, there is a plethora of results in this direction, see for instance [2, 3, 9, 21, 24]. Some of these results (regarding complexity, curvature, and change in generalized Poincaré series) have been recorded in the introduction. In the case of a regular element, they are mainly consequences of the exact sequence (1.0.1).

One would like to understand whether similar/comparable results can be obtained when f is a zero divisor. In the case when f is an exact zero divisor, such results have been previously investigated by Bergh, Celikbas, and Jorgensen [10] using methods that rely on a change of ring spectral sequence. The results of [10] provide some counterparts of above-mentioned results that are known when f is regular, with the caveat that behavior when f is an exact zero divisor may sometimes be opposite to the one in the case when f is regular, see Remark 5.2.2 for a discussion. We approach the same type of problem, but, instead of a spectral sequence, we use the two exact sequences provided by Theorem 4.2.6. While we recover many of the results of [10] (in a weaker form, because we use an additional assumption on characteristic), the additional assumption that the characteristic of the residue field is zero leads to new results, and in particular we answer one of the questions raised in [10], see Remark 5.1.7. In addition, we pursue an investigation into understanding how the two exact sequences of Theorem 4.2.6 can be used to deduce rationality results for a generalized Poincaré series of two modules.

Throughout the chapter, $(Q, \mathfrak{m}, \mathsf{k})$ denotes a local ring, $f \in Q$ is an exact zero divisor, $R = Q/(f)$ and M, N are finitely generated R-modules such that $M \otimes_R N$ has finite length. (This assumption is trivially satisfied when $N = \mathsf{k}$.) Section 5.1 investigates changes in complexity of M, N along $Q \to R$. Section 5.2 addresses vanishing of homology over Q versus vanishing of homology over R, and Section 5.3 establishes rationality results for the generalized Poincaré series of M, N, with particular attention to rings with $\mathfrak{m}^3 = 0$.

5.1 Complexity and Curvature

In this section, we compare the asymptotic behavior of betti numbers/homology via the canonical projection $Q \to R$ and $f \in Q$ is an exact zero divisor. We use the notions of complexity and curvature to quantify asymptotic behavior.

5.1.1. *Poincaré series and length complexity.* Suppose M, N are finite Q-modules with the property that $\ell(M \otimes_Q N) < \infty$ where ℓ denotes length. Then, for every nonnegative integer n the length of $\operatorname{Tor}_n^Q(M,N)$ is finite over Q. We set

$$\beta_n^Q(M,N) = \ell(\operatorname{Tor}_n^Q(M,N))$$

as defined in [10, Section 3] for example.

The (generalized) *Poincaré series* of the pair (M,N) is the formal power series

$$\mathrm{P}_{M,N}^Q(t) = \sum_{i=0}^{\infty} \beta_i^Q(M,N)t^i\,.$$

Note that $\mathrm{P}_{M,\mathsf{k}}^Q(t)$ is the Poincaré series of M over Q, usually denoted $\mathrm{P}_M^Q(t)$. The *complexity* of a formal power series $p(t) = \sum_{n\geq 0} a_n t^n$ with $a_n \in \mathbb{R}$ for all $n \geq 0$ is defined as follows:

$$\operatorname{cx}(p) = \inf\{c \in \mathbb{Z}^+ \colon \exists \alpha > 0 \text{ such that } |a_n| \leq \alpha n^{c-1},\ \forall n \gg 0\}\,.$$

In this definition, observe that we can replace $n \gg 0$ with $\forall n$. Indeed, when $\operatorname{cx}(p) = c$, we have $|a_N| \leq \alpha_1 N^{c-1}$ for some $N \gg 0$. We can set $\alpha = \max\{|a_i|, \alpha_1\}$ for $i \leq N-1$ which guarantees the inequality for all n.

The *length complexity of the pair* (M,N), denoted by $\ell\operatorname{cx}_Q(M,N)$, is defined by

$$\ell\operatorname{cx}_Q(M,N) = \operatorname{cx}(\mathrm{P}_{M,N}^Q(t))\,.$$

The *complexity* of a Q-module M is $\operatorname{cx}_Q(M) = \ell\operatorname{cx}_Q(M,\mathsf{k})$.

Lemma 5.1.2. *If p and q are formal power series with real coefficients, then*

$$\operatorname{cx}(p \cdot q) \leq \operatorname{cx}(p) + \operatorname{cx}(q)\,.$$

Proof. Let $p(t) = \sum_{i=0}^{\infty} a_n t^n, q(t) = \sum_{i=0}^{\infty} b_n t^n$, $\operatorname{cx}(p) = c$ and $\operatorname{cx}(q) = c'$. Denote the nth-coefficient of the product of the power series p and q as d_n. Notice that $d_n = \sum_{i=0}^{n} a_i b_{n-i}$.

For big enough α and α' and for all n we have

$$|a_i b_{n-i}| \leq \alpha(i^{c-1}(n-i)^{c'-1}) \leq \alpha(n^{c-1}(n-i)^{c'-1}) \leq \alpha' n^{c+c'-2}$$
$$|d_n| = \left|\sum_{i=0}^{n} a_i b_{n-i}\right| \leq \sum_{i=0}^{n} |a_i b_{n-i}| \leq n\alpha' n^{c+c'-2} = \alpha' n^{c+c'-1}\,.$$

By the definition of the complexity, we have our desired result. □

If $p(t) = \sum_{n\geq 0} a_n t^n, q(t) = \sum_{n\geq 0} b_n t^n$ are formal power series with real coefficients, coefficient-wise inequality is denoted $\preccurlyeq$ and is defined as follows:

$$p(t) \preccurlyeq q(t) \iff a_n \leq b_n \text{ for all}\, n\,.$$

Theorem 5.1.3. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with* $\operatorname{char}(\mathsf{k}) = 0$ *and (f, g) is an exact pair of zero divisors in Q. We set $R = Q/(f)$ and $S = Q/(f, g)$. Then, for every finitely generated R-modules M, N with $gN = 0$ and $\ell(M \otimes_Q N) < \infty$ we have*

$$\mathrm{P}^R_{M,N}(t) \preccurlyeq \frac{(1-t+t^2)\,\mathrm{P}^Q_{M,N}(t)}{(1-t)} \qquad \textit{and} \tag{5.1.1}$$

$$\mathrm{P}^Q_{M,N}(t) \preccurlyeq \frac{\mathrm{P}^R_{M,N}(t)}{(1-t)}\,. \tag{5.1.2}$$

Furthermore, equality holds in (5.1.2) *if and only if the maps* μ^N *and* δ^N *from the exact sequences of Theorem 4.2.6 are both equal to zero.*

Proof. Recall the following long exact sequence from Theorem 4.2.6.

$$\cdots \xrightarrow{\mu^N_{n+3}} \operatorname{Tor}^Q_n(M,N) \to \operatorname{Tor}^Q_{n+2}(M,N) \to \mathrm{H}_{n+2}(W \otimes_S N) \xrightarrow{\mu^N_{n+2}} \operatorname{Tor}^Q_{n-1}(M,N) \to \cdots \tag{5.1.3}$$

A length count gives

$$\ell(\mathrm{H}_{n+2}(W \otimes_S N)) \leq \beta^Q_{n+2}(M,N) + \beta^Q_{n-1}(M,N)$$

for all $n \geq 0$. Using these inequalities for all n, we further obtain

$$\begin{aligned}\sum_{n=-2}^{\infty} \ell(\mathrm{H}_{n+2}(W \otimes_S N))t^{n+2} &\preccurlyeq \sum_{n=-2}^{\infty} \beta^Q_{n+2}(M,N)t^{n+2} + \sum_{n=-2}^{\infty} \beta^Q_{n-1}(M,N)t^{n+2} \\ &= (1+t^3)\,\mathrm{P}^Q_{M,N}(t)\,.\end{aligned} \tag{5.1.4}$$

Similarly, we use the other long exact sequence from Theorem 4.2.6, namely

$$\cdots \xrightarrow{\delta^N_{n+3}} \operatorname{Tor}^R_{n+1}(M,N) \to \mathrm{H}_{n+2}(W \otimes_S N) \to \operatorname{Tor}^R_{n+2}(M,N) \xrightarrow{\delta^N_{n+2}} \operatorname{Tor}^R_n(M,N) \to \cdots \tag{5.1.5}$$

A length count gives

$$\ell(\mathrm{H}_{n+2}(W \otimes_S N)) \geq \beta^R_{n+2}(M,N) - \beta^R_n(M,N)\,.$$

Using these inequalities for all n, we have:

$$\begin{aligned}\sum_{n=-2}^{\infty} \ell(\mathrm{H}_{n+2}(W \otimes_S N))t^{n+2} &\succcurlyeq \sum_{n=-2}^{\infty} \beta^R_{n+2}(M,N)t^{n+2} - \sum_{n=-2}^{\infty} \beta^R_n(M,N)t^{n+2} \\ &= (1-t^2)\,\mathrm{P}^R_{M,N}(t)\,.\end{aligned} \tag{5.1.6}$$

From the results of (5.1.4) and (5.1.6), we get

$$(1-t)\,\mathrm{P}^R_{M,N}(t) \preccurlyeq (1-t+t^2)\,\mathrm{P}^Q_{M,N}(t)\,.$$

Since $0 \preccurlyeq (1-t)^{-1}$, the inequality is preserved by multiplying both sides by $(1-t)^{-1}$ and this gives the first inequality of the theorem.

To get the second inequality, we use again a length count in (5.1.3):

$$\ell(\mathrm{H}_{n+2}(W \otimes_S N)) \geq \beta^Q_{n+2}(M,N) - \beta^Q_n(M,N)$$

with equality if and only if $\mu^N_{n+2} = 0 = \mu^N_{n+3}$. Using these inequalities for all n, we further obtain

$$\begin{aligned}\sum_{n=-2}^{\infty} \ell(\mathrm{H}_{n+2}(W \otimes_S N))t^{n+2} &\succcurlyeq \sum_{n=-2}^{\infty} \beta^Q_{n+2}(M,N)t^{n+2} - \sum_{n=-2}^{\infty} \beta^Q_n(M,N)t^{n+2} \\ &= (1-t^2)\,\mathrm{P}^Q_{M,N}(t)\,,\end{aligned} \tag{5.1.7}$$

with equality if and only if $\mu^N = 0$. Similarly, a length count in (5.1.5) gives

$$\ell(\mathrm{H}_{n+2}(W \otimes_S N)) \leq \beta^R_{n+2}(M,N) + \beta^R_{n+1}(M,N)$$

with equality if and only if $\delta^N_{n+2} = 0 = \delta^N_{n+3}$. Using these inequalities for all n we further have

$$\begin{aligned}\sum_{n=-2}^{\infty} \ell(\mathrm{H}_{n+2}(W \otimes_S N))t^{n+2} &\preccurlyeq \sum_{n=-2}^{\infty} \beta^R_{n+2}(M,N)t^{n+2} + \sum_{n=-2}^{\infty} \beta^R_{n+1}(M,N)t^{n+2} \\ &= (1+t)\,\mathrm{P}^R_{M,N}(t)\,,\end{aligned} \tag{5.1.8}$$

with equality if and only if $\delta^N = 0$. From (5.1.7) and (5.1.8), we get

$$(1-t^2)\,\mathrm{P}^Q_{M,N}(t) \preccurlyeq (1+t)\,\mathrm{P}^R_{M,N}(t)\,.$$

Observing that $0 \preccurlyeq (1-t^2)^{-1}$, we then obtain the desired inequality by multiplying both sides with $(1-t^2)^{-1}$. □

Corollary 5.1.4. *With the hypotheses of Theorem 4.2.6, if $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ and $\mathfrak{m}N = 0$, then $\mu^N = 0 = \delta^N$.*

Proof. When $f \notin \mathfrak{m}^2$, it follows from [4, Theorem 6.2] that we have an equality

$$\mathrm{P}^{Q}_{M,\mathsf{k}}(t) = \mathrm{P}^{R}_{M,\mathsf{k}}(t)/(1-t)\,.$$

Obviously, if $\mathfrak{m}N = 0$, then we can replace k with N in this equality. Consequently, equality holds in (5.1.2), and Theorem 5.1.3 gives the desired conclusion. □

Remark 5.1.5. The inequality $\mathrm{P}^{Q}_{M,N}(t) \preccurlyeq \mathrm{P}^{R}_{M,N}(t)/(1-t)$ was previously proved using spectral sequences, without the characteristic assumption, in [10, Theorem 3.2].

Corollary 5.1.6. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with $\operatorname{char}(\mathsf{k}) = 0$, (f, g) an exact pair of zero divisors in Q and set $R = Q/(f)$. For every finitely generated R-modules M, N with $gN = 0$ and $\ell(M \otimes_Q N) < \infty$, we have*

$$\ell\,\mathrm{cx}_Q(M,N) - 1 \le \ell\,\mathrm{cx}_R(M,N) \le \ell\,\mathrm{cx}_Q(M,N) + 1\,.$$

Proof. Using Lemma 5.1.2, (5.1.1) and (5.1.2), we have the following inequalities of length complexities:

$$\begin{aligned}
\ell\,\mathrm{cx}_R(M,N) &\le \ell\,\mathrm{cx}_Q(M,N) + \mathrm{cx}((1-t)^{-1}) + \mathrm{cx}(1-t+t^2)\\
&= \ell\,\mathrm{cx}_Q(M,N) + 1\\
\ell\,\mathrm{cx}_Q(M,N) &\le \ell\,\mathrm{cx}_R(M,N) + \mathrm{cx}((1-t)^{-1})\\
&= \ell\,\mathrm{cx}_R(M,N) + 1\,. \qquad \square
\end{aligned}$$

Remark 5.1.7. When $f \in Q$ is a regular element, upper and lower bounds for $\mathrm{cx}_R(M)$ in terms of $\mathrm{cx}_Q(M)$ can be derived using (1.0.1) and are spelled out in the introduction. Consequently, we have $\mathrm{cx}_R(M) < \infty \iff \mathrm{cx}_Q(M) < \infty$ in this case. When f is an exact zero divisor, Bergh, Celikbas and Jorgensen [10, Corollary 3.3] provide the upper bound in Corollary 5.1.6 for $\ell\,\mathrm{cx}_Q(M,N)$ (without any assumption on the characteristic) and question whether a lower bound for $\ell\,\mathrm{cx}_Q(M,\mathsf{k}) = \mathrm{cx}_Q(M)$ can be given using $\mathrm{cx}_R(M)$. Thus Corollary 5.1.6 provides a partial answer to this question.

5.1.8. *Curvature.* Let $p(t) = \sum_n a_n t^n$ be a formal power series with real coefficients and radius of convergence $r(p)$. We define the *curvature* of $p(t)$ by

$$\mathrm{curv}(p) \coloneqq \frac{1}{r(p)} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,.$$

If M is a finite Q-module then the curvature of the module M, denoted $\mathrm{curv}_Q(M)$, is defined as the curvature of $\mathrm{P}^Q_M(t)$, see [8]. When $\ell(M \otimes_Q N) < \infty$, we define the length curvature of the pair (M,N) to be the curvature of $\mathrm{P}^Q_{M,N}(t)$.

Notice that for any two power series p and q with real coefficients, we have

$$\operatorname{curv}(p \cdot q) \leq \max\{\operatorname{curv}(p), \operatorname{curv}(q)\}\,.$$

This is a consequence of the fact that the radius of convergence of the product $p \cdot q$ is greater than or equal to the minimum of the radii of convergence of p, q.

Proposition 5.1.9. *Let $R = Q/(f)$ where $(Q, \mathfrak{m}, \mathsf{k})$ is a commutative local ring with $\operatorname{char}(\mathsf{k}) = 0$ and (f, g) is an exact pair of zero divisors in Q. Let M, N be finitely generated R-modules M, N such that $gN = 0$ and $\ell(M \otimes_Q N) < \infty$.*

If $\ell \operatorname{curv}_Q(M, N) \geq 1$ and $\ell \operatorname{curv}_R(M, N) \geq 1$ then

$$\ell \operatorname{curv}_Q(M, N) = \ell \operatorname{curv}_R(M, N)\,. \tag{5.1.9}$$

Proof. From (5.1.1), (5.1.2) and the hypothesis $\ell \operatorname{curv}_Q(M, N) \geq 1$ and $\ell \operatorname{curv}_R(M, N) \geq 1$, we get the following inequalities:

$$\begin{aligned}
\ell \operatorname{curv}_R(M, N) &\leq \max\{\ell \operatorname{curv}_Q(M, N), \operatorname{curv}(1 - t + t^2), \operatorname{curv}((1-t)^{-1})\} \\
&= \max\{\ell \operatorname{curv}_Q(M, N), 1\} \\
&= \ell \operatorname{curv}_Q(M, N) \\
\ell \operatorname{curv}_Q(M, N) &\leq \max\{\ell \operatorname{curv}_R(M, N), \operatorname{curv}((1-t)^{-1})\} \\
&= \max\{\ell \operatorname{curv}_R(M, N), 1\} \\
&= \ell \operatorname{curv}_R(M, N)\,.
\end{aligned}$$

□

Remark 5.1.10. When $f \in Q$ is a regular element, we know $\operatorname{curv}_Q(M) = \operatorname{curv}_R(M)$ when $\operatorname{pd}_Q(M) = \infty$, see [8, Proposition 4.2.5]. Our result implies similar behavior when f is an exact zero divisor, namely $\operatorname{curv}_Q(M) = \operatorname{curv}_R(M)$ when $\operatorname{pd}_Q(M) = \infty$ and $\operatorname{pd}_R(M) = \infty$

5.2 Applications to the vanishing of homology and maps induced in homology

Let $(Q, \mathfrak{m}, \mathsf{k})$ is a local ring, (f, g) is an exact pair of zero divisors in Q, $R = Q/(f)$ and M, N are finitely generated R-modules with $gN = 0$. The first result relates the vanishing of homology over R to the vanishing of homology over Q. Then, we study the vanishing of maps induced in homology by the inclusion $\mathfrak{n}N \subseteq N$. The next proposition recovers the result of [10, Theorem 2.1] and extends it, by requiring a smaller range of vanishing in the hypothesis.

Proposition 5.2.1. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with $\operatorname{char}(\mathsf{k}) = 0$, (f, g) an exact pair of zero divisors, and set $R = Q/(f)$. Let M, N be R-modules such that $gN = 0$. If there exists integers $n, m \geq 1$ with $n - m \geq 1$ and*

$$\operatorname{Tor}_i^R(M, N) = 0 \qquad \textit{for} \quad m \leq i \leq n$$

then

$$\operatorname{Tor}_{i-1}^Q(M, N) \cong \operatorname{Tor}_{i+1}^Q(M, N) \qquad \textit{for} \quad m \leq i \leq n-2$$

and there is an exact sequence:

$$\operatorname{Tor}^Q_{m-2}(M,N) \to \operatorname{Tor}^Q_m(M,N) \to \operatorname{Tor}^R_{m-1}(M,N) \to \operatorname{Tor}^Q_{m-3}(M,N).$$

Furthermore, if $m = 1$*, then*

$$M \otimes_Q N \cong \operatorname{Tor}^Q_i(M,N) \qquad \textit{for} \quad 1 \le i \le n-1.$$

Proof. Under the assumptions in the statement, Theorem 4.2.6 gives a long exact sequence

$$\operatorname{Tor}^R_{i+2}(M,N) \to \operatorname{Tor}^R_i(M,N) \to \mathrm{H}_{i+1}(W \otimes_S N) \to \operatorname{Tor}^R_{i+1}(M,N) \to \operatorname{Tor}^R_{i-1}(M,N)$$

from which we obtain:

$$\mathrm{H}_{i+1}(W \otimes_S N) \cong \begin{cases} 0 & \text{for } m \le i \le n-1 \\ \operatorname{Tor}^R_{i+1}(M,N) & \text{for } i = n \\ \operatorname{Tor}^R_i(M,N) & \text{for } i = m-1 \end{cases} \tag{5.2.1}$$

The second exact sequence of Theorem 4.2.6 is

$$\cdots \to \mathrm{H}_{i+1}(W \otimes_S N) \xrightarrow{\mu^N_{i+1}} \operatorname{Tor}^Q_{i-2}(M,N) \to \operatorname{Tor}^Q_i(M,N) \xrightarrow{\phi^N_n} \mathrm{H}_i(W \otimes_S N) \to \cdots$$

The exact sequence in the statement and the isomorphisms $\operatorname{Tor}^Q_{i-1}(M,N) \cong \operatorname{Tor}^Q_{i+1}(M,N)$ for $m \le i \le n$ follow by using (5.2.1) into this sequence.

When $m = 1$, we have thus

$$\operatorname{Tor}^Q_i(M,N) \cong \begin{cases} M \otimes_Q N & \text{when } i \text{ is even} \\ \operatorname{Tor}^Q_1(M,N) & \text{when } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq n-1$. Using again (5.2.1) in the second sequence from Theorem 4.2.6, we conclude

$$\mathrm{H}_0(W \otimes_S N) \cong M \otimes_Q N \cong \mathrm{H}_1(W \otimes_S N) \cong \mathrm{Tor}_1^Q(M,N)\,. \qquad \square$$

Remark 5.2.2. Let $(Q, \mathfrak{m})$ be a local ring and $f \in Q$ be a non-zero divisor. Set $R = Q/(f)$ and let M and N be finitely generated R-modules. If $\mathrm{Tor}_i^R(M,N) = 0$ for $m \leq i \leq n$ such that $m, n \geq 1$ and $n - m \geq 1$, then we have

$$\mathrm{Tor}_i^Q(M,N) = 0 \qquad \text{for} \quad m+1 \leq i \leq n$$

$$\mathrm{Tor}_m^Q(M,N) \cong \mathrm{Tor}_{m-1}^R(M,N)\,.$$

When we compare the above results to Proposition 5.2.1, we see that the results are opposite. In the regular case, the vanishing of homology over R results in the vanishing of homology over Q, whereas in the exact zero divisor case, it results in the non-vanishing of homology over Q.

5.2.3. *Induced maps.* Let M, N, P be Q-modules, $\phi : M \to N$ a homomorphism and U a complex of Q-modules. We let then $U \otimes_Q \phi$ and $\mathrm{Tor}_i^Q(P, \phi)$ denote the induced maps

$$U \otimes_Q \phi : U \otimes_Q M \to U \otimes_Q N$$

$$\mathrm{Tor}_i^Q(P,\phi) : \mathrm{Tor}_i^Q(P,M) \to \mathrm{Tor}_i^Q(P,N)\,.$$

Further, let

$$\nu_N : \mathfrak{m}N \to N \quad \text{and} \quad \pi_N : N \to \overline{N} = N/\mathfrak{m}N$$

denote the inclusion and the projection map respectively.

As mentioned in the proof of Corollary 5.1.4, equality holds in (5.1.2) when $\mathfrak{m}N = 0$ and $f \notin \mathfrak{m}^2$. In the next result, we identify another situation in which equality holds in (5.1.2).

Proposition 5.2.4. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a local ring with $\operatorname{char}(\mathsf{k}) = 0$ and (f, g) an exact pair of zero divisors with $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Set $R = Q/(f)$ and let M and N be R-modules such that $\mathfrak{m}^2 N = 0 = gN$.*

If $\operatorname{Tor}_i^R(M, \nu_N) = 0$ for all $i \geq 0$ then $\operatorname{Tor}_i^Q(M, \nu_N) = 0$ for all $i \geq 0$ and

$$\mathrm{P}^Q_{M,N}(t) = \frac{\mathrm{P}^R_{M,N}(t)}{1-t}\,.$$

Proof. Set $S = R/(g)$. Consider the short exact sequence of R-modules:

$$0 \to \mathfrak{m}N \xrightarrow{\nu_N} N \to \overline{N} \to 0 \tag{5.2.2}$$

where $\overline{N} = N/\mathfrak{m}N$. This sequence remains exact when tensoring with the complexes of free S-modules $\Sigma V \otimes_Q S$, W, respectively $V \otimes_Q S$. We also consider the sequence of complexes of free S-modules (4.2.1) and we tensor with $\mathfrak{m}N$, N, respectively $\overline{N}$; note the resulting sequences also remain exact. We thus obtain a commutative diagram of complexes with exact rows and columns, where the columns are induced from (5.2.2) and the rows are induced by (4.2.1). We omit writing this diagram, and we write directly, for all n, the following diagram induced in homology, cf. [13, Chapter IV, Proposition 2.1].

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta^{N}_{n+2}} & \operatorname{Tor}^R_n(M,N) & \longrightarrow & \mathrm{H}_{n+1}(W\otimes_S N) & \longrightarrow & \operatorname{Tor}^R_{n+1}(M,N) & \xrightarrow{\delta^{N}_{n+1}} & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \xrightarrow[0]{\delta^{\overline{N}}_{n+2}} & \operatorname{Tor}^R_n(M,\overline{N}) & \hookrightarrow & \mathrm{H}_{n+1}(W\otimes_S \overline{N}) & \twoheadrightarrow & \operatorname{Tor}^R_{n+1}(M,\overline{N}) & \xrightarrow[0]{\delta^{\overline{N}}_{n+1}} & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \xrightarrow[0]{\delta^{\mathfrak{m}N}_{n+1}} & \operatorname{Tor}^R_{n-1}(M,\mathfrak{m}N) & \hookrightarrow & \mathrm{H}_{n}(W\otimes_S \mathfrak{m}N) & \twoheadrightarrow & \operatorname{Tor}^R_{n}(M,\mathfrak{m}N) & \xrightarrow[0]{\delta^{\mathfrak{m}N}_{n}} & \cdots \\
 & & {\scriptstyle 0}\big\downarrow{\scriptstyle \operatorname{Tor}^R_{n-1}(M,\nu_N)} & & \big\downarrow{\scriptstyle \mathrm{H}_n(W\otimes_S\nu_N)} & & {\scriptstyle 0}\big\downarrow{\scriptstyle \operatorname{Tor}^R_{n}(M,\nu_N)} & & \\
\cdots & \longrightarrow & \operatorname{Tor}^R_{n-1}(M,N) & \longrightarrow & \mathrm{H}_{n}(W\otimes_S N) & \longrightarrow & \operatorname{Tor}^R_{n}(M,N) & \longrightarrow & \cdots
\end{array}
$$

In this diagram, all rows and columns are exact. Corollary 5.1.4 shows that the connecting homomorphisms $\delta^{\overline{N}}$ and $\delta^{\mathfrak{m}N}$ are zero. The maps $\operatorname{Tor}^R_n(M,\nu_N)$ are zero for all n, by hypothesis. Thus, some of the maps in the diagram are injective, respectively surjective, as indicated. Applying Snake lemma to the middle two rows of the commutative diagram above for all values of n, we get

$$
\mathrm{H}_n(W\otimes_S \nu_N) = 0 \qquad \text{and} \qquad \partial^N_{n+1} = 0 \qquad \text{for all } n\,. \tag{5.2.3}
$$

We now use the short exact sequences (4.2.4) and (5.2.2) to similarly get the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccc}
\xrightarrow{\mu^N_{n+3}} & \operatorname{Tor}^Q_n(M,N) & \longrightarrow & \operatorname{Tor}^Q_{n+2}(M,N) & \longrightarrow & \mathrm{H}_{n+2}(W\otimes_S N) & \xrightarrow{\mu^N_{n+2}} \\
& \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow[0]{\mu^{\overline{N}}_{n+3}} & \operatorname{Tor}^Q_n(M,\overline{N}) & \hookrightarrow & \operatorname{Tor}^Q_{n+2}(M,\overline{N}) & \twoheadrightarrow & \mathrm{H}_{n+2}(W\otimes_S \overline{N}) & \xrightarrow[0]{\mu^{\overline{N}}_{n+2}} \\
& \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow[0]{\mu^{\mathfrak{m}N}_{n+2}} & \operatorname{Tor}^Q_{n-1}(M,\mathfrak{m}N) & \hookrightarrow & \operatorname{Tor}^Q_{n+1}(M,\mathfrak{m}N) & \twoheadrightarrow & \mathrm{H}_{n+1}(W\otimes_S \mathfrak{m}N) & \xrightarrow[0]{\mu^{\mathfrak{m}N}_{n+1}} \\
& {\scriptstyle \operatorname{Tor}^Q_{n-1}(M,\nu_N)}\downarrow & & {\scriptstyle \operatorname{Tor}^Q_{n+1}(M,\nu_N)}\downarrow & & {\scriptstyle \mathrm{H}_{n+1}(W\otimes_S\nu_N)}\downarrow{\scriptstyle 0} & \\
\longrightarrow & \operatorname{Tor}^Q_{n-1}(M,N) & \longrightarrow & \operatorname{Tor}^Q_{n+1}(M,N) & \longrightarrow & \mathrm{H}_{n+1}(W\otimes_S N) & \xrightarrow{\mu^N_{n+1}}
\end{array}
$$

Corollary 5.1.4 shows that the connecting homomorphisms $\mu^{\overline{N}}$ and $\mu^{\mathfrak{m}N}$ are zero, and (5.2.3) gives $\mathrm{H}_{n+1}(W \otimes_S \nu_N) = 0$. Thus, some of the maps in the diagram are injective, respectively surjective, as indicated.

We prove by induction on n that $\operatorname{Tor}^Q_n(M, \nu_N) = 0$ for all n. The statement is obviously true when $n = -1$ and $n = -2$. Let $n \geq -1$ and assume that $\operatorname{Tor}^Q_i(M, \nu_N) = 0$ for all $i \leq n$, and in particular for $i = n - 1$. A use of the Snake Lemma for the middle two rows of the diagram gives that $\operatorname{Tor}^Q_{n+1}(M, \nu_N) = 0$. This finishes the induction. Another use of the Snake lemma for the same rows also yields $\mu^N_{n+2} = 0$ for all n.

Then Theorem 5.1.3 shows that equality must hold in (5.1.2). □

5.2.5. *Koszul modules (local setting).* A $(Q, \mathfrak{m})$-module M is said to be *Koszul* if the associated graded module $\operatorname{gr}(M)$ over the associated graded ring $\operatorname{gr}(Q)$ has a linear resolution. When M is Koszul, [26, 1.8] gives the formula

$$\mathrm{P}^Q_M(t) = \frac{\mathsf{H}_M(-t)}{\mathsf{H}_R(-t)}\,. \tag{5.2.4}$$

A similar formula is not known for generalized Poincaré series $\mathrm{P}^Q_{M,N}(t)$ when both modules M, N are Koszul. One of the few situations when it is possible to find expressions for $\mathrm{P}^Q_{M,N}(t)$ is when the square of the maximal ideal is zero, which is the hypothesis in the lemma below. This result will allow us to verify some of the hypotheses in Theorem 5.3.2, in order to further compute generalized Poincaré series over rings with the cube of the maximal ideal equal to 0.

Lemma 5.2.6. *If $(R, \mathfrak{n}, \mathsf{k})$ is a local ring with $\mathfrak{n}^2 = 0$, and M, N are R-modules such that $\mathfrak{n}(M \otimes_R N) = 0$, then $\mathrm{Tor}^R_n(M, \nu_N) = 0$ for all n and*

$$\mathrm{P}^R_{M,N}(t) = \frac{\mathsf{H}_M(-t)\,\mathsf{H}_N(-t)}{\mathsf{H}_R(-t)}.$$

Proof. Let $\varphi\colon R^a \to N$ be a surjective homomorphism and let $\psi\colon \mathfrak{n}R^a \to \mathfrak{n}N$ be the map induced by φ. Note that ψ is a surjective map of k-vector spaces, and hence it is split, implying that $\mathrm{Tor}^R_i(M, \psi)$ is surjective for all i. We have a commutative diagram, where the vertical arrow on the left is induced by the inclusion $\mathfrak{n}R^a \subseteq R^a$:

$$\begin{array}{ccc} \mathrm{Tor}^R_i(M, \mathfrak{n}R^a) & \xrightarrow{\mathrm{Tor}^R_i(M,\psi)} & \mathrm{Tor}^R_i(M, \mathfrak{n}N) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Tor}^R_i(M,\nu_N)} \\ \mathrm{Tor}^R_i(M, R^a) & \xrightarrow{\mathrm{Tor}^R_i(M,\varphi)} & \mathrm{Tor}^R_i(M, N) \end{array}$$

Since $\mathrm{Tor}^R_i(M, R^a) = 0$ when $i > 0$, it follows that $\mathrm{Tor}^R_i(M, \nu_N) = 0$ for all $i > 0$. The hypothesis that $\mathfrak{n}(M \otimes_R N) = 0$ implies that $\mathrm{Tor}^R_0(M, \nu_N) = 0$ as well.

Given the hypothesis $\mathfrak{n}^2 = 0$, it is clear from the definition that any finitely generated R-module is Koszul. Furthermore, the fact that $\mathrm{Tor}^R_i(M, \nu_N) = 0$ for all i

implies we have a short exact sequence in homology, which is induced from (5.2.2):

$$0 \to \operatorname{Tor}_n^R(M,N) \to \operatorname{Tor}_n^R(M,\overline{N}) \to \operatorname{Tor}_{n-1}^R(M,\mathfrak{n}N) \to 0\,.$$

A rank count in this exact sequence, together with the formula (5.2.4), gives

$$\begin{aligned}
\mathrm{P}^R_{M,N}(t) &= \mathrm{P}^R_{M,\overline{N}}(t) - t\,\mathrm{P}^R_{M,\mathfrak{n}N}(t) \\
&= \ell(\overline{N})\,\mathrm{P}^R_{M,\mathsf{k}}(t) - \ell(\mathfrak{n}N)t\,\mathrm{P}^R_{M,\mathsf{k}}(t) \\
&= \mathrm{P}^R_{M,\mathsf{k}}(t)\cdot\big(\ell(\overline{N}) - \ell(\mathfrak{n}N)t\big) \\
&= \frac{\mathsf{H}_M(-t)\,\mathsf{H}_N(-t)}{\mathsf{H}_R(-t)}\,.
\end{aligned}$$

□

5.3 Applications to short local rings

Let $(Q,\mathfrak{m},\mathsf{k})$ be a local Artinian ring. We say that Q is *short* if $\mathfrak{m}^3=0$. While one may perhaps expect relatively simple homological behavior over such rings, this expectation is only partially met; see the introduction of [5] for an overview of known bad behavior over such rings. An element $f\in Q$ such that $f^2=0$ and $f\mathfrak{m}=\mathfrak{m}^2$ is called a *Conca generator* in [5]. Note that rings that admit a Conca generator are necessarily short. In [5, Theorem 1.1] it is proved that if Q has a Conca generator, then $\mathrm{P}^Q_M(t)$ is rational, with denominator $\mathsf{H}_Q(-t)$, for all finitely generated R-modules M.

While not all short rings admit a Conca generator, Conca [16, Theorem 10] shows that when k is algebraically closed a generic quadratic standard graded algebra

Q with $\dim_{\mathsf{k}} Q_2 < \dim_{\mathsf{k}} Q_1$ admits a Conca generator.

All Gorenstein short rings with k algebraically closed have a Conca generator, see [5, Theorem 4.1], and in this case Menning and Şega [36, Theorem 3.1] proved that $\mathrm{P}^Q_{M,N}(t)$ is rational, with denominator $\mathsf{H}_Q(-t)$, for all finitely generated R-modules M, N. While this kind of rationality result is rather trivial when $\mathfrak{m}^2 = 0$, as seen in Lemma 5.2.6, rationality does not hold for all short rings since there exist short local rings Q for which $\mathrm{P}^Q_{\mathsf{k}}(t)$ is irrational, cf. Anick [1]. It remains an open question whether all generalized Poincaré series are rational in the case of short local rings with a Conca generator. Our work below brings some evidence towards a positive answer.

In what follows we consider short local rings that admit exact zero divisors. We first show in Lemma 5.3.1 that the existence of exact zero divisors in a short local ring restricts the Hilbert series of the ring. On the other hand, if Q is a short standard graded algebra with $\mathsf{H}_Q(t) = 1 + et + (e-1)t^2$ that has a Conca generator (which is necessarily an exact zero divisor, as seen below), it follows from [6, Proposition 5.4] that a generic linear form f is an exact zero divisor. These observations make the point that the hypotheses of our final result, Theorem 5.3.2, are satisfied in many situations.

Lemma 5.3.1. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a short local ring, and let e denote the minimal number of generators of $\mathfrak{m}$.*

1. *If there exists an exact pair* (f, g) *of zero divisors in* Q*, then*

$$\mathsf{H}_Q(t) = 1 + et + (e-1)t^2 \quad \textit{and} \quad f\mathfrak{m} = \mathfrak{m}^2\,,\ g\mathfrak{m} = \mathfrak{m}^2\,,\ f, g \notin \mathfrak{m}^2\,.$$

2. *If* $\mathsf{H}_Q(t) = 1 + et + (e-1)t^2$*, then* (f, f) *is an exact pair of zero divisors in* Q *if and only if* f *is a Conca generator.*

Proof. (1) If $e = 1$, then $\mathfrak{m} = (f) = (g)$ and hence $\mathfrak{m}^2 = 0$. The statement is thus clear. Assume now $e \geq 2$. Indeed, observe first that $g \notin \mathfrak{m}^2$, since $g \in \mathfrak{m}^2$ implies $g\mathfrak{m} = 0$, and hence $(f) = \operatorname{ann}_Q(g) = \mathfrak{m}$. This is a contradiction, since $e \geq 2$. Similarly, $f \notin \mathfrak{m}^2$

Since $\mathfrak{m}^3 = 0$, note that $\mathfrak{m}(f\mathfrak{m}) = 0$. We have a short exact sequence of k-vector spaces

$$0 \to \frac{(g) + \mathfrak{m}^2}{\mathfrak{m}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{\cdot f} f\mathfrak{m} \to 0\,.$$

Since $\dim_{\mathsf{k}} \mathfrak{m}/\mathfrak{m}^2 = e$ and

$$\dim_{\mathsf{k}} \frac{(g) + \mathfrak{m}^2}{\mathfrak{m}^2} = 1\,,$$

a rank count in this exact sequence yields the second equality below:

$$\ell(f\mathfrak{m}) = \dim_{\mathsf{k}}(f\mathfrak{m}) = e - 1\,.$$

Then, a dimension count in the short exact sequence

$$0 \to f\mathfrak{m} \hookrightarrow (f) \to \frac{(f)}{f\mathfrak{m}} \to 0$$

gives

$$\ell(f) = \ell(f\mathfrak{m}) + \ell((f)/f\mathfrak{m}) = (e-1) + 1 = e\,.$$

Similarly, we have $\ell(g) = e$. The hypothesis that (f, g) is an exact pair implies $Q/(f) \cong (g)$, and hence we have an exact sequence

$$0 \to (f) \to Q \to (g) \to 0\,.$$

A length count in this sequence gives

$$\ell(Q) = \ell(f) + \ell(g) = 2e\,.$$

Since $\ell(Q) = 1 + \dim_{\mathsf{k}}(\mathfrak{m}/\mathfrak{m}^2) + \dim_{\mathsf{k}}(\mathfrak{m}^2)$, we conclude $\dim_{\mathsf{k}}(\mathfrak{m}^2) = e - 1$. This establishes the desired formula for $\mathsf{H}_Q(t)$. Finally, the inclusion $f\mathfrak{m} \subseteq \mathfrak{m}^2$ and the fact that $\dim_{\mathsf{k}}(f\mathfrak{m}) = e - 1 = \dim_{\mathsf{k}}(\mathfrak{m}^2)$ implies $f\mathfrak{m} = \mathfrak{m}^2$. Similarly, $g\mathfrak{m} = \mathfrak{m}^2$.

(2) Assume $\mathsf{H}_Q(t) = 1 + et + (e-1)t^2$. If (f, f) is an exact pair of zero divisors, then the desired conclusion follows from (1). For the converse, assume that $f^2 = 0$ and $f\mathfrak{m} = \mathfrak{m}^2$. Since $(f) \subseteq \operatorname{ann}_Q(f)$, it suffices to show that (f) and $\operatorname{ann}_Q(f)$ have the same length.

The hypothesis that $\mathsf{H}_Q(t) = 1 + et + (e-1)t^2$ implies

$$\ell(\mathfrak{m}) = 2e - 1, \qquad \ell(f\mathfrak{m}) = \ell(\mathfrak{m}^2) = e - 1 \qquad \text{and} \qquad \ell(Q) = 2e\,.$$

Using a length count in the exact sequence

$$0 \to \operatorname{ann}_Q(f) \hookrightarrow \mathfrak{m} \xrightarrow{\cdot f} f\mathfrak{m} \to 0$$

we obtain

$$\ell(\mathrm{ann}_Q(f)) = \ell(\mathfrak{m}) - \ell(f\mathfrak{m}) = 2e - 1 - (e-1) = e\,,$$

and then a length count in the exact sequence

$$0 \to \mathrm{ann}_Q(f) \hookrightarrow Q \xrightarrow{\cdot f} (f) \to 0$$

gives

$$\ell(f) = \ell(Q) - \ell(\mathrm{ann}_Q(f)) = 2e - e = e = \ell(\mathrm{ann}_Q(f))\,. \qquad \square$$

In [5, Theorem 3.2] it is proved that if Q is a short local ring admitting a Conca generator f, then any finitely generated Q-module M with $fM = 0$ is Koszul, and thus $\mathrm{P}_M^Q(t) = \mathsf{H}_M(-t)/\,\mathsf{H}_Q(-t)$. Theorem 5.3.2 below extends this property to generalized Poincaré series, in the case when Q admits exact zero divisors (and thus a Conca generator is also an exact zero divisor, by Lemma 5.3.1).

Theorem 5.3.2. *Let $(Q, \mathfrak{m}, \mathsf{k})$ be a short local ring. Assume* $\mathrm{char}\,(\mathsf{k}) = 0$*, (f, g) is an exact pair of zero divisors and set $R = Q/(f)$. Let M, N be finitely generated R-modules such that $\mathfrak{m}(M \otimes_R N) = 0$ and $gN = 0$.*

Then

$$\mathrm{P}_{M,N}^Q(t) = \frac{\mathsf{H}_M(-t)\,\mathsf{H}_N(-t)}{\mathsf{H}_Q(-t)}\,.$$

Proof. Let e denote the minimal number of generators of $\mathfrak{m}$. By Lemma 5.3.1, we have $f \notin \mathfrak{m}^2$, $f\mathfrak{m} = \mathfrak{m}^2$ and

$$\mathsf{H}_Q(t) = 1 + et + (e-1)t^2.$$

We have then $\mathsf{H}_R(t) = 1 + (e-1)t$, and hence

$$\mathsf{H}_Q(t) = (1+t)\,\mathsf{H}_R(t).$$

Note that $\mathfrak{m}^2 N \subseteq fM = 0$. Lemma 5.2.6 gives that $\operatorname{Tor}_i^R(M, \nu_N) = 0\,, \forall i \geq 0$, and then the conclusion follows from Proposition 5.2.4 and Lemma 5.2.6. $\square$

APPENDIX A.

MACAULAY2 IMPLEMENTATION

Let $(Q, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring with $f \in \mathfrak{m}$. The process of adjoining exterior/divided power variables to construct a dg algebra resolution A of $R = Q/(f)$ is discussed in 3.1.5 for the case when f is an exact zero divisor. Given a R-module M, there exists a free resolution F of M over Q with dg module structure over the dg algebra resolution of R over Q. However, that free resolution with dg module structure need not be minimal. When $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, [8, Proposition 2.2.2] tells us that the minimal free resolution of M over Q has a semi-free dg module structure over the Koszul complex $Q\langle y \mid \partial(y) = f\rangle$. When f is regular, the Koszul complex is indeed the free resolution of R over Q. The multiplication maps in this case can be described fairly easily since the length of the Koszul complex is 1. Furthermore, Theorem 3.2.1 shows when $g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ such that $fg = 0$, the minimal free resolution of M over Q also has a dg module structure over the dg algebra $A = Q\langle y, t \mid \partial(y) = f, \partial(t) = gy\rangle$, constructed in 3.1.5. In this chapter, we write an algorithm that can be implemented in *Macaulay2* [23], which builds the multiplication maps $\sigma_{i,j}$ in Theorem 3.2.1, iteratively.

Algorithm 1. *(version I:* `multMap`*) Let Q be standard graded k-algebra with a linear* *zero divisor f and g such that $fg = 0$. Let M be a Q-module such that $fM = 0$ with (C, d) as its free resolution. Given integers i and j, the function returns the $\sigma_{m,n}$ such that $m + n \leq i + j$, basis of the free module V_k and basis of the free modules U_k for all $k \leq j + 1$ where σ, U, V are as defined in Theorem 3.2.1.*

Require: $i + j \leq \text{length}(C)$

Ensure: f has a linear annihilator g, C is a free resolution of a Q-module M with $fM = 0$.

function `multMap`(C, f, i, j)
  $Z \leftarrow \{f, g\}$
  $j \leftarrow \deg(v)$
  Initialize **MutableHashTable** Sig, BV, BU ; **List** P
  ▷ *Sigma, BV, BU denote $\sigma_{i,j}$, basis of V_i and basis of U_i respectively* ◁
  $BU(0) = BV(0) \leftarrow \text{id}(C_0)$ ▷ $\text{id}(F)$ *denotes identity matrix of size* $\dim(F)$
  for $k = 0$ to j **do**
    for $l = 1$ to $(i + j - k)$ **do**
      ▷ *We use* (3.2.5) *in the next step.* ◁
      $B \leftarrow Z[l\%2] \cdot Sig(l - 1, k) - ((-1)^{l+1} Sig(l, k - 1) \cdot d_k)$
      Solve for X in $d_{l+k} \cdot X = B$
    for $h = 1$ to $k + 1$ **do**
      Append $Sig(h, (k + 1 - h)) * BV(k + 1 - h)$ to P

▷ *Each p in P is an individual σ maps in the description of* $\mathrm{Im}(\phi_k)$ *in Theorem 3.2.1* ◁

Sum $\mathrm{Im}(p)$ for p in P to get $\mathrm{Im}(\phi_k)$ ▷ *computing the module* $\mathrm{Im}(p)$

Extend the basis of $\mathrm{Im}(\phi_k)$ to a basis of U_{k+1}

▷ *Find the basis of kernel of transpose of the matrix that describes the module* $\mathrm{Im}(\phi_k)$ ◁

$BV(k+1) \leftarrow$ Extended basis elements in the previous step.

$BU(k+1) \leftarrow$ Append $BV(k+1)$ to P

return (Sig, BU, BV) ▷ *A sequence of Hash Tables*

We further use the above function to build the minimal free resolution of M over R using Theorem 3.2.1.

Algorithm 2. *(version I:* `tateRes`*.) Let Q be standard graded k-algebra with a linear exact pair of zero divisor (f, g). Let M be a Q-module with $fM = 0$ with (C, d) as its minimal free resolution. The function returns the minimal free resolution of M over $Q/(f)$ of length 4.(Optional input:* `LenLimit => 4`*, default value can be changed).*

Ensure: f is an exact zero divisor, C is a minimal free resolution of Q module with $fM = 0$.

function `tateRes`(C, f)

$i \leftarrow 1$

$j \leftarrow$ length(C) - 1

$R \leftarrow Q/(f)$

Initialize **MutableHashTable** $invBU$, pC ; **List** D

▷ *invBU, pC denote the inverse of BU and the projection matrix* π *in Construction 3.1.2 resp.* ◁

$(Sig, BV, BU) \leftarrow$ `multMap`(C, f, i, j)

for $k = 0$ to j **do**

  if k == 0 **then**

    $invBU(1) \leftarrow BU(1)^{-1}$

    Append $d_1 \cdot BV(1)$ to D

    ▷ A^i *denotes the sub-matrix of* A *that has the first* $i+1$ *columns of* A ◁

    $pC \leftarrow BU(1)^{\dim(BV(1))} \cdot invBU(1)$

  else

    $invBU(k+1) \leftarrow BU(k+1)^{-1}$

    $pC \leftarrow BU(k+1)^{\dim(BV(k+1))} \cdot invBU(k+1)$

    Append $pC(k) \cdot d_{k+1} \cdot BV(k+1)$ to D

▷ *D is a list of matrices in Q. We convert these maps to define a chain complex in Q and further substitute it to the ring R* ◁

return Chain Complex of D in R

Example. We start by defining a ring Q that admits linear zero divisors. f and b here are the desired linear zero divisors such $fb = 0$.

```
i1 : S = ZZ/32003[f,b,c,d]
```

o1 = S

o1 : PolynomialRing

```
i2 : I = ideal(f*b, c^7+d^8)
```

o2 = ideal $(f\,b,\, d^8 + c^7)$

o2 : Ideal of S

```
i3 : Q = S/I
```

o3 = Q

o3 : QuotientRing

We define a Q-module M such that $fM = 0$. We achieve this by taking the cokernel of f along with some other elements.

```
i4 : M = cokernel matrix{{b+c,d^2,f}} -- Q module with fM = 0
```

o4 = cokernel $\begin{pmatrix} b+c & d^2 & f \end{pmatrix}$

o4 : Q-module, quotient of Q^1

We find the minimal free resolution of M over Q:

```
i5 : U = res (trim M, LengthLimit => 3)
```

o5 = $$\underset{0}{Q^1} \xleftarrow{\begin{pmatrix} b+c & f & d^2 & c^7 \end{pmatrix}} \underset{1}{Q^4} \xleftarrow{\begin{pmatrix} f & 0 & -d^2 & 0 & 0 & -c^7 & 0 & 0 \\ -c & b & 0 & -d^2 & 0 & 0 & -c^7 & 0 \\ 0 & 0 & b+c & f & d^6 & 0 & 0 & -c^7 \\ 0 & 0 & 0 & 0 & 1 & b+c & f & d^2 \end{pmatrix}} \underset{2}{Q^8} \xleftarrow{\begin{pmatrix} b & \dots & 0 \\ c & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & c^7 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & d^6 \end{pmatrix}} \underset{3}{Q^{12}}$$

o5 : ChainComplex

Finally, we use the function we defined `multMap` which takes in inputs (C, f, i, j) which computes $\sigma_{i,j}$ along with the basis of free modules U_l and V_l for $l \leq i + j$.

```
i6 : Vec = matrix{{c^2}, {d*c^3},{c+d},{f*d+b}}
```

```
-- an element in U_1 = Q^4
```

o6 = $\begin{pmatrix} c^2 \\ c^3d \\ c+d \\ f\,d+b \end{pmatrix}$

o6 : Matrix $Q^4 \longleftarrow Q^1$

```
i7 : (sig,A,B) = multMap(U,f,1,1)
```

o7 : Sequence

```
i8 : prod = sig#(1,1)*Vec  -- should be an element in U_2 = Q^8
```

o8 = $\begin{matrix} \{2\} \\ \{2\} \\ \{3\} \\ \{3\} \\ \{8\} \\ \{8\} \\ \{8\} \\ \{9\} \end{matrix} \begin{pmatrix} c^2 \\ -c^2 \\ 0 \\ c+d \\ 0 \\ 0 \\ f\,d+b \\ 0 \end{pmatrix}$

o8 : Matrix $Q^8 \longleftarrow Q^1$

Example. We define a ring Q with a linear pair of exact zero divisors. Because of [6, Proposition 5.4] a generic linear form f from a short standard graded algebra like the one below algebra with $\mathsf{H}_Q(t) = 1 + et + (e-1)t^2$ for $e \geq 2$ is an exact zero divisor.

```
i1 : needsPackage "TateAlgebra"
```

o1 = TateAlgebra

o1 : Package

```
i2 : S = ZZ/32003[a,b,c]
```

o2 = S

o2 : PolynomialRing

```
i3 : setRandomSeed 0
```

o3 = 0

```
i4 : I = trim ideal matrix {{random(2,S), random(2,S),
                            random(2,S), random(2,S)}}
--There are 6 degree 2 homogeneous elements. Quotient out 4 of them
--will get us the required Hilbert series of {1,3,2}.
```

o4 : Ideal of S

i5 : Q = S/I

o5 = Q

o5 : QuotientRing

i6 : reduceHilbert hilbertSeries Q

o6 = $\frac{1+3T+2T^2}{1}$

o6 : Expression of class Divide

We now verify whether f and g are exact pair of zero divisors:

i7 : f = a - 14488*b - 7246*c

o7 = $a - 14488\,b - 7246\,c$

o7 : Q

i8 : g = ann f

o8 = ideal $(a + 10990\,b - 12549\,c)$

o8 : Ideal of Q

i9 : ann g == ideal f

o9 = true

We define a Q-module with $fM = 0$ and find its minimal free resolution.

i10 : M = cokernel matrix{{107*b+4376*c, -5570*b + 3187*c, f}}

o10 = cokernel $\begin{pmatrix} 107\,b+4376\,c & -5570\,b+3187\,c & a-14488\,b-7246\,c \end{pmatrix}$

o10 : Q-module, quotient of Q^1

i11 : C = res(trim M, LengthLimit => 4)

o11 = $\underset{0}{Q^1} \leftarrow \underset{1}{Q^3} \leftarrow \underset{2}{Q^7} \leftarrow \underset{3}{Q^{15}} \leftarrow \underset{4}{Q^{31}}$

o11 : ChainComplex

We now define $R = Q/(f)$ and find the minimal free resolution of M over R using the in-built function `res` and our function `tateMul`. Since our function needs a free resolution to start with, the time complexity of the built-in function is computationally better.

```
i12 : R = Q/(f)
o12 = R
o12 : QuotientRing
i13 : CRtate = time tateRes(C,f,LenLimit => 4)
     -- used 0.0545271 seconds
```

o13 = $\underset{0}{R^1} \leftarrow \underset{1}{R^2} \leftarrow \underset{2}{R^4} \leftarrow \underset{3}{R^8} \leftarrow \underset{4}{R^{16}}$

```
o13 : ChainComplex
i14 : CR = time res(trim(M**R), LengthLimit => 4)
     -- used 0.00779566 seconds
```

o14 = $\underset{0}{R^1} \leftarrow \underset{1}{R^2} \leftarrow \underset{2}{R^4} \leftarrow \underset{3}{R^8} \leftarrow \underset{4}{R^{16}}$

```
o14 : ChainComplex
```

We need to verify whether our function produced an exact sequence. So, we check the exactness at each level:

```
i15 : apply({1,2,3}, i->(ker CRtate.dd_i == image CRtate.dd_(i+1)))
o15 = {true, true, true}
o15 : List
```

It is clear that `CRtate` is also a minimal free resolution of M over R since the betti numbers match `CR`; the output from `res`.

REFERENCE LIST

[1] D. J. Anick, *A counterexample to a conjecture of serre*, Ann. of Math. **115** (1982), 1–33.

[2] L. L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. **96** (1989), 71–101.

[3] L. L. Avramov and R. O. Buchweitz, *Support varieties amd cohomology over complete intersections*, Invent.math. **142** (2000), 285–318.

[4] L. L. Avramov, I. Henriques, and L. M. Şega, *Quasi-complete intersection homomorphisms*, Pure Appl. Math. Q. **9** (2013), no. 4, 579–612.

[5] L. L. Avramov, S. B. Iyengar, and L. M. Şega, *Free resolutions over short local rings*, J. London Math. Soc **78** (2008), 459–476.

[6] ______, *Short koszul modules*, Journal of Commutative Algebra **2** (2010), 249–279.

[7] L. L. Avramov, *Obstructions to the existence of multiplicative structures on minimal free resolutions*, American Journal of Mathematics **103** (1981), no. 1, 1–31.

[8] L. L. Avramov, *Infinite free resolutions*, Six Lectures on Commutative Algebra (Basel), Birkhäuser Basel, Basel, 1998, pp. 1–118.

[9] P. A. Bergh, *On the vanishing of (co)homology over local rings*, J. Pre Appl. Algebra **212** (2008), 262–270.

[10] P. A. Bergh, O. Celikbas, and D. A. Jorgensen, *Homological algebra modulo exact zero-divisors*, Kyoto Journal of Mathematics **54** (2014), 879–895.

[11] W. Bruns and H. J. Herzog, *Cohen-macaulay rings*, 2 ed., Cambridge University Press, 1998.

[12] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, American Journal of Mathematics **99** (1977), no. 3, 447–485.

[13] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.

[14] L. W. Christensen and S. Sather-Wagstaff, *A cohen-macaulay algebra has only finitely many semidualizing modules*, Math. Proc. Cambridge Philos. Soc. **3** (2008), 601–603.

[15] A. Conca, E. D. Negri, and M. E. Rossi, *Koszul algebras and regularity*, Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of His 65^{th} Birthday (2013), 285–315.

[16] A. Conca, *Gröbner bases for spaces of quadrics of low codimension*, Adv, Appl. Math **24** (2000), 111–124.

[17] S. A. Cooper and S. Sather-Wagstaff, *A somewhat gentle introduction to differential graded commutative algebra*, Connections between Algebra, Combinatorics, and Geometry (New York, NY), Springer, New York, NY, 2014, pp. 3–99.

[18] L. M. Şega and D. Sireeshan, *Dg module structures and minimal free resolutions modulo an exact zero-divisor*, 2023.

[19] ———, *The mapping cone of an eisenbud operator and applications to exact zero divisors*, 2024.

[20] D. Dugger, *Betti numbers of almost complete intersections*, Illinois journal of Mathematics **44** (2000), no. 3, 531–541.

[21] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc **260** (1980), 35–64.

[22] G. Evans and H. Charalambous, *Resolutions obtained by iterated mapping cones*, J. Algebra **176** (1995), 750–754.

[23] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www2.macaulay2.com.

[24] T. H. Gulliksen, *A change of ring theorem with applications to poincaré series and intersection multiplicity*, Math. Scand. **34** (1974), 167–183.

[25] I. Henriques and L. M. Şega, *Free resolutions over short gorenstein local rings*, Mathematische Zeitschrift **267** (2009), 645–663.

[26] J. Herzog and S. Iyengar, *Koszul modules*, J. Pure Appl. Algebra **201** (2005), 154–188.

[27] H. Holm, *Construction of totally reflexive modules from an exact pair of zero divisors*, Bull. London Math. Soc. **43** (2011), no. 2, 278–288.

[28] S. B. Iyengar, *Free resolutions and change of rings*, Journal of Algebra **190** (1997), 195–213.

[29] Y. T. J. Herzog, *Resolutions by mapping cones.*, Homology, Homotopy and Applications **4** (2002), 277–294.

[30] C. Jacobsson, *Finitely presented graded lie algebras and homomorphisms of local rings*, J. Pure Appl. Algebra **38** (1985), 243–253.

[31] A. Kustin and M. Miller, *Algebra structures on minimal resolutions of gorenstein rings of embedding codimension four*, Math. Z. **173** (1980), 171–184.

[32] A. R. Kustin, L. M. Şega, and A. Vraciu, *Minimal quasi-complete intersection ideals*, Illinois Journal of Mathematics **58** (2014), 867–889.

[33] A. R. Kustin, *Gorenstein algebras of codimension four and characteristic two*, Communications in Algebra **15** (1987), no. 11, 2417–2429.

[34] H. Matsumura, *Commutative ring theory*, 1 ed., vol. 8, Cambridge University Press, 1989.

[35] V. Mehta, *Endomorphisms of complexes and modules over golod rings*, Phd thesis, University of California, Berkeley, CA, 1976.

[36] M. C. Menning and L. M. Şega, *Cohomology of finite modules over short gorenstein rings*, Journal of Commutative Algebra **10** (2018), 63–81.

[37] S. Nasseh and S. Sather-Wagstaff, *Geometric aspects of representation theory for dg algebras: answering a question of vasconcelos*, J. London Math. Soc **96** (2017), 271–292.

[38] M. E. Rossi and L. M. Şega, *Poincaré series of modules over compressed gorenstein local rings*, Advances in Mathematics **259** (2014), 421–447.

[39] J. Shamash, *Poincaré series of a local ring*, Journal of Algebra **12** (1969), 453–470.

[40] L. Sharifan, *Minimal free resolution of monomial ideals by iterated mapping cone*, Bull. Iran. Math. Soc. **44** (2018), 1007–1024.

[41] H. Srinivasan, *The non-existence of a minimal algebra resolution despite the vanishing of avramov obstructions*, Journal of Algebra **146** (1992), no. 2, 251–266.

[42] ———, *A grade five gorenstein algebra with no minimal algebra resolutions*, J. Algebra **179** (1996), 362–379.

[43] J. Tate, *Homology of Noetherian rings and local rings*, Illinois Journal of Mathematics **1** (1957), no. 1, 14 – 27.

[44] A. Windle, *Cohomological operators on quotients by exact zero divisors*, J. Algebra Appl. **21** (2022), no. 1, 2250016.

VITA

Deepak Sireeshan was born on May 26, 1994 in Tamil Nadu, India. In 2015, he received his B.E. in Electrical and Electronics Engineering from Anna University, Chennai, India. He received a M.S. in Mathematics from the University of Missouri, Kansas City, USA, in 2020.