THE POISSON PROBLEM ON LIPSCHITZ DOMAINS

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Doctor of Philosophy

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by
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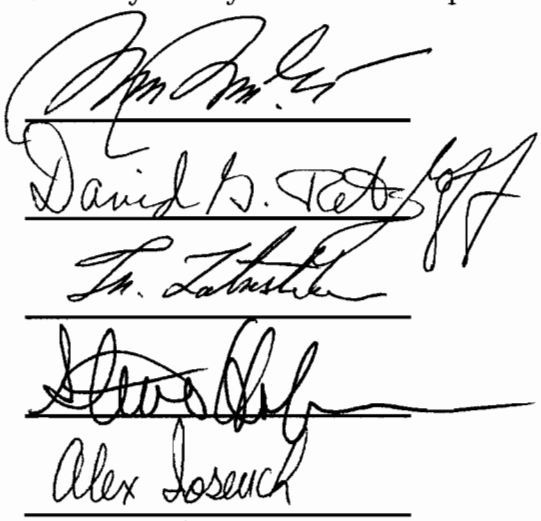

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ABSTRACT

The aim of this work is to describe the sharp ranges of indices, for which the Poisson problem for Laplacian with Dirichlet or Neumann boundary conditions is well-posed on the scales of Besov and Triebel-Lizorkin spaces on arbitrary Lipschitz domains.

The main theorems we prove extend the work of D. Jerison and C. Kenig [JFA, 95], whose methods and results are largely restricted to the case $p \geq 1$, and answer the open problem # 3.2.21 on p. 121 in C. Kenig's book in the most complete fashion. When specialized to Hardy spaces, our results provide a solution of a (strengthened form of a) conjecture made by D.-C. Chang, S. Krantz and E. Stein regarding the regularity of the Green potentials on Hardy spaces in Lipschitz domains.

The corollaries of our main results include new proofs and various extensions of: Hardy space estimates for Green potentials in convex domains due to V. Adolfsson, B. Dahlberg, S. Fromm, D. Jerison, G. Verchota and T. Wolff and the $L^p - L^q$ estimates for the gradients of Green potentials in Lipschitz domains, due to B. Dahlberg.

Contents

Chapter 1

Introduction

1.1 Description of main results

Given an open, connected domain $\Omega \subset \mathbb{R}^n$, let $\mathbf{G}_D$, $\mathbf{G}_N$ be the solution operators for the Poisson equation for the Laplacian in Ω with homogeneous Dirichlet and Neumann boundary conditions, respectively. The aim of this work is to describe the sharp ranges of indices p, q, α for which

$$\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad \mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{1.1.1}$$

and

$$\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad \mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{1.1.2}$$

are well-defined and bounded, in the case when Ω is an arbitrary, bounded Lipschitz domain. Here $B^{p,q}_{\alpha}$, $F^{p,q}_{\alpha}$ stand, respectively, for the classes of Besov and Triebel-Lizorkin spaces. Here, $F^{p,q}_{\alpha,0}(\Omega)$, $B^{p,q}_{\alpha,0}(\Omega)$ are distributions from $F^{p,q}_{\alpha}(\mathbb{R}^n)$ and $B^{p,q}_{\alpha}(\mathbb{R}^n)$, respectively which are supported in $\bar{\Omega}$. Smoothness spaces in domains, such as $F^{p,q}_{\alpha}(\Omega)$ and $B^{p,q}_{\alpha}(\Omega)$ are defined by restricting corresponding distributions from $\mathbb{R}^n$ to Ω.

When $\Omega = \mathbb{R}^n$, or even when $\partial\Omega \in C^{\infty}$, both $\partial_{x_j}\partial_{x_k}\mathbf{G}_D$ and $\partial_{x_j}\partial_{x_k}\mathbf{G}_N$, $1 \leq$

$j,k\leq n$, fall under the scope of the classical theory of singular integral operators of Calderón-Zygmund type, but this categorically fails to be the case if $\partial\Omega$ is not smooth.

We build on the work of many people who have dealt with homogeneous/inhomogenous problems for the Laplacian with Dirichlet/Neumann boundary conditions in Lipschitz domains; an excellent account can be found in [**79**]. Earlier results emphasized homogeneous problems with boundary data exhibiting an integer amount of smoothness. The corresponding estimates for the harmonic functions involve mixed norms (informally, L^∞ in the transversal direction, and L^p in the tangential direction), of the type

$$\Big\{\int_{\partial\Omega}\Big[\sup_{y\in\Gamma(x)}|\nabla^s u(y)|^p\Big]\,d\sigma_x\Big\}^{1/p}\leq C\|u\|_{F^{p,2}_s(\partial\Omega)},\qquad s=0,1,\tag{1.1.3}$$

$$\Big\{\int_{\partial\Omega}\Big[\sup_{y\in\Gamma(x)}|\nabla u(y)|^p\Big]\,d\sigma_x\Big\}^{1/p}\leq C\|\partial_\nu u\|_{L^p(\partial\Omega)},\tag{1.1.4}$$

for appropriate ranges of p. Here $\{\Gamma(x)\}_{x\in\partial\Omega}$ is a fixed family of cone-like non-tangential approach regions in Ω, with vertices at boundary points, $d\sigma$ is the canonical surface measure on $\partial\Omega$, and ν is the unit normal (defined a.e.) on $\partial\Omega$. Estimates such as (1.1.3)-(1.1.4) underpin the entire work here and play the role of end-point/limiting cases in our theory.

The theorems we prove pertaining to the operators (1.1.1) extend the work of D. Jerison and C. Kenig [**68**] whose methods and results are essentially restricted to the case $p\geq 1$. Similarly, our main result dealing with the operators (1.1.2) generalizes the work of E. Fabes, O. Mendez and M. Mitrea [**40**], and D. Zanger [**130**], and answers the open problem # 3.2.21 on p. 121 in C. Kenig's book [**79**] in

the most complete fashion.

We now turn to a more systematic description of our main results. Let $(a)_+ := \max\{a, 0\}$, $|\Omega|$ the Euclidean volume of Ω and $\partial^f_\nu u$ the normal derivative of u, normalized in a suitable fashion relative to the distribution f (this concept is fully explained in the body of the paper). The first central result in this paper reads as follows.

Theorem 1.1.1. *For each bounded, connected Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ with the following significance. Assume that $\frac{n-1}{n} < p \leq \infty$, $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$, are such that either one of the four conditions*

$$\begin{aligned}
&(I): && \tfrac{n-1}{n-1+\varepsilon} < p \leq 1 \quad \text{and} \quad (n-1)\left(\tfrac{1}{p} - 1\right) + 1 - \varepsilon < s < 1;\\
&(II): && 1 \leq p \leq \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1;\\
&(III): && \tfrac{2}{1+\varepsilon} \leq p \leq \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1;\\
&(IV): && \tfrac{2}{1-\varepsilon} \leq p \leq \infty \quad \text{and } 0 < s < \tfrac{2}{p} + \varepsilon,
\end{aligned} \tag{1.1.5}$$

is satisfied, if $n \geq 3$, and either one of the following three conditions

$$\begin{aligned}
&(I'): && \tfrac{2}{1+\varepsilon} \leq p \leq \tfrac{2}{1-\varepsilon} \quad \text{and } 0 < s < 1;\\
&(II'): && \tfrac{2}{3+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \quad \text{and } \tfrac{1}{p} - \tfrac{1+\varepsilon}{2} < s < 1;\\
&(III'): && \tfrac{2}{1-\varepsilon} < p \leq \infty \quad \text{and } 0 < s < \tfrac{1}{p} + \tfrac{1+\varepsilon}{2},
\end{aligned} \tag{1.1.6}$$

is satisfied, if $n = 2$. Then, with $\alpha := s + 1/p - 2$ and $0 < q \leq \infty$, the operators

$$\mathbf{G}_D : B^{p,q}_\alpha(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \tag{1.1.7}$$

$$\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \tag{1.1.8}$$

are well-defined and bounded. In the case of Neumann boundary conditions it is understood that $u := \mathbf{G}_N f$ *solves* $\Delta u = f|_\Omega - |\Omega|^{-1}\langle f, 1\rangle$ *in* Ω, $\partial_\nu^f u = 0$ *on* $\partial\Omega$, *and that* u *is normalized so that* $\langle u, 1\rangle = 0$.

Similar results hold for Triebel-Lizorkin spaces. More specifically, under the same assumptions on the indices p, q, s, α *as before, the operators*

$$\mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{1.1.9}$$

$$\mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{1.1.10}$$

are well-defined and bounded if $p \neq \infty$ *and*

$$\Big(\min\big\{\tfrac{1}{pn}+1,\ \tfrac{s}{n-1}+1\big\}\Big)^{-1} < q \leq \infty. \tag{1.1.11}$$

In particular, the above inequality is true when $\min\{p, 1\} \leq q \leq \infty$.

These results are sharp in the class of Lipschitz domains. For $\partial\Omega \in C^1$ *one can take* $\varepsilon = 1$.

When specialized to the Triebel-Lizorkin scale with $q = 2$ and $\alpha = 0$, these results solve a (strengthened form of a) conjecture made by D.-C. Chang, S. Krantz and E. Stein regarding the regularity of the Green potentials on Hardy spaces in Lipschitz domains. Cf. p. 130 of [**22**] where the authors write: "*For some applications it would be desirable to find minimal smoothness conditions on* $\partial\Omega$ *in order for our analysis of the Dirichlet and Neumann problems to remain valid. We do not know whether* $C^{1+\varepsilon}$ *boundary is sufficient in order to obtain* h^p_r [a Hardy space which, in our current notation corresponds to $F^{p,2}_0$] *estimates for the Dirichlet problem when* p *is near* 1." On p. 289 of [**21**] the authors make the conjecture that

$F_0^{p,2}$ estimates *continue to hold if* Ω *is of class* C^k*, where* $k > 1/p$." Here we derive such estimates when $\partial\Omega$ is Lipschitz. A sample result reads

$$\partial_{x_j}\partial_{x_k}\mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \quad \partial_{x_j}\partial_{x_k}\mathbf{G}_N : h_0^p(\Omega) \longrightarrow h^p(\Omega), \quad 1 \leq j,k \leq n, \tag{1.1.12}$$

provided that $1 - \varepsilon < p < 1$, where $\varepsilon = \varepsilon(\Omega) > 0$. This is sharp in the class of Lipschitz domains. For example, B. Dahlberg **[29]** has constructed a Lipschitz domain for which the analogue of (1.1.12) with Dirichlet boundary conditions fails for the *entire* L^p scale, $1 < p < \infty$.

It is illuminating to point out that the collection of all points with coordinates $(\alpha, 1/p)$ such that α, p are as in the statement of Theorem 1.1.1 can be identified with the two-dimensional hexagonal region depicted below:

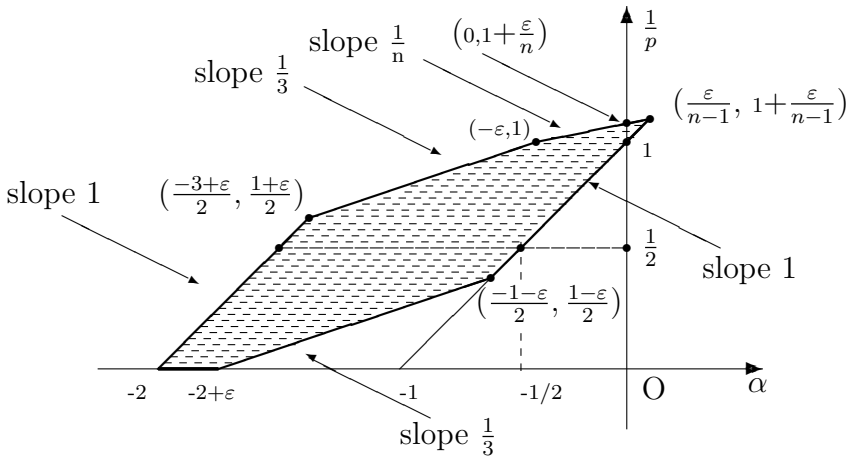

Figure 1.1

Let us denote by $F^{p,q}_{\alpha,z}(\Omega)$, $B^{p,q}_{\alpha,z}(\Omega)$ the restrictions of $F^{p,q}_{\alpha,0}(\Omega)$ and $B^{p,q}_{\alpha,0}(\mathbb{R}^n)$, respectively, to Ω. With this piece of notation we can then state:

Theorem 1.1.2. *For each bounded, connected Lipschitz domain* Ω *in* $\mathbb{R}^n$ *there*

exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ *with the following significance. Assume that* $\frac{n-1}{n} < p < \infty$, $(n-1)\big(\frac{1}{p}-1\big)_+ < s < 1$, *are such that either one of the four conditions in (1.1.5) is satisfied, if* $n \geq 3$, *and either one of the three conditions in (1.1.6) is satisfied, if* $n = 2$. *Then, with* $\alpha := s + 1/p - 2$ *and* $\min\{p,1\} \leq q < \infty$, *the operators*

$$\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2,z}(\Omega), \tag{1.1.13}$$

$$\mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2,z}(\Omega), \tag{1.1.14}$$

are isomorphisms. These ranges are sharp in the class of Lipschitz domains. For $\partial\Omega \in C^1$ *one can take* $\varepsilon = 1$.

Our approach to Theorem 1.1.1 rests on a suitable extension of the classical trace/extension theory and on the mapping properties of the harmonic layer potentials on Besov and Triebel-Lizorkin scales in Lipschitz domains. With regard to the former issue, a basic result proved in this paper is as follows. Let $B^{p,q}_s(\partial\Omega)$ denote the Besov class on the Lipschitz manifold $\partial\Omega$, obtained by transporting (via localization, involving a smooth partition of unity, and pull-back) the standard scale $B^{p,q}_s(\mathbb{R}^{n-1})$, as defined in, e.g., **[125]**.

Theorem 1.1.3. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$ *and assume that the indices* p, s *satisfy* $\frac{n-1}{n} < p \leq \infty$ *and* $(n-1)(\frac{1}{p}-1)_+ < s < 1$. *Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \quad \textit{for} \quad 0 < q \leq \infty. \tag{1.1.15}$$

Moreover, for this range of indices, Tr *is onto and has a bounded right inverse*

$$\mathrm{Ex} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{1.1.16}$$

(ii) Similar considerations hold for

$$\mathrm{Tr} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s}(\partial\Omega) \tag{1.1.17}$$

(it is understood that $q = \infty$ if $p = \infty$). In this situation, there exists a linear, bounded right inverse

$$\mathrm{Ex} : B^{p,p}_{s}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{1.1.18}$$

assuming that also (1.1.11) holds.

This type of results has a long history. Some of the forerunners are the books by Nikol'skij [**106**], Peetre [**108**] and Triebel [**124**], where the case $\Omega = \mathbb{R}^n_+$, $p \geq 1$ has been treated. Cf. also [**120**] for $p = 1$ and the discussion in [**118**]. Jawerth was the first to consider the case $p < 1$ (still for the upper-half space) in [**66**]; cf. also [**47**], [**48**], [**49**] in this regard. Adaptations to smooth domains are found in [**46**] and [**125**]. Dealing with non-smooth domains raises a new set of challenges and the case $p \geq 1$ has been artfully dealt with in [**71**] for a class of domains including those having a Lipschitz boundary; see also the excellent discussion in [**68**].

Here we extend this work by allowing a range of indices which is optimal in the context of arbitrary Lipschitz domains. In [**71**] the authors have constructed an extension operator by decomposing $\mathbb{R}^n \setminus \partial\Omega$ in Whitney cubes and gluing local averages over such cubes via a suitable partition of unity, but the argument breaks down when dealing with Besov and Triebel-Lizorkin spaces which are not locally convex. By way of comparison, our approach employs singular integral operators and – as long as $0 < s < 1$, which is natural given that $\partial\Omega$ is a Lipschitz manifold – works for the same range of indices as in the case when $\Omega = \mathbb{R}^n_+$.

As far as the class of Lipschitz domains is concerned, Theorem 1.1.3 satisfactorily settles the issue under discussion (counterexamples to (1.1.17) when $s = 1$ in C^1 domains can be found in **[68]**). It also further reinforces the philosophy that the Lipschitz domains make up the most general class of domains where a rich function theory can be developed, comparable in power and scope with that associated with the upper-half space.

We would like to mention that our extension operator also satisfies (with $a \vee b := \max\{a, b\}$)

$$\text{Ex} : F_1^{p,2}(\partial\Omega) \longrightarrow F_{1+\frac{1}{p}}^{p,p\vee 2}(\Omega), \quad 1 < p < \infty. \tag{1.1.19}$$

In particular, $F_1^{p,2}(\partial\Omega) \subseteq \text{Tr}(F_{1+1/p}^{p,p\vee 2}(\Omega))$. In this connection, see the open problem # 3.2.20 on p. 121 of **[79]**, which asks for an intrinsic characterization of $\text{Tr}(F_{3/2}^{2,2}(\Omega))$, when Ω is Lipschitz.

Having developed a satisfactory trace theory in Lipschitz domains, extending the definition of the *normal derivative*, i.e., $\partial_\nu u = \nu \cdot \text{Tr}\,[\nabla u]$, from functions in $C^1(\overline{\Omega})$ to functions belonging to Besov and Triebel-Lizorkin scales is a straightforward step if the smoothness index is large enough (as to allow for a meaningful definition of the trace and then to make sense of the dot product). However, matters are considerably more delicate if u is not regular enough so that ∇u has a trace in the classical sense. Also, recall that in the case when $\partial\Omega$ is merely Lipschitz, the outward unit normal ν is only in L^∞). Here we describe a procedure which takes into account not only the smoothness of the function u itself but the smoothness of the Laplacian of u as well. In fact, our definition of the normal derivative strongly

depends on the choice of an extension f of Δu to a distribution in $\mathbb{R}^n$ supported in $\overline{\Omega}$. The notation, $\partial_\nu^f u$, is chosen to reflect this peculiarity.

When $1 < p, q \leq \infty$, and $u \in B^{p,q}_{s+\frac{1}{p}}(\Omega)$ and $f \in B^{p,q}_{s+\frac{1}{p}-2,z}(\Omega)$ (or $u \in F^{p,q}_{s+\frac{1}{p}}(\Omega)$, $f \in F^{p,q}_{s+\frac{1}{p}-2,z}(\Omega)$, $p \neq \infty$) are such that $\Delta u = f|_\Omega$, then one can define $\partial_\nu^f u \in B^{p,q}_{s-1}(\partial\Omega) = \Big(B^{p',q'}_{1-s}(\partial\Omega)\Big)^*$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, by

$$\langle \partial_\nu^f u, \psi\rangle := \langle f, \mathrm{Ex}(\psi)\rangle + \langle \nabla u, \nabla \mathrm{Ex}(\psi)\rangle, \qquad \forall\, \psi \in B^{p',q'}_{1-s}(\partial\Omega), \tag{1.1.20}$$

where Ex is the extension operator introduced in (1.1.16). It follows that all duality pairings in the right-hand side of (1.1.20) are well-defined. Note that the definition (1.1.20) corresponds to a formal application of the standard Green's formula. This is essentially the point of view adopted in [**40**] but its applicability is limited to the range $1 < p, q \leq \infty$, as $B^{p,q}_{s-1}(\partial\Omega)$ fails to be a dual space if $\min\{p, q\} \leq 1$. We nonetheless have:

Theorem 1.1.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(1/p-1)_+ < s < 1$, $0 < q \leq \infty$. Then one can define a concept of normal derivative, extending (1.1.20), such that*

$$\Big\{(u, f) \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \oplus B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) :\, \Delta u = f|_\Omega\Big\} \ni (u, f) \mapsto \partial_\nu^f u \in B^{p,q}_{s-1}(\partial\Omega) \tag{1.1.21}$$

is well-defined, linear and bounded, i.e.,

$$\|\partial_\nu^f u\|_{B^{p,q}_{s-1}(\partial\Omega)} \leq C\Big(\|f\|_{B^{p,q}_{s+\frac{1}{p}-2,z}(\Omega)} + \|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)}\Big) \tag{1.1.22}$$

holds, and which reduces to (1.1.20) when $p, q > 1$. Furthermore, there exists a linear, bounded, right-inverse of (1.1.21).

Similar conclusions are valid in the context of Triebel-Lizorkin spaces, for the map

$$\Big\{(u,f)\in F^{p,q}_{s+\frac{1}{p}}(\Omega)\oplus F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega):\ \Delta u=f|_{\Omega}\Big\}\ni(u,f)\mapsto\partial^{f}_{\nu}u\in B^{p,p}_{s-1}(\partial\Omega), \tag{1.1.23}$$

assuming that in addition one of the following conditions holds:

$$\begin{aligned}&(1):\ \tfrac{n-1}{n}<p\leq 1,\ \tfrac{n-1}{2}\left(\tfrac{1}{p}-1\right)<\tfrac{1}{q}<\infty,\\ &(2):\ 1<p<\infty,\ 0<q\leq\infty.\end{aligned} \tag{1.1.24}$$

The above result has also pedagogical value since it anticipates the important role played by the scales $B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$, $F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$ when dealing with the Neumann Green potential $\mathbf{G}_N$ later on.

Compared with the work in [**40**], [**130**], the principal new difficulty in the present setting is that, for $p<1$, duality techniques cannot easily be implemented. Instead, our proof relies on a different representation of ∂_ν which makes essential use of atomic decompositions of $B^{p,q}_{\alpha,z}(\Omega)$ and $F^{p,q}_{\alpha,z}(\Omega)$ with atoms individually supported in $\bar{\Omega}$. This, in turn, generalizes a key result in [**21**] where the case of Hardy spaces is considered (for the case of h^1 see also [**3**]).

Having discussed the tangential and normal traces we can describe the second major result in this paper, closely related to Theorem 1.1.1). Consider the Poisson problem for the Laplacian in a domain $\Omega\subset\mathbb{R}^n$

$$\Delta u=f\ \text{ in }\ \Omega,\qquad Bu=g\ \text{ on }\ \partial\Omega, \tag{1.1.25}$$

with either Dirichlet ($Bu:=\operatorname{Tr}u$) or Neumann ($Bu:=\partial^{f}_{\nu}u$) boundary conditions.

For domains with smooth boundaries, the classical regularity theory of Agmon, Douglis and Nirenberg [**4**] has been extended in [**69**] to the scales of Besov and Triebel-Lizorkin spaces $B^{p,q}_s$ and $F^{p,q}_s$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. In the case of domains with isolated singularities, a satisfactory theory has been developed in [**56**] for data in Besov spaces with $p > 1$, and in [**82**] for suitable weighted Sobolev spaces.

The first systematic study of (1.1.25) with Dirichlet boundary conditions on Sobolev-Besov spaces on Lipschitz domains was undertaken by D. Jerison and C. Kenig [**68**] via harmonic measure techniques. Subsequently, their results have been reproved in [**40**] with an approach which relies on the method of boundary layer potentials. In particular, this allowed the authors to deal with Neumann boundary conditions as well. See also [**130**] in this regard. However, both the techniques and the main results in [**40**], [**68**], [**130**], are largely limited to the case $p \geq 1$. Note that all Besov and Triebel-Lizorkin spaces with $\min\{p, q\} < 1$ fail to be locally convex. Furthermore, as visible from (1.1.15)-(1.1.11) the nature of the trace operator changes below the critical index $p = 1$ as well.

The Theorems 1.1.5-1.1.6 below describe the most general ranges of indices for which the Poisson problem for the Laplacian equipped with Dirichlet or Neumann boundary conditions is well-posed for data in Besov and Triebel-Lizorkin spaces in Lipschitz domains.

Theorem 1.1.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, and, for $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$, consider the following*

boundary value problem:

$$\Delta u = f \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad u \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \operatorname{Tr} u = g \in B^{p,q}_{s}(\partial\Omega). \tag{1.1.26}$$

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that (1.1.26) is well-posed if the pair (s,p) satisfies either one of the four conditions (1.1.5) when $n \geq 3$, and either one of the three conditions (1.1.6) when $n = 2$.

Furthermore, the solution has an integral representation formula in terms of (harmonic) layer potential operators and satisfies natural estimates. Concretely, there exists a finite, positive constant $C = C(\Omega, p, q, s, n)$ such that

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} \leq C\|f\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,q}_{s}(\partial\Omega)}. \tag{1.1.27}$$

In the class of Lipschitz domains this result is sharp. If, however, $\partial\Omega \in C^1$ then one may take $\varepsilon = 1$.

Finally, analogous well-posedness results hold on the Triebel-Lizorkin scale, i.e., for the problem

$$\Delta u = f \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad u \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \operatorname{Tr} u = g \in B^{p,p}_{s}(\partial\Omega). \tag{1.1.28}$$

This time, it is assumed that (1.1.11) holds and that indices s, p are as before, except that $p < \infty$.

In Figure 1.2 below, the two-dimensional hexagonal region consisting of all points with coordinates $(s, 1/p)$ satisfying (1.1.5) is described.

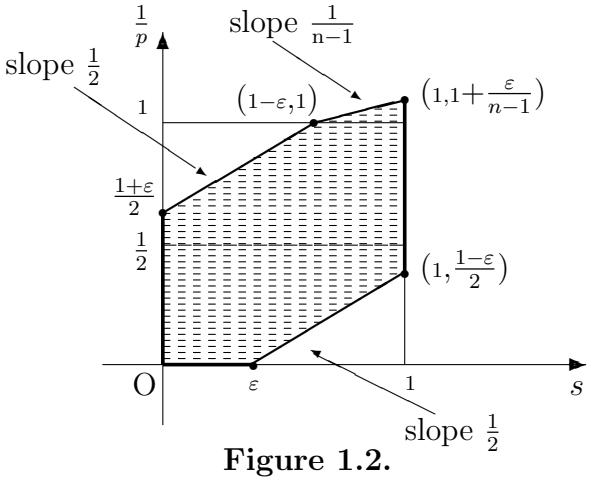

Figure 1.2.

An appropriate interpretation applies to the set of conditions (1.1.6).

Regarding the Poisson problem with Neumann boundary conditions we have:

Theorem 1.1.6. *Let Ω be a bounded, connected Lipschitz domain in $\mathbb{R}^n$ and, for $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, and $(n-1)\big(\frac{1}{p}-1\big)_+ < s < 1$, consider the following boundary value problem:*

$$\Delta u = f\Big|_{\Omega}, \quad f \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad u \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \partial^f_\nu u = g \in B^{p,q}_{s-1}(\partial\Omega), \tag{1.1.29}$$

where the data are assumed to satisfy the necessary compatibility condition

$$\langle f, 1\rangle = \langle g, 1\rangle. \tag{1.1.30}$$

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that (1.1.29) has a unique, modulo constants, solution if the pair s, p satisfies either one of the four conditions in (1.1.5) when $n \geq 3$, and either one of the three conditions in (1.1.6) when $n = 2$. In addition, the solution (subject to the normalization condition $\langle u, 1\rangle = 0$) satisfies

the estimate

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} \leq C\|f\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)} + C\|g\|_{B^{p,q}_{s-1}(\partial\Omega)}. \tag{1.1.31}$$

An analogous well-posedness result holds for the problem

$$\Delta u = f\Big|_{\Omega}, \quad f \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad u \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \partial^f_\nu u = g \in B^{p,p}_{s-1}(\partial\Omega), \tag{1.1.32}$$

assuming (1.1.11) and (1.1.30).

When $\partial\Omega \in C^1$, one can take $\varepsilon = 1$ in each case.

We also analyze the alternative ways of measuring smoothness of solutions of the Poisson problems. The classical approach includes the estimates on non-tangential maximal function and square function (area integral). However the character of the problem at hand forces consideration of the objects adapted for handling data with the fractional amount of smoothness. In this work we introduce the modified versions of the aforementioned maximal operators and establish the corresponding analogues of Theorems 1.1.5, 1.1.6. More details on the history of these results are included in the Sections 6.1 and 7.1.

A final comment is that most of the results discussed here have natural analogues at the level of symmetric, second order, strongly elliptic systems (with constant, real-valued coefficients) in Lipschitz domains in $\mathbb{R}^n$ with $n \leq 2, 3$. The last chapter of the dissertation is devoted to the detailed discussion of the Poisson problem for Lamé system of elastostatics.

The work reported on here is based on [**89**], [**88**] which have already appeared in press, as well as on [**87**], [**90**], [**91**] which are in preprint format.

1.2 Layout of dissertation

The outline of the dissertation is as follows. In Section 2 we discuss the geometry of Lipschitz domain, boundary layer potentials and basic properties of harmonic functions. Section 3 contains material pertaining to smoothness scales defined on $\mathbb{R}^n$, or on the boundary and the interior of a bounded Lipschitz domain. In particular, it contains the atomic decomposition of $B^{p,q}_{s,z}(\Omega)$ and $F^{p,q}_{s,z}(\Omega)$, extending work in [**21**].

One of the key ingredients in the proof of the well-posedness of the homogeneous Poisson boundary value problem (i.e., when $f = 0$ in Ω) is the mapping properties of singular integral operators of boundary layer type. This topic is addressed in Section 4.1. Another basic tool is the trace/extension theory which occupies Section 4.2. In Section 4.3, we review and further develop a number of functional analytic tools, well-suited for work in the context of quasi-Banach spaces. Most prominently, we identify the p-envelopes of all quasi-Banach spaces on the scales of Besov, Hardy, Triebel-Lizorkin spaces. Next, in Section 4.4, the well-posedness of the Poisson boundary value problem is considered. In particular, this section contains the proofs of Theorems 1.1.5 and 1.1.6.

Section 5 contains a detailed discussion of regularity properties of Green potentials on Lipschitz domains (including the proof of Theorem 1.1.1) along with a number of corollaries of our results. Here we present new proofs and various extensions of results such as the $L^p - L^q_1$ estimates for Green potentials due to B. Dahlberg [**29**], and the $L^p - L^p_2$ estimates for Green potentials due to V. Adolfsson

[**2**], V. Adolfsson and D. Jerison [**3**], S. Fromm [**51**], and B. Dahlberg, T. Wolff and G. Verchota (cf. the discussion on p. 1124 in [**3**]).

Sections 6 and 7 are devoted to the alternative ways of measuring smoothness of solutions of the Poisson problem. More specifically, those contain the estimates on the singular integral operators invoking the modified versions of the non-tangential maximal operator and square function. We establish the corresponding well-posedness results for the Poisson problem for Laplacian with Dirichlet and Neumann boundary conditions and discuss various connections between different types of estimates for solutions of elliptic boundary value problems in the Besov and Sobolev spaces as well as related historical references.

Finally, in Section 8 we prove well-posedness of the Poisson problem for Lamé system with Dirichlet and traction boundary data in low-dimensional Lipschitz domains and derive estimates for the elastic Green potential.

Chapter 2

Preliminaries

2.1 Notation and conventions

As is customary, $\mathbb{Z}$ and $\mathbb{N}$ are, respectively, the collection of integers and the collection of positive integers, respectively. We also set $\mathbb{N}_o := \mathbb{N} \cup \{0\}$.

Throughout the paper, $\mathbb{R}^n$ stands for the standard Euclidean space, equipped with the canonical orthonormal basis $e_j := (\delta_{jk})_k$, $1 \leq j \leq n$, and norm $|x| := \sqrt{x_1^2 + \cdots x_n^2}$, if $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Given a (measurable) set E in $\mathbb{R}^n$, we denote by $|E|$ its measure and by χ_E its characteristic function. The (open) ball centered at $a \in \mathbb{R}^n$ and having radius $r > 0$ is going to be denoted by $B(a, r)$; i.e., $B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$.

By a cube Q in $\mathbb{R}^n$ we shall always mean a set of the form $I_1 \times I_2 \times \cdots I_n$ where the I_j's are intervals of the same length, denoted $l(Q)$. If Q is a cube and $\lambda > 0$, we let λQ stand for the cube in $\mathbb{R}^n$ concentric with Q and whose side length $\lambda l(Q)$.

By $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$ we denote the j-th partial derivative in $\mathbb{R}^n$, $1 \leq j \leq n$, and by $\nabla = (\partial_1, ..., \partial_n)$ the gradient operator. Iterated partial derivatives are denoted by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index with nonnegative integer components, of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For a scalar-valued function f, $\nabla^k f$ is

the vector of all partial derivatives of order k of f, i.e., $\nabla^k f = (\partial^\alpha f)_{\alpha:\,|\alpha|=k}$.

Given $0 < p \leq \infty$, we let p' stand for the Hölder conjugate exponent of p when $1 \leq p \leq \infty$, and set $p' := \infty$ otherwise. For a number $s \in \mathbb{R}$, $[s]$ denotes its integer part, while $(s)_+$ stands for $\max\{s, 0\}$.

We let $\langle \cdot\,, \cdot \rangle$ stand for the duality pairing between a topological vector space X and its dual X^* (which should be clear from the context). In addition, we use the same piece of notation to denote the inner product in various Hilbert spaces, including $\mathbb{R}^n$.

Given an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $C^0(\Omega)$ the space of continuous real valued functions on Ω, by $C^r(\Omega)$, $r \in \mathbb{N}$, the space of r times continuously differentiable real valued functions on Ω, and set $C^\infty(\Omega) = \bigcap C^r(\Omega)$ with intersection taken over all $r \in \mathbb{N} \cup \{0\}$. Next denote by $C^r_c(\Omega)$ the space of functions $\phi \in C^r(\Omega)$ with compact support ($r \in \mathbb{N} \cup \{0\}$ or $r = \infty$), and by $D'(\Omega)$ the dual of $C^\infty_c(\Omega)$, i.e. the space of distributions in Ω. If $E \subset \Omega$ and $u \in D'(\Omega)$, we can restrict u to a distribution u_E in E by setting $\langle u_E, \phi \rangle = \langle u, \phi \rangle$ for every $\phi \in C^\infty_c(E)$. The support of u, denoted by $\operatorname{supp}(u)$, is the set of points in Ω having no open neighborhood to which the restriction of u is 0.

Also, as is customary, we let $S(\mathbb{R}^n)$ denote the Schwartz class, of smooth, rapidly decreasing functions, and by $S'(\mathbb{R}^n)$ the space of tempered distributions in $\mathbb{R}^n$.

All partial derivatives in this paper are considered (at least a priori) in the sense of distributions. The Laplacian is then given by $\Delta = \partial_1^2 + \cdots + \partial_n^2$. A function u defined in Ω is called harmonic if $\Delta u = 0$ in Ω. If $X(\Omega)$ is a quasi-Banach space of distributions in the open set $\Omega \subseteq \mathbb{R}^n$ (i.e., $X(\Omega) \subseteq D'(\Omega)$), we set

$$\mathcal{H}\cap X(\Omega):=\{u\in X(\Omega):\,\Delta u=0\text{ in }\Omega\}\tag{2.1.1}$$

which we equip with the natural quasi-norm.

Given a space $X(\Omega)$ of distributions in the open set $\Omega\subseteq\mathbb{R}^n$ we denote by $X_{\rm loc}(\Omega)$ the space of distributions u in Ω such that for every $x\in\Omega$ there exists $V(x)$ – an open set containing x – such that $u|_{V(x)}\in X(V(x))$.

Let $0<p<\infty$. By ℓ^p we denote the space of all sequences $\lambda=\{\lambda_j\}_{j=0}^{\infty}$ such that $\lambda_j\in\mathbb{R}$ for every $j\in\mathbb{N}_o$ and $\|\lambda\|_{\ell^p}:=\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}<\infty$. As usual, ℓ^∞ is the space of all sequences $\lambda=\{\lambda_j\}_{j=0}^{\infty}$ such that $\lambda_j\in\mathbb{R}$ for every $j\in\mathbb{N}_o$ and $\|\lambda\|_{\ell^\infty}:=\sup_{j\in\mathbb{N}\cup\{0\}}|\lambda_j|<\infty$.

Unless really necessary in order to avoid confusion, we shall not make notational any distinction between scalar-valued object (or spaces consisting of scalar-valued objects) and their natural vector-valued analogues.

Throughout the paper, $A\approx B$ signifies that the quotient A/B is bounded away from zero and infinity, by finite, positive constants which are independent of the relevant parameters in A,B.

Finally, we adopt the standard practice of denoting by C generic constants which may differ from one occurrence to the other and write $C=C(\kappa)$ whenever it is important to stress that C depends on a certain parameter κ.

2.2 Lipschitz domains

As a preamble, here we review some basic concepts. Recall that a function $\varphi:E\to\mathbb{R}$, $E\subset\mathbb{R}^n$ is called *Lipschitz* if there exists a finite constant $C>0$ such that

$|\varphi(x) - \varphi(y)| \leq C|x - y|$, for every $x, y \in E$. According to a classical theorem of Rademacher, for any $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz the gradient $\nabla\varphi$ exists a.e. and the best constant in the previous inequality is $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}$.

An *unbounded Lipschitz domain* Ω in $\mathbb{R}^n$ is simply the domain above the graph of Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, i.e. $\Omega = \{(x', x_n) : \varphi(x') < x_n\}$. Next, a *bounded* domain $\Omega \subset \mathbb{R}^n$, with connected boundary, is called *Lipschitz* if:

i) there exist ψ_i, $1 \leq i \leq m$, rigid motions of the Euclidean space and a family of open, upright, circular cylinders $\{Z_i\}_{i=1}^m$ in $\mathbb{R}^n$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{m} \psi_i^{-1}(Z_i). \tag{2.2.1}$$

ii) for each i, there exists a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that if $t_i Z_i$ denotes the concentric dialation of Z_i by factor $t_i = \left(1 + 10\|\nabla\varphi_i\|^2_{L^\infty(\mathbb{R}^{n-1})}\right)^{\frac{1}{2}}$, then

$$\begin{aligned} &\psi_i(\Omega \cap \psi_i^{-1}(t_i Z_i)) = \{x = (x', x_n) : \varphi_i(x') < x_n\} \cap t_i Z_i, \\ &\psi_i(\partial\Omega \cap \psi_i^{-1}(t_i Z_i)) = \{x = (x', x_n) : \varphi_i(x') = x_n\} \cap t_i Z_i, \end{aligned} \tag{2.2.2}$$

for each i.

In the sequel, we shall call $\{\mathcal{O}_i\}_{1 \leq i \leq m}$, $\mathcal{O}_i := \psi_i^{-1}(Z_i) \cap \partial\Omega$ an *atlas* for $\partial\Omega$, and we shall say that a constant depends on the Lipschitz character of Ω if its size is controlled in terms of m, the number of cylinders $\{Z_i\}_i$, the size of these cylinders and $\sup\{\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^{n-1})} : 1 \leq i \leq m\}$.

It is sometimes useful to consider the special case of a *star-like* Lipschitz domain Ω. This implies the existence of a point $x^* \in \Omega$ and a Lipschitz function $\varphi : S^{n-1} \to$

$\mathbb{R}$ with $\inf_{\omega\in S^{n-1}}\varphi(\omega)>0$ such that, in polar coordinates (ρ,ω), the domain Ω has the parametric representation $\Omega=\{x=x^*+\omega\rho:\,\omega\in S^{n-1},\,0<\rho<\varphi(\omega)\}$.

Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain (unbounded, or star-like). The *radial maximal function* of a given function $u\in C^0_{\rm loc}(\Omega)$ is defined as

$$u_{\rm rad}(x):=\sup_{t>0}|u(x+te_n)|,\qquad x\in\bar\Omega, \tag{2.2.3}$$

if Ω is an unbounded Lipschitz domain, and

$$u_{\rm rad}(x):=\sup_{t>0}|u(e^{-t}x)|,\qquad x\in\bar\Omega, \tag{2.2.4}$$

if Ω is a star-like Lipschitz domain.

It is well-known that, for a given Lipschitz domain $\Omega\subset\mathbb{R}^n$, the outward unit normal ν is well-defined at almost every boundary point, with respect to the surface measure $d\sigma$. By $L^p(\partial\Omega),0<p<\infty$, we will denote the Lebesgue space of $d\sigma$-measurable, p-th power integrable functions on $\partial\Omega$. Also, $\partial_\nu:=\langle\nu,\nabla\rangle$ will denote the directional derivative along the unit normal and $\nabla_{\rm tan}u:=\nabla u-(\partial_\nu u)\nu$ will stand for the tangential gradient of a function u defined on $\partial\Omega$. The tangential derivatives will be denoted by $\partial_{\tau_{jk}}:=\nu_j\partial_k-\nu_k\partial_j$ for $j,k=1,...,n$. For each $1<p<\infty$ we introduce the Sobolev-type space

$$L^p_1(\partial\Omega):=\{u\in L^p(\partial\Omega):\,\nabla_{\rm tan}u\in L^p(\partial\Omega)\}. \tag{2.2.5}$$

Going further, $\mathrm{Lip}\,(\partial\Omega)$ stands for the Banach space of all Lipschitz functions defined on $\partial\Omega$, i.e. measurable functions such that

$$\|f\|_{L^\infty(\partial\Omega)}+\sup_{\substack{x,y\in\partial\Omega\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|}<\infty. \tag{2.2.6}$$

Its dual, $(\mathrm{Lip}\,(\partial\Omega))'$, will be referred to as the space of distributions on $\partial\Omega$.

Pointwise restrictions to the boundary of a Lipschitz domain Ω are always taken in the nontangential limit sense. More specifically, for constant $\kappa = \kappa(\partial\Omega) > 1$, sufficiently close to 1, we define the nontangential approach regions

$$\Gamma(x) := \{y \in \Omega : \, |x-y| < \kappa \,\mathrm{dist}\,(y, \partial\Omega)\}, \qquad x \in \partial\Omega, \tag{2.2.7}$$

and, for a function $u \in C^0_{\rm loc}(\Omega)$, set

$$u\Big|_{\partial\Omega}(x) := \lim_{\substack{y\to x \\ y\in\Gamma(x)}} u(y) \tag{2.2.8}$$

at every boundary point where the limit exists.

Finally, given $\Omega \subset \mathbb{R}^n$, we let $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$ and let $\delta : \Omega \to [0, \infty)$ denote the distance function to the boundary of Ω, i.e., $\delta(x) := \mathrm{dist}\,(x, \partial\Omega)$, $x \in \mathbb{R}^n$.

2.3 Layer potentials

We continue by reviewing the definitions of the classical (harmonic) layer potentials associated with a Lipschitz domain $\Omega \subset \mathbb{R}^n$. Recall that the canonical radial fundamental solution for the Laplace operator $\Delta = \partial_1^2 + ... + \partial_n^2$ in $\mathbb{R}^n$ is

$$E(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2, \\ \frac{1}{n(n-2)\,\omega(n)} \, \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \tag{2.3.1}$$

where $\omega(n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ denotes the volume of unit ball in $\mathbb{R}^n$. We then define the single and double layer potential operators by setting

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x-y)\, f(y)\, d\sigma_y, \qquad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{2.3.2}$$

and

$$\mathcal{D}f(x) := \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{2.3.3}$$

As a classical, by now, corollary of the L^p-boundedness, $1 < p < \infty$, of the Cauchy singular integral operator on Lipschitz curves [**24**], we note the following jump formulas:

$$\partial_\nu \mathcal{S}\Big|_{\partial\Omega_\pm} = \mp\tfrac{1}{2} I + K^* \tag{2.3.4}$$

and

$$\mathcal{D}\Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{2} I + K \tag{2.3.5}$$

where the boundary traces are taken in the sense of (2.2.8) and the operators involved act on functions from $L^p(\partial\Omega)$, $1 < p < \infty$. Here

$$Kf(x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{2.3.6}$$

I stands for the identity operator and K^* is the formal adjoint of K. The above integral is taken in the principal value sense, i.e. removing Euclidean balls of radius ε and passing to the limit, $\varepsilon \to 0$. Later we will also discuss the boundary version of the single layer potential operator, i.e.

$$Sf(x) := \int_{\partial\Omega} E(x-y)\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega. \tag{2.3.7}$$

Finally, the Newtonian potential is defined as

$$\Pi f(x) := \int_{\mathbb{R}^n} E(x-y) f(y)\, dy, \qquad x \in \mathbb{R}^n. \tag{2.3.8}$$

2.4 Functions with the sub-averaging property

In this section we collect several technical results which are going to be of importance later on. For our purposes, it will be convenient to adopt the following definition.

Definition 2.4.1. *Assume that Ω is an open subset of $\mathbb{R}^n$ and that $0 < p < \infty$. A function $u \in L^p_{\text{loc}}(\Omega)$ is p-subaveraging if there exists a positive constant C with the following property:*

$$|u(x)| \leq C \left(\frac{1}{|B_r|} \int_{B_r(x)} |u(y)|^p \, dy \right)^{\frac{1}{p}} \tag{2.4.1}$$

for almost every $x \in \Omega$ and all $r \in (0, \delta(x))$.

We have:

Lemma 2.4.1. *If there exists $p_0 > 0$ such that u is p_0-subaveraging function, then u is p-subaveraging for every $p \in (0, \infty)$.*

Granted this result, it is unequivocal to refer to a function u as simply being *subaveraging* if it is p-sub-averaging for some $p \in (0, \infty)$. The optimal constants which can be used in the (2.4.1) make up what we call the subaveraging character of the function u.

Proof of Lemma 2.4.1. The proof goes back to the work of G. Hardy and J. Littlewood **[64]** (cf. also Lemma 2, pp. 172-173, in **[43]**). The case $p > p_0$ can be handled directly utilizing Hölder's inequality with $q = \frac{p}{p_0} > 1$. Henceforth we shall focus on the case when $p < p_0$.

There is no loss of generality in assuming $p_0 = 1$, $p < 1$. We may also assume (by rescaling and making a translation) that $x = 0 \in \Omega$ and $\int_{|y|<1} |u(y)|^p \, dy = 1$. The goal is then to prove the estimate $|u(0)| \leq C$ with a constant C independent of u. Continuing our series of reductions, we may assume that $|u(0)| > 1$. Next,

introduce

$$m_p(r) := \left(\int_{B_r(0)} |u(y)|^p \, dy \right)^{\frac{1}{p}}, \quad m_\infty(r) := \sup_{B_r(0)} |u(y)|, \quad r \in (0,1). \tag{2.4.2}$$

Thus, for $r \in (0,1)$,

$$m_1(r) \leq (m_p(r))^p \, (m_\infty(r))^{1-p} \leq (m_\infty(r))^{1-p}, \qquad p \in (0,1). \tag{2.4.3}$$

The last inequality holds by virtue of the trivial estimate $m_p(r) \leq m_p(1)$, valid for every $r \in (0,1)$, and the assumption $m_p(1) = 1$. On the other hand, for every $x \in \Omega$ and every $r \in (0, \delta(x))$

$$|u(x)| \leq C \, \frac{1}{r^n} \int_{B_r(x)} |u(y)| \, dy, \tag{2.4.4}$$

and, consequently,

$$|u(z)| \leq C \, \frac{1}{(r-\rho)^n} \int_{B_{r-\rho}(z)} |u(y)| \, dy \leq C \, \frac{1}{(r-\rho)^n} \int_{B_r(0)} |u(y)| \, dy \tag{2.4.5}$$

whenever $|z| = \rho$ and $0 < \rho < r$. Then for any point z^* in the ball $B_\rho(0)$ such that $|z^*| = \rho^* < \rho$, we obtain

$$|u(z^*)| \leq C \, \frac{1}{(r-\rho^*)^n} \int_{B_r(0)} |u(y)| \, dy \leq C \, \frac{1}{(r-\rho)^n} \int_{B_r(0)} |u(y)| \, dy, \tag{2.4.6}$$

which, in concert with (2.4.3), yields the estimate

$$m_\infty(\rho) \leq C \, \frac{1}{(r-\rho)^n} \, m_\infty^{1-p}(r). \tag{2.4.7}$$

To continue, set $\rho := r^\alpha$ with $\alpha > 1$ to be specified momentarily. Then (2.4.7) entails

$$\int_{1/2}^1 \log \, m_\infty(r^\alpha) \, \frac{dr}{r} \leq C + n \int_{1/2}^1 \log \, \frac{1}{(r-r^\alpha)} \, \frac{dr}{r} + (1-p) \int_{1/2}^1 \log \, m_\infty(r) \, \frac{dr}{r}. \tag{2.4.8}$$

Making the change of variables $t := r^\alpha$, one may write

$$\int_{1/2}^{1} \log\, m_\infty(r^\alpha)\, \frac{dr}{r} = \frac{1}{\alpha} \int_{(1/2)^\alpha}^{1} \log\, m_\infty(t)\, \frac{dt}{t}. \tag{2.4.9}$$

Since our assumption $|u(0)| \geq 1$ implies $m_\infty(t) \geq 1$, the right-hand side of (2.4.9) is bounded from below by

$$\frac{1}{\alpha} \int_{1/2}^{1} \log\, m_\infty(r)\, \frac{dr}{r}. \tag{2.4.10}$$

Therefore, (2.4.8)-(2.4.10) imply

$$\left(\frac{1}{\alpha} - 1 + p\right) \int_{1/2}^{1} \log\, m_\infty(r)\, \frac{dr}{r} \leq C + C \int_{1/2}^{1} \log\, \frac{1}{(r - r^\alpha)}\, \frac{dr}{r} \leq C. \tag{2.4.11}$$

Choose now $\alpha > 1$ such that $\frac{1}{\alpha} - 1 + p > 0$. Then (2.4.11) forces

$$\int_{1/2}^{1} \log\, m_\infty(r)\, dr \leq C, \tag{2.4.12}$$

and hence, $\log\, m_\infty(1/2) \leq C$ for some finite constant C independent of initial function u. In concert with the inequality $|u(0)| \leq m_\infty(1/2)$, this finishes the proof of the lemma. □

There are clear connections between the subaveraging property and reverse Hölder estimates. To illustrate this, we state the following.

Lemma 2.4.2. *Let u be a subaveraging function in a domain $\Omega \subset \mathbb{R}^n$ and assume that $0 < p, q < \infty$. Then*

$$\left(\frac{1}{|B_r|} \int_{B_r(x)} |u(y)|^q\, dy\right)^{\frac{1}{q}} \leq C \left(\frac{1}{|B_{2r}|} \int_{B_{2r}(x)} |u(y)|^p\, dy\right)^{\frac{1}{p}} \tag{2.4.13}$$

uniformly for $x \in \Omega$ and $0 < r < \delta(x)/2$, where the constant C depands only on p, q, n and the subaveraging character of u.

Proof. We write

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r(x)}|u(y)|^q\,dy &\leq C\frac{1}{|B_r|}\int_{B_r(x)}\left(\frac{1}{|B_r|}\int_{B_r(y)}|u(z)|^p\,dz\right)^{q/p}dy\\ &\leq C\frac{1}{|B_r|}\int_{B_r(x)}\left(\frac{1}{|B_r|}\int_{B_{2r}(x)}|u(z)|^p\,dz\right)^{q/p}dy\\ &= C\left(\frac{1}{|B_{2r}|}\int_{B_{2r}(x)}|u(z)|^p\,dz\right)^{q/p}\end{aligned}\tag{2.4.14}$$

from which (2.4.13) readily follows. □

To state our next result, recall the radial maximal operator introduced in §2.2. The following sharpens Lemma 3 in **[119]**.

Lemma 2.4.3. *Let $\Omega\subset\mathbb{R}^n$ be an arbitrary bounded, star-like Lipschitz domain and consider $0<q\leq p<\infty$, $sp>-1$. Then for a subaveraging function u in Ω,*

$$\left(\int_\Omega|u_{\rm rad}(x)|^p\,\delta(x)^{sp}\,dx\right)^{1/p}\leq\kappa\left(\int_\Omega|u(x)|^q\,\delta(x)^{sq+n(\frac{q}{p}-1)}\,dx\right)^{1/q}\tag{2.4.15}$$

where κ depends exclusively on p,q,s,n, the Lipschitz character of Ω and the subaveraging character of u.

Several other variants are possible. For example, a similar result holds when Ω is an unbounded Lipschitz domain in $\mathbb{R}^n$. Also, a suitable version of (2.4.15) remains valid in an arbitrary Lipschitz domain, provided the radial maximal function is approriately interpreted.

Proof of Lemma 2.4.3. The proof follows closely **[119]**. Nonetheless, some alterations allow to treat a wider range of indices so we include a complete argument. We begin by considering the Whitney decomposition of the domain Ω (cf. **[118]**).

Specifically, there exists a sequence of balls $\{B_{r_j}\}_j$ with the following properties:

$$\delta(x) \approx r_j \text{ for } x \in B_{r_j}, \qquad \Omega = \cup_j B_{r_j}, \qquad \sum_j \chi_j \le 200^{2n}, \tag{2.4.16}$$

where χ_j denotes the characteristic function of the ball B_{r_j}. Then Fatou's lemma implies

$$\int_\Omega \delta(x)^{sp} (|u|^p)_{\rm rad}(x)\, dx \le C \sum_j \int_\Omega \delta(x)^{sp} (\chi_j |u|^p)_{\rm rad}(x)\, dx, \tag{2.4.17}$$

Next, we set

$$\widehat{B}_{r_j} := \{z \in \Omega :\, z + te_n \in B_{r_j} \text{ for some } t > 0\}, \tag{2.4.18}$$

or

$$\widehat{B}_{r_j} := \{z \in \Omega :\, e^{-t} z \in B_{r_j} \text{ for some } t > 0\}, \tag{2.4.19}$$

depending on whether the Lipschitz domain Ω is unbounded or star-like, respectively. Note that $\operatorname{supp}(\chi_j |u|^p)_{\rm rad} \subseteq \widehat{B}_{r_j}$ and $\|(\chi_j |u|^p)_{\rm rad}\|_{L^\infty(\widehat{B}_{r_j})} \le \|u\|^p_{L^\infty(B_{r_j})}$. Also,

$$\int_{\widehat{B}_{r_j}} \delta(x)^{sp}\, dx \le C\, r_j^{n-1} \int_0^{cr_j} t^{sp}\, dt = C r_j^{n+sp}. \tag{2.4.20}$$

Consequently, granted the sub-averaging property of u,

$$\begin{aligned} \int_\Omega \delta(x)^{sp} (\chi_j |u|^p)_{\rm rad}(x)\, dx &\le C r_j^{n+sp} \sup_{x \in B_{r_j}} |u(x)|^p \le C\, r_j^{n+sp} \Big(\frac{1}{r_j^n} \int_{B_{2r_j}} |u(x)|^q\, dx \Big)^{p/q} \\ &\le C \Big(\int_{B_{2r_j}} (\delta(x)^{s+n(1/p-1/q)} |u(x)|)^q\, dx \Big)^{p/q}, \end{aligned} \tag{2.4.21}$$

where the last inequality holds thanks to (2.4.16). Recalling that no point of Ω is contained in more than 200^{2n} of the balls B_{2r_j}, we can combine (2.4.17) and (2.4.21). The assumption $p \ge q$ allows to sum up the integrals in (2.4.21), which leads to (2.4.15). □

Lemma 2.4.4. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and $0 < q \leq p < \infty$, $s \in \mathbb{R}$. Then for every subaveraging function u in Ω,*

$$\Big(\int_{\Omega}[\delta(x)^{s+n(\frac{1}{q}-\frac{1}{p})}|u(x)|]^p\,dx\Big)^{1/p} \leq \kappa\Big(\int_{\Omega}[\delta(x)^s|u(x)|]^q\,dx\Big)^{\frac{1}{q}} \tag{2.4.22}$$

where κ depends exclusively on p, q, s, n, and the subaveraging character of u.

Proof. To start, let us observe that for every $x \in \Omega$

$$|u(x)|^q \leq C\delta(x)^{-n-qs}\int_{B_{\delta(x)/2}(x)}(\delta(y)^s|u(y)|)^q\,dy \leq C\delta(x)^{-n-qs}\int_{\Omega}(\delta(y)^s|u(y)|)^q\,dy, \tag{2.4.23}$$

owing to the sub-averaging property of u and a simple observation, to the effect that $\delta(x) \approx \delta(y)$ for every $y \in B_{\delta(x)/2}$. Next,

$$|u(x)| = |u(x)|^{\theta}|u(x)|^{1-\theta} \leq C|u(x)|^{\theta}\delta(x)^{-(\frac{n}{q}+s)(1-\theta)}\Big(\int_{\Omega}(\delta(y)^{\sigma}|u(y)|)^q\,dy\Big)^{\frac{1-\theta}{q}}, \tag{2.4.24}$$

for every θ such that $0 < \theta \leq 1$. Consequently,

$$\delta(x)^{p\left[\sigma+n\left(\frac{1}{q}-\frac{1}{p}\right)\right]}|u(x)|^p \leq C|u(x)|^{\theta p}\,\delta(x)^{-n+\theta p(\sigma+\frac{n}{q})}\,\Big(\int_{\Omega}(\delta(y)^{\sigma}|u(y)|)^q\,dy\Big)^{(1-\theta)\frac{p}{q}}. \tag{2.4.25}$$

At this stage, we set $\theta := \frac{q}{p}$ (so that $0 < \theta \leq 1$ given that $q \leq p$) and integrate both sides of the above inequality in order to obtain (2.4.22). □

To continue, we assume that

$$L = \sum_{|\gamma|=\mu} a_{\gamma}\partial^{\gamma} \tag{2.4.26}$$

is a (homogeneous) constant-coefficient, elliptic differential operator of order $\mu \in 2\,\mathbb{N}$ in $\mathbb{R}^n$. By elliptic, we mean that there exists a finite constant $\Lambda > 1$ such that

$$\Lambda^{-1}|\xi|^{\mu} \leq \sum_{|\gamma|=\mu} a_{\gamma}\xi^{\gamma} \leq \Lambda|\xi|^{\mu}, \qquad \forall\, \xi \in \mathbb{R}^n. \tag{2.4.27}$$

Denote by $\operatorname{Ker} L$ the space of all C^{∞} functions satisfying $Lu = 0$ in Ω. We now establish some useful interior estimates; a similar result has been established in [**35**].

Lemma 2.4.5. *Let L be an elliptic differential operator as above and assume that $\Omega \subset \mathbb{R}^n$ is open. Then for each $u \in \operatorname{Ker} L$, $0 < p < \infty$, $k \in \mathbb{N}_o$, and $x \in \Omega$, $0 < r < \delta(x)$,*

$$|\nabla^k u(x)|^p \leq \frac{C}{r^{n+kp}} \int_{B_r(x)} |u(y)|^p \, dy \tag{2.4.28}$$

where $C = C(L, p, k, n) > 0$ is a finite constant. In particular,

$$u \in \operatorname{Ker} L \Longrightarrow u \text{ is subaveraging.} \tag{2.4.29}$$

Proof. Rescaling (2.4.28), there is no loss of generality in assuming that $r = 1$. In this scanario, (2.4.28) with $p = 2$ follows from Sobolev's embedding theorem and standard interior regularity results (cf., e.g., Theorem 11.1, p. 379 in [**122**]). In particular, if $u \in \operatorname{Ker} L$ then u is 2-subaveraging and, hence, (2.4.29) holds, thanks to Lemma 2.4.1. With this in hand we then write

$$r^k|\nabla^k u(x)| \leq C\left(\frac{1}{|B_r|}\int_{B_{r/2}(x)} |u(y)|^2\,dy\right)^{1/2} \leq C\left(\frac{1}{|B_r|}\int_{B_r(x)} |u(y)|^p\,dy\right)^{1/p} \tag{2.4.30}$$

by Lemma 2.4.2. □

Lemma 2.4.6. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that L is an elliptic opertor as in (2.4.26). Also, fix $0 < q \leq p < \infty$, $s \in \mathbb{R}$, and $k \in \mathbb{N}_o$. Then for any $u \in \operatorname{Ker} L$,*

$$\Big(\int_\Omega (\delta(x)^{s+k+n(\frac{1}{p}-\frac{1}{q})}|\nabla^k u(x)|)^p\,dx\Big)^{1/p} \leq C\Big(\int_\Omega (\delta(x)^s|u(x)|)^q\,dx\Big)^{1/q}, \quad (2.4.31)$$

where $C = C(L,\Omega,p,q,s,k) > 0$ is a finite constant. In particular, when $q = p$,

$$\Big(\int_\Omega (\delta(x)^{s+k}|\nabla^k u(x)|)^p\,dx\Big)^{1/p} \leq C\Big(\int_\Omega (\delta(x)^s|u(x)|)^p\,dx\Big)^{1/p} \quad (2.4.32)$$

holds for any $u \in \operatorname{Ker} L$ provided $0 < p < \infty$, $s \in \mathbb{R}$, and $k \in \mathbb{N}_o$.

Proof. Start with a Whitney decomposition of Ω into a family of balls as in (2.4.16). Integrating (2.4.28) allows us to write

$$\int_{B_{r_j}} (r_j^{s+k}|\nabla^k u(x)|)^p\,dx \leq C\int_{B_{2r_j}} (r_j^s|u(x)|)^p\,dx \quad (2.4.33)$$

which further translates into

$$\int_{B_{r_j}} (\delta(x)^{s+k}|\nabla^k u(x)|)^p\,dx \leq C\int_{B_{2r_j}} (\delta(x)^s|u(x)|)^p\,dx \quad (2.4.34)$$

for each j. This, in turn, readily yields (2.4.32) by summing over all j's. Having proved (2.4.32), the estimate (2.4.31) follows by bringing in (2.4.22) and (2.4.29) and by redenoting s by $s + n(\frac{1}{q} - \frac{1}{p})$. □

Lemma 2.4.7. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $k \in \mathbb{N}_o$, $0 < q \leq p < \infty$, and $s \in \mathbb{R}$ with $sp > -1$. Then there exists a relatively compact subset $\mathcal{O}$ of Ω such that*

$$\Big(\int_\Omega (\delta(x)^s|u(x)|)^p\,dx\Big)^{1/p} \leq C\left[\Big(\int_\Omega (\delta(x)^{s+k+n(\frac{1}{p}-\frac{1}{q})}|\nabla^k u(x)|)^q\,dx\Big)^{1/q} + \sup_{x\in\mathcal{O}}|u(x)|\right] \quad (2.4.35)$$

uniformly for $u \in \operatorname{Ker} L$. *In particular, for* $q = p$,

$$\Big(\int_{\Omega}(\delta(x)^{s}|u(x)|)^{p}\,dx\Big)^{1/p} \leq C\left[\Big(\int_{\Omega}(\delta(x)^{s+k}|\nabla^{k}u(x)|)^{p}\,dx\Big)^{1/p} + \sup_{x\in\mathcal{O}}|u(x)|\right], \tag{2.4.36}$$

uniformly for $u \in \operatorname{Ker} L$.

Once again, other versions are possible. For example, the same conclusion remains valid when Ω is an unbounded Lipschitz domain in $\mathbb{R}^n$ in which scenario u is assumed to vanish at infinity and the last terms in the right side of (2.4.35) and (2.4.36), respectively, are dropped. Prior to presenting the proof of Lemma 2.4.7 we record a result of independent interest.

Lemma 2.4.8. *Assume that* $f : \mathbb{R} \to [0,\infty]$ *is measurable,* $r > 0$ *and* $0 \leq M \leq \infty$. *Then the estimate*

$$\left(\int_{0}^{M}\left(\int_{x}^{M} y^{q}f(y)\,dy\right)^{p} x^{r-1}\,dx\right)^{1/p} \leq C(p,q,r)\left(\int_{0}^{M}(y^{q+1}f(y))^{p}y^{r-1}\,dy\right)^{1/p} \tag{2.4.37}$$

holds provided either $1 \leq p < \infty$, *or* $0 < p < 1$ *and* f *is nonincreasing.*

Proof. When $M = \infty$ and $p \geq 1$ this is just the classical Hardy inequality (cf. Appendix A.4 in **[118]**). The case $M = \infty$, $0 < p < 1$ and f nonincreasing is proved in §5 of **[119]**. The case when $M < \infty$ follows from these results by replacing f by $f\chi_{[0,M]}$. □

Proof of Lemma 2.4.7. Our argument is akin the proof of Lemma 1 on p. 656 of **[119]**. First, there is no loss of generality in assuming that Ω is star-like with respect

to the origin in $\mathbb{R}^n$, as the case of a general (bounded) Lipschitz domain follows from this via a standard localization argument. Second, an inductive argument reduces matters to proving (2.4.35) when $k = 1$. Adopt these conventions and recall the representation of starlike Lipschitz domain in polar coordinates as discussed in §2.2. We write

$$|u(x)| = \left| \int_0^\infty x \cdot (\nabla u)(xe^{-t}) e^{-t}\, dt \right| \leq C \int_0^\infty |(\nabla u)(xe^{-t})|\, dt \tag{2.4.38}$$

and then estimate

$$\begin{aligned} &\int_\Omega (\delta^s(x)|u(x)|)^p\, dx \\ &\qquad \leq C \int_{S^{n-1}} \int_0^{\phi(\omega)} (\phi(\omega) - \rho)^{sp} \Big(\int_0^\infty |(\nabla u)(e^{-t}\rho\omega)|\, dt \Big)^p \rho^{n-1}\, d\rho d\omega. \end{aligned} \tag{2.4.39}$$

At this stage, we make the change of variables $\rho = \phi(\omega)e^{-\eta}$, $\eta \in (0, \infty)$ and re-write the last integral as

$$\int_{S^{n-1}} \int_0^\infty (\phi(\omega)(1-e^{-\eta}))^{sp} \Big(\int_0^\infty |(\nabla u)(\phi(\omega)\omega e^{-t-\eta})|\, dt \Big)^p (\phi(\omega)e^{-\eta})^n\, d\eta d\omega. \tag{2.4.40}$$

If we set $\lambda = t + \eta$, $\lambda \in (\eta, \infty)$, we can further bound the above expression by

$$C \int_{S^{n-1}} \int_0^\infty (1 - e^{-\eta})^{sp} \Big(\int_\eta^\infty |(\nabla u)(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta}\, d\eta d\omega. \tag{2.4.41}$$

If we now split the integral in η into two integrals over the sets $[0, M]$ and $[M, \infty)$, for some large M, matters are reduced to controlling the following two terms:

$$I := \int_{S^{n-1}} \int_0^M \eta^{sp} \Big(\int_\eta^M |(\nabla u)(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p\, d\eta d\omega, \tag{2.4.42}$$

$$II := \int_{S^{n-1}} \int_M^\infty (1 - e^{-\eta})^{sp} \Big(\int_\eta^\infty |(\nabla u)(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta}\, d\eta d\omega. \tag{2.4.43}$$

Note that, by interior estimates such as (2.4.28), $|II| \leq C \sup\{|u(x)|^p : x \in \mathcal{O}\}$ for a suitable compact subset $\mathcal{O}$ of Ω, since the argument of ∇u stays away from $\partial\Omega$. As for I, we have

$$|I| \leq C \int_{S^{n-1}} \int_0^M \eta^{sp} \Big(\int_\eta^M (\nabla u)_{\rm rad}(\phi(\omega)\omega e^{-\lambda})\, d\lambda \Big)^p \, d\eta d\omega. \tag{2.4.44}$$

Now an application of Hardy's inequality (2.4.37) with $q = 0$ and $r = sp + 1 > 0$ leads to

$$\begin{aligned} |I| &\leq C \int_{S^{n-1}} \int_0^M (\nabla u)_{\rm rad}(\phi(\omega)\omega e^{-\lambda})^p \lambda^{sp+p}\, d\lambda d\omega \\ &\leq C \int_{\Omega\setminus\mathcal{O}} \delta(x)^{sp+p} (\nabla u)_{\rm rad}(x)^p \, dx. \end{aligned} \tag{2.4.45}$$

In the last step we have changed variables back to the original $x \in \Omega$ and, in the process, have noticed that the Jacobian of this tranformation is bounded given that $0 < \lambda < M$. The last step is to eliminate the subscript "rad" in the last integral in (2.4.45) by invoking (2.4.15). □

Chapter 3

Function spaces measuring smoothness

3.1 Quasi-Banach spaces, interpolation and stability results

The presentation in this section follows closely [**77**]. For a quasi-normed space $(X, \|\cdot\|_X)$, we denote by $\rho = \rho(X)$ its *modulus of concavity*, i.e. the smallest positive constant for which

$$\|x+y\|_X \leq \rho(X)(\|x\|_X + \|y\|_X), \qquad x, y \in X. \tag{3.1.1}$$

Note that always $\rho(X) \geq 1$. We recall the Aoki-Rolewicz theorem, which asserts that X can be given an equivalent r-norm (where $2^{1/r-1} = \rho$) i.e. a quasi-norm which also satisfies the inequality:

$$\|x+y\| \leq (\|x\|^r + \|y\|^r)^{1/r}.$$

In general a quasi-norm need not be continuous but an r-norm is continuous. We shall assume however, throughout the paper that all quasi-norms considered are continuous: in fact, of course it would suffice to consider an r-norm for suitable r.

To set the stage for adapting Calderón's original complex method of interpola-

tion to the setting of quasi-Banach spaces, we first review some basic results from the theory of analytic functions with values in quasi-Banach spaces as developed in [**128**], [**74**], [**75**].

Recall that if X is a topological vector space and U is an open subset of the complex plane then a map $f : U \to X$ is called *analytic* if given $z_0 \in U$ there exists $\eta > 0$ so that there is a power series expansion

$$f(z) = \sum_{j=0}^{\infty} (z - z_0)^n x_n, \quad x_n \in X, \text{ uniformly convergent for } |z - z_0| < \eta. \tag{3.1.2}$$

As explained in [**74**], in the context of quasi-Banach spaces, this is the most natural definition. Indeed, there are simple examples which show that complex differentiability leads to an unreasonably weaker concept of analyticity (see also [**128**] and [**6**] in this regard).

Proposition 3.1.1. *Suppose $0 < p \leq 1$ and that $m \in \mathbb{N}$ is such that $m > \frac{1}{p}$. Then there is a constant $C = C(m, p)$ so that if X is a p-normed quasi-Banach space and $f : \bar{D} \to X$ is a continuous function which is analytic on the unit disk $D := \{z : |z| < 1\}$ then for $z \in D$ we have $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, and*

$$\|f^{(n)}(0)\|_X \leq C(m + n)! \sup_{z \in D} \|f(z)\|_X. \tag{3.1.3}$$

This is Theorem 6.1 of [**74**].

Proposition 3.1.2. *Let X be a quasi-Banach space and let U be an open subset of the complex plane. Let $f_n : U \to X$ be a sequence of analytic functions. If $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on compacta then f is also analytic.*

This follows from Theorem 6.3 of [**74**].

Proposition 3.1.3. *Suppose X is a quasi-Banach space and that U is an open subset of the complex plane. Let $f : U \to X$ be a locally bounded function. Suppose there is a weaker Hausdorff vector topology τ_0 on X which is locally p-convex for some $0 < p < 1$ and such that $f : U \to (X, \tau_0)$ is analytic. Then $f : U \to X$ is analytic.*

This is Theorem 3.3 of [**77**]. It shows that, many times in practice, the ambient space (within which the interpolation process is carried out) plays only a minor role in the setup. More specifically, assume that Y is a space of distributions in which a quasi-Banach space X is continuously embedded. Then, having a X-valued function analytic for the quasi-norm topology is basically the same as requiring analyticity for the weak topology (induced on X from Y).

We are now prepared to elaborate on the complex method of interpolation for pairs of quasi-Banach spaces. Consider a compatible couple (pair) of quasi-Banach spaces X_0, X_1, i.e. X_j, $j = 0, 1$, are continuously embedded into a larger topological vector space Y, and $X_0 \cap X_1$ is dense in X_j, $j = 0, 1$. Also, let U stand for the strip $\{z \in \mathbb{C} : 0 < \Re e\, z < 1\}$.

A family $\mathcal{F}$ of functions which map U into $X_0 + X_1$ is called *admissible* provided the following axioms are satisfied:

i) $\mathcal{F}$ is a (complex) vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{F}}$ with respect to which it is complete (i.e. $\mathcal{F}$ is a quasi-Banach space);

ii) the point-evaluation mappings $\mathrm{ev}_w : \mathcal{F} \to X_0 + X_1$, $w \in U$, defined by $\mathrm{ev}_w(f) := f(w)$ are continuous;

iii) for any K compact subset of U there exists a positive constant C such that for any $w \in K$ and any $f \in \mathcal{F}$ with $f(w) = 0$, it then follows that the mapping $U \setminus \{w\} \ni z \mapsto f(z)/(z-w) \in X_0 + X_1$ extends to an element in $\mathcal{F}$ and

$$\left\| \frac{f(z)}{z-w} \right\|_{\mathcal{F}} \leq C \|f\|_{\mathcal{F}}. \tag{3.1.4}$$

These are the minimal requirements needed in order develop a reasonable interpolation theory at an abstract level. In practice, a common choice for $\mathcal{F}$, the class of admissible functions, is the space of bounded, analytic functions $f : U \to X_0 + X_1$, which is extended continuously to the closure of the strip such that the traces $t \mapsto f(j+it)$ are bounded continuous functions into X_j, $j = 0, 1$. We endow $\mathcal{F}$ with the quasi-norm

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_t \|f(it)\|_{X_0} , \sup_t \|f(1+it)\|_{X_1} \right\}. \tag{3.1.5}$$

Although natural in light of the original Calderón's theory of complex interpolation, this definition gives rise to a number of questions in the framework of quasi-Banach spaces. In particular, the continuity of evaluation functions and even the completeness of space $\mathcal{F}$ are not necessarily satisfied – due to the possible failure of the Maximum Modulus Principle. One *ad hoc* remedy is to include the quantity $\sup_{w \in U} \|f(w)\|_{X_0+X_1}$ among the terms whose maximum is considered in the right-hand side of (3.1.5), though this may lead to a less standard complex interpolation theory.

Instead, we remark that while in general no analogue of the Maximum Modulus Principle is valid, for a wide subclass of quasi-Banach spaces, called *A-convex*

(analytically convex) in [**75**], the Maximum Modulus Principle does hold. A quasi-Banach space X is *A-convex* if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have $\|P(0)\|_X \leq C \max_{|z|=1} \|P(z)\|_X$. It is shown in [**75**] that if X is *A-convex* it has an equivalent quasi-norm which is plurisubharmonic (i.e. we can insist that $C = 1$). Let us also point out that being A-convex is equivalent to the condition that

$$\max_{0<\Re e\, z<1} \|f(z)\|_X \leq C \max_{\Re e\, z=0,1} \|f(z)\|_X, \tag{3.1.6}$$

for any analytic function $f : \{z \in \mathbb{C} : 0 < \Re e\, z < 1\} \to X$ which is continuous on the closed strip $\{z \in \mathbb{C} : 0 \leq \Re e\, z \leq 1\}$.

Assuming that $X_0 + X_1$ is A-convex, the axioms (i)-(iii) are readily verified for Calderón's norm (3.1.5). We denote by $X_w = [X_0, X_1]_w$, $w \in U$, the corresponding intermediate spaces, i.e.,

$$X_w = \{x \in X_0 + X_1;\, x = f(w) \text{ for some } f \in \mathcal{F}\} \subseteq X_0 + X_1. \tag{3.1.7}$$

When equipped with $\|x\|_w := \inf\{\|f\|_{\mathcal{F}};\, f \in \mathcal{F},\, f(w) = x\}$, $x \in X_w$, the space X_w becomes quasi-Banach, in fact analytically convex, for each $w \in U$. Furthermore, the modulus of concavity of the interpolation quasi-norm $\|\cdot\|_w$ does not exceed $\rho(\mathcal{F})$, and the inclusion $X_w \hookrightarrow X_0 + X_1$ is continuous. Of course, X_w depends only on $\Re e\, w$ and we typically only consider X_θ for $0 < \theta < 1$.

We now discuss several useful criteria for analytic convexity in the context of quasi-Banach *lattices* of functions. To set the stage, assume that (Ω, Σ, μ) is a $\sigma-$finite measure space and denote by L_0 the space of all complex-valued, μ-measurable functions on Ω. Then a *quasi-Banach function space* X on (Ω, Σ, μ),

equipped with a quasi-norm $\|\cdot\|_X$ so that $(X, \|\cdot\|_X)$ is complete, is an order-ideal in the space L_0 if it contains a strictly positive function and if $f \in X$ and $g \in L_0$ with $|g| \leq |f|$ a.e. implies $g \in X$ with $\|g\|_X \leq \|f\|_X$.

Going further, a quasi-Banach lattice of functions $(X, \|\cdot\|_X)$ is called *lattice r-convex* if

$$\Big\| \Big(\sum_{j=1}^{m} |f_j|^r \Big)^{1/r} \Big\|_X \leq \Big(\sum_{j=1}^{m} \|f_j\|_X^r \Big)^{1/r} \tag{3.1.8}$$

for any finite family $\{f_j\}_{1 \leq j \leq m}$ of functions from X (see, e.g., [76]; cf. also [83], Vol. II). This implies that the space

$$[X]^r := \Big\{ f \text{ measurable; } |f|^{1/r} \in X \Big\}, \quad \text{normed by } \ \|f\|_{[X]^r} := \| \, |f|^{1/r} \|_X^r, \tag{3.1.9}$$

is a Banach function space, called the *r-convexification* of X (cf. also **[83]**, Vol. II, pp. 53-54, at least if $r > 1$). The theorem below is due to N. Kalton (**[76]**, **[75]**), though in this particular form it is stated in **[94]**.

Theorem 3.1.4. *Let X be a (complex) quasi-Banach lattice of functions and denote by κ its modulus of concavity. Then the following assertions are equivalent:*

(i) X is analytically convex;

(ii) X is lattice r-convex for some $r > 0$;

(iii) X is lattice r-convex for each $0 < r < (1 + log_2 \kappa)^{-1}$.

To state the next result, recall that, given two quasi-Banach lattices of functions $(X_j, \|\cdot\|_{X_j})$, $j = 0, 1$, the *Calderón product* $X_0^{1-\theta} X_1^{\theta}$, $0 < \theta < 1$, is

$$\begin{aligned} &X_0^{1-\theta} X_1^{\theta} := \{ h \in L_0 : \exists\, f \in X_0,\ g \in X_1 \text{ such that } |h| \leq |f|^{1-\theta} |g|^{\theta} \}, \\ &\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf \Big\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta};\ |f| \leq |f_0|^{1-\theta} |f_1|^{\theta},\ f_j \in X_j,\ j = 0, 1 \Big\}. \end{aligned} \tag{3.1.10}$$

The theorem below is proved in [**55**] and [**77**].

Theorem 3.1.5. *Let Ω be a Polish space (i.e., complete, separable, metric space) and let μ be a $\sigma-$finite Borel measure on Ω. Let X_0, X_1 be a pair of quasi-Banach function spaces on (Ω, μ). Suppose that both X_0 and X_1 are analytically convex and separable. Then $X_0 + X_1$ is analytically convex and, for each $\theta \in (0,1)$,*

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta \tag{3.1.11}$$

in the sense of equivalence of quasi-norms.

Remarks. (i) As pointed out in [**77**], the hypothesis of separability in this case is equivalent to σ-order continuity. For a general quasi-normed space X, this property asserts that a non-negative, non-increasing sequence of functions in X which converges a.e. to zero also converges to zero in the quasi-norm topology of X (cf., e.g., [83], Vol. II). An equivalent reformulation is that if $g \in X$ and $|f_n| \leq |g|$ for all n and $f_n \to f$ a.e. then $\|f_n - f\|_X \to 0$. For us, it is of interest to also note a result, proved in Theorem 1.29 of [**26**], to the effect that

$$\begin{array}{c}\text{one of the lattices } X_0,\ X_1 \\ \text{is } \sigma\text{-order continuous}\end{array} \implies \begin{array}{c}X_0^{1-\theta} X_1^\theta \text{ is } \sigma\text{-order continuous} \\ \text{for each } \theta \in (0,1).\end{array} \tag{3.1.12}$$

(ii) It has been noted in [**94**] that, for the applications we have in mind (i.e. sequence spaces), Theorem 3.1.5 continues to hold in the case when just *one* of the two quasi-Banach lattices X_0, X_1 is separable.

Our final result in the series of analytic convexity criteria is the following observation from [**94**].

Theorem 3.1.6. *Assume that (Ω, μ) is a measure space, $0 < p \leq +\infty$ and that $(X, \|\cdot\|_X)$ is an analytically convex quasi-Banach space. Then $L^p(\Omega, X)$, the space of X-valued functions which are p-th power integrable on Ω, is an analytically convex quasi-Banach space.*

Next, a family of quasi-Banach spaces $\{X_p\}_{p\in I}$, where I is an open interval, is said to be a *complex interpolation scale* if for any $p_0, p_1 \in I$

$$[X_{p_0}, X_{p_1}]_\theta = X_p, \quad \text{if } \theta \in (0,1) \text{ and } 1/p = (1-\theta)/p_0 + \theta/p_1. \tag{3.1.13}$$

Our next results states that, for bounded linear operators, the quality of being invertible, or even Fredholm, is stable on complex interpolation scales of quasi-Banach spaces.

Theorem 3.1.7. *Let X_0, X_1 be a compatible couple of quasi-Banach spaces and assume that $X_0 + X_1$ is analytically convex. Also, consider a bounded, linear operator $T : X_j \to X_j$, $j = 0, 1$. If we now set $X_\theta := [X_0, X_1]_\theta$ for each $\theta \in (0,1)$, then T induces a bounded linear operator*

$$T_\theta : X_\theta \longrightarrow X_\theta, \qquad \theta \in (0,1), \tag{3.1.14}$$

in a natural fashion.

Assume next that there exists $\theta_o \in (0,1)$ such that T_{θ_o} is an isomorphism (respectively, Fredholm). Then there exists $\varepsilon > 0$ such that T_θ continues to be isomorphism (respectively, Fredholm) whenever $|\theta - \theta_o| < \varepsilon$.

Furthermore, T_θ^{-1} agrees with $T_{\theta'}^{-1}$ on $X_\theta \cap X_{\theta'}$ for any $\theta, \theta' \in (\theta_o - \varepsilon, \theta_o + \varepsilon)$.

The version of this result for Banach spaces is contained can be found in **[117]**, **[5]**, **[116]**. (for some elementary proofs of some special cases see also **[16]** and **[110]**).

Nonetheless, since the case of quasi-Banach spaces is of paramount importance to us we include a proof. The argument below is due to N. Kalton [**73**].

Proof. The interpolation property (3.1.14) along with the stability of the quality of being Fredholmness or invertibility for linear operators on complex interpolation scales of quasi-Banach spaces have been proved already in [**77**]. Here we focus on the compatibility condition in the last part of the theorem.

Note that if T_{θ_o} is invertible then so is T_z for z in a neighborhood of θ_o in the strip U and we denote by T_z^{-1} its inverse. Now, if F is an arbitrary admissible function, there exists $G_1 \in \mathcal{F}$ satisfying the properties

$$T[G_1(\theta_o)] = F(\theta_o) \quad \text{and} \quad \|G_1\|_{\mathcal{F}} \leq 2\|F\|_{\mathcal{F}}\|T_{\theta_o}^{-1}\|. \tag{3.1.15}$$

Therefore, thanks to the axiom (iii), F can be represented as $F = TG_1 + \omega F_1$, where $F_1 \in \mathcal{F}$ and ω is a suitable analytic function with $\omega(\theta_o) = 0$ and $|\omega| < 1$ on the boundary of the strip U (for instance, $\omega(z) := \frac{1}{3}(z - \theta_o)\, e^{z^2}$ will do). Also, by (3.1.4),

$$\|F_1\|_{\mathcal{F}} \leq \kappa\|F\|_{\mathcal{F}}, \qquad \kappa := \rho(\mathcal{F})(2\|T\|\|T_{\theta_o}^{-1}\| + 1). \tag{3.1.16}$$

Iterating this process, we obtain

$$F_i = TG_{i+1} + \omega F_{i+1}, \tag{3.1.17}$$

where, for $i = 1, 2, ...,$

$$\|F_{i+1}\|_{\mathcal{F}} \leq \kappa^{i+1}\|F\|_{\mathcal{F}} \quad \text{and} \quad \|G_{i+1}\|_{\mathcal{F}} \leq 2\|F_i\|_{\mathcal{F}}\|T_{\theta_o}^{-1}\| \leq 2\kappa^i\|F\|_{\mathcal{F}}\|T_{\theta_o}^{-1}\|. \tag{3.1.18}$$

Thus,

$$F = T\Big[\sum_{k=1}^{n} \omega^{k-1} G_k\Big] + \omega^n F_n, \quad \text{for every } n \in \mathbb{N}. \tag{3.1.19}$$

Granted the estimates (3.1.18), it is not too difficult to see that there exist V, neighborhood of θ_o such that $\sum_{k=1}^{\infty} \omega^{k-1} G_k$ converges to a function G uniformly on compacta and $\omega^n F_n$ converges to 0. By Proposition 3.1.2 G is analytic as a $(X_0 + X_1)$-valued function and satisfies $T[G(z)] = F(z)$, $G(z) \in X_z$ for every $z \in V$. In particular, $T_z^{-1}[F(z)] = G(z)$ is analytic for $z \in V$.

Next, recall that $X_0 \cap X_1$ is dense in X_z for all z, and fix some $x \in X_0 \cap X_1$. From the above reasoning it follows that there exists V neighborhood of θ_o in U such that $T_z^{-1} x$, viewed as a mapping $V \ni z \mapsto T_z^{-1}[F_o(z)] \in X_0 + X_1$, where $F_o \in \mathcal{F}$ is the constant function $F_o(z) := x$, is analytic. Since this function is also independent of the imaginary part of z, we may conclude that it is a constant function. Thus, $T_z^{-1} x \in X_0 + X_1$ is independent of $z \in V$, as desired. □

In the second part of this section we discuss some general interpolation results which are going to play an important role in future considerations. Throughout the paper, we let $(\cdot\,,\,\cdot)_{\theta,q}$ stand for the standard *real interpolation* bracket. More specifically, consider a compatible couple of quasi-Banach spaces X_0, X_1. Given $a \in X_0 + X_1$ and $0 < t < \infty$ Peetre's K-functional is defined by

$$K(t, a; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x_0 \in X_0,\ x_1 \in X_1 \text{ such that } a = x_0 + x_1\}. \tag{3.1.20}$$

Then we introduce the real interpolation spaces as

$$(X_0, X_1)_{\theta,q} := \Big\{a \in X_0 + X_1;$$
$$\|a\|_{(X_0,X_1)_{\theta,q}} := \left(\int_0^{\infty} (t^{-\theta} K(t, a; X_0, X_1))^q \, \frac{dt}{t}\right)^{1/q} < \infty\Big\}. \tag{3.1.21}$$

if $0 < \theta < 1$, $0 < q < \infty$, and

$$(X_0, X_1)_{\theta,\infty} := \left\{ a \in X_0 + X_1;\ \|a\|_{(X_0,X_1)_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta} K(t, a; X_0, X_1) < \infty \right\}, \tag{3.1.22}$$

for $0 < \theta < 1$.

Some properties of resulting spaces and more details regarding the real method of interpolation can be found **[10]**, **[124]**, **[125]**.

Theorem 3.1.8. *Let X_i, Y_i, Z_i, $i = 0, 1$, be quasi-Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_i \hookrightarrow Z_i$, $i = 0, 1$ and there exists a linear operator D such that $D : X_i \to Z_i$ boundedly for $i = 0, 1$. Define the spaces*

$$X_i(D) := \{u \in X_i :\ Du \in Y_i\}, \quad i = 0, 1, \tag{3.1.23}$$

equipped with the graph norm, i.e. $\|u\|_{X_i(D)} := \|u\|_{X_i} + \|Du\|_{Y_i}$, $i = 0, 1$. Finally, suppose that there exist continuous linear mappings $G : Z_i \to X_i$ and $K : Z_i \to Y_i$ with the property $D \circ G = I + K$ on the spaces Z_i for $i = 0, 1$. Then, for each $0 < \theta < 1$ and $0 < q \leq \infty$,

$$(X_0(D), X_1(D))_{\theta,q} = \{u \in (X_0, X_1)_{\theta,q} :\ Du \in (Y_0, Y_1)_{\theta,q}\}. \tag{3.1.24}$$

Furthermore, if the spaces $X_0 + X_1$ and $Y_0 + Y_1$ are A-convex, then

$$[X_0(D), X_1(D)]_\theta = \{u \in [X_0, X_1]_\theta :\ Du \in [Y_0, Y_1]_\theta\}, \qquad \theta \in (0, 1). \tag{3.1.25}$$

Proof. We are grateful to Nigel Kalton for discussions which led to the following

argument. For the real interpolation method it is convenient to work with K-functionals. The crux of the matter is establishing the following estimate:

$$K(t,a;X_0(D),X_1(D)) \approx K(t,a;X_0,X_1) + K(t,Da,Y_0,Y_1). \tag{3.1.26}$$

One direction is, of course, trivial. For the other one, given $a \in X_0 + X_1$, let

$$a = x_0 + x_1, \;\; x_i \in X_i, \;\; \text{and} \;\; Da = y_0 + y_1, \;\; y_i \in Y_i, \quad i = 0,1, \tag{3.1.27}$$

be nearly optimal splittings so that

$$\|x_0\|_{X_0} + t\|x_1\|_{X_1} \approx K(t,a,X_0,X_1) \tag{3.1.28}$$

and

$$\|y_0\|_{Y_0} + t\|y_1\|_{Y_1} \approx K(t,Da,Y_0,Y_1). \tag{3.1.29}$$

We then define a new splitting $a = x_0' + x_1'$, where

$$x_i' := x_i - GDx_i + Gy_i, \qquad i = 0,1. \tag{3.1.30}$$

Then

$$\|x_i'\|_{X_i} \le C(\|x_i\|_{X_i} + \|y_i\|_{Y_i}), \qquad i = 0,1. \tag{3.1.31}$$

Also

$$Dx_i' = -KDx_i + y_i + Ky_i, \qquad i = 0,1, \tag{3.1.32}$$

so that

$$\|Dx_i'\|_{Y_i} \le C(\|x_i\|_{X_i} + \|y_i\|_{Y_i}), \qquad i = 0,1, \tag{3.1.33}$$

since KD maps X_i boundedly into Y_i for $i = 0, 1$, and K maps Y_i boundedly to itself, $i = 0, 1$. The estimates (3.1.31)-(3.1.32) then justify the equivalence (3.1.26).

The identity (3.1.25), regarding complex interpolation, is due to J.-L. Lions and E. Magenes (cf. [**84**]) when all spaces involved are Banach. However, their argument goes through with minor modifications for quasi-Banach spaces given the analytic convexity assumptions made in this portion of our theorem. The only thing we need to check is that is that the space $X_0(D) + X_1(D)$ is analytically convex, so that the complex interpolation method outlined in the first part of this section applies to the couple $X_0(D)$, $X_1(D)$.

In order to justify this we first note that

$$X_0(D) + X_1(D) = (X_0 + X_1)(D) := \{u \in X_0 + X_1 : \, Du \in Y_0 + Y_1\}, \quad (3.1.34)$$

where the rightmost space is equipped with the natural graph norm. Indeed, (3.1.34) follows readily from the decompositions (3.1.27), (3.1.30). Thus, it suffices to prove that $(X_0 + X_1)(D)$ is analytically convex. To this end, let $f : U \to (X_0 + X_1)(D)$ be an analytic function which extends by continuity to $\bar{U}$. Since the inclusion $\iota : (X_0 + X_1)(D) \to X_0 + X_1$ is linear and bounded, f can also be regarded as a $X_0 + X_1$-valued analytic function in U, extendible by continuity to $\bar{U}$. Similarly, the operator $D : (X_0 + X_1)(D) \to Y_0 + Y_1$ is linear and bounded, thus Df is a $Y_0 + Y_1$-valued analytic function in U, which extends by continuity to $\bar{U}$. Consequently, given that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex, we may

write

$$\max_{0<\Re e\, z<1} \|f(z)\|_{(X_0+X_1)(D)} \approx \max_{0<\Re e\, z<1} \|f(z)\|_{X_0+X_1} + \max_{0<\Re e\, z<1} \|Df(z)\|_{Y_0+Y_1}$$

$$\leq C \max_{\Re e\, z=0,1} \|f(z)\|_{X_0+X_1} + C \max_{\Re e\, z=0,1} \|Df(z)\|_{Y_0+Y_1}$$

$$\approx C \max_{\Re e\, z=0,1} \|f(z)\|_{(X_0+X_1)(D)}, \tag{3.1.35}$$

as desired. □

We continue this section with a simple, yet useful result, which is essentially folklore. First, we make a definition. Let X_0, X_1 and Y_0, Y_1 be two compatible pairs of quasi-Banach spaces. Call $\{Y_0, Y_1\}$ a *retract* of $\{X_0, X_1\}$ if there exist two bounded, linear operators $E : Y_i \to X_i$, $R : X_i \to Y_i$, $i = 0, 1$, such that $R \circ E = I$, the identity map, on each Y_i, $i = 0, 1$.

Lemma 3.1.9. *Assume that X_0, X_1 and Y_0, Y_1 are two compatible pairs of quasi-Banach spaces such that $\{Y_0, Y_1\}$ is a retract of $\{X_0, X_1\}$ (as before, the "extension-restriction" operators are denoted by E and R, respectively). Then for each $\theta \in (0, 1)$ and $0 < q \leq \infty$,*

$$[Y_0, Y_1]_\theta = R\Big([X_0, X_1]_\theta\Big) \quad \text{and} \quad (Y_0, Y_1)_{\theta,q} = R\Big((X_0, X_1)_{\theta,q}\Big). \tag{3.1.36}$$

In the case of the complex method, it is assumed that $X_0 + X_1$ is analytically convex.

As a corollary, we also have the following. Assume that (X_0, X_1) is a compatible pair of quasi-Banach spaces and that P is a common projection (i.e., a linear, bounded operator on X_i, $i = 0, 1$, such that $P^2 = P$). Then the real and complex interpolation brackets commute with the action of P, i.e.

$$[PX_0, PX_1]_\theta = P\Big([X_0, X_1]_\theta\Big) \quad \text{and} \quad (PX_0, PX_1)_{\theta,q} = P\Big((X_0, X_1)_{\theta,q}\Big), \tag{3.1.37}$$

for each $\theta \in (0,1)$ *and* $0 < q \leq \infty$. *In the case of the complex method, it is assumed that* $X_0 + X_1$ *is analytically convex.*

Remark. (i) Generally speaking, given two quasi-normed spaces X, Y and a linear, bounded operator $T : X \to Y$, by TX we shall denote its image equipped with the quasi-norm

$$\|y\|_{TX} := \inf\{\|x\|_X : x \in X \text{ such that } y = Tx\}, \qquad y \in TX. \tag{3.1.38}$$

In particular, this is the sense in which (3.1.36) and (3.1.37) should be understood.

(ii) The portion of Lemma 3.1.9 referring to real interpolation remains valid when the spaces in question are quasi-normed Abelian groups (in which case, the operators involved are assumed to be group morphisms).

Proof. The first order of business is to show that $Y_0 + Y_1$ is analytically convex (hence justifying the use of the complex method of interpolation for the pair Y_0, Y_1). One way to see this is by observing that E maps $Y_0 + Y_1$ isomorphically onto $E(Y_0 + Y_1)$ which, given that this operator has a left inverse, is a closed subspace of the analytically convex space $X_0 + X_1$. Hence, $Y_0 + Y_1$ is also analytically convex.

The remainder of the proof follows a well-known path. We, nonetheless, include the details for the convenience of the reader. Fix $\theta \in (0,1)$ and set $X_\theta = [X_0, X_1]_\theta$, $Y_\theta = [Y_0, Y_1]_\theta$. By the interpolation property for bounded linear operators, R maps X_θ to Y_θ, i.e., $R(X_\theta) \subseteq Y_\theta$. To justify the opposite inclusion, note that E takes Y_θ into X_θ which is further mapped by R into $R(X_\theta)$. Since the composition of these two applications acts as the identity operator, we may conclude that $Y_\theta \subseteq R(X_\theta)$, as desired. The proof in the case of the real interpolation method is virtually the

same and this completes the proof of the first part of the lemma.

Turning to the second part of our lemma, we note that $\{PX_0, PX_1\}$ is a retract of $\{X_0, X_1\}$ (taking E to be the inclusion and R the given projection). Thus, (3.1.37) is a corollary of what we have proved so far. □

Theorem 3.1.10. *(cf.* **[65]***) Let μ be a fixed Borel measure and denote by $L^p(\mu)$ the corresponding Lebesgue spaces. Also, let X_0, X_1 be a couple of quasi-Banach spaces and let $X_\theta = [X_0, X_1]_\theta$, $0 < \theta < 1$, be the complex interpolation intermediate spaces.*

Finally, let $\mathcal{T}$ be a sublinear operator from $X_0 + X_1$ into $L^{p_0}(\mu) + L^{p_1}(\mu)$, $1 \leq p_0, p_1 \leq \infty$, such that

$$\|\mathcal{T}x\|_{L^{p_0}(\mu)} \leq A_0\|x\|_{X_0}, \quad \text{for } x \in X_0, \tag{3.1.39}$$

and

$$\|\mathcal{T}x\|_{L^{p_1}(\mu)} \leq A_1\|x\|_{X_1}, \quad \text{for } x \in X_1, \tag{3.1.40}$$

for some finite, positive constants A_0 and A_1. Then

$$\|\mathcal{T}x\|_{L^{p_\theta}(\mu)} \leq A_0^{1-\theta}A_1^{\theta}\|x\|_{X_\theta}, \quad \text{for } \ x \in X_\theta, \text{ with } \tfrac{1}{p_\theta} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}. \tag{3.1.41}$$

We conclude our review of interpolation with (a version of) the celebrated theorem of E. Stein for analytic families of operators.

Theorem 3.1.11. *(cf.* **[19]***) Let (A_0, A_1) be an interpolation pair of complex Banach spaces and set $\mathcal{X} = A_0 \bigcap A_1$, $\mathcal{X}_\theta = [A_0, A_1]_\theta$, for $0 \leq \theta \leq 1$. Analogously, let (B_0, B_1) be another interpolation pair of complex Banach spaces; set $\mathcal{Y} = B_0 \bigcap B_1$ and $\mathcal{Y}_\theta = [B_0, B_1]_\theta$, for $0 \leq \theta \leq 1$.*

Next, let L_z be a family of linear operators defined in $\mathcal{X}$, with values in $\mathcal{Y}$, indexed by a complex parameter z, with $0 \leq \Re ez \leq 1$. Assume that $l(L_z f)$ is continuous and bounded in $0 \leq \Re ez \leq 1$, and analytic in $0 < \Re ez < 1$ for every $f \in \mathcal{X}$ and every continuous linear functional l on $\mathcal{Y}$. Assume further that for $\Re ez = 0$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_0} \leq c_0 \|f\|_{\mathcal{X}_0}, \tag{3.1.42}$$

and for $\Re ez = 1$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_1} \leq c_1 \|f\|_{\mathcal{X}_1}. \tag{3.1.43}$$

Then for $0 < \Re ez = \theta < 1$ there exists $c = c(s, q_0, q_1, c_0, c_1)$ such that

$$\|L_z f\|_{\mathcal{Y}_\theta} \leq c \|f\|_{\mathcal{X}_\theta}, \tag{3.1.44}$$

uniformly for $f \in \mathcal{X}$.

3.2 Function spaces on $\mathbb{R}^n$

We debut with a brief discussion of Hardy-type spaces in $\mathbb{R}^n$. Fix $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$, and set $\psi_t(x) := t^{-n}\psi(x/t)$ for each $t > 0$. Given a tempered distribution $u \in S'(\mathbb{R}^n)$ we define its radial maximal function and its truncated version, respectively, by setting

$$u^{++} := \sup_{0<t<\infty} |\psi_t * u|, \qquad u^{+} := \sup_{0<t<1} |\psi_t * u|. \tag{3.2.1}$$

For $0 < p < \infty$, the classical *homogeneous* Hardy space $H^p(\mathbb{R}^n)$, and its *local*

version, $h^p(\mathbb{R}^n)$, introduced in **[54]**, are then defined as

$$H^p(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \|u\|_{H^p(\mathbb{R}^n)} := \|u^{++}\|_{L^p(\mathbb{R}^n)} < \infty\}, \tag{3.2.2}$$

$$h^p(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \|u\|_{h^p(\mathbb{R}^n)} := \|u^{+}\|_{L^p(\mathbb{R}^n)} < \infty\}. \tag{3.2.3}$$

Different choices of the function ψ yield equivalent quasi-norms so (3.2.2), (3.2.3) viewed as topological spaces, are intrinsically defined.

Next, we consider local weak-Hardy spaces $h^{p,\infty}(\mathbb{R}^n)$, $0 < p < \infty$. Concretely, given a tempered distribution u in $\mathbb{R}^n$,

$$u \in h^{p,\infty}(\mathbb{R}^n) \Longleftrightarrow u^+ \in L^{p,\infty}(\mathbb{R}^n), \tag{3.2.4}$$

equipped with the natural quasi-norm. Here, as usually, $L^{p,\infty}(\mathbb{R}^n)$ denotes the weak-L^p space, $0 < p < \infty$. The connection with the standard scale of Hardy spaces is most visible in the context of real interpolation. Indeed, C. Fefferman, N. Rivière, and Y. Sagher have proved in **[41]** that

$$(h^{p_0}(\mathbb{R}^n), h^{p_1}(\mathbb{R}^n))_{\theta,\infty} = h^{p,\infty}(\mathbb{R}^n), \quad 0 < p_0, p_1 < \infty, \quad 0 < \theta < 1, \quad \tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}. \tag{3.2.5}$$

In fact, these authors proved the version of (3.2.5) corresponding to the homogeneous scale of Hardy spaces (3.2.2) but this distinction is immaterial. Indeed, the main alteration needed in the original approach is to work with the version of the grand maximal function defined on p. 76 of **[41]** in which the nontangential cone is truncated at a fixed, sufficiently large height (cf. also **[54]** in this regard).

Later on, we shall make use of Hardy-based Sobolev spaces denoted by $h^p_k(\mathbb{R}^n)$.

Whenever $0 < p < \infty$, $k \in \mathbb{N}_o$, these are given by

$$h^p_k(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \partial^\gamma u \in h^p(\mathbb{R}^n),\ \forall\, \gamma \in \mathbb{N}^n_o \text{ with } |\gamma| \leq k\}, \tag{3.2.6}$$

and are equipped with the quasi-norm $\|u\|_{h^p_k(\mathbb{R}^n)} := \sum_{|\gamma|\leq k} \|\partial^\gamma u\|_{h^p(\mathbb{R}^n)}$. For $0 < p < \infty$ and $k \in \mathbb{N}$ we also set

$$h^p_{-k}(\mathbb{R}^n) := \Big\{u \in S'(\mathbb{R}^n) : u = \sum_{|\gamma|\leq k} \partial^\gamma u_\gamma,\ u_\gamma \in h^p(\mathbb{R}^n)\ \forall\, \gamma \in \mathbb{N}^n_o \text{ with } |\gamma| \leq k\Big\} \tag{3.2.7}$$

which we equip with the natural quasi-norm $\|u\|_{h^p_{-k}(\mathbb{R}^n)} := \inf \sum_{|\gamma|\leq k} \|u_\gamma\|_{h^p(\mathbb{R}^n)}$, where the infimum is taken over all representations of u.

We now record an important identification result, whose proof is given at the end of this section.

Theorem 3.2.1. *For $0 < p \leq 1$ and $k \in \mathbb{Z}$,*

$$h^p_k(\mathbb{R}^n) = F^{p,2}_k(\mathbb{R}^n). \tag{3.2.8}$$

We continue by reviewing those aspects of the classical atomic theory of Hardy spaces in $\mathbb{R}^n$ which are useful for us. Recall that for $\frac{n}{n+1} < p \leq 1 < p_1 < +\infty$ an L^{p_1}-atom for $h^p(\mathbb{R}^n)$ is a function satisfying $\operatorname{supp}(a) \subseteq Q$ for some cube Q in $\mathbb{R}^n$ with side-length ≤ 1 and so that

$$\|a\|_{L^{p_1}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{p_1}-\frac{1}{p}} \text{ and } \int_{\mathbb{R}^n} a(x)\,dx = 0 \text{ if } l(Q) < 1. \tag{3.2.9}$$

Then

$$f \in h^p(\mathbb{R}^n) \Leftrightarrow f = \sum_j \lambda_j a_j \in S'(\mathbb{R}^n) : \{\lambda_j\}_j \in \ell^p,\ a_j \text{ is an } L^{p_1}-\text{atom for } h^p(\mathbb{R}^n),$$

(3.2.10)

and

$$\|f\|_{h^p(\mathbb{R}^n)} \approx \inf\Big\{\Big(\sum_j |\lambda_j|^p\Big)^{1/p} : \ f = \sum_j \lambda_j a_j, \ a_j \text{ is an } L^{p_1} - \text{atom for } h^p(\mathbb{R}^n), \ \{\lambda_j\}_j \in \ell^p\Big\}, \tag{3.2.11}$$

uniformly for $f \in h^p(\mathbb{R}^n)$. We shall also need an atomic decomposition result for

$$h_1^p(\mathbb{R}^n) = \{f \in h^p(\mathbb{R}^n) : \partial_j f \in h^p(\mathbb{R}^n), \ 1 \le j \le n\} \tag{3.2.12}$$

which we shall prove in the range $\frac{n}{n+1} < p \le 1$. This involves "regular" atoms, defined as follows. Let $\frac{n}{n+1} < p \le 1 < p_1 \le +\infty$. Call A an L^{p_1}-atom for $h_1^p(\mathbb{R}^n)$ if there exists some cube Q in $\mathbb{R}^n$ with $l(Q) \le 1$ such that

$$\operatorname{supp} A \subseteq Q, \qquad \|\nabla A\|_{L^{p_1}(\mathbb{R}^n)} \le |Q|^{\frac{1}{p_1}-\frac{1}{p}}. \tag{3.2.13}$$

Theorem 3.2.2. *For $\frac{n}{n+1} < p \le 1 < p_1 \le +\infty$,*

$$h_1^p(\mathbb{R}^n) = \Big\{f = \sum_j \lambda_j A_j \in S'(\mathbb{R}^n) : \{\lambda_j\}_j \in \ell^p, \ A_j \text{ is an } L^{p_1}\text{-atom for } h_1^p(\mathbb{R}^n)\Big\} \tag{3.2.14}$$

and

$$\|f\|_{h_1^p(\mathbb{R}^n)} \approx \inf\Big\{\Big(\sum_j |\lambda_j|^p\Big)^{1/p} : \ f = \sum_j \lambda_j A_j, \ \{\lambda_j\}_j \in \ell^p, \ A_j \ L^{p_1}\text{-atom for } h_1^p(\mathbb{R}^n)\Big\}, \tag{3.2.15}$$

uniformly for $f \in h_1^p(\mathbb{R}^n)$.

Once again, the proof of this result is deferred until the end of this section.

We now briefly review Besov and Triebel-Lizorkin scales in $\mathbb{R}^n$. The classical Littlewood-Paley definition of Triebel-Lizorkin and Besov spaces (see, for example, [**112**], [**126**]) has the following form. Let Ξ be the collection of all systems $\{\zeta_j\}_{j=0}^{\infty} \subset S$ with the properties

(i) there exist positive constants A, B, C such that

$$\begin{cases} \text{supp}\,(\zeta_0) \subset \{x;\, |x| \leq A\}, \\ \text{supp}\,(\zeta_j) \subset \{x;\, B2^{j-1} \leq |x| \leq C2^{j+1}\} \quad \text{if } j = 1, 2, 3..., \end{cases} \tag{3.2.16}$$

(ii) for every multi-index α there exists a positive number c_α such that

$$\sup_{x\in\mathbb{R}^n} \sup_{j\in\mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \zeta_j(x)| \leq c_\alpha, \tag{3.2.17}$$

(iii)

$$\sum_{j=0}^{\infty} \zeta_j(x) = 1 \text{ for every } x \in \mathbb{R}^n. \tag{3.2.18}$$

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$ and fix some family $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$. Also, let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^n$. If $0 < p < \infty$ then the Triebel-Lizorkin spaces are defined as

$$F_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \quad \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{3.2.19}$$

If $0 < p \leq \infty$ then the Besov spaces are defined as

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \quad \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)\|_{L^p(\mathbb{R}^n)}^q \Big)^{1/q} < \infty \Big\}. \tag{3.2.20}$$

A different choice of the system $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$ yields the same spaces (3.2.19)-(3.2.20), albeit equipped with equivalent norms.

There is an alternative version of definitions above starting with a function $\phi_0 \in S(\mathbb{R}^n)$ such that

$$\begin{cases} \int_{\mathbb{R}^n} \phi_0(x)\,dx \neq 0, \\ \phi(x) := \phi_0(x) - 2^{-n}\phi_0(x/2) \Longrightarrow L(\phi) \geq [s], \end{cases} \tag{3.2.21}$$

where $s \in \mathbb{R}$, $[s]$ denotes the integer part of s and $L(\phi)$ stands for the order up to which the moments of function ϕ vanish, i.e.

$$\int_{\mathbb{R}^n} x^{\alpha}\phi(x)\,dx = 0 \quad \text{if} \quad |\alpha| \leq L(\phi) \tag{3.2.22}$$

(with the convention that this condition is void when $L(\phi) = -1$). It is well-known that, given any $s \in \mathbb{R}$, there exist functions ϕ_0 satisfying (3.2.21). Indeed, for $s < 1$ the second condition in (3.2.21) is trivial, whereas for $s \geq 1$ we may take any function $\phi_0 \in S(\mathbb{R}^n)$ whose Fourier transform is $1 + O(|\xi|^{[s]+1})$ uniformly as $\xi \to 0$. Next, let $\phi_j(x) := 2^{jn}\phi(2^j x)$, $j \in \mathbb{N}$, denote the dyadic dilates of ϕ. The Triebel-Lizorkin space $F_s^{p,q}(\mathbb{R}^n)$ is then defined for $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, as

$$F_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \ \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj}\phi_j * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{3.2.23}$$

In a similar spirit, Besov spaces are defined for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, by

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \ \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj}\phi_j * f\|_{L^p(\mathbb{R}^n)}^q \Big)^{1/q} < \infty \Big\}. \tag{3.2.24}$$

Once again, a different choice of the function $\phi_0 \in S(\mathbb{R}^n)$ satisfying (3.2.21) yields the same spaces (3.2.23)-(3.2.24) with equivalent norms (see, e.g., the discussion in **[113]**).

It has to be noted that the two definitions are equivalent, for one can take a function ϕ_0 such that $\zeta_0(\xi) = \mathcal{F}\phi_0(\xi)$ is identically 1 if $|\xi| \leq 1$ and vanishes if $|\xi| \geq 2$. Then with the notation as above ϕ and $\zeta_j = \mathcal{F}\phi_j$ satisfy the required properties.

Theorem 3.2.3. *(cf.* **[125]***, 2.3.3) $B_s^{p,q}(\mathbb{R}^n)$ is a quasi-Banach space for $s \in \mathbb{R}$, $0 < p, q \leq \infty$ (Banach space if $1 \leq p, q \leq \infty$) and*

$$S(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n). \tag{3.2.25}$$

in the sense of continuous (topological) embeddings. Similarly, $F_s^{p,q}(\mathbb{R}^n)$ is a quasi-Banach space for $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ (Banach space if $1 \leq p < \infty$, $1 \leq q \leq \infty$) and

$$S(\mathbb{R}^n) \hookrightarrow F_s^{p,q}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n). \tag{3.2.26}$$

Furthermore, $S(\mathbb{R}^n)$ is dense in $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$ provided $s \in \mathbb{R}$ and $0 < p, q < \infty$. The class of Schwartz functions is not dense in $B_s^{\infty,q}(\mathbb{R}^n)$, $B_s^{p,\infty}(\mathbb{R}^n)$ and $F_s^{p,\infty}(\mathbb{R}^n)$ if $0 < q \leq \infty$ and $0 < p < \infty$.

For the range of indices $n/(n+1) < p, q \leq \infty$ and $n(1/p - 1)_+ < s < 1$, an intrinsic definition for membership to $B_s^{p,q}(\mathbb{R}^n)$ is obtained by requiring that

$$\|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot + t) - f(\cdot)\|_{L^p(\mathbb{R}^n)}^q}{|t|^{n+sq}} \, dt \right)^{1/q} < +\infty. \tag{3.2.27}$$

The result stated above is proved, for instance, in **[125]**, §2.5.12. There one can also find an analogous characterization of Triebel-Lizorkin spaces and intrinsic definition of spaces (both of Besov and Triebel-Lizorkin type) with larger amounts of smoothness, involving higher order differences.

Let $\mathcal{Q}_n$ stand for the standard family of dyadic cubes in $\mathbb{R}^n$, i.e.,

$$\mathcal{Q}_n := \{Q_{jk} = 2^{-j}[0,1]^n + k : \, j \in \mathbb{N}_o, \, k \in \mathbb{Z}^n\}. \tag{3.2.28}$$

Also, for an arbitrary function ψ in $\mathbb{R}^n$ and $Q \in \mathcal{Q}_n$ set

$$\psi_Q(x) := 2^{jn}\psi(2^j x - k) \quad \text{if} \quad Q = Q_{jk}, \; j \in \mathbb{N}_o, \; k \in \mathbb{Z}^n. \tag{3.2.29}$$

Following **[49]**, **[50]** the spaces $b^{p,q}_s$ and $f^{p,q}_s$ consist of sequences $\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ equipped with the quasi-norms

$$\|\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}\|_{b^{p,q}_s} := \left(\sum_{j=0}^{\infty}\Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)=2^{-j}}}[\,|Q|^{-s/n-1/2+1/p}|\lambda_Q|\,]^p\Big)^{q/p}\right)^{1/q} \tag{3.2.30}$$

$$\|\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}\|_{f^{p,q}_s} := \left\|\Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}}(|Q|^{-s/n-1/2}|\lambda_Q|\chi_Q)^q\Big)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{3.2.31}$$

where χ_Q stands for the characteristic function of Q. There is also an appropriate version of this definition when $p = +\infty$. Specifically,

$$\begin{aligned}\|\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}\|_{f^{\infty,q}_s} &:= \sup_{\substack{P\in\mathcal{Q}_n\\ l(P)\leq 1}}\left(\frac{1}{|P|}\int_P \sum_{Q\in\mathcal{Q}_n, Q\subset P}\Big(|Q|^{-1/2-s/n}|\lambda_Q|\chi_Q(x)\Big)^q dx\right)^{1/q}\\ &= \sup_{\substack{P\in\mathcal{Q}_n\\ l(P)\leq 1}}\left(\frac{1}{|P|}\sum_{Q\in\mathcal{Q}_n, Q\subset P}\Big(|Q|^{-1/2-s/n}|\lambda_Q|\Big)^q|Q|\right)^{1/q}.\end{aligned} \tag{3.2.32}$$

Observe that $J_\alpha(\{\lambda_Q\}_Q) := \{|Q|^{-\alpha/n}\lambda_Q\}_Q$ is an isomorphism between $f_s^{p,q}$ and $f_{s+\alpha}^{p,q}$ for each $\alpha \in \mathbb{R}$. Also, as is well known, each $f_s^{p,q}$ is a quasi-Banach lattice.

Finally, as in **[49]** we define f_0 as the collection of all sequences $\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\le 1}$ subject to the constraint

$$\|\{\lambda_Q\}_{Q\in\mathcal{Q}_n,l(Q)\le 1}\|_{f_0} := \Big|\bigcup_{\lambda_Q\neq 0} Q\Big| < +\infty. \tag{3.2.33}$$

While f_o is *not* a normed vector space (since the homogeneity of the "norm" may fail) it is, nonetheless, a *normed Abelian group*, since the triangle inequality $\|s + s'\|_{f_o} \le \|s\|_{f_o} + \|s'\|_{f_o}$ holds. Recall that A is a quasi-normed Abelian group, if it is an Abelian group (the group operation is denoted by $+$, the inverse of element $a \in A$ by $-a$ and the neutral element by 0), equipped with a quasi-norm $\|\cdot\|_A$ satisfying the following properties: (i) $\|a\|_A \ge 0$ and $\|a\|_A = 0$ if and only if $a = 0$, (ii) $\|-a\|_A = \|a\|_A$, (iii) $\|a+b\|_A \le C(\|a\|_A + \|b\|_A)$, where $C \ge 1$ and $a, b \in A$. For further use, it will be important to note that the real interpolation method works for quasi-normed Abelian groups in exactly the same way as in the classical case of quasi-Banach spaces, since the homogeneity condition of the quasi-norm does not play any role (see **[10]**, §§3.10-3.11 for a relevant discussion).

In the next several theorems we collect some basic properties of Besov and Triebel-Lizorkin spaces, starting with

Theorem 3.2.4 (Lifting). *Assume that* $0 < p, q \le \infty$ *and* $s \in \mathbb{R}$. *Then for any* $\mu \in \mathbb{R}$,

$$F_s^{p,q}(\mathbb{R}^n) = (I - \Delta)^\mu F_{s+\mu}^{p,q}(\mathbb{R}^n) \tag{3.2.34}$$

Also, for any $m \in \mathbb{N}$,

$$\begin{aligned} F_s^{p,q}(\mathbb{R}^n) &= \{f \in S'(\mathbb{R}^n) : \partial^\alpha f \in F_{s-m}^{p,q}(\mathbb{R}^n),\ \forall\, \alpha \text{ with } |\alpha| \le m\} \\ &= \{f \in F_{s-m}^{p,q}(\mathbb{R}^n) : \partial^\alpha f \in F_{s-m}^{p,q}(\mathbb{R}^n),\ \forall\, \alpha \text{ with } |\alpha| = m\} \end{aligned} \tag{3.2.35}$$

and

$$\begin{aligned} \|f\|_{F_s^{p,q}(\mathbb{R}^n)} &\approx \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)} \\ &\approx \|f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)} + \sum_{|\alpha|=m} \|\partial^\alpha f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)}. \end{aligned} \tag{3.2.36}$$

In particular,

$$\partial^\alpha : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_{s-|\alpha|}^{p,q}(\mathbb{R}^n) \tag{3.2.37}$$

is bounded.

Furthermore, completely analogous results hold for the scale of Besov spaces as well.

Theorem 3.2.5 (Embeddings). *For* $0 < p_0 \le p_1 \le +\infty$, $s_0, s_1 \in \mathbb{R}$, $0 < q_0 \le q_1 \le +\infty$ *with* $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, *the inclusion*

$$B_{s_0}^{p_0,q_0}(\mathbb{R}^n) \hookrightarrow B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \tag{3.2.38}$$

is continuous with dense range. Moreover, the same holds for the inclusion

$$F_{s_0}^{p_0,q_0}(\mathbb{R}^n) \hookrightarrow F_{s_1}^{p_1,q_1}(\mathbb{R}^n), \tag{3.2.39}$$

provided that either $0 < p_0 < p_1 < +\infty$, $0 < q_0, q_1 \le +\infty$ *and* $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ *or* $0 < p_0 = p_1 < +\infty$, $0 < q_0 < q_1 \le +\infty$ *and* $s_0 = s_1$.

Theorem 3.2.6 (Duality). *For $s \in \mathbb{R}$ and $0 < p, q < +\infty$, one has*

$$(B_s^{p,q}(\mathbb{R}^n))^* = B^{p',q'}_{-s+n\left(\frac{1}{p}-1\right)_+}(\mathbb{R}^n) \tag{3.2.40}$$

and

$$(F_s^{p,q}(\mathbb{R}^n))^* = F_{-s}^{p',q'}(\mathbb{R}^n), \qquad \text{if } \ p > 1, \tag{3.2.41}$$

$$(F_s^{p,q}(\mathbb{R}^n))^* = B^{\infty,\infty}_{-s+n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n), \qquad \text{if } \ p < 1. \tag{3.2.42}$$

Proofs can be found in, e.g., **[125]**, **[112]**; cf. also the references therein. Here we only want to point out that (3.2.41) with $0 < q < 1$ is not usually covered in the literature. An argument can be found in **[94]**.

For every choice of $r \in \mathbb{N}$ and $L \in \mathbb{N} \cup \{0, -1\}$ it is possible to construct

$$\begin{array}{l} \text{a "father" wavelet } \varphi \in C_0^r(\mathbb{R}^n) \text{ and a family of} \\ \text{"mother" wavelets } \psi^\ell \in C_0^r(\mathbb{R}^n),\ \ell = 1, 2, ..., 2^n - 1, \end{array} \tag{3.2.43}$$

in the sense of **[34]**, **[86]**, **[95]**, such that $L(\psi^\ell) \geq L$ for each $1 \leq \ell \leq 2^n - 1$, i.e.

$$\int_{\mathbb{R}^n} x^\alpha \psi^\ell(x)\, dx = 0 \quad \text{whenever} \quad |\alpha| \leq L,\ 1 \leq \ell \leq 2^n - 1. \tag{3.2.44}$$

In the sequel, we shall refer to these as *Daubechies wavelets*. Another variant of this construction, corresponding to the so-called *Lemarie-Meyer wavelets*, allows for φ, $\psi^\ell \in S(\mathbb{R}^n)$ and $L = \infty$. Cf. **[95]** for details. The most striking property of the wavelet functions is that

$$\{\varphi_Q : Q \in \mathcal{Q}_n,\, l(Q) = 1\} \cup \{\psi_Q^\ell : Q \in \mathcal{Q}_n,\, l(Q) \leq 1,\, 1 \leq \ell \leq 2^n - 1\}, \tag{3.2.45}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

The *wavelet coefficients* of a given tempered distribution $f \in S'(\mathbb{R}^n)$ relative to a family of Daubechies or Lemarie-Meyer wavelets, φ, ψ^ℓ, $\ell = 1, ..., 2^n - 1$, are then defined as

$$\lambda_Q^\ell(f) := \begin{cases} \langle f, \varphi_Q \rangle, & \text{if } \ l(Q) = 1, \\ \langle f, \psi_Q^\ell \rangle, & \text{if } \ l(Q) < 1, \end{cases} \tag{3.2.46}$$

for each $Q \in \mathcal{Q}_n$, $\ell = 1, ..., 2^n - 1$. The wavelet characterization of Besov and Triebel-Lizorkin spaces we are about to describe next goes back to **[47]**, **[48]**, **[50]**.

Recall that the collection of vectors x_0, x_1,..., in the space $(X, \|\cdot\|_X)$ is called Schauder basis if every vector $x \in X$ can be written in the form

$$x = \sum_{j=0}^{\infty} \lambda_j x_j, \tag{3.2.47}$$

where $\lambda_i \in \mathbb{R}$, $j = 0, 1, ...$, the series (3.2.47) converges in the $\|\cdot\|_X$ norm, i.e.

$$\lim_{N\to\infty} \Big\| x - \sum_{j=0}^{N} \lambda_j x_j \Big\|_X = 0, \tag{3.2.48}$$

and coefficients λ_i, $i = 0, 1, ...$, are uniquely determined by (3.2.47)-(3.2.48). The Schauder basis is unconditional if for every $x \in X$ the series (3.2.47) converges unconditionally to x, that is there exists $\varepsilon > 0$ and F_ε – finite subset of $\mathbb{N}$ such that for every finite $F \in \mathbb{N}_o$ containing F_ε $\|x - \sum_{j\in F} \lambda_j x_j\|_X \leq \varepsilon$.

Theorem 3.2.7. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and assume that $r \geq ([s]+1)_+$ and $L \geq \max\{[J - n - s], -1\}$, where $J := \frac{n}{\min\{1,p,q\}}$. Then for any family of Daubechies wavelets as in (3.2.43) the following is true. For each $f \in S'(\mathbb{R}^n)$ one has*

$$f \in F_s^{p,q}(\mathbb{R}^n) \Longleftrightarrow \{\lambda_Q^\ell(f)\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1} \in f_s^{p,q} \text{ for each } \ell = 1, ..., 2^n - 1, \tag{3.2.49}$$

with a naturally accompanying norm estimate.

Next, assume that $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $r \geq ([s]+1)_+$ *and* $L \geq \max\{[J - n - s], -1\}$, *where* $J := \frac{n}{\min\{1,p\}}$. *Then for any family of Daubechies wavelets as in (3.2.43) the following is true. For each* $f \in S'(\mathbb{R}^n)$ *one has*

$$f \in B^{p,q}_s(\mathbb{R}^n) \Longleftrightarrow \{\lambda^\ell_Q(f)\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1} \in b^{p,q}_s \text{ for each } \ell = 1, ..., 2^n - 1, \quad (3.2.50)$$

with a naturally accompanying norm estimate.

Finally, similar statements are true in the case when the Daubechies wavelets are replaced by Lemarie-Meyer wavelets, this time, with no restrictions on r (regularity) and L (number of vanishing moments). In either case, (3.2.45) is an unconditional Schauder basis in $F^{p,q}_s(\mathbb{R}^n)$ *if* $q < \infty$, *and in* $B^{p,q}_s(\mathbb{R}^n)$ *if* $max\{p, q\} < \infty$.

Other useful norm inequalities, found in **[109]**, **[12]**, **[13]**, **[113]**, are recorded below.

Theorem 3.2.8. *Let* $s \in \mathbb{R}$ *and consider* ϕ_0 *and* ϕ *as in (3.2.21). For each fixed* $N > 0$, *Peetre's family of maximal functions is defined for* $f \in S'(\mathbb{R}^n)$ *by*

$$\phi^*_{j,N} f(x) := \sup_{y\in\mathbb{R}^n} \frac{|(\phi_j * f)(y)|}{(1 + 2^j|x - y|)^N}, \quad x \in \mathbb{R}^n, \, j = 0, 1, 2, ... \quad (3.2.51)$$

Then for any $0 < p, q < \infty$ *one has*

$$\Big\| \Big(\sum_{j=0}^{\infty} (2^{sj} \phi^*_{j,N} f)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{F^{p,q}_s(\mathbb{R}^n)}, \quad (3.2.52)$$

$$\Big[\sum_{j=0}^{\infty} \Big(2^{sj} \|\phi^*_{j,N} f\|_{L^p(\mathbb{R}^n)} \Big)^q \Big]^{1/q} \leq C \|f\|_{B^{p,q}_s(\mathbb{R}^n)}, \quad (3.2.53)$$

provided $N \geq n/\min\{p, q\}$.

It has been long known that many classical smoothness spaces are encompassed by the Besov and Triebel-Lizorkin scales. For example,

$$C^s(\mathbb{R}^n) = B_s^{\infty,\infty}(\mathbb{R}^n), \quad 0 < s \notin \mathbb{Z}, \tag{3.2.54}$$

$$L^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \tag{3.2.55}$$

$$L_s^p(\mathbb{R}^n) = F_s^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \qquad s \in \mathbb{R}, \tag{3.2.56}$$

$$W_k^p(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \qquad k \in \mathbb{N}, \tag{3.2.57}$$

$$h^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), \quad 0 < p \leq 1. \tag{3.2.58}$$

Above, the Sobolev (or Bessel potential) spaces are defined by

$$L_s^p(\mathbb{R}^n) := \Big\{(I-\Delta)^{-s/2} g : \, g \in L^p(\mathbb{R}^n)\Big\}, \quad 1 < p < \infty, \; s \in \mathbb{R}. \tag{3.2.59}$$

With $\mathcal{F}$ denoting the Fourier transform and $\mathcal{F}^{-1}$ denoting its inverse the definition above reads

$$L_s^p(\mathbb{R}^n) := \Big\{\mathcal{F}^{-1}(1+|\xi|^2)^{-s/2}\mathcal{F} g : \, g \in L^p(\mathbb{R}^n)\Big\}, \quad 1 < p < \infty, \; s \in \mathbb{R}, \tag{3.2.60}$$

the norm is given by

$$\|f\|_{L_s^p(\mathbb{R}^n)} := \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F} f\|_{L^p}. \tag{3.2.61}$$

When smoothness is a natural number Sobolev spaces can equivalently be defined by

$$W_k^p(\mathbb{R}^n) := \Big\{f \in L^p(\mathbb{R}^n); \quad \partial^\gamma f \in L^p(\mathbb{R}^n) \; \forall \, \gamma : \, |\gamma| \leq k\Big\}, \quad 1 < p < \infty, \; k \in \mathbb{N}, \tag{3.2.62}$$

the norm being

$$\|f\|_{W^p_k(\mathbb{R}^n)} = \sum_{|\gamma|\leq k} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.63}$$

Also, $C^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$, denotes the Hölder class of functions endowed with the norm

$$\|f\|_{C^s(\mathbb{R}^n)} := \sum_{|\alpha|\leq [s]} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\alpha|=[s]} \sup_{x\neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^{s-[s]}}. \tag{3.2.64}$$

A function $f \in L^2_{\rm loc}(\mathbb{R}^n)$ belongs to the space $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2\,dx\right)^{1/2} < \infty, \tag{3.2.65}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|}\int_Q f(x)\,dx$. Next, we define the space $\mathrm{VMO}(\mathbb{R}^n)$ as the closure of continuous, compactly supported functions in $\mathrm{BMO}(\mathbb{R}^n)$. It is well-known that $(H^1(\mathbb{R}^n))^* = \mathrm{BMO}(\mathbb{R}^n)$ and $(\mathrm{VMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n)$ (see [**43**] and [**25**]).

Further, we define the local versions of the $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{VMO}(\mathbb{R}^n)$ spaces. A function $f \in L^2_{\rm loc}(\mathbb{R}^n)$ belongs to the space $\mathrm{bmo}(\mathbb{R}^n)$ if

$$\begin{aligned}\|f\|_{\mathrm{bmo}(\mathbb{R}^n)} := \sup\Big\{ &\sup_{Q:\, l(Q)\leq 1} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2\,dx\right)^{1/2}, \\ &\sup_{Q:\, l(Q)>1} \left(\frac{1}{|Q|}\int_Q |f(x)|^2\,dx\right)^{1/2}\Big\}\end{aligned} \tag{3.2.66}$$

is finite. Then, $\mathrm{vmo}(\mathbb{R}^n)$ is introduced as the closure of continuous, compactly supported functions in $\mathrm{bmo}(\mathbb{R}^n)$. Once again, $(h^1(\mathbb{R}^n))^* = \mathrm{bmo}(\mathbb{R}^n)$ and $(\mathrm{vmo}(\mathbb{R}^n))^* = h^1(\mathbb{R}^n)$ (see [**54**]). Furthermore, $\mathrm{bmo}(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ are included, respectively, in the inhomogeneous and homogeneous Triebel-Lizorkin scales, corresponding to $F^{\infty,2}_0(\mathbb{R}^n)$ and its homogeneous version.

We now proceed to record some important interpolation results. As before, by $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,q}$ we will denote the complex and real methods of interpolation, respectively. Both are applicable to compatible pairs of quasi-Banach spaces X_0, X_1, though in the case of the complex method an extra hypothesis is required, namely that $X_0 + X_1$ is analytically convex; cf. §3.1 for details.

As far as the real method of interpolation is concerned, we note the following classical result.

Theorem 3.2.9. *(cf.* **[125]***) Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(F^{p,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p,q_1}_{\alpha_1}(\mathbb{R}^n))_{\theta,q} = B^{p,q}_{\alpha}(\mathbb{R}^n), \quad 0 < p < \infty, \tag{3.2.67}$$

$$(B^{p,q_0}_{\alpha_0}(\mathbb{R}^n), B^{p,q_1}_{\alpha_1}(\mathbb{R}^n))_{\theta,q} = B^{p,q}_{\alpha}(\mathbb{R}^n), \quad 0 < p \leq \infty. \tag{3.2.68}$$

Turning to the complex method of interpolation, we have:

Theorem 3.2.10. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$. Then*

$$[F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)]_\theta = F^{p,q}_{\alpha}(\mathbb{R}^n), \tag{3.2.69}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$ then also

$$[B^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n), B^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)]_\theta = B^{p,q}_{\alpha}(\mathbb{R}^n), \tag{3.2.70}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Of course, when $p, q \geq 1$, this is well-known; cf. [**48**], [**124**]. The novelty here is the consideration of the *entire* scale $p, q > 0$. In this form, the result first appeared in [**94**]; nonetheless, in the interest of completeness, we include below a brief argument.

Proof. For a start, for our method of complex interpolation to apply here we need to ensure that

$$F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n) + F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n) \text{ is analytically convex,} \tag{3.2.71}$$

for any set of indices as in the statement of the theorem. To this end, we note that, thanks to Theorem 3.2.7, $F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n)$ and $F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)$ have a common unconditional basis, namely (3.2.45). This is then an unconditional basis in the sum space, $F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n) + F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)$. By virtue of Theorem 3.2.7, the wavelet transform, i.e., the map associating to each distribution its sequence of wavelet coefficients defined as in (3.2.46), is an isomorphism between $F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n) + F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)$ and $f^{p_0,q_0}_{\alpha_0} + f^{p_1,q_1}_{\alpha_1}$, for $\alpha_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$, $i = 0, 1$, (see also [**50**], Theorem 7.20) so that matters can be reduced to working with the latter space. That is, it suffices to show $f^{p_0,q_0}_{\alpha_0} + f^{p_1,q_1}_{\alpha_1}$, regarded as a space of real-valued functions defined on $\{Q \in \mathcal{Q}_n : l(Q) \leq 1\}$, is analytically convex. However, each space $f^{p_i,q_i}_{\alpha_i}$, $i = 0, 1$, is analytically convex (which is visible by applying Theorem 3.1.6 twice) and, hence, the desired conclusion is furnished by Theorem 3.1.5.

In a similar manner, one can prove that

$$\begin{aligned} &F^{p,q}_{\alpha}(\mathbb{R}^n) \text{ is analytically convex for } 0 < p, q \leq \infty, \ \alpha \in \mathbb{R}, \\ &\text{and separable whenever } p + q < +\infty. \end{aligned} \tag{3.2.72}$$

Having disposed of these technical aspects, Theorem 3.1.5 ensures that the intermediate interpolation spaces can be computed using Calderón's formula (3.1.11) plus the fact that

$$f_{\alpha}^{p,q} = \big(f_{\alpha_0}^{p_0,q_0}\big)^{1-\theta}\big(f_{\alpha_1}^{p_1,q_1}\big)^{\theta}, \tag{3.2.73}$$

which is proved in [**48**].

The case of the Besov scale is rather similar. Here we only want to point out that the identity

$$b_{\alpha}^{p,q} = \big(b_{\alpha_0}^{p_0,q_0}\big)^{1-\theta}\big(b_{\alpha_1}^{p_1,q_1}\big)^{\theta}, \qquad \alpha_0 \neq \alpha_1, \tag{3.2.74}$$

has been established in [**94**]. □

As far as vector-valued L^p spaces are concerned, we state the following result. Assume that A_0 and A_1 are Banach spaces and that $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$. Then

$$[L^{p_0}(A_0), L^{p_1}(A_1)]_{\theta} = L^{p^*}([A_0, A_1]_{\theta}), \tag{3.2.75}$$

where $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Here by $L^p(A) = L^p(\partial\Omega; A)$ we denote the space of all strongly measurable functions f such that

$$\int_{\partial\Omega} \|f(x)\|_A^p \, d\sigma(x) < +\infty, \tag{3.2.76}$$

for $1 \leq p < \infty$.

For $0 \leq \delta, \rho \leq 1$, $m \in \mathbb{R}$, let $S_{\rho,\delta}^m$ be the class of symbols consisting of all functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for each pair of multi-indices β, γ there

exists a constant $C_{\beta,\gamma}$ such that

$$|D_\xi^\beta D_x^\gamma p(x,\xi)| \leq C_{\beta\gamma}(1+|\xi|)^{m-\rho|\beta|+\delta|\gamma|}, \tag{3.2.77}$$

uniformly for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. For $p \in S^m_{\rho,\delta}$ we define the pseudodifferential operator $p(x,D)$ by

$$p(x,D)f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\mathcal{F}f(\xi)\, d\xi, \qquad f \in S(\mathbb{R}^n), \tag{3.2.78}$$

and write $p(x,D) \in OPS^m_{\rho,\delta}$.

The following is a consequence of Theorem 6.2.2 on p. 258 of [**126**] (cf. also Remark. 3 on p. 257 of [**126**]). A similar result is also proved in [**123**] under more restrictive conditions on the indices involved.

Theorem 3.2.11. *Let $m \in \mathbb{R}$, $0 \leq \delta < 1$ and fix $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, arbitrary. Then any $T \in OPS^m_{1,\delta}$ induces a bounded, linear operator*

$$T : F^{p,q}_\alpha(\mathbb{R}^n) \longrightarrow F^{p,q}_{\alpha-m}(\mathbb{R}^n) \tag{3.2.79}$$

whenever $0 < p < \infty$. Moreover,

$$T : B^{p,q}_\alpha(\mathbb{R}^n) \longrightarrow B^{p,q}_{\alpha-m}(\mathbb{R}^n) \tag{3.2.80}$$

boundedly, whenever $0 < p \leq \infty$.

A useful consequence of the above result, frequently used in the sequel to treat operators akin the Newtonian potential operator defined in (2.3.8), is recorded separately.

Corollary 3.2.12. *Assume that $m \in \mathbb{R}$, $m > -n$, and*

$$Tf(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(\xi) \mathcal{F}f(\xi)\, d\xi, \qquad f \in S(\mathbb{R}^n), \tag{3.2.81}$$

where, for each $\gamma \in \mathbb{N}_o$,

$$|(\partial^\gamma a)(\xi)| \leq C_\gamma |\xi|^{m-|\gamma|}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.2.82}$$

Fix $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ (viewed below as multiplication operators). Then

$$\phi T \psi : F_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow F_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{3.2.83}$$

is a bounded operator whenever $0 < p < \infty$. In fact, so is

$$\phi T \psi : B_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow B_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{3.2.84}$$

if $0 < p \leq \infty$.

Proof. Fix $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and note that $u := T(\psi f)$ is a tempered distribution whenever $f \in F_\alpha^{p,q}(\mathbb{R}^n)$. Indeed, $\langle u, g\rangle = (-1)^\mu \langle f, \psi T g\rangle$ and $\psi T g \in C_c^\infty(\mathbb{R}^n)$ for each $g \in \mathcal{S}(\mathbb{R}^n)$. Next, consider some $\theta \in S(\mathbb{R}^n)$ which is identically equal to 1 in a neighborhood of the origin and set $\eta := \mathcal{F}^{-1}\theta \in \mathcal{S}(\mathbb{R}^n)$. Then one can write

$$\mathcal{F}(u) = \theta(\xi) a(\xi) \mathcal{F}(\psi f) + (1 - \theta(\xi)) a(\xi) \mathcal{F}(\psi f) \tag{3.2.85}$$

and, hence,

$$\phi u = \phi\, (u * \eta) + \phi\, \mathcal{F}^{-1}\left((1 - \theta(\xi)) a(\xi) \mathcal{F}(\psi f)\right). \tag{3.2.86}$$

Since $\mathcal{S}'(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^n)$, the first term in the right-hand side of the above identity belongs to $C_c^\infty(\mathbb{R}^n)$. As for the second one, the membership to $F_{\alpha-m}^{p,q}(\mathbb{R}^n)$

follows from Theorem 3.2.11 as soon as we observe that $p(x,\xi) := (1-\theta(\xi))a(\xi) \in S^m_{1,0}$. This justifies the claim made about (3.2.83). Finally, (3.2.84) is proved in a similar manner. □

3.3 Decomposition of spaces on $\mathbb{R}^n$

Some powerful tools in the study of the spaces $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ are the so-called atomic decompositions. Here we review some of the main results from **[47]** and **[49]** which, in turn, build on the work of many other people.

Turning to specifics, given $s \in \mathbb{R}$, $0 < p \leq \infty$, and $Q \in \mathcal{Q}_n$, $l(Q) \leq 1$, call a function a_Q an (s,p)-atom if the following properties hold:

$$\text{(1) } \operatorname{supp}(a_Q) \subseteq 3Q, \tag{3.3.1}$$

$$\text{(2) } \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{s/n-1/p-|\gamma|/n} \text{ if } |\gamma| \leq K, \tag{3.3.2}$$

$$\text{(3) } \int_{\mathbb{R}^n} x^\gamma a_Q(x)\,dx = 0 \text{ for each } |\gamma| \leq L, \text{ whenever } |Q| < 1. \tag{3.3.3}$$

Here $K \geq ([s]+1)_+$ and $L \geq \max\{[J-n-s], -1\}$ are fixed integers, whereas the value of J is to be specified in each specific occurrence. With these conventions, the following holds.

Theorem 3.3.1. *(cf.* **[47]**, **[49]**) *Fix $s \in \mathbb{R}$, $0 < p,q \leq \infty$. Then for every $f \in B^{p,q}_s(\mathbb{R}^n)$ there exist a family of (s,p)-atoms $\{a_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ with $J := \frac{n}{\min\{1,p\}}$, and a sequence of numbers $\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ with the property that*

$$\|f\|_{B^{p,q}_s(\mathbb{R}^n)} \approx \left(\sum_{j=0}^{\infty} \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)=2^{-j}}} |\lambda_Q|^p \Big)^{q/p} \right)^{1/q}, \tag{3.3.4}$$

and such that f can be represented in the form

$$f = \sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}} \lambda_Q a_Q \quad \textit{with convergence in} \quad S'(\mathbb{R}^n). \tag{3.3.5}$$

Furthermore, there exists $C = C(s,p,n) > 0$ such that for any numerical sequence $\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ and any family of (s,p)-atoms $\{a_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ (with $J := \frac{n}{\min\{1,p\}}$), there holds

$$\Big\| \sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}} \lambda_Q a_Q \Big\|_{B^{p,q}_s(\mathbb{R}^n)} \leq C \left(\sum_{j=0}^{\infty} \Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)=2^{-j}}} |\lambda_Q|^p \Big)^{q/p} \right)^{1/q}. \tag{3.3.6}$$

A similar statement holds for Triebel-Lizorkin spaces granted that, this time, $J := \frac{n}{\min\{1,p,q\}}$ and $p < \infty$. More concretely, for any $f \in F^{p,q}_s(\mathbb{R}^n)$, (3.3.5) holds for some family of (s,p)-atoms $\{a_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ with $J := \frac{n}{\min\{1,p,q\}}$, and numerical sequence $\{\lambda_Q\}_{Q\in\mathcal{Q}_n,\, l(Q)\leq 1}$ satisfying

$$\|f\|_{F^{p,q}_s(\mathbb{R}^n)} \approx \left\| \left(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}} \Big(|Q|^{-1/p} |\lambda_Q| \chi_Q \Big)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{3.3.7}$$

Also, there exists $C = C(s,p,n) > 0$ such that

$$\Big\| \sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}} \lambda_Q a_Q \Big\|_{F^{p,q}_s(\mathbb{R}^n)} \leq C \left\| \left(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 1}} \Big(|Q|^{-1/p} |\lambda_Q| \chi_Q \Big)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{3.3.8}$$

for any numerical sequence $\{\lambda_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ and any family of (s,p)-atoms $\{a_Q\}_{Q\in\mathcal{Q}_n, l(Q)\leq 1}$ with $J := \frac{n}{\min\{1,p,q\}}$.

Following **[49]**, for $s \in \mathbb{R}$, $0 < p < p_1 < +\infty$, $0 < q \leq \infty$ we say that $r = \{r_Q\}_{Q\in\mathcal{Q}_n : l(Q)\leq 1}$ is a *p_1-atom* for $f^{p,q}_s$ if there exists $\bar{Q} \in \mathcal{Q}_n$ with $l(\bar{Q}) \leq 1$ such

that $r_Q = 0$ whenever Q is not included in $\bar{Q}$ and $\|r\|_{f_\alpha^{p_1,q}} \leq |\bar{Q}|^{1/p_1 - 1/p}$. We call the cube $\bar{Q}$ the *support* of the p_1-atom r and denote this as $\operatorname{supp} r = \bar{Q}$.

Suppose now that $\alpha \in \mathbb{R}$, $0 < p \leq 1$, $p \leq q \leq +\infty$ and $p < p_1 < +\infty$. Then (cf. [**49**], Theorem 7.2 and Remark 7.3, pp. 88-90),

$$\|s\|_{f_\alpha^{p,q}} \approx \inf \left\{ \Big(\sum_{k \in \mathbb{Z}} |\mu_k|^p \Big)^{1/p} : \, s = \sum_{k \in \mathbb{Z}} \mu_k r_k, \ r_k\text{'s are } p_1\text{-atoms for } f_\alpha^{p,q} \right\}. \quad (3.3.9)$$

Below, we present an alternative version of the atomic decomposition for Triebel-Lizorkin spaces. Given $s \in \mathbb{R}$, consider the family of functions $\{a_Q\}_{Q \in \mathcal{Q}_n, \, l(Q) \leq 1}$ satisfying:

$$(1) \ \operatorname{supp}(a_Q) \subseteq 3Q, \quad (3.3.10)$$

$$(2) \ \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-1/2 - |\gamma|/n} \ \text{ if } \ |\gamma| \leq K, \quad (3.3.11)$$

$$(3) \int_{\mathbb{R}^n} x^\gamma a_Q(x) \, dx = 0 \ \text{ if } \ |Q| < 1 \text{ and } |\gamma| \leq L. \quad (3.3.12)$$

Here $K \geq ([s] + 1)_+$ and $L \geq \max\{[J - n - s], -1\}$ with J to be specified later. In fact, $\{a_Q\}_{Q \in \mathcal{Q}_n, \, l(Q) \leq 1}$ is a collection of renormalized (s, p)-atoms.

Suppose now that $s \in \mathbb{R}$, $0 < p \leq 1$ and $p \leq q \leq \infty$, $p < p_1 < +\infty$. Consider the family of functions $\{A_Q\}_{Q \in \mathcal{Q}_n : \, l(Q) \leq 1}$ (depending on parameters s, p, q, p_1) such that

A_Q satisfies (3.3.10) $-$ (3.3.11) when $|Q| = 1$, and

$$A_Q = \sum_{\substack{Q' \in \mathcal{Q}_n \\ Q' \subset Q}} r_{Q'} a_{Q'}, \ r = \{r_{Q'}\}_{Q' \in \mathcal{Q}_n : \, l(Q') \leq 1} \text{ is a } p_1\text{-atom for } f_s^{p,q} \text{ with } \operatorname{supp} r = \bar{Q} \quad (3.3.13)$$

and $a_{Q'}$ satisfy (3.3.10) $-$ (3.3.12) with $J = \frac{n}{p}$, when $|Q| < 1$.

Theorem 3.3.2. *(cf.* **[49]***, Theorem 7.4) Let $s \in \mathbb{R}$, $0 < p \leq 1$ and $p \leq q \leq \infty$, $p < p_1 < +\infty$. Then*

$$\|f\|_{F_s^{p,q}(\mathbb{R}^n)} \approx \inf\Big\{ \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)\leq 1}} |\lambda_Q|^p \Big)^{1/p} : \\ f = \sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)\leq 1}} \lambda_Q A_Q \ \text{ and each } A_Q \text{ satisfies (3.3.13)} \Big\}. \tag{3.3.14}$$

We now discuss localization properties and the invariance of Besov and Triebel-Lizorkin spaces under Lipschitz diffeomorphisms.

Theorem 3.3.3. *Assume that*

$$0 < p, q \leq \infty, \qquad n\Big(\frac{1}{p} - 1\Big)_+ < s < 1, \tag{3.3.15}$$

and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function with Lipschitz components, mapping $\mathbb{R}^n$ onto itself in a one-to-one fashion, and such that

$$|\Phi(x) - \Phi(y)| \approx |x - y|, \ \text{uniformly for } x, y \in \mathbb{R}^n \text{ with } |x - y| \leq 1. \tag{3.3.16}$$

Finally, denote by J_Φ the Jacobian of Φ. Then the applications

$$B_s^{p,q}(\mathbb{R}^n) \ni f \mapsto f \circ \Phi \in B_s^{p,q}(\mathbb{R}^n), \tag{3.3.17}$$

$$B_{s-1}^{p,q}(\mathbb{R}^n) \ni f \mapsto (f \circ \Phi)|J_\Phi| \in B_{s-1}^{p,q}(\mathbb{R}^n), \tag{3.3.18}$$

are isomorphisms.

Furthermore, if $\Psi \in L^\infty(\mathbb{R}^n)$ is also Lipschitz then the mappings

$$B_s^{p,q}(\mathbb{R}^n) \ni f \mapsto \Psi\, f \in B_s^{p,q}(\mathbb{R}^n), \tag{3.3.19}$$

$$B_{s-1}^{p,q}(\mathbb{R}^n) \ni f \mapsto \Psi\, f \in B_{s-1}^{p,q}(\mathbb{R}^n), \tag{3.3.20}$$

are well-defined and bounded.

The same type of results holds in the case of Triebel-Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$ *if instead of (3.3.15) we assume this time*

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad n\Big(\frac{1}{\min\{p,q\}} - 1\Big)_+ < s < 1, \tag{3.3.21}$$

or

$$0 < p < \infty, \quad q = 2, \quad n\Big(\frac{1}{p} - 1\Big)_+ < s \leq 1. \tag{3.3.22}$$

Proof. The claim about the operators (3.3.17)-(3.3.18) follows from Theorem 3.3.1 and the invariance – under these transformations – of the class of atoms for the spaces at hand, up to a fixed multiplicative constant. Let us note that a similar reasoning for (3.3.17) is employed in Theorem 6.5 of [**127**]. In fact, the same approach is easily implemented in the case of (3.3.19).

In the case of (3.3.20) let a_Q be an arbitrary $(s-1,p)$-atom, where s, p, q are as in (3.3.15). Thus,

$$\operatorname{supp}(a_Q) \subseteq 3Q, \quad \|a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{\frac{s-1}{n}-\frac{1}{p}}, \text{ and } \int_{\mathbb{R}^n} a_Q(x)\,dx = 0 \text{ if } l(Q) < 1. \tag{3.3.23}$$

Now, with x_Q denoting the center of the cube Q, we split

$$a_Q\Psi = a_Q^1 + a_Q^2 := \Psi(x_Q)a_Q + [\Psi - \Psi(x_Q)]a_Q. \tag{3.3.24}$$

It is then immediate that a_Q^1 is, up to an unimportant multiplicative constant, an $(s-1,p)$-atom, and that $\|a_Q^2\|_{L^\infty(\mathbb{R}^n)} \leq C(\Psi)\,|Q|^{\frac{s}{n}-\frac{1}{p}}$, $\operatorname{supp} a_Q^2 \subseteq 3Q$. In particular,

$\|a_Q^2\|_{L^{p^*}(\mathbb{R}^n)} \leq C(\Psi)$ provided $\frac{1}{p^*} = \frac{1}{p} - \frac{s}{n}$ which, by (3.3.15), guarantees that $p^* \in (1,\infty)$. Now, for this choice of p^*, we may write the sequence of embeddings

$$L^{p^*}(\mathbb{R}^n) = F_0^{p^*,2}(\mathbb{R}^n) \hookrightarrow B_s^{p,\infty}(\mathbb{R}^n) \hookrightarrow B_{s-1}^{p,q}(\mathbb{R}^n) \tag{3.3.25}$$

where the last one is elementary and non-sharp (cf. pp. 30-31 in [**112**] for a discussion). From this, the claim about the boundedness of (3.3.20) easily follows.

The case of the Triebel-Lizorkin scale corresponding to indices as in (3.3.21) is handled in a similar manner. Hence, it remains to consider the situation when $q = 2$ and $s = 1$. The case when $q = 2$, $s = 1$ and $1 < p < \infty$ is immediate from the identifications $F_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $F_1^{p,2}(\mathbb{R}^n) = L_1^p(\mathbb{R}^n)$. Finally, for the case when $\frac{n}{n+1} < p \leq 1$, we invoke (3.2.8).

Going further, the boundedness of the map $F_1^{p,2}(\mathbb{R}^n) \ni f \mapsto f \circ \Phi \in F_1^{p,2}(\mathbb{R}^n)$ follows from (3.2.8), the atomic characterization of the space $h_1^p(\mathbb{R}^n)$ in Theorem 3.2.2, and the observation that the type of atoms under consideration (i.e., functions satisfying (3.2.13)) is invariant – up to a fixed multiplicative constant – under composition with bi-Lipschitz homeomorphisms of $\mathbb{R}^n$. Next, the boundedness of the map $F_0^{p,2}(\mathbb{R}^n) \ni f \mapsto (f \circ \Phi)|J_\Phi| \in F_0^{p,2}(\mathbb{R}^n)$ is a consequence of the fact that $F_0^{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p \leq 1$ and (3.2.9)-(3.2.10). This takes care of the analogue of (3.3.17)-(3.3.18) for the Triebel-Lizorkin scale when $q = 2$, $s = 1$ and $\frac{n}{n+1} < p \leq 1$.

At this stage, we are left proving the analogues of (3.3.19)-(3.3.20) for the Triebel-Lizorkin scale when $q = 2$, $s = 1$ and $\frac{n}{n+1} < p \leq 1$. These, however, are proved in a fashion similar to their Besov counterparts, by working with the Hardy-

type spaces $h^p_1(\mathbb{R}^n)$, $h^p(\mathbb{R}^n)$, instead and by relying on their respective atomic theories. $\square$

We are now in a position to present the

Proof of Theorem 3.2.1. The inclusion $h^p_k(\mathbb{R}^n) \hookrightarrow F^{p,2}_k(\mathbb{R}^n)$ is immediate from the definitions (3.2.6)-(3.2.7), the identity $h^p(\mathbb{R}^n) = F^{p,2}_0(\mathbb{R}^n)$, valid for $0 < p \leq 1$, and Theorem 3.2.4. In fact, the converse inclusion - at least if $k \in \mathbb{N}$ - follows in a similar manner. On the other hand, if $u \in F^{p,2}_k(\mathbb{R}^n)$ with $k < 0$ then $u_\gamma := \partial^\gamma (I - \Delta)^{-k} u \in F^{p,2}_0(\mathbb{R}^n) = h^p(\mathbb{R}^n)$ whenever $|\gamma| \leq k$ and u belongs to the linear span of $\{\partial^\gamma u_\gamma : \gamma \in \mathbb{N}^n_o, |\gamma| \leq k\}$. Thus, in either scenario, $F^{p,2}_k(\mathbb{R}^n) \hookrightarrow h^p_k(\mathbb{R}^n)$. $\square$

We continue with the

Proof of Theorem 3.2.2. We start by recalling the identification (3.2.8). Consequently, by Theorem 3.3.2, it is enough to check that every function A_Q, $Q \in \mathcal{Q}_n$, associated with $s = 1$, $q = 2$, $\frac{n}{n+1} < p \leq 1 < p_1 \leq \infty$, is also a fixed multiple of an L^{p_1}-atom for $h^p_1(\mathbb{R}^n)$. To this end, note that $\operatorname{supp} A_Q \subseteq 3Q$ for any $Q \in \mathcal{Q}_n$. Also, so we claim,

$$\|\nabla A_Q\|_{L^{p_1}(\mathbb{R}^n)} \leq C \|A_Q\|_{F^{p_1,2}_1(\mathbb{R}^n)} \leq C \|r\|_{f^{p_1,2}_1} \leq C|Q|^{\frac{1}{p_1} - \frac{1}{p}}, \quad \text{if } |Q| < 1, \tag{3.3.26}$$

where the sequence r is associated with A_Q as in (3.3.13). Indeed, the first inequality above follows from the identification $F^{p_1,2}_1(\mathbb{R}^n) = L^{p_1}_1(\mathbb{R}^n)$, the second one by (inhomogeneous space version of) Theorem 4.1, p. 60 in [**49**], and the last one by the properties of A_Q given in (3.3.13). Further, the inequality

$$\|\nabla A_Q\|_{L^{p_1}(\mathbb{R}^n)} \leq C \quad \text{with } |Q| = 1, \tag{3.3.27}$$

directly follows from the definition (3.3.13) and conditions (3.3.10)-(3.3.11).

The estimates (3.3.26)-(3.3.27) prove the right-to-left inclusion in (3.2.14). As for the opposite inclusion, we only need to observe that

$$f \text{ in the space in the right-hand side of (3.2.14)} \Rightarrow \nabla f \in h^p(\mathbb{R}^n), \qquad (3.3.28)$$

plus a natural norm estimate. This, in turn, can be check based on (3.2.10) and working with individual atoms. This concludes the proof of Theorem 3.2.2.

3.4 Besov spaces on boundaries of Lipschitz domains

There are several ways to define Besov spaces on the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$, denoted in the sequel by $B_s^{p,q}(\partial\Omega)$. One point of view is to regard $\partial\Omega$ (equipped with the intrinsic surface measure and the Euclidean distance inherited from $\mathbb{R}^n$) as a space of homogeneous type in the sense of R. Coifman and G. Weiss (cf. **[25]**). In this latter context, a theory of Besov (and Triebel-Lizorkin) spaces has been developed in **[60]**, **[61]**, **[63]**, see also references therein.

On the other hand, the scale of Besov spaces can be naturally transported from $\mathbb{R}^{n-1}$ to the boundary of a Lipschitz domain Ω via pull-back and a partition of unity. For us, this approach (which we shall adopt) has the advantage of allowing one to carry over, in a fairly direct, straightforward fashion, a great many results already proved in the Euclidean context. These include: duality and embedding results, intrinsic descriptions, atomic and molecular decompositions, as well as interpolation theorems.

Turning to specifics, whenever

$$0 < p, q \leq \infty, \qquad (n-1)\Big(\frac{1}{p} - 1\Big)_+ < s < 1 \tag{3.4.1}$$

and Ω is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, we define $B^{p,q}_s(\partial\Omega)$ as the space of locally integrable functions f for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto f(x', \varphi(x'))$ belongs to $B^{p,q}_s(\mathbb{R}^{n-1})$. Regarding Besov spaces with a negative amount of smoothness, if Ω is the domain in $\mathbb{R}^n$ above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, we agree that

$$f \in B^{p,q}_{s-1}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B^{p,q}_{s-1}(\mathbb{R}^{n-1}), \tag{3.4.2}$$

whenever $0 < p, q \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$.

These definitions then readily extend to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. The details are as follows. Recall the concept of bounded Lipschitz domain given in §2.2, along with family of cylinders $\{Z_i\}_{1\leq i\leq m}$, the family of rigid motions $\{\psi_i\}_{1\leq i\leq m}$ and the Lipschitz functions φ_i, $i = 1, 2, ..., m$ (which retain the original significance). Also, let $\{\Psi_i\}_{i=1}^m$ be a family of functions such that $\Psi_i \in C^\infty_c(\mathbb{R}^n)$, $0 \leq \Psi_i \leq 1$, $\operatorname{supp}\Psi_i \subset \psi_i^{-1}(Z_i)$ and $\sum_{i=1}^m \Psi_i = 1$ in a neighborhood of $\partial\Omega$. Then for $0 < p, q \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$ we define the space $B^{p,q}_s(\partial\Omega)$ as the collection of all locally integrable functions on $\partial\Omega$ such that

$$(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot))) \in B^{p,q}_s(\mathbb{R}^{n-1}), \quad \text{for every } i = 1, ..., m, \tag{3.4.3}$$

endowed with the norm

$$\|f\|_{B^{p,q}_s(\partial\Omega)} := \left(\sum_{i=1}^m \|(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\|^p_{B^{p,q}_s(\mathbb{R}^{n-1})}\right)^{1/p}, \tag{3.4.4}$$

with the usual convention when $p=\infty$. The space (3.4.3)-(3.4.4) is a quasi-Banach space and different choices of the particular partition of unity used in the definition lead to equivalent quasi-norms. The proof relies on the invariance of $B^{p,q}_s(\mathbb{R}^{n-1})$ (for the type of indices considered here) under Lipschitz diffeomorphisms and pointwise multiplication by Lipschitz functions – cf. Theorem 3.3.3. In fact, it is this particular result which is the source of the specific limitations on the indices p, q, s enforced throughout this section. The case when $p=q=\infty$ corresponds to the usual (non-homogeneous) Hölder spaces $C^s(\partial\Omega)$, defined by the requirement that

$$\|f\|_{C^s(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sup_{\substack{x\neq y\\ x,y\in\partial\Omega}} \frac{|f(x)-f(y)|}{|x-y|^s} < +\infty. \tag{3.4.5}$$

Retaining the notation introduced above we define $B^{p,q}_{s-1}(\partial\Omega)$ as the collection of all $f\in(\mathrm{Lip}\,(\partial\Omega))'$ such that

$$(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\in B^{p,q}_{s-1}(\mathbb{R}^{n-1}),\quad \text{for every } i=1,...,m, \tag{3.4.6}$$

which we equip with the quasi-norm

$$\|f\|_{B^{p,q}_{s-1}(\partial\Omega)} := \left(\sum_{i=1}^m \|(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\|^p_{B^{p,q}_{s-1}(\mathbb{R}^{n-1})}\right)^{1/p}. \tag{3.4.7}$$

The sense in which (3.4.6) should be interpreted is as follows. For each $1\leq i\leq m$, select $\Phi_i\in C^\infty_c(\mathbb{R}^{n-1})$, $0\leq\Phi_i\leq 1$, which is identically one in a small, open neighborhood $\mathcal{O}_i$ of $\{x'\in\mathbb{R}^{n-1}:\ \psi_i^{-1}(x',\varphi_i(x'))\in\mathrm{supp}\,\Psi_i\}$ and which vanishes outside a slightly larger open set $\widetilde{\mathcal{O}}_i$. For each $1\leq i\leq m$ there exists $F_i\in B^{p,q}_{s-1}(\mathbb{R}^{n-1})$ such that the "change of variables" formula

$$\langle f,\Psi_i\xi\rangle = \langle F_i,(\Phi_i\xi)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\rangle \tag{3.4.8}$$

holds for each $\xi \in \mathrm{Lip}(\partial\Omega)$ (the duality pairings above are those between $(\mathrm{Lip}(\partial\Omega))'$ and $\mathrm{Lip}(\partial\Omega)$, and between $B^{p,q}_{s-1}(\mathbb{R}^{n-1})$ and $(B^{p,q}_{s-1}(\mathbb{R}^{n-1}))^* \hookleftarrow \mathrm{Lip}_c(\mathbb{R}^{n-1}) := \{f \in \mathrm{Lip}\,(\mathbb{R}^{n-1}) : \text{supp}\, f \text{ compact}\}$, respectively). Then we set $(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2} := F_i$ for each $i = 1, ..., m$.

Assume next that $\Psi \in C^\infty_c(\mathbb{R}^n)$ is such that $\text{supp}\,\Psi \cap \partial\Omega$ is a relatively compact subset of $\psi^{-1}[\text{graph}\,\varphi] \cap \partial\Omega$, where ψ is an isometry of $\mathbb{R}^n$ and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. We claim that

$$f \in B^{p,q}_{s-1}(\partial\Omega) \Longrightarrow (\Psi f)(\psi^{-1}(\cdot, \varphi(\cdot)))\sqrt{1+|\nabla\varphi(\cdot)|^2} \in B^{p,q}_{s-1}(\mathbb{R}^{n-1}), \tag{3.4.9}$$

where a similar convention as in (3.4.6) applies. Indeed, with the help of (3.4.8), one can check that the expression in the right hand-side of (3.4.9) matches, in the sense of distributions in $\mathbb{R}^{n-1}$, $\sum_{i=1}^m (\Psi\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}$. Since for $f \in B^{p,q}_{s-1}(\partial\Omega)$ this belongs to $B^{p,q}_{s-1}(\mathbb{R}^{n-1})$, the claim follows. In particular, it follows from (3.4.9) that for $0 < p, q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $B^{p,q}_{s-1}(\partial\Omega)$ is a quasi-Banach space, with different partitions of unity and Lipschitz graphs used in the definition yielding equivalent quasi-norms.

It is then natural to expect the following characterization of the diagonal Besov scale on the boundary of a Lipschitz domain.

Proposition 3.4.1. *For $(n-1)/n < p < \infty$ and $(n-1)(1/p-1)_+ < s < 1$,*

$$\|f\|_{B^{p,p}_s(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n-1+sp}}\, d\sigma_x d\sigma_y\right)^{1/p}. \tag{3.4.10}$$

Proof. This follows directly from definitions and (3.2.27). □

We next record the version of Theorem 3.2.5 for the spaces we are currently considering.

Proposition 3.4.2. *Let Ω be a bounded Lipschitz domain and assume that $(n-1)/n < p_0 \leq p_1 \leq +\infty$, $(n-1)/n < q_0 \leq q_1 \leq +\infty$ $(n-1)(1/p_i - 1)_+ < s_i < 1$, $i = 0, 1$. Then, if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, the inclusions*

$$B_{s_0}^{p_0,q_0}(\partial\Omega) \hookrightarrow B_{s_1}^{p_1,q_1}(\partial\Omega), \qquad B_{s_0-1}^{p_0,q_0}(\partial\Omega) \hookrightarrow B_{s_1-1}^{p_1,q_1}(\partial\Omega), \tag{3.4.11}$$

are continuous with dense range.

Proof. This is a direct consequences of Theorem 3.2.5 and definitions. □

To state our next result, recall the definition of the conjugate exponent p' of a number $0 < p \leq \infty$.

Proposition 3.4.3. *For $0 < p, q < \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$,*

$$(B_{s-1}^{p,q}(\partial\Omega))^* = B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega), \qquad (B_s^{p,q}(\partial\Omega))^* = B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega), \tag{3.4.12}$$

where the pairing between f in a Besov space and g in its dual is a natural extension of the bilinear form $f, g \mapsto \int_{\partial\Omega} f g \, d\sigma$.

Proof. Let $\{\mathcal{O}_i\}_{i\in I}$ be a finite open cover of $\partial\Omega$ with sets which are pieces of graphs of Lipschitz functions (appropriately translated and rotated). Also, select $\Psi_i \in C_c^\infty(\mathbb{R}^n)$ such that $\sum_i \Psi_i = 1$ near $\partial\Omega$ and such that $\partial\Omega \cap \operatorname{supp}\Psi_i \subset \mathcal{O}_i$ for each $i \in I$.

If $f \in B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega)$, then (3.4.8) plus the fact that $B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+}(\mathbb{R}^{n-1}) = (B_{s-1}^{p,q}(\mathbb{R}^{n-1}))^*$ guarantee that $\Psi_i f \in (B_{s-1}^{p,q}(\partial\Omega))^*$ for each $i \in I$. This observation allows us to define the operator $\mathcal{A} : B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega) \to (B_{s-1}^{p,q}(\partial\Omega))^*$ by set-

ting $\mathcal{A}(f) := \sum_{i\in I} \Psi_i f$. This definition is independent of the partition of unity chosen at the beginning of the proof.

An inverse for this operator is constructed as follows. By once again changing variables and relying on the well-known Euclidean counterpart of (3.4.12), we eventually arrive at the conclusion that $\Lambda|_{\mathcal{O}_i} \in B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+,\mathrm{loc}}(\partial\Omega)$ for each $i \in I$, whenever $\Lambda \in (B^{p,q}_{s-1}(\partial\Omega))^*$. Accordingly, we define $\mathcal{B} : (B^{p,q}_{s-1}(\partial\Omega))^* \to B^{p',q'}_{1-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega)$ by setting $\mathcal{B}(\Lambda) := \sum_{i\in I} \Psi_i(\Lambda|_{\mathcal{O}_i})$. Once again, the definition is independent of the particular open cover and partition of unity involved. It is then routine to check that $\mathcal{A}\circ\mathcal{B} = I$, $\mathcal{B}\circ\mathcal{A} = I$ which proves the first identity in (3.4.12).

The second identity in (3.4.12) is proved in a similar manner. More specifically, assume that $f \in B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega)$. Then formula (3.4.8) written as

$$\langle g, \Psi_i f\rangle = \langle (\Psi_i g)(\psi_i^{-1}(\cdot,\varphi_i(\cdot))), (\Phi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\rangle, \quad (3.4.13)$$

and duality relation $(B^{p,q}_s(\mathbb{R}^{n-1}))^* = B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+}(\mathbb{R}^{n-1})$ assure that $\Psi_i f \in (B^{p,q}_s(\partial\Omega))^*$, $i \in I$, and hence $\mathcal{A}$ defined by $\mathcal{A}(f) := \sum_{i\in I} \Psi_i f$ is bounded operator from $B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega)$ to $(B^{p,q}_s(\partial\Omega))^*$.

Conversely, if $\Lambda \in (B^{p,q}_s(\partial\Omega))^*$ then by (3.4.13) and standard duality results in $\mathbb{R}^{n-1}$ we can show that $\Lambda|_{\mathcal{O}_i} \in B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+,\mathrm{loc}}(\partial\Omega)$ in the sense of restrictions of distributions. Then $\mathcal{B} : (B^{p,q}_s(\partial\Omega))^* \to B^{p',q'}_{-s+(n-1)(\frac{1}{p}-1)_+}(\partial\Omega)$, with $\mathcal{B}$ defined as above. As in the previous case, the observation that $\mathcal{A}\circ\mathcal{B} = I$ and $\mathcal{B}\circ\mathcal{A} = I$ finishes the proof. □

Proposition 3.4.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Also, assume*

that $0 < p, q, q_0, q_1 \leq \infty$ *and that one of the following two conditions is satisfied:*

(1) $(n-1)(1/p-1)_+ < \alpha_0 \neq \alpha_1 < 1$*;*

(2) $-1 + (n-1)(1/p-1)_+ < \alpha_0 \neq \alpha_1 < 0$*.*

Then, with $0 < \theta < 1$*,* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$*,*

$$(B^{p,q_0}_{\alpha_0}(\partial\Omega), B^{p,q_1}_{\alpha_1}(\partial\Omega))_{\theta,q} = B^{p,q}_{\alpha}(\partial\Omega). \tag{3.4.14}$$

Furthermore, if $0 < p_i, q_i \leq \infty$*,* $\alpha_0 \neq \alpha_1$*, and either one of the following two conditions*

(1') $(n-1)(1/p_i-1)_+ < \alpha_i < 1$*,* $i = 0, 1$*;*

(2') $-1 + (n-1)(1/p_i-1)_+ < \alpha_i < 0$*,* $i = 0, 1$*,*

is satisfied then

$$[B^{p_0,q_0}_{\alpha_0}(\partial\Omega), B^{p_1,q_1}_{\alpha_1}(\partial\Omega)]_\theta = B^{p,q}_{\alpha}(\partial\Omega), \tag{3.4.15}$$

where $0 < \theta < 1$*,* $\alpha := (1-\theta)\alpha_0 + \theta\alpha_1$*,* $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$*.*

Proof. For any triplet of indices s, p, q such that $0 < p, q \leq \infty$ and $(n-1)(1/p-1)_+ < s < 1$, our definitions of the Besov spaces ensure that the linear operators

$$\begin{aligned} &E_+ : B^{p,q}_s(\partial\Omega) \longrightarrow \oplus_{1\leq i\leq m} B^{p,q}_s(\mathbb{R}^{n-1}), \\ &(E_+ f)_i := (\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot))) \text{ on } \mathbb{R}^{n-1},\ 1 \leq i \leq m, \end{aligned} \tag{3.4.16}$$

and

$$\begin{aligned} &E_- : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow \oplus_{1\leq i\leq m} B^{p,q}_{s-1}(\mathbb{R}^{n-1}), \\ &(E_- f)_i := (\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2} \text{ on } \mathbb{R}^{n-1},\ 1 \leq i \leq m, \end{aligned} \tag{3.4.17}$$

are well-defined and bounded. These operators have left-inverses described as follows. Let $\Phi_i \in C^\infty_c(\mathbb{R}^n)$, $\operatorname{supp}\Phi_i \subset \psi^{-1}(t_i Z_i)$, $\Phi_i \equiv 1$ in a neighborhood of $\operatorname{supp}\Psi_i$.

Also, denote by $\pi'(x', x_n) := x'$ the Cartesian projection on the first $n-1$ coordinates in $\mathbb{R}^n$. If we now set

$$\begin{aligned} &R_+ : \oplus_{1\leq i\leq m} B_s^{p,q}(\mathbb{R}^{n-1}) \longrightarrow B_s^{p,q}(\partial\Omega) \\ &R_+\big((g)_{1\leq i\leq m}\big) := \textstyle\sum_{i=1}^m \Phi_i(\cdot)\, g_i(\pi'\circ\psi_i(\cdot)) \text{ on } \partial\Omega, \end{aligned} \tag{3.4.18}$$

and

$$\begin{aligned} &R_- : \oplus_{1\leq i\leq m} B_{s-1}^{p,q}(\mathbb{R}^{n-1}) \longrightarrow B_{s-1}^{p,q}(\partial\Omega) \\ &R_-\big((g)_{1\leq i\leq m}\big) := \textstyle\sum_{i=1}^m \Phi_i(\cdot)\, g_i(\pi'\circ\psi_i(\cdot))\Big[\sqrt{1+|\nabla\varphi_i(\pi'\circ\psi_i(\cdot))|^2}\Big]^{-1} \text{ on } \partial\Omega, \end{aligned} \tag{3.4.19}$$

a straightforward calculation shows that

$$\begin{aligned} &R_+\circ E_+ = I, \quad \text{the identity operator on} \quad B_s^{p,q}(\partial\Omega), \text{ and} \\ &R_-\circ E_- = I, \quad \text{the identity operator on} \quad B_{s-1}^{p,q}(\partial\Omega). \end{aligned} \tag{3.4.20}$$

In turn, these identities, in conjunction with Lemma l3.3 and Theorems 3.2.9-3.2.10, readily yield interpolation identities claimed in the statement of the proposition. □

We next discuss atomic and molecular decompositions of the diagonal Besov scale on $\partial\Omega$. Call $S = S_r = S_r(x)$ a *surface ball* provided $x \in \partial\Omega$ and $0 < r \leq \operatorname{diam}(\Omega)$ and $S_r = B(x,r)\cap\partial\Omega$. Also, for $\kappa > 0$ and $S_r(x)$ surface ball we write $\kappa S := B(x,\kappa r)\cap\partial\Omega$. Recall the tangential gradient $\nabla_{\tan}$ defined in §2.2. A function $a_S \in \operatorname{Lip}(\partial\Omega)$ is called an atom for $B_s^{p,p}(\partial\Omega)$, $(n-1)/n < p < \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, if

$$(1)\ \exists\, S = S_r, \text{ surface ball, such that } \operatorname{supp}(a_S)\subseteq S, \tag{3.4.21}$$

$$(2)\ \|\nabla_{\tan} a_S\|_{L^\infty(\partial\Omega)} \leq r^{s-\frac{n-1}{p}-1}. \tag{3.4.22}$$

It is useful to observe that, by the Fundamental Theorem of Calculus, (1) & (2) above also entail

$$\|a_S\|_{L^\infty(\partial\Omega)} \leq Cr^{s-\frac{n-1}{p}} \tag{3.4.23}$$

where C depends exclusively on the Lipschitz character of Ω.

Proposition 3.4.5. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and fix $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$. Then there exists a finite constant $C = C(\partial\Omega, s, p, n) > 0$ with the property that for any countable family $\{a_S\}_S$ of $B_s^{p,p}(\partial\Omega)$ atoms and any numerical sequence $\{\lambda_S\}_S \in \ell^p$,*

$$\Big\|\sum_S \lambda_S a_S\Big\|_{B_s^{p,p}(\partial\Omega)} \leq C\|\{\lambda_S\}_S\|_{\ell^p}. \tag{3.4.24}$$

Proof. Assume that each a_S is a $B_s^{p,p}(\partial\Omega)$ atom and that $\{\lambda_S\}_S \in \ell^p$. In view of the fact that

$$\Big\|\sum_S \lambda_S a_S\Big\|^p_{B_s^{p,p}(\partial\Omega)} \leq \sum_S |\lambda_S|^p \|a_S\|^p_{B_s^{p,p}(\partial\Omega)} \tag{3.4.25}$$

it suffices to check that all Besov atoms as in (3.4.21)-(3.4.22) belong to some large, fixed ball in $B_s^{p,p}(\partial\Omega)$. In turn, by Proposition 3.4.1, this comes down to verifying that, if $a_S \in \mathrm{Lip}\,(\partial\Omega)$ is as in (3.4.21)-(3.4.22), then

$$\|a_S\|_{L^p(\partial\Omega)} \leq C, \qquad \int_{\partial\Omega}\int_{\partial\Omega} \frac{|a_S(x)-a_S(y)|^p}{|x-y|^{n-1+sp}}\, d\sigma_x d\sigma_y \leq C, \tag{3.4.26}$$

for some finite $C = C(\partial\Omega) > 0$. Now, the first estimate in (3.4.26) is an immediate consequence of (3.4.23), while the second one is handled by decomposing the domain of integration giving rise to terms of the form $I := \int_S\int_S$, $II := \int_{\partial\Omega\setminus S}\int_S$

plus the other permutation. It is then easy to see that with given restrictions on indices

$$\begin{aligned} I &= \int_S \int_S \frac{|a_S(x)-a_S(y)|^p}{|x-y|^{n-1+sp}}\, d\sigma_x d\sigma_y \\ &\leq C\|\nabla_{\tan} a_S\|^p_{L^\infty(\partial\Omega)} \int_S \int_S \frac{1}{|x-y|^{n-1+sp-p}}\, d\sigma_x d\sigma_y \leq C, \end{aligned} \tag{3.4.27}$$

using (3.4.22) and

$$\begin{aligned} II &= \int_{\partial\Omega\setminus S} \int_S \frac{|a_S(x)|^p}{|x-y|^{n-1+sp}}\, d\sigma_x d\sigma_y \\ &\leq C\|a_S\|^p_{L^\infty(\partial\Omega)} \int_S \int_{\partial\Omega\setminus S} \frac{1}{|x-y|^{n-1+sp}}\, d\sigma_y d\sigma_x \leq C, \end{aligned} \tag{3.4.28}$$

using (3.4.23). □

Proposition 3.4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$. Then*

$$\begin{aligned} \|f\|_{B^{p,p}_s(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_S |\lambda_S|^p\Big)^{1/p} : \\ f = \sum_S \lambda_S a_S,\ a_S \text{ are } B^{p,p}_s(\partial\Omega) \text{ atoms, } \{\lambda_S\}_S \in \ell^p \Big\}, \end{aligned} \tag{3.4.29}$$

uniformly for $f \in B^{p,p}_s(\partial\Omega)$.

In (3.4.29), the infimum is taken over all possible representations of f as $\sum_S \lambda_S a_S$, for countable families of surface balls, and the series is assumed to converge absolutely in $L^1_{\rm loc}(\partial\Omega)$.

Proof of Proposition 3.4.6. Fix $f \in B^{p,p}_s(\partial\Omega)$ and retain the notation introduced in conjunction with (3.4.3), (3.4.4). Then (3.4.3) holds and our intention is to invoke Theorem 3.3.1 (with $p = q$ and n replaced by $n-1$) for each function

$(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot))) \in B_s^{p,q}(\mathbb{R}^{n-1})$. Note that, in this scenario, we can pick $K = 1$ and $L = -1$. Consequently, for each $1 \leq i \leq m$, there exist a sequence of coefficients $\{\lambda_Q^i\}_{Q\in\mathcal{Q}_{n-1},l(Q)\leq 1}$ and a sequence of smooth functions $\{a_Q^i\}_{Q\in\mathcal{Q}_{n-1},l(Q)\leq 1}$ such that

$$\operatorname{supp}(a_Q^i) \subseteq 3Q, \qquad \|\partial^\gamma a_Q^i\|_{L^\infty(\mathbb{R}^{n-1})} \leq |Q|^{s/(n-1)-1/p-|\gamma|/(n-1)}, \quad |\gamma| \leq 1, \tag{3.4.30}$$

and for which

$$\begin{aligned} &(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot))) = \textstyle\sum_{\substack{Q\in\mathcal{Q}_{n-1}\\ l(Q)\leq 1}} \lambda_Q^i a_Q^i, \\ &\|(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\|_{B_s^{p,p}(\mathbb{R}^{n-1})} \approx \left(\textstyle\sum_{\substack{Q\in\mathcal{Q}_{n-1}\\ l(Q)\leq 1}} |\lambda_Q^i|^p\right)^{1/p}. \end{aligned} \tag{3.4.31}$$

As before, for each $1 \leq i \leq m$, select $\Phi_i \in C_c^\infty(\mathbb{R}^{n-1})$, $0 \leq \Phi_i \leq 1$, which is identically one in a small, open neighborhood $\mathcal{O}_i$ of $\{x' \in \mathbb{R}^{n-1} : \psi_i^{-1}(x',\varphi_i(x')) \in \operatorname{supp}\Psi_i\}$ and which vanishes outside a slightly larger open set $\widetilde{\mathcal{O}}_i$.

Recall that by $\pi' : (x', x_n) \mapsto x'$ we denote the Cartesian projection onto the first $n-1$ coordinates in $\mathbb{R}^n$. Given $Q \in \mathcal{Q}_{n-1}$ with $l(Q) \leq 1$ we let $S \subset \partial\Omega$ be the smallest surface ball such that

$$\operatorname{supp}\Phi_i \cap 3Q \subseteq \pi'(\psi_i(S)) \tag{3.4.32}$$

and observe that the radius of S is at most a (fixed) multiple of $l(Q)$. Now, for a fixed scalar $\kappa > 0$, set

$$\bar{a}_S^i(\psi_i^{-1}(x',\varphi_i(x'))) := \kappa\Big(a_Q^i\Phi_i\Big)(x'), \tag{3.4.33}$$

first when $x' \in \operatorname{supp}\Phi_i$, then extend $\bar{a}_S^i$ by zero outside its support to the entire boundary $\partial\Omega$. Note that there exists $C = C(\partial\Omega)$ such that $|(\nabla_{\tan}\bar{a}_S^i)(\psi_i^{-1}(x',\varphi_i(x')))| \leq C\kappa|\nabla(a_Q^i\Phi_i\Big)(x')| \leq C\kappa[\|\nabla a_Q^i\|_{L^\infty} + \|a_Q^i\|_{L^\infty}]$. Granted that $l(Q) \leq 1$, it follows

that $\kappa = \kappa(\partial\Omega)$ can be picked so that the function just defined is a $B_s^{p,p}(\partial\Omega)$ atom. Furthermore, the choice $\lambda_S^i := \kappa^{-1}\lambda_Q^i$ yields

$$\Psi_i f = \sum_S \lambda_S^i \bar{a}_S^i, \qquad \|\Psi_i f\|_{B_s^{p,p}(\partial\Omega)} \approx \|\{\lambda_S^i\}_S\|_{\ell^p}, \tag{3.4.34}$$

for each $i \in \{1, ..., m\}$. Hence, ultimately,

$$f = \sum_{i=1}^m \sum_S \lambda_S^i \bar{a}_S^i, \qquad \|f\|_{B_s^{p,p}(\partial\Omega)} \approx \|\{\lambda_S^i\}_{S,i}\|_{\ell^p}. \tag{3.4.35}$$

This proves the left-to-right inequality in (3.4.29). The opposite direction, i.e. the right-to-left inequality in (3.4.29), follows from Proposition 3.4.5 and this finishes the proof of the proposition. □

Next, we discuss an atomic decomposition result for the space $B_{s-1}^{p,p}(\partial\Omega)$ when $(n-1)/n < p < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. First, we make a definition. For a given, fixed parameter $\eta = \eta(\partial\Omega) > 0$, call $a_S \in L^\infty(\partial\Omega)$ an atom for $B_{s-1}^{p,p}(\partial\Omega)$ if

$$(1)\ \exists\, S = S_r, \text{ surface ball, such that } \operatorname{supp}(a_S) \subseteq S, \tag{3.4.36}$$

$$(2)\ \|a_S\|_{L^\infty(\partial\Omega)} \leq r^{s-1-\frac{n-1}{p}}, \tag{3.4.37}$$

$$(3)\ \int_{\partial\Omega} a_S(x)\, d\sigma_x = 0 \ \text{ when } r < \eta. \tag{3.4.38}$$

Much as in the case of atoms satisfying (3.4.21)-(3.4.22), we have:

Proposition 3.4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$. Then there exists a finite constant $C = C(\partial\Omega, s, p, n) > 0$ with the property that for any countable family $\{a_S\}_S$ of $B_{s-1}^{p,p}(\partial\Omega)$ atoms and any numerical sequence $\{\lambda_S\}_S \in \ell^p$,*

$$\Big\|\sum_S \lambda_S a_S\Big\|_{B_{s-1}^{p,p}(\partial\Omega)} \leq C\|\{\lambda_S\}_S\|_{\ell^p}. \tag{3.4.39}$$

Proof. Consider a series $\sum_S \lambda_S a_S$ where the a_S's are $B^{p,p}_{s-1}(\partial\Omega)$ atoms and $\{\lambda_S\}_S \in \ell^p$. Much as in the case of (3.4.25), it suffices to observe that if $a_S \in L^\infty(\partial\Omega)$ is as in (3.4.36)-(3.4.38) then

$$\|a_S\|_{B^{p,p}_{s-1}(\partial\Omega)} \leq C = C(\partial\Omega) < +\infty. \tag{3.4.40}$$

To this end, there is no loss of generality in assuming that η in (3.4.38) is small enough so that any surface ball S_r with $r < \eta$ is entirely contained in a coordinate chart of the atlas $\{\mathcal{O}_i\}_i$ introduced in §2.2 and so that pull-back of any function supported in S_r, $r < \eta$, from $\partial\Omega$ to $\mathbb{R}^{n-1}$ has support contained in $3Q$ for some cube $Q \in \mathcal{Q}_n$, $l(Q) \leq 1$. In this scenario, it is easy to see that the pull-pack of any Besov atom on $\partial\Omega$ to $\mathbb{R}^{n-1}$ becomes a fixed multiple of Euclidean Besov atom, i.e. satisfies (3.3.1)-(3.3.3) with $K = L = 0$. However, the Euclidean counterpart of the estimate (3.4.40) is a consequence of Theorem 3.3.1.

If we work with an atom a_S supported in S_r, $r \geq \eta$, consider the following procedure. We take some $N \in (2r/\eta, 4r/\eta) \cap \mathbb{N}$ and collection of disjoint sets $\{A_i\}_{i=1}^N \subset \partial\Omega$ such that $\cup_{i=1}^N A_i = S_r$ and for every $i = 1, ..., N$ the set A_i is contained in the support ball of radius $r_i = 2r/N$, so that $r_i \in (\eta/2, \eta)$. Then $a_S = \sum_{i=1}^N a_S \chi_{A_i}$ and

$$\|a_S\chi_{A_i}\|_{L^\infty(\partial\Omega)} \leq \|a_S\|_{L^\infty(\partial\Omega)} \leq r^{s-1-\frac{n-1}{p}} = (N/2)^{s-1-\frac{n-1}{p}} r_i^{s-1-\frac{n-1}{p}}, \tag{3.4.41}$$

hence for $a^i_S := (2/N)^{s-1-\frac{n-1}{p}} a_S \chi_{A_i}$

$$\text{supp}\,(a^i_S) \subset S_{r_i}, \quad r_i \in (\eta/2, \eta), \tag{3.4.42}$$

$$\|a^i_S\|_{L^\infty(\partial\Omega)} \leq r_i^{s-1-\frac{n-1}{p}}. \tag{3.4.43}$$

Now the pull-back of function a_S^i, denoted by $\bar{a}_S^i$, satisfies

$$\operatorname{supp}(\bar{a}_S^i) \subset 3Q', \qquad Q' \in \mathcal{Q}_n, \ \ \kappa < l(Q') < 1, \tag{3.4.44}$$

$$\|\bar{a}_S^i\|_{L^\infty(\mathbb{R}^{n-1})} \leq l(Q')^{s-1-\frac{n-1}{p}}, \tag{3.4.45}$$

for some constant $\kappa = \kappa(\partial\Omega)$ independent of a_S^i. Therefore, there exists some cube $Q \in \mathcal{Q}_n$ with $l(Q) = 1$ such that

$$\operatorname{supp}(\bar{a}_S^i) \subset 3Q, \qquad Q \in \mathcal{Q}_n,\, l(Q) = 1, \tag{3.4.46}$$

$$\|\bar{a}_S^i\|_{L^\infty(\mathbb{R}^{n-1})} \leq C, \tag{3.4.47}$$

with C depending on $\partial\Omega$ only. By Theorem 3.3.1 the functions $\bar{a}_S^i$, $i = 1, ..., N$, are uniformly bounded in $B^{p,p}_{s-1}(\mathbb{R}^{n-1})$-norms as Euclidean Besov atoms. Then

$$\|a_S\|^p_{B^{p,p}_{s-1}(\partial\Omega)} \leq C \sum_{i=1}^{N} (N/2)^{sp-p-(n-1)} \|\bar{a}_S^i\|^p_{B^{p,p}_{s-1}(\mathbb{R}^{n-1})} \leq C N^{sp-p-n+2} \leq C, \tag{3.4.48}$$

since $sp - p - n + 2 < 0$ under current assumptions on indices, and (3.4.40) follows.

□

Proposition 3.4.8. *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists $\eta = \eta(\partial\Omega) > 0$ such that the following is true. If $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p} - 1) < s < 1$ then*

$$\|f\|_{B^{p,p}_{s-1}(\partial\Omega)} \approx \inf \Big\{ \Big(\sum_S |\lambda_S|^p \Big)^{1/p} : \\ f = \sum_S \lambda_S a_S, \ a_S \text{ are } B^{p,p}_{s-1}(\partial\Omega) \text{ atoms, } \{\lambda_S\}_S \in \ell^p \Big\}, \tag{3.4.49}$$

uniformly for $f \in B^{p,p}_{s-1}(\partial\Omega)$.

Proof. The reasoning parallels to some extent that in the proof of Proposition 3.4.6 so we shall only stress the novel aspects. We also retain the same notation (mostly pertaining to the geometric aspects of the proof) as before. Given $f \in B^{p,p}_{s-1}(\partial\Omega)$, we once again invoke Theorem 3.3.1 (with $p = q$, s replaced by $s-1$ and n replaced by $n-1$) for each function $(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2} \in B^{p,q}_s(\mathbb{R}^{n-1})$. Note that, this time, we may choose $K = 0$ and $L = 0$. Hence, for each $1 \leq i \leq m$, there exist a sequence of coefficients $\{\lambda^i_Q\}_{Q\in\mathcal{Q}_{n-1}, l(Q)\leq 1}$ and a sequence of smooth functions $\{a^i_Q\}_{Q\in\mathcal{Q}_{n-1}, l(Q)\leq 1}$ such that

$$\begin{aligned} &(i)\ \operatorname{supp}(a^i_Q) \subseteq 3Q, \quad (ii)\ \|a^i_Q\|_{L^\infty(\mathbb{R}^{n-1})} \leq |Q|^{\frac{s-1}{n-1}-\frac{1}{p}}, \\ &(iii)\ \text{if } |Q| < 1,\ \int_{\mathbb{R}^{n-1}} a^i_Q(x')\,dx' = 0, \end{aligned} \tag{3.4.50}$$

and for which

$$\begin{aligned} &(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2} = \textstyle\sum_{\substack{Q\in\mathcal{Q}_{n-1}\\ l(Q)\leq 1}} \lambda^i_Q a^i_Q, \\ &\|(\Psi_i f)(\psi_i^{-1}(\cdot, \varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\|_{B^{p,p}_s(\mathbb{R}^{n-1})} \approx \left(\textstyle\sum_{\substack{Q\in\mathcal{Q}_{n-1}\\ l(Q)\leq 1}} |\lambda^i_Q|^p\right)^{1/p}. \end{aligned} \tag{3.4.51}$$

As before, given $Q \in \mathcal{Q}_{n-1}$ with $l(Q) \leq 1$, we take $S \subset \partial\Omega$ to be the smallest surface ball such that $\operatorname{supp}\Phi_i \cap 3Q \subseteq \pi'(\psi_i(S))$ and observe that the radius of S is at most a (fixed) multiple of $l(Q)$.

The main difficulty in further implementing the rest of the proof of Proposition 3.4.6 is that, because of the vanishing moment appearing in (3.4.50), the concept of atom we are currently working with is no longer stable under (smooth) truncations. Thus, the definition (3.4.33) requires further adjustments. Specifically, for each Euclidean atom a_Q – and with x_Q denoting the center of the cube

Q– we introduce

$$\begin{aligned}
&\bar{a}_S^i(\psi_i^{-1}(x',\varphi_i(x'))) := \Phi_i(x_Q)\tfrac{a_Q^i(x')}{\sqrt{1+|\nabla\varphi_i(x')|^2}},\\
&\bar{b}_S^i(\psi_i^{-1}(x',\varphi_i(x'))) := \tfrac{\Big[\Phi_i(x')-\Phi_i(x_Q)\Big]a_Q^i(x')}{\sqrt{1+|\nabla\varphi_i(x')|^2}},
\end{aligned} \tag{3.4.52}$$

first when $x' \in \operatorname{supp}\Phi_i$, then extend $\bar{a}_S^i$ and $\bar{b}_S^i$ by zero to the remainder of $\partial\Omega$. The following properties are apparent:

$$\Psi_i f = \sum_S \lambda_Q^i(\bar{a}_S^i + \bar{b}_S^i), \quad 1 \le i \le m, \tag{3.4.53}$$

$$\|\bar{a}_S^i\|_{L^\infty(\partial\Omega)} \le Cr^{s-1-\frac{n-1}{p}}, \qquad \|\bar{b}_S^i\|_{L^\infty(\partial\Omega)} \le Cr^{s-\frac{n-1}{p}}, \tag{3.4.54}$$

$$\exists\,\kappa = \kappa(\partial\Omega) > 0 \text{ such that } \operatorname{supp}\bar{a}_S^i,\ \operatorname{supp}\bar{b}_S^i \subset \kappa S, \tag{3.4.55}$$

$$\exists\,\eta = \eta(\partial\Omega) > 0 \text{ such that } l(Q) < \eta \Rightarrow \int_{\partial\Omega} \bar{a}_S^i\,d\sigma = 0. \tag{3.4.56}$$

It follows that the $\bar{a}_S^i$'s are (up to a fixed, multiplicative factor) atoms for $B_{s-1}^{p,p}(\partial\Omega)$, whereas the $\bar{b}_S^i$'s satisfy

$$\|\bar{b}_S^i\|_{L^q(\partial\Omega)} \le Cr^{\frac{n-1}{q}}\cdot r^{s-\frac{n-1}{p}} = C, \tag{3.4.57}$$

provided $\frac{1}{q} = \frac{1}{p} - \frac{s}{n-1}$, which we shall assume. Consequently, if for each $1 \le i \le m$ we set $g^i := \sum_S \lambda_Q^i \bar{b}_S^i$ then

$$\|g^i\|_{L^q(\partial\Omega)} \le \sum_S |\lambda_Q^i|\|\bar{b}_S^i\|_{L^q(\partial\Omega)} \le C\|\{\lambda_Q^i\}\|_{\ell^1} \le C\|\{\lambda_Q^i\}\|_{\ell^p} \le \|f\|_{B_{s-1}^{p,p}(\partial\Omega)}, \tag{3.4.58}$$

where the last inequality follows from (3.4.51). Set $g := \sum_i g^i$. Based on the analysis carried out up to this stage we may conclude that f can be decomposed

in the form $f = g + \sum_S \lambda_S a_S$ where a_S are Besov atoms and

$$\|g\|_{L^q(\partial\Omega)} + \Big(\sum_S |\lambda_S|^p\Big)^{1/p} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega)}. \tag{3.4.59}$$

There remains to decompose g further. Let $a_g \in \mathbb{R}$ stand for the integral average of g on $\partial\Omega$ and consider $\frac{1}{p_o} := \frac{1}{p} + \frac{1-s}{n-1}$ so that $\frac{n-1}{n} < p_o \leq 1$ and $p_o < p$. Theorem A on p. 592 of **[25]** (and its proof starting on pp. 626 *loc. cit.*) then guarantees that $g - a_g$ can be represented in the form $\sum_j \mu_j c_j$, where $\{\mu_j\}_j \in \ell^{p_o} \hookrightarrow \ell^p$ and the c_j's are $B^{p,p}_{s-1}(\partial\Omega)$ atoms, with a natural accompanying estimate. As for the remaining piece, the constant function a_g, this can be viewed as a Besov atom with "large" support (for which no cancellation is required) and which satisfies $\|a_g\|_{L^\infty(\partial\Omega)} \leq C(\partial\Omega)\|g\|_{L^q(\partial\Omega)}$. All in all, this justifies the left-to-right inequality in (3.4.49).

Finally, the opposite inequality follows directly from Proposition 3.4.7. □

Remark. The above atomic decomposition results can, in particular, be used to show that the space defined in (3.4.6)-(3.4.7) does not depend (up to an equivalence of quasi-norms) on the choice of partition of unity.

In **[49]**, **[47]**, the authors also develop a molecular theory for Euclidean Besov spaces. Here we want to record a related result with the Euclidean space replaced by the boundary of a Lipschitz domain which is going to play an important role later on. To set the stage, fix $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain and assume that $\frac{n-1}{n} < p \leq 1$, $(n-1)(\frac{1}{p} - 1) < s < 1$. Call $m_S \in L^\infty(\partial\Omega)$ a $B^{p,p}_{s-1}(\partial\Omega)$ molecule if there exist $M > \frac{n-1}{p}$ and a surface ball S centered at $x_S \in \partial\Omega$ and having radius

$r \in (0, \operatorname{diam}\Omega)$ such that

$$(1)\ |m_S(x)| \leq r^{s-1-\frac{n-1}{p}}(1+r^{-1}|x-x_S|)^{-M+s-1} \quad \text{for } x \in \partial\Omega, \tag{3.4.60}$$

$$(2)\ \int_{\partial\Omega} m_S(x)\, d\sigma_x = 0 \ \text{ if } \ r < \eta. \tag{3.4.61}$$

Proposition 3.4.9. *Let $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$. Then, given an arbitrary bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, there exists $\eta = \eta(\partial\Omega) > 0$ such that*

$$\|f\|_{B^{p,p}_{s-1}(\partial\Omega)} \approx \inf\Big\{\Big(\sum_S |\lambda_S|^p\Big)^{1/p} : f = \sum_S \lambda_S m_S,\ m_S\text{'s are } B^{p,p}_{s-1}(\partial\Omega) \text{ molecules, } \{\lambda_S\}_S \in \ell^p\Big\}, \tag{3.4.62}$$

uniformly for $f \in B^{p,p}_{s-1}(\partial\Omega)$.

Conversely, there exists $C = C(\partial\Omega, s, p, M, n) > 0$ such that for any countable family $\{m_S\}_S$ of $B^{p,p}_{s-1}(\partial\Omega)$ molecules and any numerical sequence $\{\lambda_S\}_S \in \ell^p$,

$$\Big\|\sum_S \lambda_S m_S\Big\|_{B^{p,p}_{s-1}(\partial\Omega)} \leq C\|\{\lambda_S\}_S\|_{\ell^p}. \tag{3.4.63}$$

Proof. The left-to-right inequality in (3.4.62) is simple consequence of Proposition 3.4.8. In the opposite direction, it suffices to prove (3.4.63), a task to which we now turn. As before, it is enough to show that there exists a finite constant $C = C(\partial\Omega, s, p, n, M) > 0$ such that

$$m_S \text{ molecule for } B^{p,p}_{s-1}(\partial\Omega) \Longrightarrow \|m_S\|_{B^{p,p}_{s-1}(\partial\Omega)} \leq C. \tag{3.4.64}$$

To this end, fix m_S satisfying (3.4.60)-(3.4.61). If $\eta \leq r < \operatorname{diam}\Omega$, then $\|m_S\|_{L^\infty(\partial\Omega)} \leq C = C(\partial\Omega)$, from which (3.4.64) follows from Proposition 3.4.7 as m_S is a fixed multiple of a Besov atom with large support (hence, without

a vanishing moment condition). Therefore, it remains to analyze the case when $r < \eta$, where we adapt an argument used in **[25]** for Hardy spaces. First, split $\partial\Omega$ in dyadic annuli $\Delta_0 := S_r$, $\Delta_i := 2^i S_r \setminus 2^{i-1} S_r$, $i = 1, 2, ...$, and denote by χ_i the characteristic function of Δ_i. Next, introduce

$$m_i := m_S \chi_i - \Big(\frac{1}{\sigma(\Delta_i)} \int_{\Delta_i} m_S \, d\sigma\Big)\chi_i, \qquad i = 0, 1, ..., \tag{3.4.65}$$

so that

$$\int_{\partial\Omega} m_i \, d\sigma = 0, \qquad \operatorname{supp} m_i \subseteq S_{2^i r}. \tag{3.4.66}$$

Also, since $|x - x_S| \leq C 2^i r$ for $x \in \Delta_i$, $i = 0, 1, ...$, it follows that

$$|m_i(x)| \leq C r^{s-1-\frac{n-1}{p}} 2^{i(-M+s-1)} = C(2^i r)^{s-1-\frac{n-1}{p}} 2^{i(-M+\frac{n-1}{p})}, \quad x \in \partial\Omega. \tag{3.4.67}$$

In particular,

$$2^{i(M-\frac{n-1}{p})} m_i \ \text{ is a (fixed multiple of a) } B^{p,p}_{s-1}(\partial\Omega) \text{ atom.} \tag{3.4.68}$$

Going further, for $i = 1, 2, ...$, introduce

$$\bar{\chi}_i := \frac{\chi_i}{\sigma(\Delta_i)} - \frac{\chi_{i-1}}{\sigma(\Delta_{i-1})} \tag{3.4.69}$$

and observe that

$$\int_{\partial\Omega} \bar{\chi}_i \, d\sigma = 0, \qquad \operatorname{supp} \bar{\chi}_i \subseteq S_{2^i r}, \qquad \|\bar{\chi}_i\|_{L^\infty(\partial\Omega)} \leq C(2^i r)^{-(n-1)}. \tag{3.4.70}$$

Thus,

$$(2^i r)^{s-1-(n-1)(\frac{1}{p}-1)} \bar{\chi}_i \ \text{ is a (fixed multiple of a) } B^{p,p}_{s-1}(\partial\Omega) \text{ atom.} \tag{3.4.71}$$

On the other hand,

$$\begin{aligned}\lambda_j := \sum_{i=j}^{\infty}\int_{\Delta_i} m_S\,d\sigma \Longrightarrow |\lambda_j| &\leq C\sum_{i=j}^{\infty} r^{s-1-\frac{n-1}{p}}2^{i(-M+s-1)}(2^i r)^{n-1}\\ &\leq Cr^{s-1-(n-1)(\frac{1}{p}-1)}2^{j(-M+s-1+n-1)},\end{aligned} \tag{3.4.72}$$

while summing by parts yields

$$\sum_{i=0}^{\infty}\Big(\frac{1}{\sigma(\Delta_i)}\int_{\Delta_i} m_S\,d\sigma\Big)\chi_i = \sum_{j=0}^{\infty}(\lambda_j-\lambda_{j+1})\frac{\chi_j}{\sigma(\Delta_j)} = \sum_{j=0}^{\infty}\lambda_{j+1}\bar{\chi}_{j+1}. \tag{3.4.73}$$

We now observe that

$$m_S = \sum_{i=0}^{\infty} m_i + \sum_{j=0}^{\infty}\lambda_{j+1}\bar{\chi}_{j+1} \tag{3.4.74}$$

and that, by (3.4.72) and the fact that $M > \frac{n-1}{p}$,

$$\sum_{i=0}^{\infty}[2^{-i(M-\frac{n-1}{p})}]^p + \sum_{j=0}^{\infty}\Big[|\lambda_{j+1}|(2^{j+1}r)^{1-s+(n-1)(\frac{1}{p}-1)}\Big]^p \leq C < +\infty \tag{3.4.75}$$

for some constant $C = C(\partial\Omega, s, p, M) > 0$ independent of r. This concludes the proof of the Proposition 3.4.9. □

3.5 Triebel-Lizorkin spaces on boundaries of Lipschitz domains

Similar considerations as in the previous section apply to case of Triebel-Lizorkin spaces on the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, denoted in the sequel by $F^{p,q}_s(\partial\Omega)$. Compared with the Besov scale, the most important novel aspect here is the possibility of allowing the endpoint case $s = 1$ as part of the general discussion if $q = 2$. To discuss this in more detail, retain the same notation as in the first part of §3.4 and assume that either

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad (n-1)\Big(\frac{1}{\min\{p,q\}}-1\Big)_+ < s < 1, \tag{3.5.1}$$

or

$$\frac{n-1}{n}<p<\infty,\quad q=2,\quad s=1. \tag{3.5.2}$$

In this scenario, the Triebel-Lizorkin scale in $\mathbb{R}^{n-1}$ is invariant under pointwise multiplication by Lipschitz maps as well as composition by Lipschitz diffeomorphisms, in the precise sense described in Theorem 3.3.3.

We may therefore define the space $F^{p,q}_s(\partial\Omega)$ as the collection of all locally integrable functions on $\partial\Omega$ such that

$$(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\in F^{p,q}_s(\mathbb{R}^{n-1}),\quad \text{for every } i=1,...,m, \tag{3.5.3}$$

endowed with the norm

$$\|f\|_{F^{p,q}_s(\partial\Omega)}:=\left(\sum_{i=1}^m\|(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\|^p_{F^{p,q}_s(\mathbb{R}^{n-1})}\right)^{1/p}. \tag{3.5.4}$$

Also, $F^{p,q}_{s-1}(\partial\Omega)$ is defined as the collection of all functionals $f\in(\mathrm{Lip}\,(\partial\Omega))'$ such that

$$(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\in F^{p,q}_{s-1}(\mathbb{R}^{n-1}),\quad \text{for every } i=1,...,m, \tag{3.5.5}$$

and we equip with space the quasi-norm

$$\|f\|_{F^{p,q}_{s-1}(\partial\Omega)}:=\left(\sum_{i=1}^m\|(\Psi_i f)(\psi_i^{-1}(\cdot,\varphi_i(\cdot)))\sqrt{1+|\nabla\varphi_i(\cdot)|^2}\|^p_{F^{p,q}_{s-1}(\mathbb{R}^{n-1})}\right)^{1/p}. \tag{3.5.6}$$

As in the case of Besov spaces, the Triebel-Lizorkin spaces just introduced are quasi-Banach with different partitions of unity yielding equivalent quasi-norms. Also, as before, certain results from the Euclidean setting carry over to spaces defined on $\partial\Omega$ in a rather direct fashion. Below we list embedding, duality, and interpolation results which are useful for us later on.

Proposition 3.5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that two triplets (p_0, q_0, s_0), (p_1, q_1, s_1) simultaneously satisfy either (3.5.1) or (3.5.2). Then the inclusions*

$$F_{s_0}^{p_0,q_0}(\partial\Omega) \hookrightarrow F_{s_1}^{p_1,q_1}(\partial\Omega), \qquad F_{s_0-1}^{p_0,q_0}(\partial\Omega) \hookrightarrow F_{s_1-1}^{p_1,q_1}(\partial\Omega), \tag{3.5.7}$$

are continuous with dense range, assuming in addition that either $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ and $p_0 < p_1$ or $s_0 = s_1$, $p_0 = p_1$ and $q_0 < q_1$.

Proposition 3.5.2. *For p, q, s satisfying either (3.5.1) with $q < \infty$ or (3.5.2),*

$$\begin{aligned} &(F_s^{p,q}(\partial\Omega)^* = F_{-s}^{p',q'}(\partial\Omega), \qquad \text{if } \ p \geq 1, \\ &(F_s^{p,q}(\partial\Omega)^* = B^{\infty,\infty}_{-s+(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega), \qquad \text{if } \ p < 1, \end{aligned} \tag{3.5.8}$$

and

$$\begin{aligned} &(F_{s-1}^{p,q}(\partial\Omega)^* = F_{1-s}^{p',q'}(\partial\Omega), \qquad \text{if } \ p \geq 1, \\ &(F_{s-1}^{p,q}(\partial\Omega)^* = B^{\infty,\infty}_{1-s+(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega), \qquad \text{if } \ p < 1. \end{aligned} \tag{3.5.9}$$

In each case, the corresponding duality pairing is a natural extension of the distributional pairing on $\partial\Omega$.

Proposition 3.5.3. *Let Ω be an arbitrary, bounded Lipschitz domain in $\mathbb{R}^n$ and assume that each of the two triplets (p_0, q_0, s_0), (p_1, q_1, s_1) satisfies either (3.5.1) or (3.5.2). Then*

$$[F_{s_0}^{p_0,q_0}(\partial\Omega), F_{s_1}^{p_1,q_1}(\partial\Omega)]_\theta = F_s^{p,q}(\partial\Omega), \quad [F_{s_0-1}^{p_0,q_0}(\partial\Omega), F_{s_1-1}^{p_1,q_1}(\partial\Omega)]_\theta = F_{s-1}^{p,q}(\partial\Omega), \tag{3.5.10}$$

where $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Furthermore, assume that each of the two triplets, (p, q_0, s_0) *and* (p, q_1, s_1)*, satisfies either (3.5.1) or (3.5.2). Then, if* $s_0 \neq s_1$, $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$,

$$(F^{p,q_0}_{s_0}(\partial\Omega), F^{p,q_1}_{s_1}(\partial\Omega))_{\theta,q} = B^{p,q}_{s}(\partial\Omega), \quad (F^{p,q_0}_{s_0-1}(\partial\Omega), F^{p,q_1}_{s_1-1}(\partial\Omega))_{\theta,q} = B^{p,q}_{s-1}(\partial\Omega). \tag{3.5.11}$$

In addition, much as in the Euclidean setting, there is a natural identification between the Triebel-Lizorkin scales defined on the boundary of a bounded, Lipschitz domain Ω and various Hardy and Sobolev spaces defined on $\partial\Omega$ by viewing the boundary as a space of homogenous type (in the sense of **[25]**). In order to explain how this can be done, we need to digress momentarily and study Hardy-type spaces on $\partial\Omega$.

We start by recalling some definitions. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, assume that $\frac{n-1}{n} < p \leq 1 < p_1 \leq +\infty$ and fix $\eta = \eta(\partial\Omega) > 0$ sufficiently small. Then, for a surface ball $S = S_r \subseteq \partial\Omega$, we call $a_S \in L^\infty(\partial\Omega)$ a (p, p_1)-atom if

$$\begin{aligned} &\operatorname{supp}(a_S) \subseteq S, \quad \|a_S\|_{L^{p_1}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_1}-\frac{1}{p}\right)}, \text{ and} \\ &\int_{\partial\Omega} a_S \, d\sigma = 0 \text{ if } r < \eta. \end{aligned} \tag{3.5.12}$$

We then set

$$h^p(\partial\Omega) := \Big\{ f = \sum_S \lambda_S a_S : \{\lambda_S\}_S \in \ell^p, \ a_S \text{ is a } (p, p_1) - \text{atom} \Big\}. \tag{3.5.13}$$

and equip it with the quasi-norm

$$\|f\|_{h^p(\partial\Omega)} := \inf\Big\{ \Big(\sum_S |\lambda_S|^p\Big)^{1/p} : \\ f = \sum_S \lambda_S a_S,\ a_S \text{ is a } (p,p_1)-\text{atom},\ \{\lambda_S\}_S \in \ell^p \Big\}. \tag{3.5.14}$$

One can check that this is a "local" quasi-Banach space, in the sense that

$$h^p(\partial\Omega) \text{ is a module over } B^{\infty,\infty}_\alpha(\partial\Omega) = C^\alpha(\partial\Omega) \text{ for any } \alpha > (n-1)\Big(\tfrac{1}{p}-1\Big). \tag{3.5.15}$$

See, e.g., **[98]** for a proof. Also, as is well-known (cf. **[25]**), different choices of the parameter p_1 lead to equivalent quasi-norms and

$$(h^p(\partial\Omega))^* = B^{\infty,\infty}_{(n-1)(\frac{1}{p}-1)}(\partial\Omega) = C^{(n-1)(\frac{1}{p}-1)}(\partial\Omega). \tag{3.5.16}$$

It is also going to be useful to note that, for $\frac{n-1}{n} < p \leq 1$,

$$L^q(\partial\Omega) \hookrightarrow h^p(\partial\Omega), \qquad \forall\, q > 1. \tag{3.5.17}$$

Indeed, if $f \in L^q(\partial\Omega)$, $q > 1$, and if $a_f \in \mathbb{R}$ stands for the integral average of f on $\partial\Omega$, then (the proof of) Theorem A in **[25]** shows that $f - a_f$ can be written as $\sum_j \lambda_j a_j$ where $\{\lambda_j\}_j \in \ell^p$ and the a_j's are (p,p_1)-atoms, plus a naturally accompanying estimate. As regards the remaining piece, i.e., the constant function a_f, this can be viewed as a Hardy atom with "large" support (for which no cancellation is necessary) and which satisfies $\|a_f\|_{L^{p_1}(\partial\Omega)} \leq C(\partial\Omega, p_1, q)\|f\|_{L^q(\partial\Omega)}$. This finishes the proof of (3.5.17).

A similar construction can be carried out one level of smoothness up, leading

to the following definition:

$$h_1^p(\partial\Omega) := \Big\{ f = \sum_S \lambda_S a_S : \{\lambda_S\}_S \in \ell^p,\ a_S \text{ is a regular } (p,p_1)\text{-atom} \Big\}, \quad (3.5.18)$$

$$\|f\|_{h_1^p(\partial\Omega)} := \inf\Big\{ \Big(\sum_S |\lambda_S|^p\Big)^{1/p} : f = \sum_S \lambda_S a_S,\ \{\lambda_S\}_S \in \ell^p,\ a_S \text{ regular } (p,p_1)\text{-atom} \Big\},$$

where, for $\frac{n-1}{n} < p \leq 1 < p_1 \leq +\infty$, a function $a_S \in \mathrm{Lip}\,(\partial\Omega)$ is called a regular (p,p_1)-atom if there exists a surface ball $S = S_r$ such that

$$\operatorname{supp} a_S \subseteq S, \qquad \|\nabla_{\tan} a_S\|_{L^{p_1}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_1}-\frac{1}{p}\right)}. \quad (3.5.19)$$

Note that whenever a_S is as in (3.5.19) then, by Poincaré's inequality, we also have

$$\|a_S\|_{L^{p_1}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_1}-\frac{1}{p}\right)+1}. \quad (3.5.20)$$

We are now in a position to state and prove the following.

Proposition 3.5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$F_0^{p,2}(\partial\Omega) = \begin{cases} h^p(\partial\Omega) \text{ if } \frac{n-1}{n} < p \leq 1, \\ L^p(\partial\Omega) \text{ if } 1 < p < \infty, \end{cases} \quad (3.5.21)$$

and

$$F_1^{p,2}(\partial\Omega) = \begin{cases} h_1^p(\partial\Omega) \text{ if } \frac{n-1}{n} < p \leq 1, \\ L_1^p(\partial\Omega) \text{ if } 1 < p < \infty. \end{cases} \quad (3.5.22)$$

Proof. That (3.5.21) and (3.5.22) hold when $1 < p < \infty$ is a relatively straightforward consequence of the fact that, in this case, $F_0^{p,2}(\mathbb{R}^{n-1}) = L^p(\mathbb{R}^{n-1})$, $F_1^{p,2}(\mathbb{R}^{n-1}) = L_1^p(\mathbb{R}^{n-1})$, and the definitions (3.5.3)-(3.5.6).

Let us turn our attention to the case $\frac{n-1}{n} < p \leq 1$ of (3.5.21). Pick $f \in F_0^{p,2}(\partial\Omega)$ and proceed along the same lines as in the proof of Proposition 3.4.8. With the

aid of the identification $F_0^{p,2}(\mathbb{R}^{n-1}) = h^p(\mathbb{R}^{n-1})$ and the atomic decomposition result of the latter space, described in (3.2.10)-(3.2.11), we are able to write $f = g + \sum_S \lambda_S a_S$, where the a_S's are (p, p_1)-atoms and

$$\|g\|_{L^q(\partial\Omega)} + \Big(\sum_S |\lambda_S|^p\Big)^{1/p} \leq C\|f\|_{F_0^{p,2}(\partial\Omega)}, \tag{3.5.23}$$

provided q is selected such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n-1}$. Note that this choice entails $1 < q \leq \frac{n-1}{n-2}$ so that $g \in h^p(\partial\Omega)$ by (3.5.17). All in all, this proves that $F_0^{p,2}(\partial\Omega) \hookrightarrow h^p(\partial\Omega)$.

To prove the opposite inclusion, recall the parameter η from (3.5.12) and pick a finite atlas $\{\mathcal{O}_i\}_i$ for $\partial\Omega$ and $0 < \eta_o \leq \eta$ small enough so that any surface ball S_r has the property that $2S_r$ is entirely contained in some coordinate patch $\mathcal{O}_i$ whenever $r < \eta_o$. Then if a_S, $S = S_r$ surface ball, is a (p, p_1)-atom (i.e., it satisfies (3.5.12)), we distinguish two cases. First, if $r \geq \eta_o$, it follows that $\|a_S\|_{L^{p_1}(\partial\Omega)} \leq C(\partial\Omega, \eta_o)$ and since $L^{p_1}(\partial\Omega) = F_0^{p_1,2}(\partial\Omega) \hookrightarrow F_0^{p,2}(\partial\Omega)$, it follows that $\|a_S\|_{F_0^{p,2}(\partial\Omega)} \leq C(\partial\Omega, p, p_1, \eta)$.

In fact we claim that the same estimate is valid in the second case, i.e., when $r < \eta_0$. The idea is that when pull-backed to $\mathbb{R}^{n-1}$ in the usual fashion, a_S becomes an atom with vanishing moment for $h^p(\mathbb{R}^{n-1}) = F_0^{p,2}(\mathbb{R}^{n-1})$ so that (3.3.8) applies. Thus, altogether, $h^p(\partial\Omega) \hookrightarrow F_0^{p,2}(\partial\Omega)$, proving that $F_0^{p,2}(\partial\Omega) = h^p(\partial\Omega)$.

There remains to show that $F_1^{p,2}(\partial\Omega) = h_1^p(\partial\Omega)$ for $\frac{n-1}{n} < p \leq 1$, a task we tackle next. In one direction, we proceed as in the proof of Proposition 3.4.6 with natural alterations (most importantly, the atomic decomposition described in (3.2.14)-(3.2.15) and the identification (3.2.8) are used here), and obtain the inclusion $F_1^{p,2}(\partial\Omega) \hookrightarrow h_1^p(\partial\Omega)$. Finally, the opposite inclusion is justified in a very

similar fashion. This completes the proof of the Proposition 3.5.4. □

Remark. Proposition 3.5.4 shows that the particular parameter p_1 (with the property that $1 < p_1 \leq +\infty$) used in the definition of the atoms for the space $h^p_1(\partial\Omega)$ is largely irrelevant (in the sense that a different choice leads to the same space (3.5.18)). In [**11**], the choice $p_1 = 2$ was made, while in [**31**] the authors picked $p_1 = +\infty$ when $p = 1$.

To state our next result, recall the tangential derivative operators $\partial_{\tau_{jk}}$ from §2.2.

Proposition 3.5.5. *For* $\frac{n-1}{n} < p \leq 1$,

$$h^p_1(\partial\Omega) = \{f \in B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega) : \partial_{\tau_{jk}} f \in h^p(\partial\Omega) \text{ for every } j,k = 1,...,n\} \tag{3.5.24}$$

and

$$\|f\|_{h^p_1(\partial\Omega)} \approx \|f\|_{B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega)} + \sum_{j,k} \|\partial_{\tau_{jk}} f\|_{h^p(\partial\Omega)}, \tag{3.5.25}$$

uniformly for $f \in h^p_1(\partial\Omega)$.

Proof. Let a_S be a regular (p, p_1)-atom, so that (3.5.19) and (3.5.20) are satisfied. Then $a_S \in h^p_1(\partial\Omega) = F^{p,2}_1(\partial\Omega) \hookrightarrow B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega)$ so that $\|a_S\|_{B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega)} \leq C$, for some finite $C > 0$ independent of the particular atom. Since, after an integration by parts,

$$\int_{\partial\Omega} \partial_{\tau_{jk}} a_S \, d\sigma = 0, \qquad j,k = 1,2,...,n, \tag{3.5.26}$$

we see that $\partial_{\tau_{jk}} a_S$ is a (p,p_1)-atom for each $j,k=1,2,...,n$. In particular, the estimate $\sum_{j,k}\|\partial_{\tau_{jk}} a_S\|_{h^p(\partial\Omega)} \leq C$ holds for a finite constant independent of a_S. This proves the left-to-right inclusion in (3.5.24).

To prove the right-to-left inclusion in (3.5.24), we first observe that by classical embedding results, $B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$ for some $q>1$. Let now f be an arbitrary function belonging to the space in the right-hand side of (3.5.24) and pick $\psi \in \mathrm{Lip}\,(\partial\Omega)$. We claim that ψf belongs to the space in the right-hand side of (3.5.24) as well. Indeed, by Theorem 3.3.3, $\psi f \in B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega)$ and, for each $1\leq j,k\leq n$, Leibnitz's product rule gives $\partial_{\tau_{jk}}(\psi f) = \psi\partial_{\tau_{jk}} f + f\partial_{\tau_{jk}}\psi =: I+II$. Note that $I\in h^p(\partial\Omega)$, by (3.5.15), since $\partial_{\tau_{jk}} f\in h^p(\partial\Omega)$ and ψ is Lipschitz. Also, by the above discussion, $II\in L^q(\partial\Omega)$ for some $q>1$ since $\partial_{\tau_{jk}}\psi\in L^\infty(\partial\Omega)$. Thus, further, $II\in h^p(\partial\Omega)$ by (3.5.17).

This reasoning shows that there is no loss of generality in assuming that f has small support, say contained in some coordinate patch belonging to some atlas of $\partial\Omega$. Assuming that this is the case, we can pull f back to $\mathbb{R}^{n-1}$ giving rise to a compactly supported function $\bar{f}\in B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\mathbb{R}^{n-1}) \hookrightarrow L^q(\mathbb{R}^{n-1}) \hookrightarrow h^p(\mathbb{R}^{n-1}) = F^{p,2}_0(\mathbb{R}^{n-1})$ and such that $\nabla\bar{f}\in h^p(\mathbb{R}^{n-1}) = F^{p,2}_0(\mathbb{R}^{n-1})$. Invoking the lifting theorem (3.2.35) gives that $\bar{f}\in F^{p,2}_1(\mathbb{R}^{n-1})$ which, ultimately, proves that $f\in h^p_1(\partial\Omega)$, as desired. □

Remark. Inspection of the proof shows that the same result remains valid with the Besov space $B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega)$ replaced by the Lebesgue space $L^q(\partial\Omega)$ for any q

such that

$$\frac{1}{p}-\frac{1}{n-1}\leq\frac{1}{q}<1. \tag{3.5.27}$$

Assume that $\Omega\subset\mathbb{R}^n$ is a Lipschitz domain and that $\frac{n-1}{n}<p\leq 1$. Fix $1<p_1<\infty$ and $b>1/p-1/p_1$ and set $\theta:=(1/p-1/p_1)/b\in(0,1)$. A measurable function $m:\partial\Omega\to\mathbb{R}$ is called a (p_1,b)-molecule for $h^p(\partial\Omega)$ if either

(I): there exists $x_o\in\partial\Omega$ such that the following two conditions are satisfied

$$\Big(\int_{\partial\Omega}|m(x)|^{p_1}\,d\sigma_x\Big)^{(1-\theta)/p_1}\cdot\Big(\int_{\partial\Omega}[|x-x_o|^{(n-1)b}|m(x)|]^{p_1}\,d\sigma_x\Big)^{\theta/p_1}\leq 1, \tag{3.5.28}$$

$$\int_{\partial\Omega}m(x)\,d\sigma_x=0, \tag{3.5.29}$$

or (II):

$$\|m\|_{L^{p^1}(\partial\Omega)}\leq 1. \tag{3.5.30}$$

Proposition 3.5.6. *For any Lipschitz domain $\Omega\subset\mathbb{R}^n$, $(n-1)/n<p\leq 1$ and $1<p_1<\infty$, $b>1/p-1/p_1$, $\theta:=(1/p-1/p_1)/b\in(0,1)$, there exists a finite constant $M=M(\partial\Omega,p,p_1,b)>0$ such that*

$$\|m\|_{h^p(\partial\Omega)}\leq M \tag{3.5.31}$$

whenever m is a (p_1,b)-molecule for $h^p(\partial\Omega)$.

Proof. The proposition was proved in **[25]** with the exception of case when molecules satisfy (3.5.30) only. However, in this latter situation (3.5.31) follows immediately from (3.5.17) with $q=p_1$. □

Lemma 3.5.7. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider a bounded, linear operator*

$$T : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{3.5.32}$$

such that there exists a locally bounded function $k(x,y) : \{(x,y) : x, y \in \partial\Omega,\ x \neq y\} \to \mathbb{R}$ with the following properties:

(i) For each $f \in L^2(\partial\Omega)$,

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y)\, d\sigma_y, \qquad x \in \partial\Omega \setminus \operatorname{supp} f. \tag{3.5.33}$$

(ii) There exist $C_0, C_1 > 0$ such that

$$|k(x,y)| \leq C_0 |x-y|^{-(n-1)}, \tag{3.5.34}$$

$$|k(x,y) - k(x,y_0)| \leq C_0 \frac{|y-y_0|}{|x-y_0|^n}, \quad \text{uniformly for } |y-y_0| < C_1|x-y_0|. \tag{3.5.35}$$

If, in addition, $T^*(1) = 0$ in the sense that

$$f \in L^2(\partial\Omega) \text{ with compact support, } \int_{\partial\Omega} f\, d\sigma = 0 \Longrightarrow \int_{\partial\Omega} Tf\, d\sigma = 0, \tag{3.5.36}$$

then T extends as a bounded operator

$$T : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{3.5.37}$$

for every $\frac{n-1}{n} < p \leq 1$.

Proof. The crux of the matter is to show that T maps atoms in $h^p(\partial\Omega)$ into (a fixed multiples of) molecules in $h^p(\partial\Omega)$. This is a well-known approach and we closely follow [**25**].

Assume that a is a $(p,2)$ atom for $h^p(\partial\Omega)$ so that the conditions (3.5.12) are satisfied with some $\eta = \eta(\partial\Omega)$. If $r \geq \eta$, then $\|a\|_{L^2(\partial\Omega)} \leq \eta^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}$ and L^2-boundedness of operator T immediately implies that Ta satisfies (3.5.30) modulo some fixed multiplicative constant depending on $\partial\Omega, n, p$ and T and hence is a multiple of a $(2,b)$-molecule.

Suppose now that $r < \eta$. In that case we proceed as follows. First of all, the vanishing moment condition on a along with (3.5.36) yields (3.5.29). Going further, we use (3.5.32) and size estimates on atom to write

$$\begin{aligned}
&\Big(\int_{\partial\Omega} |Ta(x)|^2\,d\sigma_x\Big)^{(1-\theta)/2}\cdot\Big(\int_{\partial\Omega}[|x-x_o|^{(n-1)b}|Ta(x)|]^2\,d\sigma_x\Big)^{\theta/2}\\
&\qquad\leq Cr^{(1-\theta)(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}\Big(\int_{\substack{x\in\partial\Omega\\ |x-x_o|<C_1r}}[|x-x_o|^{(n-1)b}|Ta(x)|]^2\,d\sigma_x\\
&\qquad\qquad+\int_{\substack{x\in\partial\Omega\\ |x-x_o|\geq C_1r}}[|x-x_o|^{(n-1)b}|Ta(x)|]^2\,d\sigma_x\Big)^{\theta/2}\\
&\qquad\leq C+Cr^{(1-\theta)(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}\Big(\int_{\substack{x\in\partial\Omega\\ |x-x_o|\geq C_1r}}[|x-x_o|^{(n-1)b}|Ta(x)|]^2\,d\sigma_x\Big)^{\theta/2}. \qquad (3.5.38)
\end{aligned}$$

Using the vanishing moment condition on atom we obtain for every $x\in\partial\Omega\setminus\operatorname{supp} a$

$$Ta(x)=\int_{\partial\Omega}k(x,y)a(y)\,d\sigma_y=\int_{\partial\Omega}(k(x,y)-k(x,x_o))a(y)\,d\sigma_y, \qquad (3.5.39)$$

and therefore if $|x-x_o|\geq C_1r$

$$|Ta(x)|\leq C\int_{\operatorname{supp} a}\frac{|y-x_o|}{|x-x_o|^n}|a(y)|\,d\sigma_y\leq C\frac{r^{1+(n-1)\left(1-\frac{1}{p}\right)}}{|x-x_o|^n}. \qquad (3.5.40)$$

This yields that the expression in (3.5.38) is bounded by

$$C+Cr^{(1-\theta)(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}r^{\frac{\theta}{2}+(n-1)\left(1-\frac{1}{p}\right)\frac{\theta}{2}}\Big(\int_{\substack{x\in\partial\Omega\\ |x-x_o|\geq C_1r}}[|x-x_o|^{(n-1)b-n}]^2\,d\sigma_x\Big)^{\theta/2}\leq C, \qquad (3.5.41)$$

provided that the integral converges. This however is true for every $b < \frac{n+1}{2(n-1)}$. Note that such a choice of $b > \frac{1}{p} - \frac{1}{2}$ is always possible since $p < \frac{n-1}{n}$.

All in all, fixed multiple of Ta satisfies (3.5.28)–(3.5.29) with $p_1 = 2$, $\frac{1}{p} - \frac{1}{2} < b < \frac{n+1}{2(n-1)}$ and hence is a molecule for $h^p(\partial\Omega)$, as desired. □

Recall next the boundary layer potential operators K, K^* and S introduced in (2.3.6)-(2.3.7).

Proposition 3.5.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $\frac{n-1}{n} < p < \infty$, $0 \leq s \leq 1$, $0 \leq q \leq \infty$ and the triplet $\left(s, \frac{1}{p}, \frac{1}{q}\right)$ belongs to the interior of the convex hull of the following points:*

$$(0,0,0), \quad (1,0,0), \quad \left(1,0,\tfrac{1}{2}\right), \quad \left(0,0,\tfrac{1}{2}\right), \quad \left(\tfrac{1}{n},0,\tfrac{n+1}{2n}\right),$$

$$(0,1,1), \quad \left(1,\tfrac{n}{n-1},\tfrac{n}{n-1}\right), \quad \left(0,1,\tfrac{1}{2}\right), \quad \left(1,\tfrac{n}{n-1},\tfrac{1}{2}\right), \quad \left(\tfrac{2}{n+1},\tfrac{1}{n+1},0\right), \tag{3.5.42}$$

or the face that can be described by inequalities

$$0 < s < 1, \quad \max\left\{\tfrac{2}{s}, \tfrac{n-1}{1-s}\right\} < p \leq \infty, \quad q = \infty, \tag{3.5.43}$$

or one of the segments

$$1 < p < \infty, \quad s = 0, \quad q = 2, \quad \text{or} \quad \tfrac{n-1}{n} < p < \infty, \quad s = 1, \quad q = 2. \tag{3.5.44}$$

Then the operators

$$K : F^{p,q}_s(\partial\Omega) \longrightarrow F^{p,q}_s(\partial\Omega), \tag{3.5.45}$$

$$K^* : F^{p,q}_{s-1}(\partial\Omega) \longrightarrow F^{p,q}_{s-1}(\partial\Omega), \tag{3.5.46}$$

$$S : F^{p,q}_{s-1}(\partial\Omega) \longrightarrow F^{p,q}_s(\partial\Omega), \tag{3.5.47}$$

are well-defined and bounded. Furthermore, if $(n-1)/n < p \leq \infty$ *and* $(n-1)(\frac{1}{p} - 1)_+ < s < 1$, $0 < q \leq \infty$, *then the operators*

$$K : B_s^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega), \tag{3.5.48}$$

$$K^* : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s-1}^{p,q}(\partial\Omega), \tag{3.5.49}$$

$$S : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega), \tag{3.5.50}$$

are also well-defined and bounded.

Proof. The boundedness of the operators (3.5.45)-(3.5.47) is well-known when $1 < p < \infty$, $q = 2$ and $s \in \{0, 1\}$. This follows from **[39]** and **[24]**. When $\frac{n-1}{n} < p \leq 1$, $q = 2$, $s = 1$, the claim about (3.5.46) follows straight from Lemma 3.5.7, whereas the claim about (3.5.47) is a consequence of Proposition 3.5.5 (cf. also the remark following its proof) and Lemma 3.5.7.

Consider next the operator (3.5.45) in the case $\frac{n-1}{n} < p \leq 1$, $q = 2$, $s = 1$. Then $F_1^{p,2}(\partial\Omega) = h_1^p(\partial\Omega)$, and it suffices to study the action of K on a regular (p, p_1)-atom, i.e., a function $a = a_S$ satisfying (3.5.19), for some surface ball $S = S_r$ and $1 < p_1 \leq \infty$. Note that, for each fixed pair of indices $j, k \in \{1, ..., n\}$, we have $\int_{\partial\Omega} \partial_{\tau_{jk}}(Ka)\, d\sigma = 0$. Also, an integration by parts argument allows us to write

$$\partial_{\tau_{jk}}(Ka) = -\tfrac{1}{2}\partial_{\tau_{jk}} a + \sum_\ell \Big(\nu_k \partial_\ell S(\partial_{\tau_{\ell j}} a) - \nu_j \partial_\ell S(\partial_{\tau_{\ell k}} a) \Big). \tag{3.5.51}$$

Thus, since $\partial_{\tau_{jk}} a$ is an atom for $h^p(\partial\Omega)$ for any $j, k \in \{1, ..., n\}$, Lemma 3.5.7 (cf. also the remark made right after its statement), in concert with Proposition 3.5.6, proves that there exists a finite constant $C = C(\partial\Omega, p) > 0$ such that

$$\|\partial_{\tau_{jk}}(Ka)\|_{h^p(\partial\Omega)} \leq C, \qquad \forall\, j, k \in \{1, ..., n\}. \tag{3.5.52}$$

With this in hand, Proposition 3.5.5 gives that

$$\|Ka\|_{h^p_1(\partial\Omega)} \leq C, \tag{3.5.53}$$

which further implies that K maps $h^p_1(\partial\Omega)$ boundedly into itself.

The fact that the operators (3.5.45)-(3.5.47) and (3.5.48)-(3.5.50) are bounded for required range range of indices now follows by interpolation. More specifically, we could show that the operators (3.5.45)-(3.5.47) are bounded provided that s, p, q satisfy (3.5.44). Then the complex interpolation using identity (3.5.10) yields that the same operators are bounded if

$$\begin{aligned} &1 < p < \infty, \quad q = 2, \quad 0 \leq s \leq 1 \quad \text{or} \\ &\tfrac{n-1}{n} < p \leq 1, \quad q = 2, \quad (n-1)\big(\tfrac{1}{p} - 1\big) < s \leq 1. \end{aligned} \tag{3.5.54}$$

Going further, the real interpolation between the spaces with the same index p and different smoothness (cf. (3.5.11)) allows to obtain (3.5.48)-(3.5.50) for $p < \infty$ and in particular, gives (3.5.45)-(3.5.47) for

$$\tfrac{n-1}{n} < p = q < \infty, \quad 0 < (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1. \tag{3.5.55}$$

Now we again interpolate by complex method, this time between two regions described in (3.5.54) and the region described in (3.5.55), and denote the result by $\mathcal{R}_1$.

With this at hand, note that by duality the region where the operators (3.5.45)-(3.5.47) are bounded should be stable under the affine transformation $(s, 1/p, 1/q) \mapsto (1-s, 1-1/p, 1-1/q)$ when $p \geq 1$ and under the transformation $(s, p, q) \mapsto (1 - s + (n-1)(1/p - 1), \infty, \infty)$ if $p < 1$.

Consider first the case when $1 \leq p < \infty$, $1 \leq q \leq \infty$. The corresponding part of $\mathcal{R}_1$ is a convex hull of $1 \leq p < \infty$, $1 \leq q \leq \infty$ parts of (3.5.54)-(3.5.55) (stable under

the transformation caused by duality) and points $\big(\frac{n-1}{n}, 1, \frac{n-1}{2n}\big)$, $\big(\frac{n-1}{n+1}, \frac{n}{n+1}, 1\big)$ (which are mapped by aforementioned transformations to points $\big(\frac{1}{n}, 0, \frac{n+1}{2n}\big)$, $\big(\frac{2}{n+1}, \frac{1}{n+1}, 0\big)$). Another round of complex interpolation, between $\mathcal{R}_1$ and the points $\big(\frac{1}{n}, 0, \frac{n+1}{2n}\big)$, $\big(\frac{2}{n+1}, \frac{1}{n+1}, 0\big)$, finishes the proof of (3.5.45)-(3.5.47). (One can observe that the above transformation applied to the points from the parts of $\mathcal{R}_1$ corresponding to $p < 1$, $0 < q \leq \infty$ and $q < 1$, $0 < p < \infty$ does not enlarge the final polyhedron).

Finally, the argument above based on duality, in particular, gives $p = \infty$, $q = \infty$ case of (3.5.48)-(3.5.50) and hence, the $p = \infty$, $0 < q \leq \infty$ case of (3.5.48)-(3.5.50) by real interpolation. □

3.6 Function spaces in the interior of a Lipschitz domain

Given an arbitrary open subset Ω of $\mathbb{R}^n$, we denote by $f|_\Omega \in D'(\Omega)$ the restriction of a distribution $f \in D'(\mathbb{R}^n)$ to Ω. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ we then set

$$\begin{aligned} &A^{p,q}_s(\Omega) := \{f \in D'(\Omega) : \exists\, g \in A^{p,q}_s(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \\ &\|f\|_{A^{p,q}_s(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} : g \in A^{p,q}_s(\mathbb{R}^n),\ g|_\Omega = f\}, \qquad f \in A^{p,q}_s(\Omega). \end{aligned} \tag{3.6.1}$$

The convention we make in (3.6.1) is that either $A = F$ and $p < \infty$ or $A = B$, corresponding to, respectively, the definition of Besov and Triebel-Lizorkin spaces in Ω. Hardy, Sobolev (or Bessel potential) and bmo spaces are defined analogously, namely

$$\begin{aligned} &L^p_s(\Omega) := \{f \in D'(\Omega) : \exists\, g \in L^p_s(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \quad 1 < p < \infty,\ s \in \mathbb{R}, \\ &h^p(\Omega) := \{f \in D'(\Omega) : \exists\, g \in h^p(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \quad 0 < p \leq 1, \\ &\mathrm{bmo}(\Omega) := \{f \in L^2_{\mathrm{loc}}(\Omega) : \exists\, g \in \mathrm{bmo}(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \end{aligned} \tag{3.6.2}$$

equipped, in each case, with the natural, infimum-type, (quasi-)norms.

By (3.2.54)-(3.2.58), it follows that

$$C^s(\Omega) = B^{\infty,\infty}_s(\Omega), \quad 0 < s \notin \mathbb{Z}, \tag{3.6.3}$$

$$L^p(\Omega) = F^{p,2}_0(\Omega), \quad 1 < p < \infty, \tag{3.6.4}$$

$$L^p_s(\Omega) = F^{p,2}_s(\Omega), \quad 1 < p < \infty, \qquad s \in \mathbb{R}, \tag{3.6.5}$$

$$h^p(\Omega) = F^{p,2}_0(\Omega), \quad 0 < p \leq 1, \tag{3.6.6}$$

$$W^{k,p}(\Omega) = F^{p,2}_k(\Omega), \quad 1 < p < \infty, \quad k \in \mathbb{N}, \tag{3.6.7}$$

$$\mathrm{bmo}(\Omega) = F^{\infty,2}_0(\Omega), \tag{3.6.8}$$

$$B^{p,p}_s(\Omega) = F^{p,p}_s(\Omega), \quad s \in \mathbb{R}, \quad 0 < p \leq \infty, \tag{3.6.9}$$

where $C^s(\Omega)$ and $L^p(\Omega)$ are, respectively, the standard Hölder and Lebesgue spaces in Ω. It is immediate from these definitions that the restriction operator

$$\mathcal{R}_\Omega : S'(\mathbb{R}^n) \longrightarrow D'(\Omega), \qquad \mathcal{R}_\Omega f := f|_\Omega, \tag{3.6.10}$$

induces a linear and bounded operator in each of the following instances:

$$\begin{aligned} &\mathcal{R}_\Omega : B^{p,q}_s(\mathbb{R}^n) \longrightarrow B^{p,q}_s(\Omega), \quad 0 < p, q \leq \infty, \ s \in \mathbb{R}, \\ &\mathcal{R}_\Omega : F^{p,q}_s(\mathbb{R}^n) \longrightarrow F^{p,q}_s(\Omega), \quad 0 < p < \infty, \ 0 < q \leq \infty, \ s \in \mathbb{R}. \end{aligned} \tag{3.6.11}$$

Let us also note that for each $\alpha \in \mathbb{N}^n_o$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$,

$$\partial^\alpha : F^{p,q}_s(\Omega) \longrightarrow F^{p,q}_{s-|\alpha|}(\Omega) \tag{3.6.12}$$

$$\partial^\alpha : B^{p,q}_s(\Omega) \longrightarrow B^{p,q}_{s-|\alpha|}(\Omega) \tag{3.6.13}$$

are bounded operators.

We now pause to record a useful characterization of the local Hardy space $h^p(\Omega)$. First, we need some notation. Fix $\psi \in C_c^\infty(B(0,1))$ such that $\int_{B(0,1)} \psi(x)\,dx = 1$ and set $\psi_t(x) := t^{-n}\psi(x/t)$. Then the radial maximal function of a distribution u in Ω is defined as

$$u^+(x) := \sup_{0<t<\operatorname{dist}(x,\partial\Omega)} |(\psi_t * u)(x)|, \qquad x \in \Omega. \tag{3.6.14}$$

For $u \in D'(\Omega)$, $k \in \mathbb{N}_o$ and $x \in \mathbb{R}^n$ introduce

$$u^*_{k,\Omega}(x) := \sup\{|\langle u, \psi\rangle| : \psi \in \Psi_x\} \tag{3.6.15}$$

where the class Ψ_x consists of all functions $\psi \in C_c^\infty(\mathbb{R}^n)$ with the property that there exists $r = r_\psi > 0$ with $\operatorname{supp}\psi \subset B(x,r) \cap \Omega$ and $\|\partial^\gamma \psi\|_{L^\infty(\mathbb{R}^n)} \leq r^{-n-|\gamma|}$ for each $\gamma \in \mathbb{N}_o^n$, $|\gamma| \leq k$.

Theorem 3.6.1. *(cf.* **[101]**, **[100]***) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fix ψ as above and define the radial maximal function as in (3.6.14). Then, for any $0 < p \leq 1$ and any $u \in D'(\Omega)$*

$$u \in h^p(\Omega) \iff u^+ \in L^p(\Omega), \tag{3.6.16}$$

with equivalence of quasi-norms.

Furthermore, if $k \in \mathbb{N}$ and $\frac{n}{n+k} < p \leq 1$, then

$$\|u^*_{k,\Omega}\|_{L^p(\mathbb{R}^n)} \approx \|u^+\|_{L^p(\Omega)}. \tag{3.6.17}$$

In particular, a different choice of the function ψ affects the size of u^+ in $L^p(\Omega)$ only up to a fixed multiplicative constant.

Finally, similar results are valid in the range $1 < p < \infty$ provided $h^p(\Omega)$ is replaced by $L^p(\Omega)$.

In analogy with the Hardy-based Sobolev spaces in $\mathbb{R}^n$ introduced in (3.2.6)-(3.2.7), for an open set $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_o$ and $0 < p \leq 1$, we set

$$h^p_k(\Omega) = \{u \in D'(\Omega) : \partial^\gamma u \in h^p(\Omega),\ \forall\, \gamma \in \mathbb{N}^n_o \text{ with } |\gamma| \leq k\}, \tag{3.6.18}$$

equipped with the quasi-norm $\|u\|_{h^p_k(\Omega)} := \sum_{|\gamma|=k} \|\partial^\gamma u\|_{h^p(\Omega)}$, and

$$h^p_{-k}(\Omega) := \Big\{u \in D'(\Omega) : u = \sum_{|\gamma|\leq k} \partial^\gamma u_\gamma,\ u_\gamma \in h^p(\Omega)\ \forall\, \gamma \in \mathbb{N}^n_o \text{ with } |\gamma| \leq k\Big\} \tag{3.6.19}$$

equipped with $\|u\|_{h^p_{-k}(\Omega)} := \inf \sum_{|\gamma|\leq k} \|u_\gamma\|_{h^p(\Omega)}$, where the infimum is taken over all representations of u.

Theorem 3.6.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $0 < p \leq 1$ and that $k \in \mathbb{Z}$ is either negative, or else satisfies $k > n(1/p - 1)$. Then*

$$h^p_k(\Omega) = F^{p,2}_k(\Omega). \tag{3.6.20}$$

Proof. Assume first that $k < 0$ and note that the inclusion $h^p_k(\Omega) \hookrightarrow F^{p,2}_k(\Omega)$ is immediate from definitions, (3.6.12), and (3.6.6) To see the opposite inclusion fix $u \in F^{p,2}_k(\Omega)$, say $u = w|_\Omega$, $w \in F^{p,2}_k(\mathbb{R}^n)$. Since, by Theorem 3.2.1, w can be represented in the form $w = \sum_{|\gamma|\leq -k} \partial^\gamma w_\gamma$, $w_\gamma \in h^p(\mathbb{R}^n)$, it follows that $u = \sum_{|\gamma|\leq -k} \partial^\gamma (w_\gamma|_\Omega)$ and $w_\gamma|_\Omega \in h^p(\Omega)$. Consequently, $u \in h^p_k(\Omega)$, proving the right-to-left inclusion in (3.6.20).

The case when $k \in \mathbb{N}$ satisfies $k > n(1/p - 1)$ is essentially due to A. Miyachi (cf. [**101**], [**102**]

As on p. 80 of [**102**], let us temporarily introduce

$$\mathcal{W}^k_p(\Omega) := \Big\{u \in h^p(\Omega) : \partial^\gamma u \in h^p(\Omega)\ \ \forall\, \gamma \in \mathbb{N}^n_o \text{ with } |\gamma| = k\Big\} \tag{3.6.21}$$

and observe that, by virtue of the last remark in §4 of **[102]**,

$$\mathcal{W}_p^k(\Omega) = \mathcal{C}_p^k(\Omega), \tag{3.6.22}$$

where $\mathcal{C}_p^k(\Omega)$ is the space introduced by R. DeVore and R. Sharpley in §6 of **[36]**. Due to the extension resuts for the latter spaces proved in **[99]** (cf. Theorem 4/(ii) p. 1035 *loc. cit.*), it follows that $\mathcal{W}_p^k(\Omega) = \{u|_\Omega : u \in \mathcal{W}_p^k(\mathbb{R}^n)\}$ However, $\mathcal{W}_p^k(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n)$, thanks to (3.2.35) and (3.1.16) so that, altogether,

$$\mathcal{W}_p^k(\Omega) = F_k^{p,2}(\Omega). \tag{3.6.23}$$

Consequently, $h_k^p(\Omega) \hookrightarrow \mathcal{W}_p^k(\Omega) = F_k^{p,2}(\Omega)$, proving the left-to-right inclusion in (3.6.20). The opposite inclusion is a direct consequence of (3.6.12) and (3.6.6) and this completed the proof of the theorem. □

Returning to the mainstream discussion, let us single out two other types of function spaces which will play an important role for us later on. We continue to assume that Ω is an arbitrary open subset of $\mathbb{R}^n$. First, for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we set

$$\begin{aligned} &A_{s,0}^{p,q}(\Omega) := \{f \in A_s^{p,q}(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \\ &\|f\|_{A_{s,0}^{p,q}(\Omega)} := \|f\|_{A_s^{p,q}(\mathbb{R}^n)}, \quad f \in A_{s,0}^{p,q}(\Omega), \end{aligned} \tag{3.6.24}$$

where, as usual, either $A = F$ and $p < \infty$ or $A = B$. Thus, $B_{s,0}^{p,q}(\Omega)$, $F_{s,0}^{p,q}(\Omega)$ are closed subspaces of $B_{s,0}^{p,q}(\mathbb{R}^n)$ and $F_{s,0}^{p,q}(\mathbb{R}^n)$, respectively. In the same vein, we also define

$$\begin{aligned} &L_{s,0}^p(\Omega) := \{f \in L_s^p(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad 1 < p < \infty, \ s \in \mathbb{R}, \\ &h_0^p(\Omega) := \{f \in h^p(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad 0 < p \leq 1, \\ &\operatorname{bmo}_0(\Omega) := \{f \in \operatorname{bmo}(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \end{aligned} \tag{3.6.25}$$

with the norms inherited from $L^p_{s,0}(\mathbb{R}^n)$, $h^p(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ respectively. We understand $\mathrm{vmo}_0(\Omega)$ as the closure of $C_c(\Omega)$, the space of continuous functions with support in Ω, in $\mathrm{bmo}_0(\Omega)$.

Second, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we introduce

$$\begin{aligned} &A^{p,q}_{s,z}(\Omega) := \{f \in D'(\Omega) : \exists\, g \in A^{p,q}_{s,0}(\Omega) \text{ such that } g|_\Omega = f\}, \\ &\|f\|_{A^{p,q}_{s,z}(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} : g \in A^{p,q}_{s,0}(\Omega),\ g|_\Omega = f\}, \qquad f \in A^{p,q}_{s,z}(\Omega), \end{aligned} \tag{3.6.26}$$

(where, as before, $A = F$ and $p < \infty$ or $A = B$) and, in keeping with earlier conventions,

$$L^p_{s,z}(\Omega) := F^{p,2}_{s,z}(\Omega) = \{f \in D'(\Omega) : \exists\, g \in L^p_{s,0}(\Omega) \text{ such that } g|_\Omega = f\}, \tag{3.6.27}$$

if $1 < p < \infty$, $s \in \mathbb{R}$, and

$$h^p_z(\Omega) := F^{p,2}_{0,z}(\Omega) = \{f \in D'(\Omega) : \exists\, g \in h^p_0(\Omega) \text{ such that } g|_\Omega = f\}, \text{ if } 0 < p \leq 1, \tag{3.6.28}$$

once again equipped with natural, infimum-type, (quasi-)norms. Finally,

$$C^\alpha_z(\Omega) := \{u \in C^\alpha(\overline{\Omega}) : u|_{\partial\Omega} = 0\}, \quad 0 < \alpha < 1, \tag{3.6.29}$$

$$\mathrm{bmo}_z(\Omega) := \{u|_\Omega : u \in \mathrm{bmo}\,(\mathbb{R}^n) \text{ with } \mathrm{supp}\, u \subseteq \overline{\Omega}\}. \tag{3.6.30}$$

It follows that the restriction operator (3.6.10) induces linear, continuous mappings

$$\begin{aligned} &\mathcal{R}_\Omega : B^{p,q}_{s,0}(\Omega) \longrightarrow B^{p,q}_{s,z}(\Omega),\ 0 < p, q \leq +\infty,\ s \in \mathbb{R}, \\ &\mathcal{R}_\Omega : F^{p,q}_{s,0}(\Omega) \longrightarrow F^{p,q}_{s,z}(\Omega),\ 0 < p, q \leq +\infty,\ s \in \mathbb{R}, \\ &\mathcal{R}_\Omega : L^p_{s,0}(\Omega) \longrightarrow L^p_{s,z}(\Omega),\ 1 < p < +\infty,\ s \in \mathbb{R}, \\ &\mathcal{R}_\Omega : h^p_0(\Omega) \longrightarrow h^p_z(\Omega),\ 0 < p \leq 1. \end{aligned} \tag{3.6.31}$$

In many instances it is important to establish whether there is a linear, bounded, extension operator, i.e., a right inverse for the various manifestations of the $\mathcal{R}_\Omega$. In the case of (3.6.11) when Ω is an arbitrary Lipschitz domain this problem has been solved in full generality by V. Rychkov (cf. [**113**]) who proved the following.

Theorem 3.6.3. *([**113**]) Let $\Omega \subset \mathbb{R}^n$ be either a bounded Lipschitz domain, the exterior of a bounded Lipschitz domain, or an unbounded Lipschitz domain. Then there exists a linear, continuous operator $E_\Omega : D'(\Omega) \to S'(\mathbb{R}^n)$ such that whenever $0 < p, q \leq +\infty$, $s \in \mathbb{R}^n$, then*

$$E_\Omega : A^{p,q}_s(\Omega) \longrightarrow A^{p,q}_s(\mathbb{R}^n) \ \text{ boundedly, satisfying } \mathcal{R}_\Omega \circ E_\Omega f = f, \ \ \forall\, f \in A^{p,q}_s(\Omega), \tag{3.6.32}$$

for $A = B$ or $A = F$, in the latter case assuming $p < \infty$.

This type of results has many forerunners. See [**125**] and [**44**] for the case of smooth domain. The theory on extension operators on Lipschitz domains was developed by different methods bringing corresponding restrictions on indices: Calderón's method [**17**], extended by E.M. Stein in [**118**] and then used by G.A. Kalyabin in [**45**] and T. Muramatu in [**104**], allowed to consider Banach space case only; A. Seeger [**115**], R.A. DeVore, R.C. Sharpley [**37**] and A. Miyachi [**103**] relied on approximation techniques and treated the spaces consisting of locally integrable functions; finally, the results on Hardy spaces were obtained in [**100**]. An informative account of these and related matters can be found in [**127**] and [**113**].

In addition to identifications (3.6.3)–(3.6.6) we would like to discuss the spaces

$$W_k^p(\Omega) := \Big\{ f \in L^p(\Omega); \quad \partial^\gamma f \in L^p(\Omega) \ \forall\ \gamma : \ |\gamma| \leq k \Big\}, \quad 1 < p < \infty, \ k \in \mathbb{N}, \tag{3.6.33}$$

with the norm given by

$$\|f\|_{W_k^p(\Omega)} = \sum_{|\gamma| \leq k} \|\partial^\gamma f\|_{L^p(\Omega)}. \tag{3.6.34}$$

It was proved in [**17**] that there exists a bounded linear extension operator

$$\mathcal{E}_k : W_k^p(\Omega) \longrightarrow W_k^p(\mathbb{R}^n) \text{ such that } \mathcal{R}_\Omega \circ E_k f = f, \ \forall\, f \in W_k^p(\Omega). \tag{3.6.35}$$

In view of (3.6.32), (3.6.5) and (3.2.56)–(3.2.57) this yields

$$W_k^p(\Omega) = L_k^p(\Omega) = F_k^{p,2}(\Omega), \quad 1 < p < \infty, \qquad k \in \mathbb{N}. \tag{3.6.36}$$

We now discuss the interpolation of some of the Besov and Triebel-Lizorkin spaces defined in this section and aim to establish the following analogues of Theorems 3.2.9 and 3.2.10 in Lipschitz domains.

Theorem 3.6.4. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(F_{\alpha_0}^{p,q_0}(\Omega), F_{\alpha_1}^{p,q_1}(\Omega))_{\theta,q} \ = \ B_\alpha^{p,q}(\Omega), \quad 0 < p < \infty, \tag{3.6.37}$$

$$(B_{\alpha_0}^{p,q_0}(\Omega), B_{\alpha_1}^{p,q_1}(\Omega))_{\theta,q} \ = \ B_\alpha^{p,q}(\Omega), \quad 0 < p \leq \infty. \tag{3.6.38}$$

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$ then

$$[F_{\alpha_0}^{p_0,q_0}(\Omega), F_{\alpha_1}^{p_1,q_1}(\Omega)]_\theta = F_\alpha^{p,q}(\Omega), \tag{3.6.39}$$

where $0<\theta<1$, $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ *and* $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$.

If $\alpha_0,\alpha_1\in\mathbb{R}$, $\alpha_0\neq\alpha_1$, $0<p_0,p_1,q_0,q_1\leq\infty$ *and either* $p_0+q_0<\infty$ *or* $p_1+q_1<\infty$ *then also*

$$[B^{p_0,q_0}_{\alpha_0}(\Omega),B^{p_1,q_1}_{\alpha_1}(\Omega)]_\theta=B^{p,q}_{\alpha}(\Omega). \tag{3.6.40}$$

where $0<\theta<1$, $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ *and* $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$.

Finally, the same interpolation results remain valid if the spaces $B^{p,q}_s(\Omega)$, $F^{p,q}_s(\Omega)$ *are replaced by* $B^{p,q}_{s,0}(\Omega)$ *and* $F^{p,q}_{s,0}(\Omega)$, *respectively.*

Proof. Thanks to the existence of the universal, bounded, linear extension operator reviewed in Theorem 3.6.3, the identities (3.6.37)-(3.6.39) follow from (3.2.72), Lemma 3.1.9 and Theorems 3.2.9, 3.2.10.

There remains to prove the last claim made in the statement of the theorem. To this end, we note that the operators

$$\begin{aligned}&I-E_{\mathbb{R}^n\setminus\overline{\Omega}}\circ\mathcal{R}_{\mathbb{R}^n\setminus\overline{\Omega}}:F^{p,q}_s(\mathbb{R}^n)\longrightarrow F^{p,q}_{s,0}(\Omega),\\ &I-E_{\mathbb{R}^n\setminus\overline{\Omega}}\circ\mathcal{R}_{\mathbb{R}^n\setminus\overline{\Omega}}:B^{p,q}_s(\mathbb{R}^n)\longrightarrow B^{p,q}_{s,0}(\Omega),\end{aligned} \tag{3.6.41}$$

are projections onto the target spaces (i.e., are linear, bounded and idempotent). Furthermore, it is apparent that a distribution $f\in S'(\mathbb{R}^n)$ belongs to the range of $I-E_{\mathbb{R}^n\setminus\overline{\Omega}}\circ\mathcal{R}_{\mathbb{R}^n\setminus\overline{\Omega}}$ if and only if $\text{supp}\,f\subseteq\overline{\Omega}$. Thus, the second part of Lemma 3.1.9 applies and yields the desired conclusion. □

Theorem 3.6.5. *Let* Ω *be an arbitrary Lipschitz domain. For* $0<p_0\leq p_1\leq+\infty$, $s_0,s_1\in\mathbb{R}$, $0<q_0\leq q_1\leq+\infty$ *with* $s_0-\frac{n}{p_0}=s_1-\frac{n}{p_1}$, *the inclusion*

$$B^{p_0,q_0}_{s_0}(\Omega)\hookrightarrow B^{p_1,q_1}_{s_1}(\Omega) \tag{3.6.42}$$

is continuous with dense range. Moreover, the same holds for the inclusion

$$F_{s_0}^{p_0,q_0}(\Omega) \hookrightarrow F_{s_1}^{p_1,q_1}(\Omega), \tag{3.6.43}$$

provided that either $0 < p_0 < p_1 < +\infty$, $0 < q_0, q_1 \le +\infty$ *and* $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ *or* $0 < p_0 = p_1 < +\infty$, $0 < q_0 < q_1 \le +\infty$ *and* $s_0 = s_1$.

The same results remain valid if the spaces $B_s^{p,q}(\Omega)$, $F_s^{p,q}(\Omega)$ *are replaced by* $B_{s,0}^{p,q}(\Omega)$ *and* $F_{s,0}^{p,q}(\Omega)$, *respectively, or by* $B_{s,z}^{p,q}(\Omega)$ *and* $F_{s,z}^{p,q}(\Omega)$, *respectively.*

Proof. The results immediately follow from Theorem 3.2.5 and definitions of spaces (3.6.1), (3.6.24), (3.6.26). □

Proposition 3.6.6. *(cf.* [**127**]*) Let* Ω *be a bounded Lipschitz domain,* $0 < p, q \le \infty$ *and* $\max\Big(1/p - 1, n(1/p - 1)\Big) < s$. *Then extension by zero defined as*

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.6.44}$$

induces a linear and bounded operator from $B_{s,z}^{p,q}(\Omega)$ *to* $B_{s,0}^{p,q}(\Omega)$ *and, if* $p < \infty$, *from* $F_{s,z}^{p,q}(\Omega)$ *to* $F_{s,0}^{p,q}(\Omega)$.

Furthermore, if $\max\Big(1/p - 1, n(1/p - 1)\Big) < s < 1/p$ *and* $0 < p, q < \infty$, *this operator also maps* $B_s^{p,q}(\Omega)$ *to* $B_{s,0}^{p,q}(\Omega)$ *and, if* $\min\{p, 1\} \le q$, $F_s^{p,q}(\Omega)$ *to* $F_{s,0}^{p,q}(\Omega)$.

We now discuss lifting results for Besov and Triebel-Lizorkin spaces in Lipschitz domains.

Theorem 3.6.7. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$.

(i) Then, for each $\alpha > 0$ *and* $1 \le p, q \le \infty$,

$$f \in B_{\alpha+1}^{p,q}(\Omega) \Longleftrightarrow f \in L^p(\Omega) \ \text{ and } \ \nabla f \in B_\alpha^{p,q}(\Omega). \tag{3.6.45}$$

(ii) Suppose that $\alpha > 0$ and $1 < p, q < \infty$. Then

$$f \in F^{p,q}_{\alpha+1}(\Omega) \Longleftrightarrow f \in L^p(\Omega) \ \ and \ \ \nabla f \in F^{p,q}_{\alpha}(\Omega). \tag{3.6.46}$$

The first part of this theorem is proved in **[68]** (in fact, there one can also find (ii) when $q = 2$). The second part is proved in **[45]**.

For $h \in \mathbb{R}^n$, inductively define the difference operators

$$(\Delta^1_h f)(x) := f(x+h) - f(x), \qquad \Delta^{M+1}_h f := \Delta^1_h \Delta^M_h f, \quad M \in \mathbb{N}. \tag{3.6.47}$$

This definition is further adapted to the context of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ as follows:

$$(\Delta^M_{h,\Omega} f)(x) := \begin{cases} (\Delta^M_h f)(x) & \text{if } x + lh \in \Omega \text{ for } l = 0, 1, ..., M, \\ 0 & \text{otherwise,} \end{cases} \tag{3.6.48}$$

whenever f is defined on Ω and $x \in \Omega$.

Theorem 3.6.8. *(cf.* **[38]***) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fix*

$$0 < p, q \leq \infty, \qquad n\Big(\frac{1}{p} - 1\Big)_+ < s < M \in \mathbb{N} \tag{3.6.49}$$

and set $\bar{p} := \max\{p, 1\}$. Then $f \in B^{p,q}_s(\Omega)$ if and only if

$$\|f\|_{L^{\bar{p}}(\Omega)} + \left(\int_0^1 t^{-sq} \sup_{|h| \leq t} \|\Delta^M_{h,\Omega} f\|^q_{L^p(\Omega)} \, \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \tag{3.6.50}$$

and the expression in (3.6.50) is an equivalent norm on $B^{p,q}_s(\Omega)$.

3.7 Local atomic decompositions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Of course, since $B^{p,q}_{s,0}(\Omega)$, $F^{p,q}_{s,0}(\Omega)$ are closed subspaces of $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$,

respectively, distributions belonging to these spaces can be expanded into series of atoms according to Theorem 3.3.1. The novel aspect here is that, due to the extra support condition in the definitions of $B^{p,q}_{s,0}(\Omega)$, $F^{p,q}_{s,0}(\Omega)$ it is possible to ensure that the individual atoms are themselves supported in Ω. Our first main result in this regard is theorem below.

Theorem 3.7.1. *Let Ω be an unbounded Lipschitz domain in $\mathbb{R}^n$, and assume that $0 < p, q \leq +\infty$, $s \in \mathbb{R}$. Also, define $J := \frac{n}{\min\{1,p\}}$ and fix two integers, $K \geq ([s]+1)_+$ and $L \geq \max\{[J-n-s], -1\}$. Then there exists $\beta = \beta(\Omega) \in \mathbb{N}_o$ with the following significance. Any distribution $f \in B^{p,q}_{s,0}(\Omega)$ can be written in the form*

$$f = \sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)\leq 2^{-\beta}}} \lambda_Q a_Q \quad \text{with convergence in} \quad S'(\mathbb{R}^n), \tag{3.7.1}$$

where the atoms a_Q's satisfy

$$(1)\ \exists\, Q \in \mathcal{Q}_n : \operatorname{supp}(a_Q) \subseteq Q \text{ and } \rho\, Q \subset \Omega \text{ for some } \rho > 1, \tag{3.7.2}$$

$$(2)\ \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq C|Q|^{s/n-1/p-|\gamma|/n} \text{ if } |\gamma| \leq K, \tag{3.7.3}$$

$$(3)\ \int_{\mathbb{R}^n} x^\gamma a_Q(x)\, dx = 0 \text{ if } |\gamma| \leq L \text{ and } l(Q) < 2^{-\beta}, \tag{3.7.4}$$

the constant $\rho = \rho(\Omega)$ is independent of particular atom and the sequence of coefficients $\{\lambda_Q\}_Q$ is such that $\left(\sum_{j=\beta}^\infty \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)=2^{-j}}} |\lambda_Q|^p\Big)^{q/p}\right)^{1/q} < \infty$. Moreover,

$$\|f\|_{B^{p,q}_{s,0}(\Omega)} \approx \inf\Big\{ \left(\sum_{j=\beta}^\infty \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)=2^{-j}}} |\lambda_Q|^p\Big)^{q/p}\right)^{1/q} : f = \sum_{Q\in\mathcal{Q}_n,\, l(Q)\leq 2^{-\beta}} \lambda_Q a_Q \Big\}, \tag{3.7.5}$$

where the infimum is taken over all possible representations of f in a series of atoms satisfying (3.7.2)-(3.7.4).

A similar statement holds for Triebel-Lizorkin spaces granted that, this time, $J := \frac{n}{\min\{1,p,q\}}$ *and* $p < \infty$. *More specifically, any* $f \in F^{p,q}_{s,0}(\Omega)$ *can be expanded as in (3.7.1) where the atoms involved satisfy (3.7.2)-(3.7.4) and* $\Big\| \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)\leq 2^{-\beta}}} \Big(|Q|^{-1/p} |\lambda_Q| \chi_Q \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty$. *Furthermore,*

$$\|f\|_{F^{p,q}_{s,0}(\Omega)} \approx \inf\Big\{ \Big\| \Big(\sum_{\substack{Q\in\mathcal{Q}_n \\ l(Q)\leq 2^{-\beta}}} \Big(|Q|^{-1/p} |\lambda_Q| \chi_Q \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} : \, f = \sum_{Q\in\mathcal{Q}_n,\, l(Q)\leq 2^{-\beta}} \lambda_Q a_Q \Big\}, \tag{3.7.6}$$

where, again, the infimum is taken over all possible representations of f *in a series of atoms satisfying (3.7.2)-(3.7.4).*

We start by discussing a version of Calderón's reproducing formula for tempered distributions in $\mathbb{R}^n$, when the mollifiers involved are supported in a cone

$$\Gamma := \{(x', x_n) \in \mathbb{R}^n : \, |x'| < \kappa x_n\} \tag{3.7.7}$$

for some fixed $\kappa > 0$. To state this result, recall that for every function ϕ defined in $\mathbb{R}^n$ we denote by ϕ_j, $j \geq 1$, its dyadic dilates, i.e. $\phi_j(x) := 2^{jn}\phi(2^j x)$, $x \in \mathbb{R}^n$.

Lemma 3.7.2. *Let the cone* Γ *be as in (3.7.7) for some* $\kappa > 0$ *and consider an arbitrary function* $\phi_0 \in C^\infty_c(\Gamma)$ *with nonzero integral. Also, set* $\phi(x) := \phi_0(x) - 2^{-n}\phi_0(x/2)$. *Then for any* $L \in \mathbb{N}_o$ *there exist two functions,* $\psi_0 \in C^\infty_c(\Gamma)$ *and* $\psi \in C^\infty_c(\Gamma)$, *such that* L_ψ, *the order up to which moments of* ψ *vanish, is greater or equal than* L *and*

$$f = \sum_{j=0}^{\infty} \psi_j * \phi_j * f, \quad \text{with convergence in } S'(\mathbb{R}^n), \tag{3.7.8}$$

for every $f \in S'(\mathbb{R}^n)$. Furthermore, matters can be arranged so that

$$\operatorname{supp}(\psi_0) \subset \bigcup_{i=1}^{6^{L+2}} \langle i \operatorname{supp}\phi_0\rangle, \qquad \operatorname{supp}(\psi) \subset \bigcup_{i=1}^{2\times 6^{L+2}} \langle i \operatorname{supp}\phi_0\rangle, \tag{3.7.9}$$

where, for an arbitrary set $S \subseteq \mathbb{R}^n$ and $i \in \mathbb{N}$, we employ the notation

$$\langle iS\rangle := \Big\{\sum_{k=1}^{i} x_k : x_k \in S\Big\}. \tag{3.7.10}$$

Save for (3.7.9) this is a result due to V. Rychkov (cf. [**113**]) whose approach we follow closely. The reason for which we chose to present a detailed proof here is that it will be important for us to ensure that the support conditions described in (3.7.9) hold.

Proof of Lemma 3.7.2. Without loss of generality, we may assume that $\int_{\mathbb{R}^n} \phi_0(x)\,dx = 1$. Also, we denote the support of ϕ_0 by S, so that $S \subset \Gamma$. Set $g_0 := \phi_0 * \phi_0$ and $g := g_0 - 2^{-n} g_0(\cdot/2)$ so that Dirac's delta function can be represented in the form $\delta = \sum_{j=0}^{\infty} g_j$. This and simple algebraic manipulations further entail

$$\begin{aligned}
\delta &= \delta * \delta = \sum_{j=0}^{\infty} g_j * \sum_{k=0}^{\infty} g_k \\
&= g_0 * \big(g_0 + 2(\delta - g_0)\big) + \sum_{j=1}^{\infty} g_j * \big(g + 2(\delta - g_0)\big)_j \\
&= \phi_0 * \big(\phi_0 * (2\delta - g_0)\big) + \sum_{j=1}^{\infty} \phi_j * \big((\phi_0 + 2^{-n}\phi_0(\cdot/2)) * (g + 2\delta - 2g_0)\big)_j \\
&= \sum_{j=0}^{\infty} \phi_j * \Psi_j^1,
\end{aligned} \tag{3.7.11}$$

where

$$\Psi_0^1 := \phi_0 * (2\delta - g_0) \quad \text{and} \quad \Psi^1 := (\phi_0 + 2^{-n}\phi_0(\cdot/2)) * (g + 2\delta - 2g_0). \tag{3.7.12}$$

Recalling the convention (3.7.10), we may write

$$\operatorname{supp}(g_0) \subset \langle 2S\rangle, \quad \operatorname{supp}(g) \subset \langle 2S\rangle \cup \langle 4S\rangle \tag{3.7.13}$$

and, hence,

$$\operatorname{supp}(\Psi_0^1) \subseteq \bigcup_{i=1}^{3}\langle iS\rangle, \quad \operatorname{supp}(\Psi^1) \subseteq \bigcup_{i=1}^{6}\langle iS\rangle. \tag{3.7.14}$$

Note that $\int_{\mathbb{R}^n} g(x)\,dx = 0$ which forces $\int_{\mathbb{R}^n} \Psi^1(x)\,dx = 0$.

In the case $L = 0$ this finishes up the proof of the lemma (note that it actually suffices to prove (3.7.8) when f is the Dirac delta function). Otherwise, as in **[113]**, we observe that

$$\phi * \Psi^1 = \phi_0 * \Psi_0^1 - 2^{-n}(\phi_0 * \Psi_0^1)(\cdot/2), \tag{3.7.15}$$

and repeat the previous construction with $\phi_0 * \Psi_0^1$ and $\phi * \Psi^1$ playing the roles of ϕ_0 and ϕ, respectively. After $L+1$ iterations we arrive at an identity of the form

$$\delta = \sum_{j=0}^{\infty} \phi_j * \Psi_j^1 * ... * \Psi_j^{L+1}, \tag{3.7.16}$$

where $\int_{\mathbb{R}^n} \Psi^k(x)\,dx = 0$. Denote by ϕ_0^k, $k \geq 2$, the function playing a role of ϕ_0 in the k-th iteration of construction, i.e. given by formula $\phi_0^k := \phi_0^{k-1} * \Psi_0^{k-1}$ with $\phi_0^1 := \phi_0$. As inductive reasoning shows, for $k \geq 2$

$$\begin{aligned}\operatorname{supp}(\Psi_0^k) \quad &\subseteq \bigcup_{i=1}^{3}\langle i \operatorname{supp}(\phi_0^k)\rangle \subseteq \bigcup_{i=1}^{3}\langle i\,(\operatorname{supp}(\phi_0^{k-1}) + \operatorname{supp}(\Psi_0^{k-1}))\rangle \\ &\subset \bigcup_{i=1}^{3}\langle i\,(\operatorname{supp}(\Psi_0^{k-1}) + \operatorname{supp}(\Psi_0^{k-1}))\rangle \subseteq \bigcup_{i=1}^{6}\langle i \operatorname{supp}(\Psi_0^{k-1})\rangle,\end{aligned} \tag{3.7.17}$$

and hence,

$$\operatorname{supp}(\Psi_0^k) \subset \bigcup_{i=1}^{6^k}\langle iS\rangle, \qquad k = 1, 2, ... \tag{3.7.18}$$

Also for $k \geq 2$

$$\operatorname{supp}(\Psi^k) \subseteq \bigcup_{i=1}^{6} \langle i \operatorname{supp}(\phi_0^k) \rangle \subset \bigcup_{i=1}^{12} \langle i \operatorname{supp}(\Psi_0^{k-1}) \rangle \subseteq \bigcup_{i=1}^{12 \times 6^{k-1}} \langle iS \rangle. \tag{3.7.19}$$

Overall,

$$\operatorname{supp}(\Psi_0^k) \subset \bigcup_{i=1}^{6^k} \langle iS \rangle, \qquad \operatorname{supp}(\Psi^k) \subset \bigcup_{i=1}^{2 \times 6^k} \langle iS \rangle, \tag{3.7.20}$$

for every $k = 1, ..., L+1$. If we now introduce

$$\psi_0 := \Psi_0^1 * ... * \Psi_0^{L+1} \quad \text{and} \quad \psi := \Psi^1 * ... * \Psi^{L+1}, \tag{3.7.21}$$

it follows that $L_\psi \geq L$,

$$\operatorname{supp}(\psi_0) \subset \left(\bigcup_{i=1}^{6} \langle iS \rangle + \bigcup_{i=1}^{6^2} \langle iS \rangle + ... + \bigcup_{i=1}^{6^{L+1}} \langle iS \rangle \right) \subset \bigcup_{i=1}^{6+6^2+...+6^{L+1}} \langle iS \rangle \subset \bigcup_{i=1}^{6^{L+2}} \langle iS \rangle,$$

$$\operatorname{supp}(\psi) \subset \bigcup_{i=1}^{2 \times 6^{L+2}} \langle iS \rangle, \tag{3.7.22}$$

and, as in [**113**], $\delta = \sum_{j=0}^{\infty} \psi_j * \phi_j$, completing the proof of the lemma. □

Another technical prerequisite in the proof of Theorem 3.7.1 is the existence of a function ϕ_0 which satisfies (compare with (3.2.21)):

$$\begin{cases} \phi_0 \in C_c^\infty(\Gamma), \\ \int_{\mathbb{R}^n} \phi_0(x)\, dx \neq 0, \\ \phi(x) := \phi_0(x) - 2^{-n} \phi_0(x/2) \Longrightarrow L(\phi) \geq L, \end{cases} \tag{3.7.23}$$

for a given number $L \in \mathbb{N}_o$.

For the construction of such ϕ_0 we refer to Lemma 3.4 in [**21**]. Note that support of ϕ_0 can be chosen arbitrary as soon as it is contained in Γ given by (3.7.7). We will assume that

$$\operatorname{supp}(\phi_0) \subset [a, b]^{n-1} \times [a_n, b_n] \subset \Gamma, \tag{3.7.24}$$

with the values of $0 < a < b < \infty$ and $0 < a_n < b_n < \infty$ to be clarified later. The only restriction we should put for the moment is that $a_n > b\sqrt{n-1}/\kappa$ in order for the set $[a,b]^{n-1} \times [a_n, b_n]$ to be contained in Γ. Note that in particular, $0 < a < b < a_n < b_n < \infty$.

After this preamble, we are ready to begin the proof of Theorem 3.7.1 in the earnest. In a first stage, we shall treat a special case, namely when

$$\begin{aligned} &\exists\, \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ Lipschitz } : \Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \varphi(x') < x_n\}, \\ &f \in B^{p,q}_{s,0}(\Omega), \quad 0 \in \partial\Omega \text{ and } \operatorname{supp} f \subset \overline{\Omega} \cap [-1,1]^n. \end{aligned} \tag{3.7.25}$$

Furthermore, fix $\kappa > 0$ so that the cone (3.7.7) has the property that

$$x + \Gamma \subset \Omega, \qquad \forall\, x \in \overline{\Omega}. \tag{3.7.26}$$

Recall that we take $J := \frac{n}{\min\{1,p\}}$ and that the nonnegative integer L, chosen so that $L \geq \max\{[J-n-s], -1\}$, regulates the number of vanishing moments in the definition (3.7.2)-(3.7.4) of Besov atoms.

Having fixed Γ and L, we now select ϕ_0 as in (3.7.23)-(3.7.24). Then, as in Lemma 3.7.2, we construct the functions $\psi_0 \in C^\infty_c(\Gamma)$, $\psi \in C^\infty_c(\Gamma)$ with $L_\psi = L$ and

$$\begin{aligned} &\operatorname{supp}(\phi_j) \subset 2^{-j}\mathcal{O}_0, \qquad \operatorname{supp}(\psi_j) \subset 2^{-j}\mathcal{O}_1, \text{ where} \\ &\mathcal{O}_0 := [a, 2b]^{n-1} \times [a_n, 2b_n], \qquad \mathcal{O}_1 := [a, 2\times 6^{L+2} b]^{n-1} \times [a_n, 2\times 6^{L+2} b_n], \end{aligned} \tag{3.7.27}$$

and such that the reproducing formula (3.7.8) holds.

In order to continue, we need to introduce more notation. Let N be a large integer, to be specified later. For any dyadic cube

$$Q_{jk} = 2^{-j}[0,1]^n + k, \qquad j \in \mathbb{N}_o, \quad k \in \mathbb{Z}^n, \tag{3.7.28}$$

we denote by Q_{jk}^{ℓ}, $\ell = 1, 2, ..., 2^{Nn}$, the dyadic cubes obtained by further subdividing Q_{jk} into 2^{Nn} congruent pieces. Thus,

$$Q_{jk} = \bigcup_{\ell=1}^{2^{Nn}} Q_{jk}^{\ell}, \quad Q_{jk}^{\ell}\text{'s are pairwise dyadic cubes, } \; l(Q_{jk}^{\ell}) = 2^{-j-N}. \tag{3.7.29}$$

We also set (recall that $e_n = (0, ..., 0, 1) \in \mathbb{R}^n$):

$$\begin{aligned} &\Omega_j := \Big(\Omega + 2^{-j} a_n e_n\Big) \cap \Big([-1, 1]^n + 2^{-j}\mathcal{O}_0\Big), \\ &\mathcal{P}_{jk} := \{Q_{jk}^{\ell} : \, Q_{jk}^{\ell} \cap \Omega_j \neq \emptyset\}, \end{aligned} \tag{3.7.30}$$

so that for every $j \in \mathbb{N}_o$ and $k \in \mathbb{Z}^n$,

$$\begin{aligned} &\text{the cardinality of } \mathcal{P}_{jk} \text{ is } \leq 2^{Nn}, \\ &\Omega_j \cap Q_{jk} \subseteq \textstyle\bigcup_{Q_{jk}^{\ell} \in \mathcal{P}_{jk}} Q_{jk}^{\ell}. \end{aligned} \tag{3.7.31}$$

At this point, for some large M to be determined later, we write

$$\begin{aligned} f(x) &\overset{(i)}{=} \sum_{j=0}^{\infty} (\psi_j * \phi_j * f)(x) \overset{(ii)}{=} \sum_{j=0}^{M-1} (\psi_j * \phi_j * f)(x) + \sum_{j=M}^{\infty} (\psi_j * \phi_j * f)(x) \\ &\overset{(iii)}{=} A(x) + \sum_{j=M}^{\infty} \sum_{k \in \mathbb{Z}^n} \int_{Q_{jk}} \psi_j(x - y)(\phi_j * f)(y)\, dy \\ &\overset{(iv)}{=} A(x) + \sum_{j=M}^{\infty} \sum_{k \in \mathbb{Z}^n} \int_{Q_{jk} \cap \Omega_j} \psi_j(x - y)(\phi_j * f)(y)\, dy \\ &\overset{(v)}{=} A(x) + \sum_{j=M}^{\infty} \sum_{k \in \mathbb{Z}^n} \sum_{Q_{jk}^{\ell} \in \mathcal{P}_{jk}} \int_{Q_{jk}^{\ell}} \psi_j(x - y)(\phi_j * f)(y)\, dy \\ &\overset{(vi)}{=} A(x) + \sum_{Q_{jk}^{\ell}} \lambda_{Q_{jk}^{\ell}} a_{Q_{jk}^{\ell}}(x), \end{aligned} \tag{3.7.32}$$

where the following notation and conventions have been used:

$$\begin{aligned} &A(x) := \sum_{j=0}^{M-1} (\psi_j * \phi_j * f)(x), \\ &\lambda_{Q_{jk}^{\ell}} := |Q_{jk}^{\ell}|^{-s/n+1/p} \sup_{y \in Q_{jk}} |(\phi_j * f)(y)|, \\ &a_{Q_{jk}^{\ell}}(x) := \lambda_{Q_{jk}^{\ell}}^{-1} \int_{Q_{jk}^{\ell}} \psi_j(x - y)(\phi_j * f)(y)\, dy, \end{aligned} \tag{3.7.33}$$

all the cubes Q_{jk} appearing above are as in (3.7.28), the last sum in (3.7.32) is performed over all dyadic cubes Q_{jk}^{ℓ} such that $j \geq M$, $k \in \mathbb{Z}^n$, $1 \leq \ell \leq 2^{Nn}$, $Q_{jk}^{\ell} \in \mathcal{P}_{jk}$, and, in (3.7.33), we set $a_{Q_{jk}^{\ell}}(x) \equiv 0$ whenever $\lambda_{Q_{jk}^{\ell}} = 0$.

To justify the sequence of equalities in (3.7.32), we note that (i) is the reproducing formula (3.7.8), (ii) is a trivial spitting of the sum in two pieces, while (iii) follows from the definition of A, writing the convolution with ψ_j in an explicit fashion and decomposing the domain of integration, $\mathbb{R}^n$, in dyadic cubes of side-length 2^{-j}. Next, (iv) is a consequence of the fact that $\operatorname{supp}(\phi_j * f) \subseteq \operatorname{supp}\phi_j + \operatorname{supp} f$ in concert with the support condition on f from (3.7.25), the support condition on ϕ_j from (3.7.27), and the definition of Ω_j from (3.7.30).

Going further, (v) follows from the definition of $\mathcal{P}_{jk}$ in (3.7.30) and the second line in (3.7.31) (keep in mind that, for given j, k, the Q_{jk}^{ℓ}'s are pairwise disjoint). Finally, (vi) is a consequence of notation introduced in (3.7.33) and the conventions following right after.

We now study the properties of the functions and coefficients in the last expression in (3.7.32). That

$$\int_{\mathbb{R}^n} x^{\gamma} a_{Q_{jk}^{\ell}}(x)\,dx = 0 \ \text{ if } \ |\gamma| \leq L, \tag{3.7.34}$$

is an easy consequence of Fubini's Theorem plus the fact that the first L moments of ψ_j vanish. Furthermore, for any multi-index γ,

$$\begin{aligned} \left|\partial^{\gamma} a_{Q_{jk}^{\ell}}(x)\right| &\leq \lambda_{Q_{jk}^{\ell}}^{-1} 2^{j|\gamma|} \int_{Q_{jk}^{\ell}} |(\partial^{\gamma}\psi)_j(x-y)|\,|(\phi_j * f)(y)|\,dy \\ &\leq C(\psi, N, \gamma)|Q_{jk}^{\ell}|^{s/n-1/p-|\gamma|/n}, \end{aligned} \tag{3.7.35}$$

where we have utilized the fact that $\lambda_{Q_{jk}^{\ell}}^{-1} \sup_{y\in Q_{jk}^{\ell}} |(\phi_j * f)(y)| \leq C|Q_{jk}^{\ell}|^{s/n-1/p}$,

itself a consequence of the definition of $\lambda_{Q_{jk}^{\ell}}$ and the fact that $Q_{jk}^{\ell} \subset Q_{jk}$.

In order to be able to discuss the location of the support of $a_{Q_{jk}^{\ell}}$ recall the definition of $\mathcal{O}_1$ from (3.7.27) and introduce

$$\widetilde{Q}_{jk}^{\ell} := Q_{jk}^{\ell} + 2^{-j}\mathcal{O}_1. \tag{3.7.36}$$

This, (3.7.27) and the definition of $a_{Q_{jk}^{\ell}}$ in (3.7.33) then imply

$$\operatorname{supp}(a_{Q_{jk}^{\ell}}) \subseteq Q_{jk}^{\ell} + \operatorname{supp}(\psi_j) \subseteq \widetilde{Q}_{jk}^{\ell}. \tag{3.7.37}$$

Note that the appropriate choice of parameters a, b, a_n and b_n, namely such that $2 \times 6^{L+2}b - a = 2 \times 6^{L+2}b_n - a_n$, allows to ensure that $\mathcal{O}_1$ and hence $\widetilde{Q}_{jk}^{\ell}$ are cubes in $\mathbb{R}^n$. Our goal is to show that the last set above is contained in Ω whenever $Q_{jk}^{\ell} \in \mathcal{P}_{jk}$. To this end, we note that, by definition, every cube Q_{jk}^{ℓ} in $\mathcal{P}_{jk}$ intersects $\Omega + 2^{-j}a_n e_n$. Thus, there exist $C, C' > 0$, depending only on $\partial\Omega$ such that

$$\operatorname{dist}(Q_{jk}^{\ell}, \partial\Omega) \geq C2^{-j}a_n - C'2^{-j-N} \geq C''l(Q_{jk}^{\ell}) \tag{3.7.38}$$

where $C'' > 0$ can be made as large as we desire by taking N to be sufficiently large. Now, simple geometric considerations show that, granted (3.7.38), it is possible to find $x = x(Q_{jk}^{\ell}) \in \partial\Omega$ such that $Q_{jk}^{\ell} \subseteq x + \Gamma$ (where the cone Γ is defined in (3.7.7)). Thus, altogether, $\widetilde{Q}_{jk}^{\ell} \subset x + \Gamma + \Gamma \subset x + \Gamma \subset \Omega$, by (3.7.26). In fact, an inspection of the argument reveals that

$$\exists\, \rho = \rho(\partial\Omega) > 1 \ \text{ such that } \ \rho\,\widetilde{Q}_{jk}^{\ell} \subset \Omega, \quad \forall\, Q_{jk}^{\ell} \in \mathcal{P}_{jk}. \tag{3.7.39}$$

For later purposes it is important to observe that there exists $C = C(\Omega) > 0$ such that

$$k \in \mathbb{Z}^n,\ Q_{jk}^{\ell} \in \mathcal{P}_{jk} \Longrightarrow \operatorname{dist}\left(\widetilde{Q}_{jk}^{\ell}, [-1,1]^n\right) \leq C2^{-j}. \tag{3.7.40}$$

This follows from (3.7.36) and (3.7.30). In particular, given any $\varepsilon > 0$, one can choose $M = M(\varepsilon)$ large enough so that

$$j \geq M,\ k \in \mathbb{Z}^n,\ Q_{jk}^{\ell} \in \mathcal{P}_{jk} \Longrightarrow \operatorname{dist}\Big(\operatorname{supp} a_{Q_{jk}^{\ell}}, [-1,1]^n\Big) < \varepsilon. \tag{3.7.41}$$

Turning now our attention to the sequence of coefficients, given any $N_o \in \mathbb{R}$, we write

$$\sup_{y\in Q_{jk}} |(\phi_j * f)(y)| \leq C \sup_{y\in Q_{jk}} \frac{|(\phi_j * f)(y)|}{(1+2^j|x-y|)^{N_o}}, \quad \forall\, x \in Q_{jk}^{\ell}, \tag{3.7.42}$$

since $|x-y| \leq \sqrt{n}2^{-j}$. Taking the p-th power of both sides and integrating the resulting expressions with respect to x over Q_{jk} further leads to

$$|Q_{jk}| \left(\sup_{y\in Q_{jk}} |(\phi_j * f)(y)|\right)^p \leq C \int_{Q_{jk}} \left(\sup_{y\in Q_{jk}} \frac{|(\phi_j * f)(y)|}{(1+2^j|x-y|)^{N_o}}\right)^p dx. \tag{3.7.43}$$

Armed with this inequality and choosing $\beta \in \mathbb{N}$ depending on constants M and N, i.e. on Ω only, so that the maximal size of cubes $\tilde{Q}_{jk}^{l}$ corresponding to $\lambda_{Q_{jk}^{l}} \neq 0$ in the construction above is $2^{n(-\beta-1)}$, we may then estimate

$$\begin{aligned}
\left(\sum_{j=\beta+1}^{\infty}\Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)=2^{-j}}} |\lambda_Q|^p\Big)^{q/p}\right)^{1/q} &\leq C\left(\sum_{j=M}^{\infty}\Big(\sum_{k\in\mathbb{Z}^n}\sum_{Q_{jk}^{\ell}\in\mathcal{P}_{jk}} [\lambda_{Q_{jk}^{\ell}}]^p\Big)^{q/p}\right)^{1/q}\\
&\leq C\left(\sum_{j=M}^{\infty} 2^{jsq}\Big(\sum_{k\in\mathbb{Z}^n}\int_{Q_{jk}}\Big(\sup_{y\in\mathbb{R}^n}\frac{|(\phi_j * f)(y)|}{(1+2^j|x-y|)^{N_o}}\Big)^p dx\Big)^{q/p}\right)^{1/q}\\
&\leq C\Big(\sum_{j=M}^{\infty}(2^{js}\|\phi_{j,N_o}^{*} f\|_{L^p(\mathbb{R}^n)})^q\Big)^{1/q} \leq C\|f\|_{B_s^{p,q}(\mathbb{R}^n)},
\end{aligned} \tag{3.7.44}$$

where the last inequality holds for every $N_o \geq \min\{p,q\}$ according to Theorem 3.2.8.

Turning to the term $A(x)$ in (3.7.32), we observe that

$$\operatorname{supp}(A) \subset (\Omega + 2^{-M}a_n e_n) \cap \big([-1,1]^n + \cup_{j=0}^{M-1}(2^{-j}\mathcal{O}_0 + 2^{-j}\mathcal{O}_1)\big), \tag{3.7.45}$$

while the set on the right-hand side of (3.7.45) can be covered by a finite collection of disjoint cubes $\tilde{Q}_0^\ell \in \mathcal{Q}_n$, $\ell = 1, ..., \tilde{N}$ with $\tilde{N} = \tilde{N}(M, N, \Omega)$ such that $l(Q_0^\ell) = 2^{-\beta}$ for every $\ell = 1, ..., \tilde{N}$ and $\rho\, Q_0^\ell \subset \Omega$.

It is easy to check that with

$$\lambda_{Q_0^\ell} := \sup_{y \in Q_0^\ell - \cup_{j=0}^{M-1} 2^{-j}\mathcal{O}_1} |(\phi_j * f)(y)|,$$
$$A_{Q_0^\ell}(x) := \lambda_{Q_0^\ell}^{-1} A(x) \chi_{Q_0^l}(x), \tag{3.7.46}$$

the functions $A_{Q_0^\ell}$ are atoms in decomposition (3.7.1) corresponding to cubes of side-length $2^{-\beta}$ and coefficients $\lambda_{Q_0^\ell}$ are such that (3.7.44) can be refined to

$$\left(\sum_{j=\beta}^{\infty} \Big(\sum_{\substack{Q \in \mathcal{Q}_n \\ l(Q) = 2^{-j}}} |\lambda_Q|^p \Big)^{q/p} \right)^{1/q} \leq C \|f\|_{B_s^{p,q}(\mathbb{R}^n)},$$

as desired.

Note that the converse inequality

$$\|f\|_{B_{s,0}^{p,q}(\Omega)} = \|f\|_{B_s^{p,q}(\mathbb{R}^n)} \leq C \left(\sum_{j=\beta}^{\infty} \Big(\sum_{\substack{Q \in \mathcal{Q}_n \\ l(Q) = 2^{-j}}} |\lambda_Q|^p \Big)^{q/p} \right)^{1/q} \tag{3.7.48}$$

is provided by (3.3.6) in Theorem 3.3.1.

This concludes the proof in special case (3.7.25) in the case of the Besov scale $B_{s,0}^{p,q}(\Omega)$.

As for Triebel-Lizorkin spaces, we can follow the same line of reasoning starting with $J = \frac{1}{\min\{1,p,q\}}$ to define the atoms and corresponding sequences by (3.7.33). Once again, the properties of atoms are satisfied and the matters are reduced to the proof of norm equivalence. To prove it, we note that (3.7.42) implies that, for

every $N_o \in \mathbb{R}$ and every $j = M, M+1, ...,$

$$\sum_{k\in\mathbb{Z}^n}\sum_{Q_{jk}^\ell\in\mathcal{P}_{jk}}(|Q_{jk}^\ell|^{-1/p}|\lambda_{Q_{jk}^\ell}|)^q\chi_{Q_{jk}^\ell}(x) = 2^{jsq}\sum_{k\in\mathbb{Z}^n}\sum_{Q_{jk}^\ell\in\mathcal{P}_{jk}}(\sup_{y\in Q_{jk}}|(\phi_j * f)(y)|)^q\chi_{Q_{jk}^\ell}(x)$$

$$\leq C2^{jsq}\sum_{k\in\mathbb{Z}^n}\sum_{Q_{jk}^\ell\in\mathcal{P}_{jk}}\left(\sup_{y\in Q_{jk}}\frac{|(\phi_j * f)(y)|}{(1+2^j|x-y|)^{N_o}}\right)^q\chi_{Q_{jk}}(x)$$

$$\leq C2^{jsq}\left(\sup_{y\in\mathbb{R}^n}\frac{|(\phi_j * f)(y)|}{(1+2^j|x-y|)^{N_o}}\right)^q = C2^{jsq}\left(\phi_{j,N_o}^* f(x)\right)^q, \tag{3.7.49}$$

with similar estimates for $\lambda_{Q_0^\ell}$, $\ell = 1, 2, ..., \tilde{N}$, so that

$$\left\|\left(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\leq 2^{-\beta}}}\Big(|Q|^{-1/p}|\lambda_Q|\chi_Q\Big)^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \leq C\left\|\Big(\sum_{j=0}^\infty(2^{js}\phi_{j,N_o}^* f(x))^q\Big)^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\|f\|_{F_s^{p,q}(\mathbb{R}^n)}, \tag{3.7.50}$$

by invoking Theorem 3.2.8 with $N_o \geq \min\{p, q\}$. The opposite inequality is provided by Theorem 3.3.1 and this completes the argument in the case of Triebel-Lizorkin spaces in the scenario described in (3.7.25) as well. □

Corollary 3.7.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and assume that $0 < p \leq +\infty$, $s \in \mathbb{R}$. Also, define $J := \frac{n}{\min\{1,p\}}$ and fix two integers, $K \geq ([s]+1)_+$ and $L \geq \max\{[J-n-s], -1\}$. Then there exists $\beta = \beta(\Omega) \in \mathbb{N}_o$ with the following significance. Any distribution $f \in B_{s,0}^{p,p}(\Omega)$ can be written in the form*

$$f = \sum_Q \lambda_Q a_Q \quad \textit{with convergence in} \quad S'(\mathbb{R}^n), \tag{3.7.51}$$

where Q's belong to a countable collection of cubes in $\mathbb{R}^n$, the atoms a_Q's satisfy

$$(1)\ \exists\, Q \in \mathbb{R}^n \text{ such that } \operatorname{supp}(a_Q) \subseteq Q \text{ and } \rho\, Q \subset \Omega, \tag{3.7.52}$$

$$(2)\ \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq C|Q|^{s/n-1/p-|\gamma|/n} \text{ if } |\gamma| \leq K, \tag{3.7.53}$$

$$(3)\ \int_{\mathbb{R}^n} x^\gamma a_Q(x)\, dx = 0 \text{ if } |\gamma| \leq L \text{ and } l(Q) < 2^{-\beta}, \tag{3.7.54}$$

uniformly for some constant $\rho > 1$ and the sequence of coefficients $\{\lambda_Q\}_Q \in \ell^p$. Moreover,

$$\|f\|_{B^{p,p}_{s,0}(\Omega)} \approx \inf\Big\{\|\{\lambda_Q\}_Q\|_{\ell^p} :\ f = \sum_Q \lambda_Q a_Q\Big\}, \tag{3.7.55}$$

where the infimum is taken over all possible representations of f in a series of atoms satisfying (3.7.52)-(3.7.54).

Proof. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists some $\varepsilon > 0$ and finite collection of balls $B_i = B(r_i, x_i)$, $1 \leq i \leq m$, such that

$$B_0 := \overline{\Omega} \setminus \bigcup_{i=0}^m B(x_i, r_i - \varepsilon) \text{ is a relatively compact subset of } \Omega \tag{3.7.56}$$

and, for each $i = 1, 2, ..., m$, there exists an unbounded (graph-type) Lipschitz domain D_i, in suitable Cartesian coordinates, with the property that $B_i \cap \Omega = D_i \cap \Omega$. In particular, $\operatorname{dist}(\partial B_0, \partial\Omega) > 0$. Consider next a family of C^∞ functions $\{\eta_i\}_{0\leq i\leq m}$ such that

$$\operatorname{supp}(\eta_0) \subset B_0, \quad \operatorname{supp}(\eta_i) \subset B(x_i, r_i - \varepsilon), \quad 1 \leq i \leq m, \tag{3.7.57}$$

and for which

$$\sum_{i=0}^m \eta_i = 1 \text{ in a neighborhood of } \overline{\Omega}. \tag{3.7.58}$$

As regards the piece f_0, whose support is contained in $B_0 \subset \Omega$, Theorem 3.3.1 ensures that f can be written as

$$f\eta_0 = \sum_Q s_Q^0 a_Q^0, \tag{3.7.59}$$

with $\{s_Q^0\}_Q \in \ell^p$ and atoms a_Q^0 satisfying (3.3.1)-(3.3.3), and such that

$$\|\{s_Q^0\}_Q\|_{\ell^p} \leq C\|f\eta_0\|_{B_s^{p,p}(\mathbb{R}^n)} = C\|f\eta_0\|_{B_{s,0}^{p,p}(\Omega)}. \tag{3.7.60}$$

There is no loss of generality in assumption that $l(Q) \leq c_0$ for some fixed constant c_0 in decomposition (3.7.59) rather than $l(Q) \leq 1$ as stated originally in Theorem 3.3.1. Pick now $\Phi \in C_c^\infty(\Omega)$ with $\Phi \equiv 1$ on B_0 and write $f\eta_0 = \sum_Q s_Q^0(a_Q^0\Phi)$. In this form, the decomposition suits our purposes since each $a_Q^0\Phi$ is supported in a cube $Q \in \mathcal{Q}_n$, $Q \subset \Omega$, $\text{dist}\,(Q, \partial\Omega) > c\,l(Q)$ and such that $l(Q) \geq c' > 0$ for two constants c, c' depending only on Ω, B_0 and Φ, provided c_0 was suitably small. In particular, such functions satisfy (3.7.52)-(3.7.54) since the last condition never comes into play (granted that β is large).

Concerning the functions $f_1, ..., f_m$, according to what we have proved so far there exists $c = c(\partial\Omega) > 0$ and, for every $i = 1, ..., m$, there exists a decomposition

$$f\eta_i = A_i + \sum_Q \lambda_Q^i a_Q^i \tag{3.7.61}$$

with the following properties. Each A_i is a smooth, with support contained in some fixed, relatively compact subset $\mathcal{O}$ of Ω and such that $\|A_i\|_{B_s^{p,p}(\mathbb{R}^n)} \leq C\|f\|_{B_s^{p,p}(\mathbb{R}^n)}$. In particular, these functions can be handled as before. Next, the a_Q^i's are atoms in the sense of (3.7.33) such that $\text{dist}\,(\text{supp}\,(a_Q^i), \partial\Omega) \geq c\,l(Q)$ and $\text{dist}\,(\text{supp}\,(a_Q^i), \text{supp}\,(\eta_i)) < \varepsilon$. In fact, since the size of $\text{supp}\,(a_Q^i)$ is proportional to

$l(Q)$ (in the notation used in the proof of Theorem 3.7.1, the size of supp $(a_{Q^\ell_{jk}})$ is proportional to 2^{-j}), one can see from (3.7.40)-(3.7.41) that choosing M big enough we can guarantee that the cube $\rho\,Q$, where supp $(a_Q) \subseteq Q$, is entirely contained in ε-neighborhood of supp (η_i) and hence, is entirely contained in $B_i \cap \Omega$. Finally,

$$\|\{\lambda^i_Q\}_Q\|_{\ell^p} \leq C\|f\eta_i\|_{B^{p,p}_s(\mathbb{R}^n)} = C\|f\eta_i\|_{B^{p,p}_{s,0}(\Omega)}, \quad i = 1, ..., m. \tag{3.7.62}$$

All we have to do now in order to obtain the desired expansion of f in a series of Besov atoms is to paste all these decompositions via affine transformations.

The opposite inequality, $\left\| \sum_Q \lambda_Q a_Q \right\|_{B^{p,p}_{s,0}(\Omega)} \leq C\|\{\lambda_Q\}_Q\|_{\ell^p}$, is trivial as soon as we observe that the decomposition (3.7.55) is a version of decomposition in Theorem 3.3.1. The only difference is that in (3.7.55) we allow atoms of arbitrary large size, but that can be handled following the lines of (3.4.41)-(3.4.48) for $(|\gamma| = [s]+1)_+$ and then by Mean Value Theorem for the smaller values of $|\gamma|$ if necessary.

□

We now discuss a further refinement of Theorem 3.3.1 in the case of Triebel-Lizorkin spaces. To state this, we need to introduce a new concept of atom.

Assume that $s \in \mathbb{R}$, $0 < p \leq 1$ and $p \leq q \leq \infty$, $p < p_1 < +\infty$, define $J := \frac{n}{p}$ and fix an integer $L \geq \max\{[J-n-s], -1\}$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $\beta \in \mathbb{N}_o$ and $\rho > 1$ are constants depending on Ω. Under these circumstances, call a function A_Q a *rough atom for* $F^{p,q}_{s,0}(\Omega)$ if

$$(1)\ \exists\, Q \in \mathbb{R}^n \text{ such that supp } A \subseteq Q \subset \Omega \text{ and } \rho\, Q \subset \Omega, \tag{3.7.63}$$

$$(2)\ \|A\|_{F^{p_1,q}_s(\mathbb{R}^n)} \leq |Q|^{1/p_1 - 1/p}, \tag{3.7.64}$$

$$(3)\ \int_{\mathbb{R}^n} x^\gamma A(x)\, dx = 0 \text{ if } |\gamma| \leq L \text{ and } l(Q) < 2^{-\beta}. \tag{3.7.65}$$

Theorem 3.7.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that p, q, s, p_1, J, L are as above. Then there exist $\beta \in \mathbb{N}_o$ and $\rho > 1$ such that any $f \in F^{p,q}_{s,0}(\Omega)$ can be expanded in a series*

$$f = \sum_{k\in\mathbb{Z}} \lambda_k A_k \quad \text{with convergence in} \quad S'(\mathbb{R}^n), \tag{3.7.66}$$

where the atoms A_k's satisfy (3.7.63)-(3.7.65)and $\{\lambda_k\}_{k\in\mathbb{Z}} \in \ell^p$. Furthermore,

$$\|f\|_{F^{p,q}_{s,0}(\Omega)} \approx \inf\Big\{\|\{\lambda_k\}_k\|_{\ell^p};\ f = \sum \lambda_k A_k\Big\}, \tag{3.7.67}$$

where the infimum is taken over all possible representations of f in a series of atoms satisfying (3.7.63)-(3.7.65).

Proof. We focus on proving the "$\geq$" part of (3.7.67). The converse, "$\leq$" inequality follows from Lemma 3.7.5 which can be found right after the proof of this theorem.

We proceed in a series of seven steps starting with:

Step I. Recall the sequence spaces introduced in (3.2.30)-(3.2.33). As proved in [**49**], Corollary 6.5, p.85, for every $\alpha \in \mathbb{R}$ and $0 < q \leq +\infty$ we have

$$\Big((f_o, f^{\infty,q}_\alpha)_{\theta,1/(1-\theta)}\Big)^{1/\theta} = f^{p,q}_\alpha, \quad 0 < \theta < 1,\ p := \theta/(1-\theta), \tag{3.7.68}$$

where as usually $\|\cdot\|_{X^\gamma} := \|\cdot\|^\gamma_X$ for $\gamma > 0$ and some (quasi-)normed vector space X.

Step II. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. For an arbitrary subset $\mathcal{Q}'$ of $\{Q \in \mathcal{Q}_n : l(Q) \leq 1\}$ we set

$$f^{p,q}_\alpha(\mathcal{Q}') := \{\{s_Q\}_{Q\in\mathcal{Q}_n : l(Q)\leq 1} \in f^{p,q}_\alpha :\ s_Q = 0 \text{ whenever } Q \notin \mathcal{Q}'\}, \tag{3.7.69}$$

$$f_o(\mathcal{Q}') := \{\{s_Q\}_{Q\in\mathcal{Q}_n : l(Q)\leq 1} \in f_o :\ s_Q = 0 \text{ whenever } Q \notin \mathcal{Q}'\}, \tag{3.7.70}$$

equipped with the quasi-norms inherited from $f_\alpha^{p,q}$ and f_o, respectively. Then the mappings

$$R : f_\alpha^{p,q} \longrightarrow f_\alpha^{p,q}(\mathcal{Q}'), \quad \Big(R(s)\Big)_Q := s_Q \text{ if } Q \in \mathcal{Q}' \text{ and zero otherwise,} \quad (3.7.71)$$

$$E : f_\alpha^{p,q}(\mathcal{Q}') \longrightarrow f_\alpha^{p,q}, \quad \Big(E(s)\Big)_Q := s_Q, \ \forall\, Q \in \mathcal{Q}_n \text{ with } l(Q) \leq 1, \quad (3.7.72)$$

plus a set of similar definitions with $f_\alpha^{p,q}$, $f_\alpha^{p,q}(\mathcal{Q}')$ replaced by f_o and $f_o(\mathcal{Q}')$, respectively, satisfy $R \circ E = I$ in all circumstances when the composition makes sense.

Let $0 < \theta < 1$, $0 < q \leq +\infty$, $\alpha \in \mathbb{R}$, and set $p := \theta/(1-\theta)$. It follows from the above discussion, Lemma 3.1.9 and (3.7.68) that

$$\begin{aligned} \Big((f_o(\mathcal{Q}'), f_\alpha^{\infty,q}(\mathcal{Q}'))_{\theta,1/(1-\theta)}\Big)^{1/\theta} &= \Big(R(f_o, f_\alpha^{\infty,q})_{\theta,1/(1-\theta)}\Big)^{1/\theta} \\ &= R\Big((f_o, f_\alpha^{\infty,q})_{\theta,1/(1-\theta)}\Big)^{1/\theta} = f_\alpha^{p,q}(\mathcal{Q}'). \quad (3.7.73) \end{aligned}$$

Step III. Assume that $\alpha \in \mathbb{R}$, $0 < p \leq 1$ and $p \leq q \leq +\infty$. Then for every subset $\mathcal{Q}'$ of $\{Q \in \mathcal{Q}_n : l(Q) \leq 1\}$,

$$\begin{aligned} \|s\|_{f_\alpha^{p,q}(\mathcal{Q}')} \approx \inf \Big\{ \Big(\sum_{k\in\mathbb{Z}} |\mu_k|^p\Big)^{1/p} : \\ s = \sum_{k\in\mathbb{Z}} \mu_k s_k \text{ with } \|s_k\|_{f_o(\mathcal{Q}')}^{1/p} \|s_k\|_{f_\alpha^{\infty,q}(\mathcal{Q}')} \leq 1 \text{ for all } k \Big\}. \quad (3.7.74) \end{aligned}$$

Indeed, having disposed off (3.7.73), this is proved in a similar fashion to (7.5) p. 88 in [**49**].

Step IV. Recall the concept of atom for sequence space introduced in connection with decomposition (3.3.9). Suppose now that $\alpha \in \mathbb{R}$, $0 < p \leq 1$, $p \leq q \leq +\infty$ and

$p < p_1 < +\infty$. Then

$$\|s\|_{f^{p,q}_{\alpha}(\mathcal{Q}')} \approx \inf\Big\{\Big(\sum_{k\in\mathbb{Z}}|\mu_k|^p\Big)^{1/p}: \\ s=\sum_{k\in\mathbb{Z}}\mu_k r_k,\ r_k\text{'s are } p_1\text{-atoms for } f^{p,q}_{\alpha} \text{ supported in } \mathcal{Q}'\Big\}. \quad (3.7.75)$$

This is justified in a manner very similar to the proof of Theorem 7.2 on p. 88 in **[49]**. The only significant alteration is the replacement of (7.5) in that paper by (3.7.74) above.

Step V. Assume that $\Omega = \{(x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} :\ \varphi(x') < x_n\}$ where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $\varphi(0) = 0$, and fix $f \in F^{p,q}_{s,0}(\Omega)$ with $\operatorname{supp} f \subset \overline{\Omega}\cap[-1,1]^n$. From Theorem 3.7.1 and its proof we know that f can be written in the form

$$f = \sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\le 2^{-\beta}}} \lambda_Q a_Q \quad \text{with convergence in} \quad S'(\mathbb{R}^n), \quad (3.7.76)$$

where

$$\|\{\lambda_Q\}_Q\|_{f^{p,q}_s} \le C\|f\|_{F^{p,q}_s(\mathbb{R}^n)} \quad (3.7.77)$$

and

$$(1)\ \exists\, Q\in\mathcal{Q}_n \text{ such that } \operatorname{supp}(a_Q)\subseteq Q \text{ and } \rho\, Q\subset\Omega, \quad (3.7.78)$$

$$(2)\ \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \le C|Q|^{-1/2-|\gamma|/n} \text{ if } |\gamma|\le K, \quad (3.7.79)$$

$$(3)\ \int_{\mathbb{R}^n} x^\gamma a_Q(x)\,dx = 0 \text{ if } |\gamma|\le L \text{ and } l(Q) < 2^{-\beta}. \quad (3.7.80)$$

To see this, we consider the decomposition $f = \sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)\le 2^{-\beta}}} \tilde{\lambda}_Q\tilde{a}_Q$ obtained by methods of Theorem 3.7.1 and for every $Q \in \mathcal{Q}_n : l(Q) \le 2^{-\beta}$ set $a_Q := |Q|^{-s/n-1/2+1/p}\tilde{a}_Q$ and $\lambda_Q := |Q|^{s/n+1/2-1/p}\tilde{\lambda}_Q$.

Also, given any $\varepsilon > 0$ and using (3.7.41), we could arrange matters so that in addition to (3.7.78)

$$\operatorname{dist}(Q, [-1,1]^n) < \varepsilon \text{ if } j > \beta. \tag{3.7.81}$$

Let $\mathcal{Q}'$ be the subset of $\{Q \in \mathcal{Q}_n : l(Q) \leq 2^{-\beta}\}$ consisting of cubes Q satisfying (3.7.81) and condition $\rho\, Q \subset \Omega$.

We may therefore set $\lambda_Q = 0$ whenever $Q \notin \mathcal{Q}'$ without affecting (3.7.76), (3.7.77), so there is no loss of generality in assuming that $\lambda := \{\lambda_Q\}_Q \in f_s^{p,q}(\mathcal{Q}')$ to begin with. Consequently, by Step IV, there exist a numerical sequence $\{\mu_k\}_k \in \ell^p$ and, for a fixed $p < p_1 < +\infty$, a sequence of p_1-atoms for $f_s^{p,q}$, $\{r_k\}_k$, such that $\bar{Q}_k := \operatorname{supp} r_k \in \mathcal{Q}'$ and

$$\lambda = \sum_{k\in\mathbb{Z}} \mu_k r_k, \qquad \|\{\mu_k\}_k\|_{\ell^p} \approx \|\lambda\|_{f_s^{p,q}}. \tag{3.7.82}$$

Step VI. Continuing the discussion initiated in the previous step, for each $k \in \mathbb{Z}$ we set

$$A_k := \begin{cases} \sum_{Q\in\mathcal{Q}',\, Q\subseteq\bar{Q}_k} (r_k)_Q a_Q, & \text{if } l(\bar{Q}_k) < 2^{-\beta}, \\ a_{Q_k}, & \text{if } l(\bar{Q}_k) = 2^{-\beta}, \end{cases} \tag{3.7.83}$$

where $\bar{Q}_k \in \mathcal{Q}'$ for each $k \in \mathbb{Z}$. It is then clear that $f = \sum_{k\in\mathbb{Z}} \mu_k A_k$, i.e., (3.7.66) holds. Also, for each k,

$$\operatorname{supp} A_k \subseteq \bar{Q}_k \in \mathcal{Q}', \tag{3.7.84}$$

so that, in particular, $\rho\, \bar{Q}_k \subset \Omega$.

Going further, from (3.7.83) and (the inhomogeneous space version of) Theorem 4.1 p. 60 in **[49]**, if $l(\bar{Q}_k) < 2^{-\beta}$,

$$\|A_k\|_{F_s^{p_1,q}(\mathbb{R}^n)} \leq C\|r_k\|_{f_s^{p_1,q}} \leq C|\bar{Q}_k|^{1/p_1-1/p} \leq C|Q_k|^{1/p_1-1/p}, \tag{3.7.85}$$

where $C = C(p, q, s, n) > 0$ is independent of k. Concerning the atoms A_k with $l(\bar{Q}_k) = 2^{-\beta}$, note that they satisfy (3.7.78)-(3.7.79) and, hence, by the estimate (3.7.77) in the context of $F_s^{p_1,q}(\mathbb{R}^n)$,

$$\|A_k\|_{F_s^{p_1,q}(\mathbb{R}^n)} \leq C\|1_{\bar{Q}_k}\|_{f_s^{p_1,q}} \leq C|\bar{Q}_k|^{-\frac{s}{n}-\frac{1}{2}+\frac{1}{p_1}} \leq C, \tag{3.7.86}$$

where $1_{\bar{Q}_k}$ is the sequence indexed by dyadic cubes given by $(1_{\bar{Q}_k})_Q := 1$ if $Q = \bar{Q}_k$, and 0 otherwise. The last inequality above follows from observation that $l(\bar{Q}_k) = 2^{-\beta}$.

Finally, (3.7.65) follows directly from (3.7.83) and (3.7.4). Thus, altogether, each A_k is a rough atom in the sense of (3.7.63)-(3.7.65), up to an unimportant multiplicative constant.

Step VII. The argument presented up to this point proves the graph-domain version of Theorem 3.7.4, and there remains to explain how the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is handled based on what we have shown so far. This, however, is dealt with much as in the last part of the proof of Theorem 3.7.1. More concretely, any $f \in F_{s,0}^{p,q}(\Omega)$ can be written in the form $f = \sum_{i=0}^m f_i$ where $f_i \in F_{s,0}^{p,q}(\Omega)$, $\operatorname{supp} f_0 \subset \mathcal{O}$, a fixed, relatively compact subset of Ω and, for $1 \leq i \leq m$, $\operatorname{supp} f_i$ is a suitably small subset of $\overline{\Omega}$ so that the decomposition procedure described in Step VI applies to each f_i and yields

$$f_i = \sum_k \lambda_k^i A_k^i, \qquad i = 1, ..., m. \tag{3.7.87}$$

The atoms A_k^i having small support are, by (3.7.81), sufficiently close to the support of the f_i's. In particular, they are supported inside Ω in the desired fashion and can be selected without further adjustments as part of the final atomic decomposition of

the original distribution f. The atoms A_i^k with large support need only be truncated by a suitable smooth function Φ_i, which is identically one on the support of η_i, so that they now satisfy (3.7.63)-(3.7.65) relative to the original domain Ω (note that by choosing β large enough, the condition of vanishing moments never gets into play). Finally, the piece supported in a fixed compact subset of Ω, i.e., f_0, is handled as in the proof of Theorem 3.7.1. This concludes the proof of Theorem 3.7.4. □

Lemma 3.7.5. *Let $s \in \mathbb{R}$, $0 < p \leq p_1 < \infty$, $0 < q < \infty$, and fix*

$$L \in \mathbb{N}_o \cup \{-1\}, \qquad L > n\left(\frac{1}{p} - 1\right) - s - 1. \tag{3.7.88}$$

Also, let $Q \subseteq \mathbb{R}^n$ be a cube with $l(Q) \leq c_0$ and assume that $A \in F_s^{p_1,q}(\mathbb{R}^n)$ satisfies

$$\operatorname{supp} A \subseteq Q, \quad \|A\|_{F_s^{p_1,q}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{p_1}-\frac{1}{p}}, \quad \int_{\mathbb{R}^n} x^\gamma A(x)\,dx = 0,\ |\gamma| \leq L. \tag{3.7.89}$$

(As usual, the vanishing moment condition is void when $L = -1$.) Then

$$\|A\|_{F_s^{p,q}(\mathbb{R}^n)} \leq C \tag{3.7.90}$$

where $C = C(c_0, p, q, p_1, s, n) < +\infty$.

The lemma above can be considered as a version of Hölder inequality. Indeed, in the case when $p > 1$, $q = 2$, and $s = 0$, the vanishing moment condition in (3.7.89) is void and the statement in question becomes

$$1 < p \leq p_1 < \infty, \quad \operatorname{supp} A \subseteq Q, \quad \|A\|_{L^{p_1}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{p_1}-\frac{1}{p}} \quad \Longrightarrow \quad \|A\|_{L^p(\mathbb{R}^n)} \leq C. \tag{3.7.91}$$

Proof of Lemma 3.7.5. If $p_1 = p$ there is nothing to prove, so we assume that $0 < p < p_1 < \infty$. Making a translation, there is no loss of generality in assuming

that the cube Q is centered at zero. We proceed to estimate $\|A\|^p_{F^{p,q}_s(\mathbb{R}^n)}$ by using the Littlewood-Paley type definition of the norm on $F^{p,q}_s(\mathbb{R}^n)$. A similar strategy has been followed in [**62**]. Specifiaclly, recall (3.2.23) and decompose

$$\begin{aligned}\|A\|^p_{F^{p,q}_s(\mathbb{R}^n)} &= \left\| \left(\sum_{j=0}^{\infty} |2^{sj}\phi_j * A|^q \right)^{\frac{1}{q}} \right\|^p_{L^p(\mathbb{R}^n)} \\ &= \int_{cQ} \left(\sum_{j=0}^{\infty} |2^{sj}(\phi_j * A)(x)|^q \right)^{\frac{p}{q}} dx + \int_{\mathbb{R}^n \setminus cQ} \left(\sum_{j=0}^{\infty} |2^{sj}(\phi_j * A)(x)|^q \right)^{\frac{p}{q}} dx \\ &=: I + II,\end{aligned} \tag{3.7.92}$$

for a fixed $c > 0$ to be specified later. Let $r := \frac{p_1}{p} \in (1, \infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$. Hölder's inequality then implies

$$\begin{aligned}|I| &\leq \left\{ \int_{cQ} \left(\sum_{j=0}^{\infty} |2^{sj}(\phi_j * A)(x)|^q \right)^{\frac{p}{q}\cdot r} dx \right\}^{\frac{1}{r}} \cdot \left\{ \int_{cQ} 1\, dx \right\}^{\frac{1}{r'}} \\ &\leq C \left\{ \int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} |2^{sj}(\phi_j * A)(x)|^q \right)^{\frac{p_1}{q}} dx \right\}^{\frac{p}{p_1}} \cdot |Q|^{1-\frac{p}{p_1}} \\ &\leq C\|A\|^p_{F^{p_1,q}_s(\mathbb{R}^n)} \cdot |Q|^{1-\frac{p}{p_1}} \leq C,\end{aligned} \tag{3.7.93}$$

by the hypotheses on A. Thus, the contribution from I has the right order.

Before turning to II, we momentarily digress and establish more preliminary estimates. Given $\psi \in C^\infty(\mathbb{R}^n)$, $L \in \mathbb{Z}$, $x, h \in \mathbb{R}^n$, we denoted by $\mathcal{P}_L(\psi)(x,h)$ the Taylor polynomial of ψ of degree L expanded about x and evaluated at h (with the convention that $\mathcal{P}_L(\psi)(x,h) \equiv 0$ if $L < 0$). In particular, Taylor's remainder's formula gives

$$|\psi(x+h) - \mathcal{P}_L(\psi)(x,h)| \leq C(n,L)|h|^{L+1}\|\nabla^{L+1}\psi\|_{L^\infty(\mathbb{R}^n)}, \qquad L \geq -1, \tag{3.7.94}$$

uniformly in x. Also,

$$\partial_h^\gamma[\mathcal{P}_L(\psi)(x,h)] = \mathcal{P}_{L-|\gamma|}(\partial^\gamma\psi)(x,h), \qquad \forall\,\gamma\in\mathbb{N}_o^n, \quad |\gamma|\leq L+1. \tag{3.7.95}$$

Pick

$$\begin{aligned} &\xi\in C_c^\infty(2Q), \quad \xi\equiv 1 \text{ on } Q, \quad 0\leq\xi\leq 1,\\ &\|\partial^\gamma\xi\|_{L^\infty(\mathbb{R}^n)}\leq C_\gamma\, l(Q)^{-|\gamma|}, \qquad \forall\,\gamma\in\mathbb{N}_o^n. \end{aligned} \tag{3.7.96}$$

Our long-term goal is to prove the estimate

$$s\leq 0 \Longrightarrow \|\phi_j * A\|_{L^\infty(\mathbb{R}^n)}\leq C\,2^{j(n+L+1)}|Q|^{1+\frac{s}{n}-\frac{1}{p}+\frac{L+1}{n}}, \quad j=0,1,2,\ldots. \tag{3.7.97}$$

For an arbitrary multi-index $\gamma\in\mathbb{N}_o^n$ with $|\gamma|\leq L+1$ we start by estimating

$$\begin{aligned} &\|\partial^\gamma[(\phi_j(x-\cdot)-\mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)]\|_{L^\infty(\mathbb{R}^n)}\\ &\quad=\left\|\sum_{\beta\leq\gamma}\binom{\gamma}{\beta}\partial^\beta[\phi_j(x-\cdot)-\mathcal{P}_L(\phi_j)(x,\cdot)](\partial^{\gamma-\beta}\xi)\right\|_{L^\infty(\mathbb{R}^n)}\\ &\quad=\left\|\sum_{\beta\leq\gamma}\binom{\gamma}{\beta}[(\partial^\beta\phi_j)(x-\cdot)-\mathcal{P}_{L-|\beta|}(\partial^\beta\phi_j)(x,\cdot)](\partial^{\gamma-\beta}\xi)\right\|_{L^\infty(\mathbb{R}^n)}\\ &\quad\leq C(n,L)\sum_{\beta\leq\gamma} l(Q)^{-|\gamma|+|\beta|}\cdot l(Q)^{L-|\beta|+1}\cdot 2^{nj}\cdot 2^{j(L+1)}\\ &\quad\leq C(n,L)\,l(Q)^{L+1-|\gamma|}\,2^{j(n+L+1)}, \end{aligned} \tag{3.7.98}$$

uniformly in $x\in\mathbb{R}^n$. Here we have used (3.7.94), (3.7.95), (3.7.96). For $r>1$, $k\in\mathbb{N}_o$, $k\leq L+1$ we now estimate

$$\begin{aligned} &\|(\phi_j(x-\cdot)-\mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{L^r_k(\mathbb{R}^n)}\\ &\quad=\sum_{|\gamma|\leq k}\|\partial^\gamma[(\phi_j(x-\cdot)-\mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)]\|_{L^r(\mathbb{R}^n)}\\ &\quad\leq\sum_{|\gamma|\leq k}\|\partial^\gamma[(\phi_j(x-\cdot)-\mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)]\|_{L^\infty(\mathbb{R}^n)}\cdot|Q|^{\frac{1}{r}}\\ &\quad\leq C|Q|^{\frac{1}{r}}\,2^{j(n+L+1)}\sum_{|\gamma|\leq k} l(Q)^{L+1-|\gamma|}\leq C|Q|^{\frac{1}{r}}\,2^{j(n+L+1)}\,l(Q)^{L+1-k}\\ &\quad=C|Q|^{\frac{1}{r}-\frac{k}{n}+\frac{L+1}{n}}2^{j(n+L+1)} \end{aligned} \tag{3.7.99}$$

where we have used the fact that $l(Q) \leq c_0$. Next, let $0 < \alpha < L+1$, $\alpha \notin \mathbb{N}$, $r > 1$, and select $k \in \mathbb{N}_o$, $k \leq L+1$ such that $k < \alpha < k+1$. Set $\theta := \alpha - k \in (0,1)$. Then

$$\begin{aligned}
&\|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{L^r_\alpha(\mathbb{R}^n)} \\
&\quad \leq \|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|^{1-\theta}_{L^r_k(\mathbb{R}^n)} \cdot \|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|^{\theta}_{L^r_{k+1}(\mathbb{R}^n)} \\
&\quad \leq C|Q|^{\frac{1}{r}-\frac{\alpha}{n}+\frac{L+1}{n}} \cdot 2^{j(n+L+1)}.
\end{aligned} \tag{3.7.100}$$

Thus, so far,

$$\|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{L^r_\alpha(\mathbb{R}^n)} \leq C|Q|^{\frac{1}{r}-\frac{\alpha}{n}+\frac{L+1}{n}} 2^{j(n+L+1)}, \tag{3.7.101}$$

uniformly in $x \in \mathbb{R}^n$, for every $0 \leq \alpha \leq L+1$, $1 < r < \infty$. Via embeddings for Triebel-Lizorkin spaces, we finally conclude that

$$\begin{aligned}
&\|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{F^{r,t}_\alpha(\mathbb{R}^n)} \leq C|Q|^{\frac{1}{r}-\frac{\alpha}{n}+\frac{L+1}{n}} 2^{j(n+L+1)}, \\
&\text{uniformly in } x \in \mathbb{R}^n, \text{ for every } 0 \leq \alpha < L+1,\ 1 < r < \infty,\ 0 < t \leq \infty.
\end{aligned} \tag{3.7.102}$$

In fact, we claim that the following endpoint case of (3.7.102)

$$\begin{aligned}
&\|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{F^{\infty,\infty}_\alpha(\mathbb{R}^n)} \leq C|Q|^{-\frac{\alpha}{n}+\frac{L+1}{n}} 2^{j(n+L+1)}, \\
&\text{uniformly in } x \in \mathbb{R}^n, \text{ for every } 0 < \alpha < L+1,
\end{aligned} \tag{3.7.103}$$

holds as well. To see this, assume first that $\alpha \notin \mathbb{N}$, say $\alpha = k + \sigma$, $k \in \mathbb{N}_o$, $k \leq L$, $\sigma \in (0,1)$. Then, given that $\Phi := (\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)$ is supported in $2Q$,

$$\begin{aligned}
&\|(\phi_j \cdot (x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{F^{\infty,\infty}_\alpha(\mathbb{R}^n)} \\
&\qquad \leq \sum_{|\gamma|\leq k} \|\partial^\gamma \Phi\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma|=k} \left\{ \sup_{\substack{|h|\leq cl(Q) \\ z\in\mathbb{R}^n}} \frac{|\partial^\gamma\Phi(z+h) - \partial^\gamma\Phi(z)|}{|h|^\sigma} \right\} \\
&\qquad \leq \sum_{|\gamma|\leq k} \|\partial^\gamma \Phi\|_{L^\infty(\mathbb{R}^n)} + C \sum_{|\gamma|=k+1} \|\partial^\gamma \Phi\|_{L^\infty(\mathbb{R}^n)} \cdot l(Q)^{1-\sigma} \\
&\qquad \leq C|Q|^{\frac{L+1}{n}-\frac{\alpha}{n}} 2^{j(n+L+1)},
\end{aligned} \tag{3.7.104}$$

using (3.7.98) and, once again, the fact that $l(Q) \leq c_0$. The case when $\alpha \in \mathbb{N}$, $\alpha < L+1$, then follows from the case we have just proved and standard interpolation inequalities (for the real method).

Recall the smoothness index s and the integrability index p_1 in the statement of the lemma and assume for now that $s \leq 0$. If $p_1 > 1$ we write

$$\begin{aligned}
|(\phi_j * A)(x)| &= |\langle A, \phi_j(x-\cdot)\rangle| \\
&= |\langle A, (\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\rangle| \\
&\leq \|A\|_{F_s^{p_1,q}(\mathbb{R}^n)} \cdot \|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{F_{-s}^{p_1',q'}(\mathbb{R}^n)} \\
&\leq C|Q|^{\frac{1}{p_1}-\frac{1}{p}} \cdot |Q|^{\frac{1}{p_1'}+\frac{s}{n}+\frac{L+1}{n}} \cdot 2^{j(n+L+1)} \\
&= C|Q|^{1+\frac{s}{n}-\frac{1}{p}+\frac{L+1}{n}} \cdot 2^{j(n+L+1)},
\end{aligned} \tag{3.7.105}$$

uniformly in x, on account of (3.7.89) and (3.7.102). If $p_1 < 1$ then

$$(F_s^{p_1,q}(\mathbb{R}^n))^* = F^{\infty,\infty}_{-s+n\left(\frac{1}{p_1}-1\right)}(\mathbb{R}^n) \tag{3.7.106}$$

and (3.7.103) can be employed in order to conclude that, once again, (3.7.97) holds in this case. This finished the proof of (3.7.97).

Finally, let us consider the case when $p_1 = 1$. In this scenario, pick $1 < r < \infty$ and $\sigma < s$ such that $1/r - \sigma/n = 1 - s/n$. Then $\|A\|_{F_\sigma^{r,q}(\mathbb{R}^n)} \leq C|Q|^{1-1/p}$, thanks to (3.7.89) and an embedding, while $\|(\phi_j(x-\cdot) - \mathcal{P}_L(\phi_j)(x,\cdot))\xi(\cdot)\|_{F_{-\sigma}^{r',q'}(\mathbb{R}^n)} \leq C|Q|^{\frac{s}{n}+\frac{L+1}{n}} \cdot 2^{j(n+L+1)}$, by (3.7.102). Combining these two estimates and proceeding much as (3.7.105) we arrive at the conclusion that (3.7.97) continues to hold in this case as well.

Assuming that $s \leq 0$ we now proceed to estimate II. Obviously,

$$\begin{aligned} II &= \int_{\mathbb{R}^n \setminus cQ} \left(\sum_{j=0}^{\infty} |2^{js}(\phi_j * A)(x)|^q \right)^{\frac{p}{q}} dx \\ &= \int_{\mathbb{R}^n \setminus cQ} \left(\sum_{\substack{j \in \mathbb{N}_o \\ x \in \operatorname{supp}(\phi_j * A)}} |2^{js}(\phi_j * A)(x)|^q \right)^{\frac{p}{q}} dx. \end{aligned} \tag{3.7.107}$$

Note that, since $\operatorname{supp} \phi_j \subseteq B(0, c_o 2^{-j})$, the conditions that

$$x \in \mathbb{R}^n \setminus cQ \text{ and } x \in \operatorname{supp}(\phi_j * A) \Longrightarrow cl(Q) \leq |x| \leq c_o 2^{-j} + \sqrt{n}\, l(Q). \tag{3.7.108}$$

In particular, $l(Q) \leq c' 2^{-j}$ if $c > \sqrt{n}$ so that, ultimately, $|x| \leq c 2^{-j}$. This, in turn, implies that

$$j \leq \log_2 \left(\frac{c}{|x|} \right). \tag{3.7.109}$$

Then

$$\begin{aligned} II &\leq c \int_{\mathbb{R}^n \setminus cQ} \left(\sum_{j \leq \log_2\left(\frac{c}{|x|}\right)} \left[2^{js} \cdot |Q|^{1+\frac{s}{n}-\frac{1}{p}+\frac{L+1}{n}} \cdot 2^{j(n+L+1)} \right]^q \right)^{\frac{p}{q}} dx \\ &\leq c|Q|^{p\left(1+\frac{s}{n}-\frac{1}{p}+\frac{L+1}{n}\right)} \int_{\mathbb{R}^n \setminus cQ} \left(\sum_{j \leq \log_2\left(\frac{c}{|x|}\right)} 2^{jq(s+n+L+1)} \right)^{\frac{p}{q}} dx \\ &\leq c|Q|^{p\left(1+\frac{s}{n}-\frac{1}{p}-\frac{L+1}{n}\right)} \int_{\mathbb{R}^n \setminus cQ} \left(\frac{c}{|x|} \right)^{p(s+n+L+1)} dx \\ &\leq c|Q|^{p\left(1+\frac{s}{n}-\frac{1}{p}+\frac{L+1}{n}\right)} \cdot l(Q)^{n-p(s+n+L+1)} = C < \infty, \end{aligned} \tag{3.7.110}$$

if

$$p(s+n+L+1) > n. \tag{3.7.111}$$

Note that this last condition is satisfied whenever (3.7.88) holds. This finishes the proof of the lemma when $s \leq 0$.

The general case when $s \in \mathbb{R}$ is dealt with inductively. Specifically, assume first that $s < 1$. Fix A as in (3.7.89) and arrange matters so that Q is centered at the origin. Then pick $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\psi \equiv 1$ in $[-2c_o, 2c_o]^n$. Then, in the sense of tempered distributions,

$$A = \sum_{j=1}^{n} (\psi T_j \psi)(\partial_j A), \quad \text{where } \; T_j := \partial_j \Delta^{-1}. \tag{3.7.112}$$

Note that $\partial_j A$ has vanishing moments up to order $L+1$ (and (3.7.88) is satisfied with $L+1$ in place of L and $s-1$ in place of s), $\operatorname{supp}(\partial_j A) \subseteq Q$ and $\|\partial_j A\|_{F_{s-1}^{p_1,q}(\mathbb{R}^n)} \leq C\|A\|_{F_s^{p_1,q}(\mathbb{R}^n)} \leq C|Q|^{\frac{1}{p_1}-\frac{1}{p}}$. Since $s-1<0$, what we have proved so far applies and gives $\|\partial_j A\|_{F_{s-1}^{p,q}(\mathbb{R}^n)} \leq C$ for each j. Utilizing this back in (3.7.112) and invoking Corollary 3.2.12 yields $\|A\|_{F_s^{p,q}(\mathbb{R}^n)} \leq C$, as desired. Clearly, this scheme can be iterated to cover the case of an arbitrary number $s > 0$, finishing the proof of the lemma. □

3.8 Duality of spaces on domains

We start with an auxiliary result concerning density of infinitely differentiable functions supported in domain in the spaces with extra zero index.

Proposition 3.8.1. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then $\widetilde{C_c^\infty(\Omega)}$ (where tilde denotes extension by zero outside Ω) is dense both in $B_{s,0}^{p,q}(\Omega)$ and $F_{s,0}^{p,q}(\Omega)$ with $p \neq \infty$ in the case of F-spaces.*

Proof. The proof is based on local atomic decompositions of $B_{s,0}^{p,q}(\Omega)$ and $F_{s,0}^{p,q}(\Omega)$ spaces: speaking informally, we approximate given distribution by linear combinations of atoms.

To be more specific, assume that $f \in B^{p,q}_{s,0}(\Omega)$ and recall the partition of unity introduced in the proof Corollary 3.7.3. Following the same steps, we find the decompositions (3.7.59) and (3.7.61). The only difference is that this time

$$\left(\sum_{j=\beta}^{\infty}\Big(\sum_{\substack{Q\in\mathcal{Q}_n\\ l(Q)=2^{-j}}}|s^0_Q|^p\Big)^{q/p}\right)^{1/q}<\infty \quad\text{and}\quad \left(\sum_{j=\beta}^{\infty}\Big(\sum_{\substack{Q\in\mathcal{Q}^i_n\\ l(Q)=2^{-j}}}|\lambda^i_Q|^p\Big)^{q/p}\right)^{1/q}<\infty, \tag{3.8.1}$$

where $i=1,...,m$, and $\mathcal{Q}_n$ is a family of diadic cubes in the original system of coordinates and $\mathcal{Q}^i_n$, $i=1,...,m$ are families of diadic cubes in the Cartesian systems of coordinates $\mathcal{C}^i$ chosen so that $B_i\cap\Omega = D_i\cap\Omega$ and D_i is a special Lipschitz domain in $\mathcal{C}^i$. Also, $\|A_i\|_{B^{p,q}_s(\mathbb{R}^n)}\leq C\|f\|_{B^{p,q}_s(\mathbb{R}^n)}$.

Going further, for each $i=1,...,m$ and $j=\beta+1,\beta+2,...$ we consider partial sums of the series (3.7.59) and (3.7.61) in the form

$$S^j(f\eta_0):=\sum_{Q:\,2^{-j}\leq l(Q)}s^0_Qa^0_Q,\qquad S^j(f\eta_i):=A_i+\sum_{Q:\,2^{-j}\leq l(Q)}\lambda^i_Qa^i_Q. \tag{3.8.2}$$

Then by construction for each $i=1,...,m$ and $j=\beta+1,\beta+2,...$ the functions $S^j(f\eta_0)$ and $S^j(f\eta_i)$ belong to $\widetilde{C^\infty_c(\Omega)}$ and

$$\lim_{j\to\infty}S^j(f\eta_0)=f\eta_0,\quad \lim_{j\to\infty}S^j(f\eta_i)=f\eta_i \quad\text{with convergence in}\quad B^{p,q}_s(\mathbb{R}^n), \tag{3.8.3}$$

hence,

$$\lim_{j\to\infty}\left(S^j(f\eta_0)+\sum_{i=1}^m S^j(f\eta_i)\right)=f \quad\text{with convergence in}\quad B^{p,q}_{s,0}(\Omega), \tag{3.8.4}$$

and $S^j(f\eta_0)+\sum_{i=1}^m S^j(f\eta_i)\in\widetilde{C^\infty_c(\Omega)}$ for each $i=1,...,m$, $j=\beta+1,\beta+2,...$, as desired.

The same argument can be used to cover the case $f \in F^{p,q}_{s,0}(\Omega)$ with the only modification addressing the sequence spaces in (3.8.1). □

Proposition 3.8.2. *Let Ω be a bounded Lipschitz domain. Then with $s \in \mathbb{R}$, $1 < p, q \leq \infty$, $A \in \{B, F\}$ (and $p \neq \infty$ when $A = F$), consider the spaces*

$$\begin{aligned} \mathbb{A}^{p,q}_s(\Omega) \;:=\; \big\{ \Lambda : C^\infty_c(\Omega) \to \mathbb{R};\ \Lambda \text{ is linear and} \\ \exists\, C > 0 \text{ such that } |\Lambda(\phi)| \leq C \|\tilde{\phi}\|_{A^{p',q'}_{-s}(\mathbb{R}^n)}\ \forall\, \phi \in C^\infty_c(\Omega) \big\} \end{aligned} \tag{3.8.5}$$

Then there exists an isomorphism

$$\Upsilon : A^{p,q}_s(\Omega) \xrightarrow{iso} \mathbb{A}^{p,q}_s(\Omega). \tag{3.8.6}$$

Proof. For $u \in A^{p,q}_s(\Omega)$ we define $\Lambda_u(\phi) := \langle U, \tilde{\phi} \rangle$ for every $\phi \in C^\infty_c(\Omega)$ and $U \in A^{p,q}_s(\mathbb{R}^n)$ such that $U|_\Omega = u$. Then the operator

$$\Upsilon : A^{p,q}_s(\Omega) \ni u \longmapsto \Lambda_u \in \mathbb{A}^{p,q}_s(\Omega), \tag{3.8.7}$$

is linear and bounded. It is also one-to-one, for if $\Lambda_u = 0$ then $U|_\Omega = 0$ in $D'(\Omega)$ and hence $u = 0$ in Ω.

To see that it is onto consider some $\Lambda \in \mathbb{A}^{p,q}_s(\mathbb{R}^n)$. Then Λ extends to a bounded linear functional on the closure of $\widetilde{C^\infty_c(\Omega)}$ in $A^{p',q'}_{-s}(\mathbb{R}^n)$. Since the latter is a closed subspace of $A^{p',q'}_{-s}(\mathbb{R}^n)$, by Hahn-Banach theorem there is $U \in A^{p,q}_s(\mathbb{R}^n)$ such that $\Lambda(\phi) = \langle U, \tilde{\phi} \rangle$ for every $\phi \in C^\infty_c(\Omega)$. Then $U|_\Omega \in A^{p,q}_s(\Omega)$ and $\Upsilon(U|_\Omega) = \Lambda$, as desired. □

Theorem 3.8.3. *Let Ω be a bounded Lipschitz domain. Then there exist isomor-*

phisms

$$\Phi : (B^{p,q}_{s,0}(\Omega))^* \xrightarrow{iso} B^{p',q'}_{-s}(\Omega), \qquad s\in\mathbb{R},\ 0<p,q<\infty,$$

$$\Phi : (F^{p,q}_{s,0}(\Omega))^* \xrightarrow{iso} F^{p',q'}_{-s}(\Omega), \qquad s\in\mathbb{R},\ 1<p<\infty,\ 0<q<\infty,$$

$$\Phi : (F^{p,q}_{s,0}(\Omega))^* \xrightarrow{iso} B^{\infty,\infty}_{-s+n\left(\frac{1}{p}-1\right)}(\Omega), \qquad s\in\mathbb{R},\ 0<p<1,\ 0<q<\infty, \tag{3.8.8}$$

and isomorphisms

$$\Psi : (B^{p,q}_{s}(\Omega))^* \xrightarrow{iso} B^{p',q'}_{-s,0}(\Omega), \qquad s\in\mathbb{R},\ 0<p,q<\infty,$$

$$\Psi : (F^{p,q}_{s}(\Omega))^* \xrightarrow{iso} F^{p',q'}_{-s,0}(\Omega), \qquad s\in\mathbb{R},\ 1<p<\infty,\ 0<q<\infty,$$

$$\Psi : (F^{p,q}_{s}(\Omega))^* \xrightarrow{iso} B^{\infty,\infty}_{-s+n\left(\frac{1}{p}-1\right),0}(\Omega), \qquad s\in\mathbb{R},\ 0<p<1,\ 0<q<\infty. \tag{3.8.9}$$

For $s\in\mathbb{R}$ and $1<p,q<\infty$ the spaces $B^{p,q}_{s}(\Omega)$, $F^{p,q}_{s}(\Omega)$ and $B^{p,q}_{s,0}(\Omega)$, $F^{p,q}_{s,0}(\Omega)$ are reflexive.

Proof. We will proceed in several steps, starting with

The proof of (3.8.8). Let us focus on the statement for Besov spaces, the rest follows verbatim. To this end, consider some functional $\Lambda : B^{p,q}_{s,0}(\Omega)\to\mathbb{R}$ and recall the projection operators from (3.6.41):

$$P := I - E_{\mathbb{R}^n\setminus\overline{\Omega}}\circ\mathcal{R}_{\mathbb{R}^n\setminus\overline{\Omega}} : B^{p,q}_{s}(\mathbb{R}^n)\longrightarrow B^{p,q}_{s,0}(\Omega). \tag{3.8.10}$$

Then $\widehat{\Lambda} := \Lambda\circ P : B^{p,q}_{s}(\mathbb{R}^n)\to\mathbb{R}$ is a linear bounded functional on $B^{p,q}_{s}(\mathbb{R}^n)$. By Theorem 3.2.6 we can write $\widehat{\Lambda}\in B^{p',q'}_{-s}(\mathbb{R}^n)$. Now denote by Φ the following application

$$\Phi : (B^{p,q}_{s,0}(\Omega))^* \ni \Lambda\longmapsto\widehat{\Lambda}|_{\Omega}\in B^{p',q'}_{-s}(\Omega), \tag{3.8.11}$$

then Φ is linear and bounded by construction.

Finally, there exists an inverse of Φ defined in the following way. For $u \in B^{p',q'}_{-s}(\Omega)$ one can take any $U \in B^{p',q'}_{-s}(\mathbb{R}^n)$ such that $U|_\Omega = u$ and then set

$$\Phi^{-1}(u) : B^{p',q'}_{-s}(\Omega) \ni u \longmapsto \Lambda_u \in (B^{p,q}_{s,0}(\Omega))^*, \tag{3.8.12}$$

where

$$\Lambda_u(v) := \langle U, v\rangle \text{ for every } v \in B^{p,q}_{s,0}(\Omega). \tag{3.8.13}$$

Note that by Theorem 3.8.1 for every $v \in B^{p,q}_{s,0}(\Omega)$ there exists a sequence of functions $\{\phi_j\}_{j=0^\infty}$ from $C^\infty_c(\Omega)$ such that $\{\widetilde{\phi}_j\}^\infty_{j=0}$ converges to v in the norm of $B^{p,q}_s(\mathbb{R}^n)$. Therefore if $U \in B^{p',q'}_{-s}(\mathbb{R}^n)$ is such that $U|_\Omega = 0$ then

$$\langle U, v\rangle = \lim_{j\to\infty} \langle U, \widetilde{\phi}_j\rangle = \lim_{j\to\infty} \langle U|_\Omega, \phi_j\rangle = 0, \tag{3.8.14}$$

so that Φ^{-1} is well-defined. Clearly it is linear and bounded.

To prove that Φ^{-1} is an inverse of Φ consider first some $u \in B^{p',q'}_{-s}(\Omega)$. Then for every $\phi \in C^\infty_c(\Omega)$

$$\langle \Phi \circ \Phi^{-1}(u), \phi\rangle = \langle \widehat{\Phi^{-1}(u)}|_\Omega, \phi\rangle = \langle \widehat{\Phi^{-1}(u)}, \widetilde{\phi}\rangle = \langle \Phi^{-1}(u), P\widetilde{\phi}\rangle = \langle \Phi^{-1}(u), \widetilde{\phi}\rangle \tag{3.8.15}$$

and for every $U \in B^{p',q'}_{-s}(\mathbb{R}^n)$ such that $U|_\Omega = u$ the expression above is equal to

$$\langle U, \widetilde{\phi}\rangle = \langle U|_\Omega, \phi\rangle = \langle u, \phi\rangle, \tag{3.8.16}$$

as desired, hence, $\Phi \circ \Phi^{-1} = I$, identity operator, on $B^{p',q'}_{-s}(\Omega)$. Note that it is enough consider $\phi \in C^\infty_c(\Omega)$ in the argument above by Theorem 3.8.1.

Conversely, for every $\Lambda \in (B^{p,q}_{s,0}(\Omega))^*$ and every $v \in B^{p,q}_{s,0}(\Omega)$

$$\langle (\Phi^{-1} \circ \Phi)(\Lambda), v\rangle = \langle \Phi^{-1}(\widehat{\Lambda}|_\Omega), v\rangle = \langle \widehat{\Lambda}, v\rangle = \langle \Lambda, v\rangle, \tag{3.8.17}$$

so that $\Phi^{-1} \circ \Phi = I$ on $(B^{p,q}_{s,0}(\Omega))^*$.

The proof of (3.8.9). Once again, we concentrate on the first statement in 3.8.9, the second and the third one can be proved following the same lines. For every $\Lambda \in (B^{p,q}_{s}(\Omega))^*$ we can define $\Psi(\Lambda) := \Lambda \circ \mathcal{R}_\Omega$. By definition $\Psi(\Lambda)$ is a linear bounded functional on $B^{p,q}_s(\mathbb{R}^n)$ and as such belongs to the dual of the latter space $(B^{p,q}_s(\mathbb{R}^n))^* = B^{p',q'}_{-s}(\mathbb{R}^n)$. Also, it is not hard to see that $\Psi(\Lambda)$ is supported in $\bar{\Omega}$ in the sense that for every $u \in B^{p,q}_s(\mathbb{R}^n)$ such that $\operatorname{supp} u \subseteq \mathbb{R}^n \setminus \Omega$ the application $\Psi(\Lambda)u = 0$. Hence,

$$\Psi : (B^{p,q}_s(\Omega))^* \ni \Lambda \longmapsto \Lambda \circ \mathcal{R}_\Omega \in B^{p',q'}_{-s,0}(\Omega), \tag{3.8.18}$$

linear and bounded, as desired.

Conversely, we define

$$\Psi^{-1} : B^{p',q'}_{-s,0}(\Omega) \ni u \longmapsto \Lambda_u \in (B^{p,q}_s(\Omega))^*, \tag{3.8.19}$$

where

$$\Lambda_u(v) := \langle u, V\rangle \text{ for every } v \in B^{p,q}_s(\Omega) \text{ and } V \in B^{p,q}_s(\mathbb{R}^n) \text{ such that } V|_\Omega = v. \tag{3.8.20}$$

The definition above does not depend on the choice of V for given v, since if $V_1 \in B^{p,q}_s(\mathbb{R}^n)$ and $V_2 \in B^{p,q}_s(\mathbb{R}^n)$ are such that $V_1|_\Omega = V_2|_\Omega$ then

$$\langle u, V_1\rangle = \lim_{j\to\infty} \langle \tilde{\phi}_j, V_1\rangle = \lim_{j\to\infty} \langle \tilde{\phi}_j, V_2\rangle = \langle u, V_2\rangle, \tag{3.8.21}$$

where $\{\phi_j\}_{j=0^\infty}$ are functions from $C_c^\infty(\Omega)$ such that the sequence $\{\tilde{\phi}_j\}_{j=0}^\infty$ converges to u in $B_{-s,0}^{p',q'}(\Omega)$. So Ψ^{-1} is well-defined, linear and bounded.

Let us check now that Ψ and Ψ^{-1} are inverses of each other. For $\Lambda \in (B_s^{p,q}(\Omega))^*$ and $u \in B_s^{p,q}(\Omega)$ (with some $U \in B_s^{p,q}(\mathbb{R}^n)$ such that $U|_\Omega = u$)

$$\langle \Psi^{-1} \circ \Psi(\Lambda), u\rangle = \langle \Psi(\Lambda), U\rangle = \langle \Lambda, \mathcal{R}_\Omega U\rangle = \langle \Lambda, u\rangle, \tag{3.8.22}$$

as desired.

Conversely, consider some $u \in B_{-s,0}^{p',q'}(\Omega)$ and $V \in B_s^{p,q}(\mathbb{R}^n)$ we can write

$$\langle \Psi(\Psi^{-1}(u)), V\rangle = \langle \Psi^{-1}(u), V|_\Omega\rangle = \langle u, V\rangle, \tag{3.8.23}$$

which finishes the argument.

The proof of reflexiveness. Consider some $\Lambda \in (B_{s,0}^{p,q}(\Omega))^*$ and $\xi \in (B_{-s}^{p',q'}(\Omega))^*$, $s \in \mathbb{R}$, $1 < p, q < \infty$. Then denoting by Φ^* the formal adjoint of Φ we have

$$\langle \Phi^*(\xi), \Lambda\rangle = \langle \xi, \Phi\Lambda\rangle = \langle \xi, \widehat{\Lambda}|_\Omega\rangle = \langle \xi \circ \mathcal{R}_\Omega, \widehat{\Lambda}\rangle = \langle \xi \circ \mathcal{R}_\Omega, \Lambda\rangle = \langle \Psi(\xi), \Lambda\rangle, \tag{3.8.24}$$

so that $\Phi^* = \Psi$ which along with (3.8.8) and (3.8.9) proves the last statement of the theorem. Once again, the case of Triebel-Lizorkin spaces follows verbatim. □

Theorem 3.8.4. *Let Ω be a bounded Lipschitz domain. Then*

$$(h^1(\Omega))^* = \mathrm{bmo}_0(\Omega), \quad (h_0^1(\Omega))^* = \mathrm{bmo}(\Omega) \quad \textit{and} \quad (\mathrm{vmo}_0(\Omega))^* = h^1(\Omega), \tag{3.8.25}$$

$$\Big(h^p(\Omega)\Big)^* = \{u \in C^\alpha(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\Omega}\}, \quad \tfrac{n}{n+1} < p < 1, \ \ \alpha = n\big(\tfrac{1}{p} - 1\big). \tag{3.8.26}$$

Proof. The series of equalities (3.8.25) were proved in [**8**], the second one of those also follows from the same argument as the one used in Theorem 3.8.3 for (3.8.8).

The equality (3.8.26) goes back to [**20**] and is also a part of Theorem 3.8.3. Specifically, it coincides with (3.8.9) for $s = 0$, $q = 2$, $\frac{n}{n+1} < p < 1$. □

3.9 Spaces of null-solutions of elliptic operators

Let L be a (homogeneous) constant-coefficient, elliptic differential operator of order $\mu \in 2\,\mathbb{N}$ in $\mathbb{R}^n$, as in (2.4.26). Also, fix a (possibly unbounded) Lipschitz domain $\Omega \subset \mathbb{R}^n$, and recall that $\operatorname{Ker} L$ stands for the space of functions satisfying $Lu = 0$ in Ω. Finally, for $0 < p < \infty$ and $\alpha \in \mathbb{R}$, we denote by $\mathbb{H}^p_\alpha(\Omega; L)$ the space of functions $u \in \operatorname{Ker} L$ subject to

$$\|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} := \|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha\rangle-1} \|\nabla^j u\|_{L^p(\Omega)} < \infty, \tag{3.9.1}$$

where $\langle\alpha\rangle$ is the smallest nonnegative integer greater than or equal to α, i.e.,

$$\langle\alpha\rangle := \begin{cases} \alpha \text{ if } \alpha \in \mathbb{N}_o, \\ [\alpha] + 1 \text{ if } \alpha > 0,\ \alpha \notin \mathbb{N}, \\ 0 \text{ if } \alpha < 0. \end{cases} \tag{3.9.2}$$

Above, recall that ∇^j stands for vector of all mixed-order partial derivatives of order j and that $[\cdot]$ is the integer-part function. When Ω is bounded, an equivalent quasi-norm on $\mathbb{H}^p_\alpha(\Omega; L)$ is given by

$$\|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sup_{x\in\mathcal{O}} |u(x)|, \tag{3.9.3}$$

where $\mathcal{O}$ denotes some fixed compact subset of Ω. Indeed, using the interior estimates the lower order term in the right-hand side of (3.9.3) can be controlled by $C\|u\|_{L^p(\Omega)}$, with some constant C depending only on the domain. As for the converse, (2.4.35) shows that for every $0 \leq j \leq \langle\alpha\rangle - 1$

$$\begin{aligned} \|\nabla^j u\|_{L^p(\Omega)} &\leq C\left[\Big(\int_\Omega (\delta(x)^{\langle\alpha\rangle-j}|\nabla^{\langle\alpha\rangle}u(x)|)^p\,dx\Big)^{1/p} + \sup_{x\in\mathcal{O}}|u(x)|\right] \\ &\leq C\left[\Big(\int_\Omega (\delta(x)^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u(x)|)^p\,dx\Big)^{1/p} + \sup_{x\in\mathcal{O}}|u(x)|\right], \end{aligned} \tag{3.9.4}$$

where the last inequality rests on the observation that $\delta^{\langle\alpha\rangle-j} \leq C\delta^{\langle\alpha\rangle-\alpha}$, whenever $0 \leq j \leq \langle\alpha\rangle - 1$, on the bounded domain Ω.

For each fixed operator L as before, the spaces $\mathbb{H}^p_\alpha(\Omega; L)$ are amenable to complex and real interpolation. Specifically, we have:

Theorem 3.9.1. *Let L a constant-coefficient, elliptic differential operator as in (2.4.26). Also, suppose Ω is a bounded, star-like Lipschitz domain in $\mathbb{R}^n$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p < \infty$, $0 < \theta < 1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then the following interpolation formulas hold with equivalent norms:*

$$(\mathbb{H}^p_{\alpha_0}(\Omega; L), \mathbb{H}^p_{\alpha_1}(\Omega; L))_{\theta,p} = \mathbb{H}^p_\alpha(\Omega; L), \tag{3.9.5}$$

$$[\mathbb{H}^p_{\alpha_0}(\Omega; L), \mathbb{H}^p_{\alpha_1}(\Omega; L)]_\theta = \mathbb{H}^p_\alpha(\Omega; L). \tag{3.9.6}$$

Proof. The identity (3.9.5) is proved in [**119**] for $n = 2$ and $L = \overline{\partial}$, the Cauchy-Riemann operator in the plane, and the proof goes through vitually verbatim. In fact, (3.9.6) is also proved in [**119**] (once again, when $L = \overline{\partial}$), but for a different concept of complex interpolation. For the reasoning in [**119**] to work in our case (i.e., for the complex method of interpolation described in §3.1), we need to check two more things, namely:

(i) given $\alpha_0, \alpha_1 \in \mathbb{R}$ and $0 < p < \infty$, the space $\mathbb{H}^p_{\alpha_0}(\Omega; L) + \mathbb{H}^p_{\alpha_1}(\Omega; L)$ is analytically convex, and

(ii) given $f \in L^p(\Omega) \cap \operatorname{Ker} L$, $0 < p < \infty$, the $L^p(\Omega) \cap \operatorname{Ker} L$-valued mapping

$$\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\} \ni \zeta \mapsto R_\zeta f \in L^p(\Omega) \cap \operatorname{Ker} L, \text{ where} \tag{3.9.7}$$

$$(R_\zeta f)(x) := \frac{(-1)^m}{\Gamma(\zeta+m)} \int_0^\infty t^{\zeta+m-1} \frac{d^m}{dt^m}\Big(e^{-t} f(e^{-t}x)\Big)\, dt, \quad x \in \Omega, \tag{3.9.8}$$

is analytic, in the sense of (3.1.2). Here, Ω is assumed to be star-like with respect to the origin in $\mathbb{R}^n$, $\Gamma(\cdot)$ is the Gamma function and $m \in \mathbb{N}_o$. Note that the definition of the fractional intergration operator R_ζ is, in fact, independent of m, as an m-fold integration by parts shows.

Condition (i) is required since, according to the setup in discussed in §3.1, the sum of the two quasi-Banach spaces in question must be analytically convex. The fact that the map (3.9.7) has a complex derivative at every point in the right half-plane is proved in Lemma 9 on p. 663 in [**119**]. Nonetheless, as pointed out right after (3.1.2), this concept of analyticity is too weak for our method of complex interpolation to work. Thus, instead of pointwise complex differentiability, we have to insist on a power series condition such as (3.1.2), hence the necessity of (ii). Granted (i)-(ii), the same argument as in [**119**] applies and yields (3.9.6) (strictly speaking, this result is proved for analytic functions, [**119**] but the argument goes through with only minor alterations in the case of harmonic functions).

We now turn to the proof of (i). Since $\mathbb{H}^p_{\alpha_0}(\Omega; L) + \mathbb{H}^p_{\alpha_1}(\Omega; L) = \mathbb{H}^p_{\alpha_0}(\Omega; L)$ if $-\infty < \alpha_0 < \alpha_1 < +\infty$, this claim reduces to showing that

$$\mathbb{H}^p_\alpha(\Omega; L) \text{ is analytically convex, } \forall\, \alpha \in \mathbb{R},\ \forall\, p \in (0, \infty). \tag{3.9.9}$$

To justify (3.9.9), first introduce

$$L^p(\Omega, \delta^s) := \Big\{ u \in L^p_{\rm loc}(\Omega) : \int_\Omega |u(x)|^p \,{\rm dist}\,(x, \partial\Omega)^{sp}\, dx < +\infty \Big\}, \tag{3.9.10}$$

equipped with the obvious quasi-norm. Next, we note that the map

$$\mathbb{H}^p_\alpha(\Omega; L) \ni u \mapsto (\nabla^{\langle\alpha\rangle} u,\, \nabla^{\langle\alpha\rangle - 1} u,\, ...\, u) \in L^p(\Omega, \delta^{\langle\alpha\rangle - \alpha}) \oplus L^p(\Omega) \oplus \cdots \oplus L^p(\Omega)$$

(3.9.11)

identifies $\mathbb{H}^p_\alpha(\Omega; L)$ with a closed subspace of $L^p(\Omega, \delta^{\langle\alpha\rangle-\alpha}) \oplus L^p(\Omega) \oplus \cdots \oplus L^p(\Omega)$. Since, by Theorem 3.1.6, the latter space is analytically convex, it follows that $\mathbb{H}^p_\alpha(\Omega; L)$ is analytically convex as well.

Next, consider the analyticity claim made in (ii) about the operator (3.9.7)-(3.9.8). We shall choose $m = 1$ and drop the factor in front of the integral so that, for a fixed $f \in L^p(\Omega) \cap \operatorname{Ker} L$, it suffices to consider

$$\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\} \ni \zeta \mapsto \int_0^\infty t^\zeta e^{-t}\Big(1 + x \cdot (\nabla f)(e^{-t}x)\Big)\, dt, \quad x \in \Omega. \tag{3.9.12}$$

Continuing our series of reductions, it will suffice to assume that $0 < p < 1$ and justify that, for a fixed $f \in L^p(\Omega) \cap \operatorname{Ker} L$,

$$\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\} \ni \zeta \mapsto A_\zeta f(\cdot) := \int_0^\infty t^\zeta e^{-t}(\nabla f)(e^{-t}\cdot)\, dt \tag{3.9.13}$$

is a $L^p(\Omega)$-valued analytic function (the lower order term in (3.9.12) is handled similarly).

To this end, we fix $\zeta_o \in \mathbb{C}$ with $\operatorname{Re}\zeta_o > 0$ and formally write

$$A_\zeta f = \sum_{j=0}^\infty (\zeta - \zeta_o)^j\, f_j \tag{3.9.14}$$

where

$$f_j(x) := \frac{1}{j!}\int_0^\infty (\ln t)^j\, t^{\zeta_o}\, e^{-t}(\nabla f)(e^{-t}x)\, dt, \qquad x \in \Omega,\ \ j \in \mathbb{N}_o. \tag{3.9.15}$$

There remains to show that $f_j \in L^p(\Omega)$ and that the series (3.9.14) converges uniformly for $|\zeta - \zeta_o| < \eta$, provided that $\eta > 0$ is small enough. This, in turn, will

follow as soon as we show that there exists a finite constant $\kappa = \kappa(\zeta_o, p, f) > 0$ and some $C > 0$ such that

$$\|f_j\|_{L^p(\Omega)} \leq C\kappa^j, \qquad \forall\, j \in \mathbb{N}_o. \tag{3.9.16}$$

To see this, we first make the elementary observation that, for each $\varepsilon > 0$, there exists a finite constant $C_\varepsilon > 0$ such that for each $j \in \mathbb{N}_o$,

$$\frac{t^\varepsilon |\ln t|^j}{j!} \leq (C_\varepsilon)^j \ \text{ if } \ 0 < t \leq 1, \quad \text{and} \quad \frac{|\ln t|^j e^{-\varepsilon t}}{j!} \leq (C_\varepsilon)^j \ \text{ if } \ 1 < t < \infty. \tag{3.9.17}$$

Indeed, in the second inequality in (3.9.17), use $|\ln t|^j \leq t^j$ for $t \in (1, \infty)$ and the fact that the mapping $(0, \infty) \ni t \mapsto t^j e^{-\varepsilon t} \in (0, \infty)$ has a global maxim at $t = j/\varepsilon$, to write for each $t \in (1, \infty)$

$$\frac{|\ln t|^j e^{-\varepsilon t}}{j!} \leq \varepsilon^{-j} \frac{j^j e^{-j}}{j!} \leq C\Big(\frac{1}{\varepsilon}\Big)^j, \qquad j \in \mathbb{N}, \tag{3.9.18}$$

where the last step uses Stirling's formula to the effect that $j! \approx j^j e^{-j}$. Finally, the first inequality in (3.9.17) reduces to what we have just proved by making the substitution $t = e^{-\tau}$, $\tau \in (0, \infty)$.

Turning to the proof of (3.9.16) in the earnest, we note that, by assumption, there exists a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}$ with $\inf\{\varphi(\omega) : \omega \in S^{n-1}\} > 0$ such that $\Omega = \{\omega\rho : \omega \in S^{n-1},\, 0 < \rho < \varphi(\omega)\}$. As a consequence of the Mean Value Theorem for null-solutions of L, there exists a small, compact neighborhood $\mathcal{O}$ of $0 \in \Omega$ such that, whenever $0 < p < \infty$,

$$\int_\Omega |u(x)|^p \, dx \leq C \int_{\Omega \setminus \mathcal{O}} |u(x)|^p \, dx, \quad \text{ uniformly for } u \in \operatorname{Ker} L. \tag{3.9.19}$$

Note that $Lf_j = 0$ in Ω for every $j \in \mathbb{N}_o$ so that, for some $M > 0$ large enough,

$$\begin{aligned}
\int_{\Omega} |f_j(x)|^p \, dx &\leq C \int_{\Omega\setminus\mathcal{O}} |f_j(x)|^p \, dx \\
&\leq C \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} |f_j(\rho\omega)|^p \rho^{n-1} \, d\rho d\omega \qquad (3.9.20) \\
&\leq (C_\varepsilon)^j \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} \left(\int_0^\infty t^{\operatorname{Re}\zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)(e^{-t}\rho\omega)| \, dt \right)^p \rho^{n-1} \, d\rho d\omega \\
&\leq (C_\varepsilon)^j \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} \left(\int_0^\infty t^{\operatorname{Re}\zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)_{\rm rad}(e^{-t}\rho\omega)| \, dt \right)^p d\rho d\omega.
\end{aligned}$$

Above, we picked $0 < \varepsilon < \min\{1, \operatorname{Re}\zeta_o\}$ and then made use of the estimate (3.9.17). Also, $(\cdot)_{\rm rad}$ is the maximal radial operator introduced in (2.2.4).

To proceed, change variables such that $\rho = \varphi(\omega)e^{-s}$, $s \in (0, M)$, and $\lambda = s + t$ so that $\lambda \in (s, \infty)$; in particular, $\lambda > t > \lambda - M$. Then the last expression in (3.9.20) is

$$\begin{aligned}
&\leq (C_\varepsilon)^j \int_{S^{n-1}} \int_0^M \left(\int_s^\infty \lambda^{\operatorname{Re}\zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)_{\rm rad}(e^{-\lambda}\varphi(\omega)\omega)| \, d\lambda \right)^p ds \, d\omega \\
&\leq (C_\varepsilon)^j \int_{S^{n-1}} \int_0^\infty \left(\lambda^{\operatorname{Re}\zeta_o - \varepsilon + 1} e^{-(1-\varepsilon)t} |(\nabla f)_{\rm rad}(e^{-\lambda}\varphi(\omega)\omega)| \right)^p d\lambda \, d\omega \\
&\leq (C_\varepsilon)^j \left\{ \int_{S^{n-1}} \int_0^M \left(\lambda^{\operatorname{Re}\zeta_o - \varepsilon + 1} e^{-(1-\varepsilon)t} |(\nabla f)_{\rm rad}(e^{-\lambda}\varphi(\omega)\omega)| \right)^p d\lambda \, d\omega + \|f\|^p_{L^p(\Omega)} \right\}.
\end{aligned}$$

The second estimate above is based on Hardy's inequality (in the version discussed in Lemma 1 on p. 670 of [**119**]). Also, in the last inequality above, we have used the fact that $(\nabla f)_{\rm rad}(e^{-\lambda}\varphi(\omega)\omega) \leq C\|f\|_{L^p(\Omega)}$, uniformly for $\lambda > M$ and $\omega \in S^{n-1}$, given that $Lf = 0$ in Ω and that $e^{-\lambda}\varphi(\omega)\omega$ lies, for this range of parameters, in a fixed, compact subset of Ω.

There remains to handle the double integral inside the brackets. For this we

make the change of variables $r = e^{-\lambda}\varphi(\omega)$ so that $d\lambda \approx r^{n-1}dr$ and note that

$$\delta(r\omega) = \operatorname{dist}(r\omega, \partial\Omega) = \varphi(\omega) - r \approx e^{-\lambda} - 1 \approx \lambda, \tag{3.9.21}$$

uniformly for $\lambda \in (0, M)$. Thus, the double integral in question is majorized by

$$C\int_{\Omega}[\delta|(\nabla f)_{\rm rad}|]^p\,dx \leq C\int_{\Omega}[\delta|\nabla f|]^p\,dx \leq C\int_{\Omega}|f|^p\,dx, \tag{3.9.22}$$

by Lemma 2.4.3 and Theorem 2.4.4.

This completes the justification of (3.9.16) and finishes the proof of the theorem. □

Another type of harmonic spaces on Ω can be defined as the collection of all harmonic functions belonging to $F^{p,q}_s(\Omega)$ or $B^{p,q}_s(\Omega)$. In the rest of the paper theses spaces will be denoted by $\mathcal{H}\cap F^{p,q}_s(\Omega)$ and $\operatorname{Ker} L\cap B^{p,q}_s(\Omega)$, respectively.

Theorem 3.9.2. *Consider an elliptic, homogeneous, constant coefficient differential operator L as in (2.4.26) of degree $\mu < n$, and fix a bounded Lipschitz domain Ω in $\mathbb{R}^n$. Also, assume that $0 < q_0, q_1, q \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$, and set $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then, if $0 < p < \infty$,*

$$(\operatorname{Ker} L\cap F^{p,q_0}_{\alpha_0}(\Omega), \operatorname{Ker} L\cap F^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} = \operatorname{Ker} L\cap B^{p,q}_{\alpha}(\Omega), \tag{3.9.23}$$

and if $0 < p \leq \infty$,

$$(\operatorname{Ker} L\cap B^{p,q_0}_{\alpha_0}(\Omega), \operatorname{Ker} L\cap B^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} = \operatorname{Ker} L\cap B^{p,q}_{\alpha}(\Omega). \tag{3.9.24}$$

Next, let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then

$$[\operatorname{Ker} L\cap F^{p_0,q_0}_{\alpha_0}(\Omega), \operatorname{Ker} L\cap F^{p_1,q_1}_{\alpha_1}(\Omega)]_{\theta} = \operatorname{Ker} L\cap F^{p,q}_{\alpha}(\Omega). \tag{3.9.25}$$

Finally, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ *and either* $p_0 + q_0 < \infty$ *or* $p_1 + q_1 < \infty$, *then*

$$[\operatorname{Ker} L \cap B^{p_0,q_0}_{\alpha_0}(\Omega), \mathcal{H} \cap B^{p_1,q_1}_{\alpha_1}(\Omega)]_\theta = \mathcal{H} \cap B^{p,q}_{\alpha}(\Omega), \tag{3.9.26}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. Set $X_i := F^{p_i,q_i}_{\alpha_i}(\Omega)$, $Z_i := F^{p_i,q_i}_{\alpha_i-\mu,0}(Q)$ and $Y_i = F^{p_i,q_i}_{\alpha_i-\mu,0}(Q \setminus \overline{\Omega})$, where Q stands for some open cube in $\mathbb{R}^n$ containing $\overline{\Omega}$ and $i = 1, 2$.

We aim to use Theorem 3.1.8, so the first order of business is to check A-convexity for the spaces $X_0 + X_1$ and $Y_0 + Y_1$. However by the argument of Theorem 3.6.4 the sum of spaces $X_0 + X_1$ is analytically convex. As for the sum $Y_0 + Y_1$, we can define the operators

$$\begin{aligned} P_1 &:= I - E_{\mathbb{R}^n \setminus \overline{Q}} \circ \mathcal{R}_{\mathbb{R}^n \setminus \overline{Q}} : F^{p_i,q_i}_{\alpha_i-\mu}(\mathbb{R}^n) \longrightarrow F^{p_i,q_i}_{\alpha_i-\mu,0}(Q), \\ P_2 &:= I - E^Q_\Omega \circ \mathcal{R}_\Omega : F^{p_i,q_i}_{\alpha_i-\mu,0}(Q) \longrightarrow F^{p_i,q_i}_{\alpha_i-\mu,0}(Q \setminus \overline{\Omega}), \end{aligned} \tag{3.9.27}$$

where $i = 0, 1$ and E^Q_Ω denotes the Rychkov's extension operator truncated near $\overline{\Omega}$ so that it maps the functions from $F^{p_i,q_i}_{\alpha_i-\mu}(\Omega)$ to those supported in the cube Q, i.e. belonging to Z_i. Both P_1 and P_2 are common projections for the corresponding spaces and hence Lemma 3.1.9 applies and shows that $Y_0 + Y_1$ are analytically convex.

Consider the operator $D := L \circ E^Q_\Omega$. One can observe now that, in the notation of Theorem 3.1.8, $X_i(D) = \operatorname{Ker} L \cap F^{p_i,q_i}_{\alpha_i}(\Omega)$ for $i = 1, 2$. We now let G stand for the composition $\mathcal{R}_\Omega \circ \Pi_L$ where Π_L is the Newtonian potential operator associated with L (defined as in (3.2.81) for $a(\xi) := L(-i\xi)^{-1}$) and, as usual, $\mathcal{R}_\Omega$ is the operator

of restriction to Ω). Then $G : Z_i \to X_i$ boundedly, thanks to Corollary 3.2.12, granted that $\mu < n$. Furthermore, if I stands for the identity operator, then for every test function $\psi \in C_c^\infty(\Omega)$ and for every distribution $u \in Z_i$,

$$\begin{aligned}\langle (D \circ G - I)u, \psi\rangle &= \langle L \circ E_\Omega^Q \circ \mathcal{R}_\Omega \circ \Pi_L u\,,\, \psi\rangle - \langle u, \psi\rangle \\ &= (-1)^\mu \langle \Pi_L u, L\psi\rangle - \langle u, \psi\rangle = 0. \end{aligned} \tag{3.9.28}$$

Hence, $K := D \circ G - I$ is a bounded linear operator from Z_i to Y_i and $D \circ G = I + K$ on Z_i by construction. Then (3.9.23) and (3.9.25) follow from Theorem 3.1.8. A similar argument works for the harmonic Besov scale and this finishes the proof of the theorem. □

Theorem 3.9.3. *Let L be an elliptic, homogeneous, constant coefficient differential operator L as in (2.4.26) of degree $\mu < n$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$\mathbb{H}^p_\alpha(\Omega; L) = F^{p,2}_\alpha(\Omega) \cap \operatorname{Ker} L, \tag{3.9.29}$$

$$\mathbb{H}^p_\alpha(\Omega; L) = B^{p,p}_\alpha(\Omega) \cap \operatorname{Ker} L, \tag{3.9.30}$$

for each $\alpha \in \mathbb{R}$ and each $0 < p < \infty$.

Even though we are ultimately interested in the case when $L = \Delta$, our proof dictates that we consider a general operator L instead of just the Laplacian. Let us point out that, when $L = \Delta$, the identity (3.9.29) for $\alpha \geq 0$, $1 \leq p \leq \infty$, and the identity (3.9.30) for $\alpha > 0$, $1 < p < \infty$, have been established by C. Kenig and D. Jerison in **[68]** using some deep properties of harmonic functions. In the case of the Besoov scale, their approach uses the intrinsic characterizations of these spaces

in terms of finite difference operators of the sort described in Theorem 3.6.8. In particular, some of the limitations in the range of indices one may consider in this scenario are inherited from here.

Prior to presenting the proof of Theorem 3.9.3, we isolate one important aspect in the proposition below.

Proposition 3.9.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix $0 < p \leq 1$. Then for each function u which is sub-averaging in Ω,*

$$u \in L^p(\Omega) \Longleftrightarrow u \in h^p(\Omega). \tag{3.9.31}$$

Proof. Let us deal with the direct implication. Since membership to $h^p(\Omega)$ is a local property, it suffices to check that every $x_o \in \partial\Omega$ has an open neighborhood $\mathcal{O} \subset \mathbb{R}^n$ such that $u \in h^p(\mathcal{O} \cap \Omega)$. To see this, identify x_o with the origin in $\mathbb{R}^n$ and consider a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\varphi(0) = 0$ and for which there exist $M, R > 0$ with

$$\Sigma_R := \{(x', \varphi(x')) : x' \in \mathbb{R}^{n-1},\ |x'| < R\} \subset \partial\Omega, \tag{3.9.32}$$

$$D_{R,M} := \{x + te_n : x \in \Sigma_R,\ 0 < t < 2M\} \subset \Omega.$$

As pointed out before, it suffices to check that $u \in h^p(D_{R,M})$. To this end, note that there exists a vertical, circular, truncated cone Γ with vertex at $0 \in \mathbb{R}^n$ such that the cone $\Gamma(x) := x + \Gamma$ is contained in $\bar{\Omega}$ whenever $x \in \overline{D_{R,M}}$. We fix $\psi \in C_c^\infty(B(0,1))$ such that $\int_{B(0,1)} \psi(x)\,dx = 1$ and set $\psi_t(x) := t^{-n}\psi(x/t)$. Furthermore, we shall assume that $\operatorname{supp}\psi \subset B(0,1) \cap (-\Gamma)$. Then, for each $x \in$

$D_{R,M}$ and $0 < t < \operatorname{dist}(x, \partial D_{R,M})$, we may write

$$|(\psi_t * u)(x)| = \left| \int_\Omega u(x-y)\psi_t(y)\, dy \right| \leq \|u\|_{L^\infty(\Gamma(x)\cap D_{R,M})} \int |\psi_t(y)|\, dy \leq \mathcal{N}u(x), \tag{3.9.33}$$

where $\mathcal{N}$ denotes the non-tangential maximal operator, i.e.

$$(\mathcal{N}u)(x) := \|u\|_{L^\infty(\Gamma(x))}, \qquad x \in \overline{D_{R,M}}. \tag{3.9.34}$$

Consequently,

$$u^+(x) \leq \mathcal{N}u(x) \quad \text{for every} \quad x \in D_{R,M}, \tag{3.9.35}$$

where, in the present context,

$$u^+(x) := \sup_{0<t<\operatorname{dist}(x,\partial D_{R,M})} |(\psi_t * u)(x)|, \qquad x \in D_{R,M}. \tag{3.9.36}$$

The next step is to show that

$$\|\mathcal{N}u\|_{L^p(D_{R,M})} \leq C\|u\|_{L^p(\Omega)}. \tag{3.9.37}$$

Here we adapt an argument from [**119**]. Begin by considering the Whitney decomposition of Ω, i.e. a countable family of balls $\{B_j\}_j$ which cover Ω, whose concentric doubles have finite overlap, and such that $\operatorname{diam} B_j \approx \operatorname{dist}(B_j, \partial\Omega)$ (cf. [**118**]). Then Fatou's lemma implies

$$\int_{D_{R,M}} \mathcal{N}(|u|^p)(x)\, dx \leq C \sum_j \int_{D_{R,M}} \mathcal{N}(\chi_j |u|^p)(x)\, dx, \tag{3.9.38}$$

where χ_j denotes the characteristic function of the ball B_j. Next, we set

$$\widehat{B}_j := \{x \in D_{R,M} : \Gamma(x) \cap B_j \neq \emptyset\} \tag{3.9.39}$$

and note that $\mathcal{N}(\chi_j|u|^p) = 0$ on $D_{R,M}\setminus\widehat{B}_j$ and that $\|\mathcal{N}(\chi_j|u|^p)\|_{L^\infty(\widehat{B}_j)} \leq \|u\|^p_{L^\infty(B_j)}$. Also, $|\widehat{B}_j| \leq C|B_j|$. Consequently,

$$\int_{D_{R,M}} \mathcal{N}(\chi_j|u|^p)(x)\,dx \leq C|B_j| \sup_{x\in B_j} |u(x)|^p \leq C \int_{2B_j} |u(x)|^p\,dx. \tag{3.9.40}$$

where the last inequality follows from the sub-averaging property of u. Recalling that no point of Ω is contained in more than a fixed number of the balls $2B_j$, (3.9.37) follows from (3.9.38) and (3.9.40).

Now, (3.9.35) and (3.9.37) give that $u^+ \in L^p(D_{R,M})$ so that $u \in h^p(D_{R,M})$, as desired. This proves the left-to-right implication in (3.9.31).

To see the converse implication, assume that $u \in h^p(\Omega)$ is sub-averaging. Then fix $\psi \in C^\infty_c(B(0,1))$ such that $\int_{B(0,1)} \psi(x)\,dx = 1$ and write

$$|u(x)| = \lim_{t\to 0} |(u * \psi_t)(x)| \leq u^+(x), \qquad \text{a.e.} \quad x \in \Omega, \tag{3.9.41}$$

by Lebesgue's Differentiation Theorem applied to the locally integrable function u. Since $u^+ \in L^p(\Omega)$, this forces $u \in L^p(\Omega)$, as wanted. □

We are now ready to discuss the

Proof of Theorem 3.9.3. It is enough to consider the case when Ω is a starlike Lipschitz domain. Indeed, every bounded Lipschitz domain can be covered by a finite collection of star-shaped Lipschitz domains and the spaces involved are local in nature (i.e., their elements behave naturally under restrictions). In this scenario, consider first the left-to right inclusion in (3.9.29), i.e.

$$\mathbb{H}^p_\alpha(\Omega; L) \hookrightarrow F^{p,2}_\alpha(\Omega) \cap \operatorname{Ker} L. \tag{3.9.42}$$

The plan is to check this whenever $\alpha \in \mathbb{Z}$ and $|\alpha|$ is large. The full range of indices for (3.9.42) will then be covered by interpolation (here, Theorem 3.9.1 and Theorem 3.9.2 are used). To begin with, we observe that

$$\operatorname{Ker} L \cap L^p(\Omega) \hookrightarrow h^p(\Omega), \qquad 0 < p < \infty. \tag{3.9.43}$$

This is, of course, trivial if $1 < p < \infty$ and it follows from Proposition 3.9.4 if $0 < p \leq 1$ (the fact that null-solutions of L are sub-averaging follows from standard interior estimates).

Consider first the case when $\alpha \in \mathbb{N}$ satisfies $\alpha > n(\frac{1}{p} - 1)_+$. Now, $u \in \mathbb{H}^p_\alpha(\Omega; L)$, entails u, $\nabla^\alpha u \in L^p(\Omega) \cap \operatorname{Ker} L$ so u, $\nabla^\alpha u \in h^p(\Omega)$ by (3.9.43). Thus, $u \in h^p_\alpha(\Omega) = F^{p,2}_\alpha(\Omega)$ by Theorem 3.6.2 (and its proof), which justifies the embedding (3.9.42) in the case we are considering.

Next, we prove (3.9.42) when $\alpha \in \mathbb{Z}$ is negative. For the sake of clarity, let us deal with the situation when $\alpha = -1$ and then indicate the alterations necessary to treat the general case. Thus, fix $u \in \mathbb{H}^p_{-1}(\Omega; L)$ so that $Lu = 0$ in Ω and u, $\delta u \in L^p(\Omega)$. Our goal is to show that $u \in F^{p,2}_{-1}(\Omega)$.

Retaining the notation introduced in the proof of Proposition 3.9.4, it suffices to prove a local version of this claim, i.e. that $u \in F^{p,2}_\alpha(D_{R,M})$. To this end, for each w defined in $D_{R,M}$ we set

$$v(x) := -\int_0^M u(x + te_n)\, dt, \qquad x \in D_{R,M/2}, \tag{3.9.44}$$

so that $Lv = 0$ in $D_{R,M/2}$ and

$$\partial_n v(x) = u(x) - u(x + Me_n), \qquad x \in D_{R,M/2}, \tag{3.9.45}$$

In particular, granted the current assumptions on u, we may conclude that $\delta\,\partial_n v \in L^p(D_{R,M/2})$. For the purpose of the calculation below, we introduce

$$v_{\rm rad}(x) := \sup\{|v(x+te_n)| : 0<t<M\}, \qquad x\in D_{R,M/2}, \tag{3.9.46}$$

and denote by "l.o.t." (lower order terms) expressions which can be dominated in terms of the supremum of $|u|$ on compact subsets of Ω. With $d\sigma$ standing for the surface measure on the graph of φ, we may then write

$$\begin{aligned}
\int_{D_{R,M/2}} |v(x)|^p\,dx &\leq C\int_{\partial\Omega\cap\partial D_{R,M/2}}\int_0^{M/2} |v(x',\varphi(x')+r)|^p\,d\sigma(x')dr \\
&\leq C\int_{\partial\Omega\cap\partial D_{R,M/2}}\int_0^{M/2}\left(\int_0^M |\partial_n v(x',\varphi(x')+r+t)|\,dt\right)^p d\sigma(x')dr + \text{ l.o.t.} \\
&\leq C\int_{\partial\Omega\cap\partial D_{R,M/2}}\int_0^{M/2}\left(\int_r^M |\partial_n v(x',\varphi(x')+s)|\,ds\right)^p d\sigma(x')dr + \text{ l.o.t.} \\
&\leq C\int_{\partial\Omega\cap\partial D_{R,M/2}}\int_0^{M/2}\left(\int_r^M |(\partial_n v)_{\rm rad}(x',\varphi(x')+s)|\,ds\right)^p d\sigma(x')dr + \text{ l.o.t.} \\
&\leq C\int_{\partial\Omega\cap\partial D_{R,M/2}}\int_0^{M/2} (r|(\partial_n v)_{\rm rad}(x',\varphi(x')+r)|)^p\,d\sigma(x')dr + \text{ l.o.t.} \\
&\leq C\int_{D_{R,M/2}} \delta(x)^p|(\partial_n v)_{\rm rad}(x)|^p\,dx + \text{ l.o.t.} \\
&\leq C\int_{D_{R,M/2}} \delta(x)^p|\partial_n v(x)|^p\,dx + \text{ l.o.t.} \ <+\infty
\end{aligned} \tag{3.9.47}$$

where the last inquality is a minor variation on the theme of Lemma 2.4.3 (considered here with $q=p$ and $s=1$). Hence, $v\in L^p(D_{R,M/2})$ and, thus, $v\in h^p(D_{R,M/2})$ by Proposition 3.9.4. Consequently, $\partial_n v\in h^p_{-1}(D_{R,M/2})$ so that $u\in h^p_{-1}(D_{R,M/2})$ as well. Finally, $u\in F^{p,2}_{-1}(D_{R,M/2})$, by Theorem 3.6.2, which is what we desired to show.

When $\alpha=-k$, $k=2,3,...$, we take $v:=\mathcal{I}^k u$ so that $\partial_n^k v = u+\text{l.o.t.}$ in $D_{R,M}$.

Instead of (3.9.47), by proceeding inductively we obtain this time

$$\int_{D_{R,M/2}} |v(x)|^p\,dx \leq C\int_{D_{R,M/2}} \delta(x)^{kp}|\partial_n^k v(x)|^p\,dx + \text{ l.o.t. } < +\infty, \tag{3.9.48}$$

which serves our purposes well. This finishes the proof of (3.9.42) when $\alpha \in \mathbb{R}$ and $0 < p < \infty$.

Next, consider the right-to-left inclusion in (3.9.29), i.e.

$$F^{p,2}_{\alpha}(\Omega) \cap \operatorname{Ker} L \hookrightarrow \mathbb{H}^p_{\alpha}(\Omega; L), \tag{3.9.49}$$

which we shall prove first under the assumption that $\alpha \in \mathbb{N}$ is sufficiently large, say $\alpha > n(1/p-1)_+$. In this case, $F^{p,2}_{\alpha}(\Omega) = h^p_{\alpha}(\Omega)$ and if $u \in h^p_{\alpha}(\Omega) \cap \operatorname{Ker} L$ then u, $\partial^{\alpha} u \in L^p(\Omega)$ by Proposition 3.9.4. Thus, $u \in \mathbb{H}^p_{\alpha}(\Omega; L)$ and (3.9.49) follows.

Finally, we prove (3.9.49) in the case when $\alpha \in \mathbb{Z}$ is negative. In this scenario, fix $\xi \in C^{\infty}_c(\mathbb{R}^n)$, $\xi \equiv 1$ in a neighborhood of $\bar{\Omega}$, and select $N \in \mathbb{N}$ sufficiently large so that $\alpha + N\mu > n(1/p-1)_+$. In order to continue, recall that Π_L is the operator defined as in (3.2.81) for $a(\xi) := L(-i\xi)^{-1}$, and that E_{Ω} is Rychkov's extension operator from Theorem 3.6.3. Then, given an arbitrary $u \in F^{p,2}_{\alpha}(\Omega) \cap \operatorname{Ker} L$, we remark that

$$v := [(\xi \Pi_L \xi)^N (E_{\Omega} u)]\Big|_{\Omega} \Longrightarrow v \in F^{p,2}_{\alpha+N\mu}(\Omega) \cap \operatorname{Ker} L^{N+1} \hookrightarrow \mathbb{H}^p_{\alpha}(\Omega; L^{N+1}) \tag{3.9.50}$$

where the inclusion is a consequence of our assumptions on N and of what we have proved already. Thus, $v \in \mathbb{H}^p_{\alpha}(\Omega; L^{N+1})$ which futher enatils $\nabla^{\alpha+N\mu} v \in L^p(\Omega)$. In concert with Lemma 2.4.6, this implies that $\delta^{-\alpha}\nabla^{N\mu} v \in L^p(\Omega)$. Since $L^N v = u$, this yields $\delta^{-\alpha} u \in L^p(\Omega)$. With this in hand, the conclusion that $u \in \mathbb{H}^p_{\alpha}(\Omega; L)$ readily follows and this finishes the proof of (3.9.49).

This completes the treatment of the Triebel-Lizorkin scale. In the case of Besov spaces we utilize what we have proved so far and real interpolation (cf. Theorem 3.9.1 and Theorem 3.9.2). □

Chapter 4

The Poisson problem for Laplacian in Besov and Triebel-Lizorkin spaces

4.1 Mapping properties of singular integral operators

4.1.1 Double layer potential, single layer potential and similar operators

We start with the result describing mapping properties on Besov spaces of integral operator modeled upon the harmonic double layer.

Theorem 4.1.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ (bounded or unbounded). Consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma_y, \qquad x \in \Omega, \tag{4.1.1}$$

satisfying the following conditions:

$$(1) \quad T1 = const, \tag{4.1.2}$$

$$(2) \quad |\nabla_x^k k(x,y)| \leq C|x-y|^{-(n+k-1)}, \quad k = 1, 2, ..., N, \tag{4.1.3}$$

for some positive integer N. Then, with $\delta := \operatorname{dist}(\cdot, \partial\Omega)$,

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Tf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Tf\|_{L^p(\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{4.1.4}$$

granted that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

Proof. Consider first the case $p = 1$, when we shall prove that

$$\|\delta^{k-1-s}|\nabla^k Tf|\,\|_{L^1(\Omega)} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)}. \tag{4.1.5}$$

To this end, recall from Proposition 3.4.1 that

$$\|f\|_{B_s^{1,1}(\partial\Omega)} \approx \|f\|_{L^1(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n-1+s}}\, d\sigma_x d\sigma_y. \tag{4.1.6}$$

The estimate we seek has local character. Thus, using a partition of unity, we may assume that the support of f is included in a coordinate patch where $\partial\Omega$ is represented by the graph of the Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Assuming that this is the case, we make a change of variables and set $\tilde{f}(x) := f(x, \phi(x))$, extended by zero outside of the support. In particular, $\tilde{f} \in B_s^{1,1}(\mathbb{R}^{n-1})$.

Thanks to (4.1.2), $\nabla^k T$ annihilates constants. In concert with the assumption (4.1.3) on the kernel, this implies that

$$\begin{aligned}\int_\Omega \delta^{k-1-s}(x)|\nabla^k Tf(x)|dx &\leq C\int_0^\infty t^{k-1-s}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|\tilde{f}(x')-\tilde{f}(y')|}{(|x'-y'|^2+t^2)^{\frac{n+k-1}{2}}}\, dx'dy' \\ &\leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\tilde{f}(x')-\tilde{f}(y')|\Big(\int_0^\infty \frac{t^{k-1-s}}{(|x'-y'|+t)^{n+k-1}}dt\Big)\, dx'dy'. \end{aligned} \tag{4.1.7}$$

Making the change of variables $r = \frac{t}{|x'-y'|}$, $r \in (0,\infty)$, $dt = |x'-y'|dr$, we can further bound the innermost integral above by

$$\int_0^\infty \frac{t^{k-1-s}}{(|x'-y'|+t)^{n+k-1}}\, dt = |x'-y'|^{-s+1-n}\int_0^\infty \frac{r^{k-1-s}}{(1+r)^{n+k-1}}\, dr \leq C|x'-y'|^{-s+1-n}$$

(4.1.8)

for some finite constant $C = C(k, s, n) > 0$. Thus,

$$\begin{aligned}\int_{\Omega} \delta^{k-1-s}(x)|\nabla^k Tf(x)|\,dx &\leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|\tilde{f}(x')-\tilde{f}(y')|}{|x'-y'|^{n+s-1}}\,dx'dy' \\ &\leq C\|\tilde{f}\|_{B_s^{1,1}(\mathbb{R}^{n-1})} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)} \qquad (4.1.9)\end{aligned}$$

as desired. This completes the proof of (4.1.4) in the case $p = 1$.

Next we turn our attention to the case $p = \infty$. The goal is to show that

$$\|\delta^{k-s}|\nabla^k Tf|\|_{L^\infty(\Omega)} \leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \qquad (4.1.10)$$

To this end, let x^* denote the point on $\partial\Omega$ such that $|x - x^*| = \delta(x)$. Then we may write

$$\nabla^k Tf(x) = \int_{\partial\Omega} \nabla_x^k k(x, y)(f(y) - f(x^*))\,d\sigma_y. \qquad (4.1.11)$$

Since $f \in B_s^{\infty,\infty}(\partial\Omega) = C^s(\partial\Omega)$ we have $|f(y) - f(x^*)| \leq \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}|y - x^*|^s$, $y \in \partial\Omega$. With this in mind and recalling the assumptions (4.1.3) on the kernel, we split the last integral into two parts, I_1, I_2, corresponding to $y \in S_{cr}(x^*)$ and $y \in \partial\Omega \setminus S_{cr}(x^*)$, respectively, where $r := |x - x^*|$ and $c = c(\partial\Omega) > 0$ is a suitable constant. We have:

$$\begin{aligned}|I_1| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}\int_{S_{cr}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\,d\sigma_y \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}\int_{S_{cr}(x^*)} |x-y|^{s-n+1-k}\,d\sigma_y \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}\int_{S_{cr}(x^*)} r^{s-n+1-k}\,d\sigma_y \leq C\,r^{s-k}\,\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \qquad (4.1.12)\end{aligned}$$

On the other hand,

$$\begin{aligned}
|I_2| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{cr}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\, d\sigma_y \\
&\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{cr}(x^*)} |x^*-y|^{s-n+1-k}\, d\sigma_y \\
&\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{cr}^{\infty} \rho^{s-1-k}\, d\rho \leq C\, r^{s-k}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \qquad (4.1.13)
\end{aligned}$$

These inequalities complete the argument for the case $p=\infty$.

Now the estimate (4.1.4) in the case when $1 \leq p \leq \infty$ follows from what we have proved so far and Theorem 3.1.11. The details are as follows. Consider the family of operators

$$L_z f := \delta^{k-1+z-[(1-z)s_0+zs_1]} |\nabla^k T f|, \qquad (4.1.14)$$

so that

$$\Re ez = 0 \Rightarrow |L_0 f| = \delta^{k-1-s_0} |\nabla^k T f|,$$

$$\Re ez = 1 \Rightarrow |L_1 f| = \delta^{k-s_1} |\nabla^k T f|.$$

Our results for $p=1$ and $p=\infty$ lead to the conclusion that the operators

$$L_0 : B_{s_0}^{1,1}(\partial\Omega) \to L^1(\Omega),$$

$$L_1 : B_{s_1}^{\infty,\infty}(\partial\Omega) \to L^\infty(\Omega),$$

are well-defined and bounded for any $s_0, s_1 \in (0,1)$. Pick $0 < s_0 < s_1 < 1$, otherwise arbitrary, so that (3.4.15) applies. In this scenario, Theorem 3.1.11 can be used and we may conclude that

$$\delta^{k-\frac{1}{p}-s} |\nabla^k T f| : B_s^{p,p}(\partial\Omega) \longrightarrow L^p(\Omega) \qquad (4.1.15)$$

is also well defined and bounded for any $s \in (0,1)$ and any $p \in [1,\infty]$. This takes care of estimating the top term in the left-hand side of (4.1.4). The remaining terms can be handle in a simpler, more straightforward fashion so we omit the argument.

There remains to treat the case when $\frac{n-1}{n} < p < 1$, in which scenario (by once again focusing only the top term in the left-hand side of (4.1.4)) it is sufficient to prove the existence of some finite constant $C > 0$ such that

$$\int_{\Omega} \Big(\delta(x)^{k-\frac{1}{p}-s} |\nabla^k T a(x)| \Big)^p \, dx \leq C \tag{4.1.16}$$

for every $B^{p,p}_s(\partial\Omega)$-atom a.

Assume that a is as in (3.4.21)-(3.4.23), where $S = S_r = B(x_o, r) \cap \partial\Omega$, and re-scale $\tilde{a} = r^{\tau} a$, with τ to be specified later. Then, by Hölder's inequality,

$$\begin{aligned}
&\int_{B(x_o,2r)\cap\Omega} \Big(\delta(x)^{k-\frac{1}{p}-s} |\nabla^k T a(x)| \Big)^p \, dx \\
&= r^{-\tau p} \int_{B(x_o,2r)\cap\Omega} \delta(x)^{kp-1-sp} |\nabla^k T\tilde{a}(x)|^p \, dx \\
&\leq r^{-\tau p} \left(\int_{B(x_o,2r)\cap\Omega} \delta(x)^{k-s-\frac{1}{p}+\frac{z}{p}} |\nabla^k T\tilde{a}(x)| \, dx \right)^p \cdot \left(\int_{B(x_o,2r)\cap\Omega} \delta(x)^{\frac{-z}{1-p}} \, dx \right)^{1-p} \\
&\leq C \, r^{-\tau p + \frac{-z}{1-p} + n} \left(\int_{\Omega} \delta(x)^{k-s-\frac{1}{p}+\frac{z}{p}} |\nabla^k T\tilde{a}(x)| \, dx \right)^p \\
&\leq C \, r^{-\tau p + \frac{-z}{1-p} + n} \|\tilde{a}\|_{B^{1,1}_{-1+s+\frac{1-z}{p}}(\partial\Omega)},
\end{aligned} \tag{4.1.17}$$

where we assume that $z < 1-p$ (needed in the second inequality), and that $-1+s+\frac{1-z}{p} \in (0,1)$ (needed in the last inequality). The restrictions just imposed on z are satisfied whenever z belongs to the nondegenerate interval $(1-p(2-s), 1-p)$.

Going further, the key is to select τ so that $\tilde{a} = r^{\tau p} a$ is $B^{1,1}_{-1+s+\frac{1-z}{p}}(\partial\Omega)$-atom.

An appropriate choice is $\tau = \frac{n-z}{p} - n$, in which case

$$\int_{B(x_o,2r)\cap\Omega} \Big(\delta(x)^{k-\frac{1}{p}-s}|\nabla^k Ta|\Big)^p\,dx \leq C\,r^{-\tau p}r^{(1-p)\left(\frac{-z}{1-p}+n\right)} = C < \infty. \tag{4.1.18}$$

Next, we turn our attention to the contribution away from the support of the atom. For notational simplicity we will further assume that $x_o = 0$. Then

$$|\nabla^k Ta(x)| \leq C\int_{S_r} \frac{|a(y)|}{|x-y|^{n+k-1}}\,d\sigma_y \leq C\frac{r^{s+(n-1)(1-\frac{1}{p})}}{|x|^{n+k-1}} \quad \text{if } x\in\Omega\setminus B(0,2r). \tag{4.1.19}$$

At this point we pull-back everything to Euclidean model to obtain

$$\begin{aligned}
&\int_{\Omega\setminus B(0,2r)} \delta^{(k-s)p-1}|\nabla^k Ta(x)|^p\,dx \\
&\qquad\leq C\,r^{sp+(n-1)(p-1)}\int_{|x'|^2+t^2\geq 4r^2}\frac{t^{(k-s)p-1}}{(|x'|+t)^{(n+k-1)p}}\,dx'dt \\
&\qquad\leq C\int_{|x'|^2+t^2\geq 1}\frac{t^{(k-s)p-1}}{(|x'|+t)^{(n+k-1)p}}\,dx'dt,
\end{aligned} \tag{4.1.20}$$

after rescaling. Now the desired conclusion follows the fact that the last integral above is finite, an issue addressed below.

First, consider the case when integration is performed over the set described by $|x'|\leq t$ and $|x'|^2+t^2\geq 1$. Denoting the corresponding integral by I_1 and making the change of variables $x' = ty'$ with $t\in[\frac{1}{2},\infty)$ and $y'\in\mathbb{R}^{n-1}$, we compute

$$I_1 = \int_{1/2}^{\infty} t^{-sp+(n-1)(1-p)-1}\,dt\,\cdot\int_{\mathbb{R}^{n-1}}\frac{1}{(|y'|+1)^{(n+k-1)p}}\,dy' \tag{4.1.21}$$

which is finite since both integrals above are convergent under the current assumptions on p and s. As for the contribution from $t\leq|x'|$, $|x'|^2+t^2\geq 1$, which we denote by I_2, we remark that these conditions force $|x'|\geq 1/2$. With $t := \rho|x'|$ we

obtain

$$I_2 = \int_0^\infty \frac{\rho^{(k-s)p-1}}{(\rho+1)^{(n+k-1)p}}\,d\rho \cdot \int_{|x'|\geq 1/2} |x'|^{-sp-np+p}\,dx', \tag{4.1.22}$$

which is finite as well. This completes the proof of (4.1.4) for the full range of indices. □

The next result gives an analogue of Theorem 4.1.1 for single layer-like integral operators.

Theorem 4.1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator*

$$Rf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma_y, \qquad x \in \Omega, \tag{4.1.23}$$

whose kernel satisfies the conditions

$$|\nabla_x^k \nabla_y^j k(x,y)| \leq C|x-y|^{-(n-2+k+j)}, \quad j = 0, 1, \tag{4.1.24}$$

for $k = 1, 2, ..., N$, where N is some positive integer. Then

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Rf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Rf\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega)}, \quad k = 1, 2, ..., N, \tag{4.1.25}$$

granted that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

In the proof of this theorem we shall make use of an auxiliary result, addressing the end-point cases of (4.1.25).

Lemma 4.1.3. *(cf. Lemmas 7.2-7.3 in* **[97]***) Retain the same assumptions as in Theorem 4.1.2. Then, for each $k = 1, ..., N$, the operator (4.1.23) satisfies the*

estimates

$$\|\delta^{k-s-1}|\nabla^k Rf|\,\|_{L^1(\Omega)}+\sum_{j=0}^{k-1}\|\nabla^j Rf\|_{L^1(\Omega)}\leq C\|f\|_{(B^{\infty,\infty}_{1-s}(\partial\Omega))^*} \tag{4.1.26}$$

$$\|\delta^{k-s}|\nabla^k Rf|\,\|_{L^\infty(\Omega)}+\sum_{j=0}^{k-1}\|\nabla^j Rf\|_{L^\infty(\Omega)}\leq C\|f\|_{B^{\infty,\infty}_{s-1}(\partial\Omega)}. \tag{4.1.27}$$

Proof of Theorem 4.1.2. Since $B^{1,1}_{s-1}(\partial\Omega)\hookrightarrow\Big(B^{1,1}_{s-1}(\partial\Omega)\Big)^{**}\hookrightarrow\Big(B^{\infty,\infty}_{1-s}(\partial\Omega)\Big)^{*}$, we can invoke Theorem 3.1.11 much as in the proof of Theorem 4.1.1 in order to conclude conclude that

$$\delta^{k-\frac{1}{p}-s}|\nabla^k Tf|:B^{p,p}_{s-1}(\partial\Omega)\longrightarrow L^p(\Omega) \tag{4.1.28}$$

is a linear and bounded operator for $1\leq p\leq\infty$ and $0<s<1$.

Thus, it remains to consider the case when $(n-1)/n<p<1$ and $(n-1)(1/p-1)<s<1$. Similarly to Theorem 4.1.1, our goal is to establish an estimate of the form

$$\int_\Omega\Big(\delta(x)^{k-\frac{1}{p}-s}|\nabla^k Ra(x)|\Big)^p\,dx\leq C \tag{4.1.29}$$

where a is a $B^{p,p}_{s-1}(\partial\Omega)$-atom and constant C does not depend on a.

To see this, let $a=a_S$ be as in (3.4.36)-(3.4.38), where $S=S_r=B(x_o,r)\cap\partial\Omega$, $x_o\in\partial\Omega$, $r>0$, is a surface ball. In a neighborhood of the support of the atom, with $\tilde{a}:=r^\tau a$, $c>0$ and τ,z to be specified later, we write

$$\begin{aligned}
&\int_{B(x_o,cr)\cap\Omega}\Big(\delta(x)^{k-\frac{1}{p}-s}|\nabla^k Ra(x)|\Big)^p\,dx\\
&\qquad\leq C\,r^{-\tau p}\left(\int_{B(x_o,cr)\cap\Omega}\delta(x)^{k-s-\frac{1}{p}+\frac{z}{p}}|\nabla^k R\tilde{a}(x)|\,dx\right)^p\,\left(\int_{B(x_o,cr)\cap\Omega}\delta(x)^{\frac{-z}{1-p}}\,dx\right)^{1-p}\\
&\qquad\leq C\,\|\tilde{a}\|_{B^{1,1}_{s-2+\frac{1-z}{p}}(\partial\Omega)}\leq C,
\end{aligned} \tag{4.1.30}$$

by Hölder's inequality, and (4.1.28) with $p = 1$ in the last step. For this program to work (i.e, for the last constant above to be finite) it suffices to take $\tau := \frac{n-z}{p} - n$ and pick $z \in (1 - p(2 - s), 1 - p)$.

There remains to estimate the contribution away from the support of the atom, a task we shall now consider, starting with the case when $r \geq \eta > 0$ (with η as in the paragraph containing the display (3.4.36)-(3.4.38); in particular, such atoms do not necessarily have a vanishing integral). Since Ω is a bounded domain there exists a constant $\kappa = \kappa(\Omega, \eta) > 0$ such that $\Omega \subset B(x_o, \kappa r)$ whenever $x_o \in \partial\Omega$ and $r \geq \eta$. Then the estimate (4.1.30) with κ in place of c will do.

Turning to the case when $r < \eta$ and assuming that $x_o = 0$ we note, based on the vanishing moment condition of the atom, that for each $x \in \Omega \setminus B(0, cr)$,

$$\begin{aligned}
|\nabla^k Ra(x)| &= \Big| \int_{S_r} [\nabla_x^k k(x,y) - \nabla_x^k k(x,0)] a(y)\, d\sigma_y \Big| \\
&= \Big| \int_{S_r} \int_0^1 \frac{d}{d\theta} \Big[\nabla_x^k k(x, (1-\theta)y) \Big] a(y)\, d\theta\, d\sigma_y \Big| \\
&\leq C\|a\|_{L^\infty(\partial\Omega)} \Big| \int_{S_r} |y| \max_{\theta\in(0,1)} |\nabla_x^k \nabla_y k(x, (1-\theta)y)|\, d\sigma_y \Big| \\
&\leq C \frac{r^{s+(n-1)(1-\frac{1}{p})}}{|x|^{n+k-1}}, \qquad (4.1.31)
\end{aligned}$$

where, in the last step, we have used $|x - (1-\theta)y| \approx |x|$ and $|y| \leq r$ in the current context. With this in hand and proceeding much as in the end-game of the proof of Theorem 4.1.1 we can show that

$$\int_{\Omega\setminus B(x_o,cr)} \Big(\delta(x)^{k-\frac{1}{p}-s} |\nabla^k Ra(x)| \Big)^p \, dx \leq C < \infty, \qquad (4.1.32)$$

with C independent of the particular atom. The lower order terms in (4.1.25) are handled in a simpler, more direct fashion and this completes the proof of the

theorem. □

Corollary 4.1.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $\frac{n-1}{n} < p \leq \infty$, and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$. Then*

$$\mathcal{D} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{4.1.33}$$

$$\mathcal{S} : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega) \tag{4.1.34}$$

are well-defined, bounded operators. Furthermore,

$$\mathcal{D} : B^{p,p}_s(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{4.1.35}$$

$$\mathcal{S} : B^{p,p}_{s-1}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega) \tag{4.1.36}$$

are also well-defined and bounded provided $p \neq \infty$ and one of the following two conditions holds:

$$(i) \quad (n-1)(\tfrac{1}{p}-1)_+ < s \leq \tfrac{n-1}{np} \quad \text{and} \quad \left(\tfrac{s}{2(n-1)} + \tfrac{1}{2p} + \tfrac{1}{2}\right)^{-1} < q \leq \infty, \tag{4.1.37}$$

$$(ii) \quad \tfrac{n-1}{np} \leq s < 1 \quad \text{and} \quad \left(\tfrac{1}{p}\left(1 - \tfrac{n-1}{2n}\right) + \tfrac{1}{2}\right)^{-1} < q \leq \infty. \tag{4.1.38}$$

Proof. The case $p = q$ for (4.1.33)–(4.1.34), (4.1.35)–(4.1.36) and the case $q = 2$ for (4.1.35)–(4.1.36) follow immediately from Theorems 4.1.1, 4.1.2 and 3.9.3. The full range of indices is then covered by interpolation and embedding results. □

4.1.2 The normal derivative of the Newtonian potential

Theorem 4.1.5. *Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and supposed p, q, s are fixed such that $\frac{n-1}{n} < p \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q \leq \infty$. Then*

$$\partial_\nu \Pi : B^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega), \tag{4.1.39}$$

$$\partial_\nu \Pi : F^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \quad \text{if } p \neq \infty, \tag{4.1.40}$$

are well-defined, linear and bounded operators.

Proof. We start with implication (4.1.40) for $\frac{n-1}{n} < p \leq 1$, $(n-1)(1/p-1) < s < 1$ and $p \leq q \leq \infty$. By Proposition 3.4.9 and Theorem 3.7.4 it is enough to show that $\partial_\nu \Pi$ maps rough $F^{p,q}_{s+1/p-2,0}(\Omega)$-atoms to $B^{p,p}_{s-1}(\partial\Omega)$-molecules.

Note that current restrictions on indices imply that rough $F^{p,q}_{s+1/p-2,0}(\Omega)$-atoms satisfy (3.7.63)–(3.7.65) with $L \geq 0$. Consider first such a rough atom A supported in a Whitney cube $Q \subset \Omega$, with center $x_Q \in Q$ and pick $x_S \in \partial\Omega$ such that $|x_Q - x_S| = \operatorname{dist}(x_Q, \partial\Omega)$. Then set $m := \partial_\nu \Pi(A)$ on $\partial\Omega$ which, so we claim, is a molecule for $B^{p,p}_{s-1}(\partial\Omega)$ concentrated about the surface ball $S := B(x_S, l(Q)) \cap \partial\Omega$.

The claim will be justified by checking (3.4.60)-(3.4.61). Take the vanishing moment condition, required when $l(Q)$ is small. Then A has one vanishing moment and

$$\int_{\partial\Omega} m \, d\sigma = \int_{\partial\Omega} \partial_\nu \Pi A \, d\sigma = \int_\Omega \Delta \Pi A \, dx = \int_\Omega A \, dx = \int_{\mathbb{R}^n} A \, dx = 0, \quad (4.1.41)$$

by Green's formula and the support condition on A.

Turning to size estimates, we observe that m can be expressed in the form (recall that x_Q is the center of Q),

$$m(x) = \int_Q \Big(\frac{\langle x-y, \nu(x)\rangle}{|x-y|^n} - \frac{\langle x - x_Q, \nu(x)\rangle}{|x-x_Q|^n} \Big) \xi(y) A(y) \, dy, \quad (4.1.42)$$

for some $\xi \in C^\infty_c(\Omega)$ such that $\xi \equiv 1$ on Q, ξ vanishes outside some small neighborhood cQ, $c = c(\Omega) > 1$, $0 \leq \xi \leq 1$, and $|\nabla \xi| \leq C l(Q)^{-1}$.

For the range of indices we are currently working with,

$$F^{p_1,q}_{s+1/p-2}(\mathbb{R}^n) \hookrightarrow L^{p_2}_{-1}(\mathbb{R}^n), \qquad \text{if} \quad s + \tfrac{1}{p} - 2 - \tfrac{n}{p_1} = -1 - \tfrac{n}{p_2}, \quad (4.1.43)$$

where $p_1 > 1$ is the index appearing in (3.7.64), chosen sufficiently close to 1, and $p_2 > p_1$. Also, $(L^{p_2}_{-1}(\mathbb{R}^n))^* = L^{p_2'}_1(\mathbb{R}^n)$, so that (4.1.42) together with (4.1.43) and (3.7.64) imply

$$|m(x)| \leq C\|F_x\|_{L^{p_2'}_1(\mathbb{R}^n)}\|A\|_{L^{p_2}_{-1}(\mathbb{R}^n)} \leq C|Q|^{\frac{1}{p_1}-\frac{1}{p}}\|F_x\|_{L^{p_2'}_1(\mathbb{R}^n)}, \tag{4.1.44}$$

where

$$F_x(y) := \Big(\frac{\langle x-y,\nu(x)\rangle}{|x-y|^n} - \frac{\langle x-x_Q,\nu(x)\rangle}{|x-x_Q|^n}\Big)\xi(y), \quad y \in \mathbb{R}^n. \tag{4.1.45}$$

We can see that

$$|\nabla F_x(y)| \leq C\,\frac{1}{|x-y|^n}\,|\xi(y)| + C\Big|\frac{\langle x-y,\nu(x)\rangle}{|x-y|^n} - \frac{\langle x-x_Q,\nu(x)\rangle}{|x-x_Q|^n}\Big|\,|\nabla\xi(y)| =: I+II. \tag{4.1.46}$$

By the Mean Value Theorem,

$$II \leq C|y-x_Q| \sup_{z\in[y,x_Q]}\left|\nabla_z\frac{\langle x-z,\nu(x)\rangle}{|x-z|^n}\right|\,|\nabla\xi(y)| \leq Cl(Q)\sup_{z\in[y,x_Q]}\frac{1}{|x-z|^n}\,|\nabla\xi(y)|, \tag{4.1.47}$$

so that

$$II \leq C\sup_{z\in[y,x_Q]}\frac{1}{|x-z|^n}, \tag{4.1.48}$$

since $|\nabla\xi| \leq \frac{C}{l(Q)}$. Using the property that Q is a Whitney cube for Ω and keeping in mind that $y \in cQ$, $x \in \partial\Omega$, $z \in [y, x_Q]$, some elementary geometry leads to the conclusion that $|x-x_Q| \leq C|x-z|$. Consequently,

$$II \leq Cl(Q)^{-n}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}. \tag{4.1.49}$$

The same reasoning shows that a similar estimate holds for I so that, altogether,

$$\|\nabla F_x\|_{L^{p_2'}(\mathbb{R}^n)} \leq Cl(Q)^{-\frac{n}{p_2}} \left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}. \tag{4.1.50}$$

Similarly,

$$\|F_x\|_{L^{p_2'}(\mathbb{R}^n)} \leq Cl(Q)^{1-\frac{n}{p_2}} \left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n} \leq Cl(Q)^{-\frac{n}{p_2}} \left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}, \tag{4.1.51}$$

where the last inequality rests on the observation that $l(Q)$ is bounded by the diameter of the domain Ω. Then by (4.1.44), (4.1.50) and (4.1.51)

$$|m(x)| \leq Cl(Q)^{s-1-\frac{n-1}{p}} \left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}. \tag{4.1.52}$$

Now, by definition, $|x_Q-x_S| = \operatorname{dist}(x_Q,\partial\Omega)$, so that $|x-x_S| \leq |x-x_Q|+|x_Q-x_S| \leq 2\,|x-x_Q|$ for every $x\in\partial\Omega$. If we now set $r := l(Q)$, then

$$1+\frac{|x-x_Q|}{r} \geq 1+\frac{1}{2}\,\frac{|x-x_S|}{r} \geq \frac{1}{2}\left(1+\frac{|x-x_S|}{r}\right) \tag{4.1.53}$$

which entails

$$|m(x)| \leq C\,r^{s-1-\frac{n-1}{p}} \left(1+\frac{|x-x_S|}{r}\right)^{-n}. \tag{4.1.54}$$

This proves (3.4.60) with $M := n+s-1 > \frac{n-1}{p}$ and justifies the claim that m is a molecule for $B^{p,p}_{s-1}(\partial\Omega)$ concentrated about the surface ball $S = S_r(x_S)$. At this stage, Proposition 3.4.9 applies and yields that, for $\frac{n-1}{n} < p \leq 1$ and $(n-1)(1/p-1) < s < 1$, the operator (4.1.40) is well-defined and bounded, first for $p \leq q \leq \infty$, and then for the complementary range, $0 < q \leq p$, by embeddings.

To further expand this range, we shall rely on the observation that the normal derivative of Newtonian potential can be viewed as the adjoint of the harmonic

double layer. Then, Corollary 4.1.4, the duality results in Theorem 3.8.3 and interpolation with what we have just proved allows us to cover the more range of indices described in the statement of the theorem.

Finally, the claim made about the operator (4.1.39) is a consequence of the boundedness of (4.1.40), duality reasoning described in the paragraph above (in particular, contributing to the case $p=\infty$) and interpolation. □

Below we present an alternative argument handling (4.1.39) for the range $\frac{n-1}{n}<p=q\leq 1$ and $(n-1)(1/p-1)_+<s<1$. This time we can work with the decomposition of $B^{p,p}_{s+1/p-2,0}(\Omega)$ spaces to smooth atoms addressed in Corollary 3.7.3. This time, the goal is to show that $\partial_\nu\Pi$ maps smooth $B^{p,p}_{s+1/p-2,0}(\Omega)$-atoms to $B^{p,p}_{s-1}(\partial\Omega)$-molecules.

Let a be a smooth $B^{p,p}_{s+1/p-2,0}(\Omega)$ – atom with the support in Whitney cube Q centered at $x_Q\in\Omega$ and consider $m:=\partial_\nu\Pi a$.

Assume first that $l(Q)$ is small. Following (4.1.41) one can verify the vanishing moment condition for m. As for the size estimates, (4.1.42) allows to write for every $x\in\partial\Omega$

$$\begin{aligned}|m(x)|&\leq\int_Q|y-x_Q|\sup_{z\in[y,x_Q]}\left|\nabla_z\frac{\langle x-z,\nu(x)\rangle}{|x-z|^n}\right||a(y)|\,dy\\&\leq Cl(Q)\int_Q\sup_{z\in[y,x_Q]}\frac{1}{|x-z|^n}|a(y)|\,dy\leq Cl(Q)^{n+s-1-\frac{n}{p}+\frac{1}{p}}|x-x_Q|^{-n}\\&\leq Cl(Q)^{s-1-\frac{n}{p}+\frac{1}{p}}\left(\frac{|x-x_Q|}{l(Q)}\right)^{-n}\leq Cl(Q)^{s-1-\frac{n}{p}+\frac{1}{p}}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n},\end{aligned}\tag{4.1.55}$$

where we used the Mean Value Theorem in the first inequality, and condition $\operatorname{dist}(Q,\partial\Omega)\approx l(Q)$ in the third and the last one. Choosing $x_S\in\partial\Omega$ such that

$\operatorname{dist}(x_Q, \partial\Omega) = |x_Q - x_S|$ and following (4.1.52)–(4.1.54) we finish the proof for $l(Q)$ small.

If, on the other side, $l(Q) > \eta_0$ for some $\eta_0 = \eta_0(\partial\Omega)$ fixed, then (3.4.60) is equivalent to the boundedness of $\|m\|_{L^\infty(\partial\Omega)}$ by the constant independent of molecule. However,

$$|m(x)| \leq C \int_Q \frac{1}{|x-y|^{n-1}} |a(y)|\, dy \leq C l(Q)^{s-1-\frac{n}{p}+\frac{1}{p}} \leq C, \tag{4.1.56}$$

as desired.

Corollary 4.1.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then*

$$\mathcal{D} : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \quad 0 < q \leq \infty, \tag{4.1.57}$$

$$\mathcal{D} : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \text{ if } p \neq \infty \text{ and } \Big(\min\big\{\tfrac{1}{pn}+1, \tfrac{s}{n-1}+1\big\}\Big)^{-1} < q \leq \infty \tag{4.1.58}$$

are well-defined, bounded, linear operators.

Proof. The claim about (4.1.57) was already proved in Corollary 4.1.4. Going further, we rely on the fact that the double layer potential is adjoint of the normal derivative of Newtonian potential, duality results and Theorem 4.1.5 to verify that the operator (4.1.58) is bounded for $0 < s < 1$, $1 < p < \infty$, $1 \leq q \leq \infty$. Then interpolation with the case $p = q \leq 1$ of (4.1.57) yields the desired conclusion about (4.1.58) for the full range of indices. □

4.2 Tangential and normal traces

4.2.1 Traces of Besov and Triebel-Lizorkin spaces

Proof of Theorem 1.1.3. To get started, we shall prove a related result to the effect that the restriction to the boundary map extends as a bounded operator

$$\widetilde{\mathrm{Tr}} : B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n) \longrightarrow B^{p,q}_{s}(\partial\Omega). \tag{4.2.1}$$

By a routine density argument, it suffices to show that there exists a finite constant $C = C(\partial\Omega, p, q, s) > 0$ such that

$$\|\widetilde{\mathrm{Tr}} f\|_{B^{p,q}_{s}(\partial\Omega)} \leq C\|f\|_{B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)}, \qquad f \in B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n). \tag{4.2.2}$$

To this end, using a smooth partition of unity, there is no loss of generality in assuming that Ω is a graph Lipschitz domain. Consider first the diagonal case $p = q \leq 1$. Fix $f \in B^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)$ which, according to Theorem 3.3.1, admits the representation $f = \sum_{Q\in\mathcal{Q}_n : l(Q)\leq 1} s_Q a_Q$, where the sequence of coefficients $\{s_Q\}_Q$ belongs to ℓ^p and the atoms a_Q's satisfy

$$\mathrm{supp}\,(a_Q) \subseteq 3Q, \qquad \|\nabla a_Q\|_{L^\infty(\partial\Omega)} \leq l(Q)^{s+\frac{1}{p}-\frac{n}{p}-1}. \tag{4.2.3}$$

Furthermore, it can be assumed that this decomposition is nearly optimal in the sense that

$$\|f\|_{B^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)} \approx \Big(\sum_{Q\in\mathcal{Q}_n : l(Q)\leq 1} |s_Q|^p\Big)^{1/p}. \tag{4.2.4}$$

Then we define

$$\widetilde{\mathrm{Tr}} f := \sum_{\substack{Q\in\mathcal{Q}_n,\, l(Q)\leq 1\\ Q\cap\partial\Omega\neq\emptyset}} s_Q\Big(a_Q\Big|_{\partial\Omega}\Big) \tag{4.2.5}$$

and observe that, for each $Q \in \mathcal{Q}_n$, $l(Q) \leq 1$, $Q \cap \partial\Omega \neq \emptyset$, the function $a_Q|_{\partial\Omega}$ satisfies the properties (3.4.21)-(3.4.22). Consequently, by virtue of (3.4.29) and (4.2.4),

$$\begin{aligned} \|\widetilde{\mathrm{Tr}} f\|_{B^{p,p}_s(\partial\Omega)} &\leq C\Big(\sum_{\substack{Q\in\mathcal{Q}_n,\, l(Q)\leq 1\\ Q\cap\partial\Omega\neq\emptyset}} |s_Q|^p\Big)^{1/p} \\ &\leq C\Big(\sum_{Q\in\mathcal{Q}_n:\, l(Q)\leq 1} |s_Q|^p\Big)^{1/p} \approx \|f\|_{B^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)}, \end{aligned} \tag{4.2.6}$$

proving (4.2.2).

To show that the trace operator we have just defined does not depend on the particular atomic decomposition of the function f it suffices to show that

$$f_j \in B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n),\ f_j \to 0 \text{ in } B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n) \Longrightarrow \widetilde{\mathrm{Tr}} f_j \to 0 \text{ in } (\mathrm{Lip}(\partial\Omega))^*. \tag{4.2.7}$$

To see this, we note that under the current assumptions on the indices involved, there exist $1 < p^* \leq \infty$ and $s^* \in (0,1)$ such that $B^{p,p}_{s+1/p}(\mathbb{R}^n) \hookrightarrow B^{p^*,p^*}_{s^*+1/p^*}(\mathbb{R}^n)$. It follows that $f_j|_\Omega \to 0$ in $B^{p^*,p^*}_{s^*+1/p^*}(\Omega)$ and, further, that $\mathrm{Tr}_{JW}(f_j|_\Omega) \to 0$ in $B^{p^*,p^*}_{s^*}(\partial\Omega)$ as $j \to \infty$, where Tr_{JW} stands here for the trace operator constructed by A. Jonsson and H. Wallin in [**71**]. Since $\widetilde{\mathrm{Tr}}$ agrees with the composition $\mathrm{Tr}_{JW} \circ \mathcal{R}_\Omega$ on $B^{p^*,p^*}_{s^*+1/p^*}(\mathbb{R}^n)$ (as it is clear by considering their action on the partial sums associated with the atomic series representation of functions in $B^{p^*,p^*}_{s^*+1/p^*}(\mathbb{R}^n)$), it ultimately follows that $\widetilde{\mathrm{Tr}} f_j \to 0$ in $B^{p^*,p^*}_{s^*}(\partial\Omega)$ and, hence, in $(\mathrm{Lip}(\partial\Omega))^*$. This justifies (4.2.7) and proves that the operator (4.2.5) is well-defined. This proves the claim made about the operator (4.2.1) in the case when $0 < p = q \leq 1$ and $(n-1)(1/p-1) < s < 1$.

Going further, when $p = q = \infty$ and $0 < s < 1$, (3.6.3) gives the identifications

$B_s^{\infty,\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n)$, $B_s^{\infty,\infty}(\partial\Omega) = C^s(\partial\Omega)$, in which case the boundedness of the operator (4.2.1) is evident. Now the boundedness of the operator (4.2.1) in the general case, i.e. when $0 < p \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$, $0 < q \leq \infty$, follows from what we have proved so far and interpolation: by complex method yielding (4.2.1) for $0 < p = q \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$ and then by real method for the full range of indices.

Having dealt with the operator (4.2.1), we now define

$$\mathrm{Tr} := \widetilde{\mathrm{Tr}} \circ E_\Omega : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \tag{4.2.8}$$

where $E_\Omega : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$ is the universal extension operator introduced in (3.6.32). Clearly, the trace operator just defined is linear and bounded. Furthermore, it coincides with the ordinary restriction to the boundary for functions in $C^\infty(\overline{\Omega})$.

We now claim that its definition is independent of the particular extension operator E_Ω used in (4.2.8). By linearity, proving this amounts to showing that $\widetilde{\mathrm{Tr}}$ maps the space $B^{p,q}_{s+\frac{1}{p},0}(\Omega)$ to zero. This, in turn, is a consequence of the definition (4.2.5) in concert with the fact, proved in Theorem 3.7.1, that functions in $B^{p,q}_{s+\frac{1}{p},0}(\Omega)$ admit an atomic decomposition with atoms supported strictly inside Ω. This finishes the proof of the fact that the trace operator is well-defined and bounde on the Besov scale.

Dealing with Triebel-Lizorkin spaces turns out to be more delicate as this case brings into focus the peculiarity that the image of the trace of Triebel-Lizorkin spaces does not depend on index q. The proof presented below builds on an idea

first used in [**49**] where the authors handled the case of the upper-half space. To get started, we introduced the operator

$$\widetilde{\mathrm{Tr}} : F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n) \longrightarrow B^{p,p}_{s}(\partial\Omega), \tag{4.2.9}$$

defined by (4.2.5) whenever $f \in F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$ has the atomic decomposition $f = \sum_{Q\in\mathcal{Q}_n:\, l(Q)\leq 1} s_Q a_Q$ where the atoms a_Q's satisfy (3.3.1)-(3.3.3) and

$$\|f\|_{F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)} \approx \left\| \Big(\sum_{\substack{Q\in\mathcal{Q}_n,\, l(Q)\leq 1 \\ Q\cap\partial\Omega\neq\emptyset}} (|Q|^{-1/p}|s_Q|\chi_Q)^q \Big)^{1/q} \right\|_{L^p(\mathbb{R}^n)}; \tag{4.2.10}$$

cf. Theorem 3.3.1. To prove that the above definition does not depend on the particular atomic decomposition of the function to which it is applied, we proceed as before after noting that, granted the current assumptions on the indices, there exist $1 < p^* < \infty$ and $0 < s^* < 1$ such that $F^{p,q}_{s+1/p}(\Omega) \hookrightarrow L^{p^*}_{s^*+1/p^*}(\Omega)$ and we use the fact that the Jonsson-Wallin trace operator maps $L^{p^*}_{s^*+1/p^*}(\Omega)$ boundedly to $B^{p^*,p^*}_{s^*}(\partial\Omega)$.

Next, we note that the operator (4.2.9) is bounded if $0 < q \leq p$ since, for this range, $F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n) \hookrightarrow B^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)$. As for the case $p < q \leq \infty$, we will show that for every $f \in F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$ there exists $f' \in F^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)$ such that $\widetilde{\mathrm{Tr}} f = \widetilde{\mathrm{Tr}} f'$. To see this, based on Theorem 3.3.1 we may write $f = \sum_{Q\in\mathcal{Q}_n:\, l(Q)\leq 1} s_Q a_Q$, where the a_Q's satisfy (3.3.1)-(3.3.3) and (4.2.10) holds. It follows that the a_Q's are also atoms for the space $F^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)$, since $\min\{1,p\} = \min\{1,p,q\}$. Also,

$$f' := \sum_{\substack{Q\in\mathcal{Q}_n,\, l(Q)\leq 1 \\ Q\cap\partial\Omega\neq\emptyset}} s_Q a_Q \Longrightarrow \widetilde{\mathrm{Tr}} f = \widetilde{\mathrm{Tr}} f'. \tag{4.2.11}$$

Next, for a fixed $m \in \mathbb{N}$, sufficiently large, and for every cube $Q \in \mathcal{Q}_n$, $l(Q) \leq 1$ and such that $Q \cap \partial\Omega \neq \emptyset$ we define

$$E_Q^m := \{x \in Q : \operatorname{dist}(x, \partial\Omega) > \sqrt{n}\, 2^{-m}\, l(Q)\}. \tag{4.2.12}$$

Since both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ have the cone property, it follows that there exists a constant $c = c(\partial\Omega, m) > 0$ such that

$$Q \in \mathcal{Q}_n, \quad l(Q) \leq 1, \quad Q \cap \partial\Omega \neq \emptyset \Longrightarrow |E_Q^m| \geq c|Q|. \tag{4.2.13}$$

Also, there exists $m' \in \mathbb{N}$, depending exclusively on $\partial\Omega$, n and m, such that if $Q, Q' \in \mathcal{Q}_n$ and $Q' \subseteq Q$, $l(Q') \leq 2^{-m'}\, l(Q)$ then $Q' \cap E_Q^m = \emptyset$. The upshot of this observation is the conclusion that the family of sets $\{E_Q^m\}_{Q \in \mathcal{Q}_n,\, l(Q) \leq 1}$ has finite overlap. That is, there exists a finite constant $N = N(\partial\Omega, n, m) > 0$ such that

$$\sum_{\substack{Q \in \mathcal{Q}_n,\, l(Q) \leq 1 \\ Q \cap \partial\Omega \neq \emptyset}} \chi_{E_Q^m} \leq N. \tag{4.2.14}$$

At this stage we may invoke Proposition 2.7 on p. 51 in [**49**] in order to conclude that

$$\begin{aligned}
&\left\| \Big(\sum_{\substack{Q \in \mathcal{Q}_n,\, l(Q) \leq 1 \\ Q \cap \partial\Omega \neq \emptyset}} (|Q|^{-1/p} |s_Q| \chi_Q)^q \Big)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad \approx \left\| \Big(\sum_{\substack{Q \in \mathcal{Q}_n,\, l(Q) \leq 1 \\ Q \cap \partial\Omega \neq \emptyset}} (|Q|^{-1/p} |s_Q| \chi_{E_Q^m})^q \Big)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad \approx \left\| \sum_{\substack{Q \in \mathcal{Q}_n,\, l(Q) \leq 1 \\ Q \cap \partial\Omega \neq \emptyset}} (|Q|^{-1/p} |s_Q|)^p \chi_{E_Q^m} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}} \approx \Big\| \{s_Q\}_{Q \in \mathcal{Q}_n : l(Q) \leq 1} \Big\|_{\ell^p}.
\end{aligned} \tag{4.2.15}$$

Thus, by Theorem 3.3.1, $f' \in B^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n) = F^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)$ and

$$\|f'\|_{F^{p,p}_{s+\frac{1}{p}}(\mathbb{R}^n)} \approx \|f\|_{F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)}. \tag{4.2.16}$$

Now, for $p < q \leq \infty$, the fact that the operator (4.2.9) is bounded follows from (4.2.16), (4.2.11) and (4.2.2).

Once we have justified that the operator (4.2.1) is well-defined and bounded, we set

$$\mathrm{Tr} := \widetilde{\mathrm{Tr}} \circ E_\Omega : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega) \tag{4.2.17}$$

where, as before, $E_\Omega : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$ is Rychkov's universal extension operator (3.6.32). Now, the trace operator (4.2.17) is linear and bounded and fact that its definition is independent of the particular extension operator E_Ω used in (4.2.17) is proved as in the case of Besov spaces, by relying on the (second part of) Theorem 3.7.1. This shows that the trace operator is well-defined and bounded on the Triebel-Lizorkin scale.

As far as the existence of right-inverses for these trace operators is concerned, first recall the mapping properties of double layer potential established in Corollary 4.1.6. In the remaining of the proof we shall work both with the domain $\Omega_+ := \Omega$ as well as the complement of its closure, $\Omega_- := \mathbb{R}^n \setminus \Omega$. Let $\mathcal{D}^\pm$ be the harmonic double layer operators associated with Ω_+ and Ω_-, respectively. The key ingredient in our approach to the construction of the right-inverse for the trace is going to be the jump-formula

$$\mathrm{Tr} \circ \mathcal{D}^\pm = \pm\tfrac{1}{2} I + K \quad \text{on} \quad B^{p,q}_s(\partial\Omega), \tag{4.2.18}$$

which, so we claim, is valid whenever $0 < (n-1)(1/p-1)_+ < s < 1$ and $0 < q \leq \infty$. Indeed, by what we have proved so far about the trace operator, Corollary 4.1.6 and Proposition 3.5.8, both sides of the equality in (4.2.18) are bounded operators on $B^{p,q}_s(\partial\Omega)$ so it suffices to check this identity for a dense subclass of $B^{p,q}_s(\partial\Omega)$. This, however, is easily checked based on (2.3.5).

To proceed, consider the operator

$$\mathrm{Ex} := \mathcal{D}^+ - \mathcal{R}_{\Omega_+} \circ E_{\Omega_-} \circ \mathcal{D}^-, \tag{4.2.19}$$

where $E_{\Omega_\pm} : A^{p,q}_{s+\frac{1}{p}}(\Omega_\pm) \to A^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$, $A \in \{B, F\}$, are Rychkov's universal extension operators and

Then, thanks to Corollary 4.1.6, for $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$

$$\mathrm{Ex} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad 0 < q \leq \infty, \tag{4.2.20}$$

$$\mathrm{Ex} : B^{p,p}_s(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad p \neq \infty, \quad \Big(\min\big\{\tfrac{1}{pn}+1,\, \tfrac{s}{n-1}+1\big\}\Big)^{-1} < q \leq \infty. \tag{4.2.21}$$

Furthermore, by the jump relations (4.2.18),

$$\mathrm{Tr} \circ \mathrm{Ex} = (\tfrac{1}{2}I + K) - (-\tfrac{1}{2}I + K) = I, \tag{4.2.22}$$

i.e., Ex is a linear, bounded, right-inverse for the trace operator on Besov and Triebel-Lizorkin spaces. □

Corollary 4.2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix two indices p, s such that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then*

$$\mathcal{S} : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad 0 < q \leq \infty, \tag{4.2.23}$$

$$\mathcal{S}: B^{p,p}_{s-1}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \text{if } p\neq\infty \text{ and } \Big(\min\big\{\tfrac{1}{pn}+1,\,\tfrac{s}{n-1}+1\big\}\Big)^{-1} < q \leq \infty. \tag{4.2.24}$$

Proof. A formal calculation shows that the adjoint operator for the harmonic single layer is $S^*g := \widetilde{\mathrm{Tr}}(\phi\Pi\psi(g))$, where $\phi,\psi\in C^\infty_c(\mathbb{R}^n)$ are such that $\phi=\psi=1$ in a neighborhood of $\overline{\Omega}$ and $\widetilde{\mathrm{Tr}}$ is as in (4.2.9). Then Theorem 1.1.3, Corollary 3.2.12 and duality yield (4.2.24) for $0<s<1$, $1<p<\infty$, $1\leq q\leq\infty$. Now (4.2.23) and (4.2.24) follow by interpolation with the boundedness results stated in Corollary 4.1.4) (in fact, we only need the case $p=q\leq 1$). □

For $\Omega\subset\mathbb{R}^n$ Lipschitz domain, let us consider the spaces

$$\overset{\circ}{A}{}^{p,q}_s(\Omega) := \text{ the closure of } C^\infty_c(\Omega) \text{ in } A^{p,q}_s(\Omega), \qquad 0<p,q\leq\infty,\ s\in\mathbb{R}, \tag{4.2.25}$$

where, as usual, $A=F$ or $A=B$. Then Proposition 3.1 in [**127**] ensures that

$$\overset{\circ}{A}{}^{p,q}_s(\Omega) = A^{p,q}_s(\Omega) = A^{p,q}_{s,z}(\Omega), \qquad A\in\{F,B\}, \tag{4.2.26}$$

whenever $0<p,q<\infty$, $\max\Big(1/p-1, n(1/p-1)\Big)<s<1/p$, and $\min\{p,1\}\leq q<\infty$ in the case $A=F$. Other cases of interest are considered below.

Proposition 4.2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$\overset{\circ}{F}{}^{p,q}_s(\Omega) = F^{p,q}_{s,z}(\Omega) \tag{4.2.27}$$

provided

$$\begin{aligned} &0<p<\infty, \quad \min\{1,p\}\leq q<\infty, \quad \textit{and}\\ &\exists\, k\in\mathbb{N}_o \ \textit{ so that } \ \max\Big(\tfrac{1}{p}-1, n\big(\tfrac{1}{p}-1\big)\Big) < s-k < \tfrac{1}{p}. \end{aligned} \tag{4.2.28}$$

Furthermore,

$$\overset{\circ}{B}{}^{p,q}_{s}(\Omega) = B^{p,q}_{s,z}(\Omega) \tag{4.2.29}$$

whenever

$$0 < p, q < \infty \ \text{ and } \ \exists\, k \in \mathbb{N}_o \ \text{ so that } \ \max\Big(\frac{1}{p}-1, n\big(\frac{1}{p}-1\big)\Big) < s-k < \frac{1}{p}. \tag{4.2.30}$$

Proof. Since, by Proposition 3.8.1, $\widetilde{C^\infty_c(\Omega)}$ is dense in $F^{p,q}_{s,0}(\Omega)$ and since the operator of restriction to Ω maps the latter space boundedly onto $F^{p,q}_{s,z}(\Omega)$, it follows that

$$F^{p,q}_{s,z}(\Omega) \hookrightarrow \overset{\circ}{F}{}^{p,q}_{s}(\Omega), \qquad 1 < p, q < \infty, \ s \in \mathbb{R}. \tag{4.2.31}$$

The opposite inclusion is contained in (4.2.26) when $k = 0$ so it suffices to work under the assumption that $k \in \mathbb{N}$. Also, there is no loss of generality in assuming that Ω is the domain in $\mathbb{R}^n$ lying above the graph of a Lipschitz function. In particular, there exists an infinite, upright circular cone Γ, with vertex at the origin in $\mathbb{R}^n$ such that $x + \Gamma \subset \Omega$ for every $x \in \bar{\Omega}$. In this context, we bring in Calderón's extension operator $\mathcal{E}_k$. This was originally introduced in [**17**] as

$$\begin{aligned} \mathcal{E}_k f(x) \ := \ & \int_{S^{n-1}} \int_0^\infty \phi(\rho\omega)\rho^{n-1}\Big\{\Big(\frac{\partial}{\partial\rho}\Big)^k [f(x-\rho\omega)]\Big\}^{\sim} d\rho\, d\omega \\ & - \int_{S^{n-1}} \int_0^\infty \psi(\rho\omega) \widetilde{f(x-\rho\omega)}\, d\rho\, d\omega, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{4.2.32}$$

where tilde (as well as $\{\cdots\}^{\sim}$) denotes extension by zero outside Ω, $\psi := (\partial/\partial\rho)^k[\rho^{n-1}\phi(\rho\omega)]$ and $\phi \in C^\infty(\mathbb{R}^n\backslash\{0\})$ is homogeneous of degree $-n+k$ for $0 < |x| \ll 1$, has bounded support, vanishes outside the cone $-\Gamma$, and is normalized such that

$$\int_{S^{n-1}} \rho^{n-k}\phi(\rho\omega)\, d\omega = \frac{1}{(k-1)!} \quad \text{for } \ \rho \ \text{ small.} \tag{4.2.33}$$

Above, $x = \rho\omega$ with $\rho := |x|$ and $\omega := x/|x| \in S^{n-1}$, is the polar representation of $x \in \mathbb{R}^n \setminus \{0\}$. Note that $\psi \in C_c^\infty(\mathbb{R}^n)$ and $0 \notin \operatorname{supp}\psi$. We shall also tacitly assume that the function f is always supported in a bounded subset of $\bar{\Omega}$. The operator (4.2.32) enjoys the following properties. First,

$$f \in C^\infty(\bar{\Omega}) \Longrightarrow \mathcal{E}_k f\Big|_\Omega = f \ \text{ in } \Omega. \tag{4.2.34}$$

This was actually proved in [**17**]. Second,

$$f \in C_c^\infty(\Omega) \Longrightarrow \mathcal{E}_k f = \tilde{f} \ \text{ in } \mathbb{R}^n, \tag{4.2.35}$$

which can be seen much as (4.2.34), given that tilde commutes with differentiation in the first integrand in (4.2.32). Following [**17**], observe that

$$\Big(\frac{\partial}{\partial\rho}\Big)^k = \sum_{|\gamma|=k} \frac{k!}{\gamma!}\omega^\gamma \Big(\frac{\partial}{\partial x}\Big)^\gamma, \qquad x = \rho\omega, \tag{4.2.36}$$

so that by actually carrying out the differentiation with respect to ρ in (4.2.32) we arrive at

$$\begin{aligned}
\mathcal{E}_k f \ &= \ (-1)^k \int_{S^{n-1}} \int_0^\infty \phi(\rho\omega)\Big[\sum_{|\gamma|=k} \frac{k!}{\gamma!}\omega^\gamma \Big\{\Big(\frac{\partial}{\partial x}\Big)^\gamma f(x-\rho\omega)\Big\}^{\sim}\Big]\rho^{n-1}\,d\rho\,d\omega \\
&\quad - \int_{S^{n-1}} \int_0^\infty \frac{\psi(\rho\omega)}{\rho^{n-1}} \widetilde{f(x-\rho\omega)}\rho^{n-1}\,d\rho\,d\omega \\
&= \ \sum_{|\gamma|=k} \varphi_\gamma * f_\gamma + \xi * \tilde{f},
\end{aligned} \tag{4.2.37}$$

where

$$\varphi_\gamma := (-1)^k(k!/\gamma!)\omega^\gamma\phi(\rho\omega), \quad f_\gamma := \widetilde{(\partial/\partial x)^\gamma f}, \quad \xi := \psi(x)/|x|^{n-1}. \tag{4.2.38}$$

Note that each $\varphi_\gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ has bounded support and is homogeneous of degree $-n+k$ for $0 < |x| \ll 1$. Based on this, it follows from (a version of)

Theorem 3.2.11 that the corresponding convolution operators, i.e. $f \mapsto \varphi_\gamma * f$, for $|\gamma| = k$, are smoothing of order k. Also, $\xi \in C_c^\infty(\mathbb{R}^n)$ so that $\xi_* \cdot$ is smoothing of any order. Finally, granted (4.2.28), there exists $r \in (1, \infty)$ so that $F_s^{p,q}(\Omega) \hookrightarrow L^r(\Omega)$. Consequently, for each $f \in C^\infty(\bar{\Omega})$ of bounded support,

$$\begin{aligned}
\|\mathcal{E}_k f\|_{F_s^{p,q}(\mathbb{R}^n)} &\leq C \sum_{|\gamma|=k} \|\varphi_\gamma * f_\gamma\|_{F_s^{p,q}(\mathbb{R}^n)} + C\|\xi * \tilde{f}\|_{F_s^{p,q}(\mathbb{R}^n)} \\
&\leq C \sum_{|\gamma|=k} \|\widetilde{\partial^\gamma f}\|_{F_{s-k}^{p,q}(\mathbb{R}^n)} + C\|\tilde{f}\|_{L^r(\mathbb{R}^n)} \\
&\leq C \sum_{|\gamma|=k} \|\partial^\gamma f\|_{F_{s-k}^{p,q}(\Omega)} + C\|f\|_{L^r(\Omega)} \\
&\leq C\|f\|_{F_s^{p,q}(\Omega)}. \qquad (4.2.39)
\end{aligned}$$

The third inequality uses the fact that $\max\left(\frac{1}{p} - 1, n\left(\frac{1}{p} - 1\right)\right) < s - k < \frac{1}{p}$ and $0 < \min\{1, p\} \leq q < \infty$ which, in turn, ensure that the extension by zero from Ω to $\mathbb{R}^n$ is a bounded operator with preservation of class; cf. Proposition 3.6.6. Thus, by density,

$$\mathcal{E}_k : F_s^{p,q}(\Omega) \longrightarrow F_s^{p,q}(\mathbb{R}^n) \qquad (4.2.40)$$

is a bounded operator whenever the indices are as in (4.2.28), which, by (4.2.34), satisfies

$$\mathcal{E}_k f\Big|_\Omega = f, \qquad \forall\, f \in F_s^{p,q}(\Omega),\ p, q, s \text{ as in (4.2.28)}. \qquad (4.2.41)$$

Thanks to (4.2.35), we also have

$$\mathcal{E}_k : \overset{\circ}{F}{}_s^{p,q}(\Omega) \longrightarrow F_{s,0}^{p,q}(\Omega), \quad p, q, s \text{ as in (4.2.28)}. \qquad (4.2.42)$$

It R_Ω is the operator of restriction to Ω, it follows from (4.2.40)-(4.2.42) that the

composition $R_\Omega \circ \mathcal{E}_k$ acts at the inclusion

$$\overset{\circ}{F}{}^{p,q}_{s}(\Omega) \hookrightarrow F^{p,q}_{s,z}(\Omega), \qquad p,q,s \text{ as in (4.2.28)}. \tag{4.2.43}$$

Along with (4.2.43), this proves (4.2.27), as desired.

The case of Besov spaces is treated analogously and this finishes the proof of the proposition. □

Theorem 4.2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $\min\{1,p\} \leq q < \infty$. Then*

$$F^{p,q}_{s+1/p,z}(\Omega) = \{f \in F^{p,q}_{s+1/p}(\Omega) : \operatorname{Tr} f = 0\} \tag{4.2.44}$$

and

$$C^\infty_c(\Omega) \hookrightarrow F^{p,q}_{s+1/p,z}(\Omega) \quad \textit{densely.} \tag{4.2.45}$$

Furthermore, a similar result is valid for the scale of Besov spaces. More specifically, if $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q < \infty$, then

$$B^{p,q}_{s+1/p,z}(\Omega) = \{f \in B^{p,q}_{s+1/p}(\Omega) : \operatorname{Tr} f = 0\} \tag{4.2.46}$$

and

$$C^\infty_c(\Omega) \hookrightarrow B^{p,q}_{s+1/p,z}(\Omega) \quad \textit{densely.} \tag{4.2.47}$$

Proof. By definition, if $f \in F^{p,q}_{s+1/p,z}(\Omega)$, then there exists $u \in F^{p,q}_{s+1/p,0}(\Omega)$ such that $f = u|_\Omega$. It follows that $f \in F^{p,q}_{s+1/p}(\Omega)$ and $\operatorname{Tr} f = \operatorname{Tr} u = 0$, proving the left-to-right inclusion in (4.2.44). Conversely, assume that $f \in F^{p,q}_{s+1/p}(\Omega)$ has $\operatorname{Tr} f = 0$. Since, given the current assumption on the indices, there exists $r \in (1,\infty)$ and

$\alpha \in (1/r, 1+1/r)$ such that $F^{p,q}_{s+1/p}(\Omega) \hookrightarrow L^r_\alpha(\Omega)$, it follows that f belongs to the subspace of functions from $L^r_\alpha(\Omega)$ with zero-trace. By Corollary 3.11 in **[68]**, $C^\infty_c(\Omega)$ is dense in this latter space. With this in hand and relying on (4.2.35), written for $k=1$, plus the fact that Calderón's extension operator $\mathcal{E}_1$ maps $L^r_\alpha(\Omega)$ to $L^r_\alpha(\mathbb{R}^n)$ (seen much as in (4.2.39)) ultimately leads to the conclusion that $\mathcal{E}_1(f) = \tilde{f}$ in $\mathbb{R}^n$.

Now, given that $f \in F^{p,q}_{s+1/p}(\Omega)$ and since $\max\Big(1/p-1, n(1/p-1)\Big) < s+1/p-1 < 1/p$ we may conclude, based on Proposition 3.6.6 and by once again reasoning as in (4.2.39), that $\mathcal{E}_1(f) \in F^{p,q}_{s+1/p}(\mathbb{R}^n)$. Thus, $\tilde{f} \in F^{p,q}_{s+1/p,0}(\mathbb{R}^n)$ so that, ultimately, $f = \tilde{f}|_\Omega \in F^{p,q}_{s+1/p,z}(\Omega)$, as desired. This finishes the proof of (4.2.44). Now (4.2.45) follows from (4.2.44) and (4.2.27). Indeed, if $f \in F^{p,q}_{s+1/p,z}(\Omega)$ then, by (4.2.27), there exists a sequence $\varphi_j \in C^\infty_c(\Omega)$ such that $\varphi_j \to f$ in $F^{p,q}_{s+1/p}(\Omega)$ and further, by (4.2.44), in $F^{p,q}_{s+1/p,z}(\Omega)$. The case of Besov spaces is treated analogously and this finishes the proof of the theorem. □

4.2.2 The normal derivative

We start with the discussion of normal derivative for *harmonic functions*. Consider first a special Lipschitz domain $\Omega \in \mathbb{R}^n$ lying above the graph of Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. For every function u harmonic in Ω one can define a system of harmonic conjugates

$$\begin{aligned} u_j(x) &:= -\int_0^\infty (\partial_j u)(x+te_n)\,dt, \quad j=1,..,n-1, \\ u_n(x) &:= u(x), \end{aligned} \tag{4.2.48}$$

where $e_n = (0,...,0,1) \in \mathbb{R}^n$. It is not hard to show that elements of system (4.2.48)

satisfy the following properties:

$$\Delta u_j = 0 \quad \text{for every} \quad j = 1, ..., n, \tag{4.2.49}$$

$$\partial_i u_j = \partial_j u_i \quad \text{for every} \quad i, j = 1, ..., n, \tag{4.2.50}$$

$$\sum_{j=1}^{n} \partial_j u_j = 0. \tag{4.2.51}$$

Theorem 4.2.4. *Assume that Ω is a special Lipschitz domain in $\mathbb{R}^n$. Then normal derivative defined on harmonic functions in Ω as*

$$\partial_\nu^\Delta u := \sum_{j=1}^{n} \partial_{\tau_{jn}} \operatorname{Tr} u_j, \tag{4.2.52}$$

is a linear bounded operator

$$\partial_\nu^\Delta : \mathbb{H}^p_{s+\frac{1}{p}}(\Omega) \to B^{p,p}_{s-1}(\partial\Omega), \tag{4.2.53}$$

provided $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$.

Here $\{u_j\}_{j=1}^n$ is a system of harmonic conjugates (4.2.48).

Proof. To justify the definition given above, let us consider an arbitrary function $\Phi \in C_c^\infty(\mathbb{R}^n)$ and $u \in C^\infty(\overline{\Omega})$ harmonic. Then

$$\begin{aligned}
&\int_{\partial\Omega} \partial_\nu^\Delta u(x)\Phi(x)\, d\sigma_x \\
&= \int_{\partial\Omega} \sum_{j=1}^{n} \partial_{\tau_{jn}} \operatorname{Tr} u_j(x)\Phi(x)\, d\sigma_x = \int_{\partial\Omega} \sum_{j=1}^{n} (\nu_j \partial_n - \nu_n \partial_j) \operatorname{Tr} u_j(x)\Phi(x)\, d\sigma_x \\
&= \int_\Omega \sum_{j=1}^{n} \partial_j(\partial_n u_j \Phi)(x)\, dx - \int_\Omega \sum_{j=1}^{n} \partial_n(\partial_j u_j \Phi)(x)\, dx,
\end{aligned} \tag{4.2.54}$$

where the last equality is obtained via integration by parts. Resting on (4.2.49)-

(4.2.51), we further get

$$
\begin{aligned}
&= \int_\Omega \sum_{j=1}^n \partial_j(\partial_n u_j \Phi)(x)\, dx \\
&= \int_\Omega \sum_{j=1}^n \partial_n u_j(x)\partial_j\Phi(x)\, dx = \int_\Omega \sum_{j=1}^n \partial_j u_n(x)\partial_j\Phi(x)\, dx \\
&= \int_{\partial\Omega} \sum_{j=1}^n \nu_j(x) \mathrm{Tr}\, \partial_j u(x)\Phi(x)\, d\sigma_x = \int_{\partial\Omega} \langle \nu(x), \mathrm{Tr}\, \nabla u(x)\rangle \Phi(x)\, d\sigma_x, \qquad (4.2.55)
\end{aligned}
$$

as desired.

Turning to (4.2.53), we prove the claim to the effect that

$$
u \in \mathbb{H}^p_\alpha(\Omega), \quad \alpha \geq 0, \qquad (4.2.56)
$$

yields

$$
u_j \in \mathbb{H}^p_\alpha(\Omega) \quad \text{and} \quad \sum_j \|u_j\|_{\mathbb{H}^p_\alpha(\Omega)} \leq C\|u\|_{\mathbb{H}^p_\alpha(\Omega)}. \qquad (4.2.57)
$$

Indeed, for every $j = 1, ..., n-1$

$$
\begin{aligned}
&\int_\Omega \left(\delta(x)^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u_j(x)|\right)^p\, dx \\
&\leq C\int_\Omega \delta(x)^{p(\langle\alpha\rangle-\alpha)} \left(\int_0^\infty |(\nabla^{\langle\alpha\rangle+1}u)(x+te_n)|\, dt\right)^p\, dx \\
&\leq C\int_{\mathbb{R}^{n-1}}\int_0^\infty r^{p(\langle\alpha\rangle-\alpha)}(x) \left(\int_0^\infty |(\nabla^{\langle\alpha\rangle+1}u)((x',\varphi(x')+r+t))|\, dt\right)^p\, drdx', \\
&\leq C\int_{\mathbb{R}^{n-1}}\int_0^\infty r^{p(\langle\alpha\rangle-\alpha)}(x) \left(\int_r^\infty |\nabla^{\langle\alpha\rangle+1}u|^*((x',\varphi(x')+\lambda))\, d\lambda\right)^p\, drdx' \qquad (4.2.58)
\end{aligned}
$$

via pull-back to Euclidean coordinates $(x', r) \in \mathbb{R}^{n-1}\times(0,\infty)$ and subsequent change of variables $\lambda := t + r$. Here * denotes the radial maximal function defined in (2.2.3). Now one can invoke Hardy's inequality (cf. **[119]** for the case $p < 1$ – it is applicable since $|\nabla^{\langle\alpha\rangle+1}u|^*((x',\varphi(x')+\lambda))$ is non-increasing and nonnegative

function of λ and $p(\langle\alpha\rangle-\alpha)>-1$). Then (4.2.58) is dominated by

$$C\int_{\mathbb{R}^{n-1}}\int_0^\infty \lambda^{p(\langle\alpha\rangle-\alpha+1)}\left(|\nabla^{\langle\alpha\rangle+1}u|^*((x',\varphi(x')+\lambda))\right)^p\,d\lambda dx'$$
$$\leq C\int_\Omega \delta(x)^{p(\langle\alpha\rangle-\alpha+1)}\left(|\nabla^{\langle\alpha\rangle+1}u|^*(x)\right)^p\,dx. \tag{4.2.59}$$

This, in turn, is controlled in terms of

$$C\int_\Omega \delta(x)^{p(\langle\alpha\rangle-\alpha+1)}\left(|\nabla^{\langle\alpha\rangle+1}u(x)|\right)^p\,dx\leq C\int_\Omega \delta(x)^{p(\langle\alpha\rangle-\alpha)}\left(|\nabla^{\langle\alpha\rangle}u(x)|\right)^p\,dx, \tag{4.2.60}$$

invoking Lemma 2.4.3 and Lemma 3.1.9.

The lower order terms in $\mathbb{H}^p_\alpha(\Omega)$-norm $\sum_{i=0}^{\langle\alpha\rangle-1}\|\nabla^i u_j\|_{L^p(\Omega)}$, $j=0,1,...,n-1$ can be estimated by $C\sum_{i=0}^{\langle\alpha\rangle-1}\|\nabla^i u\|_{L^p(\Omega)}$ along the same lines.

We proved that $\{u_j\}_{j=1}^n$ belong to the space $\mathbb{H}^p_{s+\frac{1}{p}}(\Omega)$ and therefore, to the space $B^{p,p}_{s+\frac{1}{p}}(\Omega)$ by Theorem 3.9.3. Routine check on the level of atoms can be used to verify that

$$\nabla_{tan}\circ \mathrm{Tr}: B^{p,p}_{s+1/p}(\qquad\qquad Omega)\longrightarrow B^{p,p}_{s-1}(\partial\Omega). \tag{4.2.61}$$

Together with Theorem 1.1.3 this finishes the proof. □

Now we turn our attention to the case of a Lipschitz domain Ω which is starlike with respect to the origin. Assume that u is a harmonic function on Ω and define system of harmonic conjugates $\{v_{jk}\}_{j,k=1}^n$ in the following way:

$$v:=\sum_{j,k=1}^n v_{jk}\,dx_j\wedge dx_k=2\sum_{j<k}v_{jk}\,dx_j\wedge dx_k,$$
$$v_{jk}(x):=-\frac{1}{2}\int_0^1 t^{n-2}[x_j(\partial_k u)(tx)-x_k(\partial_j u)(tx)]\,dt, \tag{4.2.62}$$

where 'wedge' is to be interpreted as the exterior product of forms and $d := \sum_{j=1}^{n} \partial_j dx_j \wedge$ denotes the standard exterior derivative operator in $\mathbb{R}^n$, with formal adjoint d^*. It can be shown that $dv = 0$, $d^*v = du$ and hence, $-(dd^* + d^*d)v = 0$ so that each v_{jk} is harmonic. Indeed,

$$v(x) = -\frac{1}{2}\int_0^1 t^{n-2} d(\|x\|^2) \wedge (du)(tx) dt$$

which implies $dv = 0$.

Going further, with δ_{jk} denoting the usual Kronecker symbol, we have

$$\begin{aligned}
d^*v(x) &= -2\sum_{j,k} \partial_k v_{kj}(x)\, dx_j \\
&= \sum_{j,k} \left\{ \int_0^1 t^{n-2} \Big[\delta_{jk}(\partial_k u)(tx) + tx_j(\partial_k^2 u)(tx) - (\partial_j u)(tx) - tx_k(\partial_k \partial_j u)(tx) \Big]\, dt \right\} dx_j \\
&= \sum_{j} \left\{ \int_0^1 \Big[t^{n-2}(n-1)(\partial_j u)(tx) + t^{n-1} \nabla(\partial_j u)(tx) \cdot x \Big]\, dt \right\} dx_j \\
&= \sum_{j} \left\{ \int_0^1 \frac{d}{dt} \Big[t^{n-1}(\partial_j u)(tx) \Big]\, dt \right\} dx_j \\
&= \sum_{j} \partial_j u(x)\, dx_j
\end{aligned}$$

where, in the last step, we integrate by parts in t. Therefore, $d^*v = du$ and in particular, $-(dd^* + d^*d)v = 0$.

Theorem 4.2.5. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ which is starlike with respect to the origin. Then normal derivative defined on harmonic functions in Ω as*

$$\partial_\nu^{\Delta} u := \sum_{j,k} \partial_{\tau_{jk}}[\operatorname{Tr} v_{jk}] = 2\sum_{j<k} \partial_{\tau_{jk}}[\operatorname{Tr} v_{jk}] \tag{4.2.63}$$

is a linear bounded operator

$$\partial_\nu^\Delta : \mathbb{H}^p_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \tag{4.2.64}$$

provided $\frac{n-1}{n} < p \leq \infty$ *and* $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

Here $\{v_{jk}\}_{j,k=1}^n$ *is a system of harmonic conjugates (4.2.62).*

Heuristically, the identity $du = d^*v$ yields

$$\begin{aligned} \partial_\nu u &= -2\sum_j \left(\nu_j \sum_k \partial_k v_{kj} \right) = \sum_{j,k} \nu_j \partial_k v_{jk} + \sum_{j,k} \nu_j \partial_k v_{jk} \\ &= \sum_{j,k} \nu_j \partial_k v_{jk} - \sum_{j,k} \nu_k \partial_j v_{jk} = \sum_{j,k} \partial_{\tau_{jk}} v_{jk}, \end{aligned} \tag{4.2.65}$$

so that (4.2.63) naturally defines normal derivative.

Proof. Similarly to the case of unbounded domain, we claim that $u \in \mathbb{H}^p_\alpha(\Omega)$, $\alpha \geq 0$, yields $v \in \mathbb{H}^p_\alpha(\Omega)$ with the appropriate norm control. Recall the representation of starlike Lipschitz domain in polar system of coordinates as given in §2.2. Then

$$\begin{aligned} &\sum_{j,k} \int_\Omega (\delta(x)^{\langle\alpha\rangle-\alpha} |\nabla^{\langle\alpha\rangle} v_{jk}(x)|)^p \, dx \\ &\leq C \int_{S^{n-1}} \int_0^{\phi(\omega)} (\phi(\omega)-\rho)^{p(\langle\alpha\rangle-\alpha)} \Big(\int_0^1 t^{n-2} \rho |\nabla^{\langle\alpha\rangle+1} u(t\rho\omega)| \, dt \Big)^p \rho^{n-1} \, d\rho d\omega. \end{aligned} \tag{4.2.66}$$

Going further, we subsequently change the variables $\rho = \phi(\omega)e^{-s}$, $s \in (0,\infty)$; $t = e^{-r}$, $r \in (0,\infty)$; and then $\lambda = r+s$, $\lambda \in (s,\infty)$ to majorize the expression (4.2.66) by

$$\begin{aligned} &C \int_{S^{n-1}} \int_0^\infty (1-e^{-s})^{p(\langle\alpha\rangle-\alpha)} \Big(\int_s^\infty e^{-\lambda(n-1)} |\nabla^{\langle\alpha\rangle+1} u(\phi(\omega)\omega e^{-\lambda})| \, d\lambda \Big)^p e^{s(np-n-2p)} \, ds d\omega \\ &= C \int_{S^{n-1}} \int_0^\infty (1-e^{-s})^{p(\langle\alpha\rangle-\alpha)} \Big(\int_s^\infty e^{-\lambda(n-1)} |\nabla^{\langle\alpha\rangle+1} u(\phi(\omega)\omega e^{-\lambda})| (\chi_{[M,\infty)}(s) + \\ &+ \chi_{(0,M]}(s)\chi_{[M,\infty)}(\lambda) + \chi_{(0,M]}(s)\chi_{(s,M]}(\lambda)) \, d\lambda \Big)^p \times \\ &\times e^{s(np-n-2p)} \, ds d\omega, \end{aligned}$$

where M is some large constant. One can see that the parts of integral above corresponding to $\chi_{[M,\infty)}(s)$ and $\chi_{(0,M]}(s)\chi_{[M,\infty)}(\lambda)$ do not exceed $C\|u\|^p_{L^p(\Omega)}$ owing to interior estimates since the argument of $\nabla^{k+2}u$ stays away from $\partial\Omega$. As for the third one, it is bounded by

$$\begin{aligned} &C\int_{S^{n-1}}\int_0^M s^{p(\langle\alpha\rangle-\alpha)}\Big(\int_s^M |\nabla^{\langle\alpha\rangle+1}u(\phi(\omega)\omega e^{-\lambda})|\,d\lambda\Big)^p\,dsd\omega \\ &\le C\int_{S^{n-1}}\int_0^\infty s^{p(\langle\alpha\rangle-\alpha)}\Big(\int_s^M |\nabla^{\langle\alpha\rangle+1}u|^*(\phi(\omega)\omega e^{-\lambda})\,d\lambda\Big)^p\,dsd\omega. \end{aligned} \tag{4.2.67}$$

Next we employ Hardy's inequality to estimate it by

$$C\int_{S^{n-1}}\int_0^M \lambda^{p(\langle\alpha\rangle-\alpha+1)}\Big(|\nabla^{\langle\alpha\rangle+1}u|^*(\phi(\omega)\omega e^{-\lambda})\Big)^p\,d\lambda d\omega. \tag{4.2.68}$$

Changing variables back to original, we observe that the Jacobian is equivalent to some finite constant whenever $0<\lambda<M$, so that the integral above is bounded by

$$C\int_\Omega \delta(x)^{(\langle\alpha\rangle-\alpha+1)p}|\nabla^{\langle\alpha\rangle+1}u(x)|^p\,dx, \tag{4.2.69}$$

and (2.4.32) finishes the proof. □

Corollary 4.2.6. *Assume that Ω is a starlike Lipschitz domain in $\mathbb{R}^n$ and $\frac{n-1}{n}<p\le\infty$, $(n-1)(\frac{1}{p}-1)_+<s<1$. Then*

$$\partial_\nu^\Delta : \mathcal{H}\cap B^{p,q}_{s+\frac{1}{p}}(\Omega)\longrightarrow B^{p,q}_{s-1}(\partial\Omega),\quad \text{if}\quad 0<q\le\infty, \tag{4.2.70}$$

$$\partial_\nu^\Delta : \mathcal{H}\cap F^{p,q}_{s+\frac{1}{p}}(\Omega)\longrightarrow B^{p,p}_{s-1}(\partial\Omega),\quad \textit{if}\quad p\ne\infty\quad \textit{and } q \textit{ satisfies } (1.1.24), \tag{4.2.71}$$

where ∂_ν^Δ stands for the normal derivative of harmonic function on starlike domain defined by (4.2.63).

Proof. For $p = q$ in (4.2.70)–(4.2.71) and $q = 2$ in (4.2.71) the result is a consequence of properties of normal derivative proved in Theorem 4.2.5 and equivalence of spaces of harmonic functions in Theorem 3.9.3. The larger range of indices in the statement of the corollary will follow by interpolation as soon as we shall show that the definition (4.2.63) coincides with (1.1.20) when $1 < p < \infty$, $1 < q \leq \infty$, $0 < s < 1$.

We now turn to the proof of this compatibility condition, starting with the case of Triebel-Lizorkin spaces. When $1 < p < \infty$, $1 < q \leq \infty$, and $u \in \mathcal{H} \cap F^{p,q}_{s+\frac{1}{p}}(\Omega)$ we temporarily set $\partial_\nu^{\Delta'} u$ for the functional

$$\langle \partial_\nu^{\Delta'} u, \psi \rangle := \langle \nabla u, \nabla \Psi \rangle, \qquad \psi \in B^{p',p'}_{1-s}(\partial\Omega), \tag{4.2.72}$$

where $\Psi \in F^{p',q'}_{1-s+1/p'}(\Omega)$ is such that $\operatorname{Tr} \Psi = \psi$ (since the trace operator (1.1.17) is onto, there exists at least one such Ψ). Note that the pairing in the right-hand side is meaningful by Theorem 3.8.3 and (4.2.26). Going further, we claim that, given $\psi \in B^{p',p'}_{1-s}(\partial\Omega)$, the expression in the right-hand side of (4.2.72) is actually independent of the choice of $\Psi \in F^{p',q'}_{1-s+1/p'}(\Omega)$ with $\operatorname{Tr} \Psi = \psi$.

To see this, it suffices, by linearity to treat the case when $\psi = 0$, i.e. show that

$$\langle \nabla u, \nabla \Psi \rangle = 0, \qquad \Psi \in F^{p',q'}_{1-s+1/p'}(\partial\Omega),\ \operatorname{Tr} \Psi = 0. \tag{4.2.73}$$

Indeed, if $\Psi \in F^{p',q'}_{1-s+1/p'}(\partial\Omega)$ has $\operatorname{Tr} \Psi = 0$ then, by (4.2.44) and (4.2.45), there exists a sequence $\varphi_j \in C^\infty_c(\Omega)$ such that $\varphi_j \to \Psi$ in $F^{p',q'}_{1-s+1/p'}(\partial\Omega)$. Consequently,

$$\langle \nabla u, \nabla \Psi \rangle = \lim_j \langle \nabla u, \nabla \varphi_j \rangle = -\lim_j \langle \Delta u, \varphi_j \rangle = 0, \tag{4.2.74}$$

since u is harmonic. This proves (4.2.73) and, at this stage, we may conclude that

$\partial_\nu^{\Delta'} u$ is a well-defined element in $B^{p,p}_{s-1}(\partial\Omega) = \Big(B^{p',p'}_{1-s}(\partial\Omega)\Big)^*$. Thus,

$$\partial_\nu^{\Delta'} : \mathcal{H} \cap F^{p,q}_{s+\frac{1}{p}}(\Omega) \to B^{p,p}_{s-1}(\partial\Omega), \qquad 1 < p < \infty,\ 1 < q \leq \infty\ 0 < s < 1, \tag{4.2.75}$$

is well-defined and bounded.

There remains to show that the operators (4.2.75) and (4.2.71) act in a compatible fashion. To this end, fix $1 < p < \infty$, $0 < s < 1$ and let $u \in \mathcal{H} \cap F^{p,2}_{s+\frac{1}{p}}(\Omega)$ and denote by $\{v_{jk}\}_{j,k=1}^n$ the system of conjugate harmonic functions introduced in (4.2.62). In particular,

$$\begin{aligned} &v_{jk} \in \mathcal{H} \cap F^{p,2}_{s+\frac{1}{p}}(\Omega), \quad \forall\, j,k, \\ &v_{jk} = -v_{kj}, \quad \forall\, j,k, \\ &\textstyle\sum_k \partial_k v_{jk} = \frac{1}{2}\partial_j u, \quad \forall\, j, \end{aligned} \tag{4.2.76}$$

Then, if $\Psi \in C^\infty(\bar\Omega)$ and $\psi = \Psi|_{\partial\Omega}$, we may write

$$\begin{aligned} \langle \partial_\nu^{\Delta} u, \psi\rangle &= \sum_{j,k} \langle \partial_{\tau_{jk}}[\operatorname{Tr} v_{jk}], \psi\rangle \\ &= \sum_{j,k} \Big\{ \langle \partial_j\partial_k v_{jk}, \Psi\rangle + \langle \partial_k v_{jk}, \partial_j\Psi\rangle \Big\} - \sum_{j,k} \Big\{ \langle \partial_k\partial_j v_{jk}, \Psi\rangle + \langle \partial_j v_{jk}, \partial_k\Psi\rangle \Big\} \\ &= \tfrac{1}{2}\sum_j \langle \partial_j u, \partial_j\Psi\rangle + \tfrac{1}{2}\sum_k \langle \partial_k u, \partial_k\Psi\rangle = \langle \nabla u, \nabla\Psi\rangle \\ &= \langle \partial_\nu^{\Delta'} u, \psi\rangle. \end{aligned} \tag{4.2.77}$$

Since the collection of all such ψ's is dense in $B^{p',p'}_{1-s}(\partial\Omega)$, it follows that $\partial_\nu^{\Delta'} u = \partial_\nu^{\Delta} u$ in $B^{p,p}_{s-1}(\partial\Omega)$, as desired. The case of (4.2.70) is similar and this completes the proof of the corollary. □

To define the normal derivative on general bounded Lipschitz domain we first prove an auxiliary lemma.

Lemma 4.2.7. *Assume that Ω is a starlike Lipschitz domain and $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$. Then for every function $u \in \mathcal{H} \cap B^{p,q}_{s+\frac{1}{p}}(\Omega)$*

$$\partial^{\Delta}_{\nu} u = \lim_{t \to 1^-} \partial_\nu u_t, \quad \text{in} \quad B^{p,q}_{s-1}(\partial\Omega), \tag{4.2.78}$$

where $u_t := u(t\cdot)$.

The similar statement holds for $u \in \mathcal{H} \cap F^{p,q}_{s+\frac{1}{p}}(\Omega)$ with convergence in the space $B^{p,p}_{s-1}(\partial\Omega)$ and $p \neq \infty$, q satisfying (1.1.11).

Proof. First of all, $u_t = u(t\cdot) \in C^\infty(\overline{\Omega})$ and hence the normal derivative on the right-hand side of (4.2.78) is well-defined in the classical sense, that is $\partial_\nu u_t = \langle \nu, \operatorname{Tr} \nabla u_t \rangle$.

Consider $\{v_{jk}\}_{j,k=1}^n$ – a system of harmonic conjugates (4.2.62) for function u. It is not hard to see that $\{(v_{jk})_t\}_{j,k=1}^n$ represents a system of harmonic conjugates for u_t and the latter is harmonic in a neighborhood of $\overline{\Omega}$ for every $t < 1$. According to (4.2.65)

$$\partial_\nu u_t = \partial^{\Delta}_{\nu} u_t = \sum_{j<k} \partial_{\tau_{jk}} \operatorname{Tr} (v_{jk})_t, \tag{4.2.79}$$

where $v_{jk} \in \mathcal{H} \cap B^{p,q}_{s+\frac{1}{p}}(\Omega)$ resting on the proof of Theorem 4.2.5. More precisely, it was proved in Theorem 4.2.5 that the "operator of taking harmonic conjugate" maps $\mathbb{H}^p_{s+\frac{1}{p}}(\Omega)$ to $\mathbb{H}^p_{s+\frac{1}{p}}(\Omega)$ and hence, $\mathcal{H} \cap B^{p,p}_{s+\frac{1}{p}}(\Omega)$ to $\mathcal{H} \cap B^{p,p}_{s+\frac{1}{p}}(\Omega)$ by Theorem 3.9.3. The general case $0 < q \leq \infty$ follows by real interpolation of $\mathbb{H} \cap B^{p,p}_{s+\frac{1}{p}}(\Omega)$ spaces.

Further, for every $j = 1, ..., n$ and $k = 1, ..., n$ the function $(v_{jk})_t$ converges to v_{jk} in the space $\mathcal{H} \cap B^{p,q}_{s+\frac{1}{p}}(\Omega)$ as $t < 1$ approaches 1. Hence, $\operatorname{Tr}(v_{jk})_t$ converges to $\operatorname{Tr} v_{jk}$ in $B^{p,q}_s(\partial\Omega)$ by Theorem 1.1.3 and $\partial^{\Delta}_{\nu} u_t$ converges to $\partial^{\Delta}_{\nu} u$, as desired. $\square$

Consider now an arbitrary bounded Lipschitz domain Ω and consider $\Psi \in C^\infty(\overline{\Omega})$ with the small support so that there exists an open set $D \subset \mathbb{R}^n$ containing support of Ψ with the property that $D \cap \Omega$ is starlike. Then define

$$\langle \partial_\nu^\Delta u, \psi \rangle := \langle \partial_\nu^\Delta \left(u|_{D\cap\Omega} \right), \tilde{\psi} \rangle, \tag{4.2.80}$$

where $\psi = \operatorname{Tr} \Psi$ and $\tilde{\psi}$ is an extension of ψ to $\partial\Omega$ by 0.

It has to be proved that such such definition does not depend on choice of the set $D \subset \mathbb{R}^n$. However in view of Lemma 4.2.7

$$\begin{aligned} \langle \partial_\nu \left(u|_{D\cap\Omega} \right), \tilde{\psi} \rangle &= \lim_{t\to 1^-} \langle \partial_\nu^\Delta \left(u|_{D\cap\Omega} \right)_t, \tilde{\psi} \rangle \\ &= \lim_{t\to 1^-} \sum_{j=1}^n \int_{\partial\Omega} \nu_j(x) \operatorname{Tr} \partial_j \left(u|_{D\cap\Omega} \right)_t (x) \tilde{\psi}(x)\, d\sigma_x. \end{aligned} \tag{4.2.81}$$

Observe that $\Psi \in C_c^\infty(D)$ and $\left(u|_{D\cap\Omega} \right)_t \in C^\infty(D \cap \overline{\Omega})$ is harmonic function in a neighborhood of $\overline{\Omega}$ for each $t < 1$. Then the last expression above is equal by the Green's theorem to

$$\lim_{t\to 1^-} \langle \nabla \left(u|_{D\cap\Omega} \right)_t, \nabla\Psi \rangle = \langle \nabla u|_{D\cap\Omega}, \nabla\Psi \rangle = \langle \nabla u, \nabla\Psi \rangle, \tag{4.2.82}$$

since $supp(\Psi) \subset D$. Hence (4.2.80) is independent of the choice of D and defines normal derivative of harmonic function on general bounded Lipschitz domain unambiguously.

Proof of Theorem 1.1.4. We start with the existence and boundedness of operator of normal derivative. For every pair (u, f) satisfying the left-hand side of (1.1.21) or (1.1.23) one can assign

$$\partial_\nu^f u := \partial_\nu \Pi f + \partial_\nu^\Delta (u - \mathcal{R}_\Omega \Pi f), \tag{4.2.83}$$

where Π, as before, stands for Newtonian potential, and ∂_ν^Δ is normal derivative defined by (4.2.80) (observe that $u - \mathcal{R}_\Omega \Pi f$ is harmonic function in Ω).

Assume that (u, f) belongs to the left-hand side of (1.1.21) or (1.1.23). Then the equality (4.2.83) combined with (4.2.70)–(4.2.71) and (4.1.39) leads to (1.1.21), (1.1.23) for the desired range of indices.

The next order of business is to construct linear bounded right inverse for the operator ∂_ν^f. We will denote $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^n \setminus \Omega$ and corresponding objects will be distinguished by super-index + and −, respectively. Assume that $g \in B^{p,q}_{s-1}(\partial\Omega)$ and set

$$u := -\mathcal{S}^+ g + \mathcal{R}_{\Omega^+} \circ E_{\Omega^-} \circ \mathcal{S}^- g; \tag{4.2.84}$$

$$f := \Delta \left(E_{\Omega^-} \circ \mathcal{S}^- g \right). \tag{4.2.85}$$

It is not hard to see that the mapping properties of operators involved imply $u \in B^{p,q}_{s+\frac{1}{p}}(\Omega)$ and $f \in B^{p,q}_{s+\frac{1}{p}-2}(\mathbb{R}^n)$. Also, $\Delta u = f|_{\Omega^+}$ and

$$f|_{\Omega^-} = \mathcal{R}_{\Omega^-} \circ \Delta \left(E_{\Omega^-} \circ \mathcal{S}^- g \right) = \Delta \left(\mathcal{R}_{\Omega^-} \circ E_{\Omega^-} \circ \mathcal{S}^- g \right) = \Delta \mathcal{S}^- g = 0, \tag{4.2.86}$$

so that evidently (u, f) belongs to left-hand side of (1.1.21) with appropriate norm control. Similarly if $g \in B^{p,p}_{s-1}(\partial\Omega)$ then the pair (u, f) belongs to the left-hand side of (1.1.23).

In the remaining of the proof we will show that the operator defined in (4.2.84)-(4.2.85) is right inverse for the normal derivative ∂_ν^f defined in (4.2.83), i.e. $\partial_\nu^f(u) = g$ for arbitrary $g \in B^{p,q}_{s-1}(\partial\Omega)$. We present the argument for u, f in Besov spaces. The situation when (u, f) belongs to the left-hand side of (1.1.23) can be handled along the same lines.

To start consider the following representation formula:

$$\partial_\nu^f(u) = -\partial_\nu^0(\mathcal{S}^+ g) + \partial_\nu^f(\mathcal{R}_{\Omega^+} \circ E_{\Omega^-} \circ \mathcal{S}^- g). \tag{4.2.87}$$

Regarding the first term, we will show that

$$\partial_\nu^0(\mathcal{S}^+ g) = -\left(\tfrac{1}{2} I - K^*\right) g. \tag{4.2.88}$$

Since the operators on right and left-hand side of the equality above are both endomorphisms of $B_{s-1}^{p,q}(\partial\Omega)$, it is enough to consider $g \in L^2(\partial\Omega)$. But

$$\partial_\nu^0(\mathcal{S}^+ g) = \partial_\nu^\Delta(\mathcal{S}^+ g), \tag{4.2.89}$$

and hence for every ψ allowed in (4.2.80) there holds:

$$\begin{aligned}\langle \partial_\nu^0(\mathcal{S}^+ g), \psi\rangle &= \int_{\partial\Omega} \partial_\nu^\Delta((\mathcal{S}^+ g)|_{D\cap\Omega})(x)\tilde{\psi}(x)\, d\sigma_x \\ &= \int_{\partial\Omega} \langle \nu(x), \operatorname{Tr} \nabla(\mathcal{S}^+ g)|_{D\cap\Omega}(x)\rangle \tilde{\psi}(x)\, d\sigma_x = \int_{\partial\Omega} \left(\tfrac{1}{2} I - K^*\right) g(x)\tilde{\psi}(x)\, d\sigma_x, \end{aligned} \tag{4.2.90}$$

owing to jump relations (2.3.4). Note that $g \in L^2(\partial\Omega)$ implies that non-tangential maximal function $\mathcal{N}(\nabla(\mathcal{S}^+ g)|_{D\cap\Omega}) \in L^2(\partial\Omega)$ and also $\mathcal{N}(\nabla(\mathcal{S}^+ g)|_{D\cap\Omega})_j \in L^2(\partial\Omega)$ with $((\mathcal{S}^+ g)|_{D\cap\Omega})_j$, $j = 1, ..., n$ denoting harmonic conjugates. Hence $\operatorname{Tr}(\nabla(\mathcal{S}^+ g)|_{D\cap\Omega})_j$ belongs to $L^2(\partial\Omega)$ for $j = 1, ..., n$, justifying the calculations behind (4.2.90).

As far as the second term in (4.2.87) is concerned, we will prove that

$$\partial_\nu^f(\mathcal{R}_{\Omega^+} \circ E_{\Omega^-} \circ \mathcal{S}^- g) = \left(\tfrac{1}{2} I + K^*\right) g, \tag{4.2.91}$$

which combined with (4.2.90) leads to $\partial_\nu^f(u) = g$ for arbitrary $g \in B_{s-1}^{p,q}(\partial\Omega)$, as desired.

Fix some function $\Psi \in C_c^\infty(\mathbb{R}^n)$ and domain $D \in \mathbb{R}^n$ as in (4.2.80). We will assume that $D \cap \Omega$ is starlike with respect to 0. Also, let $v := E_{\Omega^-} \circ \mathcal{S}^- g$. For arbitrary $t > 0$

$$v_t := v(t\cdot) \in B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n); \tag{4.2.92}$$

$$f_t := \Delta v_t \in B^{p,q}_{s+\frac{1}{p}-2}(\mathbb{R}^n). \tag{4.2.93}$$

In addition,

$$\mathcal{R}_{\Omega^-}(f_t) = \Delta(\mathcal{R}_{\Omega^-}(v_t)) = \Delta(\mathcal{R}_{\Omega^-}(v))_t = t^2(\Delta\,\mathcal{R}_{\Omega^-}(v))_t = 0, \tag{4.2.94}$$

and hence $f_t \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$ for every $t > 0$. Furthermore, as t tends to 1, v_t converges to v in $B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$, $\mathcal{R}_\Omega v_t$ converges to $\mathcal{R}_\Omega v$ in $B^{p,q}_{s+\frac{1}{p}}(\Omega)$ and f_t converges to f in $B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$. This implies that

$$\partial^f_\nu(\mathcal{R}_\Omega v) = \lim_{t\to 1} \partial^{f_t}_\nu(\mathcal{R}_\Omega v_t), \tag{4.2.95}$$

in the sense of convergence in space defined on the left-hand side of (1.1.21). Observe that by classical embedding theorems

$$\begin{gathered} B^{p,q}_{s+\frac{1}{p}}(\Omega) \hookrightarrow B^{p_0,q_0}_{s_0+\frac{1}{p_0}}(\Omega), \quad B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) \hookrightarrow B^{p_0,q_0}_{s_0+\frac{1}{p_0}-2,0}(\Omega), \\ B^{p,q}_{s-1}(\partial\Omega) \hookrightarrow B^{p_0,q_0}_{s_0-1}(\partial\Omega), \end{gathered} \tag{4.2.96}$$

for $p_0 > p$ and $q_0 \geq q$ provided $s - \frac{n-1}{p} = s_0 - \frac{n-1}{p_0}$, in particular, we can choose $p_0, q_0 > 1$. This allows to consider

$$\langle \partial^{f_t}_\nu(\mathcal{R}_\Omega v_t), \psi\rangle := \langle f_t, \Psi\rangle + \langle \nabla\mathcal{R}_\Omega v_t, \nabla\Psi\rangle, \qquad \forall\, \psi \in B^{p_0',q_0'}_{1-s_0}(\partial\Omega) \quad \text{with} \quad \Psi = \operatorname{Ex}\psi, \tag{4.2.97}$$

in particular for Ψ and ψ as chosen above. Going further, we denote by subindex ε the convolution with standard mollifier, that is $(v)_\varepsilon := v * \phi_\varepsilon$, where

$$\phi \in C_c^\infty(\mathbb{R}^n) \quad \text{is such that } \int_{\mathbb{R}^n} \phi(x)\,dx = 1 \quad \text{and} \quad \phi_\varepsilon := \varepsilon^{-n}\phi(\cdot/\varepsilon),\ \varepsilon > 0. \tag{4.2.98}$$

Then $(v_t)_\varepsilon$ and $(f_t)_\varepsilon$ are both smooth function in $\mathbb{R}^n$, moreover, as ε tends to 0, $(v_t)_\varepsilon$ converges to (v_t) in $B^{p,q}_{s+\frac{1}{p}}(\mathbb{R}^n)$.

Consequently, we can apply integration by parts to obtain

$$\begin{aligned}\langle \nabla\mathcal{R}_\Omega v_t, \nabla\Psi\rangle &= \lim_{\varepsilon\to 0}\int_\Omega \nabla\mathcal{R}_\Omega(v_t)_\varepsilon(x)\nabla\Psi(x)\,dx \\ &= -\lim_{\varepsilon\to 0}\int_\Omega \Delta\mathcal{R}_\Omega(v_t)_\varepsilon(x)\Psi(x)\,dx + \lim_{\varepsilon\to 0}\int_{\partial\Omega}\partial_\nu(v_t)_\varepsilon(x)\psi(x)\,d\sigma_x. \end{aligned}\tag{4.2.99}$$

As for the first term in the sum above observe that $v|_{\Omega^-}$ is harmonic function and $supp(\Psi) \subset D$, therefore Δv_t vanishes near the boundary of Ω and hence, so does $\Delta(v_t)_\varepsilon$ for ε small enough. Then

$$-\lim_{\varepsilon\to 0}\int_\Omega \Delta\mathcal{R}_\Omega(v_t)_\varepsilon(x)\Psi(x)\,dx = -\lim_{\varepsilon\to 0}\int_\Omega \Delta(v_t)_\varepsilon(x)\Psi(x)\,dx = -\langle f_t, \Psi\rangle. \tag{4.2.100}$$

On the other hand,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial\Omega}\partial_\nu(v_t)_\varepsilon(x)\psi(x)\,d\sigma_x &= \int_{\partial\Omega}\partial_\nu(v_t)(x)\psi(x)\,d\sigma_x \\ &= \int_{\partial\Omega}\langle\nu(x), \mathrm{Tr}^-\,\nabla(v_t)(x)\rangle\psi(x)\,d\sigma_x, \end{aligned}\tag{4.2.101}$$

where the super-index $^-$ in notation Tr^- indicates that the trace is taken from the Ω^--side. Combining (4.2.99)-(4.2.101) leads to

$$\begin{aligned}\langle \partial^f_\nu(\mathcal{R}_\Omega v), \psi\rangle &= \lim_{t\to 1}\int_{\partial\Omega}\langle\nu(x), \mathrm{Tr}^-\,\nabla(v_t)(x)\rangle\psi(x)\,d\sigma_x \\ &= \lim_{t\to 1}\int_{\partial\Omega}\langle\nu(x), t\nabla(v)(tx)\rangle\psi(x)\,d\sigma_x = \langle\left(\tfrac{1}{2}I + K^*\right)g, \psi\rangle. \end{aligned}\tag{4.2.102}$$

As before, it is enough to consider $g \in L^2(\mathbb{R}^n)$ to justify the calculation above.

Finally (4.2.91) and (4.2.92) allow to conclude the existence of linear bounded right inverse to ∂_ν^f with the required mapping properties. □

Corollary 4.2.8. *For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$*

$$\partial_\nu S|_{\partial\Omega_\pm} = \mp\frac{1}{2}I + K^* \text{ on } B^{p,q}_{s-1}(\partial\Omega), \tag{4.2.103}$$

provided $\frac{n}{n+1} < p \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q \leq \infty$. Here, as before, the $\pm$ sign corresponds to approach from inside or outside of the domain Ω.

Proof. In view of Theorems 4.2.4, 4.2.5 and boundedness of K^* on appropriate spaces the operators on the right hand side and left hand side of the equality (4.2.103) are continuous. Also, (4.2.103) holds on the space $L^2(\partial\Omega)$ and therefore, on $B^{p,q}_{s-1}(\partial\Omega)$. □

4.3 Envelopes of quasi-Banach spaces

4.3.1 General set-up

Let $0 < p \leq 1$. A set S in a vector space X is called p-convex if S coincides with its p-convex hull, that is

$$S = \Big\{\sum_{j=1}^n \lambda_j a_j \,:\, a_j \in S,\ \lambda_j \geq 0,\ \sum_{j=1}^n \lambda_j^p \leq 1,\ n \in \mathbb{N}\Big\}. \tag{4.3.1}$$

Also, call S absolutely p-convex if S coincides with its absolutely p-convex hull defined as

$$\Big\{\sum_{j=1}^n \lambda_j a_j \,:\, a_j \in S,\ \sum_{j=1}^n |\lambda_j|^p \leq 1,\ n \in \mathbb{N}\Big\}. \tag{4.3.2}$$

Let X be a linear, topological space and, for each $0 < p \leq 1$, let $W_{X,p}$ be the absolutely p-convex hull of the unit ball in X. Set

$$|||x|||_p := \inf\Big\{\lambda > 0 \,:\, x/\lambda \in W_{X,p}\Big\}. \tag{4.3.3}$$

Recall that X^*, the dual of X, is defined as the collection of all linear, continuous functionals on X. Here the continuity of linear functional $f : X \to \mathbb{R}$ has to be interpreted in the following way:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \|x\|_X < \delta \Rightarrow |f(x)| < \varepsilon. \tag{4.3.4}$$

If X is locally bounded, with a topology given by a quasi-norm $\|\cdot\|_X$, the above condition is equivalent to

$$\|f\| := \sup\{|f(x)| : x \in X \text{ such that } \|x\|_X \leq 1\} < +\infty. \tag{4.3.5}$$

We say that X^* separates the points in X if $x = 0 \Leftrightarrow f(x) = 0,\ \forall\, f \in X^*$. Throughout the paper, all vector spaces considered are assumed to be quasi-normed, with duals separating the points.

Proposition 4.3.1. *If X is a quasi-normed space whose dual separates its points, then (4.3.3) is a p-norm: $|||x|||_p = 0$ if and only if $x = 0$, $|||\lambda x|||_p = |\lambda|\,|||x|||_p$, and*

$$|||x + y|||_p^p \leq |||x|||_p^p + |||y|||_p^p, \quad \forall\, x, y \in X. \tag{4.3.6}$$

Proof. Since the first two properties of the norm are trivially satisfied, we go directly to proof of an appropriate version of triangle inequality given by (4.3.6).

Fix arbitrary points x and y in the vector space X. Then for every $\varepsilon > 0$ there exist positive numbers λ_x and λ_y with the following significance:

$$|||x|||_p < \lambda_x < |||x|||_p + \varepsilon, \qquad |||y|||_p < \lambda_y < |||y|||_p + \varepsilon, \tag{4.3.7}$$

and

$$\lambda_x^{-1/p} x \in W_{X,p}, \qquad \lambda_y^{-1/p} y \in W_{X,p}. \tag{4.3.8}$$

With this in hand, one can write

$$\begin{aligned} \frac{x+y}{(\lambda_x+\lambda_y)^{1/p}} &= \frac{\lambda_x^{1/p}}{(\lambda_x+\lambda_y)^{1/p}} \frac{x}{\lambda_x^{1/p}} + \frac{\lambda_y^{1/p}}{(\lambda_x+\lambda_y)^{1/p}} \frac{y}{\lambda_y^{1/p}} \\ &:= \alpha_1 \frac{x}{\lambda_x^{1/p}} + \alpha_2 \frac{y}{\lambda_y^{1/p}}. \end{aligned} \tag{4.3.9}$$

Clearly, the coefficients α_1 and α_2 satisfy $|\alpha_1|^p + |\alpha_2|^p = 1$ and therefore, $\frac{x+y}{(\lambda_x+\lambda_y)^{1/p}} \in W_{X,p}$. Hence,

$$|||x+y|||_p^p \leq \lambda_x + \lambda_y < |||x|||_p^p + |||y|||_p^p + 2\varepsilon. \tag{4.3.10}$$

Taking arbitrary small value of ε, we complete the argument. □

For each X as above, we denote by $\mathcal{E}_p(X)$ the *p-envelope* of X, defined as the completion of X in the quasi-norm $|||\cdot|||_p$.

Remark. (i) The $|||\cdot|||_p$ "norm" generates a locally p-convex topology, weaker than the original topology on X.

(ii) $\mathcal{E}_p(X)$ is a p-Banach space, i.e. a complete metric space whose metric is naturally induced by a p-norm.

(iii) When $p = 1$, $\mathcal{E}_p(X)$ corresponds to the so-called Banach envelope of X. For details and additional information on these matters we refer to [**94**].

Proposition 4.3.2. *Any bounded, linear operator $T : X \longrightarrow Y$ extends to a bounded operator $\hat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y)$.*

Proof. Recall that for every space X its p-envelope is completion of X in the norm given by Minkowski functional (4.3.3). Hence, $\mathcal{E}_p(X)$ can be considered as the space of equivalence classes of Cauchy sequences relative to the norm (4.3.3), more precisely,

$$[\{x_n\}_{n=1}^{\infty}] \in \mathcal{E}_p(X) \quad \Longleftrightarrow$$

$$x_n \in X\, \forall\, n \in \mathbb{N} \,\text{and}\, \forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} :\ \forall\, n \geq N \quad |||x_n - x_{n+1}|||_p < \varepsilon, \tag{4.3.11}$$

with the convention that two sequences $\{x'_n\}$ and $\{x''_n\}$ belong to the same class if

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} : \quad \forall\, n \geq N \quad |||x'_n - x''_n|||_p < \varepsilon. \tag{4.3.12}$$

Then the norm in the envelope space is given by

$$\|[\{x_n\}_{n=1}^{\infty}]\|_{\mathcal{E}_p(X)} := \lim_{n\to\infty} |||x_n|||_p. \tag{4.3.13}$$

Now we can define the extension of T in the form

$$\hat{T}\left([\{x_n\}_{n=1}^{\infty}]\right) := [\{Tx_n\}_{n=1}^{\infty}]\,. \tag{4.3.14}$$

There are two issues to be resolved: unambiguity of such definition

$$[\{x_n\}_{n=1}^{\infty}] = [\{y_n\}_{n=1}^{\infty}] \text{ in } \mathcal{E}_p(X) \quad \Longrightarrow \quad [\{Tx_n\}_{n=1}^{\infty}] = [\{Ty_n\}_{n=1}^{\infty}] \text{ in } \mathcal{E}_p(Y) \tag{4.3.15}$$

and boundedness of the operator T. We will start with the second one.

Indeed, if $[\{x_n\}_{n=1}^{\infty}] \in \mathcal{E}_p(X)$ then (4.3.11) holds, in particular,

$$\inf\Big\{\lambda > 0 :\ \frac{1}{\lambda}\,\|x_n - x_{n+1}\|_X < 1\Big\} < \varepsilon \quad \text{for every } n \geq N. \tag{4.3.16}$$

However, owing to the boundedness of the operator T on initial spaces,

$$\frac{1}{\lambda}\,\|Tx_n - Tx_{n+1}\|_Y \le \frac{C}{\lambda}\,\|x_n - x_{n+1}\|_X, \tag{4.3.17}$$

hence,

$$\Big\{\lambda>0\,:\;\frac{1}{\lambda C}\,\|Tx_n - Tx_{n+1}\|_Y<1\Big\} \quad\supseteq\quad \Big\{\lambda>0\,:\;\frac{1}{\lambda}\,\|x_n - x_{n+1}\|_X<1\Big\}, \tag{4.3.18}$$

and in view of (4.3.16)

$$\inf\Big\{\lambda>0\,:\;\frac{1}{\lambda}\,\|Tx_n - Tx_{n+1}\|_Y<1\Big\} < C\varepsilon \quad \text{for every } n\ge N. \tag{4.3.19}$$

This yields $[\{Tx_n\}_{n=1}^{\infty}]\in\mathcal{E}_p(Y)$ and

$$\begin{aligned}\|[\{Tx_n\}_{n=1}^{\infty}]\|_{\mathcal{E}_p(Y)} &= \lim_{n\to\infty}\||Tx_n|\|_p = \lim_{n\to\infty}\inf\Big\{\lambda>0\,:\;\frac{1}{\lambda}\,\|Tx_n\|_Y<1\Big\}\\ &\le C\lim_{n\to\infty}\inf\Big\{\lambda>0\,:\;\frac{1}{\lambda}\,\|x_n\|_X<1\Big\} \le C\,\|[\{x_n\}_{n=1}^{\infty}]\|_{\mathcal{E}_p(X)}\,,\end{aligned} \tag{4.3.20}$$

invoking the argument above.

Regarding the unambiguity of the definition (4.3.14), for every two sequences $\{x_n'\}_{n=1}^{\infty}$ and $\{x_n''\}_{n=1}^{\infty}$ from the same equivalence class (4.3.12) holds. Also the argument above shows that for every $\varepsilon>0$

$$\||Tx_n' - Tx_n''|\|_p \le C\||x_n' - x_n''|\|_p < c\varepsilon \tag{4.3.21}$$

for values of n large enough. Therefore, $\{Tx_n'\}_{n=1}^{\infty}$ and $\{Tx_n''\}_{n=1}^{\infty}$ belong to the same equivalence class, as desired. □

Remark. (i) If X is a p-Banach space to begin with, then $\mathcal{E}_p(X)=X$. Indeed, observe that the absolutely p-convex hull of unit ball in a p-Banach space actually

coincides with the unit ball and hence the norm given by (4.3.3) coincides with the norm in original space.

(ii) $\mathcal{E}_p(X)$ should be thought of as the "smallest" locally p-convex topological space containing X. In fact, if X is locally bounded, then $\mathcal{E}_p(X)$ is the "smallest" p-Banach space containing X in the sense that every p-Banach space Y with the property $X \subset Y$ necessarily contains $\mathcal{E}_p(X)$. It is justified by the remark above and Proposition 4.3.2, since the inclusion $\iota : X \to Y$ extends to a linear bounded operator $\hat{\iota} : \mathcal{E}_p(X) \to \mathcal{E}_p(Y)$. The latter is also the inclusion, for it coincides with identity on the densely included subspace, and finally, $\mathcal{E}_p(Y) = Y$.

(iii) It is almost immediate that for every $0 < p < p^* \leq 1$ the envelope $\mathcal{E}_{p^*}(\ell^p) = \ell^{p^*}$.

Since ℓ^{p^*} is a p^*-Banach space containing ℓ^p the inclusion $\mathcal{E}_{p^*}(\ell^p) \subseteq \ell^{p^*}$ holds automatically in concert with the remark made above. Below we will show that the converse is also true.

Assume that $x = (x_1, x_2, ...) \in \ell^{p^*}$. Then we can consider $x = 2\|x\|_{\ell^{p^*}} \sum_{i=1}^{\infty} \lambda_i X_i$, where $X_i := (0, ..., 0, 1/2, 0, ...)$ with $1/2$ on the i-th place and $\lambda_i := (\|x\|_{\ell^{p^*}})^{-1} x_i$. Clearly, X_i belongs to unit ball in ℓ^p for every $i \in \mathbb{N}$. Also, the sequence of coefficients $\{\lambda_i\}_{i=1}^{\infty} = (\|x\|_{\ell^{p^*}})^{-1} \{x_i\}_{i=1}^{\infty} \in \ell^{p^*}$ with the ℓ^{p^*}-norm less than or equal to 1. Then the sequence $(2\|x\|_{\ell^{p^*}})^{-1} \{x_i\}_{i=1}^{\infty}$ belongs to the p^*-convex hull of unit ball in ℓ^p and $\|x\|_{\mathcal{E}_{p^*}(\ell^p)} \leq 2\|x\|_{\ell^{p^*}}$ yielding $\mathcal{E}_{p^*}(\ell^p) \supseteq \ell^{p^*}$.

Proposition 4.3.3. *For each* $0 < p \leq 1$,

$$\mathcal{E}_p(X)^* = X^*. \tag{4.3.22}$$

Proof. Since $X \hookrightarrow \mathcal{E}_p(X)$, the mapping $\Phi : (\mathcal{E}_p(X))^* \to X^*$, defined by $\Phi(f) := f|_X$ is well defined, linear and bounded. Also, by Proposition 4.3.2, any $f \in X^*$ extends to a unique element $\hat{f} \in (\mathcal{E}_p(X))^*$, and we define $\Psi : X^* \to (\mathcal{E}_p(X))^*$ by $\Psi(f) := \hat{f}$. Then, clearly, $\Phi(\Psi(f)) = \hat{f}|_X = f$.

Also, for any $f \in (\mathcal{E}_p(X))^*$, the fact that $\widehat{(f|_X)}|_X = f|_X$, in concert with the dense embedding $X \hookrightarrow \mathcal{E}_p(X)$, implies that $\widehat{(f|_X)} = f$. Thus, $\Psi(\Phi(f)) = f$.

This proves that Ψ and Φ are inverses to each other, justifying (4.3.22). □

Remark. Fix $0 < p \leq p^* \leq 1$. Since $W_{X,p} \subseteq W_{X,p^*}$, it follows that $|||x|||_{p^*} \leq |||x|||_p$ for any $x \in X$. In particular, any Cauchy sequence in $|||\cdot|||_p$ is also Cauchy in $|||\cdot|||_{p^*}$. This implies (in concert with the abstract definition of completion as the vector space of equivalence classes of Cauchy sequences) that the identity map $\iota : X \to X$ extends to a linear, bounded operator $\iota : \mathcal{E}_p(X) \to \mathcal{E}_{p^*}(X)$. However, even though $\mathcal{E}_p(X)^* = \mathcal{E}_{p^*}(X)^* (= X^*)$, there are no guarantees that this map is one-to-one! Also, generally speaking, the dual of $\mathcal{E}_p(X)$ may not separate its points.

Theorem 4.3.4. *Assume that X and Y are quasi-Banach spaces and $0 < p \leq 1$. Consider an operator $T : X \longrightarrow Y$. If it is an isomorphism (onto, compact, Fredholm operator), then so is its extension $\hat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y)$. In the case of Fredholm operator the index is also preserved.*

Moreover, if T has a finite dimensional cokernel, then the same holds for $\hat{T}$.

Proof. Applying Proposition 4.3.2 to T and T^{-1}, one can easily confirm that if T is an isomorphism then so is $\hat{T}$.

Concerning the issue of ontoness, we proceed as follows. Consider $y \in \mathcal{E}_p(Y)$

satisfying $\|y\|_{\mathcal{E}_p(Y)} \leq 1$. There exists $\tilde{y} \in W_{Y,p}$ with the property $\|y-\tilde{y}\|_{\mathcal{E}_p(Y)} \leq 1/2$. Then $\tilde{y}$ can be represented in the form $\tilde{y} = \sum \lambda_j y_j$ with $\|y_j\|_Y \leq 1$ and $\sum |\lambda_j|^p \leq 1$.

If operator T is onto, then there exists a positive number k such that $Tx_j = y_j$ and $\|x_j\|_X \leq k$ for every j. Set $x := \sum \lambda_j x_j \in X$ and observe that $\|x_j/k\|_X \leq 1$ for every j and $\sum |\lambda_j|^p \leq 1$, yielding $x/k = \sum \lambda_j x_j/k \in W_{X,p}$. Thus, $\|x/k\|_{\mathcal{E}_p(X)} \leq 1$.

By construction, $Tx = y$. In particular, $\|y - \hat{T}x\|_{\mathcal{E}_p(Y)} \leq 1/2$, so that we can invoke Lemma 2.4 from [**77**] to conclude that the operator $\hat{T}$ is onto, as desired.

The proof of other properties of operator T mentioned in the statement of the Theorem is a standard argument, closely following that of Theorem 4 in [**94**]. We omit the details. □

Theorem 4.3.5. *Let E, F be two quasi-Banach spaces whose duals separate points, and fix $0 < p \leq 1$. Suppose that F is a p-Banach space, $E \subseteq F$ and the inclusion $\iota : E \hookrightarrow F$ is continuous with dense range. Then F is p-envelope of E if and only if E has a good approximation of identity, in the sense that here exists a sequence of operators $\{A_m\}_m$, $A_m : E \longrightarrow E$, such that*

$$\|A_m x\|_{\mathcal{E}_p(E)} \leq C\|x\|_F, \text{ uniformly in } m, \text{ for each } x \in E, \tag{4.3.23}$$

and

$$\|A_m x - x\|_E \to 0 \text{ as } m \to \infty, \text{ for each fixed } x \in E. \tag{4.3.24}$$

Proof. The "only if" part of the criterion is trivial, for approximation of identity is simply realized by the identity operator.

Turning to the "if" part, observe that by Proposition 4.3.2, the inclusion ι :

$E \hookrightarrow F$ extends to a bounded, linear operator

$$\hat{\imath} : \mathcal{E}_p(E) \longrightarrow F \tag{4.3.25}$$

Our goal is to show that this extension is, in fact, an isomorphism.

A moment's reflection shows that to prove that this operator is one-to-one it is enough to verify that

$$x_j \in E,\ \{x_j\}_j \text{ Cauchy in } \mathcal{E}_p(E),\ x_j \to 0 \text{ in } F \Rightarrow x_j \to 0 \text{ in } \mathcal{E}_p(E). \tag{4.3.26}$$

Granted this, and recalling the approximate identity operators A_m, we write

$$\|x_j\|_{\mathcal{E}_p(E)} \leq C\|x_j - A_m x_j\|_{\mathcal{E}_p(E)} + C\|A_m x_j\|_{\mathcal{E}_p(E)}. \tag{4.3.27}$$

Under the current hypotheses on $\{x_j\}_j$ and $\{A_m\}_m$, given any $\varepsilon > 0$ there exists j_ε such that $C\|x_{j_\varepsilon}\|_F \leq \varepsilon$ and $C\|x_{j_\varepsilon} - A_m x_{j_\varepsilon}\|_E \leq \varepsilon/2$ provided m is large enough. Thus, $\|A_m x_{j_\varepsilon}\|_{\mathcal{E}_p(E)} \leq C\|x_{j_\varepsilon}\|_F \leq \varepsilon/2$ and $C\|A_m x_{j_\varepsilon}\|_{\mathcal{E}_p(E)} \leq C\|A_m x_{j_\varepsilon}\|_E \leq \varepsilon/2$. Hence, by (4.3.24), $\|x_{j_\varepsilon}\|_{\mathcal{E}_p(E)} \leq \varepsilon$. This proves that $\{x_j\}_j$ contains a subsequence which is convergent to zero in $\mathcal{E}_p(E)$. Being Cauchy, then the entire sequence converges to zero in $\mathcal{E}_p(E)$, as desired.

Turning to the ontoness of (4.3.25), observe that by current hypotheses for every $x \in E$ and every $m \in \mathbb{N}$

$$\|x\|_{\mathcal{E}_p(E)} \leq C\|A_m x - x\|_{\mathcal{E}_p(E)} + C\|A_m x\|_{\mathcal{E}_p(E)} \leq C\|A_m x - x\|_E + C\|x\|_F. \tag{4.3.28}$$

Since m can be chosen arbitrary large, this forces $\|x\|_{\mathcal{E}_p(E)} \leq C\|x\|_F$ for every $x \in E$ and hence for some constant $c > 0$

$$E \cap B_F(0,c) \subset E \cap B_{\mathcal{E}_p(E)}(0,1) \subset \hat{\imath}(E \cap B_{\mathcal{E}_p(E)}(0,1)). \tag{4.3.29}$$

But E is densely embedded into the space F and therefore,

$$B_F(0,c) \subset \overline{\hat{\imath}(B_{\mathcal{E}_p(E)}(0,1))}^{\|\cdot\|_F}. \tag{4.3.30}$$

Since the set on the left-hand side of (4.3.30) is a neighborhood of 0 in F, it follows from the Open Mapping Theorem on p. 9 in [**78**] that $\hat{\imath}$ in (4.3.25) is onto. □

Theorem 4.3.6. *Let X_i, $i = 1, 2$, be arbitrary quasi-Banach spaces and Y_i, $i = 1, 2$, be p-Banach spaces, $0 < p \leq 1$, so that the inclusions $X_i \hookrightarrow Y_i$ are well-defined and continuous. Next, assume that $T \in \mathcal{L}(Y_1, Y_2)$ is well-defined, continuous and has a continuous inverse to the right as well as its restriction $T|_{X_1} \in \mathcal{L}(X_1, X_2)$.*

Then, if $\mathcal{E}_p(X_1) = Y_1$, we also have $\mathcal{E}_p(X_2) = Y_2$.

Proof. Owing to the functorial properties of p-envelopes investigated above, in particular, Proposition 4.3.2, operators $T : X_1 \to X_2$ and $R : X_2 \to X_1$ extend to linear bounded operators $\hat{T} : \mathcal{E}_p(X_1) \to \mathcal{E}_p(X_2)$ and $\hat{R} : \mathcal{E}_p(X_2) \to \mathcal{E}_p(X_1)$, respectively. Consider an operator $T \circ \hat{R} : \mathcal{E}_p(X_2) \to Y_2$. It is not hard to see that $T \circ \hat{R}|_{X_2} = T \circ R$ is identity on X_2. Since X_2 is densely embedded into $\mathcal{E}_p(X_2)$, one can conclude by continuity that $T \circ \hat{R} : \mathcal{E}_p(X_2) \to Y_2$ is a continuous inclusion. The similar reasoning applies to operator $\hat{T} \circ R : Y_2 \to \mathcal{E}_p(X_2)$ – identity on X_2 and hence, a bounded inclusion from Y_2 to $\mathcal{E}_p(X_2)$. Finally, the combination of the last two statements allows to deduce that $Y_2 = \mathcal{E}_p(X_2)$. □

Lemma 4.3.7. *Let E, F be two quasi-Banach spaces whose duals separate points, and fix $0 < p \leq 1$. If $F = \mathcal{E}_p(E)$ then the closure in F of the absolutely p-convex hull of $B_E(0,1)$, the unit ball of E, is a neighborhood of origin in F.*

Proof. The proof relies on the observation that

$$\|x\|_F < 1 \Rightarrow \exists\, \lambda \in (0,1) \text{ such that } x/\lambda \in W \Rightarrow x \in \lambda W \subseteq W \subseteq \overline{W}^{\|\cdot\|_F}, \quad (4.3.31)$$

where W denotes the absolutely p-convex hull of $B_E(0,1)$. □

Theorem 4.3.8. *Suppose E and F are quasi-Banach space, F is p-Banach, $p \leq 1$, and the inclusion $\iota : E \to F$ is continuous with the dense range. Then $\mathcal{E}_p(E) = F$ if and only if for every p-Banach space Z every linear bounded operator $S : E \to Z$ factors through F, i.e. there exists a linear bounded operator $\tilde{S} : F \to Z$ such that $S = \tilde{S} \circ \iota$.*

Proof. Assume that $\mathcal{E}_p(E) = F$ and recall that Z is p-Banach, hence $\mathcal{E}_p(Z) = Z$. Then the linear bounded operator S extends by Proposition 4.3.2 to $\tilde{S} : \mathcal{E}_p(E) = F \to \mathcal{E}_p(Z) = Z$, as desired.

As for the converse, take $Z = \mathcal{E}_p(E)$ and $S : E \to \mathcal{E}_p(E)$ to be the inclusion. Since $\tilde{S} : F \to \mathcal{E}_p(E)$ is continuous, we deduce that $\|\tilde{S}\|_{\mathcal{E}_p(E)} \leq C\|x\|_F$ for every $x \in F$ and in particular, for every $x \in E$. But $S = \tilde{S} \circ \iota$ is an inclusion and therefore for every $x \in E$ we obtain $\tilde{S}x = x$. According to Theorem 4.3.5 this yields $\mathcal{E}_p(E) = F$ taking $A_m \equiv \tilde{S}$. □

Corollary 4.3.9. *Suppose E is a quasi-Banach space and $0 < p < p^* \leq 1$. Then $\mathcal{E}_{p^*}(\mathcal{E}_p(E)) = \mathcal{E}_{p^*}(E)$.*

Proof. To start observe that every p^*-Banach space is also p-Banach for $p < p^*$ in concert with our definitions.

Resting on the theorem above, for every p^*-Banach space Z and every linear bounded operator $S : E \to Z$ there exists a linear bounded operator $S_1 : \mathcal{E}_p(E) \to$

Z such that $S = \iota_1 \circ S_1$, where ι_1 is the inclusion of E into $\mathcal{E}_p(E)$. Further, there exists $S_2 : \mathcal{E}_{p^*}(\mathcal{E}_p(E)) \to Z$ such that $S_1 = \iota_2 S_2$ with the inclusion operator $\iota_2 : \mathcal{E}_p(E) \to \mathcal{E}_{p^*}(\mathcal{E}_p(E))$. Then the assumptions of Theorem 4.3.8 are fulfilled with $\tilde{S} := S_2$ since $\iota_1 \circ \iota_2 : E \to \mathcal{E}_{p^*}(\mathcal{E}_p(E))$ is also the continuous inclusion. □

4.3.2 The p-envelopes of Besov and Triebel-Lizorkin spaces

Theorem 4.3.10. *Assume that* $0 < p^* \le 1$ *and* $s^* := s + n(1/p^* - 1/p)$. *Then*

$$\mathcal{E}_{p^*}(B^{p,q}_s(\mathbb{R}^n)) = B^{p^*,p^*}_{s^*}(\mathbb{R}^n) \quad \textit{and} \quad \mathcal{E}_{p^*}(F^{p,q}_s(\mathbb{R}^n)) = B^{p^*,p^*}_{s^*}(\mathbb{R}^n), \tag{4.3.32}$$

if $0 < p, q \le p^* \le 1$*;*

$$\mathcal{E}_{p^*}(F^{p,q}_s(\mathbb{R}^n)) = B^{p^*,p^*}_{s^*}(\mathbb{R}^n), \tag{4.3.33}$$

if $0 < p < p^* \le 1$, $p^* \le q < \infty$*;*

$$\mathcal{E}_{p^*}(B^{p,q}_s(\mathbb{R}^n)) = B^{p,p^*}_{s}(\mathbb{R}^n), \tag{4.3.34}$$

if $0 < q < p^* \le 1$, $p^* \le p < \infty$.

Proof. Since all necessary inclusions are satisfied by classical embedding theorems we can turn to the approximate identity property.

Given a real-valued function φ and, for $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, a dyadic cube $Q = Q_{\nu,k}$,

$$Q_{\nu,k} := \Big\{ x \in \mathbb{R}^n : \, 2^{-\nu} k_i \le x_i \le 2^{-\nu}(k_i + 1), \; 1 \le i \le n \Big\}, \tag{4.3.35}$$

we set, as usual, $\varphi_Q(x) := 2^{\nu n/2} \varphi(2^\nu x - k)$.

Let $\{\varphi^\ell\}_\ell$ be a family of "mother wavelets" and, for each $m \in \mathbb{N}$, assign

$$A_m f := \sum_{\ell=1}^{2^n - 1} \sum_{|\nu| \le m} \sum_{\operatorname{dist}(Q,0) \le m} \langle f, \varphi^\ell_Q \rangle \varphi^\ell_Q, \tag{4.3.36}$$

i.e. a truncated wavelet expansion of f.

According to [**47**], [**49**],

$$\|f\|_{B^{p^*,p^*}_{s^*}(\mathbb{R}^n)} \approx \sum_{\ell=1}^{2^n-1} \Big\| \{\tilde{f}^\ell_Q\}_Q \Big\|_{\ell^{p^*}} \tag{4.3.37}$$

where

$$\tilde{f}^\ell_Q := |Q|^{1/p-1/2} \langle f, \varphi^\ell_Q \rangle. \tag{4.3.38}$$

Also,

$$\tilde{\varphi}^\ell_Q := |Q|^{1/2-1/p} \varphi^\ell_Q \tag{4.3.39}$$

is an atom for the spaces $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ for all values of q (provided the family of wavelets was a priory chosen to have a suitable number of vanishing moments), and

$$A_m f = \sum_{\ell=1}^{2^n-1} \sum_{|\nu|\le m} \sum_{\text{dist}(Q,0)\le m} \tilde{f}^\ell_Q \tilde{\varphi}^\ell_Q. \tag{4.3.40}$$

Consequently, if $\lambda := \sum_{\ell=1}^{2^n-1} \|\{\tilde{f}^\ell_Q\}_Q\|_{\ell^{p^*}}$, then $A_m f/\lambda$ belongs to the absolute p^*-convex hull of the unit ball for all $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$, $0 < q \le \infty$, and

$$\|A_m f\|_{\mathcal{E}_{p^*}(B^{p,q}_s(\mathbb{R}^n))} \le \lambda \approx \|f\|_{B^{p^*,p^*}_{s^*}(\mathbb{R}^n)}, \tag{4.3.41}$$

$$\|A_m f\|_{\mathcal{E}_{p^*}(F^{p,q}_s(\mathbb{R}^n))} \le \lambda \approx \|f\|_{B^{p^*,p^*}_{s^*}(\mathbb{R}^n)}, \tag{4.3.42}$$

uniformly in m. This justifies (4.3.23) in the present context.

Since it is well known that

$$A_m f \longrightarrow f \text{ as } m \to \infty, \text{ in } B^{p,q}_s(\mathbb{R}^n) \quad \text{and} \quad F^{p,q}_s(\mathbb{R}^n), \tag{4.3.43}$$

(cf., e.g., Theorem 7.20 in [**50**]), it follows that the analogue of (4.3.24) also holds.

The proof of (4.3.32) and (4.3.33) is therefore finished.

Turning to (4.3.34), we follow the same pattern to get expansion (4.3.40) and this time

$$\|f\|_{B_s^{p,p^*}(\mathbb{R}^n)} \approx \sum_{\ell=1}^{2^n-1} \Big\| \{\tilde{f}_Q^\ell\}_Q \Big\|_{b^{p,p^*}} = \sum_{\ell=1}^{2^n-1} \left(\sum_{j=0}^{\infty} \Big(\sum_{Q:\, l(Q)=2^{-j}} |\tilde{f}_Q^\ell|^p \Big)^{p^*/p} \right)^{1/p^*}. \quad (4.3.44)$$

Going further, we can rewrite (4.3.40) in the form

$$A_m f = \sum_{\ell=1}^{2^n-1} \sum_{|\nu|\le m} \left(\sum_{\mathrm{dist}(Q,0)\le m} (\tilde{f}_Q^\ell)^p \right)^{1/p} \left[\sum_{\mathrm{dist}(Q,0)\le m} \left(\sum_{\mathrm{dist}(Q,0)\le m} (\tilde{f}_Q^\ell)^p \right)^{-1/p} \tilde{f}_Q^\ell \tilde{\varphi}_Q^\ell \right]. \quad (4.3.45)$$

Now it is not hard to see that the functions in the brackets are elements from the unit ball of $B_s^{p,q}(\mathbb{R}^n)$ and with

$$\lambda := \sum_{\ell=1}^{2^n-1} \Big\| \{\tilde{f}_Q^\ell\}_Q \Big\|_{b^{p,p^*}} = \sum_{\ell=1}^{2^n-1} \Big\| \Big\{ (\tilde{f}_Q^\ell)^p \Big)^{1/p} \Big\}_\nu \Big\|_{\ell^{p^*}}, \quad (4.3.46)$$

$A_m f/\lambda$ belongs to the absolute p^*-convex hull of the unit ball in $B_s^{p,q}(\mathbb{R}^n)$ and

$$\|A_m f\|_{\mathcal{E}_{p^*}(B_s^{p,q}(\mathbb{R}^n))} \le \lambda \approx \|f\|_{B_s^{p,p^*}(\mathbb{R}^n)}, \quad (4.3.47)$$

as desired. □

In what follows we study p-envelopes of the Besov and Triebel-Lizorkin spaces on Lipschitz domain.

Theorem 4.3.11. *All formulas for p-envelopes of $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$ spaces remain valid for corresponding spaces on Lipschitz domain, $B_s^{p,q}(\Omega)$, $B_{s,0}^{p,q}(\Omega)$ and $F_s^{p,q}(\Omega)$, $F_{s,0}^{p,q}(\Omega)$, respectively.*

Proof. Let us consider the spaces $B^{p,q}_s(\Omega)$ and $F^{p,q}_s(\Omega)$ first. The result follows from Theorem 4.3.10 thanks to existence of Rychkov's extension operator.

We present the details for the case (4.3.33) of Theorem 4.3.10 only, with the obvious modifications for other situations. Choose $X_1 := F^{p,q}_s(\mathbb{R}^n)$ and $Y_1 := \mathcal{E}_{p^*}(F^{p,q}_s(\mathbb{R}^n)) = B^{p^*,p^*}_{s^*}(\mathbb{R}^n)$. Similarly, $X_2 = F^{p,q}_s(\Omega)$ and $Y_1 := B^{p^*,p^*}_{s^*}(\Omega)$. Then the scheme suggested in Theorem 4.3.6 can be utilized in the present context, invoking the operator $T := \mathcal{R}_\Omega$ and its bounded right inverse $R := E_\Omega$, constructed by Rychkov in [**113**].

As for the spaces $B^{p,q}_{s,0}(\Omega)$ and $F^{p,q}_{s,0}(\Omega)$, we once again turn our attention to Theorem 4.3.6. We pass from the spaces on $\mathbb{R}^n$, namely, $X_1 := F^{p,q}_s(\mathbb{R}^n)$ and $Y_1 := \mathcal{E}_{p^*}(F^{p,q}_s(\mathbb{R}^n)) = B^{p^*,p^*}_{s^*}(\mathbb{R}^n)$ to corresponding spaces on domain, $X_2 := F^{p,q}_{s,0}(\Omega)$ and $Y_2 = B^{p^*,p^*}_{s^*,0}(\Omega)$ via the projection operator $T := I - E_{\mathbb{R}^n\setminus\overline{\Omega}} \circ \mathcal{R}_{\mathbb{R}^n\setminus\overline{\Omega}}$ with the bounded inverse to the right given by the natural inclusion $R := \iota$ acting from the spaces of compactly supported functions to the corresponding spaces on $\mathbb{R}^n$. □

4.3.3 Applications: invertibility of boundary operators

Theorem 4.3.12. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then the operators*

$$\tfrac{1}{2}I + K : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega), \tag{4.3.48}$$

$$-\tfrac{1}{2}I + K^* : \{f \in B^{p,q}_{s-1}(\partial\Omega);\ \langle f, 1\rangle = 0\} \longrightarrow \{f \in B^{p,q}_{s-1}(\partial\Omega);\ \langle f, 1\rangle = 0\}, \tag{4.3.49}$$

are isomorphisms provided the pair of indices $(s, 1/p)$ satisfies one of the conditions in (1.1.5), and $0 < q \leq \infty$.

Proof. The operator $\frac{1}{2}I+K$ is invertible on $B^{p,p}_s(\partial\Omega)$ spaces for all indices satisfying the set of conditions $(II) - (IV)$ from (1.1.5) by the results in [**40**].

Going further, in Proposition 3.5.8 it was shown that K is a bounded mapping on $h^p_1(\partial\Omega)$ for each $(n-1)/n < p \leq 1$. Based on the invertibility of $\frac{1}{2}I + K$ on $h^1_1(\partial\Omega)$ (established in [**31**]) and the perturbation argument developed in [**77**], it follows that this operator is also invertible on $h^p_1(\partial\Omega)$ for $1-\varepsilon < p \leq 1$, for some small $\varepsilon = \varepsilon(\partial\Omega) > 0$.

Observe that, in terms of the Triebel-Lizorkin scale, the results mentioned above amount to the invertibility of $\frac{1}{2}I + K$ on $F^{p,2}_1(\partial\Omega)$ for $p \in (1-\varepsilon, 1]$. Now we can take the envelopes of $F^{p,2}_1(\partial\Omega)$ spaces to extend this result to all spaces $B^{p,p}_s(\partial\Omega)$ with p and s satisfying conditions (I) in (1.1.5).

At this stage, the real method of interpolation allows us to cover the full range of $0 < q \leq \infty$.

Concerning the invertibility of the operator $-\frac{1}{2}I + K^*$, an argument very similar in spirit to that utilized above works; the only difference is that, this time, we start with the invertibility of $-\frac{1}{2}I + K^*$ on

$$L^p(\partial\Omega)_c = F^{p,2}_0(\partial\Omega)_c, \qquad 1 < p < 2+\varepsilon,$$

$$L^p_{-1}(\partial\Omega)_c := \Big(L^{p'}_1(\partial\Omega)/\mathbb{R}\Big)^* = F^{p,2}_{-1}(\partial\Omega)_c, \qquad 2-\varepsilon < p < \infty,\ 1/p + 1/p' = 1,$$

$$h^p(\partial\Omega)_c = F^{p,2}_0(\partial\Omega)_c, \qquad 1-\varepsilon < p \leq 1.$$

Above, for a space $X \subset (\mathrm{Lip}(\partial\Omega))^*$, we set $X_c := \{f \in X : \langle f, 1\rangle = 0\}$. We omit routine details. □

4.4 Poisson boundary value problems

Proof of Theorem 1.1.5. Consider the following integral representation formula for the solution:

$$u = \mathcal{R}_\Omega \circ \Pi \circ E_\Omega f + \mathcal{D} \circ \left(\frac{1}{2} I + K\right)^{-1} \circ (g - \mathrm{Tr}\,(\mathcal{R}_\Omega \circ \Pi \circ E_\Omega f)) \ \text{ in } \Omega, \tag{4.4.1}$$

where, as before, $\mathcal{R}_\Omega$ and E_Ω stand for restriction and extension operator acting between Lipschitz domain Ω and $\mathbb{R}^n$ (cf. (3.6.11) and (3.6.32)), $\mathcal{D}$ and Π are, respectively, double layer and Newtonian potential (cf.(2.3.3), (2.3.8)) and K is a boundary operator defined in (2.3.6). Then the well-posedness for (1.1.26), (1.1.28) rests on the mapping properties of the layer potentials discussed in §4.1 (see Corollary 4.1.6 and Corollary 3.2.12), invertibility of $\frac{1}{2}I + K$ on $B^{p,q}_s(\partial\Omega)$ (cf. Theorem 4.3.12) and the uniqueness argument presented below.

We claim that the problems (1.1.26) and (1.1.28) with data $f = g = 0$ have a unique solution $u \equiv 0$ for the range of indices discussed in Theorem 1.1.5. Indeed, the case $q = p \geq 1$ was treated in **[68]**. As far as the range $q = p \leq 1$ is concerned, according to the classical embedding theorems, $u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$ necessarily yields $u \in B^{1,1}_{1+\delta}(\Omega)$ granted that $p \leq 1$ and $\frac{1}{p} - \frac{s+1/p}{n} \geq 1 - \frac{1+\delta}{n}$, in particular, for δ close to 1. But according to **[68]** $\Delta u = 0$, $u \in B^{1,1}_{1+\delta}(\Omega)$, $u|_{\partial\Omega} = 0$ guarantee $u \equiv 0$ on Ω. Similarly the uniqueness argument for off-diagonal case of Besov and Triebel Lizorkin spaces ($p \neq q$) can be handled. □

Proof of Theorem 1.1.6. Consider the following integral representation formula for

the solution:

$$u = \mathcal{R}_\Omega \circ \Pi f + \mathcal{S} \circ \left(-\frac{1}{2} I + K^*\right)^{-1} \circ (g - \partial_\nu \Pi f) \ \text{ in } \Omega, \tag{4.4.2}$$

where $\mathcal{S}$ is a single layer potential (cf.(2.3.2)) and K^* denotes an adjoint of K. Now we can summarize the main ingredients necessary for well-posedness of Neumann boundary value problem.

First of all, the existence and estimates follow from the properties of single layer and Newtonian potential (cf. Corollary 4.2.1, Theorem 4.1.5 and Corollary 3.2.12). The invertibility of the operator $-\frac{1}{2}I + K^*$ was proved in Theorem 4.3.12.

As for the uniqueness, arguing as in Theorem 1.1.5, the proof reduces to the uniqueness of solution for Neumann boundary value problem with $u \in B^{1,1}_{1+\delta}(\Omega)$, $\partial_\nu u|_{\partial\Omega} \in B^{1,1}_{\delta-1}(\partial\Omega)$ for some δ close to 1 and satisfying $\delta - 1 + s \leq (n-1)(1-1/p)$, developed in **[97]**. □

Corollary 4.4.1. *For every function u defined on the bounded Lipschitz domain Ω in $\mathbb{R}^n$*

$$u \in \mathbb{H}^p_{s+\frac{1}{p}}(\Omega) \Longleftrightarrow u \in \mathcal{H} \cap F^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{4.4.3}$$

where $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q \leq \infty$ satisfies (1.1.11).

Proof. The proof rests on the observation that every harmonic function can be written in the form

$$u = \mathcal{D}(\operatorname{Tr} u) - \mathcal{S}(\partial_\nu u), \tag{4.4.4}$$

Theorem 3.9.3 and mapping properties of the operators involved (see Corollaries 4.1.6, 4.2.1, Theorems 1.1.3 and 1.1.4). □

Chapter 5

Estimates for Green potentials in non-smooth domains

5.1 Regularity of Green potentials in general Lipschitz domains

Recall from the introduction that $\mathbf{G}_D$, $\mathbf{G}_N$ are the Green operators for the Laplacian equipped with homogeneous Dirichlet and Neumann boundary conditions, respectively. In this section we concentrate on the mapping properties of $\mathbf{G}_D$, $\mathbf{G}_N$ on Lipschitz domains.

Proof of Theorem 1.1.1. The result is a direct consequence of Theorems 1.1.5-1.1.6, given the trace properties discussed in §4.2. □

Remark I. If α, p and q are as in the statement of Theorem 1.1.1, $1 < p < \infty$, then

$$\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2,z}(\Omega), \qquad \mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2,z}(\Omega) \tag{5.1.1}$$

are bounded operators. To see this, recall from **[68]** that $F^{p,2}_{\sigma,z}(\Omega) = \{u \in F^{p,2}_{\sigma}(\Omega) : \operatorname{Tr} u = 0\}$, whenever $1/p < \sigma < 1 + 1/p$ and $1 < p < \infty$ (cf. Proposition 3.3, p. 177 *loc. cit*). With tilde denoting extension by zero in $\mathbb{R}^n \backslash \overline{\Omega}$, we may therefore conclude

that $F^{p,2}_{\alpha}(\Omega) \ni f \mapsto \widetilde{\mathbf{G}_D f} \in F^{p,2}_{\alpha,0}(\Omega)$ is bounded, under the current assumptions made on the indices α, p. Then the claim made about the operators (5.1.1) follows from this and interpolation.

Next, recall that the Triebel-Lizorkin scale $F^{p,q}_{\alpha}(\mathbb{R}^n)$ reduces to the *local* Hardy class $h^p(\mathbb{R}^n)$ if $0 < p < \infty$, $q = 2$, $\alpha = 0$ and the latter coincides with the Lebesgue class $L^p(\mathbb{R}^n)$ if $1 < p < \infty$. As before for $0 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ Lipschitz domain, we set $h^p(\Omega) := F^{p,2}_0(\Omega)$ and $h^p_0(\Omega) := F^{p,2}_{0,0}(\Omega)$ (so that $h^p(\Omega) = h^p_0(\Omega) = L^p(\Omega)$ if $1 < p < \infty$). These Hardy spaces on domains have been previously introduced and studied in **[7]**, **[21]**, **[22]**, **[23]**, **[70]**, and **[100]**. Finally, let ∇^k, $k = 0, 1, 2, ...$, stand for generic partial derivatives of order k.

Corollary 5.1.1. *Retain the notation and hypotheses made in Theorem 1.1.1 on the indices involved and the domain Ω. Then*

$$\nabla^2\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha}(\Omega), \qquad \nabla^2\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_{\alpha}(\Omega) \tag{5.1.2}$$

if $0 < q \leq \infty$, and

$$\nabla^2\mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha}(\Omega), \qquad \nabla^2\mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_{\alpha}(\Omega) \tag{5.1.3}$$

if (1.1.11) holds and $p \neq \infty$, are well-defined, bounded operators.

In particular, there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that the operators

$$\nabla^2\mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \quad \text{whenever } \tfrac{n}{n+\varepsilon} < p < 1, \tag{5.1.4}$$

$$\nabla^2\mathbf{G}_N : h^p_0(\Omega) \longrightarrow h^p(\Omega), \quad \text{whenever } \tfrac{n}{n+\varepsilon} < p < 1, \tag{5.1.5}$$

are well-defined and bounded. Once again, one can take $\varepsilon = 1$ whenever Ω has a C^1 boundary, in which case the range of p's in (5.1.4)-(5.1.5) becomes $\frac{n}{n+1} < p < 1$.

The second part of the corollary – including the very last statement about C^1 domains – is an extension of a result proved by D.-C. Chang, S.G. Krantz and E.M. Stein in [**21**] for domains Ω with a C^∞ boundary. Their method invokes a reflection procedure across hyperplanes tangent to the boundary and, as such, makes essential use of the smoothness of $\partial\Omega$ (the differentiability properties of the reflection function closely mirror those of the boundary of the domain itself). Let us also mention that an alternative proof of (5.1.4) in C^∞ domains, based on the Boutet de Monvel calculus of pseudodifferential operators, is in [**69**]; cf. the discussion on p. 32 *loc. cit.*

That one cannot allow $p \leq \frac{n}{n+1}$ even if the domain is smooth has already been observed in [**23**]; see [**23**] for the appropriate function spaces which permit such an extension in smooth domains. The condition $p < 1$ is also optimal. Indeed, in [**68**], D. Jerison and C. Kenig have constructed an example of a bounded domain Ω with C^1 boundary and a function $f \in C_c^\infty(\Omega)$ such that $\nabla^2 \mathbf{G}_D f \notin L^1(\Omega)$, which sharpens a counterexample due to B. Dahlberg (cf. [**29**]). In particular, for this domain,

$$\nabla^2 \mathbf{G}_D : h^1(\Omega) \longrightarrow h^1(\Omega) \tag{5.1.6}$$

fails and, hence, so does

$$\nabla^2 \mathbf{G}_D : L^p(\Omega) \longrightarrow L^p(\Omega) \tag{5.1.7}$$

for each $1 < p < \infty$ (otherwise this and (5.1.4) would imply (5.1.6) via complex interpolation). As mentioned in §1.1, (5.1.4)-(5.1.5) answer a question asked by D.-C. Chang, S. Krantz and E. Stein in [**21**], [**22**]. Our solution is very much in

line with the comment made on p. 130 of [**22**] to the effect that "*The literature for the Dirichlet and Neumann problems for domains with Lipschitz boundaries (see* [**79**]*) teaches us that when the boundary is only Lipschitz then one can expect favorable behavior for a restricted range of p. It would be of interest to explore similar phenomena vis à vis to the Hardy spaces introduced here.*"

Remark II. Let $G_D(x,y)$ denote the integral kernel of the operator $\mathbf{G}_D$ in the domain Ω, that is, $\mathbf{G}_D f(x) = \int_\Omega G_D(x,y) f(y)\,dy$. Then the claims made in Corollary 5.1.1 about $\nabla^2\mathbf{G}_D$ correspond to mapping properties for the application $u \mapsto \int_\Omega \nabla_x \nabla_x G_D(x,y) f(y)\,dy$. A closely related object is the operator $\mathbf{J}f(x) := \int_\Omega \nabla_x \nabla_y G_D(x,y) f(y)\,dy$, $x \in \Omega$, considered, e.g., in [**51**] and [**67**]. This, in turn, is intimately connected with the solution operator for the problem

$$-\Delta\, u = \operatorname{div} \vec{f} \ \text{ in } \Omega, \ \ u|_{\partial\Omega} = 0. \tag{5.1.8}$$

Since $\mathbf{J} = -\nabla\mathbf{G}_D \circ \operatorname{div}$, all the results pertaining to $\mathbf{G}_D$ in Theorem 1.1.1 can be translated in terms of $\mathbf{J}$.

Remark III. More can be said when extra geometrical information about the domain Ω is available. For example, it is well known that

$$\nabla^2\mathbf{G}_D : F^{2,2}_\alpha(\Omega) \longrightarrow F^{2,2}_\alpha(\Omega) \quad \text{for } -1 < \alpha < \tfrac{\pi}{\omega} - 1, \tag{5.1.9}$$

where $\Omega \subset \mathbb{R}^2$ is a *nonconvex* polygonal domain with $\omega \in (\pi, 2\pi)$ denoting the measure of its largest re-entrant corner. Note that this allows the inclusion of a point with coordinates $(\alpha_o, \frac{1}{2})$, for some $\alpha_o > -\frac{1}{2}$, alongside the hexagon depicted in Figure 1.1. Accordingly, interpolating between (5.1.9) and (5.1.3) shows that (5.1.6) does hold in this particular case. Similar results hold for $\nabla^2\mathbf{G}_N$.

Another important subclass of Lipschitz domains is discussed in the section below.

5.2 The case of convex domains

According to an old result, proved by J. Kadlec [**72**] in the 60's (cf. also [**121**]),

$$\nabla^2 \mathbf{G}_D : L^2(\Omega) \longrightarrow L^2(\Omega) \tag{5.2.1}$$

is well-defined and bounded whenever Ω is a bounded convex domain in $\mathbb{R}^n$. The relatively straightforward method of proof (involving successive integrations by parts) makes essential use of the Hilbert character of L^2 and does not extend to L^p with $p \neq 2$.

In the 90's, B. Dahlberg, G. Verchota and T. Wolff used this L^2-result and elements of the Calderón-Zygmund theory to derive the weak-type estimate:

$$\nabla^2 \mathbf{G}_D : L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega). \tag{5.2.2}$$

See [**2**], [**3**] and [**51**], for a discussion. In turn, this can be used to conclude that

$$\nabla^2 \mathbf{G}_D : L^p(\Omega) \longrightarrow L^p(\Omega), \qquad 1 < p \leq 2, \tag{5.2.3}$$

is bounded, by interpolation with Kadlec's L^2-result via the real method. The weak-$(1,1)$ and L^p boundedness properties of the second derivatives of the Green potential were reproved by S. Fromm ([**51**]) and have been utilized in the sequel to establish higher regularity properties of solutions in convex domains (cf. also [**52**] in this regard).

Another proof of (5.2.2) was given by V. Adolfsson in [**2**] where he established atomic estimates amounting to

$$\nabla^2 \mathbf{G}_D : h^1(\Omega) \longrightarrow L^1(\Omega), \tag{5.2.4}$$

and obtained the L^p result (5.2.3) by interpolating between (5.2.1) and (5.2.4) via the complex method. The atomic estimate alluded to above was obtained by relying on the L^2 theory and the asymptotics at infinity for null-solutions of elliptic PDE's with L^∞ coefficients due to J. Serrin and H. Weinberger (an idea pioneered by B. Dahlberg and C. Kenig in [**31**]).

The analogue of (5.2.1) in the case of Neumann boundary conditions has been known since the mid 70's (cf. [**59**], [**56**]), but the question of establishing L^p estimates has only been resolved in the 90's by V. Adolfsson and D. Jerison. In [**3**], they proved that for any bounded convex domain Ω in $\mathbb{R}^n$,

$$\nabla^2 \mathbf{G}_N : h^1_0(\Omega) \longrightarrow L^1(\Omega), \tag{5.2.5}$$

$$\nabla^2 \mathbf{G}_N : L^p(\Omega) \longrightarrow L^p(\Omega), \quad 1 < p \leq 2. \tag{5.2.6}$$

Once again, their proof of (5.2.5) made essential use of the deep regularity theory for null-solutions of elliptic equations with bounded, measurable coefficients. Adolfsson and Jerison then proved (5.2.6) based on (5.2.5) and a non-standard interpolation method (working at the level of dual spaces). They also proved that the end-point $p = 2$ is sharp.

As a result of our Hardy space estimate with $p < 1$ we can give new, conceptually simple, proofs to all of the above results, as well as several new end-point estimates. Recall the definitions and main properties of local weak-Hardy

spaces outlined in §3.2. As is well-known, while the spaces $h^{p,\infty}(\mathbb{R}^n)$ coincide with $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$, these two scales are not comparable when $0 < p \leq 1$. For nice functions f in $\mathbb{R}^n$ it is nonetheless true that

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C_n \|f\|_{h^{1,\infty}(\mathbb{R}^n)}. \tag{5.2.7}$$

See the discussion in [**42**]. It is also more or less folklore that

$$h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n) \hookrightarrow h^{1,\infty}(\mathbb{R}^n). \tag{5.2.8}$$

For an arbitrary Lipschitz domain Ω in $\mathbb{R}^n$ and $0 < p < \infty$ we set

$$h^{p,\infty}(\Omega) := \{u \in \mathcal{D}'(\Omega) : \exists\, v \in h^{p,\infty}(\mathbb{R}^n) \text{ such that } v|_\Omega = u\}, \tag{5.2.9}$$

equipped with the natural (quasi-)norm. The important remark we want to make here is as follows. Owing to the existence of a universal extension operator $E_\Omega : F^{p,q}_\alpha(\Omega) \to F^{p,q}_\alpha(\mathbb{R}^n)$ (cf. (3.6.32)), it is straightforward to check that the analogs of (5.2.11)-(5.2.8) continue to hold for spaces defined in Lipschitz domains.

It is interesting to observe that the L^2 boundedness of $\nabla^2\mathbf{G}_D$ and $\nabla^2\mathbf{G}_N$ is available for domains that are not necessarily convex, but satisfy a uniform outer ball condition (cf. [**1**], [**96**]), so that the reasoning above can be generalized.

Theorem 5.2.1. *Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ satisfying a uniform outer ball condition (for instance, convex or with a $C^{1,1}$ boundary). Then in addition to the results of Theorem 1.1.1 (with ε retaining the same significance)*

the following operators

$$\nabla^2\mathbf{G}_D : h^1(\Omega) \longrightarrow h^1(\Omega), \tag{5.2.10}$$

$$\nabla^2\mathbf{G}_D : h^{p,\infty}(\Omega) \longrightarrow h^{p,\infty}(\Omega), \ \tfrac{n}{n+\varepsilon} < p \leq 1, \tag{5.2.11}$$

$$\nabla^2\mathbf{G}_N : h^1_0(\Omega) \longrightarrow h^1(\Omega), \tag{5.2.12}$$

$$\nabla^2\mathbf{G}_N : L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega) \tag{5.2.13}$$

are bounded. In particular, $\mathbf{G}_D : h^1(\Omega) \longrightarrow C^o(\overline{\Omega})$ *and* $\mathbf{G}_N : h^1_0(\Omega) \longrightarrow C^o(\overline{\Omega})$ *for* $n = 2$.

As will become apparent shortly, the above results contain and strengthen (5.2.2)-(5.2.6).

Indeed, (5.2.10)-(5.2.11) follow, respectively, by complex and real interpolation between (5.1.4)-(5.1.5) and the L^2-result (5.2.1). In turn, (5.2.10) immediately gives (5.2.4) since $h^1(\Omega) \hookrightarrow L^1(\Omega)$, while (5.2.11), with $p = 1$, in concert with (the domain versions of) (5.2.7)-(5.2.8) yields

$$\|\nabla^2\mathbf{G}_D f\|_{L^{1,\infty}(\Omega)} \leq C\|\nabla^2\mathbf{G}_D f\|_{h^{1,\infty}(\Omega)} \leq C\|f\|_{h^{1,\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}, \tag{5.2.14}$$

uniformly, for reasonable functions f. A density argument then gives (5.2.2). Complex interpolation between (5.2.1) and (5.2.10) implies (5.2.3). To justify (5.2.12), we interpolate between $h^p_0(\Omega)$, with $1 - p > 0$ sufficiently small, and the L^2-level, in order to allow $p = 1$ as well.

Let us also point out that (5.2.12) covers (5.2.5), via a trivial inclusion, and that an interpolation result similar in spirit to the one just carried out above readily yields (5.2.6).

Finally, (5.2.13) is proved analogously, by relying on the real method of interpolation and by observing that $L^1(\Omega) \hookrightarrow \{u \in h^{1,\infty}(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\Omega}\}$.

5.3 Estimates for the gradient of Green potential

The classical $L^p - L^q$ estimates for Green potentials in the half-space have been extended to Lipschitz domains by B. Dahlberg in [**29**], where he proved that

$$\nabla \mathbf{G}_D : L^p(\Omega) \to L^q(\Omega), \qquad 1 < p < p_n + \varepsilon, \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{n}, \tag{5.3.1}$$

$$\nabla \mathbf{G}_D : L^1(\Omega) \to L^{\frac{n}{n-1},\infty}(\Omega), \tag{5.3.2}$$

where $\varepsilon = \varepsilon(\Omega) > 0$ and $p_n := \frac{3n}{n+3}$ if $n \geq 3$ and $p_n := \frac{4}{3}$ if $n = 2$. Thus, informally, $\mathbf{G}_D$ behaves like a fractional integral operator of order one but only for a more restricted range of indices than what the classical Sobolev-Hardy-Littlewood theorem would normally warrant. In particular, if $G_D(x, y)$ denotes the Green function for Ω, then, as opposed to the case when $\partial\Omega$ is smooth, the estimate $|\nabla_x G_D(x, y)| \leq C|x - y|^{-(n-1)}$, $x, y \in \Omega$, may fail for general Lipschitz domains.

A new proof of (5.3.1) was given in [**68**] based on a suitable functional analytic calculus for the Dirichlet Laplacian developed by the authors. Since both the Jerison-Kenig solution as well as Dahlberg's original argument ultimately depend on the maximum principle and on positivity, the results in these papers are restricted to the case of Dirichlet boundary conditions. The methods of the current paper allow for a uniform approach, which also yields stronger results. Recall that the Hardy space h^p is isomorphic to L^p whenever $p > 1$. A collection of results then reads as follows.

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that if $p(n,\varepsilon) := \left(\frac{n+3}{3n} - \frac{\varepsilon}{3}\right)^{-1}$ for $n \geq 3$, and $p(n,\varepsilon) = \frac{4}{3-\varepsilon}$ for $n = 2$, then for every $\frac{n}{n+\varepsilon} < p < p(n,\varepsilon)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, the operators*

$$\nabla \mathbf{G}_D : h^p(\Omega) \longrightarrow L^q(\Omega), \qquad \nabla \mathbf{G}_N : h^p_0(\Omega) \longrightarrow L^q(\Omega), \tag{5.3.3}$$

$$\nabla \mathbf{G}_D : L^1(\Omega) \longrightarrow L^{\frac{n}{n-1},\infty}(\Omega), \qquad \nabla \mathbf{G}_N : L^1(\Omega) \longrightarrow L^{\frac{n}{n-1},\infty}(\Omega), \tag{5.3.4}$$

are well-defined and bounded. In particular, (compare with (5.3.2))

$$\nabla \mathbf{G}_D : h^1(\Omega) \longrightarrow L^{\frac{n}{n-1}}(\Omega), \qquad \nabla \mathbf{G}_N : h^1_0(\Omega) \longrightarrow L^{\frac{n}{n-1}}(\Omega) \tag{5.3.5}$$

are bounded operators. The ranges of indices are optimal in the class of Lipschitz domains. For $\partial\Omega \in C^1$ one may take $\varepsilon = 1$.

To see this, for $n \geq 3$, $1 < p < \infty$, $s \in \mathbb{R}$, consider the first operator in (5.3.3). Let $f \in L^p(\Omega)$ and construct $w \in L^p_2(\Omega)$ such that $\Delta w = f$ and $\|w\|_{L^p_2(\Omega)} \leq C\|f\|_{L^p(\Omega)}$ (for instance, take w to be the restriction to Ω of the Newtonian potential of f, extended by zero to $\mathbb{R}^n$). Then $\mathbf{G}_D f = w - u$, where u is the solution of $\Delta u = 0$ in Ω, $\operatorname{Tr} u = \operatorname{Tr} w$ on $\partial\Omega$. Note that $w \in L^p_2(\Omega) \hookrightarrow L^q_1(\Omega)$ if $1/q = 1/p - 1/n$ and that, accordingly, $\operatorname{Tr} w \in B^{q,q}_{1-1/q}(\partial\Omega)$. Theorem 1.1.5 then implies that $u \in L^q_1(\Omega)$ as well, granted that the point with coordinates $(1-1/q, q)$ belongs to the hexagon in Figure 1.2. A simple analysis shows that this is always the case whenever $\frac{3}{2+\varepsilon} < q < \frac{3}{1-\varepsilon}$, for some $\varepsilon = \varepsilon(\Omega) > 0$. The bottom line is that

$$f \in L^p(\Omega) \Longrightarrow \mathbf{G}_D f \in L^q_1(\Omega) \quad \text{if} \quad \tfrac{3}{2+\varepsilon} < q < \tfrac{3}{1-\varepsilon}, \ \ \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{n}. \tag{5.3.6}$$

Next, (1.1.9) and classical embeddings readily give

$$\mathbf{G}_D : h^p(\Omega) \longrightarrow L^{p^*}_{1,z}(\Omega) \quad \text{if} \quad \tfrac{n}{n+\varepsilon} < p < 1, \ \ \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n}. \tag{5.3.7}$$

Interpolating by the complex method between (5.3.6) and (5.3.7) then yields (5.3.3) in full, as long as $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and $q < \frac{3}{1-\varepsilon}$. Note that the last condition precisely amounts to asking that $1 < p < p(n, \varepsilon)$, for some $\varepsilon \in (0, 1]$, as desired.

The claim about the first operators in (5.3.5), (5.3.4) are also immediate consequences of (5.3.6), (5.3.7) and interpolation by the complex and the real method, respectively. The reasoning for $\mathbf{G}_N$ is very similar.

Two remarks are in order here. First, the Hardy estimates from §5.1 have once again played a major part in the proof. Second, the same circle of ideas can be used to prove similar results for the scales of Besov and Triebel-Lizorkin as well. We omit the details. Instead, we record a few consequences of our results and the duality theorems from [**20**].

Corollary 5.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that if $n < p < \frac{n}{1-\varepsilon}$ and $\alpha = 1 - \frac{n}{p}$, then the operators*

$$\mathbf{G}_D : L^p_{-1}(\Omega) \longrightarrow C^\alpha_z(\Omega), \qquad \mathbf{G}_N : L^p_{-1,0}(\Omega) \longrightarrow C^\alpha(\overline{\Omega}) \tag{5.3.8}$$

are well-defined and bounded. If $\partial\Omega \in C^1$ one can take $\varepsilon = 1$.

Furthermore, if Ω satisfies a uniform exterior ball condition, then the operators

$$\mathbf{G}_D : L^n_{-1}(\Omega) \longrightarrow \mathrm{bmo}_z(\Omega), \qquad \mathbf{G}_N : L^n_{-1,0}(\Omega) \longrightarrow \mathrm{bmo}(\Omega). \tag{5.3.9}$$

are also bounded.

Indeed, the claim about the operator $\mathbf{G}_D$ in (5.3.8) follows from (5.3.7) and duality, using Theorems 3.8.3, 3.8.4. The other operators are treated in a similar fashion.

Chapter 6

Measuring smoothness of solutions by means of the non-tangential maximal function

6.1 Motivation and statement of main results

The well-posedness of the classical Dirichlet problem for the Laplacian in a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, i.e.

$$\Delta u = 0 \text{ in } \Omega, \qquad u\Big|_{\partial\Omega} = f, \tag{6.1.1}$$

is well-understood at the moment for the datum f in various standard smoothness spaces. More specifically, the case $f \in L^p(\partial\Omega)$ has been treated by B. Dahlberg and C. Kenig in **[27]**, **[31]**, where they have established the estimate

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p(\partial\Omega)} \tag{6.1.2}$$

for the optimal range $2-\varepsilon < p < \infty$, with $\varepsilon = \varepsilon(\partial\Omega)$ (for the case of C^1 domains see **[39]**, **[28]**). Hereafter, $\mathcal{N}$ is the classical non-tangential maximal operator defined by

$$\mathcal{N}(u)(x) = \|u\|_{L^\infty(\gamma(x))}, \qquad x \in \partial\Omega, \tag{6.1.3}$$

where $\gamma(x) := \{y \in \Omega;\, |x-y| < \kappa\, \mathrm{dist}\,(y, \partial\Omega)\}$, with $\kappa = \kappa(\Omega) > 1$ fixed, is a cone-like non-tangential approach region with vertex at x.

More smoothness for the boundary data translates into more smoothness for the solution and, in the case when f belongs to the Sobolev space $L^p_1(\partial\Omega)$, G. Verchota [**129**], and B. Dahlberg and C. Kenig [**31**] have shown that

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p_1(\partial\Omega)} \tag{6.1.4}$$

for the optimal (dual) range $1 < p < 2+\varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$.

The well-posedness of (6.1.1) for boundary data in spaces with a fractional amount of smoothness, such as functions in the Besov class $B^{p,p}_s(\partial\Omega)$, has been first studied by D. Jerison and C. Kenig [**68**], where they have proved the *a priori* estimate

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C(\partial\Omega, p, s)\|f\|_{B^{p,p}_s(\partial\Omega)} \tag{6.1.5}$$

for solutions of (6.1.1), whenever $0 < s < 1$ and $1 \leq p \leq \infty$ satisfy either of the following three conditions:

$$\begin{aligned}
&\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1;\\
&1 \leq p < \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1;\\
&\tfrac{2}{1-\varepsilon} < p \leq \infty \quad \text{and} \quad 0 < s < \tfrac{2}{p} + \varepsilon.
\end{aligned} \tag{6.1.6}$$

A more geometric way of understanding the conditions (6.1.6) is to identify the space $B^{p,p}_s(\partial\Omega)$ with the point in $\mathbb{R}^2$ with coordinates $(s, 1/p)$. In this scenario, the validity region for (6.1.5) becomes the hexagonal region below:

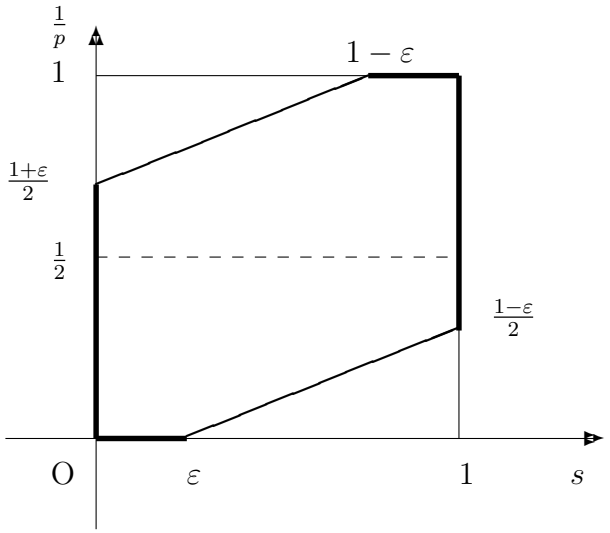

Figure 6.1.

Though exquisite in their elegance and sharpness, the main results in [**68**] are limited to the case $p \geq 1$ and to Dirichlet boundary conditions.

In this chapter we are interested in measuring the smoothness of u in (6.1.1) when $f \in B_s^{p,p}(\partial\Omega)$ in a new way, by relying on the Besov-based non-tangential maximal function

$$\mathcal{N}_s^q(u)(x) := \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}, \qquad x \in \partial\Omega. \tag{6.1.7}$$

One of the main virtues of this approach is that it allows us to consider boundary data from Besov spaces with $p < 1$. The issue of studying elliptic boundary value problems in non-smooth domains with data in quasi-Banach spaces arose in C. Kenig's book [**79**] (cf. Problem 3.2.10 on p. 119 *loc. cit.*). As we shall see momentarily, the consideration of (6.1.7) also allows us to treat the case of Neumann boundary conditions.

To state our first main result, set $(a)_+ := \max\{a, 0\}$.

Theorem 6.1.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f \in B_s^{p,p}(\partial\Omega), \\ \mathcal{N}_s^q(u) \in L^p(\partial\Omega), \end{cases} \tag{6.1.8}$$

where it is assumed that

$$\tfrac{n-1}{n} < p \leq q, \quad \tfrac{n}{1-s} < q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < s < 1. \tag{6.1.9}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (6.1.8) is well-posed if, in addition to (6.1.9), the indices s, p also satisfy one of the following three conditions:

$$\begin{aligned} &\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1; \\ &\tfrac{2}{2+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1; \\ &\tfrac{2}{1-\varepsilon} < p \leq \infty \quad \text{and } 0 < s < \tfrac{2}{p} + \varepsilon. \end{aligned} \tag{6.1.10}$$

Furthermore, the solution has the integral representation formula

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1} f] \quad \text{in } \Omega, \tag{6.1.11}$$

and satisfies

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B_s^{p,p}(\partial\Omega)}. \tag{6.1.12}$$

It is illuminating to point out that the conditions (6.1.10) amount to the membership of the point with coordinates $(s, 1/p)$ to the pentagon below:

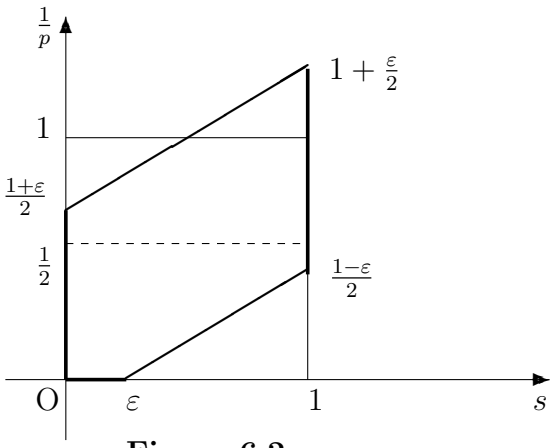

Figure 6.2.

A few remarks are in order here. For starters, we note that the accompanying estimate (6.1.12) formally agrees with (6.1.2) and (6.1.4). That is, the latter can be thought of as a limiting case of (6.1.12), corresponding to $q = \infty$ and $s = 0$, $s = 1$, respectively. In fact, it was this observation which gave us the original impetus for measuring the smoothness of the solution using the Besov-based non-tangential maximal function (6.1.7) when the data is in $B_s^{p,p}(\partial\Omega)$.

Second, the concept of trace $u|_{\partial\Omega}$ is well-defined for the problem under discussion. Indeed, since $\mathcal{N}_s^q(u) \in L^p(\partial\Omega)$ and $B_{s+\frac{n}{q}}^{q,q}(\gamma(x)) \hookrightarrow C^s(\gamma(x)) = C^s(\overline{\gamma(x)})$, the Hölder class, it follows that $u|_{\gamma(x)} \in C^s(\overline{\gamma(x)})$ for a.e. $x \in \partial\Omega$. In particular, it makes sense to talk about $u|_{\partial\Omega}$ as a non-tangential limit, i.e.

$$u(x) := \lim_{y\in\gamma(x)} u(y), \qquad \text{for } a.e.\ x \in \partial\Omega, \tag{6.1.13}$$

and the corresponding estimate $\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)}$ holds.

Third, regular elliptic boundary problems in C^∞ domains with data in Besov spaces with $p < 1$ were first treated by H. Triebel in [**125**]. Subsequently, these

results have been extended by J. Franke and T. Runst in [**46**]. A good account of these developments can be found in Chapter 3 of [**112**].

As alluded to before, there is a version of Theorem 6.1.1 in the case when a Neumann boundary condition is considered. Specifically, we have the following:

Theorem 6.1.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \partial_\nu u\Big|_{\partial\Omega} = f \in B^{p,p}_{-s}(\partial\Omega), \quad \langle f, 1\rangle = 0, \\ \mathcal{N}^q_{1-s}(u) \in L^p(\partial\Omega), \end{cases} \tag{6.1.14}$$

where

$$\tfrac{n-1}{n} < p \leq q, \quad \tfrac{n}{s} < q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < 1-s < 1. \tag{6.1.15}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (6.1.14) has a unique, modulo constants, solution if $(1-s, 1/p)$ belongs to the pentagonal region in Figure 6.2.

In addition, the solution has the integral representation formula

$$u = \mathcal{S}[(-\tfrac{1}{2}I + K^*)^{-1} f] + \text{const}, \tag{6.1.16}$$

and (after a normalization) satisfies the estimate

$$\|\mathcal{N}^q_{1-s}(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B^{p,p}_{-s}(\partial\Omega)}. \tag{6.1.17}$$

One of the main ingredients in the proofs of Theorems 6.1.1-6.1.2 is establishing estimates for potential-like singular integral operators. These results are summarized in the next theorem.

Theorem 6.1.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma(y), \qquad x \in \Omega. \tag{6.1.18}$$

Assume that

$$T1 = const \tag{6.1.19}$$

and

$$|\nabla_x k(x,y)| \leq C|x-y|^{-n}. \tag{6.1.20}$$

Then

$$\|\mathcal{N}_s^q(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)} \tag{6.1.21}$$

whenever $\frac{n-1}{n} < p \leq q$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $\frac{n}{1-s} < q \leq \infty$.

It should be noted that the class of operators satisfying (6.1.19)-(6.1.20) contains the double layer potential as well as the Cauchy operator (in two and higher dimensions; cf. §6.2 for a discussion). A similar theorem also holds for single layer-like operators. Concretely, we have the following:

Theorem 6.1.4. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator $Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma(y)$. Suppose its integral kernel satisfies*

$$|\nabla_x \nabla_y^j k(x,y)| \leq C|x-y|^{-(n-1+j)} \quad \text{for } j = 0, 1. \tag{6.1.22}$$

Then

$$\|\mathcal{N}_{1-s}^q(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_{-s}^{p,p}(\partial\Omega)} \tag{6.1.23}$$

if $\frac{n-1}{n} < p \leq q$, $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$, and $\frac{n}{s} < q \leq \infty$.

Our methods are quite general and flexible and, in principle, can be adapted to other situations of interest. For example, appropriate versions of Theorems 6.1.1-6.1.2 hold for the Lamé system of elastostatics in Lipschitz domains of dimension ≤ 3. Indeed, in this situation, the analogues of (6.1.2), (6.1.4) have been established in **[32]** via atomic estimates, which are also the main ingredients in the proof of Theorem 4.3.12. Also, the Theorems 6.1.3-6.1.4 readily extend to the vector-valued case.

6.2 Mapping properties of singular integral operators

We debut with the

Proof of Theorem 6.1.3. The basic idea of the proof is to consider the sublinear operator

$$\mathcal{T}f = \mathcal{N}_s^q(Tf) : B_s^{p,p}(\partial\Omega) \to L^p(\partial\Omega) \tag{6.2.1}$$

and interpolate – using Theorem 3.1.10 – between $(n-1)/n < p \leq 1$ and $p = q \in [1, \infty]$. We proceed by analyzing several cases starting with:

Case 1. $1 \leq p = q \leq \infty$**.** The estimate (6.1.21) will be a consequence of the following sequence of inequalities

$$\|\mathcal{N}_s^p(Tf)\|_{L^p(\partial\Omega)} \leq C\|\delta^{1-(s+\frac{1}{p})}|\nabla Tf|\|_{L^p(\Omega)}, \tag{6.2.2}$$

$$\|\delta^{1-(s+\frac{1}{p})}|\nabla Tf|\|_{L^p(\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{6.2.3}$$

each of which will be proved separately (recall that δ denotes the distance to $\partial\Omega$). In fact, the inequality (6.2.2) is proved later, in (ii) of Corollary 6.3.3.

The estimate (6.2.3) will be proved in three steps. The idea is to obtain the result for $p = q = 1$, then for $p = q = \infty$ and, finally, use Stein's interpolation theorem for analytic families of operators (Theorem 3.1.11) to cover the range in between.

Consider first the case $p = q = 1$. We shall prove that if

$$|\nabla_x^k k(x,y)| \leq C|x-y|^{-(n+k-1)}, \quad k = 1, 2, ..., N, \tag{6.2.4}$$

for some positive integer N, then

$$\|\delta^{k-1-s}|\nabla^k Tf|\,\|_{L^1(\Omega)} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)}, \tag{6.2.5}$$

(this result is stated in a slightly more general form than the one we need, which would correspond to the case $N = 1$).

Turning to the actual proof of (6.2.5), first recall the intrinsic characterization of the Besov space $B_s^{1,1}(\partial\Omega)$

$$\|f\|_{B_s^{1,1}(\partial\Omega)} = \|f\|_{L^1(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n-1+s}} d\sigma(x)d\sigma(y). \tag{6.2.6}$$

The estimate we seek has local character. Thus, using a partition of unity, we may assume that the support of f is included in a coordinate patch where $\partial\Omega$ is represented by the graph of the Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Assuming that this is the case, we make a change of variables and set $\tilde{f}(x) := f(x, \phi(x))$, extended by zero outside of the support. In particular, $\tilde{f} \in B_s^{1,1}(\mathbb{R}^{n-1})$.

Thanks to (6.1.19) we have that $\nabla^k T$ annihilates constants. In concert with the assumption (6.2.4) on the kernel, this implies that

$$\int_\Omega \delta^{k-1-s}(x)|\nabla^k Tf(x)|dx \tag{6.2.7}$$

can be controlled by a multiple of

$$\int_0^\infty t^{k-1-s}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\tilde{f}(x)-\tilde{f}(y)|}{(|x-y|^2+t^2)^{\frac{n+k-1}{2}}}\,dxdy. \tag{6.2.8}$$

In turn, this can be majorized by

$$C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}|\tilde{f}(x)-\tilde{f}(y)|\Big(\int_0^\infty\frac{t^{k-1-s}}{(|x-y|+t)^{n+k-1}}dt\Big)\,dxdy. \tag{6.2.9}$$

Making the change of variables $r=\frac{t}{|x-y|}$, $r\in(0,\infty)$, $dt=|x-y|dr$, we can further bound the innermost integral above by

$$\int_0^\infty\frac{t^{k-1-s}}{(|x-y|+t)^{n+k-1}}dt=|x-y|^{-s+1-n}\int_0^\infty\frac{r^{k-1-s}}{(1+r)^{n+k-1}}dr\leq C|x-y|^{-s+1-n} \tag{6.2.10}$$

for some finite constant $C=C(k,s,n)$. Thus,

$$\begin{aligned}\int_\Omega\delta^{k-1-s}(x)|\nabla^k Tf(x)|\,dx &\leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^{n+s-1}}\,dxdy\\ &\leq C\|\tilde{f}\|_{B_s^{1,1}(\mathbb{R}^{n-1})}\leq C\|f\|_{B_s^{1,1}(\partial\Omega)}\end{aligned} \tag{6.2.11}$$

as desired; this completes the proof of (6.2.5) in the case $p=q=1$.

Next we turn our attention to the case $p=q=\infty$. The goal is to show that $\delta^{k-s}|\nabla^k Tf|\in L^\infty(\Omega)$ for $f\in B_s^{\infty,\infty}(\partial\Omega)$, with appropriate control of the norms. To this end, let x^* denote the point on $\partial\Omega$ such that $|x-x^*|=\delta(x)$. Then we may write

$$\nabla^k Tf(x)=\int_{\partial\Omega}\nabla_x^k k(x,y)(f(y)-f(x^*))\,d\sigma(y). \tag{6.2.12}$$

Since $f\in B_s^{\infty,\infty}(\partial\Omega)=C^s(\partial\Omega)$ we have $|f(y)-f(x^*)|\leq\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}|y-x^*|^s$, $y\in\partial\Omega$. With this in mind and recalling the assumptions (6.2.4) on the kernel,

we split the last integral into two parts, I_1, I_2, corresponding to $y \in S_{100r}(x^*)$ and $y \in \partial\Omega \setminus S_{100r}(x^*)$, respectively, where $r := |x - x^*|$. We have

$$\begin{aligned} |I_1| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} |x-y|^{s-n+1-k}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} r^{s-n+1-k} d\sigma(y) \leq C\, r^{s-k}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \end{aligned} \tag{6.2.13}$$

On the other hand,

$$\begin{aligned} |I_2| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} |x^*-y|^{s-n+1-k}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{100r}^{\infty} \rho^{s-1-k} d\rho \leq C\, r^{s-k}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \end{aligned} \tag{6.2.14}$$

These inequalities complete the argument for the case $p = q = \infty$.

Now the estimate

$$\delta^{k-\frac{1}{p}-s}|\nabla^k Tf| \in L^p(\Omega) \quad \text{for} \quad f \in B_s^{p,p}(\partial\Omega), \tag{6.2.15}$$

$$k = 1, 2, ..., N, \quad s \in (0,1), \quad p \in [1,\infty],$$

will follow from Theorem 3.1.11. The details are as follows. Consider the family of operators

$$L_z f := \delta^{k-1+z-[(1-z)s_0+zs_1]}|\nabla^k Tf|, \tag{6.2.16}$$

so that

$$\Re ez = 0 \Rightarrow |L_0 f| = \delta^{k-1-s_0}|\nabla^k Tf|,$$

$$\Re ez = 1 \Rightarrow |L_1 f| = \delta^{k-s_1}|\nabla^k Tf|.$$

Our results for $p = q = 1$ and $p = q = \infty$ lead to the conclusion that the operators

$$L_0 : B^{1,1}_{s_0}(\partial\Omega) \to L^1(\Omega),$$

$$L_1 : B^{\infty,\infty}_{s_1}(\partial\Omega) \to L^\infty(\Omega),$$

are well-defined and bounded. In turn, thanks to Theorem 3.1.11, and standard complex interpolation results for the Besov scale (cf. **[10]**; here $s_0 \neq s_1$ is needed), we may conclude that

$$\delta^{k-\frac{1}{p}-s}|\nabla^k Tf| : B^{p,p}_s(\partial\Omega) \to L^p(\Omega) \tag{6.2.17}$$

is also well defined and bounded granted that $s \in (0,1)$ and $p \in [1,\infty]$.

Case 2. $(n-1)/n < p \leq 1$**.** In this situation we use the atomic characterization (3.4.29) of Besov spaces. As is well-known, the desired estimate, i.e.

$$\|\mathcal{N}^q_s(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega)}, \tag{6.2.18}$$

is a direct consequence of

$$\|\mathcal{N}^q_s(Ta)\|_{L^p(\partial\Omega)} \leq C, \tag{6.2.19}$$

for every $B^{p,p}_s(\partial\Omega)$-atom a, with a finite constant $C > 0$ independent of the atom. Turning our attention to (6.2.19), we will split the discussion into two parts: near and away from $\operatorname{supp} a$.

Near the support of a we employ a rescaling technique. Specifically,

$$\mathcal{N}^q_s(Ta)(x) = \|Ta\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}$$

can be bounded by

$$r^{-\frac{n-1}{p}}\|T\tilde{a}\|_{B^{q,q}_{s+\frac{n}{q}}(\Omega)},$$

where $\tilde{a} := r^{\frac{n-1}{p}}a$ is a $B^{q,q}_{s+\frac{n-1}{q}}(\partial\Omega)$ atom.

Upon noticing that $s+\frac{n-1}{q} \in (0,1)$ and recalling that $q>1$, we may now invoke what we have already proved in the case $1 \leq p \leq q$, i.e. (6.2.17), along with Theorem 3.9.3, in order to conclude that

$$r^{-\frac{n-1}{p}}\|T\tilde{a}\|_{B^{q,q}_{s+\frac{n}{q}}(\Omega)} \leq Cr^{-\frac{n-1}{p}}\|\tilde{a}\|_{B^{q,q}_{s+\frac{n-1}{q}}(\partial\Omega)} < Cr^{-\frac{n-1}{p}}.$$

The constant $C = C(\partial\Omega, n, s, p, q)$ above does not depend on the particular atom a. Then $\mathcal{N}^q_s(Ta) \leq Cr^{-\frac{n-1}{p}}$ pointwise on S_{10r} (where $\operatorname{supp} a \subseteq S_r$). Consequently,

$$\|\mathcal{N}^q_s(Ta)\|_{L^p(S_{10r})} \leq C, \tag{6.2.20}$$

which is of the right order.

Next, let us estimate the contribution away from $\operatorname{supp} a$. Without any loss of generality we may assume that $0 \in \partial\Omega$ and $\operatorname{supp} a \subset S_r(0)$. For every $y, z \in \Omega$ we have

$$\begin{aligned} |Ta(y)-Ta(z)| &= \Big|\int_{\partial\Omega}(k(y,w)-k(z,w))a(w)\,d\sigma(w)\Big| \\ &= \Big|\int_{\partial\Omega}\int_0^1 \frac{d}{dt}k((1-t)y+tz,w)dt\ a(w)\,d\sigma(w)\Big|. \end{aligned}$$

Keeping in mind the hypotheses (6.1.20) we make on the kernel, the last integral above is bounded by

$$C|z-y|\int_{\partial\Omega}\int_0^1 \frac{dt}{[((1-t)y+tz)^2+w^2]^{n/2}}\,|a(w)|\,d\sigma(w).$$

This, in turn, can be majorized by

$$C|z-y|r^{s+(1-\frac{1}{p})(n-1)}\int_0^1 \frac{dt}{[(1-t)y+tz]^n}, \tag{6.2.21}$$

where r is the radius of the minimal surface ball containing the support of the atom a. Relying on the intrinsic characterization of Besov spaces, we may write

$$\Big(\mathcal{N}_s^q(Ta)(x)\Big)^q=\int_{\gamma(x)}\int_{\gamma(x)}\frac{|Ta(y)-Ta(z)|^q}{|y-z|^{2n+sq}}\,dydz+\|Ta\|^q_{L^q(\gamma(x))}. \tag{6.2.22}$$

The last term above is residual and we leave the details of estimating it to the interested reader. As regards the next-to-the-last one, using (6.2.21) we estimate it by

$$Cr^{q(s+(1-\frac{1}{p})(n-1))}\int_{\gamma(x)}\int_{\gamma(x)}\Big(|y-z|^{1-s-\frac{2n}{q}}\int_0^1\frac{dt}{((1-t)y+tz)^n}\Big)^q\,dydz. \tag{6.2.23}$$

Thus, matters have been reduced to estimating the above integral. A general result to that effect (whose proof is postponed for a little while) is recorded below.

Lemma 6.2.1. *For $0<s<1$ there holds*

$$\int_{\gamma(x)}\int_{\gamma(x)}\Big(|y-z|^{1-s-\frac{2n}{q}}\int_0^1\frac{dt}{[(1-t)y+tz]^n}\Big)^q\,dydz\leq C\frac{1}{|x|^{sq+(n-1)q}}, \tag{6.2.24}$$

where the constant C is independent of x and $q>\frac{n}{1-s}$.

Returning to the main stream discussion, i.e. the sequence of inequalities (6.2.22)-(6.2.23), we may therefore bound

$$\int_{x\in\partial\Omega\setminus S_{10r}}(\mathcal{N}_s^q(Ta)(x))^p\,d\sigma(x) \tag{6.2.25}$$

by

$$C\, r^{(s+(1-\frac{1}{p})(n-1))p} \int_{|x|\geq 10r} \frac{dx}{|x|^{(s+n-1)p}} = C\, r^{[s+(1-\frac{1}{p})(n-1)]p} \int_{10r}^{\infty} \frac{\rho^{n-2}d\rho}{\rho^{(s+n-1)p}} = const, \tag{6.2.26}$$

provided the last integral converges. This is indeed the case, granted that $s > (n-1)(\frac{1}{p}-1)$, which is part of our assumptions. This observation finishes the proof in the case when $(n-1)/n < p \leq 1$, modulo the

Proof of Lemma 6.2.1. Pulling everything back to the Euclidean model, it suffices to show (after a translation and a rotation) that

$$\int_{\gamma(0)}\int_{\gamma(0)} \Big(\int_0^1 \frac{|y-z|^{1-s-\frac{2n}{q}}\, dt}{(1-t)^n y_n^n + t^n z_n^n + 1} \Big)^q dydz \leq C \frac{1}{|x|^{sq+(n-1)q}}. \tag{6.2.27}$$

By symmetry, there is no loss of generality in assuming that $y_n \leq z_n$, which allows us to bound the corresponding piece by

$$\int_{\gamma(0)}\int_{\gamma(0)} \Big(\int_0^1 \frac{|y-z|^{1-s-\frac{2n}{q}}\, dt}{t^n z_n^n + 1} \Big)^q dydz. \tag{6.2.28}$$

Switching to polar coordinates $y - z = r\omega$ where $r \in (0,\infty)$ and $\omega \in S_{n-1}$, we may rewrite (6.2.28) as

$$\int_{\gamma(0)} \Big[\int_0^1 \frac{dt}{t^n z_n^n + 1} \Big]^q \int_0^{cz_n} r^{n-1+q-sq-2n} dr dz \tag{6.2.29}$$

which, for $q > \frac{n}{1-s}$, is controlled by

$$C \int_0^{\infty} z_n^{q-sq-n} \Big[\int_0^1 \frac{dt}{t^n z_n^n + 1} \Big]^q dz_n. \tag{6.2.30}$$

After the change of variables $\tau = tz_n$, (6.2.30) becomes

$$C \int_0^{\infty} z_n^{-sq-n} \Big[\int_0^{z_n} \frac{d\tau}{\tau^n + 1} \Big]^q dz_n.$$

Upon noticing that the inner integral is $\mathcal{O}(1)$ as $z_n \to \infty$ and $\mathcal{O}(z_n)$ as $z_n \to 0$, we see that the entire expression amounts to a finite constant as long as $q > (n-1)/(1-s)$. This is, however, covered by the current assumptions, so the proof of Lemma 6.2.1 is finished. □

Case 3. $1 < p < q$**.** The idea is to use Theorem 3.1.10 for the operator $\mathcal{T}$ defined in (6.2.1), which is obviously sublinear and satisfies the conditions (3.1.39)-(3.1.40) for $p_0 = 1$, $p_1 = q$, $X_0 := B_s^{1,1}(\partial\Omega)$, and $X_1 := B_s^{q,q}(\partial\Omega)$.

Invoking a complex interpolation result (cf. **[125]**, or the diagonal case of Proposition 3.4.4) it follows that

$$\mathcal{T}f = \mathcal{N}_s^q(Tf) : B_s^{p,p}(\partial\Omega) \to L^p(\partial\Omega) \tag{6.2.31}$$

is bounded for every $p \in [1, q]$, whenever (6.1.9) holds. The proof of Theorem 6.1.3 is therefore completed. □

Remark. It is illuminating to sketch an alternative approach, most easily described in the case when Ω is the (unbounded) domain in $\mathbb{R}^n$ lying above the graph of a real-valued Lipschitz function. In this situation, fix an upright, circular cone Γ with vertex at the origin so that $\gamma(x) = x + \Gamma$, for each $x \in \partial\Omega$, and introduce the linear operator

$$\Big(\mathcal{Q}f(x)\Big)(z) := (Tf)(x+z), \qquad x \in \partial\Omega,\ z \in \Gamma.$$

The estimate (6.1.21) is then equivalent to the boundedness of

$$\mathcal{Q} : B_s^{p,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega, B_{s+\frac{n}{q}}^{q,q}(\Gamma))$$

whenever $\frac{n-1}{n} < p \leq q$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $\frac{n}{1-s} < q \leq \infty$. Note that the

reasonings in Cases 1 and 2 ensure the boundedness of the above operator when $0 \leq (n-1)(\frac{1}{p}-1) < s < 1$, $\frac{n}{1-s} < q \leq \infty$, and when $p = q = \infty$, $0 < s < 1$. The remaining cases are covered by interpolating by the real method between $p = 1$ and $p = q = \infty$. In the process, the classical result

$$\Big(L^{p_0}(A_0)\,,\, L^{p_1}(A_1)\Big)_{\theta,q} = L^q((A_0, A_1)_{\theta,q})$$

if $p_0, p_1 \in [1, \infty]$, $1/q = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$ is used; cf. **[85]**.

We continue with the

Proof of Theorem 6.1.4. Again, we proceed in a sequence of steps.

Case 1. $1 \leq p = q \leq \infty$**.** Using the estimates for $p = q = 1$ and $p = q = \infty$ which have been proved in **[97]**, Lemma 7.2, p.35 and Lemma 7.3, p.37 respectively, and Theorem 3.1.11, we conclude that

$$\delta^{k-\frac{1}{p}+s-1}|\nabla^k Tf| : B^{p,p}_{-s}(\partial\Omega) \to L^p(\Omega), \tag{6.2.32}$$

whenever

$$|\nabla^k_x \nabla^j_y k(x,y)| \leq C|x-y|^{-(n+k+j-2)}, \quad k = 1, 2, ..., N, \quad j = 0, 1, \tag{6.2.33}$$

for some positive integer N.

Going further, the estimate

$$\|\mathcal{N}^p_{1-s}(Tf)\|_{L^p(\partial\Omega)} \leq C\|\delta^{s-\frac{1}{p}}|\nabla Tf|\|_{L^p(\Omega)} \tag{6.2.34}$$

which follows from (6.2.2) by simply replacing s with $1-s$, then implies (6.1.23) in the case when $1 \leq p = q \leq \infty$.

Case 2. $(n-1)/n < p \leq 1$**.** Similarly to Theorem 6.1.3, it is sufficient to show that

$$\|\mathcal{N}^q_{1-s}(Ta)\|_{L^p(\partial\Omega)} \leq C \tag{6.2.35}$$

for every $B^{p,p}_{-s}(\partial\Omega)$-atom a, for a finite constant C independent of the particular atom.

Near the support of a we use the same technique as in Theorem 6.1.3, i.e. re-scale a, originally an atom in $B^{p,p}_{-s}(\partial\Omega)$, to $\tilde{a} := r^{(n-1)/p}a$, an atom in $B^{q,q}_{-s-\frac{1-n}{q}}(\partial\Omega)$. Invoking the mapping properties of single layer-like operators proved in Case 1 and, much as before, Theorem 3.9.3, we can show that the analogue of (6.2.20) holds in this case.

Away from $\operatorname{supp} a$ we proceed, in broad outline, as in Theorem 6.1.3 although the analytical details are different (most notably, atoms have a vanishing moment condition in this case). To get started, for every $y, z \in \Omega$ we write

$$|Ta(y) - Ta(z)| = \Big| \int_{supp\ a} \int_0^1 \frac{d}{dt} k((1-t)y + tz, w)dt\ a(w)\, d\sigma(w) \Big|.$$

Since atoms in $B^{p,p}_{-s}(\partial\Omega)$ have a vanishing moment condition, this implies that the expression in the right side can be further written as

$$\Big| \int_{supp\ a} \int_0^1 (z-y)[(\nabla_x k((1-t)y + tz, w) - \nabla_x k((1-t)y + tz, 0))]dt\ a(w) d\sigma(w) \Big|$$

where it is assumed that $0 \in \partial\Omega$ and $\operatorname{supp} a \subseteq S_r(0)$. Then the last integral above is equal to

$$\Big|\int_{supp\ a}\int_0^1 (z-y)\Big[\int_0^1 \frac{d}{d\theta}(\nabla_x k((1-t)y+tz,(1-\theta)w))d\theta\Big]\,dt\ a(w)d\sigma(w)\Big|.$$

Using our assumptions on the kernel, this expression can be further bounded by

$$\max_{\theta\in[0,1]}\Big|\int_{supp\ a}\int_0^1 \frac{|z-y||w|}{|((1-t)y+tz)-(1-\theta)w|^n}dt\ a(w)d\sigma(w)\Big|.$$

If we now observe that $|((1-t)y+tz)-(1-\theta)w|\approx|(1-t)y+tz|$ and $|w|<r$, we may invoke the size conditions on atoms to eventually arrive at

$$|Ta(y)-Ta(z)|\leq Cr^{1-s-(n-1)(\frac{1}{p}-1)}\int_0^1\frac{|z-y|}{|(1-t)y+tz|^n}dt. \tag{6.2.36}$$

Using the estimate (6.2.36), the proof for the case $(n-1)/n<p\leq 1$ can be finished in the same way as it was done in Theorem 6.1.3. We omit the details.

Case 3. $1<p<q$**.** With the help of Theorem 3.1.10, this follows from what we have proved so far via interpolation between the cases $p=1$ and $p=q$. This completes the proof of the theorem. □

6.3 Connections with classical smoothness spaces in Ω

The goal of this section is to highlight connections between our Besov-based non-tangential maximal function and the standard Besov scale in a Lipschitz domain Ω.

Theorem 6.3.1. *Assume that* $1<p<q\leq\infty$, $\frac{n}{1-s}<q\leq\infty$ *and* $0<s<1$. *Then for every Lipschitz domain* Ω *there exists* $C=C(\Omega,p,q,s)>0$ *such that*

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)}\leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{6.3.1}$$

for every harmonic function u in Ω.

The estimate (6.3.1) shows that the solution of (6.1.1) taken in the sense of Jerison and Kenig [**68**], i.e. $u \in B^{p,p}_{s+1/p}(\Omega)$, is automatically a solution in the sense of Theorem 6.1.1 for the range of indices specified above. This also serves as evidence of the appropriateness of our choice of the Besov-based non-tangential maximal function as a way of measuring smoothness.

Proof: The proof is based on ideas from complex analysis in higher dimensions, mostly inspired by the Clifford algebra formalism. In order to be able to follow the main steps more easily, we shall first carry out the proof in the somewhat simpler geometrical context of a domain $\Omega \subset \mathbb{R}^n$ lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$.

Let $\{u_j\}_{1\leq j\leq n}$ be a system of conjugate harmonic functions in Ω with $u = u_n$. For example, we may take (4.2.48). Our first claim is that if

$$u \in B^{p,p}_{\alpha+k}(\Omega), \quad k \in \mathbf{N} \cup \{0\}, \quad \alpha \in (0,1), \tag{6.3.2}$$

then

$$u_j \in B^{p,p}_{\alpha+k}(\Omega) \quad \text{and} \quad \sum_j \|u_j\|_{B^{p,p}_{\alpha+k}(\Omega)} \leq C\|u\|_{B^{p,p}_{\alpha+k}(\Omega)}. \tag{6.3.3}$$

Indeed, using Theorem 3.9.3, it is enough to control, for each $1 \leq j \leq n$,

$$\int_\Omega \Big(\delta(x)^{1-\alpha}|\nabla^{k+1}u_j(x)|\Big)^p\,dx \tag{6.3.4}$$

which, after the change of variables $x = x' + re_n$, $r > 0$, $x' \in \partial\Omega$, is bounded by

$$C\int_{\partial\Omega}\int_0^\infty (r^{1-\alpha}|\nabla^{k+1}u_j(x'+re_n)|)^p\,drdx' \tag{6.3.5}$$

$$\leq C\int_{\partial\Omega}\int_0^\infty r^{(1-\alpha)p}\Big[\int_0^\infty |\nabla^{k+2}u(x'+(r+t)e_n)|\,dt\Big]^p\,drdx'. \tag{6.3.6}$$

The last inequality above utilizes the fact that

$$|\nabla^{k+1}u_j(x)|\leq C\int_0^\infty |\nabla^{k+2}u(x+te_n)|\,dt. \tag{6.3.7}$$

If we now put $\lambda=r+t$, $\lambda>r$, $d\lambda=dt$, the last integral becomes

$$\int_{\partial\Omega}\int_0^\infty r^{(1-\alpha)p}\Big[\int_r^\infty |\nabla^{k+2}u(x'+\lambda e_n)|\,d\lambda\Big]^p\,drdx'. \tag{6.3.8}$$

This, using Hardy's inequality, can be further bounded by

$$C\int_{\partial\Omega}\int_0^\infty |\nabla^{k+2}u(x'+\lambda e_n)|^p\lambda^{(1-\alpha)p}d\lambda dx' \tag{6.3.9}$$

and, invoking Theorem 2.4.6, by

$$C\int_\Omega \delta(x)^{(1-\alpha)p}|\nabla^{k+1}u(x)|^p\,dx. \tag{6.3.10}$$

The sequence of inequalities (6.3.4)-(6.3.10) along with Theorem 3.9.3 complete the proof of (6.3.2)-(6.3.3).

Next, define

$$\{E_j\}_{1\leq j\leq n}\subset M_{2^n\times 2^n}(\mathbb{R}^n) \tag{6.3.11}$$

to be a set of matrices (of type $2^n\times 2^n$ with real entries) such that the following properties hold

$$E_jE_k=-E_kE_j,\quad \text{if}\quad j\neq k, \tag{6.3.12}$$

$$E_j^2=-I_{2^n},\quad \forall j=1,..,n, \tag{6.3.13}$$

where I_N is the $N \times N$ identity matrix. For each $k = 1, 2, ..., n$, $\{E_j^k\}_{j=1}^k$ are inductively defined by setting

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for $1 \leq k \leq n-1$, $1 \leq j \leq k$

$$E_j^{k+1} := \begin{pmatrix} E_j^k & 0 \\ 0 & -E_j^k \end{pmatrix},$$
$$E_{k+1}^{k+1} := \begin{pmatrix} 0 & -I_{2^k} \\ I_{2^k} & 0 \end{pmatrix}.$$

Finally, take E_j to be E_j^n, for $j = 1, 2, ..., n$. It is easy to see that the E_j's just constructed satisfy the properties (6.3.12)-(6.3.13).

An appropriate *Dirac*-type operator in the current context is

$$D := \sum_{j=1}^n E_j \partial_j. \tag{6.3.14}$$

Consider the matrix-valued function

$$F := \sum_{j=1}^n u_j E_j : \Omega \longrightarrow M_{2^n \times 2^n}(\mathbb{R}^n). \tag{6.3.15}$$

A direct calculation shows that $D^2 = -\Delta$ and $DF = 0$, i.e. F is *analytic* (relative to D).

The next order of business is to define a suitable *Cauchy*-type operator. With this goal in mind, recall that the fundamental solution for Laplacian is

$$\Gamma_{Lap}(x) = \frac{-1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, \tag{6.3.16}$$

if $n \geq 3$, where ω_n is the surface area of unit sphere (the modifications for the case $n = 2$ are minor). A fundamental solution for the Dirac operator D is

$$\Gamma_{Dir}(x) = -D\Gamma_{Lap}(x) = -\frac{1}{\omega_n} \sum_{j=1}^n \frac{x_j}{|x|^n} E_j. \tag{6.3.17}$$

If we denote $\overline{x} := \sum_{j=1}^{n} x_j E_j$ we arrive at

$$\Gamma_{Dir}(x) = -\frac{1}{n\omega_n}\frac{\overline{x}}{|x|^n}. \tag{6.3.18}$$

By methods similar to those in classic complex analysis, one may derive Cauchy's reproducing formula

$$F(x) = \mathcal{C}(F|_{\partial\Omega})(x), \qquad x \in \Omega, \tag{6.3.19}$$

granted that $DF = 0$ in Ω. Here

$$\mathcal{C}f(x) := -\frac{1}{n\omega_n}\int_{\partial\Omega}\frac{\overline{x}-\overline{y}}{|x-y|^n}\cdot\overline{\nu}(y)\cdot f(y)\,d\sigma(y), \quad x \in \Omega, \tag{6.3.20}$$

where 'dot' denotes matrix multiplication, and f is a matrix-valued function defined on $\partial\Omega$.

Note that the Cauchy operator (6.3.20) satisfies the conditions (6.1.19), (6.1.20) and, therefore, obeys (6.1.21). This leads to the following estimate

$$\|\mathcal{N}_s^q(F)\|_{L^p(\partial\Omega)} \leq C\|\mathrm{Tr}\, F\|_{B_s^{p,p}(\partial\Omega)}, \tag{6.3.21}$$

where Tr the trace operator, defined in (1.1.15). In turn, the last expression above is bounded by

$$\|F\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C\sum_{j=1}^{n}\|u_j\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{6.3.22}$$

which, using (6.3.3), can be controlled by a multiple of

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}. \tag{6.3.23}$$

Therefore it remains to show that

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}_s^q(F)\|_{L^p(\partial\Omega)} \tag{6.3.24}$$

which, in concert with the sequence of inequalities (6.3.21)-(6.3.24), would complete the argument.

To 'extract' $u = u_n$ out of F and obtain (6.3.24) we observe that

$$\sum_{k=1}^{n-1} E_k F E_k = \sum_{k=1}^{n-1}(\sum_{j=1}^{n} u_j E_j - u_k E_k) = (n-1)F - 2(F - u_n E_n). \tag{6.3.25}$$

Therefore,

$$u = u_n = -\frac{1}{2} E_n \Big[\sum_{k=1}^{n-1} E_k F E_k - (n-3)F\Big] \tag{6.3.26}$$

which obviously implies (6.3.24).

After this preamble, we are ready to tackle the technically more involved case of a general (bounded) Lipschitz domain. In fact, since the estimate we seek is local in character, there is no loss of generality in assuming that $\Omega \subset \mathbb{R}$ is a *star-like* Lipschitz domain with respect to the origin. Fix a harmonic function $u \in B^{p,p}_{k+\alpha}(\Omega)$, $k \in \mathbf{N} \cap \{0\}$, $\alpha \in (0,1)$, and define a system of harmonic conjugates following (4.2.62).

Our first goal is to show that $v \in B^{p,p}_{k+\alpha}(\Omega, \Lambda^2 \mathbb{R}^n)$ with norm controlled by that of u. By Theorem 3.9.3 it suffices to show that

$$\delta^{\alpha-1} |\nabla^{k+1} v_{jk}| \in L^p(\Omega), \quad \forall j,k. \tag{6.3.27}$$

In order to prove (6.3.27) we assume that, in polar coordinates, Ω can be described as the set $\{x = \omega\rho;\, \omega \in S^{n-1},\, 0 < \rho < \phi(\omega)\}$ for some Lipschitz function $\phi : S^{n-1} \to \mathbb{R}$ with $\inf \phi > 0$. Dropping lower order terms, we can write

$$\sum_{j,k} \int_\Omega (\delta^{1-\alpha}(x) |\nabla^{k+1} v_{jk}(x)|)^p dx$$

$$\leq C \int_{S^{n-1}} \int_0^{\phi(\omega)} (\phi(\omega) - \rho)^{p(1-\alpha)} \Big(\int_0^1 |\nabla^{k+2} u(t\rho\omega)|\, dt \Big)^p \rho^{n-1}\, d\rho d\omega. \tag{6.3.28}$$

At this stage, making the change of variables $\rho = \phi(\omega)e^{-\eta}$, $\eta \in (0,\infty)$ and $t = e^{-r}$, $r \in (0,\infty)$, we can re-write the last integral as

$$\int_{S^{n-1}} \int_0^\infty (\phi(\omega)(1-e^{-\eta}))^{p(1-\alpha)} \Big(\int_0^\infty |\nabla^{k+2} u(\phi(\omega)\omega e^{-r-\eta})|\, dr \Big)^p (\phi(\omega)e^{-\eta})^n \, d\eta d\omega. \tag{6.3.29}$$

If we set $\lambda = r + \eta$, $\lambda \in (\eta, \infty)$, we can bound the above expression by

$$C \int_{S^{n-1}} \int_0^\infty (1-e^{-\eta})^{p(1-\alpha)} \Big(\int_\eta^\infty |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta} \, d\eta d\omega. \tag{6.3.30}$$

If we now split the integral in η into two integrals over the sets $[0, M]$ and $[M, \infty)$, for some large M, matters are reduced to controlling

$$I := \int_{S^{n-1}} \int_0^M \eta^{p(1-\alpha)} \Big(\int_\eta^M |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p \, d\eta d\omega \tag{6.3.31}$$

$$II := \int_{S^{n-1}} \int_M^\infty (1-e^{-\eta})^{p(1-\alpha)} \Big(\int_\eta^\infty |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta} \, d\eta d\omega. \tag{6.3.32}$$

Note that, by interior estimates, $|II| \leq C\|u\|_{L^p(\Omega)}$ since the argument of $\nabla^{k+2}u$ stays away from $\partial\Omega$. As for I,

$$|I| \leq C \int_{S^{n-1}} \int_0^\infty \eta^{p(1-\alpha)} \Big(\int_\eta^M |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p \, d\eta d\omega. \tag{6.3.33}$$

Now an application of Hardy's inequality (cf., e.g., [**118**]) leads to

$$\begin{aligned} |I| &\leq \int_{S^{n-1}} \int_0^M |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|^p \lambda^{p(2-\alpha)} \, d\lambda d\omega \\ &\leq C \int_{\Omega\setminus\mathcal{O}} \delta^{(2-\alpha)p}(x) |\nabla^{k+2} u(x)|^p \, dx, \end{aligned} \tag{6.3.34}$$

for a suitable compact subset $\mathcal{O}$ of Ω. In the last step we have changed variables back to the original $x \in \Omega$ and, in the process, have noticed that the Jacobian is bounded given that $0 < \lambda < M$. This proves that the norm of v in $B^{p,p}_{k+\alpha}(\Omega)$

can be controlled by that of u in any star-like Lipschitz domains. The case of a general (bounded) Lipschitz domain follows from what we have proved so far via a standard localization argument.

Now we use an argument similar to that employed in the case of the unbounded domain lying above the graph of a real-valued Lipschitz function to finish the proof of (6.3.1) in the current context. This time, we choose to work with the Dirac operator $D := d + d^*$ (so that $D^2 = -\Delta$) and the 'analytic' function $F := u - \sum_{j<k} v_{jk} dx_j \wedge dx_k$ (in fact, a differential form of inhomogeneous degree two).

Once again, there is a Cauchy integral operator $\mathcal{C}$ (which falls under the scope of the analysis in §6.2), a Cauchy reproducing formula, etc. We leave the details to the interested reader. □

Theorem 6.3.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that $0 < s < 1$, $s + \frac{n}{q} \notin \mathbf{N}$. Then*

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)} \quad \text{provided } 1 \leq q \leq p < \infty, \tag{6.3.35}$$

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \geq C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)} \quad \text{provided } 1 \leq p \leq q < \infty, \tag{6.3.36}$$

uniformly for u harmonic in Ω and $s + \frac{n}{q} > \frac{n-1}{p}$.

Proof. Consider first the case when $p \geq q$ and set $k := [s + \frac{n}{q}]$, $\varepsilon := s + \frac{n}{q} - k$. We have

$$\begin{aligned}
\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} &= \Big[\int_{\partial\Omega}\Big(\int_{\gamma(x)} \operatorname{dist}(y,\partial\gamma(x))^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{p}{q}}\,d\sigma(x)\Big]^{\frac{1}{p}} \\
&\leq C\Big[\int_{\partial\Omega}\Big(\int_{\Omega} \chi_{\gamma(x)}(y)\delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{p}{q}}d\sigma(x)\Big]^{\frac{1}{p}} \\
&\approx \Big\|\int_{\Omega} \chi_{\gamma(x)}(y)\delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\,dy\Big\|^{\frac{1}{q}}_{L^{\frac{p}{q}}_x(\partial\Omega)}.
\end{aligned} \tag{6.3.37}$$

Since we are currently assuming that $p \geq q$, this last expression can be bounded by

$$C\Big(\int_{\Omega}\Big\|\chi_{\gamma(x)}(y)\delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^{q}\Big\|_{L^{\frac{p}{q}}_{x}(\partial\Omega)}\,dy\Big)^{\frac{1}{q}}$$
$$\approx \Big(\int_{\Omega}\delta(y)^{(1-\varepsilon+\frac{n-1}{p})q}|\nabla^{k+1}u(y)|^{q}\,dy\Big)^{\frac{1}{q}}. \tag{6.3.38}$$

By Theorem 2.4.6 the last integral above can be controlled by

$$C\Big(\int_{\Omega}\delta(y)^{(1-\varepsilon+\frac{n-1}{p}-j)q}|\nabla^{k+1-j}u(y)|^{q}\,dy\Big)^{\frac{1}{q}}, \tag{6.3.39}$$

where

$$j := \begin{cases} [\frac{n-1}{p}], & \text{if } \varepsilon > \{\frac{n-1}{p}\}, \\ [\frac{n-1}{p}]+1, & \text{if } \varepsilon < \{\frac{n-1}{p}\}. \end{cases} \tag{6.3.40}$$

Using Theorem 3.9.3, the expression (6.3.39) is further controlled by

$$C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)}, \tag{6.3.41}$$

as desired.

We now turn to the analysis of the remaining case, i.e. $p \leq q$. Since u is harmonic, we may write

$$\|\mathcal{N}^{q}_{s}(u)\|_{L^{p}(\partial\Omega)} \geq C\Big[\int_{\partial\Omega}\Big(\int_{\gamma_{\theta}(x)}\operatorname{dist}(y,\partial\gamma_{\theta}(x))^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^{q}\,dy\Big)^{\frac{p}{q}}\,d\sigma(x)\Big]^{\frac{1}{p}}. \tag{6.3.42}$$

Changing the aperture of non-tangential cones θ to $\theta/2$, we may bound the last expression from below by

$$\geq\ C\Big[\int_{\partial\Omega}\Big(\int_{\gamma_{\theta/2}(x)}\delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^{q}\,dy\Big)^{\frac{p}{q}}\,d\sigma(x)\Big]^{\frac{1}{p}}$$
$$\approx\ \Big[\Big\|\int_{\partial\Omega}\chi_{\gamma_{\theta/2}(x)}(y)\delta(y)^{(1-\varepsilon)p}|\nabla^{k+1}u(y)|^{p}\Big\|_{L^{\frac{q}{p}}_{y}(\Omega)}\,d\sigma(x)\Big]^{\frac{1}{p}}. \tag{6.3.43}$$

Since $p \leq q$, this can be further bounded by

$$C\Big\| \int_{\partial\Omega} \chi_{\gamma_{\theta/2}(x)}(y)\delta(y)^{(1-\varepsilon)p}|\nabla^{k+1}u(y)|^p\, d\sigma(x)\Big\|^{\frac{1}{p}}_{L^{\frac{q}{p}}_y(\Omega)}$$
$$\approx \Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p})q}|\nabla^{k+1}u(y)|^q\, dy\Big)^{\frac{1}{q}}. \tag{6.3.44}$$

By Theorem 2.4.6 the last integral is controlled (modulo lower order terms) by a multiple of

$$\Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p}-j)q}|\nabla^{k+1-j}u(y)|^q\, dy\Big)^{\frac{1}{q}}, \tag{6.3.45}$$

where j is defined as in (6.3.40). Finally, we may bound the above expression from below by

$$C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)}, \tag{6.3.46}$$

as wanted. This finishes the proof of Theorem 6.3.2. □

Corollary 6.3.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$.*

(i) If $1 \leq p \leq \infty$, $0 < s < 1$, and $s + n/p \notin \mathbf{N}$, then

$$\|\mathcal{N}^p_s(u)\|_{L^p(\partial\Omega)} \approx \|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{6.3.47}$$

uniformly for u harmonic function in Ω.

(ii) If $0 < s < 1$ and $n/(1-s) < p \leq \infty$, then

$$\|\mathcal{N}^p_s(u)\|_{L^p(\partial\Omega)} \leq C\|\delta^{1-(s+\frac{1}{p})}|\nabla u|\|_{L^p(\Omega)} \tag{6.3.48}$$

for any C^1 function u in Ω (not necessarily harmonic).

Proof. The first claim follows directly from Theorem 6.3.2 by making $p = q$. The second claim is a consequence of the (first half of the) proof of Theorem 6.3.2, in which we take $p = q$ (note that our current assumptions force $k = 0$). □

Theorem 6.3.4. *For any Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any $0 < s < 1$, $0 < p \leq \infty$, there exists $C > 0$ so that*

$$\|\delta^{1-s}|\nabla u|\|_{L^{\frac{pn}{n-1}}(\Omega)} \leq C\|\mathcal{N}_s^\infty(u)\|_{L^p(\partial\Omega)} \tag{6.3.49}$$

for any C^1 function u in Ω. In particular,

$$\mathcal{N}_s^\infty(u) \in L^p(\partial\Omega) \Longrightarrow u \in B_s^{\frac{pn}{n-1},\frac{pn}{n-1}}(\Omega) \tag{6.3.50}$$

for the range of indices as in (6.1.9).

A comment is in order here. While it is not clear whether the implication

$$\mathcal{N}_s^\infty(u) \in L^p(\partial\Omega) \Longrightarrow u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$$

holds for general indices, the above theorem shows that this is nonetheless true after an embedding, since $B^{p,p}_{s+\frac{1}{p}}(\Omega) \hookrightarrow B_s^{\frac{pn}{n-1},\frac{pn}{n-1}}(\Omega)$.

Proof of Theorem 6.3.4. For $\lambda \in (0,\infty)$ set

$$E_\lambda := \{x \in \partial\Omega;\ \mathcal{N}_{B_s^{\infty,\infty}}(u)(x) > \lambda\}, \tag{6.3.51}$$

$$F_\lambda := \{x \in \Omega;\ \delta(x)^{1-s}|\nabla u(x)| > \lambda\}. \tag{6.3.52}$$

The idea is to show that there exist two constants $C, c > 0$ such that

$$|F_\lambda| \leq C|E_{c\lambda}|^{\frac{n}{n-1}}, \quad \forall\, \lambda > 0, \tag{6.3.53}$$

where $|E|$ is the (surface) measure of the set E. Then, since for a general measure μ

$$\|f\|_{L^p_\mu}^p = \int_0^\infty p\lambda^{p-1}\mu(\{|f| > \lambda\})\, d\lambda, \tag{6.3.54}$$

we can write

$$\begin{aligned}\|u\|_{B_s^{\frac{pn}{n-1},\frac{pn}{n-1}}(\Omega)} &\leq C\|\delta^{1-s}|\nabla u|\|_{L^{\frac{np}{n-1}}(\Omega)} \\ &= \Big(\int_0^\infty \frac{np}{n-1}\lambda^{\frac{np}{n-1}}|F_\lambda|\,d\lambda\Big)^{\frac{n-1}{np}} \\ &\leq C\Big(\int_0^\infty \lambda^{\frac{np}{n-1}-1}|E_\lambda|^{\frac{n}{n-1}}\,d\lambda\Big)^{\frac{n-1}{np}}. \end{aligned} \tag{6.3.55}$$

Note that

$$\lambda^p|E_\lambda| \leq \int_{E_\lambda}(\mathcal{N}_{B_s^{\infty,\infty}}(u)(x))^p\,d\sigma(x) \leq \|\mathcal{N}_s^\infty(u)\|^p_{L^p(\partial\Omega)}, \tag{6.3.56}$$

so that

$$|E_\lambda| \leq \lambda^{-p}\|\mathcal{N}_s^\infty(u)\|^p_{L^p(\partial\Omega)}. \tag{6.3.57}$$

Therefore, the last expression in (6.3.55) can be further bounded by

$$\Big(\int_0^\infty \lambda^{\frac{np}{n-1}-1-\frac{p}{n-1}}|E_\lambda|\,d\lambda\Big)^{\frac{n-1}{np}}\|\mathcal{N}_s^\infty(u)\|^{\frac{1}{n}}_{L^p(\partial\Omega)} = \|\mathcal{N}_s^\infty(u)\|_{L^p(\partial\Omega)}, \tag{6.3.58}$$

as desired.

In order to prove (6.3.53), we shall work with 'tent' regions

$$T(\mathcal{O}) := \Omega\setminus\Big[\bigcup_{x\in\partial\Omega\setminus\mathcal{O}}\gamma(x)\Big] \tag{6.3.59}$$

associated with arbitrary open subsets $\mathcal{O}$ of $\partial\Omega$. The idea is that (6.3.53) is going to be a consequence of the simple inclusion

$$F_\lambda \subseteq T(E_{c\lambda}), \quad \forall\,\lambda>0, \tag{6.3.60}$$

used in concert with a general fact, to the effect that

$$|T(\mathcal{O})| \leq C\,|\mathcal{O}|^{n/(n-1)}, \quad \forall\,\mathcal{O}\text{ open subset of }\partial\Omega. \tag{6.3.61}$$

In turn, (6.3.61) is seen by decomposing $\mathcal{O}$ into a disjoint union of Whitney cubes $\{Q_k\}_k$ (considering $\partial\Omega$ as a space of homogeneous type), so that $T(\mathcal{O}) \subset \cup_k T(c\,Q_k)$ for some constant $c = c(\partial\Omega) > 0$, and then writing

$$\begin{aligned} |T(\mathcal{O})| &\leq \sum_k |T(c\,Q_k)| \leq C \sum_k |Q_k|^{n/(n-1)} \\ &\leq C\Big[\sum_k |Q_k|\Big]^{n/(n-1)} = C\,|\mathcal{O}|^{n/(n-1)}. \end{aligned} \tag{6.3.62}$$

This proves (6.3.49) and finishes the proof of the Theorem 6.3.4. □

6.4 End of proofs of the main results

Existence, integral representation formulas and estimates in Theorem 6.1.1 and Theorem 6.1.2 follow from what we have proved so far. There remains to establish uniqueness, which we address next.

Theorem 6.4.1. *If u is a solution of (6.1.8) with $f = 0$, then $u \equiv 0$ in Ω.*

Proof. To begin with, from standard embedding results for Besov spaces in Lipschitz domain, we may infer that

$$\mathcal{N}_s^{\infty}(u) \in L^p(\partial\Omega). \tag{6.4.1}$$

Consider next a sequence of domains Ω_j such that for every j

$$\Omega_j \subset \Omega, \quad \Omega_j \subset \Omega_{j+1}, \quad \partial\Omega_j \in C^{\infty}, \quad \Omega_j \nearrow \Omega, \tag{6.4.2}$$

and so that the Lipschitz character of $\partial\Omega_j$ is bounded by some constant M, which is independent of j. The details of the construction of such a family of domains can be found in [**129**]. We adopt the following notational convention: objects labeled

by j are in relation to Ω_j much as those originally constructed in connection with Ω. In particular,

$$\|\mathcal{N}^{\infty}_{s,j}(u)\|_{L^p(\partial\Omega_j)} \leq C\|\mathcal{N}_{B^{\infty,\infty}_s}(u)\|_{L^p(\partial\Omega)}, \tag{6.4.3}$$

with constant C independent of j.

Next, we make the claim that, for the range of indices specified in the statement of the theorem, for every $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$ such that

$$\|Tr_j u\|_{B^{p,p}_{s-\varepsilon}(\partial\Omega_j)} \leq C_\varepsilon \|\mathcal{N}^{\infty}_{s,j}(u)\|_{L^p(\partial\Omega_j)}, \tag{6.4.4}$$

for $j = 1, 2, ...$

To see this, using (6.4.3), we write

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(z)| + |u(z) - u(y)| \\ &\leq C|x-z|^s \|u\|_{B^{\infty,\infty}_s(\gamma(x))} + |z-y|^s \|u\|_{B^{\infty,\infty}_s(\gamma(y))}. \end{aligned} \tag{6.4.5}$$

It is easy to show that $|x - z| \leq C|x - y|$ as well as $|z - y| \leq C|x - y|$, where $C = C(\partial\Omega)$. Therefore, the last expression in (6.4.5) can be bounded by

$$C|x-y|^s [\|u\|_{B^{\infty,\infty}_s(\gamma(x))} + \|u\|_{B^{\infty,\infty}_s(\gamma(y))}]. \tag{6.4.6}$$

Then,

$$\begin{aligned} &\Big[\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+\alpha p}}\, d\sigma(x) d\sigma(y)\Big]^{\frac{1}{p}} \\ &\qquad \leq C\Big[\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\mathcal{N}^{\infty}_{s,j}(u)(x)|^p}{|x-y|^{n-1+(\alpha-s)p}}\, d\sigma(x) d\sigma(y)\Big]^{\frac{1}{p}}. \end{aligned} \tag{6.4.7}$$

Since Ω is bounded, the last integral above is majorized by $C\|\mathcal{N}^{\infty}_{s,j}(u)\|_{L_p(\partial\Omega)}$ for every $\alpha < s$. The proof of (6.4.4) is completed by taking $\alpha := s - \varepsilon$ and recalling (6.4.3).

By classical embedding results, for every $\delta < s$ such that $1 - \frac{\delta}{n-1} > \frac{1}{p} - \frac{s}{n-1}$ we have

$$\|Tr_j u\|_{B^{1,1}_{\delta}(\partial\Omega_j)} \leq C\|Tr_j u\|_{B^{p,p}_{s-\varepsilon}(\partial\Omega_j)}, \tag{6.4.8}$$

where the constant C is independent of j. Based on (6.4.3)-(6.4.4), the right side of (6.4.8) is controlled by

$$\|\mathcal{N}^{\infty}_{s}(u)\|_{L^p(\partial\Omega)} < +\infty. \tag{6.4.9}$$

On the other hand, we may write, thanks to the estimates in **[68]**, that

$$\|u\|_{B^{1,1}_{1+\delta}(\Omega_j)} \leq C\|Tr_j u\|_{B^{1,1}_{\delta}(\partial\Omega_j)}, \tag{6.4.10}$$

where, once again, the constant C is independent of j. Since the expression in the left side converges to $\|u\|_{B^{1,1}_{1+\delta}(\Omega)}$ as $j \to \infty$, we eventually arrive at the conclusion that

$$\|u\|_{B^{1,1}_{1+\delta}(\Omega)} < +\infty. \tag{6.4.11}$$

However, by **[68]**, a harmonic function $u \in B^{1,1}_{1+\delta}(\Omega)$, with δ sufficiently close to 1, is necessarily zero if $\operatorname{Tr} u \equiv 0$ on $\partial\Omega$. This proves uniqueness for (6.1.8) in the case $p \leq 1$.

In the case $p \geq 1$, since $u|_{\partial\Omega} \in B^{p,p}_{s-\varepsilon}(\partial\Omega)$, it follows that

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1}(u|_{\partial\Omega})] \in B^{p,p}_{s-\varepsilon+\frac{1}{p}}(\Omega),$$

for every ε sufficiently small, whenever $(s, 1/p)$ lies in the hexagon depicted in Figure 6.1. In this case, the desired conclusion about the uniqueness of the solution follows even more directly from **[68]**. Theorem 6.4.1 is therefore proved. □

Theorem 6.4.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. If a function u solves (6.1.14) with $f = 0$, then, necessarily, u is constant in Ω.*

Proof. Using the same argument as in Theorem 6.4.1 for $p < 1$, with smoothness $1 - s$ instead of s, it can be shown that the size restriction (6.1.17) imposed on solutions of (6.1.14) implies

$$u \in B^{1,1}_{1+\delta}(\Omega) \tag{6.4.12}$$

for some δ close to 1. Matters can also be arranged so that δ satisfies $\delta < 1 - s$ and $1 - \frac{\delta}{n-1} > \frac{1}{p} - \frac{1-s}{n-1}$. By the work in **[97]**, this entails

$$\partial_\nu u \in B^{1,1}_{\delta-1}(\partial\Omega) \tag{6.4.13}$$

and the desired uniqueness (modulo constant) result for $p < 1$ will follow from **[97]** as soon as we show that

$$B^{p,p}_{-s}(\partial\Omega) \hookrightarrow B^{1,1}_{\delta-1}(\partial\Omega). \tag{6.4.14}$$

In turn, this latter embedding can be checked at the level of atoms; we omit the straightforward details.

For $p > 1$ the same argument as in Theorem 6.4.1 can be used to show that $u \in B^{p,p}_{1-s-\varepsilon+\frac{1}{p}}(\Omega)$ for $(1-s, 1/p)$ in the hexagon in Figure 6.1. However, this entails $\partial_\nu u \in B^{p,p}_{-s-\varepsilon}(\partial\Omega)$ and, further, $u = \mathcal{S}[(-\frac{1}{2}I + K^*)^{-1}(\partial_\nu u|_{\partial\Omega})] \in B^{p,p}_{1-s-\varepsilon+\frac{1}{p}}(\Omega)$ for every ε sufficiently small and every $(s, 1/p')$ in the hexagon in Figure 6.1. Observing that the hexagon in Figure 6.1 is invariant under the affine transformation $(s, 1/p') \mapsto (1 - s, 1/p)$ allows us to finish the proof. □

Chapter 7

Square Function Estimates for Singular Integrals and Applications to Partial Differential Equations

7.1 Motivation and statement of main results

We continue the study of the Dirichlet boundary-value problem for the Laplacian

$$\Delta u = 0 \text{ in } \Omega, \qquad u\Big|_{\partial\Omega} = f, \tag{7.1.1}$$

on Lipschitz domain Ω with a special emphasis on the well-posedness of the problem at hand for data in specific smoothness spaces. In previous sections we already discussed the estimates for solutions of the problem (7.1.1)

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p(\partial\Omega)}, \qquad 2-\varepsilon < p < \infty, \tag{7.1.2}$$

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p_1(\partial\Omega)}, \qquad 1 < p < 2+\varepsilon, \tag{7.1.3}$$

established in **[27]**, **[31]**, **[129]**. Here $\varepsilon = \varepsilon(\partial\Omega) > 0$ is a small constant.

Another result of capital importance is equivalence of the non-tangential max-

imal operator and the classical area integral

$$\mathcal{S}(u)(x) := \left(\int_{\gamma(x)} \operatorname{dist}(y, \partial\Omega)^{2-n} |\nabla u(y)|^2 \, dy \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega, \tag{7.1.4}$$

often referred to as the *square-function* of u in the literature, for harmonic functions in Ω. More specifically,

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \approx \|\mathcal{S}u\|_{L^p(\partial\Omega)}, \qquad 0 < p < \infty, \tag{7.1.5}$$

uniformly for u harmonic in Ω (the "$\leq$" inequality requires that u is appropriately normalized). This striking theorem, due to B. Dahlberg [**30**], extends the ground-breaking work of D. Burkholder, R. Gundy and M. Silverstein [**15**], [**14**], and C. Fefferman and E. Stein [**43**], in the case when $\Omega = \mathbb{R}^n_+$. In particular,

$$\|\mathcal{S}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \qquad 2 - \varepsilon < p < \infty, \tag{7.1.6}$$

$$\|\mathcal{S}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p_1(\partial\Omega)}, \qquad 1 < p < 2 + \varepsilon, \tag{7.1.7}$$

for solutions of (7.1.1).

In 1995 D. Jerison and C. Kenig [**68**] undertook a thorough study of (7.1.1) on Sobolev-Besov scales. One of their main results, proved by means of subtle estimates for the harmonic measure associated with Ω, is that

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C(\partial\Omega, p, s)\|f\|_{B^{p,p}_s(\partial\Omega)}, \tag{7.1.8}$$

provided the integrability exponent p, and the order of differentiability s, satisfy one of the following sets of conditions:

$$\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \text{ and } 0 < s < 1; \quad 1 \leq p < \tfrac{2}{1+\varepsilon} \text{ and } \tfrac{2}{p} - 1 - \varepsilon < s < 1;$$

$$\tfrac{2}{1-\varepsilon} < p \leq \infty \text{ and } 0 < s < \tfrac{2}{p} + \varepsilon, \tag{7.1.9}$$

where $\varepsilon = \varepsilon(\Omega) > 0$.

Let us point out that although the range (7.1.9) is sharp in the class of Lipschitz domains if one insists that $1 \leq p \leq \infty$, there are suitable versions of (7.1.8) which continue to hold for certain pairs of indices (s, p) with $p < 1$. In our earlier work [**87**], we have produced such an extension of the main result in [**68**] based on a modification of the non-tangential maximal operator, i.e.

$$\mathcal{N}_s^q(u)(x) := \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}, \qquad x \in \partial\Omega. \tag{7.1.10}$$

One attractive feature of this approach is its flexibility. For example, this allowed us to consider boundary data from certain Besov spaces with $p < 1$ and to treat the case of Neumann boundary conditions. Furthermore, this is, in principle, applicable to the case of systems as well.

The goal of the present paper is to continue this work by examining the role of the square-function in this context. The main question which we address here is that of the effectiveness of objects like (7.1.4) for measuring the size/smoothness of solutions of (7.1.1) for $f \in B^{p,p}_s(\partial\Omega)$.

We are able to carry out this program by introducing an appropriate analogue of the area integral (7.1.4). Specifically, for each $0 \leq s \leq 1$ we define the *modified square-function*

$$\mathcal{S}_s^q(u)(x) := \left(\int_{\gamma(x)} \delta(y)^{q-n-qs} |\nabla u(y)|^q \, dy \right)^{\frac{1}{q}}, \qquad x \in \partial\Omega, \ 0 < q < \infty, \tag{7.1.11}$$

where δ denotes the distance to the boundary. Note that this definition agrees with that of the classical square function in the case $s = 0$, $q = 2$. As customary, the

appropriate version for $q = \infty$ is given by

$$\mathcal{S}_s^{\infty}(u)(x) := \sup_{y\in\gamma(x)} \delta(y)^{1-s}|\nabla u(y)|, \qquad x \in \partial\Omega. \tag{7.1.12}$$

Anticipating notation to be introduced later, our main result reads as follows:

Theorem 7.1.1. *For a bounded Lipschitz domain Ω in $\mathbb{R}^n$, consider the following Dirichlet boundary-value problem:*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f \in B_s^{p,p}(\partial\Omega), \\ \mathcal{S}_s^q(u) \in L^p(\partial\Omega), \end{cases} \tag{7.1.13}$$

where

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < s < 1. \tag{7.1.14}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (7.1.13) is well-posed whenever the indices satisfy (7.1.14) and one of the following conditions:

$$\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \text{ and } 0 < s < 1; \quad \tfrac{2}{2+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \text{ and } \tfrac{2}{p} - 1 - \varepsilon < s < 1;$$

$$\tfrac{2}{1-\varepsilon} < p \leq \infty \text{ and } 0 < s < \tfrac{2}{p} + \varepsilon. \tag{7.1.15}$$

The solution has the integral representation formula

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1} f] \text{ in } \Omega, \tag{7.1.16}$$

and satisfies the estimate

$$\|\mathcal{S}_s^q(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B_s^{p,p}(\partial\Omega)}. \tag{7.1.17}$$

A similar well-posedness result is valid for data in the Sobolev space $L^p_s(\partial\Omega)$ for a suitable range of indices; cf. the discussion in §7.4.

Here and elsewhere, $\mathcal{D}$ denotes the double layer potential operator and K stands for its (principal-value) boundary version; cf. the discussion in §2.3.

As it will be shown later, the fact that $\mathcal{S}^q_s(u) \in L^p(\partial\Omega)$ entails the existence of the nontangential trace

$$u\Big|_{\partial\Omega}(x) = \lim_{y\in\gamma(x)} u(y) \tag{7.1.18}$$

at almost every boundary point x. Since, heuristically, multiplying by δ^α amounts to (fractional) integration of order α, our main estimate, (7.1.17), can be viewed as the natural intermediate analogue of (7.1.6) and (7.1.7).

In the proof of Theorem 7.1.1, the following square-function estimate for singular integral operators (patented after the harmonic double layer (2.3.3)) plays a crucial role.

Theorem 7.1.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma_y, \qquad x \in \Omega, \tag{7.1.19}$$

whose kernel satisfies

$$|\nabla_x k(x,y)| \leq C|x-y|^{-n}, \qquad x \in \Omega,\ y \in \partial\Omega. \tag{7.1.20}$$

Also, assume that

$$T1 = const. \tag{7.1.21}$$

Then

$$\|\mathcal{S}_s^q(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)} \tag{7.1.22}$$

provided

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \qquad (n-1)(\tfrac{1}{p}-1)_+ < s < 1. \tag{7.1.23}$$

In turn, the proof of the above theorem, occupying the bulk of Section 5.3, rests on atomic estimates and interpolation.

It should be noted that our methods are flexible enough to allow us to treat an analogue of Theorem 7.1.1 for the Neumann boundary-value problem; the details are presented in §7.5. Let us also point out that an appropriate analogue of the classical g-function can also be used to measure size/smoothness in the present context. We briefly elaborate on this point in the last part of §7.4. Future plans include dealing with the case of the Lamé system of elastostatics and with parabolic PDE's in a similar setting.

7.2 Preliminary results

Our first result amounts to saying that the membership of the modified square function to L^p does not depend on the particular choice of the family of non-tangential cones.

Lemma 7.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider two families of non-tangential approach regions*

$$\{\gamma(x)\}_{x\in\partial\Omega}, \quad \{\gamma'(x)\}_{x\in\partial\Omega}, \text{ such that } \gamma'(x) \subseteq \gamma(x), \ \forall\, x \in \partial\Omega. \tag{7.2.1}$$

These are defined as

$$\gamma(x) := \{y \in \Omega;\, |x-y| < \kappa\, \mathrm{dist}\,(y, \partial\Omega)\}, \qquad \kappa = \kappa(\Omega) > 1, \tag{7.2.2}$$

corresponding to two choices of the parameter κ.

Next, denote by $\mathcal{S}^q_{s,\gamma}$, $\mathcal{S}^q_{s,\gamma'}$ *the modified square-function (7.1.11) associated with (7.2.1), and fix* $0 < p \leq q \leq \infty$, $0 \leq s \leq 1$. *Then, for each* $u \in C^1_{loc}(\Omega)$,

$$\|\mathcal{S}^q_{s,\gamma'}(u)\|_{L^p(\partial\Omega)} \approx \|\mathcal{S}^q_{s,\gamma}(u)\|_{L^p(\partial\Omega)}, \tag{7.2.3}$$

with constants depending only on $p, q, \partial\Omega$, *and the geometric characteristics of the families (7.2.1).*

Proof. This follows along the lines of similar results discussed in §3 of [**18**], or §3 of [**80**], with minor alterations. □

Proposition 7.2.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and fix* $\mathcal{K}$ *a compact subset of* Ω. *Then for every harmonic function* u *in* Ω,

$$\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)} + \sup_{x\in\mathcal{K}} |u(x)| \approx \|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)}, \tag{7.2.4}$$

if, in addition to (7.1.23), $q > \frac{n}{1-s}$. *Also,*

$$\|\mathcal{S}^\infty_0(u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}(u)\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{7.2.5}$$

Proof. Recall the Besov-based non-tangential maximal function (7.1.10). Thanks to Theorem 3.9.3, for each $q > \frac{n}{1-s}$ and u harmonic in Ω, we may write

$$\begin{aligned}
\mathcal{N}^q_s(u)(x) &= \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))} \\
&\approx \left(\int_{\gamma(x)} \mathrm{dist}(y, \partial\gamma(x))^{q-qs-n} |\nabla u(y)|^q\, dy\right)^{1/q} + \sup_{x\in\mathcal{K}} |u(x)|, \\
&\leq C\, \mathcal{S}^q_s(u)(x) + \sup_{x\in\mathcal{K}} |u(x)|,
\end{aligned} \tag{7.2.6}$$

uniformly for $x \in \partial\Omega$. Here we have used the fact that $\operatorname{dist}(y, \partial\gamma(x)) \leq C\delta(y)$ for every $y \in \gamma(x)$.

Turning to the opposite inequality, choose a second family of cones γ' so that (7.2.1) holds. In particular,

$$\operatorname{dist}(y, \partial\gamma'(x)) \leq C\delta(y) \leq C \operatorname{dist}(y, \partial\gamma(x)), \tag{7.2.7}$$

uniformly for $x \in \partial\Omega$ and $y \in \gamma'(x)$. Going further, for every $x \in \partial\Omega$, we have

$$\begin{aligned} \mathcal{S}^q_{s,\gamma'}(u)(x) &= \left(\int_{\gamma'(x)} \delta(y)^{q-qs-n} |\nabla u(y)|^q \, dy \right)^{1/q} \\ &\leq C \left(\int_{\gamma(x)} \operatorname{dist}(y, \partial\gamma(x))^{q-qs-n} |\nabla u(y)|^q \, dy \right)^{1/q} \leq C\mathcal{N}^q_s(u)(x), \end{aligned} \tag{7.2.8}$$

thanks to our assumption on q, s and Theorem 3.9.3. With this in hand, Lemma 7.2.1 may now be used to finish the proof of (7.2.4).

Concerning the inequality (7.2.5), standard interior estimates for harmonic functions allows us to write, at every point $x \in \partial\Omega$,

$$\mathcal{S}^\infty_{0,\gamma'}(u)(x) = \sup_{y \in \gamma'(x)} \delta(y) |\nabla u(y)| \leq C \sup_{y \in \gamma'(x)} \left(\frac{\delta(y)}{\delta_{\partial\gamma}(y)} \sup_{z \in B(y, c\delta_{\partial\gamma}(y))} |u(z)| \right), \tag{7.2.9}$$

where the constant $c = c(\gamma, \gamma') < 1$ is chosen so that the ball $B(y, c\delta_{\partial\gamma}(y)) \subset \gamma(x)$ for every $y \in \gamma'(x)$. Then the last expression above does not exceed

$$C \sup_{y \in \gamma(x)} |u(y)| = C\mathcal{N}(u)(x), \tag{7.2.10}$$

as desired. □

Remark. It is useful to point out here that, for any function $u \in C^1(\Omega)$,

$$\mathcal{S}^\infty_1(u) = \mathcal{N}(\nabla u), \tag{7.2.11}$$

which follows by unraveling definitions.

As dictated by the representation formula (7.1.16), at the heart of our strategy for treating the problem (7.1.13) is establishing the estimate (7.1.17) when $u = \mathcal{D}f$. This, among other things, is the object of §7.3 below.

7.3 Square-function estimates on Besov scales

We debut with the

Proof of Theorem 7.1.2. First, assume $p = q$ so that

$$\|\mathcal{S}_s^q(Tf)\|_{L^q(\partial\Omega)}^q = \int_{\partial\Omega}\int_{\Omega} \chi_{\gamma(x)}(y)\,\delta(y)^{q-n-qs}|\nabla Tf(y)|^q\,dyd\sigma_x. \tag{7.3.1}$$

Since

$$\int_{\partial\Omega} \chi_{\gamma(x)}(y)\,d\sigma_x \approx \delta(y)^{n-1}, \tag{7.3.2}$$

uniformly for $y \in \Omega$, the expression on the right side of (7.3.1) is further equivalent to

$$\int_{\Omega} \delta(y)^{q-1-qs}|\nabla Tf(y)|^q\,dy. \tag{7.3.3}$$

Let us now recall the estimate

$$\|\delta^{1-s-\frac{1}{q}}|\nabla Tf|\,\|_{L^q(\Omega)} \leq C_{s,q}\|f\|_{B_s^{q,q}(\partial\Omega)}, \quad \tfrac{n-1}{n} < q \leq \infty,\ (n-1)(\tfrac{1}{q}-1)_+ < s < 1, \tag{7.3.4}$$

established in Theorem 4.1.1, which is valid for any integral operator satisfying (7.1.20) and (7.1.21). Then the desired estimate (i.e. (7.1.22) with $p = q$) follows readily from (7.3.1)-(7.3.3). This completes the proof for $p = q$.

Turning our attention to the case when $\frac{n-1}{n} < p \leq \min\{1, q\}$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$, we utilize the atomic characterization of Besov spaces given in (3.4.29); thus, the goal is to analyze the action of the operator T on an individual atom.

Suppose a is a $B_s^{p,p}(\partial\Omega)$-atom with $\operatorname{supp} a \subseteq S_r(x^*)$, for some $x^* \in \partial\Omega$ and $r > 0$. We first seek a pointwise estimate on $\mathcal{S}_s^q(Tf)$ away from $\operatorname{supp} a$. Concretely, assume that $x \notin S_{10r}(x^*)$, then

$$|\nabla Ta(y)| \leq \int_{z\in S_r(x^*)} |\nabla_y k(y,z)||a(z)|\, d\sigma_z \leq C\frac{r^{s-\frac{n-1}{p}+n-1}}{|x^*-y|^n}, \tag{7.3.5}$$

for each $y \in \gamma(x)$, owing to the size condition imposed on atom a, and a basic geometrical observation to the effect that $|x^*-y| \approx |z-y|$ for $y \in \gamma(x)$, $x \notin S_{10r}(x^*)$ and $z \in S_r(x^*)$. Consequently, for $x \notin S_{10r}(x^*)$,

$$\mathcal{S}_s^q(Tf)(x) \leq C\, r^{s-\frac{n-1}{p}+n-1} \left(\int_{\gamma(x)} \delta(y)^{q-n-qs} \frac{1}{|y-x^*|^{qn}}\, dy \right)^{\frac{1}{q}}. \tag{7.3.6}$$

By locally flattening $\partial\Omega$ to $\mathbb{R}^{n-1}$ via a bi-Lipschitz change of coordinates – so that $x^* \in \partial\Omega$ corresponds to $0 \in \mathbb{R}^{n-1}$ – we see that the right side of (7.3.6) is bounded by

$$Cr^{s-\frac{n-1}{p}+n-1} \left(\int_0^\infty \int\limits_{\substack{|y'-x|\leq \kappa t \\ y'\in\mathbb{R}^{n-1}}} \frac{t^{q-n-qs}}{(|y'|+t)^{qn}}\, dy'dt \right)^{\frac{1}{q}}, \tag{7.3.7}$$

where C depends only on the Lipschitz character of Ω. In fact, since $|x| + t \leq C(|y'|+t)$ uniformly on the domain of integration, the last integral can be controlled further in terms of

$$\int_0^\infty \frac{t^{q-n-qs+n-1}}{(|x|+t)^{qn}}\, dt. \tag{7.3.8}$$

Introducing $t=\tau|x|$ permits us to bound (7.3.8) by

$$C\frac{1}{|x|^{qn+qs-q}}\int_0^\infty\frac{\tau^{q-qs-1}}{(1+\tau)^{qn}}\,d\tau\leq C\frac{1}{|x|^{qn+qs-q}}. \tag{7.3.9}$$

Therefore,

$$\mathcal{S}_s^q(Tf)(x)\leq Cr^{s-\frac{n-1}{p}+n-1}\frac{1}{|x-x^*|^{n+s-1}},\qquad\forall\,x\notin S_{10r}(x^*). \tag{7.3.10}$$

In particular,

$$\int_{\partial\Omega\setminus S_{10r}(x^*)}|\mathcal{S}_s^q(Tf)(x)|^p\,d\sigma_x\leq C\,r^{p(s-\frac{n-1}{p}+n-1)}\int_{\partial\Omega\setminus S_{10r}(x^*)}\frac{d\sigma_x}{|x-x^*|^{p(n+s-1)}}\leq C \tag{7.3.11}$$

for some finite constant C, independent of the atom.

We next examine the contribution from $x\in S_{10r}(x^*)$. In this scenario, we employ a rescaling technique allowing us to invoke the estimates obtained in the case $p=q$. More specifically, Hölder's inequality gives

$$\int_{S_{10r}(x^*)}|\mathcal{S}_s^q(Ta)(x)|^p\,d\sigma_x\leq C_p\,r^{(n-1)(1-\frac{p}{q})}\left(\int_{S_{10r}(x^*)}|\mathcal{S}_s^q(Ta)(x)|^q\,d\sigma_x\right)^{\frac{p}{q}} \tag{7.3.12}$$

for every $0<p\leq q$. Now let us define $\tilde{a}:=r^{(n-1)(\frac{1}{p}-\frac{1}{q})}\,a$, which is easily seen to be a $B_s^{q,q}(\partial\Omega)$-atom. With this piece of notation, the right side of (7.3.12) simply reads

$$\left(\int_{S_{10r}(x^*)}|\mathcal{S}_s^q(T\tilde{a})(x)|^q\,d\sigma_x\right)^{\frac{p}{q}}. \tag{7.3.13}$$

In view of the results established for $p=q$, the last integral is bounded by the constant independent of the atom $\tilde{a}\in B_s^{q,q}(\partial\Omega)$, as desired.

Thus, at this stage,

$$\int_{\partial\Omega}|\mathcal{S}_s^q(Ta)(x)|^p\,d\sigma_x\leq C<+\infty \tag{7.3.14}$$

for some constant $C > 0$, independent of the atom $a \in B^{p,p}_s(\partial\Omega)$, proving (7.1.22) for $(n-1)/n < p \leq \min\{1, q\}$ and $(n-1)(1/p-1) < s < 1$. This completes the argument provided $q \leq 1$. Concerning the situation $q \geq 1$, the full range of indices (7.1.23) is covered by interpolation with the case $p = q$ (treated earlier) via the complex method. □

Corollary 7.3.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then for each triplet of indices s, p, q satisfying (7.1.23) there exists a finite constant $C = C(\Omega, s, p, q) > 0$ with the following significance.*

Whenever $u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$ is harmonic in Ω,

$$\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}. \tag{7.3.15}$$

Sketch of Proof. The key steps are as follows. First, it is possible to represent u in the form $u = \mathcal{C}(\operatorname{Tr} u)$ where $\operatorname{Tr} u \in B^{p,p}_s(\partial\Omega)$ is the trace of u on $\partial\Omega$, and $\mathcal{C}$ is a Cauchy-type singular integral operator whose properties closely mirror those of the harmonic double layer (cf. §6.3 for details in the case when $p > 1$). Then (7.3.15) follows from the mapping properties of $\mathcal{C}$ established in Theorem 7.1.2, in concert with the estimate $\|\operatorname{Tr} u\|_{B^{p,p}_s(\partial\Omega)} \leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}$, valid for the range of indices considered here. □

Later on, when discussing the Neumann problem, it will be useful to know the mapping properties of a class of singular integrals whose kernels are patented after that of the harmonic single layer. Our main result in this regard is as follows.

Theorem 7.3.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the integral operator*

$$Rf(x) = \int_{\partial\Omega} r(x,y) f(y)\, d\sigma_y, \qquad x \in \Omega, \tag{7.3.16}$$

such that

$$|\nabla_x \nabla_y^j r(x,y)| \leq C|x-y|^{-(n-1+j)}, \quad j=0,1,\ x\in\Omega,\ y\in\partial\Omega. \tag{7.3.17}$$

Then

$$\|\mathcal{S}^q_{1-s}(Rf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p}_{-s}(\partial\Omega)}, \tag{7.3.18}$$

whenever

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < 1-s < 1. \tag{7.3.19}$$

Proof. We use the same general strategy as in the proof of Theorem 7.1.2, although certain key technical details are different. To get started, consider the case $p=q$; much as in (7.3.1)-(7.3.3), with $1-s$ in place of s, we obtain

$$\|\mathcal{S}^q_{1-s}(Rf)\|^q_{L^q(\partial\Omega)} \approx \int_\Omega \delta(y)^{-1+qs}|\nabla Rf(y)|^q\,dy. \tag{7.3.20}$$

In concert with the estimate

$$\|\delta^{s-\frac{1}{q}}|\nabla Rf|\,\|_{L^q(\partial\Omega)} \leq C_{s,q}\|f\|_{B^{q,q}_{-s}(\partial\Omega)}, \quad \tfrac{n-1}{n} < q \leq \infty,\ (n-1)(\tfrac{1}{q}-1)_+ < 1-s < 1, \tag{7.3.21}$$

established in §4.1 under the assumptions (7.3.16)-(7.3.17), this readily yields (7.3.18) for $p=2$.

As far as the range $\frac{n-1}{n} < p \leq \min\{1,q\}$ is concerned, matters can once again be reduced to studying the action of operator R on $B^{p,p}_{-s}(\partial\Omega)$-atom a supported in $S_r(x^*)$, $x^*\in\partial\Omega$. Away from the support, for every $x\notin S_{10r}(x^*)$ and every $y\in\gamma(x)$ one computes

$$|\nabla Ra(y)| = \Big|\int_{S_r(x^*)} [\nabla_y r(y,z) - \nabla_y r(y,x^*)]a(z)\,d\sigma_z\Big|, \tag{7.3.22}$$

thanks to the vanishing moment condition imposed on $B^{p,p}_{-s}(\partial\Omega)$-atom a. To continue, we estimate the expression in the brackets:

$$\int_0^1 \frac{d}{d\theta}\nabla_y r(y,\theta z+(1-\theta)x^*)\,d\theta \leq |z-x^*| \max_{\theta\in(0,1)} |\nabla_y\nabla_z r(y,\theta z+(1-\theta)x^*)|. \quad (7.3.23)$$

In the current scenario, $|y-\theta z-(1-\theta)x^*|\approx|y-x^*|$, $|z-x^*|\leq r$ and, therefore, (7.3.22) is dominated by

$$C\frac{r^{1-s+(n-1)(1-\frac{1}{p})}}{|y-x^*|^n}. \quad (7.3.24)$$

With this in hand, the contribution corresponding to integrating away from the support of the atom can be controlled as in (7.3.5)-(7.3.11) in the proof of Theorem 7.1.2, with $1-s$ in place of s.

Finally, the contribution near the support of the atom turns out to be bounded similarly to (7.3.12)-(7.3.14) by rescaling the original atom to a $B^{q,q}_{-s}$-atom, i.e. $\tilde{a}:=r^{(n-1)(\frac{1}{p}-\frac{1}{q})}a$, and then invoking the $p=q$ results. □

7.4 Square-function estimates on Sobolev spaces

We first record and prove a general interpolation result.

Lemma 7.4.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that $0\leq s\leq 1$, $1<p<\infty$, $1\leq q\leq\infty$. Then for every linear operator T mapping functions defined on $\partial\Omega$ to functions in $C^1(\Omega)$, the three-dimensional region*

$$\mathcal{O}:=\{(s,1/p,1/q)\in[0,1]\times(0,1)\times[0,1];\ \mathcal{S}^q_s(Tf):L^p_s(\partial\Omega)\to L^p(\partial\Omega)\} \quad (7.4.1)$$

is geometrically convex.

Proof. We shall only deal with the model case when Ω is the (unbounded) domain lying above the graph of a real-valued Lipschitz function; the simple adaptations to the bounded case are left to the interested reader.

Assume that the points $(s_0, 1/p_0, 1/q_0)$ and $(s_1, 1/p_1, 1/q_1)$ belong to $\mathcal{O}$. For $\Re e\, z \in [0,1]$ let us introduce

$$(\mathcal{L}_z f)(x)(y) := \delta(y+x)^{1-\left(\frac{n}{q_0}(1-z)+\frac{n}{q_1}z\right)-\left(s_0(1-z)+s_1 z\right)}\nabla T f(y+x), \quad x \in \partial\Omega,\ y \in \Gamma, \tag{7.4.2}$$

where Γ is a fixed cone in $\mathbb{R}^n$ such that $\gamma(x) = x + \Gamma$ for each $x \in \partial\Omega$. Then

$$\mathcal{S}_s^q(T(\cdot)) : L_s^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{7.4.3}$$

if and only if

$$\mathcal{L}_z : L_s^p(\partial\Omega) \longrightarrow L^p(\partial\Omega; L^q(\Gamma)). \tag{7.4.4}$$

Observing that $\mathcal{L}_z$ represents an analytic family of operators allows us to invoke a suitable version of Stein's interpolation theorem (cf. [**19**]) along with the complex interpolation formulae for vector-valued spaces (3.2.75) to complete the proof. □

We proceed further by restricting our attention to the case of the harmonic double layer.

Proposition 7.4.2. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|\mathcal{S}_s^q(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L_s^p(\partial\Omega)} \tag{7.4.5}$$

provided one of the following conditions is satisfied:

$$
\begin{aligned}
&(i) \quad 1<p<\infty,\ \max\left\{p,\frac{p}{p-1}\right\}<q\leq\infty \ \text{ and } 0<s<1;\\
&(ii) \quad 2\leq q\leq p<\infty \ \text{ and } 0<s<1+\frac{2}{p}-\frac{2}{q};\\
&(iii) \quad 1<p\leq 2,\ 2\leq q\leq\frac{p}{p-1} \ \text{ and } 0<s<3-\frac{2}{p}-\frac{2}{q}.
\end{aligned}
\tag{7.4.6}
$$

Proof. The proof is based on the classical estimates

$$
\begin{aligned}
\|\mathbb{S}_0^2(\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L^p(\partial\Omega)}, \quad 1<p<\infty,\\
\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L^p(\partial\Omega)}, \quad 1<p<\infty,\\
\|\mathcal{N}(\nabla\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L^p_1(\partial\Omega)}, \quad 1<p<\infty,
\end{aligned}
\tag{7.4.7}
$$

used in concert with (7.2.5), (7.2.11), and the inequality

$$
\|\mathbb{S}_s^q(\mathcal{D}(f))\|_{L^q(\partial\Omega)}\leq C\|f\|_{L^q_s(\partial\Omega)}, \quad 0<s<1,\ \frac{n-1}{n}<q\leq\infty,
\tag{7.4.8}
$$

corresponding to (7.1.22) with $p=q$. The full range of indices described in the statement of the proposition is then obtained by invoking Lemma 7.4.1. □

To state our next result, set $a\wedge b:=\min\{a,b\}$.

Corollary 7.4.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then*

$$
\|\mathbb{S}_s^2(\mathcal{D}f)\|_{L^p(\partial\Omega)}\leq C\|f\|_{B_s^{p,p\wedge 2}(\partial\Omega)}
\tag{7.4.9}
$$

whenever

$$
\tfrac{n-1}{n}<p<\infty, \qquad (n-1)(\tfrac{1}{p}-1)_+<s<\min\{1,\tfrac{2}{p}\}.
\tag{7.4.10}
$$

Proof. The case $\frac{n-1}{n} < p \leq 2$ corresponds precisely to Theorem 7.1.2, whereas the case $2 < p < \infty$ is covered by Proposition 7.4.2, thanks to the classical embedding $B_s^{p,2}(\partial\Omega) \hookrightarrow L_s^p(\partial\Omega)$ for $0 < s < 1$, $2 \leq p < \infty$. □

We conclude this section by stating the analogue of Proposition 7.4.2 and Corollary 7.4.3 in the context of the harmonic single layer. The proofs of these results require only minor alterations compared to the proof presented in the case of the double layer, and are left to the interested reader.

Proposition 7.4.4. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|\mathbb{S}_{1-s}^q(\mathcal{S}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p_{-s}(\partial\Omega)} \tag{7.4.11}$$

provided one of the following conditions is satisfied:

$$\begin{aligned}
&(i) \quad 1 < p < \infty,\ \max\left\{p, \frac{p}{p-1}\right\} < q \leq \infty \ \text{ and } 0 < s < 1;\\
&(ii) \quad 2 \leq q \leq p < \infty \ \text{ and } 0 < 1 - s < 1 + \frac{2}{p} - \frac{2}{q};\\
&(iii) \quad 1 < p \leq 2,\ 2 \leq q \leq \frac{p}{p-1} \ \text{ and } 0 < 1 - s < 3 - \frac{2}{p} - \frac{2}{q}.
\end{aligned} \tag{7.4.12}$$

Corollary 7.4.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|\mathbb{S}_{1-s}^2(\mathcal{S}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p\wedge 2}_{-s}(\partial\Omega)} \tag{7.4.13}$$

whenever

$$\tfrac{n-1}{n} < p < \infty, \qquad (n-1)(\tfrac{1}{p} - 1)_+ < 1 - s < \min\{1, \tfrac{2}{p}\}. \tag{7.4.14}$$

Remark. There is also a natural modification of the classical Littlewood-Paley g-function. In the case of an unbounded domain $\Omega \subset \mathbb{R}^n$, lying above the graph of

a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, this is given by

$$g_s^q(u)(x) := \left(\int_0^\infty t^{q-1-qs} |\nabla u(x + te_n)|^q \, dt \right)^{1/q}, \quad x \in \partial\Omega. \tag{7.4.15}$$

The appropriate variant for the case of a bounded Lipschitz domain can be defined as

$$g_s^q(u)(x) := \left(\int_0^\eta t^{q-1-qs} |\nabla u(x + t\theta(x))|^q \, dt \right)^{1/q}, \tag{7.4.16}$$

where θ is some unit transversal field to $\partial\Omega$ and $\eta > 0$ is a fixed parameter, depending on the geometric characteristics of Ω.

Many of the results developed in this section have suitable versions in terms of (7.4.15)-(7.4.16). In particular, the Theorems 7.1.2 and 7.3.2 have natural analogues in this context. One result we would like to single out is as follows. For every harmonic function u defined on Lipschitz domain Ω, we have

$$\|g_s^q(u)\|_{L^p(\partial\Omega)} \leq C \|\mathcal{S}_s^q(u)\|_{L^p(\partial\Omega)}, \quad \text{if } \ 0 < p, q < \infty, \tag{7.4.17}$$

$$\|g_s^q(u)\|_{L^p(\partial\Omega)} \geq C \|\mathcal{S}_s^q(u)\|_{L^p(\partial\Omega)}, \quad \text{if } \ 1 < q \leq p < \infty. \tag{7.4.18}$$

We leave the details to the interested reader (cf. [**118**] p. 90-91 and [**80**] p. 154-155 for a related discussion).

7.5 Applications to partial differential equations

We are now ready to present the last details in the

Proof of Theorem 7.1.1. To start, fix two appropriate families of nontangential cones, γ, γ', as in (7.2.1) and set

$$\dot{\mathcal{N}}_{s,\gamma'}^\infty(u)(x) := \sup\Big\{ \tfrac{|u(z_1)-u(z_2)|}{|z_1-z_2|^s};\ z_1, z_2 \in \gamma'(x) \Big\}, \quad x \in \partial\Omega. \tag{7.5.1}$$

In this context, we claim that there exists $C = C(q, \partial\Omega) > 0$ so that for every point $x \in \partial\Omega$ and every harmonic function u in Ω,

$$\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) \leq C\, \mathcal{S}^{q}_{s}(u)(x), \qquad \frac{n-1}{n} < q \leq \infty. \tag{7.5.2}$$

Indeed, by (a slight variation of) Theorem 3.9.3 and the Mean Value Formula for harmonic functions,

$$\begin{aligned}\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) &\leq C \sup_{z\in\gamma'(x)} \operatorname{dist}(z, \partial\gamma'(x))^{1-s} |\nabla u(z)| \\ &\leq C \sup_{z\in\gamma'(x)} \Big(\operatorname{dist}(z, \partial\gamma(x))^{q-n-qs} \int_{2|z-y|\leq \operatorname{dist}(z,\partial\gamma(x))} |\nabla u(y)|^{q}\, dy \Big)^{\frac{1}{q}}. \end{aligned} \tag{7.5.3}$$

Next, observe that $z \in \gamma'(x)$, $2|y-z| \leq \operatorname{dist}(z, \partial\gamma(x))$ entail

$$\operatorname{dist}(y, \partial\gamma(x)) \leq C \operatorname{dist}(z, \partial\gamma(x)) \leq C \operatorname{dist}(z, \partial\Omega), \tag{7.5.4}$$

so that, utilizing this back in (7.5.3), we arrive at

$$\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) \leq C \Big(\int_{\gamma(x)} \delta(y)^{q-n-qs} |\nabla u(y)|^{q}\, dy \Big)^{\frac{1}{q}} = C\, \mathcal{S}^{q}_{s}(u)(x), \tag{7.5.5}$$

as claimed.

Next, the inequality (7.5.2) leads to the conclusion that

$$\begin{aligned} \mathcal{S}^{q}_{s}(u) \in L^{p}(\partial\Omega) \;&\Longrightarrow\; \dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) < +\infty \text{ for a.e. } x \in \partial\Omega \\ &\Longrightarrow\; u \in C^{s}(\overline{\gamma'(x)}) \text{ for a.e. } x \in \partial\Omega \\ &\Longrightarrow\; \text{there exists } \lim_{y\in\gamma'(x)} u(y) \text{ for a.e. } x \in \partial\Omega. \end{aligned} \tag{7.5.6}$$

This justifies the claim made in connection with (7.1.11) in §7.1 to the effect that

$$\Delta u = 0 \text{ in } \Omega \;\&\; \mathcal{S}^{q}_{s} u \in L^{p}(\partial\Omega),\ 0 < s < 1, \Longrightarrow \exists\, u\Big|_{\partial\Omega} \in L^{p}(\partial\Omega). \tag{7.5.7}$$

Equally important, the above reasoning shows that, for $0 < s < 1$,

$$\mathcal{S}_s^q u \in L^p(\partial\Omega) \Longrightarrow \mathcal{N}_{s,\gamma'}^{\infty}(u) \in L^p(\partial\Omega). \tag{7.5.8}$$

Now, the implication

$$\Delta u = 0 \text{ in } \Omega,\ \mathcal{N}_s^{\infty}(u) \in L^p(\partial\Omega),\ u\Big|_{\partial\Omega} = 0 \Longrightarrow u \equiv 0 \text{ in } \Omega \tag{7.5.9}$$

has been proved in Theorem 6.1 of [**87**], granted that s, p are as in (7.1.14)-(7.1.15). Clearly, this takes care of the uniqueness statement in Theorem 7.1.1.

Finally, existence is seen by taking u as in (7.1.16). The issue of the invertibility of the operator $\frac{1}{2}I + K$ on the spaces $B_s^{p,p}(\partial\Omega)$ for the range of indices described in Theorem 7.1.1 is addressed in Theorem 4.3.12, while the fact that $\mathcal{S}_s^q(u) \in L^p(\partial\Omega)$ follows from our results in §7.3. □

We conclude with a brief discussion pertaining to Neumann boundary conditions.

Theorem 7.5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \partial_\nu u\Big|_{\partial\Omega} = f \in B_{-s}^{p,p}(\partial\Omega),\ \int_{\partial\Omega} f\, d\sigma = 0, \\ \mathcal{S}_{1-s}^q u \in L^p(\partial\Omega), \end{cases} \tag{7.5.10}$$

where $\frac{n-1}{n} < p \leq q \leq \infty$, and $(n-1)(\frac{1}{p} - 1)_+ < 1 - s < 1$.

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (7.5.10) is well-posed if any of the three conditions in (7.1.15) is satisfied with $1 - s$ in place of s.

Furthermore, the solution has the integral representation formula

$$u = \mathcal{S}[(-\tfrac{1}{2}I + K^*)^{-1} f] + const \quad \text{in } \Omega, \tag{7.5.11}$$

and obeys the estimate

$$\|\mathcal{S}^{q}_{1-s}(u)\|_{L^{p}(\partial\Omega)} \leq C(\partial\Omega,p,q,s)\|f\|_{B^{p,p}_{-s}(\partial\Omega)}. \tag{7.5.12}$$

The main steps in the proof of Theorem 7.5.1 closely mirror those taken in the proof of Theorem 7.1.1. Indeed, the issue of existence of the solution for (7.5.10) is covered by the representation formula (7.5.11) and Theorem 1.5 in [**87**] to the effect that the operator $\frac{1}{2}I+K^{*}$ is invertible on the class of spaces under discussion. As far as uniqueness is concerned, the key step is to observe that, thanks to (7.5.8),

$$\Delta u=0 \text{ on } \Omega, \qquad \mathcal{S}^{q}_{1-s}(u)\in L^{p}(\partial\Omega) \Longrightarrow \mathcal{N}^{\infty}_{1-s}(u)\in L^{p}(\partial\Omega). \tag{7.5.13}$$

However, it was shown in [**87**] that, under our current assumptions made on s and p,

$$\Delta u=0 \text{ in } \Omega,\ \mathcal{N}^{\infty}_{1-s}(u)\in L^{p}(\partial\Omega),\ \partial_{\nu}u\Big|_{\partial\Omega}=0 \Longrightarrow u\equiv const \text{ in } \Omega. \tag{7.5.14}$$

Taking u as in (7.5.11) yields existence, given that the operator $-\frac{1}{2}I+K^{*}$ is invertible on the subspace of $B^{p,p}_{-s}(\partial\Omega)$, consisting of functions with vanishing moment, for the range of indices in question (Theorem 4.3.12). Note that the square-function estimates in the last part of §7.3 guarantee that (7.5.12) holds. □

Chapter 8

The Poisson problem for the Lamé system on low dimensional Lipschitz domains

8.1 Motivation and statement of main results

Consider the Lamé operator of linear elastostatics in $\mathbb{R}^3$,

$$\mathcal{L}\vec{u} := \mu\Delta\vec{u} + (\lambda+\mu)\nabla(\operatorname{div}\vec{u}), \quad \mu > 0,\ \lambda > -\tfrac{2}{3}\mu. \tag{8.1.1}$$

In this paper we study the well-posedness of the Poisson problems for the system of elastostatics equipped with either Dirichlet or Neumann-type boundary conditions in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ (see §8.6 for the two-dimensional case):

$$\mathcal{L}\vec{u} = \vec{f} \text{ in } \Omega, \qquad \operatorname{Tr}\vec{u} = \vec{g} \text{ on } \partial\Omega, \tag{8.1.2}$$

$$\mathcal{L}\vec{u} = \vec{f} \text{ in } \Omega, \qquad \partial_\nu\vec{u} = \vec{g} \text{ on } \partial\Omega. \tag{8.1.3}$$

Hereafter Tr stands for the *trace* map on $\partial\Omega$, and ∂_ν is *traction conormal* defined by

$$\partial_\nu\vec{u} := \lambda(\operatorname{div}\vec{u})\nu + \mu[\nabla\vec{u} + \nabla\vec{u}^t]\nu \tag{8.1.4}$$

where ν is the unit normal to $\partial\Omega$ and the superscript t indicates transposition (in this case, of the matrix $\nabla\vec{u} = (\partial_j u^\alpha)_{j,\alpha}$).

Relying on the method of layer potentials and suitable Rellich-Nečas-Payne-Weinberger formulas, the boundary value problems (8.1.2)-(8.1.3) with $\vec{f} = 0$ and $\vec{g} \in L^p(\partial\Omega)$, $2 - \varepsilon < p < 2 + \varepsilon$, have been treated (in all space-dimensions) by B. Dahlberg, C. Kenig, and G. Verchota (cf. [**33**]). In the three-dimensional setting, these results have been subsequently extended to optimal ranges of p's ($2 - \varepsilon < p \leq \infty$ for the Dirichlet boundary condition, and $1 < p < 2 + \varepsilon$ for the traction boundary condition, with $\varepsilon = \varepsilon(\partial\Omega) > 0$) in [**32**]. More recently, the results for the Dirichlet problem (i.e., (8.1.2) with $\vec{f} = 0$ and $\vec{g} \in L^p(\partial\Omega)$) have been further extended in dimension $n \geq 4$ to the range $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ by Z. Shen in [**114**] (though determining the optimal range of p's remains an open problem at the moment). In all these works, nontangential maximal function estimates are sought for the solution $\vec{u}$.

Following the breakthrough in the case of the Dirichlet Laplacian in [**68**], as well as the subsequent developments in [**40**], a study of (8.1.2)–(8.1.3) on Sobolev-Besov spaces on low-dimensional Lipschitz domains has been initiated in [**94**], where the optimal ranges of indices have been identified under assumption that all spaces involved are *Banach* (roughly speaking, this amounts to the requirement that all the integrability exponents are greater than one). Here we take the next natural step and extend the scope of this study as to allow the consideration of Besov and Triebel-Lizorkin spaces for the full range of indices $0 < p, q \leq \infty$. This builds on the work in [**89**] where we have recently dealt with the case of the Laplacian.

Our current goals are also motivated by questions regarding the regularity of the Green potentials associated with the Lamé system and here we prove new mapping properties of these potentials on L^p and Hardy spaces. For example, we show that (any) two derivatives on the elastic Green potential with Dirichlet boundary condition yield an operator bounded on $L^p(\Omega)$, $1 < p \leq 2$, if $\Omega \subset \mathbb{R}^2$ is a convex domain and $|\lambda| < \sqrt{3}\mu$, and on the Hardy space $h^1(\Omega)$ if $\Omega \subset \mathbb{R}^3$ is a convex polyhedron.

Before stating our main results we briefly elaborate on notation and terminology. We refer to, e.g., [**127**] for definitions and basic properties of the Besov and Triebel-Lizorkin scales, $B^{p,q}_\alpha(\mathbb{R}^n)$, $F^{p,q}_\alpha(\mathbb{R}^n)$, $n \geq 2$. An open, connected set $\Omega \subset \mathbb{R}^3$ is called a *Lipschitz domain* if its boundary can be locally described by means of Lipschitz graphs (in appropriate systems of coordinates); see, e.g., [**118**] for a more detailed discussion. Given $\Omega \subset \mathbb{R}^3$ Lipschitz and $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, we set

$$\begin{aligned} &B^{p,q}_\alpha(\Omega) := \{u \in D'(\Omega) : \exists\, v \in B^{p,q}_\alpha(\mathbb{R}^3) \text{ with } v|_\Omega = u\}, \\ &B^{p,q}_{\alpha,0}(\Omega) := \{u \in B^{p,q}_\alpha(\mathbb{R}^3) : \operatorname{supp} u \subseteq \overline{\Omega}\}, \end{aligned} \tag{8.1.5}$$

with similar definitions for $F^{p,q}_\alpha(\Omega)$ and $F^{p,q}_{\alpha,0}(\Omega)$. Here, $D(\Omega)$ denotes the collection of test functions in Ω (equipped with the usual inductive limit topology), while $D'(\Omega)$ stands for the space of distributions in Ω. Finally, if $|s| < 1$, $B^{p,q}_s(\partial\Omega)$ stands for the Besov class on the Lipschitz manifold $\partial\Omega$, obtained by transporting (via a partition of unity and pull-back) the standard scale $B^{p,q}_s(\mathbb{R}^2)$. Typically, we shall work with vector-valued distributions, such as $\vec{u} = (u^1, u^2, u^3)$, etc; however, even when warranted, our notation for the various function spaces employed in this paper does not emphasize the vector nature of the objects involved (this will

eventually be clear from the context).

To state our first result, for each $a \in \mathbb{R}$ set $(a)_+ := \max\{a, 0\}$.

Theorem 8.1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, and, for $\frac{2}{3} < p \leq \infty$, $0 < q \leq \infty$, $2\big(\frac{1}{p} - 1\big)_+ < s < 1$, consider the following boundary value problem:*

$$\mathcal{L}\vec{u} = \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{g} \in B^{p,q}_{s}(\partial\Omega). \tag{8.1.6}$$

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that (8.1.6) is well-posed if the pair (s, p) satisfies one of the following three conditions:

$$(I): \qquad \tfrac{2}{2+\varepsilon} < p \leq \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1; \tag{8.1.7}$$

$$(II): \qquad \tfrac{2}{1+\varepsilon} \leq p \leq \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1;$$

$$(III): \qquad \tfrac{2}{1-\varepsilon} \leq p \leq \infty \quad \text{and} \quad 0 < s < \tfrac{2}{p} + \varepsilon.$$

Furthermore, the solution has an integral representation formula in terms of (elastic) layer potential operators and satisfies the natural estimate

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|\vec{g}\|_{B^{p,q}_{s}(\partial\Omega)}. \tag{8.1.8}$$

Finally, an analogous well-posedness result holds on the Triebel-Lizorkin scale, i.e., for the problem

$$\mathcal{L}\vec{u} = \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{g} \in B^{p,p}_{s}(\partial\Omega). \tag{8.1.9}$$

This time, it is assumed that $p \neq \infty$ and

$$\Big(\min\big\{\tfrac{1}{3p} + 1,\ \tfrac{s}{2} + 1\big\}\Big)^{-1} < q \leq \infty. \tag{8.1.10}$$

In particular, the above inequality is true when $\min\{p, 1\} \leq q \leq \infty$.

In the class of Lipschitz domains this result is sharp, but one may take $\varepsilon = 1$ *if* $\partial\Omega \in C^1$.

Figure 8.1 below depicts the two-dimensional pentagonal region consisting of all points with coordinates $(s, 1/p)$ satisfying (8.1.7):

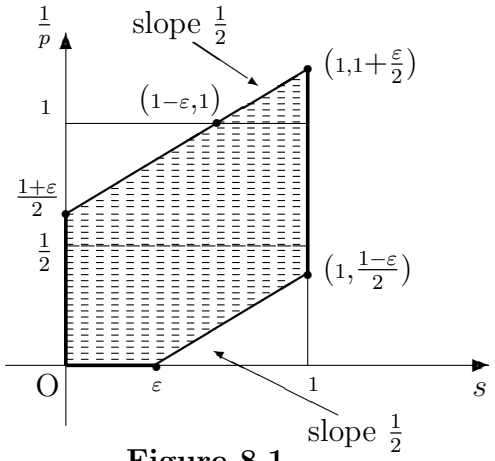

Figure 8.1.

To formulate our main result regarding the Poisson problem with traction boundary conditions we introduce one more piece of notation. Let Ψ be the six-dimensional linear space of vector-valued functions $\vec{\psi}$ in $\mathbb{R}^3$ satisfying

$$\partial_i \psi^j + \partial_j \psi^i = 0, \qquad 1 \leq i, j \leq 3, \tag{8.1.11}$$

(note that each $\vec{\psi} \in \Psi$ is a null-solution of $\mathcal{L}$ and $\partial_\nu \vec{\psi} = 0$), and for a space of vector-valued distributions $\mathcal{X}$ on a set $\mathcal{O}$, define

$$\Psi^{\perp} \cap \mathcal{X} := \{\vec{u} \in \mathcal{X} : \langle \vec{u}, \vec{\psi}|_{\mathcal{O}} \rangle = 0, \ \forall\, \vec{\psi} \in \Psi\}. \tag{8.1.12}$$

Theorem 8.1.2. *Let* Ω *be a bounded, connected Lipschitz domain in* $\mathbb{R}^3$ *and, for* $\frac{2}{3} < p \leq \infty$, $0 < q \leq \infty$, *and* $2\big(\frac{1}{p} - 1\big)_+ < s < 1$, *consider the following boundary*

value problem:

$$\mathcal{L}\vec{u}=\vec{f}\Big|_{\Omega},\quad \vec{f}\in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega),\quad \vec{u}\in\Psi^{\perp}\cap B^{p,q}_{s+\frac{1}{p}}(\Omega),\quad \partial_{\nu}^{\vec{f}}\vec{u}=\vec{g}\in B^{p,q}_{s-1}(\partial\Omega), \tag{8.1.13}$$

where the data are assumed to satisfy the necessary compatibility condition

$$\langle\vec{f},\vec{\psi}\rangle=\langle\vec{g},\vec{\psi}\rangle,\qquad\forall\,\vec{\psi}\in\Psi. \tag{8.1.14}$$

Then there exists $\varepsilon=\varepsilon(\Omega)\in(0,1]$ such that (8.1.13) has a unique solution if the pair s,p satisfies one of the three conditions in (8.1.7). In addition, the solution has an integral representation formula in terms of (elastic) layer potential operators and satisfies the estimate

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)}\leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)}+C\|\vec{g}\|_{B^{p,q}_{s-1}(\partial\Omega)}. \tag{8.1.15}$$

An analogous well-posedness result holds for the problem

$$\mathcal{L}\vec{u}=\vec{f}\Big|_{\Omega},\quad \vec{f}\in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega),\quad \vec{u}\in\Psi^{\perp}\cap F^{p,q}_{s+\frac{1}{p}}(\Omega),\quad \partial_{\nu}^{\vec{f}}\vec{u}=\vec{g}\in B^{p,p}_{s-1}(\partial\Omega), \tag{8.1.16}$$

assuming (8.1.14), (8.1.10) and $p\neq\infty$. When $\partial\Omega\in C^1$, one can take $\varepsilon=1$ in each case.

Above, $\partial_{\nu}^{\vec{f}}\vec{u}$ stands for a suitable concept of conormal derivative which takes into account both the field $\vec{u}$ as well as $\vec{f}$. See §8.3 for a detailed definition.

Our strategy for dealing with the problems (8.1.6), (8.1.9), (8.1.13), (8.1.16), is to rely on integral representation formulas of the solution in terms of elastic layer

potential operators. This approach brings into focus the properties of singular integral operators involved in such representations. In this particular, we study the boundedness of the elastic layer potentials on Besov and Triebel-Lizorkin scales in §8.2, as well as the invertibility of some of their boundary traces in §8.4. As a key prerequisite, in §8.3 we include a discussion of sharp results for traction derivatives on Lipschitz domains. The properties of the Green potentials are then analyzed in §8.5. Finally, in §8.6, the two-dimensional setting is briefly discussed.

8.2 Estimates for singular integral operators

Even though we shall work in the three-dimensional setting, all results proved in this section have natural versions in $\mathbb{R}^n$ with $n \geq 2$. Fix a Lipschitz domain $\Omega \subset \mathbb{R}^3$ and denote by $d\sigma$ and ν the surface measure and the unit normal on $\partial\Omega$, respectively. We start by reviewing the definitions of the elastic layer potentials associated with Ω. Recall that the standard fundamental solution for the system of elastostatics is given by Kelvin matrix $\Gamma = (\Gamma_{\alpha\beta})_{\alpha,\beta}$, where for each $1 \leq \alpha, \beta \leq 3$,

$$\Gamma_{\alpha\beta}(x) := \frac{3}{8\pi}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right)\frac{\delta_{\alpha\beta}}{|x|} + \frac{3}{8\pi}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right)\frac{x_\alpha x_\beta}{|x|^3}, \quad x = (x_\alpha)_\alpha \in \mathbb{R}^3. \tag{8.2.1}$$

Here and elsewhere, $\delta_{\alpha\beta}$ stands for the standard Kronecker symbol. For vector valued densities on $\partial\Omega$, the *single (elastic) layer potential* operator is defined by

$$\mathcal{S}\vec{f}(x) := \int_{\partial\Omega} \Gamma(x-y)\,\vec{f}(y)\,d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \partial\Omega, \tag{8.2.2}$$

where $d\sigma$ stands for the canonical surface measure on $\partial\Omega$. Also, the *double (elastic) layer potential* operator is given by

$$\mathcal{D}\vec{f}(x) := \int_{\partial\Omega} \big(\partial_{\nu_y}\Gamma(x-y)\big)^t \ \vec{f}(y)\, d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \partial\Omega. \tag{8.2.3}$$

Here the operator ∂_ν (defined in (8.1.4)) applies to each column of the matrix Γ. The well-known jump-relations for the operators (8.2.2)-(8.2.3) then read:

$$\begin{aligned}\lim_{\substack{x\to z\\ x\in\gamma_\pm(z)}} \mathcal{D}\vec{f}(x) &= \pm\tfrac{1}{2}\vec{f}(z) + \text{p.v.} \int_{\partial\Omega} \big(\partial_{\nu_y}\Gamma(z-y)\big)^t \ \vec{f}(y)\, d\sigma(y) \\ &=: \big(\pm\tfrac{1}{2}I + K\big)\,\vec{f}(z), \quad z \in \partial\Omega,\end{aligned} \tag{8.2.4}$$

and

$$\lim_{\substack{x\to z\\ x\in\gamma_\pm(z)}} \partial_\nu\mathcal{S}\vec{f}(x) = \big(\mp\tfrac{1}{2}I + K^*\big)\,\vec{f}(z), \quad z \in \partial\Omega, \tag{8.2.5}$$

where K^* denotes the formal adjoint of K. In (8.2.4)-(8.2.5), $\gamma_\pm(z)$ are suitable nontangential approach regions with vertex at $z \in \partial\Omega$, which are contained in $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3 \setminus \bar{\Omega}$, respectively. See, e.g., [DKV] for more details.

Finally, the elastic Newtonian potential is defined as

$$\Pi\vec{f}(x) := \int_{\mathbb{R}^n} \Gamma(x-y)\vec{f}(y)\, dy, \qquad x \in \mathbb{R}^3. \tag{8.2.6}$$

With regard to this operator, we first note the following consequence of known results about the mapping properties of pseudo-differential operators on Besov and Triebel-Lizorkin scales.

Proposition 8.2.1. *Suppose that* $0 < p < \infty$, $0 < q \leq \infty$, $3\big(\frac{1}{\min\{1,p,q\}} - 1\big) < \alpha$, *and fix two arbitrary scalar-valued functions* $\phi, \psi \in C_c^\infty(\mathbb{R}^3)$. *Then*

$$\phi\Pi\psi : F^{p,q}_{\alpha-2}(\mathbb{R}^3) \longrightarrow F^{p,q}_{\alpha}(\mathbb{R}^3) \tag{8.2.7}$$

is a bounded operator. Furthermore, for $0 < p, q \leq \infty$ *and* $3\big(\frac{1}{\min\{1,p\}} - 1\big) < \alpha$,

$$\phi\Pi\psi : B^{p,q}_{\alpha-2}(\mathbb{R}^3) \longrightarrow B^{p,q}_{\alpha}(\mathbb{R}^3) \tag{8.2.8}$$

is also bounded.

We shall now state two theorems whose proofs are intertwined and will be presented together. We would like to remark that the argument presented in §4.1 can also be applied in this situation, however we present an alternative proof invoking specific properties of the Lamé system.

Theorem 8.2.2. *Consider a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^3$ *and assume that* p, q, s *are fixed such that* $\frac{2}{3} < p \leq \infty$, $2(\frac{1}{p} - 1)_+ < s < 1$ *and* $0 < q \leq \infty$. *Then*

$$\partial_\nu\Pi : B^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega), \tag{8.2.9}$$

$$\partial_\nu\Pi : F^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \quad \text{if } p \neq \infty, \tag{8.2.10}$$

are well-defined, linear and bounded operators.

Theorem 8.2.3. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$ *and assume that the indices* p, s *satisfy* $\frac{2}{3} < p \leq \infty$ *and* $2(\frac{1}{p} - 1)_+ < s < 1$. *Then*

$$\mathcal{D} : B^{p,q}_{s}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \qquad \mathcal{S} : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{8.2.11}$$

are well-defined, bounded, linear operators for each $0 < q \leq \infty$.

The same holds for

$$\mathcal{D} : B^{p,p}_{s}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \qquad \mathcal{S} : B^{p,p}_{s-1}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega), \tag{8.2.12}$$

provided $p \neq \infty$ *and (8.1.10) is satisfied.*

Proof of Theorems 8.2.3-8.2.2. This parallels arguments developed for harmonic layer potentials in §4.1, so we shall present a sketch in which only the novel aspects are emphasized. We proceed in a series of steps starting with

Step I. Consider first the case when $1 \leq p = q \leq \infty$ for Besov spaces, and $1 < p < \infty$, $q = 2$, for Triebel-Lizorkin spaces. For a proof of Theorems 8.2.3-8.2.2 under these assumptions, see **[94]**.

Step II. For $0 < p < \infty$, $\alpha \in \mathbb{R}$, introduce the space

$$\begin{aligned} &\mathbb{L}^p_\alpha(\Omega) := \{\vec{u} \in L^p(\Omega) : \mathcal{L}\vec{u} = 0 \text{ and } \|\vec{u}\|_{\mathbb{L}^p_\alpha(\Omega)} < \infty\}, \text{ where} \\ &\|\vec{u}\|_{\mathbb{L}^p_\alpha(\Omega)} := \|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}\vec{u}|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha\rangle-1} \|\nabla^j\vec{u}\|_{L^p(\Omega)}. \end{aligned} \tag{8.2.13}$$

Throughout the paper, $\langle\alpha\rangle$ will denote the smallest nonnegative integer greater than or equal to α, and $\nabla^j\vec{u}$ will stand for vector of all mixed-order partial derivatives of order j of the components of $\vec{u}$. Also, here and elsewhere, we set $\delta(x) := \text{dist}(x, \partial\Omega)$ for $x \in \Omega$.

The claim we make in this scenario is that

$$\mathcal{D} : B^{p,p}_s(\partial\Omega) \longrightarrow \mathbb{L}^p_{s+\frac{1}{p}}(\Omega), \qquad \mathcal{S} : B^{p,p}_{s-1}(\partial\Omega) \longrightarrow \mathbb{L}^p_{s+\frac{1}{p}}(\Omega), \tag{8.2.14}$$

are bounded operators provided $\frac{2}{3} < p \leq \infty$ and $2(\frac{1}{p} - 1)_+ < s < 1$. This is proved by means of atomic decomposition results for Besov spaces and pointwise estimates on the kernel of the elastic layer potentials in question. The reader is referred to §4.1 for details.

Step III. The following embedding holds:

$$\mathbb{L}^p_\alpha(\Omega) \hookrightarrow F^{p,2}_\alpha(\Omega), \quad \text{if} \quad 0 < p < \infty, \quad \alpha \geq 0. \tag{8.2.15}$$

This has been established in **[68]** for $1 < p < \infty$, so the novelty here is the consideration of the range $0 < p \leq 1$, on which we shall focus subsequently. First, we verify that, if $0 < p \leq 1$,

$$\mathcal{L}\vec{u} = 0 \text{ in } \Omega \ \& \ \vec{u} \in L^p(\Omega) \implies \vec{u} \in h^p(\Omega), \tag{8.2.16}$$

where membership to the local Hardy space $h^p(\Omega)$ can be characterized as follows. Fix $\psi \in C_c^\infty(B(0,1))$ such that $\int_{B(0,1)} \psi(x)\,dx = 1$ and set $\psi_t(x) := t^{-n}\psi(x/t)$. Then for any $0 < p \leq 1$ and any $\vec{u} \in D'(\Omega)$

$$\vec{u} \in h^p(\Omega) \iff (\vec{u})^+(x) := \sup_{0<t<\delta(x)} |(\psi_t * \vec{u})(x)| \in L^p(\Omega). \tag{8.2.17}$$

With an eye on (8.2.16) we first observe that null-solutions of $\mathcal{L}$ satisfy a sub-averaging property. That is, there exists a finite, positive constant C such that if $\mathcal{L}\vec{u} = 0$ in Ω then

$$|\vec{u}(x)| \leq \frac{C}{r^3} \int_{B_r(x)} |\vec{u}(y)|\,dy, \qquad \forall\, x \in \Omega,\ 0 < r < \delta(x). \tag{8.2.18}$$

Indeed, it was proved in **[81]** that there exist two constants C_1 and C_2 such that for every null-solution $\vec{u} = (u^\alpha)_\alpha$ of the Lamé system in an open domain Ω in $\mathbb{R}^3$

$$u^\alpha(x) = \frac{1}{4\pi\rho^2} \int_{|y-x|=\rho} \sum_\beta \Big(C_1\delta_{\alpha\beta} + C_2 \frac{x_\alpha - y_\alpha}{|x-y|}\frac{x_\beta - y_\beta}{|x-y|} \Big) u^\beta(y)\,d\sigma_y, \tag{8.2.19}$$

for $x \in \Omega$, $0 < \rho < \delta(x)$, $1 \leq \alpha \leq 3$. Multiply both sides above by ρ^2 then integrate in ρ for $0 \leq \rho \leq r$. We get

$$u^\alpha(x) = \frac{3}{4\pi r^3} \int_{B_r(x)} \sum_\beta \Big(C_1\delta_{\alpha\beta} + C_2 \frac{x_\alpha - y_\alpha}{|x-y|}\frac{x_\beta - y_\beta}{|x-y|} \Big) u^\beta(y)\,dy, \tag{8.2.20}$$

for $x \in \Omega$, $0 < r < \delta(x)$, $1 \leq \alpha \leq 3$, which readily yields (8.2.18). Making a dilation (to reduce matters to $r = 1$) and differentiating under the integral sign,

we also obtain, in a similar fashion,

$$|\nabla^k \vec{u}(x)| \leq \frac{C_k}{r^{3+k}} \int_{B_r(x)} |\vec{u}(y)|\, dy, \quad x \in \Omega,\ 0 < r < \delta(x),\ k = 0, 1, 2, ... \tag{8.2.21}$$

It is known (going back to an old proof of Hardy and Littlewood) that the sub-averaging property (8.2.18) entails

$$|\vec{u}(x)|^p \leq \frac{C_p}{r^3} \int_{B_r(x)} |\vec{u}(y)|^p\, dy, \quad x \in \Omega,\ 0 < r < \delta(x),\ 0 < p < \infty. \tag{8.2.22}$$

Thus, for each $0 < p < \infty$, $k = 0, 1, 2, ...$, $x \in \Omega$ and $0 < r < \delta(x)$ we may write

$$\begin{aligned} |\nabla^k \vec{u}(x)| &\leq \frac{C_k}{r^{3+k}} \int_{B_{r/2}(x)} |\vec{u}(y)|\, dy \leq \frac{C_k}{r^{3+k}} \int_{B_{r/2}(x)} \left(\frac{C_p}{r^3} \int_{B_{r/2}(y)} |\vec{u}(z)|^p\, dz \right)^{1/p} dy \\ &\leq \frac{C_k}{r^{3+k}} \int_{B_{r/2}(x)} \left(\frac{C_p}{r^3} \int_{B_r(x)} |\vec{u}(z)|^p\, dz \right)^{1/p} dy \\ &\leq \frac{C_k}{r^k} \left(\frac{C_p}{r^3} \int_{B_r(x)} |\vec{u}(y)|^p\, dy \right)^{1/p} \end{aligned} \tag{8.2.23}$$

so that for each $0 < p < \infty$ and $k = 0, 1, ...$,

$$|\nabla^k \vec{u}(x)|^p \leq \frac{C_{p,k}}{r^{3+kp}} \int_{B_r(x)} |\vec{u}(y)|^p\, dy, \quad x \in \Omega,\ 0 < r < \delta(x). \tag{8.2.24}$$

In particular, by (8.2.24) and Fubini's theorem,

$$\int_\Omega |\delta(x)^k \nabla^k \vec{u}(x)|^p\, dx \leq C \int_\Omega \left(\delta(x)^{-3} \int_{B_{\delta(x)/2}(x)} |\vec{u}(y)|^p\, dy \right) dx \leq C \int_\Omega |\vec{u}(x)|^p\, dx \tag{8.2.25}$$

since $\delta(x) \approx \delta(y)$ uniformly for $y \in B_{\delta(x)/2}(x)$.

Next, observe that $\mathcal{L}\vec{u} = 0$ in Ω implies $\Delta^2 \vec{u} = 0$ in Ω. Bring in the mean-value theorem for bi-harmonic functions in $\mathbb{R}^3$:

$$\vec{u}(x) = \frac{1}{4\pi\rho^2} \int_{|y-x|=\rho} \vec{u}(y)\, d\sigma_y - \frac{\rho^2}{6} (\Delta \vec{u})(x), \quad x \in \Omega,\ 0 < \rho < \delta(x), \tag{8.2.26}$$

and pick a non-negative, radial function $\psi \in C_c^\infty(B(0,1))$ such that $\int_{B_1(0)} \psi(x)\,dx = 1$. We then write

$$\begin{aligned}
(\vec{u})^+(x) &= \sup_{0<t<\delta(x)} \frac{1}{t^3} \left| \int_{B_t(x)} \psi\left(\frac{|x-y|}{t}\right) \vec{u}(y)\,dy \right| \\
&= \sup_{0<t<\delta(x)} \frac{1}{t^3} \left| \int_0^t \psi\left(\frac{\rho}{t}\right) \int_{|x-z|=t} \vec{u}(z)\,d\sigma_z d\rho \right| \\
&\leq C \sup_{0<t<\delta(x)} \frac{1}{t} \int_0^t \psi\left(\frac{\rho}{t}\right) \Big(|\vec{u}(x)| + t^2 |\nabla^2 \vec{u}(x)| \Big)\,d\rho \\
&\leq C\Big(|\vec{u}(x)| + \delta(x)^2 |\nabla^2 \vec{u}(x)| \Big), \qquad x \in \Omega. \qquad (8.2.27)
\end{aligned}$$

Therefore,

$$\|(\vec{u})^+\|_{L^p(\Omega)} \leq C\|\vec{u}\|_{L^p(\Omega)} + C\|\delta^2 \nabla^2 \vec{u}\|_{L^p(\Omega)} \leq C\|\vec{u}\|_{L^p(\Omega)}, \qquad (8.2.28)$$

which, in view of (8.2.17), proves (8.2.16). Granted the identification $F_0^{p,2}(\Omega) = h^p(\Omega)$ for $0 < p \leq 1$, this justifies (8.2.15) when $\alpha = 0$. This argument can be also carried out when $\alpha \in \mathbb{N}$ is large, thanks to the Hardy space extension results of A. Miyachi [**103**].

There remains to interpolate between these partial results. To this end, we note that with $0 < p < \infty$ fixed, the spaces $\mathbb{L}^p_\alpha(\Omega)$, indexed by $\alpha \in \mathbb{R}$, form a complex interpolation scale. With analytic functions (of several complex variables) in place of elastic fields, this has been proved in [**119**] but essentially the argument goes through since it only uses the sub-averaging properties of the elastic fields. This finishes the proof of (8.2.15).

Step IV. In concert, Steps II and III justify the boundedness of the operators (8.2.12) for $q = 2$, provided $\frac{2}{3} < p \leq \infty$ and $2(\frac{1}{p} - 1)_+ < s < 1$. With this in hand,

interpolation with the results in Step I allows to further extend the range of q's for which the operators (8.2.12) are bounded.

Step V. The fact that the operator in (8.2.10) is well-defined and bounded when $\frac{2}{3} < p \leq 1$, $2(\frac{1}{p} - 1) < s < 1$ and $0 < q \leq \infty$ can be proved using an atomic decomposition result for the space $F^{p,q}_{s+1/p-2,0}(\Omega)$ in which the atoms are supported in interior Whitney cubes for Ω. This, in turn, is proved in §3.7 (cf. also the decomposition results for Hardy spaces from [**21**]).

Step VI. Formally, the operators $\partial_\nu \Pi$ and $\mathcal{D}$ are dual to each other (as an inspection of their kernels shows). In fact, so are $\mathrm{Tr} \circ \Pi$ and $\mathcal{S}$, where the trace operator is analyzed in the next section. Dualize the results in Step IV (when $1 < p, q < \infty$) and interpolate with the results in Step V, then finally dualize this new round of results (again, when $1 < p, q < \infty$) and interpolate with the results in Step IV. Close inspection of these arguments shows that this procedure yields precisely the range of indices advertised in Theorems 8.2.3-8.2.2 for the operators (8.2.10) and (8.2.12).

Step VII. Finally, the results on Besov scales follow from what we have proved so far and real interpolation. □

8.3 Conormal derivative

When dealing with conormal derivatives, we adopt a slightly more general point of view and assume that the Lamé system (8.1.1) is written in the form

$$(\mathcal{L}\vec{u})^\alpha = \sum_\beta \sum_{j,k} a^{\alpha\beta}_{jk} \partial_j \partial_k u^\beta, \quad \alpha = 1, 2, 3, \tag{8.3.1}$$

for some (real-valued, constant) coefficient tensor, i.e. some 9×9 matrix $A:=\left(a_{jk}^{\alpha\beta}\right)_{j,k,\alpha,\beta}$. The coefficient tensor A is not uniquely determined by the initial system of PDE's and each choice gives rise to a conormal derivative, i.e.

$$(\partial_\nu\vec{u})^\alpha=\sum_\beta\sum_{j,k}a_{jk}^{\alpha\beta}\nu_j\partial_k u^\beta,\quad \alpha,\beta,j,k=1,2,3. \tag{8.3.2}$$

Evidently, the traction conormal (8.1.4) is of this type, by choosing

$$a_{jk}^{\alpha\beta}:=\lambda\delta_{\alpha j}\delta_{\beta k}+\mu(\delta_{\alpha\beta}\delta_{jk}+\delta_{\alpha k}\delta_{\beta j}),\quad \alpha,\beta,j,k=1,2,3, \tag{8.3.3}$$

which we shall assume from now on.

Our goal is to make sense of (8.3.2) on the boundary of a Lipschitz domain $\Omega\subset\mathbb{R}^3$ when the field $\vec{u}$ is not regular enough in Ω in order to make sense of $\mathrm{Tr}\,(\partial_k u^\beta)$. The approach we propose will take into account not only the smoothness of the field $\vec{u}$ itself but the smoothness of $\mathcal{L}\vec{u}$ as well. For starters, when $1<p,q<\infty$, $0<s<1$, $\vec{u}\in F^{p,q}_{s+\frac{1}{p}}(\Omega)$ and $\vec{f}\in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$ are such that $\mathcal{L}\vec{u}=\vec{f}|_\Omega$, then we define $\partial_\nu^{\vec{f}}\vec{u}\in F^{p,q}_{s-1}(\partial\Omega)=\left(F^{p',q'}_{1-s}(\partial\Omega)\right)^*$, $1/p+1/p'=1$, $1/q+1/q'=1$, by

$$\begin{aligned}\langle\partial_\nu^{\vec{f}}\vec{u},\vec{\psi}\rangle\;&:=\;\langle\vec{f}\,,\,\mathrm{Ex}(\vec{\psi})\rangle+\mu\langle\nabla\vec{u}+\nabla\vec{u}^{\,t},\nabla\mathrm{Ex}(\vec{\psi})+\nabla(\mathrm{Ex}(\vec{\psi}))^t\rangle\\&\qquad+\lambda\langle\mathrm{div}\,\vec{u}\,,\,\mathrm{div}\,\mathrm{Ex}(\vec{\psi})\rangle,\qquad\forall\,\vec{\psi}\in B^{p',q'}_{1-s}(\partial\Omega),\end{aligned} \tag{8.3.4}$$

where Ex is the (vector-valued version of the) extension operator (1.1.18). Note that all pairings of distributions in Ω are meaningful, so this procedure does yield a definition of the traction conormal, though limited to the range $1<p,q<\infty$ (since, e.g., $B^{p,q}_{s-1}(\partial\Omega)$ fails to be a dual space if $\min\{p,q\}\leq 1$). We nonetheless have:

Theorem 8.3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and assume that $\frac{2}{3} < p \leq \infty$ and $2(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$. Then one can define a concept of traction derivative such that*

$$\Big\{(\vec{u},\vec{f}) \in B^{p,q}_{s+\frac{1}{p}}(\Omega)\oplus B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) : \mathcal{L}\vec{u} = \vec{f}|_{\Omega}\Big\} \ni (\vec{u},\vec{f}) \mapsto \partial^{\vec{f}}_{\nu}\vec{u} \in B^{p,q}_{s-1}(\partial\Omega) \tag{8.3.5}$$

is well-defined, linear and bounded, i.e.,

$$\|\partial^{\vec{f}}_{\nu}\vec{u}\|_{B^{p,q}_{s-1}(\partial\Omega)} \leq C\Big(\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)} + \|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)}\Big) \tag{8.3.6}$$

holds, and which reduces to (8.3.4) when $p, q > 1$.

Finally, similar conclusions are valid in the context of Triebel-Lizorkin spaces, for the map

$$\Big\{(\vec{u},\vec{f}) \in F^{p,q}_{s+\frac{1}{p}}(\Omega)\oplus F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) : \mathcal{L}\vec{u} = \vec{f}|_{\Omega}\Big\} \ni (\vec{u},\vec{f}) \mapsto \partial^{\vec{f}}_{\nu}\vec{u} \in B^{p,p}_{s-1}(\partial\Omega), \tag{8.3.7}$$

assuming that s, p are as before, $p \neq \infty$, and $\Big(\min\big\{\frac{1}{3p}+1,\,\frac{s}{2}+1\big\}\Big)^{-1} < q < \frac{2}{3}\Big(\frac{1}{p}-1\Big)^{-1}_{+}$.

Proof. We shall break down the argument into a series of steps.

Step I. Firstly, we introduce an operator

$$\partial^{\mathcal{L}}_{\nu} : \mathbb{L}^p_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \tag{8.3.8}$$

which is well-defined, linear and bounded whenever $\frac{2}{3} < p < \infty$ and $2(\frac{1}{p}-1)_+ < s < 1$, and which reduces to (8.1.4) when $\vec{u}$ is sufficiently regular (say, $\vec{u} \in C^{\infty}(\overline{\Omega})$).

To this end, consider the system of elastostatics represented in the form (8.3.1) and suppose that $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function such that $\varphi(0) = 0$ and, for

some $M, R > 0$,

$$\Sigma_R := \{(x', \varphi(x')) : x' \in \mathbb{R}^2,\ |x'| < R\} \subset \partial\Omega, \tag{8.3.9}$$

$$D_{R,M} := \{x + te_3 : x \in \Sigma_R,\ 0 < t < 2M\} \subset \Omega.$$

For every vector field $\vec{u} = (u^1, u^2, u^3)$ such that $\mathcal{L}\vec{u} = 0$ in Ω we set

$$u_j^\alpha(x) := -\int_0^M (\partial_j u^\alpha)(x + te_3)\, dt, \quad 1 \le j, \alpha \le 3, \quad x \in D_{R,M/2}, \tag{8.3.10}$$

$$\vec{v}_j := (u_j^\alpha)_{1\le\alpha\le 3} \text{ in } D_{R,M/2},\ 1 \le j \le 3, \tag{8.3.11}$$

where $e_3 = (0, 0, 1) \in \mathbb{R}^3$. Then the following identities can be easily checked:

$$\partial_k u_j^\alpha = \partial_j u_k^\alpha \text{ in } D_{R,M/2} \quad \text{for every} \quad \alpha, j, k = 1, 2, 3, \tag{8.3.12}$$

$$u_3^\alpha(x) = u^\alpha(x) - u^\alpha(x + Me_3) \quad \forall\, x \in D_{R,M/2}, \quad \forall\, \alpha = 1, 2, 3, \tag{8.3.13}$$

$$\mathcal{L}\vec{v}_j = 0 \text{ in } D_{R,M/2} \quad \text{for every} \quad j = 1, 2, 3. \tag{8.3.14}$$

Next, we prove an incisive claim to the effect that, generally speaking,

$$\vec{u} \in \mathbb{L}^p_\theta(\Omega), \quad 0 < p < \infty,\ \theta \ge 0 \implies \sum_{j=1}^3 \|\vec{v}_j\|_{\mathbb{L}^p_\theta(D_{R,M/2})} \le C\|\vec{u}\|_{\mathbb{L}^p_\theta(\Omega)}. \tag{8.3.15}$$

Indeed, since $\operatorname{dist}(x, \partial D_{R,M/2}) \le \delta(x) = \operatorname{dist}(x, \partial\Omega)$, uniformly for $x \in D_{R,M/2}$, we may write

$$\begin{aligned}
&\int_{D_{R,M/2}} \Big(\operatorname{dist}(x, \partial D_{R,M/2})^{\langle\theta\rangle-\theta} |\nabla^{\langle\theta\rangle}\vec{v}_j(x)|\Big)^p\, dx \\
&\le C \int_{D_{R,M/2}} \delta(x)^{p(\langle\theta\rangle-\theta)} \left(\int_0^M |(\nabla^{\langle\theta\rangle+1}\vec{u})(x + te_3)|\, dt\right)^p dx \\
&\le C \int_{\substack{x'\in\mathbb{R}^2\\ |x'|<R}} \int_0^M r^{p(\langle\theta\rangle-\theta)} \left(\int_0^M |(\nabla^{\langle\theta\rangle+1}\vec{u})((x', \varphi(x') + r + t))|\, dt\right)^p dr dx' \\
&\le C \int_{\substack{x'\in\mathbb{R}^2\\ |x'|<R}} \int_0^{2M} r^{p(\langle\theta\rangle-\theta)} \left(\int_r^{2M} |\nabla^{\langle\theta\rangle+1}\vec{u}|^*((x', \varphi(x') + \lambda))\, d\lambda\right)^p dr dx'
\end{aligned} \tag{8.3.16}$$

by changing variables, first $D_{R,M/2} \ni x = (x', \varphi(x') + r) \mapsto (x', r) \in \{x' \in \mathbb{R}^2 : |x'| < R\} \times (0, M)$, and, second $\lambda := t + r \in (0, 2M)$. Above, $(\cdot)^*$ denotes the maximal radial operator defined (for a function w defined in $D_{R,M}$) as

$$w(x) := \sup_{t>0:\, x+te_n \in D_{R,M}} |w(x + te_n)|, \qquad x \in D_{R,M}. \tag{8.3.17}$$

This subtlety (i.e., the consideration of the radial maximal operator) is only needed in order to be able to use Hardy's inequality when $p \leq 1$ (when $p > 1$, we shall disregard $(\cdot)^*$, i.e., interpret it as the identity operator). This is because the ordinary Hardy's inequality continues to hold in the L^p-setting even when $0 < p \leq 1$ provided the function involved is non-increasing (which is certainly the case for $|\nabla^{\langle\theta\rangle+1}\vec{u}|^*$ in the transversal direction); cf., e.g., the appendix in [**119**] for a precise statement. Invoking Hardy's inequality just alluded to, we can further dominate the last expression in (8.3.16) by

$$\begin{aligned} & C \int_{\substack{x' \in \mathbb{R}^2 \\ |x'|<R}} \int_0^{2M} \lambda^{p(\langle\theta\rangle-\theta+1)} \left(|\nabla^{\langle\theta\rangle+1}\vec{u}|^*((x', \varphi(x') + \lambda)) \right)^p \, d\lambda dx' \\ & \qquad \leq C \int_{D_{R,M}} \delta(x)^{p(\langle\theta\rangle-\theta+1)} \left(|\nabla^{\langle\theta\rangle+1}\vec{u}|^*(x) \right)^p \, dx. \end{aligned} \tag{8.3.18}$$

To continue, we consider a Whitney decomposition of Ω (cf. [**118**]). Specifically, there exists a family of balls $B_j = B_{r_j}(z_j)$, with $z_j \in \Omega$ and $\delta(z_j) \approx r_j$, which cover Ω and such that their concentric doubles have finite overlap. Then Fatou's lemma implies

$$\begin{aligned} & \int_{D_{R,M}} \delta(x)^{p(\langle\theta\rangle-\theta+1)} \left(|\nabla^{\langle\theta\rangle+1}\vec{u}|^*(x) \right)^p \, dx \\ & \qquad \leq \sum_j \int_{D_{R,M}} \delta(x)^{p(\langle\theta\rangle-\theta+1)} \left(\chi_{B_j \cap D_{R,M}} |\nabla^{\langle\theta\rangle+1}\vec{u}|^p \right)^* (x) \, dx. \end{aligned} \tag{8.3.19}$$

Note that $(\chi_{B_j\cap D_{R,M}}|\nabla^{\langle\theta\rangle+1}\vec{u}|^p)^*$ is supported in a set of measure $\leq Cr_j^3$. Also, on that set, $\delta(x)^{p(\langle\theta\rangle-\theta+1)} \leq Cr_j^{p(\langle\theta\rangle-\theta+1)}$. Finally, $\|(\chi_{B_j\cap D_{R,M}}|\nabla^{\langle\theta\rangle+1}\vec{u}|^p)^*\|_{L^\infty(D_{R,M})} \leq \|\nabla^{\langle\theta\rangle+1}\vec{u}\|^p_{L^\infty(Bj)}$. All in all,

$$\int_{D_{R,M}} \delta(x)^{p(\langle\theta\rangle-\theta+1)} \left(\chi_{B_j\cap D_{R,M}}|\nabla^{\langle\theta\rangle+1}\vec{u}|^p\right)^*(x)\,dx \leq C\,r_j^{3+p(\langle\theta\rangle-\theta+1)}\|\nabla^{\langle\theta\rangle+1}\vec{u}\|^p_{L^\infty(Bj)}$$
$$\leq C\,r_j^{p(\langle\theta\rangle-\theta)}\int_{B_{2r_j}(z_j)}|\nabla^{\langle\theta\rangle}\vec{u}(x)|^p\,dx \tag{8.3.20}$$

where the last inequality follows from the sub-averaging property of $\nabla^{\langle\theta\rangle}\vec{u}$ (note that this is still a null-solution for the constant coefficient operator $\mathcal{L}$, so (8.2.24) applies). Since the family $\{B_{2r_j}(z_j)\}_j$ has finite overlap and $\delta(x) \approx r_j$ for $x \in B_{2r_j}(z_j)$, adding up inequalities of the form (8.3.20) in j and recalling (8.3.19) leads to the conclusion that

$$\int_{D_{R,M}} \delta(x)^{p(\langle\theta\rangle-\theta+1)} \left(|\nabla^{\langle\theta\rangle+1}\vec{u}|^*(x)\right)^p\,dx \leq C\int_\Omega \delta(x)^{p(\langle\theta\rangle-\theta)}|\nabla^{\langle\theta\rangle}\vec{u}(x)|^p\,dx. \tag{8.3.21}$$

This readily implies (8.3.15).

After this preamble, we can introduce the operator $\partial^{\mathcal{L}}_\nu$ acting on $\vec{u} \in \mathbb{L}^p_{s+\frac{1}{p}}(\Omega)$ by

$$(\partial^{\mathcal{L}}_\nu\vec{u})^\alpha\Big|_{\Sigma_R} := \sum_{\beta,j,k} a^{\alpha\beta}_{jk}\,\partial_{\tau_{j3}}\Big[\mathrm{Tr}\,u^\beta_k\Big]\Big|_{\Sigma_R} + \sum_{\beta,j,k} a^{\alpha\beta}_{jk}\nu_j(\partial_k u^\beta)(\cdot+Me_3)\Big|_{\Sigma_R}, \quad 1\leq\alpha\leq 3, \tag{8.3.22}$$

where the u^β_j's are as in (8.3.10) and $\partial_{\tau_{jk}} := \nu_j\partial_k - \nu_k\partial_j$, $j,k=1,2,3$, are tangential derivatives. Note that

$$\partial_{\tau_{jk}} \circ \mathrm{Tr} : F^{p,2}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \qquad 1\leq j,k\leq 3, \tag{8.3.23}$$

are (by Theorem 1.1.3) well-defined and bounded operators, so the first sum in (8.3.22) is meaningful. The second sum above is harmless (in fact, in L^∞) since the first-order partial derivatives of u^β are evaluated away from $\partial\Omega$. Then (8.3.22) can be used to define $\partial_\nu^{\mathcal{L}}\vec{u}$ globally on $\partial\Omega$, by gluing the various local definitions together using a smooth partition of unity.

By (8.2.15), it follows that the operator (8.3.8) is bounded. There remains to show that its action is compatible with (8.1.4) in the case when $\vec{u}\in C^\infty(\overline{\Omega})$, i.e. that, in this situation, the formula (8.3.22) coincides with the definition of conormal (traction) derivative (8.3.2). Indeed, for $\alpha=1,2,3$, we may write, based on (8.3.12)-(8.3.14), that

$$\begin{aligned}\sum_{\beta,j,k}a_{jk}^{\alpha\beta}\nu_j\partial_k u^\beta &= \sum_{\beta,j,k}a_{jk}^{\alpha\beta}\nu_j\Big(\partial_3 u_k^\beta+\partial_k u^\beta(\cdot+Me_3)\Big)\\ &= \sum_{\beta,j,k}a_{jk}^{\alpha\beta}\Big(\nu_j\partial_3-\nu_3\partial_j\Big)u_k^\beta+\sum_{\beta,j,k}a_{jk}^{\alpha\beta}\nu_j\partial_k u^\beta(\cdot+Me_3)\\ &= \sum_{\beta,j,k}a_{jk}^{\alpha\beta}\partial_{\tau_{j3}}u_k^\beta+\sum_{\beta,j,k}a_{jk}^{\alpha\beta}\nu_j\partial_k u^\beta(\cdot+Me_3),\end{aligned}\tag{8.3.24}$$

where in the second step we have used the fact that

$$\sum_\beta\sum_{j,k}a_{jk}^{\alpha\beta}\partial_j u_k^\beta(x)=-\int_0^M\left(\sum_\beta\sum_{j,k}a_{jk}^{\alpha\beta}\partial_j\partial_k u^\beta\right)(x+te_n)\,dt=0.\tag{8.3.25}$$

This finishes the proof of the claim made in Step I.

Step II. We next claim that

$$\mathbb{L}^p_\alpha(\Omega)=\{\vec{u}\in F^{p,2}_\alpha(\Omega):\,\mathcal{L}\vec{u}=0\text{ in }\Omega\}\ \text{ if }\ 0<p<\infty\text{ and }\alpha>3(\tfrac{1}{p}-1)_+.\tag{8.3.26}$$

This, however, is proved using similar arguments to those employed in the proof of (8.2.15); cf. §3.9 for details in the case of harmonic functions. In particular,

(8.3.26) shows that

$$\partial_\nu^{\mathcal{L}} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \tag{8.3.27}$$

is a bounded operator provided $\frac{2}{3} < p < \infty$, $2(\frac{1}{p}-1)_+ < s < 1$ and $q = 2$. By embeddings, the same is true for $0 < q \leq 2$ and, by the discussion just prior to the statement of Theorem 8.3.1, for $1 < q < \infty$ granted that $1 < p < \infty$. Interpolating between these various partial results allows us to conclude that the operator (8.3.27) is bounded for the range of indices specified in the statement of the theorem in connection with (8.3.7).

Step III. Finally, assume that $\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega)$ and $\vec{f} \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$ are such that $\mathcal{L}\vec{u} = \vec{f}|_\Omega$. We then set

$$\partial_\nu^{\vec{f}}\vec{u} := \partial_\nu \Pi\vec{f} + \partial_\nu^{\mathcal{L}}\Big(\vec{u} - (\Pi\vec{f})|_\Omega\Big) \tag{8.3.28}$$

where Π stands for the elastic Newtonian potential (8.2.6) and $\partial_\nu^{\mathcal{L}}$ is the operator (8.3.8), (8.3.27). Now the properties of $(\vec{u},\vec{f}) \mapsto \partial_\nu^{\vec{f}}\vec{u}$ which have been advertised in the statement of the theorem can be checked directly from (8.3.28) keeping in mind the mapping properties of the operators involved. This finishes the proof of the claims made about (8.3.7). The case of (8.3.5) follows from this and real interpolation. □

8.4 Boundary integral operators and proofs of main results

Recall the linear space of vector-valued functions Ψ introduced in §8.1 and the convention (8.1.12).

Theorem 8.4.1. *For each bounded Lipschitz domain Ω in $\mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\partial\Omega) \in (0,1]$ with the following significance. Let $\frac{2}{3} < p \leq \infty$ and $2(\frac{1}{p}-1)_+ < s < 1$ be so that one of the conditions in (8.1.7) is satisfied. Then the operators*

$$\tfrac{1}{2}I + K : \quad B_s^{p,p}(\partial\Omega) \longrightarrow B_s^{p,p}(\partial\Omega), \tag{8.4.1}$$

$$-\tfrac{1}{2}I + K^* : \quad \Psi^{\perp} \cap B_{s-1}^{p,p}(\partial\Omega) \longrightarrow \Psi^{\perp} \cap B_{s-1}^{p,p}(\partial\Omega) \tag{8.4.2}$$

are invertible.

Proof. When $1 \leq p, q \leq \infty$ this is proved in **[94]**. An inspection of the argument there shows that it is possible to further extend this range, by means of the interpolation techniques developed in Chapter 3, in the fashion indicated in the statement of the theorem. □

We are now ready to present the

Proof of Theorems 8.1.1-8.1.2. The solution of the Poisson problem with Dirichlet boundary conditions admits the following integral representation formula

$$\vec{u} = (R_\Omega \circ \Pi \circ E_\Omega)\vec{f} + \mathcal{D} \circ \Big(\tfrac{1}{2}I + K\Big)^{-1}\Big(\vec{g} - \mathrm{Tr}\,(R_\Omega \circ \Pi \circ E_\Omega \vec{f})\Big) \ \text{ in } \Omega, \tag{8.4.3}$$

where R_Ω and E_Ω are, respectively, the restriction from $\mathbb{R}^3$ to Ω and the extension from Ω to $\mathbb{R}^3$, with preservation of class (cf. **[127]** for a discussion and references). Then the relevant properties of $\vec{u}$ can be read off (8.4.3), given that all operators involved are well-understood. The uniqueness part is a consequence of embeddings and the results for $1 \leq p, q \leq \infty$ from **[94]**.

As for Theorem 8.1.2, we proceed similarly, starting with the integral represen-

tation formula

$$\vec{u} = (R_\Omega \circ \Pi)\vec{f} + \mathcal{S} \circ \left(-\tfrac{1}{2}I + K^*\right)^{-1}\left(\vec{g} - \partial_\nu \Pi \vec{f}\right) \text{ in } \Omega. \tag{8.4.4}$$

The condition (8.1.14) ensures that $\vec{g} - \partial_\nu \Pi \vec{f} \in \Psi^\perp \cap B^{p,p}_{s-1}(\partial\Omega)$ so Theorem 8.4.1 applies. □

8.5 Regularity of elastic Green potentials in Lipschitz domains

Given $\Omega \subset \mathbb{R}^3$ bounded, Lipschitz domain, we let $\mathbf{G}_D$, $\mathbf{G}_N$ denote the solution operators for the Poisson problem for the Lamé operator with homogeneous Dirichlet and traction boundary conditions, respectively. In the case of the traction problem, it is understood that the given datum $\vec{f}$ is first normalized so that $\langle \vec{f}, \vec{\psi} \rangle = 0$ for each $\vec{\psi} \in \Psi$.

Theorem 8.5.1. *For each bounded, connected Lipschitz domain Ω in $\mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ with the following significance. Assume that $\frac{2}{3} < p \leq \infty$, $2(\frac{1}{p} - 1)_+ < s < 1$ are such that one of the three conditions in (8.1.7) is satisfied. Then, with $\alpha := s + \frac{1}{p} - 2$, the operators*

$$\mathbf{G}_D : B^{p,q}_\alpha(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega) \quad \text{if } \ 0 < q \leq \infty, \tag{8.5.1}$$

$$\mathbf{G}_D : F^{p,q}_\alpha(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega) \quad \text{if (8.1.10) holds and } p \neq \infty, \tag{8.5.2}$$

are well-defined and bounded. Similar results hold for traction boundary conditions, i.e. for

$$\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega) \quad \text{if } \ 0 < q \leq \infty, \tag{8.5.3}$$

$$\mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega) \quad \text{if (8.1.10) holds and } p \neq \infty. \tag{8.5.4}$$

These results are sharp in the class of Lipschitz domains. For $\partial\Omega \in C^1$ one can take $\varepsilon = 1$.

The two-dimensional pentagonal region consisting of all points with coordinates $(\alpha, 1/p)$ such that α, p are as in the statement of this theorem has the following shape:

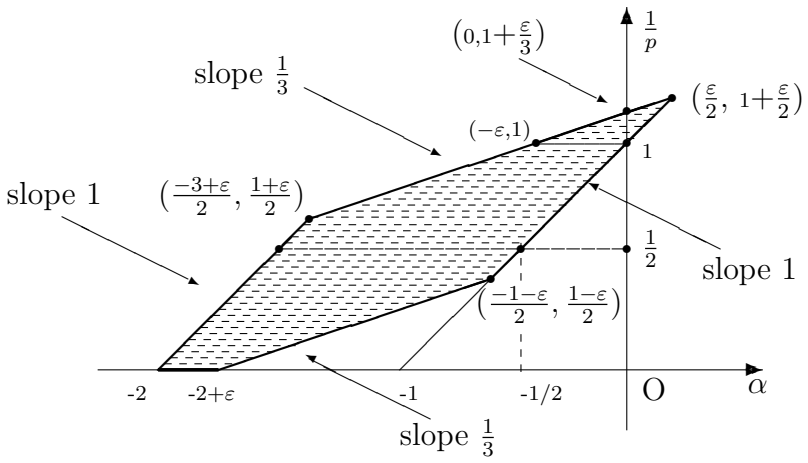

Figure 8.2.

Theorem 8.5.1 is a direct consequence of Theorems 8.1.1-8.1.2, given the results discussed in §§8.2-8.4.

Recall that the Triebel-Lizorkin space $F^{p,q}_{\alpha}(\mathbb{R}^3)$ reduces to the *local* Hardy class $h^p(\mathbb{R}^3)$ if $0 < p < \infty$, $q = 2$, $\alpha = 0$ (as is well-known, the latter coincides with the Lebesgue class $L^p(\mathbb{R}^3)$ if $1 < p < \infty$). Consequently, for $0 < p < \infty$ and $\Omega \subset \mathbb{R}^3$ Lipschitz domain, we set $h^p(\Omega) := F^{p,2}_0(\Omega)$ and $h^p_0(\Omega) := F^{p,2}_{0,0}(\Omega)$ (so that $h^p(\Omega) = h^p_0(\Omega) = L^p(\Omega)$ if $1 < p < \infty$).

Corollary 8.5.2. *Retain the notation and hypotheses made in Theorem 8.5.1 on*

the indices involved and the domain Ω. Then

$$\nabla^2\mathbf{G}_D : B^{p,q}_\alpha(\Omega) \longrightarrow B^{p,q}_\alpha(\Omega), \qquad \nabla^2\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_\alpha(\Omega) \tag{8.5.5}$$

if $0 < q \leq \infty$, and

$$\nabla^2\mathbf{G}_D : F^{p,q}_\alpha(\Omega) \longrightarrow F^{p,q}_\alpha(\Omega), \qquad \nabla^2\mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_\alpha(\Omega) \tag{8.5.6}$$

if (8.1.10) holds and $p \neq \infty$, are well-defined, bounded operators.

In particular, there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that the operators

$$\nabla^2\mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \quad \text{whenever } \tfrac{3}{3+\varepsilon} < p < 1, \tag{8.5.7}$$

$$\nabla^2\mathbf{G}_N : h^p_0(\Omega) \longrightarrow h^p(\Omega), \quad \text{whenever } \tfrac{3}{3+\varepsilon} < p < 1, \tag{8.5.8}$$

are well-defined and bounded. Once again, one can take $\varepsilon = 1$ whenever Ω has a C^1 boundary, in which case the range of p's in (8.5.7)-(8.5.8) becomes $\frac{3}{4} < p < 1$.

The second part of the corollary is related to a result proved by D.-C. Chang, S.G. Krantz and E.M. Stein in [**21**] regarding the Green potentials associated with the Laplacian on C^∞ domains. In [**89**] we have successfully dealt with the Laplacian on arbitrary Lipschitz domains, thus answering in the affirmative a conjecture made by Stein at al in [**21**]. In this paper, we take the first steps in the direction of further extending this circle of ideas to elliptic systems.

To state our next result, let $W^p_k(\Omega)$ denote the usual L^p-based Sobolev space of order (of smoothness) k in Ω. Also, by $\overset{\circ}{W}{}^p_k(\Omega)$ we denote the closure of C^∞, compactly supported functions in $W^p_k(\Omega)$ and by $L^{p,\infty}(\Omega)$ the usual weak-L^p space in Ω.

Corollary 8.5.3. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, and denote by $G_D(x,y)$ the integral kernel of the elastic Dirichlet Green potential operator $\mathbf{G}_D$ associated with Ω. Then $\nabla G_D(x,\cdot) \in L^{3/2,\infty}(\Omega)$ and $\delta(\cdot)^{-1}G_D(x,\cdot) \in L^{3/2,\infty}(\Omega)$, uniformly for in $x \in \Omega$. In particular, $G_D(x,\cdot) \in \overset{\circ}{W}{}^p_1(\Omega)$, uniformly in $x \in \Omega$, for any $1 < p < \frac{3}{2}$.*

Proof. For each $k = 0, 1, 2, ...$ and $1 < p < \infty$, introduce the weak Sobolev space

$$W^{p,\infty}_k(\Omega) := \{f \in L^{p,\infty}(\Omega) : \partial^\alpha f \in L^{p,\infty}(\Omega),\ |\alpha| \leq k\},$$

plus a similar definition for $W^{p,\infty}_k(\mathbb{R}^3)$ with Ω replaced by $\mathbb{R}^3$. We shall proceed in several steps.

Step I. We claim that, with $(\cdot,\cdot)_{\theta,q}$ denoting the usual real-interpolation bracket,

$$\Big(W^{p_0}_k(\mathbb{R}^3), W^{p_1}_k(\mathbb{R}^3)\Big)_{\theta,\infty} = W^{p,\infty}_k(\mathbb{R}^3),$$

for each $k = 0, 1, 2, ...$, $1 < p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$.

To see this, bring in the Bessel potential of complex order z, acting on a tempered distribution f according to

$$\widehat{J_z f}(\xi) := (1 + 4\pi^2|\xi|^2)^{-z/2}\hat{f}(\xi),$$

where 'hat' denotes the Fourier transform. As is well-known (cf. **[17]**), J_k maps $L^p(\mathbb{R}^3)$ isomorphically onto $W^p_k(\mathbb{R}^3)$ for $1 < p < \infty$ (with J_{-k} as inverse). On the one hand, this and real interpolation immediately yield that $J_k : L^{p,\infty}(\mathbb{R}^3) \longrightarrow \Big(W^{p_0}_k(\mathbb{R}^3), W^{p_1}_k(\mathbb{R}^3)\Big)_{\theta,\infty} = W^{p,\infty}_k(\mathbb{R}^3)$ is an isomorphism whenever $1 < p_0, p_1 < \infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. On the other hand, proceeding much as

in the proof of Theorem 7 in [**17**] we obtain that, under the same assumptions on the indices, $J_k : L^{p,\infty}(\mathbb{R}^3) \longrightarrow W_k^{p,\infty}(\mathbb{R}^3)$ is an isomorphism as well. All in all, the claim made at the beginning of Step I follows.

Step II. We shall now show that

$$\Big(W_k^{p_0}(\Omega), W_k^{p_1}(\Omega)\Big)_{\theta,\infty} = W_k^{p,\infty}(\Omega),$$

for each $k = 0, 1, 2, ...$, $1 < p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$.

The left-to-right inclusion is trivial and, granted the result in Step I, the key step in proving the opposite one is the claim that Stein's extension operator $E : W_k^p(\Omega) \to W_k^p(\mathbb{R}^3)$ (cf. [**118**]) also extends as a bounded operator from $W_k^{p,\infty}(\Omega)$ into $W_k^{p,\infty}(\mathbb{R}^3)$.

To see this, we note the commutator identity $\partial^\alpha(Ef) = E(\partial^\alpha f) + \sum_{|\beta|\leq|\alpha|} T_{\alpha\beta}(\partial^\beta f)$ where the linear operators $T_{\alpha\beta}$ map $L^p(\Omega)$ boundedly into $L^p(\mathbb{R}^3)$ for $1 < p < \infty$. This latter claim is seen as in p. 174 of [**68**] and p. 187 of [**118**]. By real interpolation we may therefore conclude that $E : W_k^{p,\infty}(\Omega) \to W_k^{p,\infty}(\mathbb{R}^3)$ boundedly, as desired. This concludes the discussion in Step II.

Step III. Consider now the linear operator

$$T\vec{u}(x) := \vec{u}(x) - \mathcal{D}\Big(\Big(\tfrac{1}{2}I + K\Big)^{-1}\Big(\operatorname{Tr}\vec{u}\Big)\Big)(x), \quad x \in \Omega,$$

which, we claim, maps $W_1^{p,\infty}(\Omega)$ boundedly into itself for any $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$.

By Step II and real interpolation, it suffices to show that $T : W_1^p(\Omega) \to W_1^p(\Omega)$ boundedly for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$. This, in turn, follows from the mapping properties of the operators involved. Indeed, Theorem 1.1.3 ensures that $\operatorname{Tr}\vec{u} \in B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$ if

$\vec{u} \in W_1^p(\Omega)$, Theorem 8.4.1 guarantees that the operator $\frac{1}{2}I + K$ is invertible on $B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$ for any $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$, and Theorem 8.2.3 implies that $\mathcal{D}$ maps this latter space into $F_1^{p,2}(\Omega) = W_1^p(\Omega)$. This finishes the proof of the claim made in Step III.

Step IV. Recall the matrix $\Gamma = (\Gamma_{\alpha\beta})_{\alpha,\beta}$ with entries as in (8.2.1). To prove the first claim made in the statement of the corollary, we make the observation that, in the y variable, the functions $|x-y|^{-1}$, $\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^3}$, $1 \leq \alpha, \beta \leq 3$, belong to $W_1^{\frac{3}{2},\infty}(\Omega)$ uniformly in $x \in \Omega$. Thus, $\Gamma(x - \cdot) \in W_1^{\frac{3}{2},\infty}(\Omega)$, uniformly in $x \in \Omega$ and since $G_D(x,y) = T(\Gamma(x-\cdot))(y)$ we may conclude with the aid of Step III that $\nabla G_D(x,\cdot) \in L^{3/2,\infty}(\Omega)$, uniformly in $x \in \Omega$.

Step V. The proof of the fact that $\delta(\cdot)^{-1} G_D(x,\cdot) \in L^{3/2,\infty}(\Omega)$, uniformly in $x \in \Omega$, follows a similar pattern. The departure point is the observation that $T : W_1^p(\Omega) \to \overset{\circ}{W}{}_1^p(\Omega)$ is well-defined and bounded for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ which, in turn, yields that $W_1^{p,\infty}(\Omega) \ni \vec{u} \mapsto T(\vec{u})/\delta \in L^{p,\infty}(\Omega)$ is a bounded assignment (for the same range of p's), by Hardy's inequality and real interpolation. This finishes the proof of the corollary. □

Of course, there is a similar result for the (matrix-valued) function $G_N(x,y)$, the integral kernel of the operator $\mathbf{G}_N$.

More can be said when extra geometrical information about the domain Ω is available. For example, according to Theorems 5.4-5.5 in [**92**], [**58**] and Theorem 6.1 in [**105**],

$$\nabla^2 \mathbf{G}_D : L^2(\Omega) \longrightarrow L^2(\Omega), \tag{8.5.9}$$

whenever Ω is a *convex polyhedron* in $\mathbb{R}^3$. By interpolation with (8.5.7) we see that for a convex polyhedron $\Omega \subset \mathbb{R}^3$,

$$\nabla^2 \mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \qquad \tfrac{3}{3+\varepsilon} < p \leq 2. \tag{8.5.10}$$

In particular, $\nabla^2 \mathbf{G}_D$ maps $L^p(\Omega)$ boundedly to itself for $1 < p \leq 2$. A similar set of conclusions applies to the operator $\nabla^2 \mathbf{G}_N$. More specifically, with Ω as before,

$$\nabla^2 \mathbf{G}_N : h^p_0(\Omega) \longrightarrow h^p(\Omega), \qquad \tfrac{3}{3+\varepsilon} < p < \tfrac{3}{2}, \tag{8.5.11}$$

where the end-point $3/2$ is as in **[93]**.

If we denote by $L^{1,\infty}(\Omega)$ the standard weak-L^1 space on Ω, then interpolating between (8.5.9)–(8.5.11) and the results in Corollary 8.5.2 further yields the following.

Theorem 8.5.4. *Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedron. Then the following operators are bounded:*

$$\nabla^2 \mathbf{G}_D : h^1(\Omega) \longrightarrow h^1(\Omega), \qquad \nabla^2 \mathbf{G}_N : h^1_0(\Omega) \longrightarrow h^1(\Omega), \tag{8.5.12}$$

$$\nabla^2 \mathbf{G}_D : L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega), \qquad \nabla^2 \mathbf{G}_N : L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega). \tag{8.5.13}$$

We conclude this section with a brief discussion of the mapping properties of *gradient* of the elastic Green potentials. Recall that the classical $L^p - L^q$ estimates for gradient of the Green potential associated with Laplacian in the half-space have been extended to Lipschitz domains by B. Dahlberg in **[29]**. In **[89]** we gave new proofs to Dahlberg's estimates and extended his main results to other types of function spaces. The theorem below, pertaining to the mapping properties of the

Green potentials associated with the system of elastostatics, can be proved in a similar manner, starting with Theorems 8.1.1-8.1.2.

Theorem 8.5.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. There exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that for every $\frac{3}{3+\varepsilon} < p < \frac{3}{2-\varepsilon}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$, the operators*

$$\nabla \mathbf{G}_D : h^p(\Omega) \to L^q(\Omega), \qquad \nabla \mathbf{G}_N : h^p_0(\Omega) \to L^q(\Omega), \tag{8.5.14}$$

$$\nabla \mathbf{G}_D : L^1(\Omega) \to L^{3/2,\infty}(\Omega), \qquad \nabla \mathbf{G}_N : L^1(\Omega) \to L^{3/2,\infty}(\Omega), \tag{8.5.15}$$

are well-defined and bounded. In particular,

$$\nabla \mathbf{G}_D : h^1(\Omega) \to L^{3/2}(\Omega), \qquad \nabla \mathbf{G}_N : h^1_0(\Omega) \to L^{3/2}(\Omega) \tag{8.5.16}$$

are bounded operators. The ranges of indices are optimal in the class of Lipschitz domains. For $\partial\Omega \in C^1$ one may take $\varepsilon = 1$.

8.6 The two-dimensional setting

All our main results discussed so far continue to hold in the two-dimensional setting, albeit with possibly different conditions imposed on the intervening indices (p, q, s), when Ω is a Lipschitz domain in the *two-dimensional* Euclidean space $\mathbb{R}^2$.

Somewhat more specifically, the conditions (8.1.7) should be replaced with the demand that the indices $\frac{1}{2} < p \leq \infty$ and $(\frac{1}{p} - 1)_+ < s < 1$ satisfy any of the following three conditions:

$$\begin{aligned}
&(I'): && \tfrac{2}{1+\varepsilon} \leq p \leq \tfrac{2}{1-\varepsilon} \ \text{ and } \ 0 < s < 1; \\
&(II'): && \tfrac{2}{3+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \ \text{ and } \ \tfrac{1}{p} - \tfrac{1+\varepsilon}{2} < s < 1; \\
&(III'): && \tfrac{2}{1-\varepsilon} < p \leq \infty \ \text{ and } \ 0 < s < \tfrac{1}{p} + \tfrac{1+\varepsilon}{2}.
\end{aligned} \tag{8.6.1}$$

Also, the two-dimensional analogue of (8.1.10) takes the form

$$\Big(\min\big\{\tfrac{1}{2p}+1,\, s+1\big\}\Big)^{-1} < q \leq \infty. \tag{8.6.2}$$

With the aforementioned alterations, Theorems 8.1.1,8.1.2,8.5.1, Corollary 8.5.2 remain valid for bounded Lipschitz domains in $\mathbb{R}^2$.

In closing, we want to point out that $\nabla^2\mathbf{G}_D : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a bounded operator if either $\Omega \subset \mathbb{R}^2$ is a convex polygon (cf. p. 149 in [**58**], [**9**]), or the Lamé moduli satisfy $|\lambda| < \sqrt{3}\mu$ and $\Omega \subset \mathbb{R}^2$ is a bounded convex domain ([**57**]). In particular, we may state the following:

Theorem 8.6.1. *Assume that the Lamé moduli satisfy* $|\lambda| < \sqrt{3}\mu$ *and that* $\Omega \subset \mathbb{R}^2$ *is a bounded convex domain. Then*

$$\nabla^2\mathbf{G}_D : L^p(\Omega) \longrightarrow L^p(\Omega) \tag{8.6.3}$$

is a bounded operator for each $1 < p \leq 2$. *In this setting, the operator* $\nabla^2\mathbf{G}_D$ *also maps* $h^1(\Omega)$ *boundedly to itself. Thus, by embeddings,* $\mathbf{G}_D$ *maps* $h^1(\Omega)$ *boundedly into* $C(\bar{\Omega})$, *the space of continuous functions on* $\bar{\Omega}$.

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