RELATIVE SUBREPRESENTATION THEOREM FOR A FINITE CENTRAL EXTENSION OF A REDUCTIVE GROUP

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CHENGYU JIAO

Dr. William Banks, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Chengyu Jiao, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Professor William Banks

Professor Dan Edidin

Professor Calin Chindris

Professor Petros Valettas

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Chengyu Jiao

Dr. William Banks, Dissertation Supervisor

ABSTRACT

Jacquet's subrepresentation theorem asserts that any irreducible admissible representation of a reductive p-adic group is a subrepresentation of $\mathrm{Ind}_P^G(\tau)$, where P is a parabolic subgroup of G and τ is a cuspidal representation. Kato and Takano extended this theorem to the H-relatively cuspidal case in [KT08]. In this dissertation, we work on the level of finite central extensions, and extend Kato and Takano's results to the finite central extension case.

Chapter 1

Introduction

We shall start by talking about the setup in the reductive case discussed in [KT08], and we shall clarify all definitions in the following section. Let G be a reductive p-adic group over a local field F. Let Z be the F-split component. Jacquet's subrepresentation theorem states that any irreducible admissible representation (π, V) of G can be embedded into an induced representation from a parabolic subgroup. Kato and Takano took this result and worked on the following scenario: For a reductive G, let σ be an involution on G, and let H be the σ-fixed subgroup of G, i.e. $H = \{g \in G \mid \sigma(g) = g\}$. A parabolic subgroup P is called σ-split if P and $\sigma(P)$ are opposite parabolic subgroups. A representation (π, V) of G is called *H-distinguished* if the space of *H-invariant linear forms on V*, defined as $(V^*)^H = \{\lambda : V \to \mathbb{C} \mid \lambda(\pi(h)v) = \lambda(v) \text{ for all } v \in V, h \in H\}$, is nonzero. A representation (π, V) is called H-relative cuspidal if the (H, λ)-matrix coefficients of π are compactly supported modulo ZH, where the (H, λ)-matrix coefficients are defined as $\varphi_{\lambda,v}(x) = \langle \lambda, \pi(x^{-1})v \rangle$ for $\lambda \in (V^*)^H$, $v \in V$ and $x \in G$.

Recall that for a reductive G, a representation (π, V) of G, and a compact open subgroup $K \subset G$, we denote $\{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$, the space of K-fixed vectors, by V^K. A representation (π, V) is called a *smooth* representation if for all $v \in V$, there exists a compact open subgroup K such that $v \in V^K$. A smooth representation is called *admissible* if the dimension of V^K is finite for every open compact K.

The main theorem Kato and Takano showed in [KT08, Theorem 7.1] is the following:

Theorem 1.1 (Relative subrepresentation theorem for the reductive case)**.** *An H-distinguished irreducible and admissible representation of G is isomorphic to a subrepresentation of a induced representation of $\pi^\#$, where $(\pi^\#, V^\#)$ is an irreducible and $M \cap H$-relative cuspidal representation of M for a σ-split parabolic subgroup $P = MU$.*

One natural question to ask is that given the finite central extension $\widetilde{G}$ of a reductive group G, do we have a similar result? This dissertation aims to answer the above question by extending the result from [KT08] to the finite central extension case.

We say $(\widetilde{G}, p)$ is a *finite central extension* of G if:

1. $p : \widetilde{G} \to G$ is a surjective homomorphism of topological groups.

2. $\mathscr{A} = ker(p)$ is a finite subgroup of the center of $\widetilde{G}$.

3. p is a topological covering.

Let $\widetilde{G} = p^{-1}(G)$ be a finite central extension of G. Let H be the σ-invariant subgroup of G. We furthermore assume there is a lifting from H to $\widetilde{G}$. We fix this specific lifting $s_{\widehat{H}}$ and denote the lifting of H by $\widehat{H}$. A parabolic subgroup $\widetilde{G}$ is in the form of $\widetilde{P} = \widetilde{M}\widehat{U}$. Let us adapt the definition of admissible representation to $\widetilde{G}$. For $\widetilde{G}$, a representation (π, V) of $\widetilde{G}$ and a compact open subgroup $\bar{K} \subset \widetilde{G}$, we denote all $\bar{K}$ fixed vectors by $V^{\bar{K}}$. A representation (π, V) is called a *smooth* representation if for all $v \in V$, there exists a compact open subgroup $\bar{K}$ such that $v \in V^{\bar{K}}$. It is called *admissible* if the dimension of $V^{\bar{K}}$ is finite for every open compact $\bar{K}$. After adapting all other definitions from reductive G to $\widetilde{G}$, the main theorem is:

Theorem 1.2 (Relative subrepresentation theorem for a finite central extension)**.** *Let (π, V) be an irreducible admissible $\widehat{H}$-distinguished representation of $\widetilde{G}$. Then, there exists a σ-split parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$ of $\widetilde{G}$ and an irreducible $\widetilde{M} \cap \widehat{H}$-relatively cuspidal representation $\pi^{\#}$ of $\widetilde{M}$ such that π is a subrepresentation of the induced representation of $\pi^{\#}$ from $\widetilde{P}$ to $\widetilde{G}$.*

To prove the relative subrepresentation theorem for a finite central extension, this dissertation is divided into five main sections.

In Chapter 2, we adapt numerous definitions from the reductive G to our finite central extension $\widetilde{G}$, including the involution σ, the $(\widehat{H}, \lambda)$-matrix coefficient of a representation where λ is a $\widehat{H}$-invariant linear form, and so on. We have an important proposition at the end, which states:

Proposition 1.3. *Any finitely generated $(\widehat{H}, \lambda)$-cuspidal representation of $\widetilde{G}$ has a nontrivial $\widehat{H}$-distinguished irreducible quotient representation.*

In Chapter 3, we work on a special type of parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$ of $\widetilde{G}$ such that $\widetilde{P} = \rho^{-1}(P)$ and $P = \sigma(P)$. This type of $\widetilde{P}$ is called σ-split. We also define a specific family of compact open subgroups that not only have Iwahori factorization, which is a property that holds for congruence subgroups (this term is generally used in [Ber92] and [Bum97]) in the reductive case, but also invariant under σ. One important fact we use from [BJ] is that for a family of $\{K\}$ with Iwahori factorization on G, $\{\widehat{K}\}$ also has an Iwahori factorization on $\widetilde{G}$.

In Chapter 4, we first introduce the Jacquet module $(\pi_{\widetilde{P}}, V_{\widetilde{P}})$ of a representation (π, V) on $\widetilde{G}$. Then, we define and show some properties of a special map

$$r_{\widetilde{P}} : (V^*)^{\widehat{H}} \longrightarrow (V^*_{\widetilde{P}})^{\widetilde{M} \cap \widehat{H}},$$

such that $\langle r_{\widetilde{P}}(\lambda), \bar{v} \rangle = \langle \lambda, v \rangle$, where $v \in V$ is the canonical lift of $\bar{v} \in \pi_{\widetilde{P}}$. This map has many useful properties. At the end of the chapter, for another parabolic subgroup $\widetilde{Q}$ of $\widetilde{G}$ that is contained in $\widetilde{P}$, we study how to commute $r_{\widetilde{P}}$ with $r_{\widetilde{Q}}$. This mapping plays a big role in our proofs in Chapter 5 and 6.

In Chapter 5, we prove an important theorem:

Theorem 1.4. *Let (π, V) be an admissible $\widehat{H}$-distinguished representation of $\widetilde{G}$ and $\lambda \in (V^*)^{\widehat{H}}$. Then, $r_{\widetilde{P}}(\lambda) = 0$ for every proper σ-split parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ if and only if (π, V) is $(\widehat{H}, \lambda)$-relatively cuspidal.*

The "if" part of the proof is more straightforward, and we mostly use the work from Chapters 2 - 4. To show the "only if" part, we have to take a detour. Let $P_0 = M_0 U_0$ be the minimal parabolic subgroup of G. Pick $\gamma \in M_0 H$. We study $\widetilde{Q} = \widetilde{\gamma^{-1} P \gamma}$ and the relationship between $\widetilde{P}$ and $\widetilde{Q}$.

Chapter 6 gives the statement and the proof of the main theorem.

In the appendix, the first section is a motivating example for this dissertation. It is an example that Banks, Levy and Sepanski gave in [BLS99]. This gives us a concrete example of the finite central extension.

The second section of the appendix gives a proof the result at the end of Chapter 2 which was mentioned above.

One might expect the result to be trivial since $\widetilde{G}$ and G have a deep connection through p. However, representations on them have no trivial correlations and are less studied in general. Also, as shown in Chapter 2, the only $\widetilde{H}$-invariant linear form is 0. Therefore, we do not study $\widetilde{H}$-invariant linear forms. We mainly work on $\widehat{H}$-invariant forms instead.

There are numerous reasons we care about $\widehat{H}$-distinguished representations of $\widetilde{G}$ and H-distinguished representations of G. These representations are the basic objects of harmonic analysis on the space $\widetilde{G}/\widehat{H}$ or G/H, respectively. The classification of these representations is a fundamental problem. Furthermore, in some studies of automorphic forms, such as [Jac97], these representations are proven to be useful tools. The result over the real field has been well developed. Kato and Takano tried to extend the theory to reductive G, which is not developed too much at the time of their writing. Since their paper came out, it helped many studies such as distinguished generic representations of $GL(n)$ in [Mat10] and Speh representations in [Smi20]. In our case of finite central extension $\widetilde{G}$, these representations are even less studied compared with the reductive and real cases.

Throughout this dissertation there are numerous citations, and I want to quickly mention what their purposes are. [KT08] works on our main theorem for a reductive G. On the level of $\widetilde{G}$, [BJ] did a lot of work on the finite central extensions when they extended Langlands quotient theorem. Parabolic subgroups are a special kind of subgroup that are fundamental to the study of reductive groups. We use a lot of information on these subgroups, most of which are found in [Ber92] and [Bum97]. As for parabolic subgroups of a finite central extension $\widetilde{G}$, [BJ] has some useful and important results.

1.1 Notations and definitions

This section's purpose is to recall the basic definitions used in this dissertation, and to introduce some common notations used throughout.

Let F be a local field, i.e., a commutative non-discrete locally compact field. As [Wei73] summarized, there are only two cases: If F has characteristic 0, then F is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or finite algebraic extensions of the fields $\mathbb{Q}_p$ (p-adic numbers) for any prime p. For the other case where F has characteristic p, there exists $q = p^n$ for some positive integer n, such that F is isomorphic to a finite extension of $\mathbb{F}_q((T))$, i.e. the Laurent series over $\mathbb{F}_q$.

Recall that a linear algebraic group $\mathbf{G}$ is called *reductive* over a field F if the unipotent radical of $\mathbf{G}$, i.e. the largest connected, smooth, normal, unipotent subgroups, is simply $\{1\}$. Let $\mathbf{G}$ be a connected reductive group over F. We denote the F-rational points of $\mathbf{G}$ by G or $G(F)$. This type of group G is often called a reductive p-adic group. For a homomorphism $\sigma : G \to G$, we say it is an (F-)involution of G if $\sigma(\sigma(g)) = g$ for all $g \in G$.

Definition 1.5. We say $(\widetilde{G}, p)$ is a *finite central extension* of G if:

1. $p : \widetilde{G} \to G$ is a surjective homomorphism of topological groups.

2. $\mathscr{A} = ker(p)$ is a finite subgroup of the center of $\widetilde{G}$.

3. p is a topological covering, i.e., there is an open neighborhood O of 1_G and a homeomorphism $j : p^{-1}(O) \to O \times \mathscr{A}$ such that $pr_1 \circ j = p$ on $p^{-1}(O)$. where pr_1 is the projection map from $O \times \mathscr{A}$ to O.

Given a finite central extension $\widetilde{G}$ of G. A *section* of p is a continuous map

$s : G \to \widetilde{G}$ such that $p \circ s = id_G$ where p is the projection of $\widetilde{G}$ onto G, and $s(1_G) = 1_{\mathscr{A}}$. If the section s is a homomorphism, then we say s is a splitting, and the central extension $\widetilde{G}$ splits. In this case, for the group structure, $\widetilde{G} \cong G \times \mathscr{A}$. Recall a mapping $c : G \times G \to \mathscr{A}$ is called a *2-cocycle* if

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_1, g_2g_3)c(g_2, g_3)$$

and $c(1_G, 1_G) = 1_{\mathscr{A}}$. The multiplication on $\widetilde{G}$ is

$$(g_1, a_1)(g_2, a_2) = (g_1g_2, c(g_1, g_2)a_1a_2).$$

Let H be a subgroup of G. If there exists a continuous homomorphism $s_H : H \to \widetilde{G}$ such that $p \circ s_H = id_H$, then we say H lifts to $\widetilde{G}$, and $s_H(H) = \widehat{H}$ is a *lifting* of H.

From the definition, simply applying p^{-1} gives us that for $A, B \subset G$, $\widetilde{AB} = \widetilde{A}\widetilde{B}$ and $\widetilde{A/B} = \widetilde{A}/\widetilde{B}$.

Throughout the dissertation, when picking an arbitrary subgroup of $\widetilde{G}$, we often look at an arbitrary subgroup $\widetilde{B} \subset \widetilde{G}$. Note $\widetilde{B}$ is not necessarily $p^{-1}(B)$ for some $B \subset G$.

Recall that for a non-trivial homomorphism $\varepsilon : Z(\widetilde{G}) \longrightarrow \mathbb{C}$ where $Z(\widetilde{G})$ is the center of $\widetilde{G}$, a representation (π, V) is called (ε-)*genuine* if for all $\widetilde{z} \in Z(\widetilde{G})$, $\pi(\widetilde{z}) = \varepsilon(\widetilde{z})$.

Denote the space of all linear forms on V by V^*. For a genuine representation (π, V) of $\widetilde{G}$ and a subgroup $\widetilde{B} \subset \widetilde{G}$, the space $(V^*)^{\widetilde{B}}$ is the collection of all $\widetilde{B}$-invariant linear forms on V. That is

$$(V^*)^{\widetilde{B}} = \{\lambda : V \to \mathbb{C} |\ \lambda(\pi(\widetilde{b})v) = \lambda(v) \text{ for all } v \in V, \widetilde{b} \in \widetilde{B}\}.$$

For reductive G, recall that for a representation (π, V) of G and a compact open subgroup $K \subset G$, we denote all K fixed vectors by $V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$. A representation (π, V) is called a *smooth* representation if for all $v \in V$, there exists a compact open subgroup K such that $v \in V^K$. A smooth representation is called *admissible* if the dimension of V^K is finite for every open compact K.

Similarly, for $\widetilde{G}$, a representation (π, V) of $\widetilde{G}$ and a compact open subgroup $\bar{K} \subset \widetilde{G}$, we denote all $\bar{K}$ fixed vectors by $V^{\bar{K}}$. A representation (π, V) is called a *smooth* representation if for all $v \in V$, there exists a compact open subgroup $\bar{K}$ such that $v \in V^{\bar{K}}$. It is called *admissible* if the dimension of $V^{\bar{K}}$ is finite for every open compact $\bar{K}$.

For a subgroup $\widetilde{B} \subset \widetilde{G}$, a representation (π, V) is called $\widetilde{B}$-*distinguished* if for all $\widetilde{b} \in \widetilde{B}, v \in V$,

$$\{\lambda \in V^* \mid \langle \lambda, \pi(\widetilde{b})v \rangle = \langle \lambda, v \rangle\} \neq \{0\}.$$

Chapter 2

Representations of a finite central extension

In this section, we review some important definitions from finite central extensions and adapt definitions of relative cuspidality to our case.

Throughout this dissertation, we are interested in the following setup: Given a reductive group G and an involution σ. Let $H = \{g \in G \mid \sigma(g) = g\}$, and $\widetilde{G}$ is a central extension of G that splits. We further more assume that there is a lifting from H to $\widetilde{G}$. Note that H is isomorphic to $\widehat{H}$. We denote this lifting by $s_H : H \to \widehat{H}$. Lastly, we denote $p^{-1}(H) = \widetilde{H}$. This gives us the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{A} & \hookrightarrow & \widetilde{G} & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & & & \uparrow & \nwarrow^{s_H} & \uparrow & & \\ & & & & \widetilde{H} & \longrightarrow & H & & \end{array}$$

For any subgroup $J \subset G$ which lifts to $\widetilde{G}$ with a corresponding lifting s_J, define $\widetilde{\sigma} : \widehat{J} \to \widehat{J}$ by $\widetilde{\sigma}(s_J(j)) = s_J(\sigma(j))$ for $j \in J$. One can check that this is an involution, but it only takes elements in subsets of G which lifts to $\widetilde{G}$. Since $\widetilde{\sigma}$ plays the role of σ on the level of $\widetilde{G}$, we just denote it by σ if it is clear from the context.

Recall that for a genuine representation (π, V) of $\widetilde{G}$ and a subgroup $\widetilde{B} \subset \widetilde{G}$, the space $(V^*)^{\widetilde{B}}$ is the collection of all $\widetilde{B}$-invariant linear forms on V. That is

$$(V^*)^{\widetilde{B}} = \{\lambda : V \to \mathbb{C} |\ \lambda(\pi(\widetilde{b})v) = \lambda(v) \text{ for all } v \in V, \widetilde{b} \in \widetilde{B}\}.$$

Throughout this dissertation, we often look at $\lambda \in (V^*)^{\widehat{H}}$ instead of $(V^*)^{\widetilde{H}}$. It is because of the following lemma:

Lemma 2.1. *For any genuine representation (π, V) of $\widetilde{G}$ over $\mathbb{C}$, the space $(V^*)^{\widetilde{H}}$ is zero.*

Proof. From the exact sequence earlier, as a group, $\widetilde{G}$ has the same structure as $G \times \mathscr{A}$. Take $(1, a) \in \widetilde{G}$. Then $(1, a^{-1})$ is also in $\widetilde{G}$. For any $\lambda \in (V^*)^{\widetilde{H}}$ and $v \in V$, we have $\lambda(\ \pi((1, a))(v)\ ) = \lambda(\ \pi((1, a^{-1}))(v)\ )$, which results in $(\varepsilon(1, a) - \varepsilon(1, a^{-1}))\,\lambda(v) = 0$. As long as the representation is not trivial, we have $\lambda(v) = 0$. Therefore, $(V^*)^{\widetilde{H}} = \{0\}$. ∎

In the reductive case ([KT08]), we denote the F-split component, i.e. the largest F-split torus in the center of $\mathbf{G}$, by $\mathbf{Z}$. Taking their F-rational points, we have $Z \subset G$. Naturally, we have $\widetilde{Z} \subset \widetilde{G}$. The problem is that $\widetilde{Z}$ is not necessarily central. To fulfill our goal, we look at $\widetilde{Z}'$ which is the intersection of $\widetilde{Z}$ and the center of $\widetilde{G}$.

Definition 2.2. Fix a smooth and $\widehat{H}$-distinguished representation (π, V). For $\lambda \in (V^*)^{\widehat{H}}$ and $v \in V$, we define a function on $\widetilde{G}$ by

$$\varphi_{\lambda,v}(x) = \langle \lambda, \pi(x^{-1})v \rangle$$

for $x \in \widetilde{G}$. We call such functions the $(\widehat{H}, \lambda)$-*matrix coefficients* of π.

Lemma 2.3. *Let A be a subgroup of G. Then $\widetilde{AH} = \widetilde{A}\widehat{H}$.*

Proof. It is clear that $\widetilde{A}\widehat{H} \subset \widetilde{AH}$. To show $\widetilde{AH} \subset \widetilde{A}\widehat{H}$, denote the lift from H to $\widehat{H}$ by s_H, i.e., h lifts to $(h, s_H(h))$. Let $(ah, b) \in \widetilde{AH}$ where $a \in A, h \in H, b \in \mathscr{A}$. Then it can be viewed as a product of $(h, s_H(h)) \in \widehat{H}$ and $(a, (c(z,h))^{-1}\, b\, (s_H(h))^{-1}) \in \widetilde{A}$. This shows $\widetilde{AH} \subset \widetilde{A}\widehat{H}$. Therefore, $\widetilde{AH} = \widetilde{A}\widehat{H}$. ■

Recall that for a representation (π, V) of $\widetilde{G}$ and a compact open subgroup $\bar{K} \subset \widetilde{G}$, (π, V) is called a smooth representation if for all $v \in V$, there exists a compact open subgroup $\bar{K}$ such that $v \in V^{\bar{K}}$. It is furthermore called *admissible* if the dimension of $V^{\bar{K}}$ is finite for every open compact $\bar{K}$.

Definition 2.4. For $\lambda \in (V^*)^{\widehat{H}}$, an admissible representation π is $(\widehat{H}, \lambda)$-*relatively cuspidal* if all $(\widehat{H}, \lambda)$-matrix coefficients of π are compactly supported modulo $\widetilde{Z}'\widehat{H}$. A representation (π, V) is said to be $\widehat{H}$-*relatively cuspidal* if it is $(\widehat{H}, \lambda)$-relatively cuspidal for all $\lambda \in (V^*)^{\widehat{H}}$.

We need the following proposition, but the proofs and definitions are a bit unrelated with the rest of the dissertation. Thus, the proof is shown in the appendix.

Proposition 2.5. *Any finitely generated* $(\widehat{H}, \lambda)$*-cuspidal representation of* $\widetilde{G}$ *has a nontrivial* $\widehat{H}$*-distinguished irreducible quotient representation.*

Chapter 3

σ-split parabolic subgroups

We have some important definitions of split torus that we need to define for our finite central extension.

Definition 3.1. According to [KT08], an F-split torus $S \subset G$ is said to be (σ, F)-*split* if $\sigma(s) = s^{-1}$ for all $s \in S$. For our case, we start with $S \subset G(E)$, and we have $\widetilde{S} = p^{-1}(S) \subset \widetilde{G}$. In [KT08], S_0 is the maximal (σ, F)-*split* torus of G and $A_\emptyset$ is the maximal F-split torus of G containing S_0. Similar to the definition of $\widetilde{Z}'$, we apply p^{-1} and intersect with the center of $\widetilde{M}$, we get $\widetilde{S_0}'$ and $\widetilde{A}'$.

Recall that on the level of reductive groups, with A a maximal torus in G, the *roots* are the non-trivial weights for A under the adjoint action on the Lie group of G. Denote the set of roots of A in G by $\Phi = \Phi(G, A)$. Let Φ^+ be the positive roots and $\Phi_\sigma = \{\alpha \in \Phi \mid \sigma(\alpha) = \alpha\}$. By [HH98, 1.6], we can find a basis Δ of Φ such that α is not in Φ^+ for all $\alpha \in \Phi^+ \backslash \Phi_\sigma$. Recall again that a subgroup of G is called a *parabolic* subgroup if it contains either a Borel subgroup of G or a conjugate of a Borel subgroup, where Borel group is a maximal Zariski-connected solvable subgroup. For such Δ mentioned above, there is a corresponding *minimal parabolic subgroup* P_0, and any parabolic subgroup P containing P_0 is said to be *standard with respect to* $(S_0, A_\emptyset, \Delta)$.

Unlike the reductive case G, we are not blessed with a corresponding Lie Algebra for our finite central extension $\widetilde{G}$. However, it is shown in [BJ] that every parabolic subgroup of $\widetilde{G}$ is in the form of $\widetilde{P} = p^{-1}(P)$. Therefore, we say $\widetilde{P}$ is standard with respect to $(S_0, A_\emptyset, \Delta)$ if P is standard with respect to $(S_0, A_\emptyset, \Delta)$. For the rest of the dissertation, we look at $\widetilde{P}$ standard with respect to a fixed $(S_0, A_\emptyset, \Delta)$ unless stated otherwise.

Recall that a parabolic subgroup P of a reductive group G can be decomposed as $P = MU$ where U is the unipotent radical, i.e. the maximal unipotent subgroup of P, and M is the Levi subgroup, i.e. the maximal reductive subgroup of P. We also have a modulus character δ_P for this parabolic subgroup P. The parabolic opposite, $\bar{P} = M\bar{U}$ is the unique parabolic subgroup of G such that $P \cap \bar{P} = M$. By [BJ], the parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ has the decomposition $\widetilde{P} = \widetilde{M}\widehat{U}$ where $\widetilde{M} = p^{-1}(M)$ and $\widehat{U}$ is the canonical lifting of U. (This lifting is constructed in the appendix I of [MW95].) We still call $\widetilde{M}$ the Levi part and $\widehat{U}$ the unipotent part. As of the parabolic opposite, for visual clarity, we denote the parabolic opposite of $\widehat{U}$ by $\widehat{U}^- = \bar{\widehat{U}}$, and similarly $\widetilde{P}^- = \bar{\widetilde{P}}$, $\widetilde{M}^- = \bar{\widetilde{M}}$.

For a subset $I \subset \Delta$, let $P_I = M_I U_I$ be a standard parabolic subgroup of G corresponding to I. On the level of $\widetilde{G}$, we have $\widetilde{P}_I = \widetilde{M}_I \widehat{U}_I$. The corresponding maximal (σ, F)-split torus and F-split torus is denoted by $\widetilde{S}'_I$ and $\widetilde{A}'_I$, respectively. Denote the center of $\widetilde{M}$ by $Z(\widetilde{M})$, then for a positive $\varepsilon \leq 1$, we also look at these two important sets:

$$\widetilde{A}'^{-}_I(\varepsilon) = \left\{\widetilde{a} = p^{-1}(a) \mid a \in A_I \text{ and } |a^\alpha|_F \leq \varepsilon \text{ for all } \alpha \in \Delta \backslash I\right\} \cap Z(\widetilde{M})$$

and

$$\widetilde{S}_I'^{-}(\varepsilon) = \{\widetilde{s} = p^{-1}(s) \mid s \in S_I \cap A_I^{-}(\varepsilon)\} \cap Z(\widetilde{M})$$
$$= \{\widetilde{s} = p^{-1}(s) \mid s \in S_I \text{ and } |s| \le \varepsilon\} \cap Z(\widetilde{M}).$$

Correspondingly,

$$\widetilde{A}_I'^{+}(\varepsilon) = \left\{\widetilde{a} = p^{-1}(a) \mid a \in A_I \text{ and } |a^{\alpha}|_F \ge \varepsilon \text{ for all } \alpha \in \Delta \backslash I\right\} \cap Z(\widetilde{M}),$$

$$\widetilde{S}_I'^{+}(\varepsilon) = \{p^{-1}(s) \mid s \in S_I \text{ and } |s| \ge \varepsilon\} \cap Z(\widetilde{M}).$$

For convenience, if it is clear from the context, we skip writing I and just write $\widetilde{A}'^{-}(\varepsilon), \widetilde{S}'^{-}(\varepsilon)$ and $\widetilde{S}'^{+}(\varepsilon)$.

Definition 3.2. $\widetilde{P}$ is said to be *σ-split* if $\widetilde{P}$ and $\widetilde{\sigma(P)}$ are opposite. In this case, $\widetilde{P} \cap \widetilde{\sigma(P)} = \widetilde{M}$, and $\widehat{\sigma(U)}$ is the opposite of $\widehat{U}$. For $I \subset \Delta$, we say I is σ-split subset of Δ if $\widetilde{P}_I$ is σ-split.

For the special case where I is the minimal σ-split subset of Δ, we denote $\widetilde{P}_I$ by $\widetilde{P}_0$. This is a minimal σ-split parabolic subgroup of $\widetilde{G}$.

Now, we find a sequence of open compact subgroups of $\widetilde{G}$ with certain properties called Iwahori factorization. We usually refer to these open compact subgroups as congruence subgroups. However, we need to construct a sequence of open compact subgroups that is standard with respect $(S_0, A_{\emptyset}, \Delta)$ and furthermore stable under σ.

For a compact open subgroup $\bar{K}$ of $\widetilde{G}$ and parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$, we denote $\bar{K}_{\widehat{U}} = \bar{K} \cap \widehat{U}$. Similarly, denote $\bar{K}_{\widetilde{M}} = \bar{K} \cap \widetilde{M}$ and $\bar{K}_{\widehat{U}^{-}} = \bar{K} \cap \widehat{U}^{-}$.

Definition 3.3. We say $\bar{K}$ has an *Iwahori factorization* with respect to $\widetilde{P}$ if both of the followings are true:

1. The product map $\bar{K}_{\widehat{U}^-}\bar{K}_{\widetilde{M}}\bar{K}_{\widehat{U}} \to \bar{K}$ is an isomorphism.

2. For every $a \in \widetilde{A}_I'^{-}(1)$, $a\bar{K}_{\widehat{U}}a^{-1} \subset \bar{K}_{\widehat{U}}, a^{-1}\bar{K}_{\widehat{U}^-}a \subset \bar{K}_{\widehat{U}^-}$.

To begin with, for $\widetilde{G}$, [BJ, 2.11] looked at $\{\widehat{K}'_n\}_{n\geq 0}$, where each $\widehat{K}'_n$ is lifted from K_n on the level of G.

Remark 3.4. For $P = MU$ and K defined above, [BJ] showed that

1. $\widehat{K \cap M} = \widehat{K} \cap \widetilde{M}$.

2. $\widehat{K \cap U} = \widehat{K} \cap \widehat{U}$.

3. $\widehat{K \cap U^-} = \widehat{K} \cap \widehat{U}^-$.

   Similar to the proof of 3 shown in [BJ], we also have

4. $\widehat{H \cap M} = \widehat{H} \cap \widetilde{M}$.

5. $\widehat{H \cap Z'} = \widehat{H} \cap \widetilde{Z}'$.

Then, they showed the following proposition:

Proposition 3.5. *For a minimal parabolic subgroup P_0, let $\widetilde{P}$ be the parabolic subgroup containing $\widetilde{P}_0$. There is a decreasing sequence $\{\widehat{K}'_n\}_{n\geq 0}$ of open compact subgroups of $\widetilde{G}$ satisfying the following properties:*

1. *$\{\widehat{K}'_n\}_{n\geq 0}$ forms a open neighborhood basis of identity in $\widetilde{G}$.*

2. *Every $\widehat{K}'_n$ is a normal subgroup of $\widehat{K}'_0$.*

3. *$\{\widehat{K}'_n\}_{n\geq 0}$ has an Iwahori factorization with respect to $\widetilde{P}$.*

4. *$\{\widehat{K}'_n \cap \widetilde{M}\}_{n\geq 0}$ has an Iwahori factorization with respect to $\widetilde{M}$.*

We take a step further, and want to have something similar when dealing with parabolic subgroups standard with respect to $(S_0, A_\emptyset, \Delta)$ and invariant under σ. Thus we need the following lemma:

Lemma 3.6. *For $(S_0, A_\emptyset, \Delta)$, let $\{\widehat{K}'_n\}_{n\geq 0}$ be the same as above, and $\widetilde{P}$ be a σ-split parabolic subgroup standard with respect to $(S_0, A_\emptyset, \Delta)$. Let $\widehat{K}_n = \widehat{K}'_n \cap \sigma(\widehat{K}'_n)$. Then $\{\widehat{K}_n\}_{n\geq 0}$ satisfy the following properties:*

1. *Each $\widehat{K}_n$ is σ-invariant.*

2. *Every $\widehat{K}_n$ is a normal subgroup of $\widehat{K}_0$.*

3. *For each $\widehat{K} = \widehat{K}_n$, the product map $\widehat{K}_{\widehat{U}^-}\widehat{K}_{\widetilde{M}}\widehat{K}_{\widehat{U}} \to \widehat{K}$ is an isomorphism. Also, for every $s \in \widetilde{S}_I'^{-}(1)$, $s\widehat{K}_{\widehat{U}}s^{-1} \subset \widehat{K}_{\widehat{U}}, s^{-1}\widehat{K}_{\widehat{U}^-}s \subset \widehat{K}_{\widehat{U}^-}$. (Note: This is basically Iwahori factorization, except that the last part holds for $\widetilde{S}'$ instead of $\widetilde{A}'$. We still call this Iwahori factorization when the context is clear.)*

4. *$\{\widehat{K}_n \cap \widetilde{M}\}$ has an Iwahori factorization with respect to $\widetilde{M}$.*

Proof. Part 1 is obvious directly by the definition of $\widehat{K}_n$. For part 2, to show it is normal, take any $\widehat{k} \in \widehat{K}_n$ and $g \in \widehat{K}_0$. We look at $g\widehat{k}g^{-1}$. It is in $\widehat{K}'_n$ by proposition 3.5, and we can rewrite $\widehat{k} = \sigma(\widehat{k}'), g = \sigma(g')$ for some $\widehat{k}' \in \widehat{K}'_n, g' \in \widehat{K}'_0$. This means $g\widehat{k}g^{-1} = \sigma(g'\widehat{k}'g'^{-1}) \in \sigma(\widehat{K}_n)$. This shows $g\widehat{k}g^{-1} \in \widehat{K}_n$, showing it is normal. For part 3, it is sufficient to show the map is surjective. Take $\widehat{k} \in \widehat{K}$. By proposition 3.5, since $\widehat{k}$ and $\sigma(\widehat{k})^{-1}$ can be viewed as an element of $\widehat{K}'_n$, we can decompose them as $\widehat{k} = u_1^- m_1 u_1$, and $\sigma(\widehat{k})^{-1} = u_2^- m_2 u_2$. We rewrite the second equality to be $\widehat{k} = \sigma(u_2)^{-1}\sigma(m_2)^{-1}\sigma(u_2^-)^{-1}$. This gives us

$$\sigma(u_2)u_1^- = \sigma(m_2)^{-1}\sigma(u_2^-)^{-1}u_1^{-1}m_1^{-1} \in \widehat{U}^- \cap \widetilde{P}$$

which contains nothing but the identity element. Therefore, we get that $u_1^- = \sigma(u_2)^{-1}$. Furthermore, $u_1^- = \sigma(u_2)^{-1} \in (\widehat{U}^- \cap \widehat{K}') \cap (\sigma(\widehat{U}) \cap \sigma(\widehat{K}')) = \widehat{U}^- \cap \widehat{K} = \widehat{K}_{\widehat{U}^-}$ Similarly, $u_1 = \sigma(u_2^-)^{-1} \in K_{\widehat{U}}$ and $m_1 = \sigma(m_2)^{-1} \in K_{\widetilde{M}}$. Since $\widetilde{S}_I'(1)$ is the (σ, F)-split component of $\widetilde{A}_I'(1)$, the second half of part 3 is obvious.

■

Chapter 4

Invariant Linear forms on Jacquet Modules

In this section, let $\widehat{H}$-invariant linear form λ on the vector space of an admissible representation (π, V) of $\widetilde{G}$ with σ-split parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$. Let $(\pi_{\widetilde{P}}, V_{\widetilde{P}})$ be the Jacquet module of (π, V) along $\widetilde{P}$, where $V_{\widetilde{P}} = V/V(\widehat{U})$, $\pi_{\widetilde{P}}$ is a $\widetilde{M}$-representation obtained by restricting π from $\widetilde{G}$ to $\widetilde{M}$, and $V(\widehat{U})$ is the subspace of V spanned by $\pi(u)v - v, u \in \widehat{U}, v \in V$. Let $j_{\widetilde{P}}$ be the canonical projection from V to $V_{\widetilde{P}}$. Let δ_P be the modular character of P.

Recall in the reductive case, for an admissible representation (π_0, V_0) of G, and $P = MU$ a parabolic subgroup of G, we denote the Jacquet module by (π_{0P}, V_{0P}) and the corresponding canonical projection by j_P. For $v \in V_0, m \in M$, in [KT08], the Jacquet module is normalized if

$$\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi_0(m)v).$$

We want to construct a similar "modulus" for our case to normalize our $(\pi_{\widetilde{P}}, V_{\widetilde{P}})$. Let us start with Haar measure. For a locally compact topological group G with a left Haar measure dg, the covering group $\widetilde{G}$ is also locally compact, meaning it also has a left Haar measure, unique up to a constant. Then we choose a left Haar measure

$d\widetilde{g}$ on $\widetilde{G}$ such that

$$meas_{\widetilde{G}}(\widehat{K}) = meas_G(K).$$

For $\widetilde{c} \in \widetilde{G}$, define $\delta_{\widetilde{G}}$ so that:

$$\int_{\widetilde{G}} f(\widetilde{c}^{-1}\widetilde{g}\widetilde{c})d\widetilde{g} = \left(\delta_{\widetilde{G}}(\widetilde{c})\right)^2 \int_{\widetilde{G}} f(\widetilde{g})d\widetilde{g}.$$

In [BJ, p. 12], it is shown that for a parabolic P, $\widetilde{a} \in \widetilde{A}'^{-}_{\widetilde{M}}$ and $a = p(\widetilde{a})$, one has

$$\delta_{\widetilde{P}}(\widetilde{a}) = \delta_P(a).$$

This is the modulus we use to normalize, giving us

$$\pi_{\widetilde{P}}(\widetilde{m})j_{\widetilde{P}}(v) = \delta_{\widetilde{P}}^{-1/2}(\widetilde{m})j_{\widetilde{P}}(\pi(\widetilde{m})v). \tag{1}$$

For now on, we assume $(\pi_{\widetilde{P}}, V_{\widetilde{P}})$ is normalized. The next goal is to construct a linear form $r_{\widetilde{P}}(\lambda)$ that works the same way as $\lambda \in (V^*)^{\widehat{H}}$, but on the Jacquet module $(\pi_{\widetilde{P}}, V_{\widetilde{P}})$ along $\widetilde{P}$, i.e., we want to construct a mapping

$$r_{\widetilde{P}} : (V^*)^{\widehat{H}} \longrightarrow (V^*_{\widetilde{P}})^{\widetilde{M}\cap\widehat{H}}.$$

In order to do that, we start with Casselman's canonical pairing from [KT08] and [Cas95]. For an open compact subgroup $\widehat{K}$ of $\widetilde{G}$, denote all $\widehat{K}$-fixed vectors, $\{v \in V : \pi(\widehat{k})v = v \text{ for all } \widehat{k} \in \widehat{K}\}$, by $V^{\widehat{K}}$. For $v \in V$, define $P_{\widehat{K}} : V \to V^{\widehat{K}}$ by

$$P_{\widehat{K}}(v) = \frac{1}{meas_{\widetilde{G}}(\widehat{K})} \int_{\widehat{K}} \pi(\widehat{k})v d\widehat{k}.$$

Next, we want to show that for any $s \in \widetilde{S}'$, there is an isomorphism $P_{\widehat{K}}(\pi(s)V^{\widehat{K}}) \longrightarrow (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}$. In [BJ, p. 13], it is shown that for a compact subgroup $\widehat{U}_1$ of $\widehat{U}$, we define $V(\widehat{U}_1) = \{v \in V \mid \int_{\widehat{U}_1} \pi(\widetilde{u})v \; d\widetilde{u} = 0\}$. Then $V(\widehat{U}) = \bigcup V(\widehat{U}_1)$ where $\widehat{U}_1$ runs over all compact open subgroups of $\widehat{U}$. Now for a given $\bar{v} \in V_{\widetilde{P}}$, take a family $\{\widehat{K}\}$ such that $\bar{v} \in (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}$. Now, choose an open compact subgroup $\widehat{U}_1$ of $\widehat{U}$ such that $V^{\widehat{K}} \cap V(\widehat{U}) \subset V(\widehat{U}_1)$. For $\varepsilon < 1$, [KT08, 2.8] showed for $s \in S^-(\varepsilon)$, $sU_1s^{-1} \subset K_U$. By applying p^{-1} to both side, we have for $\widetilde{s} \in \widetilde{S}'^-(\varepsilon)$, $\widetilde{s}\widehat{U}_1\widetilde{s}^{-1} \subset \widehat{K}_{\widehat{U}}$. Because $\widetilde{S}'^-(\varepsilon) \subset \widetilde{A}'^-(\varepsilon)$, using the same argument of [Cas95, 4.1.4], we have the above isomorphism

$$P_{\widehat{K}}(\pi(\widetilde{s})V^{\widehat{K}}) \xrightarrow{\cong} (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}} \tag{4.1}$$

via the canonical projection $j_{\widetilde{P}} : V \to V_{\widetilde{P}}$. This allow us to define $v \in P_{\widehat{K}}(\pi(s)V^{\widehat{K}})$ as the *canonical lift* of $\bar{v} \in V_{\widetilde{P}}$ if $j_{\widetilde{P}}(v) = \bar{v}$.

Here are some related remarks adapted from [KT08, section 5] by doing similar calculations.

Remark 4.1. Let $v \in P_{\widehat{K}}(\pi(s)V^{\widehat{K}})$ be the canonical lift of $\bar{v} \in V_{\widetilde{P}}$, and λ a $\widehat{H}$-invariant linear form,

1. For another compact subgroup $\widehat{K}' \subset \widehat{K}$, if v' is another canonical lift of $\bar{v}$ with respect to $\widehat{K}'$, we have that

$$v' \in V^{\widehat{K}_{\widetilde{M}}\widehat{K}_{\widehat{U}^-}} \text{ and } v = P_{\widehat{K}}(v') = P_{\widehat{K}_{\widehat{U}}}(v')$$

2. For two canonical lifts v and v' of $\bar{v}$, we have

$$\langle \lambda, v \rangle = \langle \lambda, v' \rangle.$$

3. For $v \in V^{\widehat{K}_{\widetilde{M}}\widehat{K}_{\widehat{U}^-}}$, we have

$$\langle \lambda, v \rangle = \langle \lambda, P_{\widehat{K}_{\widehat{U}}}(v) \rangle$$

The second part of the previous remark allows us to define a linear from $r_{\widetilde{P}}(\lambda)$ on $V_{\widetilde{P}}$ by

$$\langle r_{\widetilde{P}}(\lambda), \bar{v}\rangle \;=\; \langle \lambda, v\rangle,$$

where $v \in V$ is the canonical lift of $\bar{v} \in V_{\widetilde{P}}$.

Proposition 4.2. *Let λ be an $\widehat{H}$-invariant linear form on an admissible representation (π, V) of $\widetilde{G}$, and let $\widetilde{P} = \widetilde{M}\widehat{U}$ be a σ-split parabolic subgroup of $\widetilde{G}$ with (σ, F)-split component $\widetilde{S}'^{-}$. For each $v \in V$, there exists a positive real number $\varepsilon \le 1$ such that*

$$\langle \lambda, \pi(\widetilde{s})v\rangle = \delta_{\widetilde{P}}(\widetilde{s})^{1/2}\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s}) j_{\widetilde{P}}(v)\rangle$$

for all $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$.

Additionally, if there exists a λ' as a linear form on $V_{\widetilde{P}}$ such that for any $\widetilde{s} \in \widetilde{S}'^{-}, v \in V$,

$$\langle \lambda, \pi(\widetilde{s})v\rangle = \delta_{\widetilde{P}}(\widetilde{s})^{1/2}\langle \lambda', \pi_{\widetilde{P}}(\widetilde{s}) j_{\widetilde{P}}(v)\rangle$$

for all $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$, then $\lambda' = r_{\widetilde{P}}(\lambda)$.

Proof. We fix a $v \in V$, and choose an open compact subgroup $\widehat{K}$ such that $v \in V^{\widehat{K}}$. Now, choose an open compact subgroup $\widehat{U}_1$ of $\widehat{U}$ such that $V^{\widehat{K}} \cap V(\widehat{U}) \subset V(\widehat{U}_1)$. As discussed above, there exists a $\varepsilon < 1$ such that for $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$, $\widetilde{s}\widehat{U}_1\widetilde{s}^{-1} \subset \widehat{K}_{\widehat{U}}$. By the Iwahori factorization with respect to $\widetilde{P}$, $v \in V^{\widehat{K}} \cong V^{\widehat{K}_{\widehat{U}}\widehat{K}_{\widetilde{M}}\widehat{K}_{\widehat{U}^-}}$, we have $\pi(\widetilde{s})v \in V^{\widehat{K}_{\widetilde{M}}\widehat{K}_{\widehat{U}^-}}$, and by the first part of the previous remark,

$$j_{\widetilde{P}}(\pi(\widetilde{s})v) = j_{\widetilde{P}}(P_{\widehat{K}}(\pi(\widetilde{s})v)) = j_{\widetilde{P}}(P_{\widehat{K}_{\widehat{U}}}(\pi(\widetilde{s})v)).$$

On the other hand, from the above (4.1), $j_{\widetilde{P}}(v) \in (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}$. Because we know $\widetilde{s}$ is central in $\widetilde{M}$, we know $\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)$ is still in $(V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}$ and by the backward direction of equation (1),

$$j_{\widetilde{P}}(\pi(\widetilde{s})v) = \delta_{\widetilde{P}}^{1/2}\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v) \in (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}.$$

Combining the above two equations, we know $P_{\widehat{K}_{\widehat{U}}}(\pi(\widetilde{s})v)$ is a canonical lift of $\delta_{\widetilde{P}}^{1/2}\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)$. According to the definition we have

$$\delta_{\widetilde{P}}(\widetilde{s})^{1/2}\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = \langle \lambda, j_{\widetilde{P}}(P_{\widehat{K}_{\widehat{U}}}(\pi(\widetilde{s})v))\rangle = \langle \lambda, \pi(\widetilde{s})v\rangle$$

where the last equality is from (3) of the previous remark. This finishes the proof of the first half. To show the uniqueness, let λ' be a linear form on $V_{\widetilde{P}}$ such that for any $\widetilde{s} \in \widetilde{S}', v \in V$,

$$\langle \lambda, \pi(\widetilde{s})v\rangle = \delta_{\widetilde{P}}(\widetilde{s})^{1/2}\langle \lambda', \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle$$

for all $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$. Then by the first half, we know

$$\langle \lambda', \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = \langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle.$$

This shows that $\lambda' \circ \pi_{\widetilde{P}}(\widetilde{s}) = r_{\widetilde{P}}(\lambda) \circ \pi_{\widetilde{P}}(\widetilde{s})$ for all $j_{\widetilde{P}}(V) = V_{\widetilde{P}}$. Let $\widetilde{s}$ vary in all of $\widetilde{S}'^{-}$, we get $\lambda' = r_{\widetilde{P}}(\lambda)$, finishing the proof. ∎

It can be checked that for $\lambda \in (V^*)^{\widehat{H}}$, $r_{\widetilde{P}}(\lambda) \in (V_{\widetilde{P}}^*)^{\widetilde{M}\cap\widehat{H}}$. This allows us to view $r_{\widetilde{P}}$ as a mapping from $(V^*)^{\widehat{H}}$ to $(V_{\widetilde{P}}^*)^{\widetilde{M}\cap\widehat{H}}$.

Let $\widetilde{P} = \widetilde{M}\widehat{U}$ be a σ-split parabolic subgroup of $\widetilde{G}$. It is known that σ-split parabolic subgroup of $\widetilde{M}$ are of the form $\widetilde{M \cap Q} = \widetilde{M} \cap \widetilde{Q}$ where Q is a σ-split parabolic subgroup of G contained in P. Naturally $\widetilde{Q}$ is a σ-split parabolic subgroup of $\widetilde{G}$ contained in $\widetilde{P}$. Let L be the Levi subgroup of $M \cap Q$ and denote $p^{-1}(L)$ by $\widetilde{L}$. We say $\widetilde{L}$ is the Levi subgroup of the σ-split parabolic subgroup of $\widetilde{M} \cap \widetilde{Q}$. Then we have the following for $r_{\widetilde{P}}, r_{\widetilde{M}\cap\widetilde{Q}}$ and $r_{\widetilde{Q}}$.

Proposition 4.3. *For P and Q as above,*

$$r_{\widetilde{M}\cap\widetilde{Q}} \circ r_{\widetilde{P}} = r_{\widetilde{Q}}.$$

Proof. Let $\lambda \in (V^*)^{\widehat{H}}$. Let $\widetilde{S}'_{\widetilde{Q}}$ be the (σ, F)-split component of $\widetilde{Q}$. Apply proposition 4.2 to $\lambda \in (V^*)^{\widehat{H}}$ first, and then $r_{\widetilde{P}}(\lambda) \in (V^*_{\widetilde{P}})^{\widetilde{M}\cap\widehat{H}}$, and we can find a positive real $\varepsilon < 1$ for $v \in V$ such that both equations

$$\langle \lambda, \pi(s)v \rangle = \delta^{1/2}_{\widetilde{P}}(\widetilde{s})\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle,$$

$$\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = \delta^{1/2}_{\widetilde{M}\cap\widetilde{Q}}(\widetilde{s})\langle r_{\widetilde{M}\cap\widetilde{Q}}(r_{\widetilde{P}}(\lambda)), \pi_{\widetilde{M}\cap\widetilde{Q}}(\widetilde{s})j_(\widetilde{M}\cap\widetilde{Q})(j_{\widetilde{P}}(v))\rangle,$$

hold for any $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon) \cap \widetilde{S}'^{\,-}_{\widetilde{Q}}(\varepsilon) = \widetilde{S}'^{\,-}_{\widetilde{Q}}(\varepsilon)$.

Plugging the second equation into the first equation,

$$\langle \lambda, \pi(s)v \rangle = \delta^{1/2}_{\widetilde{P}}(\widetilde{s})\delta^{1/2}_{\widetilde{M}\cap\widetilde{Q}}(\widetilde{s})\langle r_{\widetilde{M}\cap\widetilde{Q}}(r_{\widetilde{P}}(\lambda)), \pi_{\widetilde{M}\cap\widetilde{Q}}(\widetilde{s})j_{\widetilde{M}\cap\widetilde{Q}}(j_{\widetilde{P}}(v))\rangle.$$

Notice that for $\widetilde{s} \in \widetilde{S}'^{-}$, which is a subgroup of $\widetilde{A}'^{-}$, we mentioned earlier that $\delta_{\widetilde{P}}(\widetilde{s}) = \delta_P(s)$ where $s = p(\widetilde{s})$. Therefore,

$$\delta_{\widetilde{P}}(\widetilde{s})\delta_{\widetilde{M}\cap\widetilde{Q}}(\widetilde{s}) = \delta_P(s)\delta_{M\cap Q}(s) \overset{\text{[KT08, p. 25]}}{=\!=\!=\!=} \delta_Q(s) = \delta_{\widetilde{Q}}(\widetilde{s}).$$

Also, because we identified $(V_{\widetilde{P}})_{\widetilde{M}\cap\widetilde{Q}}$ with $V_{\widetilde{Q}}$ earlier, it is easy to see for the canonical projection, $j_{\widetilde{M}\cap\widetilde{Q}}(j_{\widetilde{P}}(v)) = j_{\widetilde{Q}}(v)$.

Plugging these two equations in, we get

$$\langle \lambda, \pi(\widetilde{s})v \rangle = \delta^{1/2}_{\widetilde{Q}}(\widetilde{s})\langle r_{\widetilde{M}\cap\widetilde{Q}}(r_{\widetilde{P}}(\lambda)), \pi_{\widetilde{Q}}(\widetilde{s})\rangle$$

This is the uniqueness part of previous proposition, thus finishing the proof. ∎

Chapter 5

Relative cuspidality

The goal of this section is to prove the following theorem.

Theorem 5.1. *Let (π, V) be an admissible $\widehat{H}$-distinguished representation of $\widetilde{G}$ and $\lambda \in (V^*)^{\widehat{H}}$. Then, $r_{\widetilde{P}}(\lambda) = 0$ for every proper σ-split parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ if and only if (π, V) is $(\widehat{H}, \lambda)$-relatively cuspidal.*

Recall that from section 2, $(\widetilde{\pi}, \widetilde{V})$ is said to be $(\widehat{H}, \lambda)$-relatively cuspidal if the support of the $(\widehat{H}, \lambda)$-coefficient $\varphi_{\lambda,v}(x) = \langle \lambda, \pi(x^{-1}v) \rangle$ is compact modulo $\widetilde{Z}'\widehat{H}$ for all $v \in V$.

Let $\widetilde{Z_\sigma^-} = \{z \in \widetilde{Z}' \mid \sigma(z) = z^{-1}\}$. [KT08] showed that $Z/\ Z_\sigma^-\ (Z \cap H)$ is finite. It follows that $\widetilde{Z}'/\ \widetilde{Z_\sigma^-}\ (\widetilde{Z}' \cap \widehat{H})$ is finite. This means (π, V) is $(\widehat{H}, \lambda)$-relatively cuspidal if and only if $\varphi_{\lambda,v}(x)$ is compact modulo $\widetilde{Z_\sigma^-}\widehat{H}$ for all $v \in V$.

Lemma 5.2. *For a compact subgroup $\widetilde{C}$ of $\widetilde{G}$, $\widetilde{Z_\sigma^-}\widetilde{C}\widehat{H}\ \cap\ \widetilde{S}'^{+}(1)$ is contained in a subset of $\widetilde{S}'^{+}(1)$ that is compact module $\widetilde{Z_\sigma^-}$.*

In [KT08], the reductive version of this lemma is proved, and we simply apply p^{-1} to get our result.

Now we prove the "if" part of 5.1.

Proof. Let $\widetilde{P} = \widetilde{M}\widehat{U}$ be a proper σ-split parabolic subgroup of $\widetilde{G}$ with the (σ, F)-split component $\widetilde{S}'$. Assume (π, V) is $(\widehat{H}, \lambda)$-relative cuspidal. Let $\widehat{K}_n$ be one of the open compact subgroup of $\widetilde{G}$ from section 3. Since $V^{\widehat{K}_n}$ is finite dimensional, we can take a compact subset $\widetilde{C}$ of $\widetilde{G}$ such that the support of $\varphi_{\lambda,v}$ is contained in $\widetilde{Z_\sigma^-}\widetilde{C}\widehat{H}$ for all $v \in V$. Due to the compactness from the previous lemma, we know there is a small real ε' such that for $\alpha \in \Delta \backslash I$,

$$\widetilde{Z_\sigma^-}\widetilde{C}\widehat{H} \ \cap \ \widetilde{S}'^{+}(1) \subset \{\widetilde{s} = p^{-1}(s) \in \widetilde{S}' \mid 1 \leq |s^\alpha|_F < \varepsilon'^{\ -1}\}.$$

Note that for $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon')$, $\widetilde{s}^{-1} \notin \widetilde{Z_\sigma^-}\widetilde{C}\widehat{H} \ \cap \ \widetilde{S}'^{+}(1)$. Therefore, for all $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon')$ and $v \in V$,

$$\langle \lambda, \pi(\widetilde{s})v \rangle = \varphi_{\lambda,v}(\widetilde{s}^{-1}) = 0.$$

Additionally, by proposition 4.2, we know we can choose a small real ε'' such that for $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon'')$,

$$\delta_{\widetilde{P}}(\widetilde{s})^{1/2}\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = \langle \lambda, \pi(\widetilde{s})v\rangle.$$

Choosing $\varepsilon = \min(\varepsilon', \varepsilon'')$, we know

$$\langle r_{\widetilde{P}}(\lambda), \pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = 0$$

for all $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$ and $v \in V^{\widehat{K}}$. This shows $r_{\widetilde{P}}(\lambda)$ is zero on $j_{\widetilde{P}}(V^{\widehat{K}}) = (V_{\widetilde{P}})^{\widehat{K}_{\widetilde{M}}}$. Finally, we run the same argument for all $\{\widehat{K}\}$, we know $r_{\widetilde{P}}(\lambda) = 0$ on $V_{\widetilde{P}}$. ∎

To prove the "only if" part, we start on the level of a reductive group G. Let $P_0 = M_0U_0$ be the minimal σ-split parabolic corresponding to a fixed (S_0, A_0, Δ) and H be the σ-fixed subgroup of G. Let $P = MU$ be a parabolic subgroup of G with (σ, F)-split torus S. We fix an element $\gamma \in (\mathbf{M_0H})(F) = M_0H$. As [KT08] constructed in section 6.5, denote $m_\gamma = \gamma(\sigma(\gamma))^{-1}$, and there is an F-involution σ_γ on G defined as

$$\sigma_\gamma(g) = m_\gamma \sigma(g) m_\gamma^{-1}.$$

It can be easily checked that the σ_γ-fixed subgroup of G is identified with $\gamma H \gamma^{-1}$, and we denote $H_\gamma = \gamma H \gamma^{-1}$. Now we take it to the level of $\widetilde{G}$, and work on $\widehat{H}$ and $\widehat{H}_\gamma$. $\widetilde{S}'$ is also a (σ_γ, F)- split torus and a σ-split parabolic subgroup $\widetilde{P}$ is also σ_γ-split. This allows us to adapt the definition of $r_{\widetilde{P}}$ by changing σ to σ_γ and $\widehat{H}$ to $\widehat{H}_\gamma$, and we get to view $r_{\widetilde{P}}$ as a mapping from $(V^*)^{\widehat{H}_\gamma}$ to $(V^*)^{\widetilde{M} \cap \widehat{H}_\gamma}$.

On the other hand, let $Q = \gamma^{-1} P \gamma$. From the definitions, this is still a σ-split parabolic with (σ, F)-split component $\gamma^{-1} S \gamma$. As a parabolic subgroup, we have $Q = M_\gamma U_\gamma$, where $M_\gamma = \gamma^{-1} M \gamma$ and similarly for $U_\gamma = \gamma^{-1} U \gamma$. Again taking everything to the level of $\widetilde{G}$, we get $\widetilde{Q} = \widetilde{M_\gamma}\widehat{U_\gamma}$, and we have a corresponding $r_{\widetilde{Q}} : (V^*)^{\widehat{H}} \to (V^*)^{\widetilde{M_\gamma} \cap \widehat{H}}$. Now, we have two $r_{\widetilde{P}}$ and $r_{\widetilde{Q}}$, one from a different involution and one from a different parabolic subgroup. Let $\pi_{\widetilde{\gamma}}$ is an isomorphism that maps $V(\widehat{U})$ to $V(\widehat{U_\gamma})$.

Lemma 5.3. *There is a mapping $f_\gamma : (V^*)^{\widehat{H}} \to (V^*)^{\widehat{H}_\gamma}$ such that for every $\lambda \in (V^*)^H, v \in V$,*

$$\langle r_{\widetilde{P}}(f_\gamma(\lambda)), j_{\widetilde{P}}(v)\rangle = \langle r_{\widetilde{Q}}(\lambda), j_{\widetilde{Q}}(\pi_{\widetilde{\gamma}}(v))\rangle.$$

Proof. Denote $\rho^{-1}(\gamma) = \widetilde{\gamma}$. From $\pi_{\widetilde{\gamma}}$, we can induce a isomorphism

$$\pi_{\widetilde{P},\widetilde{Q}} : V_{\widetilde{P}} = V/V(\widehat{U}) \to V/V(\widehat{U_\gamma}) = V_{\widetilde{Q}}$$

and clearly $\pi_{\widetilde{P},\widetilde{Q}}(j_{\widetilde{P}}(v)) = j_{\widetilde{Q}}(\pi_{\widetilde{\gamma}}(v))$ (2). Now, let $\lambda' = r_{\widetilde{Q}}(\lambda) \circ \pi_{\widetilde{P},\widetilde{Q}}$, and we want to show that this λ' satisfies the uniqueness part of 4.2. Let $\varepsilon < 1$, and $\widetilde{s} \in \widetilde{S}'^{-}(\varepsilon)$, we look at the following equations:

$$\begin{aligned}
\delta_{\widetilde{P}}^{1/2}(\widetilde{s})\langle\lambda',\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle &= \langle r_{\widetilde{Q}}(\lambda),\pi_{\widetilde{P},\widetilde{Q}}\left(\delta_{\widetilde{P}}^{1/2}(\widetilde{s})\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\right)\rangle \\
&= \langle r_{\widetilde{Q}}(\lambda),\pi_{\widetilde{P},\widetilde{Q}}(j_{\widetilde{P}}(\pi(\widetilde{s})v))\rangle \text{ by equation (1)} \\
&= \langle r_{\widetilde{Q}}(\lambda),j_{\widetilde{Q}}(\pi_{\widetilde{\gamma}}(\pi(\widetilde{s})v))\rangle \text{ by equation (2)} \\
&= \langle r_{\widetilde{Q}}(\lambda),\delta_{\widetilde{Q}}^{1/2}(\widetilde{\gamma}^{-1}\widetilde{s}\widetilde{\gamma})\pi_{\widetilde{Q}}(\widetilde{\gamma}^{-1}\widetilde{s}\widetilde{\gamma})j_{\widetilde{Q}}(\pi_{\widetilde{\gamma}}(v))\rangle \text{ by (1)} \\
\text{applying 4.2 to } \widetilde{Q} \quad &= \langle\lambda,\pi(\widetilde{\gamma}^{-1}\widetilde{s}\widetilde{\gamma})\pi_{\widetilde{\gamma}}(v)\rangle.
\end{aligned}$$

Define $f_\gamma(\lambda)$ as $\langle f_\gamma\lambda,\pi(\widetilde{s})v\rangle = \langle\lambda,\pi(\widetilde{\gamma}^{-1}\widetilde{s}\widetilde{\gamma})\pi_{\widetilde{\gamma}}(v)\rangle$.

In summary,

$$\delta_{\widetilde{P}}^{1/2}(\widetilde{s})\langle\lambda',\pi_{\widetilde{P}}(\widetilde{s})j_{\widetilde{P}}(v)\rangle = \langle f_\gamma\lambda,\pi(\widetilde{s})v\rangle,$$

and by the uniqueness of 4.2, we showed that $\lambda' = r_{\widetilde{Q}}(\lambda)\circ\pi_{\widetilde{P},\widetilde{Q}}$ is the same with $r_{\widetilde{P}}(f_\gamma\lambda)$, i.e.,

$$\langle r_{\widetilde{P}}(f_\gamma\lambda),j_{\widetilde{P}}(v)\rangle = \langle r_{\widetilde{Q}}(\lambda),\pi_{\widetilde{P},\widetilde{Q}}(j_{\widetilde{P}}(v))\rangle = \langle r_{\widetilde{Q}}(\lambda),j_{\widetilde{Q}}(\pi_{\widetilde{\gamma}}(v))\rangle$$

finishing the proof.

∎

Lemma 5.4. *Let π, V be an irreducible admissible representation of $\widetilde{G}$. Let $\widetilde{P}_I = \widetilde{M_I}\widehat{U}_I$ be a standard parabolic subgroup with respect to a fixed $(S_0, A_\emptyset, \Delta)$ and σ-split $I \in \Delta$ as discussed in section 2. Take $\gamma \in M_0H$ and $\widetilde{\Omega}$ compact subset of $\widetilde{G}$. Take $\lambda \in (V^*)^{\widehat{H}}$. If $r_{\widetilde{\gamma^{-1}P\gamma}}(\lambda) = 0$, then for each $v \in V$, there exists a positive $\varepsilon < 1$ such that the matrix coefficient $\varphi_{\lambda,v} = 0$ on $\widetilde{\Omega}\widetilde{S_I^+(\varepsilon)}\widetilde{\gamma}\widehat{H}$.*

Proof. Take $\widetilde{\omega} \in \widetilde{\Omega}, \widetilde{s} \in \widetilde{S_I^-(\varepsilon)}, \widehat{h} \in \widehat{H}$,

$$
\begin{aligned}
\varphi_{\lambda,v}(\widetilde{\omega}\widetilde{s}^{-1}\gamma\widehat{h}) &= \langle \lambda, \pi(\widehat{h}^{-1})\pi_{\widetilde{\gamma}}\pi(\widetilde{s})\pi(\widetilde{\omega})v\rangle \\
\text{since } \lambda \in (V^*)^{\widehat{H}} \quad &= \langle \lambda, \pi(\widehat{h}^{-1})\pi_{\widetilde{\gamma}}\pi(\widetilde{s})\pi(\widetilde{\omega})v\rangle \\
&= \langle f_\gamma \lambda, \pi(\widetilde{s})\pi(\widetilde{\omega}^{-1})v\rangle \\
\text{since } f_\gamma(\lambda) \in (V^*)^{\widehat{H}_\gamma}, \text{by 4.2} \quad &= \delta_{\widetilde{P}}^{1/2}\langle r_{\widetilde{P}}\left(f_\gamma(\lambda)\right), \pi_{\widetilde{P}}(s) j_{\widetilde{P}}(\pi(\widetilde{\omega}^{-1})v)\rangle \\
\text{by equation (1)} \quad &= \langle r_{\widetilde{P}}\left(f_\gamma(\lambda)\right), j_{\widetilde{P}}(\pi(\widetilde{s})\pi(\widetilde{\omega}^{-1})v)\rangle \\
\text{by the previous lemma} \quad &= \langle r_{\widetilde{\gamma^{-1}P\gamma}}\left(\lambda\right), j_{\gamma^{-1}\widetilde{P}\gamma}(\pi_{\widetilde{\gamma}}\pi(\widetilde{s})\pi(\widetilde{\omega}^{-1})v)\rangle \\
\text{by assumption} \quad &= 0
\end{aligned}
$$

■

Here is the proof of the "only if" part:

Proof. Assume $r_{\widetilde{P}}(\lambda) = 0$ for all σ-split parabolic subgroup $\widetilde{P}$. Recall that we want to show the support of $\varphi_{\lambda,v}(x)$ is compact modulo $\widetilde{Z_\sigma^-}\widehat{H}$ for all $v \in V$. [KT08, section 3] showed a decomposition of $G = \Omega S_0^+ \Gamma H$. By applying p^{-1}, we have $\widetilde{G} = \widetilde{\Omega}\widetilde{S_0^+}\widetilde{\Gamma}\widehat{H}$, where Ω is a compact subset of G and Γ is a finite subset of $M_0 H$. By the previous lemma, we run through all $\gamma \in \Gamma$ and σ-split subset $I \subset \Delta$. As a result, let $\varepsilon < 1$ be the smallest ε in the previous lemma, then for any $\widetilde{s} \in \widetilde{S}_0'^-(1) \cap \widetilde{S}_I'^-(\varepsilon)$, $\varphi_{\lambda,v}$ restricted to $\widetilde{\Omega}\widetilde{s}^{-1}\widetilde{\Gamma}\widehat{H}$ is identically zero. This means that for any $\widetilde{s} \in \widetilde{S}_0'^-(1)$, if $\varphi_{\lambda,v}$ restricted to $\widetilde{\Omega}\widetilde{s}^{-1}\widetilde{\Gamma}\widehat{H}$ is not identically zero, then $\widetilde{s}$ is not in $\widetilde{S}_I'^-(\varepsilon)$ for any σ-split $I \in \Delta$. Therefore, the support of $\varphi_{\lambda,v}$ is the union of $\widetilde{\Omega}\widetilde{s}^{-1}\widetilde{\Gamma}\widehat{H}$ where s runs over all

$$\{\widetilde{s} \in \widetilde{S}_0'^-(1) \mid \varepsilon < |s^\alpha| < 1 \text{ for } \alpha \in \Delta\backslash\Delta_\sigma\},$$

which is finite modulo $\widetilde{Z_\sigma^-}\widetilde{S}_0'^-$. Therefore $\varphi_{\lambda,v}(x)$ is compact modulo $\widetilde{Z_\sigma^-}\widehat{H}$ for all v.

■

Chapter 6

Relative subrepresentation Theorem for a finite central extension

To understand the statement of our main theorem, let us recall the definition of induced representation. For a parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$ of $\widetilde{G}$, let $(\pi^\#, V^\#)$ be a representation on $\widetilde{M}$. Then the induced representation $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\pi^\#)$ is a representation of $\widetilde{G}$ acting on the space

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(V^\#) = \{\pi : \widetilde{G} \to V^\# \mid \pi \text{ smooth and } \pi(\widehat{u}\widetilde{m}\widetilde{g}) = \delta_{\widetilde{P}}^{1/2}(\widetilde{m})\pi^\#(\widetilde{m})\pi(\widetilde{g})\}$$

by right translation.

Theorem 6.1. *Let (π, V) be an irreducible admissible $\widehat{H}$-distinguished representation of $\widetilde{G}$. Then, there exists a σ-split parabolic subgroup $\widetilde{P} = \widetilde{M}\widehat{U}$ of $\widetilde{G}$ and an irreducible $\widetilde{M} \cap \widehat{H}$-relatively cuspidal representation $\pi^\#$ of $\widetilde{M}$ such that π is a subrepresentation of* $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\pi^\#)$.

Proof. We use induction on the dimension d of the maximal (σ, F)-split tori of $\widetilde{G}/\widetilde{Z}'$. If $d = 0$, it is shown in [HW93, Propositon 4.3] that $\widetilde{G}/\widetilde{Z}'\widehat{H}$ is compact, therefore every admissible $\widehat{H}$-distinguished representation is $\widehat{H}$-relatively cuspidal.

Assume now $d > 0$. If (π, V) is $\widehat{H}$-relative cuspidal, then there is nothing to prove. If not, then there exists a $\lambda \in (V^*)^{\widehat{H}}$ such that π is not $(\widehat{H}, \lambda)$-cuspidal. By theorem 5.1, we know there exists a proper σ-split subgroup $\widetilde{P}$ of $\widetilde{G}$ such that $r_{\widetilde{P}}(\lambda) \neq 0$. Find a minimal among these $\widetilde{P}$, and we call this minimal one $\widetilde{Q} = \widetilde{M'}\widehat{U}'$. It is known that every parabolic subgroups of $\widetilde{M'}$ is in the form of $\widetilde{M'} \cap \widetilde{Q}'$ where $\widetilde{Q}'$ is a parabolic subgroup contained in $\widetilde{Q}$. By proposition 4.3, $r_{\widetilde{M'} \cap \widetilde{Q}'}(r_{\widetilde{Q}}(\lambda)) = r_{\widetilde{Q}'}(\lambda)$. Since $\widetilde{Q}$ is the minimal parabolic subgroup that $r_{\widetilde{Q}}(\lambda) \neq 0$, we have $r_{\widetilde{Q}'}(\lambda) = 0$. Now, invoking 5.1 again, $r_{\widetilde{M'} \cap \widetilde{Q}'}(r_{\widetilde{Q}}(\lambda)) = 0$ means $\pi_{\widetilde{Q}}$ is $(\widetilde{M'} \cap \widehat{H}, r_{\widetilde{Q}}(\lambda))$-relatively cuspidal. Proposition 2.5 gives an irreducible subquotient ρ'. By the Frobenius reciprocity,

$$\mathrm{Hom}_{\widetilde{G}}(\pi, \mathrm{Ind}_{\widetilde{Q}}^{\widetilde{G}}(\rho')) \simeq \mathrm{Hom}_{\widetilde{M'}}(\pi_{\widetilde{Q}}, \rho'))$$

and therefore we have an embedding of π into $\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{G}}(\rho')$. For the induction step, denote $\widetilde{A}''$ to be the F-split component of $\widetilde{M'}$. Then, looking at the maximal (σ, F)-split tori in $\widetilde{M'}/\widetilde{A}''$, the dimension is strictly less than d. Therefore, applying the induction hypothesis to ρ', there is a σ-split parabolic subgroup of $\widetilde{G}$, denoted by $\widetilde{P}$, where $\widetilde{M'} \cap \widetilde{P} = \widetilde{M}_0 \widehat{U}_0$ is a σ-split parabolic subgroup of $\widetilde{M'}$ and ρ is a $\widetilde{M}_0 \cap \widehat{H}$-relatively cuspidal representation, such that ρ' is a subrepresentation of $\mathrm{Ind}_{\widetilde{M'} \cap \widetilde{P}}^{\widetilde{M'}}(\rho)$. Finally, this means π is a subrepresentation of $\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{G}}(\mathrm{Ind}_{\widetilde{M'} \cap \widetilde{Q}}^{\widetilde{M'}}(\rho))$, which is isomorphic to $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\rho)$. This finishes the induction step, and the proof is complete.

■

Appendix A

Appendix

A.1 Appendix 1: A example of such finite central extension

In this section, we describe one example of a specific finite central extension $\widetilde{G}$ that was mentioned in [BLS99].

Definition A.1. A map $c : F^\times \times F^\times \to \mathscr{A}$ is called a *Steinberg Symbol* if for any $x, y, z \in F^\times$, it satisfies:

1. $c(x, y)c(xy, z) = c(x, yz)c(y, z)$
2. $c(1, 1) = 1_{\mathscr{A}}$, $c(x, y) = c(x^{-1}, y^{-1})$
3. $c(x, y) = c(x, (1 - x)y)$ for $x \neq 1$

If c furthermore satisfies that $c(xy, z) = c(x, z)c(y, z)$, we say it is *bilinear.*

Given an Steinberg symbol $c : F^\times \times F^\times \to \mathscr{A}$, we have an important theorem (cf. [BLS99], Theorem 1.5)

Theorem A.2. *When c is bilinear, there exists a central extension*

$$1 \to \mathscr{A} \hookrightarrow \widetilde{G(F)} \to G(F) \to 1$$

When c is a second order Hilbert symbol, it is shown that for non-Archimedean local field E, we have this exact sequence

$$1 \to \mu_2 \hookrightarrow \widetilde{G(E)} \to G(E) \to 1$$

For a quadratic extension E/F, $G(F)$ is a subset of $G(E)$ and there exists a splitting s of $G(F)$. This gives us the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \hookrightarrow & \widetilde{G(E)} & \xrightarrow{p} & G(E) & \longrightarrow & 1 \\
 & & & & \uparrow & \overset{s}{\nwarrow} & \uparrow & & \\
 & & & & \widetilde{G(F)} & \longrightarrow & G(F) & &
\end{array}$$

where $\widetilde{G(E)}$, as a group, is isomorphic to $G(E) \times \mu_2$.

Furthermore, we have an involution σ on G. For $g \in G(E)$, it is a E rational point of $\mathbf{G}$, and σ takes each coordinate to its conjugate regarding the quadratic extension E/F. It is easy to check that $G(F) = \{g \in G \mid \sigma(g) = g\}$. These are all the criteria we need for our $\widetilde{G}$, and conclude this concrete example.

A.2 Appendix 2: Proof of Proposition 2.5

This section focus on prove the following proposition:

Proposition A.3. *Any finitely generated $(\widehat{H}, \lambda)$-cuspidal representation of $\widetilde{G}$ has a nontrivial $\widehat{H}$-distinguished irreducible quotient representation.*

Let us start with some definitions.

Let $C^\infty(\widetilde{G}/\widehat{H})$ denote the space of all $\widehat{H}$-invariant locally constant complex valued functions on $\widetilde{G}$, on which $\widetilde{G}$ acts by left translation. Now we can define $T_\lambda(v) = \mu_{\lambda,v}$

as a map from V to $C^\infty(\widetilde{G}/\widehat{H})$, for $\lambda \in (V^*)^{\widehat{H}}$ and $v \in V$. Let $C_0^\infty(\widetilde{G}/\widehat{H})$ denote the subspace of $C^\infty(\widetilde{G}/\widehat{H})$ consisting of all functions compactly supported modulo $\widetilde{Z}'\widehat{H}$.

For a fixed quasi-character ω of $\widetilde{Z}'$ which is trivial on $\widetilde{Z}' \cap \widehat{H}$, $C_{0,\omega}^\infty(\widetilde{G}/\widehat{H})$ denote the space of all $\varphi \in C^\infty(\widetilde{G}/\widehat{H})$ such that

$$\varphi(zg) = \omega^{-1}(z)\varphi(g)$$

for $z \in \widetilde{Z}'$ and $g \in \widetilde{G}$

Definition A.4. For an admissible representation (π, V) of $\widetilde{G}$ and a quasi-character ω of $\widetilde{Z}'$, let

$$V_\omega = \{v \in V |\, \pi(z)v = \omega(z)v \text{ for all } z \in \widetilde{Z}'\}$$

We call a subrepresentation (π', V') of (π, V) to be an ω-subrepresentation of (π, V) if V' is contained in V_ω. If furthermore $V' = V_\omega$, we say V' is an ω-representation.

Lemma A.5. *Let (π, V) be an admissible representation of $\widetilde{G}$ of finite length. Then there exists a nontrivial quotient representation of V which is isomorphic to an ω-subrepresentation of V for some quasi-character ω of $\widetilde{Z}'$.*

Proof. From [Cas95, 2.1.9], we can decompose V into $\bigoplus_\omega V_{\omega,\infty}$, where ω runs over a finite set of quasi-characters of $\widetilde{Z}'$, and

$$V_{\omega,\infty} = \bigcup_n V_{\omega,n}$$

where

$$V_{\omega,n} = \{v \in V | (\pi(z) - \omega(z))^n v = 0 \text{ for all } z \in \widetilde{Z}'\}.$$

It suffices to consider the case $V = V_{\omega,\infty}$ for a fixed quasi-character ω. If V is simply $V_{\omega,1} = V_\omega$, then there is nothing to prove. If not, then there exist elements

$z_0 \in \widetilde{Z}'$ such that

$$\pi(z_0)v_0 - \omega(z_0)v_0 \neq 0.$$

Now we consider $\phi_1 = (\pi(z_0) - \omega(z_0))$ from V to V. This is a nonzero *G-morphism* with nontrivial kernel. We look at the image of ϕ_1, V_1. It is a proper G-submodule of V. If $V_1 \subset V_\omega$, then $V/Ker(\phi_1) \cong V_1$ gives the desired quotient, then we are finished. If not, we can find $z_1 \in \widetilde{Z}'$ and $v_1 \in V_1$ such that

$$\pi(z_1)v_1 - \omega(z_1)v_1 \neq 0.$$

We look at $\phi_2 = (\pi(z_1) - \omega(z_1))|_{V_1}$. Similarly, let V_2 be the image of ϕ_2, which is a proper G-submodule of V_1. If V_2 is not a subset of V_ω, we repeat the same process, giving us a decreasing sequence

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq ...$$

Since V is of finite length, we must have $V_k \subset V_\omega$. Then, $V/Ker(\phi_k \circ ... \circ \phi_1) \cong V_k$ gives the desired quotient.

■

Let $\mathcal{X}(\widetilde{G}, \mathbb{R}_+^\times)$ denote the positive valued character of $\widetilde{G}$. Let

$$\mathcal{X}(\widetilde{G}/\widehat{H}, \mathbb{R}_+^\times) = \{\chi \in \mathcal{X}(\widetilde{G}, \mathbb{R}_+^\times) \mid \chi|_H = 1\}.$$

We define $\mathcal{X}(\widetilde{Z}', \mathbb{R}_+^\times)$ and $\mathcal{X}(\widetilde{Z}' \cap \widehat{H}, \mathbb{R}_+^\times)$ in the same way.

Lemma A.6.

1. *The restriction map*

$$res : \mathcal{X}(\widetilde{G}, \mathbb{R}_+^\times) \to \mathcal{X}(\widetilde{Z}', \mathbb{R}_+^\times)$$

$$res(\chi) = \chi|_{\widetilde{Z}'}$$

is bijective.

2. res in part 1 sends $\mathcal{X}(\widetilde{G}/\widehat{H}, \mathbb{R}_+^\times)$ to $\mathcal{X}(\widetilde{Z}' \cap \widehat{H}, \mathbb{R}_+^\times)$ bijectively.

Proof. Part 1 follows from [5.2.5 Cas95]. For part 2, for any χ in $\mathcal{X}(\widetilde{G}/\widehat{H}, \mathbb{R}_+^\times)$, it is obvious that $\chi|_{\widetilde{Z}'}$ lands in $\mathcal{X}(\widetilde{Z}'/\widetilde{Z}' \cap \widehat{H}, \mathbb{R}_+^\times)$. Therefore it is automatically one to one by part 1.

To show it is onto, pick $\omega \in \mathcal{X}(\widetilde{Z}'/\widetilde{Z}' \cap \widehat{H}, \mathbb{R}_+^\times)$, and by part 1, we can pick a $\chi \in \mathcal{X}(\widetilde{G}, \mathbb{R}_+^\times)$ such that $\chi|_{\widetilde{Z}'} = \omega$. Now for any $z \in \widetilde{Z}'$, we have $\sigma(z) \in \widehat{H}$ and $z\sigma(z) \in \widetilde{Z}' \cap \widehat{H}$. Thus $\omega(z\sigma(z)) = 1$. This means $\omega \circ \sigma = \omega^{-1}$. Now, consider

$$(\chi \circ \sigma)|_{\widetilde{Z}'} = (\chi|_{\widetilde{Z}'}) \circ \sigma = \omega \circ \sigma = \omega^{-1} = (\chi|_{\widetilde{Z}'})^{-1} = \chi^{-1}|_{\widetilde{Z}'}.$$

Again, by part 1, this means $\chi \circ \sigma = \chi^{-1}$. For any $h \in \widehat{H}, \chi^{-1}(h) = \chi \circ \sigma(h) = \chi(h)$, this shows $\chi(h) \equiv 0$ for $h \in \widehat{H}$. Thus $\chi \in \mathcal{X}(\widetilde{G}/\widehat{H}, \mathbb{R}_+^\times)$, showing it is onto.

As we showed it is both onto and one to one, this finishes the proof of part 2. ■

Here is the proof of proposition 2.5 in the main paper.

Proof. Let (π, V) be a finitely generated $(\widehat{H}, \lambda)$-cuspidal representation of $\widetilde{G}$ with a fixed $\lambda \in (V^*)^{\widehat{H}}$. By definition, $T_\lambda(V) \in C_0^\infty(\widetilde{G}/\widehat{H})$. By [Cas95, 6.3.10], because the quotient $V/Ker(T_\lambda)$ is finitely generated, it also has to be of finite length. Directly applying lemma A.5,we can get a quotient representation (π', V') of $V/Ker(T_\lambda)$ that is isomorphic to an ω-subrepresentation of $V/Ker(T_\lambda) \cong T_\lambda(V) \subset C_0^\infty(\widetilde{G}/\widehat{H})$ for some quasi character ω of $\widetilde{Z}'/\widetilde{Z}' \cap \widehat{H}$. We can view (π', V') as a subrepresentation of

$C^\infty_{0,\omega}(\widetilde{G}/\widehat{H})$. Now consider $|\omega|^{-1} \in \mathcal{X}(\widetilde{Z}'/\widetilde{Z}' \cap \widehat{H}, \mathbb{R}^\times_+)$, and by the part 2 of previous lemma, there is a $\chi \in \mathcal{X}(\widetilde{G}/\widehat{H}, \mathbb{R}^\times_+)$ such that $\chi|_{\widetilde{Z}'} = |\omega|^{-1}$. Then now we can consider the representation χ, π' on V' defined by

$$(\chi\pi')(g) = \chi(g)\pi'(g)$$

for all $g \in \widetilde{G}$. It can be regarded as a subrepresentation of $C^\infty_{0,\omega_u}(\widetilde{G}/\widehat{H})$, where $\omega_u = \omega \cdot |\omega|^{-1}$. Note this ω_u is clearly unitary. By [KT08, 1.8] and [Cas95, 2.1.14], we can decompose $\omega \cdot \pi'$ into a direct sum of finitely many irreducible subrepresentations. By the same decomposition, we have π' as direct sum of irreducible subrepresentation. ■

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VITA

Chengyu Jiao was born on March 16th, 1996, in Qingdao, China. He attended elementary school and high school in his hometown. During his last year of high school, he became an exchange student and studied in Carterville, Illinois.

After graduation, he attend the University of Missouri as an undergraduate student in mathematics. He was also working part time as a math tutor. He recieved his Bachelors degree in 2018 and decided to continue his academic research as a math graduate student in the University of Missouri. He also worked as an instructor.

He started researching with Professor Shurichiro Takeda in the field of representation theory and algebraic number theory. After Profrssor Takeda left the University of Missouri, he finished his dissertation with Professor William Banks. He plans to graduate in the Spring semester of 2023.