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ON THE
'CONSIDERATION OF SOME SPECIAL POINTS OF THE CALCULUS

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On the Consideration of Some Special Points of
the Calculus.

The subject of this paper is the study of the four problems of

- (1) integration of a series term by term;
- (2) differentiation of a series term by term;
- (3) reversal of the order of integration in a double integral;
- (4) differentiation under the sign of integration.

In certain classes of cases these four important problems are but different forms of the same problem, a problem in double limits, and it is of advantage to recognize them as such. If $f(x, y)$ is a function of the two independent variables x, y then

$$\lim_{y=0} \left[\lim_{x=0} f(x, y) \right] \text{ and } \lim_{x=0} \left[\lim_{y=0} f(x, y) \right]$$

may or may not be the same thing.

For example: $f(x, y) = \frac{x-y}{x+y}$

Here the first double limit has the value -1 , the second the value $+1$.

1. The Integration of Series Term by Term.

Let the series

$$u_1(x) + u_2(x) + \dots$$

converge toward a limit, $f(x)$, when the real variable x lies in the interval (a, b) ;

$$f(x) = u_1(x) + u_2(x) + \dots, \quad a \leq x \leq b,$$

and let $f(x)$, $u_i(x)$ be continuous functions of x throughout this interval. Let

$$s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x);$$

$s_n(x)$ is also continuous. Then

$$\int_a^b f(x) dx \text{ means } \int_a^b \left[\lim_{n \rightarrow \infty} s_n(x) \right] dx,$$

$$\text{while } \int_a^b u_1(x) dx + \int_a^b u_2(x) dx + \dots$$

$$\text{means } \lim_{n \rightarrow \infty} \left[\int_a^b s_n(x) dx \right].$$

Since the integral is the limit of a sum, we have here to do with a double limit. The question of whether a series can be integrated term by term is then the question of whether it is allowable to reverse the order in a certain double limit — the question of whether the equation

$$\int_a^b \left[\lim_{n \rightarrow \infty} s_n(x) \right] dx = \lim_{n \rightarrow \infty} \left[\int_a^b s_n(x) dx \right]$$

is true.

$$\text{For example, let } s_n(x) = n x e^{-n x^2}, \quad u_1(x) = s_1(x),$$

$$u_n(x) = s_n(x) - s_{n-1}(x), \quad n > 1.$$

$$\text{Then } f(x) = \lim_{n \rightarrow \infty} s_n(x) = 0 \text{ for all values of } x \text{ and hence}$$

$$\int_0^1 \left[\lim_{n \rightarrow \infty} s_n(x) \right] dx = 0.$$

$$\text{On the other hand } \int_0^1 s_n(x) dx = \frac{1}{2} (1 - e^{-n}) \text{ and}$$

$$\lim_{n \rightarrow \infty} \left[\int_0^1 s_n(x) dx \right] = \frac{1}{2}.$$

Thus this series, although a series of continuous functions converging absolutely toward a continuous limit, cannot be integrated term by term.

A series which can be integrated term by term is

$$\text{this: let } s_n(x) = \frac{n^2 x}{1 + n^3 x^2}$$

$$\lim_{u \rightarrow \infty} S_n(x) = \lim_{u \rightarrow \infty} \left[\frac{u^2 x}{1 + u^2 x^2} \right] = 0; \text{ hence } \int_0^1 \left[\lim_{u \rightarrow \infty} S_n(x) \right] dx = 0;$$

$$\int_0^1 S_n(x) dx = \int_0^1 \frac{u^2 x}{1 + u^2 x^2} dx = \left[\frac{1}{2n} \log \left(x^2 + \frac{1}{n^2} \right) \right]_0^1$$

$$\text{Hence } \lim_{u \rightarrow \infty} \int_0^1 S_n(x) dx = 0$$

$$\text{Therefore } \lim_{u \rightarrow \infty} \int_0^1 S_n(x) dx = \int_0^1 \left[\lim_{u \rightarrow \infty} S_n(x) \right] dx.$$

We shall first consider uniformly convergent series.

Definition of Uniform Convergence. The series

$$u_1(x) + u_2(x) + \dots$$

(or the function $S_n(x)$) is said to be uniformly convergent in the interval (a, b) if, the positive quantity ε having been chosen at pleasure, it is then possible to choose m so that

$$|S_{m+p}(x) - S_m(x)| < \varepsilon \quad p = 1, 2, \dots$$

no matter what value x may have in the interval (a, b) . That which is essential in this definition is the order of the choice of ε, m, x , namely: first ε , secondly m , the inequality then holding for (thirdly) any x in the interval.

Theorem: A series of continuous functions uniformly convergent in the interval (a, b) is itself a continuous function in that interval.

The establishment of this theorem makes it necessary to show that the positive quantity ε having been chosen at pleasure, δ can then be so determined

$$\text{that } |f(x_0 + h) - f(x_0)| < \varepsilon \quad |h| < \delta.$$

$$f(x) = u_0(x) + u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

Let us write

$$f(x) = u_0 + u_1 + u_2 + \dots + u_n + R_n.$$

By hypothesis we can take n large enough to make $|R_n| < \varepsilon$, x being anywhere in the interval (a, b) , and ε designating a quantity taken in advance as small as we please. We have then

$$f(x') - f(x) = u'_0 + u'_1 + \dots + u'_n - (u_0 + u_1 + u_2 + \dots + u_n) + R'_n - R_n,$$

whenever $\hat{x}$ ^{and x'} be, with $|R'_n - R_n| < 2\varepsilon$.

The function $u_0 + u_1 + \dots + u_n$, formed of a limited number of terms, is continuous; accordingly, we can take x' sufficiently near x to make the difference

$$(u'_0 + u'_1 + u'_2 + \dots + u'_n) - (u_0 + u_1 + u_2 + \dots + u_n)$$

less than ε in absolute value, and consequently

$$|f(x') - f(x)| < 3\varepsilon,$$

if x' is sufficiently near x ; the function is then continuous.

The above theorem is virtually a theorem stating the equality of two double limits. It says that if $S_n(x)$ converges uniformly toward $f(x)$, then $\lim_{x=x_0} f(x) = f(x_0)$, and this equation can be written in the form:

$$\lim_{x=x_0} \left[\lim_{n=\infty} S_n(x) \right] = \lim_{n=\infty} \left[\lim_{x=x_0} S_n(x) \right].$$

That all functions defined by convergent series of continuous functions are not themselves continuous may be seen by the following example:

$$S_n(x) = x^{\frac{1}{2n-1}}$$

$$\text{when } x > 0 \quad x^{\frac{1}{2n-1}} = e^{\frac{1}{2n-1} \log x} \quad \text{and} \quad \lim_{n=\infty} S_n(x) = 1;$$

when $x=0$, $\ln(x)=0$, $\lim_{n \rightarrow \infty} \ln(x) = 0$;

when $x < 0$, $x^{\frac{1}{2n-1}} = -(-x)^{\frac{1}{2n-1}} = -e^{\frac{1}{2n-1} \log(-x)}$ and $\lim_{n \rightarrow \infty} \ln(x) = -1$.

Thus

$$\begin{aligned} f(x) &= 1, \text{ when } x > 0 \\ &= 0, \text{ when } x = 0 \\ &= -1, \text{ when } x < 0; \end{aligned}$$

thus the principle that "whatever is true of the variable is true of the limit" is shown to be false.

It follows at once from the above Theorem that, if a convergent series of continuous functions is discontinuous, it must converge non-uniformly.

Test for Uniform Convergence. A sufficient condition for the uniform convergence of the series

$$u_1(x) + u_2(x) + \dots$$

is the following:

If a set of constant positive quantities $M_1, M_2, \dots$ can be found such that (1) $|u_i(x)| \leq M_i$, $a \leq x \leq b$, and (2) the series $\sum_{i=1}^{\infty} M_i$ converges, then the u -series will converge uniformly. For, ε being given, n can always be so determined that

$$M_{m+1} + M_{m+2} + \dots + M_{m+k} < \varepsilon, \quad k=1, 2, \dots$$

and hence

$$|u_{m+1}(x) + u_{m+2}(x) + \dots + u_{m+k}(x)| < \varepsilon, \quad k=1, 2, \dots$$

$$a \leq x \leq b.$$

This test shows at once that a power series

$$a_0 + a_1 x + a_2 x^2 + \dots$$

is uniformly convergent within any interval (a, b) included within its interval of convergence $(-r, r)$:

$-r < a < b < r$. For if h is a positive quantity less than

x , but greater than the greater of the two quantities $|a|$, $|b|$, then the series $a_0 + a_1 h + a_2 h^2 + \dots$ converges absolutely and $|a_i x^i| < |a_i| h^i$, $a \leq x \leq b$.

Hence a power series represents a continuous function within (not necessarily inclusive of the boundary of) its interval of convergence.

It is well to notice here that uniform convergence has nothing to do with absolute convergence. Thus the non-uniformly convergent series $\sum_n(x) = x^{\frac{1}{2n-1}}$, discussed above, is absolutely convergent for all values of x . On the other hand, from any absolutely convergent series a conditionally convergent series having the same value can be constructed by adding to the terms of the absolutely convergent series respectively the terms of the conditionally convergent series

$$0 = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \dots$$

The condition for uniform convergence will not thereby be affected.

Uniformly Convergent Series Integrable Term by Term.

Theorem: A uniformly convergent series can be integrated term by term in any finite interval.

We have here to prove that

$$\lim_{n \rightarrow \infty} \int_a^b S_n(x) dx = \int_a^b \left[\lim_{n \rightarrow \infty} S_n(x) \right] dx = \int_a^b f(x) dx.$$

$$\text{Now } f(x) = S_n(x) + r_n(x)$$

where $S_n(x)$ and $r_n(x)$ denote the sum of the first n terms of the series and sum of the remaining terms respectively.

7.

$$\int_a^b f(x) dx = \int_a^b S_n(x) dx + \int_a^b r_n(x) dx$$

$$\left| \int_a^b r_n(x) dx \right| \leq \int_a^b |r_n(x)| dx < 2\varepsilon (b-a)$$

since by hypothesis $|r_n(x)| < 2\varepsilon$

$n \geq m$,

$$\text{Hence } \lim_{n \rightarrow \infty} \int_a^b r_n(x) dx = 0$$

$$\text{and } \lim_{n \rightarrow \infty} \int_a^b S_n(x) dx = \int_a^b f(x) dx.$$

But this theorem does not hold for an integral one of whose limits is infinite, as is shown by the example

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \dots \quad 0 \leq x.$$

This series converges uniformly for all positive values of x .

But the term-by-term integral from a positive lower limit a to infinity is a divergent series.

The series

$$-2x e^{-x^2} = \sum_{n=1}^{\infty} -2x \left(\frac{1}{n^2} e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}} \right)$$

is an example of a uniformly convergent series which is integrable through an infinite interval, but the integral of which cannot be obtained by summing the integrals of its terms.

We have

$$S_n(x) = -2x e^{-x^2} + \frac{2x}{2^2} e^{-\frac{x^2}{2^2}} - \frac{2x}{2^2} e^{-\frac{x^2}{2^2}} + \frac{2x}{3^2} e^{-\frac{x^2}{3^2}} - \frac{2x}{3^2} e^{-\frac{x^2}{3^2}} + \dots$$

$$- \dots - \frac{2x}{n^2} e^{-\frac{x^2}{n^2}} + \frac{2x}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}}.$$

$$= -2x e^{-x^2} + \frac{2x}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}}.$$

$$r_n(x) = -\frac{2x}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}}$$

$$|r_n(x)| \leq \frac{\sqrt{2}}{(n+1)\sqrt{e}}.$$

The uniformity of convergence follows from the fact that the second member of the last relation is independent of x . By integrating this series term by term from 0 to x , we obtain

$$\sum_{n=1}^{\infty} \left(e^{-\frac{x^2}{n^2}} - e^{-\frac{x^2}{(n+1)^2}} \right) \text{ which converges to } e^{-x^2} - 1 = \int_0^x -2x e^{-x^2} dx$$

only when x is finite, being equal to zero when x is infinite.

The integration of an infinite series through an infinite interval is a question of triple limits. Uniformity of convergence on the part of the given series enables us to write

$$\lim_{b=\infty} \int_a^b \lim_{n=\infty} f_n(x) dx = \lim_{b=\infty} \lim_{n=\infty} \int_a^b f_n(x) dx$$

but it does not enable us to write

$$\lim_{b=\infty} \lim_{n=\infty} \int_a^b f_n(x) dx = \lim_{n=\infty} \lim_{b=\infty} \int_a^b f_n(x) dx.$$

A sufficient condition for the latter relation would be the uniform convergence in the infinite interval of the series resulting from the term-by-term integration of the given series. This condition has the disadvantage of not being immediately applicable to the given series. The following, however, gives a sufficient formal condition free from this disadvantage.

Let

$$S = u_1 + u_2 + \dots$$

be a series whose terms are functions of x , admitting of determinate integrals in the interval (a, b) — Then

limits having the power to become infinite. If the remainder R_n , left after stopping at the n^{th} term of the series, can be put in the form

$$R_n = \varepsilon_n f(x)$$

$f(x)$ being a positive function of x , whose integral between a and b is finite, and ε_n a factor which uniformly approaches zero when n approaches infinity, we will have

$$\int_a^b S dx = \int_a^b u_1 dx + \int_a^b u_2 dx + \dots$$

We have

$$\int_a^b S dx = \int_a^b u_1 dx + \int_a^b u_2 dx + \dots + \int_a^b u_n dx + \int_a^b R_n dx.$$

It suffices then to show that the complementary term

$$\int_a^b R_n dx = \int_a^b \varepsilon_n f(x) dx$$

approaches zero when n approaches ∞ . Now we can take n large enough to make $|\varepsilon_n|$ in the whole field constantly less than a quantity ε , taken as small as we please, and the maximum-minimum theorem will give

$$\left| \int_a^b \varepsilon_n f(x) dx \right| < \varepsilon \int_a^b f(x) dx,$$

a quantity which approaches zero.

Uniformity of convergence is an essential condition to the validity of these demonstrations.

The following are examples where this condition^{not} being fulfilled, the theorem fails to hold.

1st. Consider the series S which has for its general term

$$u_n = x(1-x^2)^{n-1}.$$

For $x=0$, it reduces to zero. For other values of x comprised between $+1$ and -1 , S is a convergent geometric progres

tion, and has for its value $\frac{1}{x}$, a quantity which approaches ∞ if x approaches 0. The series S is then discontinuous for $x=0$, even though it be convergent and its terms continuous functions.

The convergence here is not uniform. Let S_n be the sum of the first n terms of the series. We have

$$S - S_n = R_n = \frac{(1-x^2)^n}{x},$$

and whatever n be, we can always find a value for x small enough to make R_n greater than any given quantity ε .

2nd. Consider the series $S = u_1 + u_2 + \dots + u_n + \dots$, where $u_1 = x e^{-x^2}$, $\dots$, $u_n = n x e^{-n x^2} - (n-1) x e^{-(n-1)x^2}$

The sum S_n of the first n terms is $n x e^{-n x^2}$ and approaches zero, whatever be the value of x , when $n = \infty$. The series then has the value zero, whatever x be.

We have, accordingly $\int_0^b S dx = 0$; but, on the other hand $\int_0^b S_n dx = \int_0^b n x e^{-n x^2} dx = -\frac{1}{2} \int_0^b d e^{-n x^2} = -\frac{1}{2} [e^{-n x^2}]_0^b = \frac{1}{2} [1 - e^{-n b^2}]$ and for $n = \infty$, $\lim_{n=\infty} \int_0^b S_n dx = \frac{1}{2}$.

Our theorem then fails for this series. The convergence again is not uniform.

We have $S - S_n = R_n = -n x e^{-n x^2}$, and if we put $x = \frac{1}{\sqrt{n}}$, we get $|R_n| = \frac{\sqrt{n}}{e} > \frac{1}{e}$. It is then impossible to choose n so that for every value of x , $|R_n|$ will be $< \varepsilon$, when ε is $< \frac{1}{e}$.

These results, i.e. the failure of our theorem to apply in these cases, are explained by the fact that the con-

vergence is not uniform. However, we propose to show that uniformity of convergence is not an absolutely essential condition for the integration of a series term by term.

Non-uniform Convergence and a General Proposition.

after what has been said we must know that a series converges uniformly between the assigned limits in order to be sure whether this series can be subjected to term-by-term integration. Yet if this hypothesis is not fulfilled, in the case of certain singularities it may be made possible to apply fixed term-by-term integration between the prescribed limits to the given series. But we must not overlook the fact that when the condition for uniform convergence is not satisfied, it can very well happen at times, that the sum of the series is continuous, whether the series of integrals of its terms diverges or is indeterminate; or when the latter series converges, that it deviates from the integral of the sum of the given series.

For example, let the terms of a series $\sum_{n=1}^{\infty} u_n$ be finite and continuous between α and β and allow of a re-direction such that they take the form $u_n = v_n - v_{n+1}$, where v_n (like u_n) are finite and continuous between α and β ; vanish for $x = \beta$, and approach zero as n increases without limit, but not with uniform velocity for different values of x ; then the series is not uniformly convergent between α and β , although the sum v_1 is a finite and continuous function.

Its integral between α and x is $\int_{\alpha}^x v_1 dx$, while the series of

integrals $\sum_1^{\infty} \int_{\alpha}^x u_n dx$ has for the sum of its first n terms, the difference $\int_{\alpha}^x v_1 dx - \int_{\alpha}^x v_{n+1} dx$. When therefore the integral $\int_{\alpha}^x v_{n+1} dx$ does not have zero for a limiting value on account of a singularity present in v_n , then is the term-by-term integration not applicable; on the contrary the series of integrals $\sum_1^{\infty} \int_{\alpha}^x u_n dx$ can be undetermined or diverge according to the value of the integral $\int_{\alpha}^x v_{n+1} dx$.

Summable series show such behavior.

If for example the functions v_n , which serve for the reduction of u_n , are given through the equation

$$v_n = \frac{k_n g'_n(x)}{1 + g_n^2(x)}$$

in which k_n indicates a function of n , which increases indefinitely with n , and $g_n(x)$, likewise its derivative $g'_n(x)$, a finite and continuous function between α and β , which vanishes for $x = \alpha$, yet for values of x different from α , increases without limit with n , although v_n approaches zero, we get

$$\int_{\alpha}^x v_{n+1} dx = k_{n+1} \tan^{-1} g_{n+1}(x)$$

and

$$\int_{\alpha}^x v_1 dx = k_1 \tan^{-1} g_1(x),$$

where by $\tan^{-1} z$ is indicated in general the smallest element whose tan is z . If therefore $k_{n+1} \tan^{-1} g_{n+1}(x)$ neither is zero nor approaches zero, then is the sum of the series built up out of the integrals of the terms of the

given series

$$\sum_1^{\infty} \left[\frac{k_n g'_n(x)}{1 + g_n^2(x)} - \frac{k_{n+1} g'_{n+1}(x)}{1 + g_{n+1}^2(x)} \right]$$

not identical with the integral K , then $\sum' f_n(x)$; on the contrary the series may be undetermined or diverge.

If as a special case $f_n(x) = h_n(x-\alpha)^2$ wherein h_n is a positive number which increases without limit with n , then for values of x different from α

$$\lim \tan^{-1} f_n(x) = \frac{\pi}{2}.$$

The series of integrals converges therefore, but does not always fall together with the integral of the sum of the given series, when K_n has a fixed and finite limit different from zero. If, however for example $K_n = \sqrt{h_n}$, the series of integrals then diverges, while, — as a further example — the same series for $K_n = \sin h_n$ is indeterminate for an infinite number of forms of the function h_n .

If in another case

$$V_n = \frac{K_n f_n'(x)}{1 + f_n(x)}$$

and K_n can again increase with n without limit, while $f_n(x)$ — as likewise its derivative $f_n'(x)$ — vanishes for $x = \alpha$, but these functions pass through the remaining values constantly finite and continuous, and are so constituted that V_n either is zero or has zero for its limit when n increases indefinitely, then does

$$\int_{\alpha}^x V_{n+1} dx = K_{n+1} \log [1 + f_{n+1}(x)].$$

According to the behavior of the limit of $K_n \log [1 + f_n(x)]$ for $n = \infty$, does the series of integrals $\sum_{n=1}^{\infty} \int_{\alpha}^x V_n dx$ differ from the integral of the sum of the original series $\sum V_n$

and can diverge or be indeterminate, notwithstanding the fact that the original series $\sum_1^{\infty} u_n$ converges constantly and its sum $\frac{K_1 G_1'(x)}{1 + G_1(x)}$ for every value of x between α and β is finite and continuous.

If again, for example, $g_n(x) = h_n(x - \alpha)^2$ under the condition that when n increases without limit, K_n approaches zero and h_n increases through positive values towards infinity, then is

$$\begin{aligned} \sum_1^{\infty} \int_{\alpha}^x u_n dx - \int_{\alpha}^x dx \sum_1^{\infty} u_n &= -K_{n+1} \log [1 + h_{n+1} (x - \alpha)^2] \\ &= -K_{n+1} \log h_{n+1} - K_{n+1} \log \left[\frac{1}{h_{n+1}} + (x - \alpha)^2 \right]. \end{aligned}$$

The last term on the right side of this equation has zero for its limit when n increases indefinitely. If, therefore,

for example $K_n = \frac{1}{\log(h_n)}$, then the series of integrals

$\sum_1^{\infty} \int_{\alpha}^x u_n dx$ converges, but does not have for its sum the integral $\int_{\alpha}^x dx \sum_1^{\infty} u_n$ of the sum of the given series.

If, however, $K_n = \frac{\sin h_n}{\log(h_n)}$, then the series is indeterminate

for an infinite number of forms of the function h_n 's

and, finally, for $K_n = \frac{1}{\sqrt{\log(h_n)}}$, it diverges.

Series which represent discontinuous functions are non-uniformly convergent. Hence, in order to show that uniformity of convergence is not an absolutely essential condition in all possible cases that a series may be integrated term by term, we propose

to show that in the case of certain singularities a series representing a function discontinuous in a certain interval may be integrated term by term throughout that interval.

We assume first the truth of the following proposition:

* If $f(x)$ becomes infinite between α and β only in a finite or infinitely great set of points (points of the first order and adapts itself to integration in the intervals in which it is finite; and if moreover $g(x)$ is a function which is constantly finite and continuous in the interval (α, β) , and, disregarding the constant $-g(a)$ which depends on the magnitude a , represents the value of the determinate integral $\int_a^x f(x) dx$ for all values of x and a which include between them no infinity-points of $f(x)$, then the given function $f(x)$ is integrable in the whole interval (α, β) and the equation

$$\int_c^x f(x) dx = g(x) - g(c)$$

is true for all values of c and x between α and β (α and β included).

The hypothesis of our general proposition consists of the following points:

First: The series $\sum_{n=1}^{\infty} u_n$ converges in all points of a finite interval (α, β) at most with the exception of a finite

* V. Dini, see. 233.

or infinitely great set of points of the first order, in which all or certain terms become themselves infinitely great or indeterminate or in which the given series diverges or is indeterminate.

Second: The terms $u_1, u_2, \dots, u_n, \dots$, if to them in the points in which they are indeterminate, arbitrary values be assigned, may at once adapt themselves to integration between α and β .

Third: The series of integrals $\sum_1^\infty \int_\alpha^x u_n dx$ converges and represents a function finite and continuous between α and β .

Fourth: The term-by-term integration is applicable to the series $\sum_1^\infty u_n$ in any interval in which it is finite and determinate.

Then our conclusion, the proof for which follows, is this:

The sum of this series $\sum_1^\infty u_n$, if to it in the points in which it is indeterminate, arbitrary values be assigned, is integrable in the whole interval (α, β) and its integral for any value of x between α and β (α and β included) can be found by the term-by-term integration in virtue of the equation

$$\int_\alpha^x dx \sum_1^\infty u_n = \sum_1^\infty \int_\alpha^x u_n dx.$$

$F(x)$, by which we will designate under the above suppositions the function represented by the series of integrals $\sum_1^\infty \int_\alpha^x u_n dx$ satisfies the conditions of the Theorem assumed above in regard to the integral $\int_\alpha^x dx \sum_1^\infty u_n$.

hence it turns out the same thing, as is manifestly possible; and moreover it is extended to the case where the function here designated as $f(x)$ is indeterminate between α and β in a finite or infinitely great set of points. $\sum_1^\infty u_n$ is therefore integrable in the whole interval (α, β) and the Equation (A) is a true one for every value of x between α and β (α and β included).

2. The Differentiation of Series Term by Term.

Let the series

$$u_1(x) + u_2(x) + \dots \quad a \leq x \leq b,$$

converge and let its value be denoted by $f(x)$. Moreover $u_i(x)$ shall have a derivative for all values of x in the interval. Two cases arise and they must be sharply distinguished from each other.

Case A. The function $f(x)$ has a derivative for every value of x in the interval (a, b) .

Case B. The function $f(x)$ has, for certain or all values of x in the interval (a, b) no derivative.

An example of the latter case is the following:

$$S_n(x) = (1 - \cos^n x) \cdot x \sin \frac{1}{x}, \quad x \neq 0; \quad S_n(0) = 0;$$

$$f(x) = \lim_{n \rightarrow \infty} S_n(x) = x \sin \frac{1}{x}, \quad x \neq 0; \quad f(0) = 0;$$

Thus $f(x)$ has no derivative when $x=0$, although it is continuous here as elsewhere. $S_n(x)$ always has a continuous derivative.

Case A is of more importance in some respects, for this is the case in which a function whose properties

are more or less known and which in particular is known to be continuous and to have a continuous derivative, is developed into a series of simple functions. We shall confine ourselves to this case. The question we ask ourselves is:

When will the derivative of the series be given by the series of derivatives, i.e.; when will

$$f'(x) = u_1'(x) + u_2'(x) + \dots$$

be a true equation?

Taking $f'(x)$ to mean $\frac{d}{dx} \left[\lim_{n \rightarrow \infty} S_n(x) \right]$ and $\sum_{i=1}^{\infty} u_i'(x)$ to mean $\lim_{n \rightarrow \infty} \left[\frac{d}{dx} S_n(x) \right]$ the question is whether

$$\frac{d}{dx} \left[\lim_{n \rightarrow \infty} S_n(x) \right] = \lim_{n \rightarrow \infty} \left[\frac{d}{dx} S_n(x) \right]$$

Since differentiation is a limiting process, we have again before us the question of the equality of two double limits. Geometrically the left hand double limit means the slope of the limiting curve $y = f(x)$, while the right hand double limit means the limit approached by the slope of the approximation curve $y = S_n(x)$. These are not necessarily the same as is easily seen.

For example, let $S_n(x) = u x e^{-u x^2}$;

$$\text{then } \frac{d}{dx} S_n(x) \Big|_{x=0} = u(1-2ux) e^{-ux^2} \Big|_{x=0} = u$$

and the slope of the approximation curves at the origin increases without limit. Thus the corresponding series of the derivatives is divergent in this case. But $f(x) = 0$,

and hence $f'(x) = 0$.

This result, however, is not caused by the non-uniformity of convergence as might be supposed.

Consideration of the approximation curves in the case that

$$S_n(x) = n \sin^2 \pi x e^{-n \sin^2 \pi x}$$

shows that, although the convergence here is non-uniform, their slope approaches zero for every value of x , and the corresponding series of derivatives converges toward the right value, zero.

On the other hand, the u -series may converge uniformly and still not be capable of being differentiated term by term. For example take

$$S_n(x) = x e^{-u^2 x^2}, \quad f(x) = 0; \quad \frac{d}{dx} S_n(x) \Big|_{x=0} = 1;$$

$$\text{Hence } f'(0) = 0 \text{ but } \lim_{u \rightarrow \infty} \frac{d}{dx} S_n(x) \Big|_{x=0} = 1.$$

Geometrically, the approximation curves $y = x e^{-u^2 x^2}$ are all similar to the limiting curve $y = x e^{-x^2}$, only reduced in scale. The convergence here is uniform.

The series whose n^{th} term is $\frac{\sin[(1.2. \dots n)x]}{1.2. \dots n}$ is a noteworthy example of a series which is uniformly convergent and which has for terms continuous functions admitting of continuous derivatives, but with the series of derivatives divergent.

We conclude from these examples that the uniform convergence of a series is neither a necessary nor a sufficient condition that it may be differentiated

term by term.

The following gives a sufficient condition for the differentiation of a series term by term.

Let the series

$$u_1(x) + u_2(x) + \dots$$

converge for at least one value of x , x_0 , in the interval (a, b) and let $u_i(x)$ be continuous and have a continuous derivative $u'_i(x)$ in this interval; finally let the series of the derivatives

$$u'_1(x) + u'_2(x) + \dots$$

converge uniformly. Then the u -series converges for all values of x in the interval (a, b) ; and if its value be denoted by $f(x)$, $f(x)$ has a continuous derivative, given by differentiating the u -series term by term:

$$f'(x) = u'_1(x) + u'_2(x) + \dots$$

For let the value of the u' -series be denoted by $g(x)$:

$$g(x) = u'_1(x) + u'_2(x) + \dots$$

Then $g(x)$ by $\quad$ is a continuous function of x , and by

$$\begin{aligned} \int_{x_0}^x g(x) dx &= \int_{x_0}^x u'_1(x) dx + \int_{x_0}^x u'_2(x) dx + \dots \\ &= [u_1(x) - u_1(x_0)] + [u_2(x) - u_2(x_0)] + \dots \end{aligned}$$

The series $u_1(x_0) + u_2(x_0) + \dots$ is convergent, hence $u_1(x) + u_2(x) + \dots$ is convergent, and if its value is denoted by $f(x)$ then

$$\int_{x_0}^x g(x) dx = f(x) - f(x_0)$$

The left hand side of this equation is a continuous

function having as its derivative the continuous function $g(x)$. Hence $f(x)$ is continuous and

$$f'(x) = g(x) \quad \text{q. e. d.}$$

This Theorem affords a simple proof that a power series can be differentiated term by term. It also finds application in Case B.

As an application of this Theorem we may consider the series

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots;$$

it is convergent between -1 and $+1$; we have

$$f'(x) = 1 - x^2 + x^4 - x^6 + \dots;$$

this latter series is a decreasing geometric progression whose ratio is $-x^2$. We have then

$$f'(x) = \frac{1}{1+x^2}$$

and from this we get the development

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots.$$

This development is valid for x between -1 and $+1$.

For $x=1$, the series of the second member, being convergent, will represent $\tan^{-1} 1$, that is to say $\frac{\pi}{4}$.

The above Theorem is not particularly well adapted to Case A; it assumes more than is given in requiring the series of the derivatives to converge uniformly and does not make use of all that is given, namely that the u -series converges and that its value is a continuous function having a contin-

uous derivative. The following proposition conforms more closely to the data.

Necessary and Sufficient Condition for the Differentiation of Series Term by Term:

If the sum $f(x)$ of a convergent series $\sum u_n$, whose terms admit of finite and determinate derivatives for $x=a$, shall have a finite and determinate derivative and this derivative shall be the sum of the series $\sum u'_n$, the derivatives of $u_1, u_2, \dots, u_n$ then it is doubly requisite and sufficient that

1st: The series $\sum u'_n$ of the derivatives must be convergent;

2nd: For every positive and indefinitely small number σ it must be possible to find a number ε positive and different from zero, so that for every special value of δ which is numerically less than ε and for which the point $a+\delta$ will fall within the interval $(a-\varepsilon_1, a+\varepsilon_2)$, a finite (with n variable) number m exists, which is not smaller than a given number m' and for which the three magnitudes

$$\sum_1^m \left[\frac{u_n(a+\delta) - u_n(a)}{\delta} - u'_n \right], \quad \frac{R_m(a+\delta)}{\delta}, \quad \frac{R_m(a)}{\delta}$$

are at once numerically each less than σ .

Proof: If these conditions are fulfilled then with regard to the convergence of the series $\sum u'_n$ a finite number m' may then be ascertained such that for $n \geq m'$, R'_n , the remainder of the series, is numerically less than σ . Since

we may always put

$$\frac{f(a+\delta) - f(a)}{\delta} - \sum u'_n = \sum_{1}^{\infty} \left[\frac{u_n(a+\delta) - u_n(a)}{\delta} - u'_n \right] + \frac{R_n(a+\delta)}{\delta} - \frac{R_n(a)}{\delta} - R'_n,$$

so we must take care ^{only} to choose for n only a number which is greater than n' and for which in consideration of the value obtained for δ it satisfies the above mentioned conditions, in order to convince ourselves at once, that for all proper values of δ less than ε which can be taken into consideration, the magnitude

$$\frac{f(a+\delta) - f(a)}{\delta} - \sum u'_n$$

in absolute value remains smaller than $\frac{\sigma}{4}$.

From this it naturally follows, that, if the aforesaid conditions are fulfilled

$$\lim_{\delta} \frac{f(a+\delta) - f(a)}{\delta} = \sum u'_n.$$

If, conversely, $f(x)$ for $x=a$ have a finite and continuous derivative, which is the sum of the series $\sum u'_n$, then is this latter convergent. There exists then not only a number n' , such that if $n \geq n'$, $R'_n < \frac{\sigma}{4}$ in absolute value, but there is also a number ε , positive and different from zero, such that for all values of δ which are numerically smaller than ε and can come into consideration, the difference

$$\frac{f(a+\delta) - f(a)}{\delta} - \sum u'_n$$

is likewise numerically smaller than $\frac{\sigma}{4}$. We bear in mind also that in consequence of the convergence of $\sum u'_n$ for all values of x between $a-\varepsilon$ and $a+\varepsilon$, which fall

within the interval $(a - \varepsilon_1, a + \varepsilon_2)$, and for every value of δ which can be considered, manifestly a corresponding number $m \geq m'$ exists, for which the two quantities $\frac{R_m(a+\delta)}{\delta}$, $\frac{R_m(a)}{\delta}$ are smaller than $\frac{\sigma}{4}$; so we enunciate ourselves, that for these values of δ and m numerical

$$\text{by } \sum_{n=1}^{\infty} \left[\frac{u_n(a+\delta) - u_n(a)}{\delta} - u'_n \right] < \sigma,$$

wherein the proposition is completely proved.

3. Reversal of the Order of Integration in a Double Integral

We are to deal here with an important class of double integrals, namely:

$$\int_{x_0}^x dx \int_0^{\infty} f(x, y) dy$$

where $f(x, y)$ is a continuous function of the two independent variables x, y throughout the region $a \leq x \leq b, y \geq 0$.

The question here is again one of the equality of two double limits; that is whether

$$\int_{x_0}^x dx \int_0^{\infty} f(x, y) dy = \int_0^{\infty} dy \int_{x_0}^x f(x, y) dx$$

is a true equation. We can reduce this question to that of integrating a series term by term, namely the series $\int_0^{\infty} f(x, y) dy = \int_0^1 f(x, y) dy + \int_1^2 f(x, y) dy + \dots$

If we take the integral of the sum of this series, $\int_0^{\infty} f(x, y) dy$, between the limits x and x_0 we get the double limit which

is the left hand member of the above equation, namely $\int_{x_0}^x dx \int_0^\infty f(x, y) dy$. However the sum of the integrals of the terms of the series $(\int_{x-1}^x f(x, y) dy)$ forming the second member of the equation, between the same limits, has for its limit

$$\lim_{n \rightarrow \infty} \left[\int_{x_0}^x dx \int_0^n f(x, y) dy \right].$$

$$\text{But } \lim_{n \rightarrow \infty} \left[\int_{x_0}^x dx \int_0^n f(x, y) dy \right] = \lim_{n \rightarrow \infty} \left[\int_0^n dy \int_{x_0}^x f(x, y) dx \right].$$

The interchange in the order of integration in the case of the integrals in brackets is allowable here because the integral is continuous and the limits of integration finite.

$$\text{But } \lim_{n \rightarrow \infty} \left[\int_0^n dy \int_{x_0}^x f(x, y) dx \right] = \int_0^\infty dy \int_{x_0}^x f(x, y) dx$$

which latter expression is the second member of the above equation between two double limits.

Hence we have reduced the question as to whether

$$\int_{x_0}^x dx \int_0^\infty f(x, y) dy = \int_0^\infty dy \int_{x_0}^x f(x, y) dy$$

is a true equation, to the question of integrating the series

$$\int_0^\infty f(x, y) dy = \int_0^1 f(x, y) dy + \int_1^2 f(x, y) dy + \dots$$

term by term.

The following is an example in which the reversal of the order of integration is not allowable.

$$\text{Let } f(x, y) = \frac{\partial}{\partial y} \frac{3x^2 y^3}{1+x^6 y^6}.$$

Then $\int_{x_0}^x dx \int_0^\infty f(x, y) dy = 0;$

but $\int_0^\infty dy \int_{x_0}^x f(x, y) dx = \pi,$ if $x_0 < 0 < x.$

If $x_0 = 0,$ $\int_0^\infty dy \int_{x_0}^x f(x, y) dx = \frac{\pi}{2}, 0, -\frac{\pi}{2}$

respectively, according as $x > 0, = 0, < 0.$

Conversely, the problem of integrating a series of continuous functions

$$f(x) = u_1(x) + u_2(x) + \dots$$

term by term can be reduced to that of reversing the order of integration in the double integral:

$$\int_{x_0}^x dx \int_0^\infty f(x, y) dy,$$

where $f(x, y)$ is a continuous function of the two independent variables x, y throughout the region $a \leq x \leq b, y \geq 0.$

For the purposes of this demonstration, the truth of the following proposition, the arithmetical proof of which is not difficult, will be assumed:

----- If $u_1(x) + u_2(x) + \dots$ is any series of functions continuous throughout the interval $(a, b); a \leq x \leq b;$ then there exists a function $f(x, y)$ of the two independent variables $x, y,$ continuous throughout the region $a \leq x \leq b, y \geq 0,$ and such that

$$u_i(x) = \int_{i-1}^i f(x, y) dy.$$

Moreover, if the u -series converges toward a limit $f(x)$,

$$f(x) = \int_0^\infty f(x, y) dy.$$

The question is whether

$$\int_{x_0}^x f(x) \cdot dx = \int_{x_0}^x u_1(x) dx + \int_{x_0}^x u_2(x) dx + \dots$$

It follows from the above proposition that the right hand side of this Equation $\int_{x_0}^x f(x) dx$ can be written in the form

$$\lim_{n \rightarrow \infty} \left[\int_{x_0}^x dx \int_0^n f(x, y) dy \right],$$

which, however, is equal to

$$\lim_{n \rightarrow \infty} \left[\int_0^n dy \int_{x_0}^x f(x, y) dx \right] = \int_0^\infty dy \int_{x_0}^x f(x, y) dx.$$

Hence the question is reduced to that of determining whether

$$\int_{x_0}^x dx \int_0^\infty f(x, y) dy = \int_0^\infty dy \int_{x_0}^x f(x, y) dx.$$

is a true Equation.

In addition to the foregoing treatment of this question we propose to take it up for investigation independent of the question of integrating a series term by term.

Starting anew, then, let a, A, b, B be finite constants; $f(x, y)$ a function of two variables x, y continuous in the rectangle formed by the lines $x=a$, $x=A$, $y=b$, $y=B$. We have the Equation

$$\int_b^B dy \int_a^A f dx = \int_a^A dx \int_b^B f dy,$$

since the two members of the Equation are only two

different expressions of the double integral $\int_R f dx dy$.

But if one of the limits, a, A, b, B , becomes infinite, or if $f(x, y)$ ceases to be continuous in the whole field of integration, (This demonstration being no longer applicable) the equation () can cease to be exact, as the following examples show:

$$\text{Let } V = \tan^{-1} \frac{y}{x}, \quad \frac{\partial^2 V}{\partial x \partial y} = f(x, y)$$

$$a = b = 0, \quad A = B = 1.$$

We have

$$\frac{\partial V}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial V}{\partial y} = \frac{x}{x^2 + y^2},$$

$$\int_0^1 dy \int_0^1 \frac{\partial^2 V}{\partial x \partial y} dx = \int_0^1 dy \left[\frac{\partial V}{\partial y} \right]_0^1 = \int_0^1 \frac{dy}{1+y^2} = \left[\tan^{-1} y \right]_0^1 = \frac{\pi}{4}$$

$$\int_0^1 dx \int_0^1 \frac{\partial^2 V}{\partial x \partial y} dy = \int_0^1 dx \left[\frac{\partial V}{\partial x} \right]_0^1 = \int_0^1 \frac{-dx}{1+x^2} = -\frac{\pi}{4}.$$

We next consider the function

$$Ax^m + Bx^{m-1} + \dots + Kx + L,$$

$A, B, \dots, K, L$, being constants, which for the sake of simplicity we shall suppose real.

$$\text{Put } x = \rho(\cos \varphi + i \sin \varphi).$$

The function will take the form

$$P + iQ,$$

in future, for the sake of brevity,

$$P = A\rho^m \cos m\varphi + B\rho^{m-1} \cos (m-1)\varphi + \dots + K\rho \cos \varphi + L$$

$$Q = A\rho^m \sin m\varphi + B\rho^{m-1} \sin (m-1)\varphi + \dots + K\rho \sin \varphi.$$

$$\text{Let } V = \tan^{-1} \frac{P}{Q}.$$

We deduce from this

$$\frac{\partial V}{\partial \rho} = \frac{Q \frac{\partial P}{\partial \rho} - P \frac{\partial Q}{\partial \rho}}{P^2 + Q^2}; \quad \frac{\partial V}{\partial \varphi} = \frac{Q \frac{\partial P}{\partial \varphi} - P \frac{\partial Q}{\partial \varphi}}{P^2 + Q^2}; \quad \frac{\partial^2 V}{\partial \rho \partial \varphi} = \frac{M}{(P^2 + Q^2)^2},$$

M designating an integral function of ρ and the sine and cosine of multiples of φ .

We are going to show that, in designating by R a quantity sufficiently large, the two integrals

$$\int_0^R d\rho \int_0^{2\pi} \frac{\partial^2 V}{\partial \rho \partial \varphi} d\varphi \quad \text{and} \quad \int_0^{2\pi} d\varphi \int_0^R \frac{\partial^2 V}{\partial \rho \partial \varphi} d\rho$$

will not have the same value.

The first integral is equal to

$$\int_0^R d\rho \left(\frac{\partial V}{\partial \varphi} \right)_0^{2\pi} = 0,$$

since, $\frac{\partial V}{\partial \rho}$ taking the same value for $\varphi=0$ and $\varphi=2\pi$, the quantity to be integrated is zero.

The second integral has for its value

$$\int_0^{2\pi} d\varphi \left(\frac{\partial V}{\partial \rho} \right)_0^R.$$

Now for $\rho=0$, we get $Q=0$ and $\frac{\partial Q}{\partial \varphi}=0$ whence $\frac{\partial V}{\partial \varphi}=0$. For $\rho=R$, we get, in writing only the terms of highest degree in R ,

$$P = A R^m \cos m\varphi + \dots,$$

$$Q = A R^m \sin m\varphi + \dots;$$

$$\frac{\partial P}{\partial \varphi} = -m A R^m \sin m\varphi - \dots,$$

$$\frac{\partial Q}{\partial \varphi} = m A R^m \cos m\varphi + \dots;$$

$$\text{whence } \frac{\partial V}{\partial \varphi} = \frac{-m A^2 R^{2m}}{A^2 R^{2m} + \dots} = -m + \varepsilon,$$

ε being a quantity very small when R is very great.

The second integral will have then for its value

$$\int_0^{2\pi} d\varphi (-m + \varepsilon) \text{ or obviously, } \int_0^{2\pi} -m d\varphi = -2m\pi,$$

a quantity different from zero.

The two functions just considered, having different values, necessarily then the function $\frac{\partial^2 V}{\partial \rho \partial \varphi}$ is discontinuous in the field of integration. But it is equal to $\frac{M}{(P^2 + Q^2)^2}$, where M, P, Q , are functions evidently continuous. It can then become discontinuous only if its denominator vanishes. There exists then a set of values of ρ and φ which make P and Q vanish at the same time. Then there exists a value $\rho(\cos \varphi + i \sin \varphi)$ of the variable x which causes the polynomial $Ax^m + Bx^{m-1} + \dots + L$ to vanish.

We have thus demonstrated anew the fundamental proposition of the theory of equations, that every algebraic equation has a root.

We propose to assign a set of conditions sufficient to allow with certainty the reversal of the order of two successive integrations.

Suppose, at first, a, A, b, B finite, and grant:

- (1st) that the points C of the field for which the function f ceases to be continuous are all situated on a limited number of continuous arcs $P_1 Q_1, \dots, P_n Q_n$ along each of which x on the one hand, and y on the other, are constantly increasing or constantly decreasing;
- (2nd) that the integral $\int_x^{x+\lambda} f dx$, taken along a segment parallel to x , situated in any manner whatever in the field, and of which the length λ does not exceed a

fixed number λ , always has a determinate value of which the modulus is constantly less than a quantity ε_2 , depending solely on λ and approaching zero with it;

----- (3rd) that the analogous integral $\int_y^{y+\lambda} f dy$, taken along a parallel to y , has likewise, a determinate value, of modulus less than a quantity ε'_2 analogous to ε_2 .

Under these conditions we shall show: 1st that

$$\int_b^B dy \int_a^A f dx \quad \text{and} \quad \int_a^A dx \int_b^B f dy$$

have finite and determinate values; and (2nd) that they are equal.

We remark at the outset that along each of the arcs PQ , x is a continuous function of y , always increasing or always decreasing, and, reciprocally. The points of this arc are comprised within the strip limited by the two parallels y and $y+dy$ to the axis of x infinitely near to each other will have abscissae which differ by infinitesimals; and if we suppose $dy < \delta$, δ designating an infinitesimal, they will all be contained in the interior of a rectangle $\mathcal{R}$ of height dy and of length less than λ , λ being a quantity which we can make decrease at will at the same time with δ .

Each of ^{the} n arcs $P_1Q_1, \dots, P_nQ_n$ will give place to a similar rectangle (if always it meets the strip of height dy we are considering). If two of the rectangles thus formed encroach upon each other or are contiguous, we shall collect them into a single rectangle, whose length will be less

than 2ϵ . Continuing thus, we see that all the points of the arcs $P_1 Q_1, \dots, P_n Q_n$ contained in the strip will be inside the rectangles $r_1, r_2, \dots$ of which the number m is at most equal to n , and such that the sum of their lengths is less than $n\epsilon$.

We conclude that the integral $\int_a^A f(x, y) dx$ is a continuous function of y . It is composed (1st) of the integrals taken along the segments of the line of integration enclosed within the rectangles $r_1, \dots, r_m$. The modulus of each of these will be $< \epsilon_{n2}$, and hence it will be the same for the modulus of the corresponding portions of the integral

$$\int_a^A f(x, y+dy) dx;$$

(2nd) of the $m+1$ integrals taken along the segments of the line of integration outside the rectangles. These will be continuous functions of y since $f(x, y)$ is continuous without the rectangles. The modulus of the increase of each of them, when y changes to $y+dy$, will be less than an arbitrary quantity ϵ , if dy is small enough.

We will have then

$$\left| \int_a^A f(x, y+dy) dx - \int_a^A f(x, y) dx \right| < 2m\epsilon_{n2} + (m+1)\epsilon.$$

Now, if we make dy decrease indefinitely we shall be able to make decrease at the same time ϵ, δ , and consequently, ϵ_{n2} ; our proposition is therefore established.

The expression $\int_b^B dy \int_a^A f(x)$, the integral of a continuous function of y , will be finite and determinate.

The same demonstration would evidently apply to the integral $\int_a^A dx \int_b^B f(y)$.

It remains to prove these integrals equal.

With this end in view, we break up the field of integration by parallels to the axes of coördinates into elementary rectangles of side less than δ . Let us designate generally by e those of these rectangles which contain no point in common with the arcs PQ ; by e' the others, which meet at least one of these arcs.

The integral $\int_b^B dy \int_a^A f(x)$ is evidently a sum of analogous integrals having respectively for a field the diverse elements e and e' . Moreover, within the elements e , where the function f remains continuous, the double integral $\iint f(x,y) dx dy$ exists and represents the value of $\int dy \int f(x)$; we shall have then

$$\int_b^B dy \int_a^A f(x) = \sum_e \iint f(x,y) dx dy + \sum_{e'} \int dy \int f(x).$$

We are going to show that the last term of this Equation approaches zero with δ .

Let e'_k be those of the elements e' that are comprised between the consecutive parallels to x , y_k and $y_k + dy_k$. Those of the elements e'_k which meet the same arc PQ will be contiguous and their union will furnish a rectangle ρ entirely contained — if the parallels to y are sufficiently near each other — within the rectangle Ω of length λ less than 2 , which after the preceding contains within its interior all the points common to PQ and the strip considered. To each of the arcs $P_1Q_1, \dots, P_nQ_n$ which meet

the strip there corresponds thus a rectangle ρ . Collecting into a single rectangle those which are contiguous or encroach upon each other, we see that the elements e'_k group themselves in rectangles $\rho_1, \rho_2, \dots$ in number at most equal to n and such that the sum of their lengths is less than $n\delta$.

The sum of the integrals $\int dy \int f dx$ relative to the elements e'_k is evidently equal to the sum of the same integrals relative to the various rectangles $\rho_1, \rho_2, \dots$. Now, for each of these, the integral $\int f dx$, having its modulus less than ϵ_{n2} , that of the integral $\int dy \int f dx$ will be less than

$$\int_{y_k}^{y_k + dy_k} \epsilon_{n2} dy = \epsilon_{n2} dy_k.$$

We have then

$$\left| \sum_{e'_k} \int dy \int f dx \right| < n \epsilon_{n2} \sum dy_k,$$

and accordingly

$$\left| \sum_{e'} \int dy \int f dx \right| \leq \sum_k \left| \sum_{e'_k} \int dy \int f dx \right| < n \epsilon_{n2} \sum_k dy_k < n \epsilon_{n2} (B-b)$$

an expression whose limit is equal to zero.

It results from what we have just established that we have

$$\int_b^B dy \int_a^A f dx = \lim_{\delta=0} \sum_e \int f dx dy.$$

We should obtain, by an identical process, the same result for the integral $\int_a^A dx \int_b^B f dy$. These two integrals are therefore equal.

The theorem which has just been established will still be effective, if we suppose: 1st, That conditions (1) and (2) above, without necessarily being satisfied in the rectangle (a, A, b, B) are satisfied in the whole rectangle $(a, A, b + \eta, B - \eta')$ however small be the positive quantities η, η' ; 2nd That

condition (3) be satisfied in the rectangle (a, A, b, B) .

Now, the last two terms of the expression

$$\int_b^B f(x, y) dy = \int_{b+\eta}^{B-\eta'} f(x, y) dy + \int_b^{b+\eta} f(x, y) dy + \int_{B-\eta'}^B f(x, y) dy$$

have, by hypothesis, a modulus $< \varepsilon_2$ if η and η' are less than ℓ , whatever x be; the sum of their variations, when we change x to $x + dx$, is therefore less in absolute value than $4\varepsilon_2$.

In the first place, the first term $\int_{b+\eta}^{B-\eta'} f(x, y) dy$ is a continuous function of x . Hence $\int_b^B f(x, y) dy$ will be equally continuous, since, in taking first ℓ and then dx sufficiently small, we can make each of the terms of its variation decrease as much as we wish.

The integral $\int_a^A dx \int_b^B f(x, y) dy$ will therefore have a determinate value, which will be the limit of the integral $\int_a^A dx \int_{b+\eta}^{B-\eta'} f(x, y) dy$, since the difference between the two integrals will be equal to $\int_a^A dx \left(\int_b^{b+\eta} f(x, y) dy + \int_{B-\eta'}^B f(x, y) dy \right)$, an expression of which the modulus is less than $\int_a^A 2\varepsilon_2 dx = 2\varepsilon_2(A-a)$ and accordingly, approaches zero with ℓ .

But we have

$$\int_a^A dx \int_{b+\eta}^{B-\eta'} f(x, y) dy = \int_{b+\eta}^{B-\eta'} dy \int_a^A f(x, y) dx;$$

whose limit — the existence of which we have just established — is therefore by definition nothing else than the integral $\int_b^B dy \int_a^A f(x, y) dx$.

It is moreover evident that the theorem will remain in force for the whole field broken up into a limited number of rectangles of which each, considered separately, satisfies the preceding conditions.

We now pass to the case where the field is infinite. We suppose a, A, b remaining fixed, we can make B ^{indefinitely} increase without altering the validity of the theorem.

We grant, at the outset, that for all values of x comprised between a and A , the integral $\int_B^\infty f dy$ has a determinate value, whose modulus is less than $\varepsilon_B g(x)$, the first factor ε_B being limited and approaching zero uniformly for $B = \infty$ in the whole interval from $x = a$ to $x = A$ (or at least in every portion ^{any one of} of this interval which does not contain certain isolated points $c, c', \dots$); $g(x)$ designating, on the other hand, a function of x , positive, equally limited between a and A and admitting in this interval of a finite integral.

Under these conditions, the integral $\int_b^\infty f dy$ will represent between a and A a function of x continuous (save perhaps in exceptional points $c, c', \dots$). Let x be any point whatever different from these; we shall have

$$\dots \int_b^\infty f dy = \int_b^B f dy + \int_B^\infty f dy$$

For all the values of x sufficiently near that which we are considering, $g(x)$ will be less than a fixed quantity M , and ε_B can, in taking B large enough, be made less than an arbitrary number ε . The variation of the term $\int_B^\infty f dy$ in passing from the point x to a point infinitely near, $x + dx$, will therefore have its modulus less than $2M\varepsilon$. On the other hand $\int_b^B f dy$ being continuous, the modulus of its variation will be $< \varepsilon$ if dx is small enough.

The modulus of the variation of $\int_b^\infty f dy$ will therefore be less than $(2M + 1)\varepsilon$ and will thus decrease as much as

we wish, in making, first, the auxiliary quantity B increase, and then taking δx small enough.

If therefore there exist in the field aA no exceptional points $c, c', \dots$, as we have supposed, $\int_b^{\infty} f dy$ will be continuous in the whole field, and the integral

$$\int_a^A dx \int_b^{\infty} f dy$$

will have a determinate value.

In every case, moreover, where the above integral, or this one $\int_b^{\infty} dy \int_a^A f dx = \lim_{B=\infty} \int_b^B dy \int_a^A f dx = \lim_{B=\infty} \int_a^A dx \int_b^B f dy$ is determinate, they both are, and they are equal. By integrating equation (Z) from $x=a$ to $x=A$ and making B approach ∞ , we shall get

$$\int_a^A dx \int_b^{\infty} f dy = \lim_{B=\infty} \int_a^A dx \int_b^B f dy + \lim_{B=\infty} \int_a^A dx \int_B^{\infty} f dy.$$

It will suffice, therefore, to ~~show~~ establish our proposition, to show that the expression

$$\int_a^A dx \int_B^{\infty} f dy$$

has zero for its limit for $B=\infty$.

So this end, therefore, we isolate in the field aA the partial intervals of length $< \delta$ containing respectively in their interior the various points $c, c', \dots$.

We shall consider the one of these intervals $\alpha\alpha'$ containing the point c , for example.

The modulus of the integral $\int_B^{\infty} f dy$ will be less than $\epsilon_B g(x)$ and accordingly less than $\mu g(x)$, μ designating the maximum of the function ϵ_B , which

by supposition is limited. The integral $\int_a^{a'} dx \int_B^\infty f dy$ taken in this interval will therefore have a modulus less than $\mu \int_a^{a'} g(x) dx$, which approaches zero with δ , since $g(x)$ is by supposition integrable.

Therefore, by taking δ small enough, we can make the sum of the integrals relative to the small intervals which we have isolated, decrease as much as we wish. Indicate by D the remainder of the field; E_D approaches zero uniformly when B approaches ∞ ; then by taking B large enough, we can make E_D constantly less than an arbitrary quantity ε ; we have accordingly,

$$\left| \int_D dx \int_b^\infty f dy \right| < \varepsilon \int_D g(x) dx < \varepsilon \int_a^A g(x) dx,$$

a quantity which approaches zero with ε .

The equation

$$\int_a^A dx \int_b^\infty f dy = \int_b^\infty dy \int_a^A f dx$$

being established under the preceding conditions, let us make A approach ∞ . We grant: 1st That for the values of y between b and ∞ , the integral $\int_a^\infty f dx$ has a determinate value, of modulus less than $E_A \psi(y)$, E_A remaining always limited and approaching zero uniformly when A approaches ∞ , in the whole interval b to ∞ (save perhaps in the neighborhood of certain exceptional points, finite in number) and $\psi(y)$ being a positive function of y ,

whose integral between b and d is finite. Reasoning similar in all respects to the preceding will show that if one of the integrals

$$\int_b^d dy \int_a^c f dx, \quad \int_a^c dx \int_b^d f dy$$

is determinate (which will always happen if there be no exceptional points) they will be equal.

4. Derivation under the Sign of Integration.

Let $f(x, y, \dots, \alpha)$ be a function of the variables $x, y, \dots, \alpha$ which remains continuous so long as $x, y, \dots$ remain within a region D and α within a region Δ .

Let E be a region limited and completely within D ; for every value of α within Δ , the integral

$$I = \int_E f(x, y, \dots, \alpha) dx$$

will have a determinate value; it is then a function of α .

This function is continuous; since, if we change α to $\alpha + \Delta\alpha$, it will take an increment

$$\begin{aligned} \Delta I &= \int_E f(x, y, \dots, \alpha + \Delta\alpha) dx - \int_E f(x, y, \dots, \alpha) dx \\ &= \int_E \Delta f(x, y, \dots, \alpha) dx, \end{aligned}$$

whose modulus cannot exceed LE ; L being the maximum of the modulus of $\Delta f(x, y, \dots, \alpha)$.

Now, let δ be a quantity whose modulus is less than the distance of α from the boundary of Δ ; the point $\alpha + \Delta\alpha$ will remain within Δ , as long as $|\Delta\alpha| \leq \delta$. The function f will therefore remain continuous as long as $x, y, \dots$ move within E and the parameter varies between $\alpha - \delta$ and $\alpha + \delta$. This set of values being evidently limited and con-

plete, the continuity of f will be uniform; hence L , and accordingly ΔI , will approach zero with $\Delta \alpha$.

Let us suppose further that f considered as a function of α alone ($x, y, \dots$ keeping constant values) admits of a derivative; this will then be a new function of $x, y, \dots; \alpha$, which we can designate by $f'_\alpha(x, y, \dots, \alpha)$, and we shall have

$$\Delta f(x, y, \dots, \alpha) = f'_\alpha(x, y, \dots, \alpha + \theta \Delta \alpha) \Delta \alpha,$$

θ being variable, but always comprised between 0 and 1.

If we grant now that f'_α be continuous in the same region in which we have supposed f continuous, this expression will be of the form

$$[f'_\alpha(x, y, \dots, \alpha) + R] \Delta \alpha,$$

R approaching zero uniformly with $\Delta \alpha$, in the whole region E .

We shall therefore have

$$\begin{aligned} \frac{\Delta I}{\Delta \alpha} &= \int_E [f'_\alpha(x, y, \dots, \alpha) + R] d\alpha \\ &= \int_E f'_\alpha(x, y, \dots, \alpha) d\alpha + \int_E R d\alpha. \end{aligned}$$

Let L_1 be the maximum of $|R|$ in E ; then we have

$$|\int_E R d\alpha| \leq L_1 E,$$

a quantity which approaches zero with $\Delta \alpha$. We have therefore

$$\lim \frac{\Delta I}{\Delta \alpha} = \int_E f'_\alpha(x, y, \dots, \alpha) d\alpha.$$

The integral I admits then of a derivative, represented by the integral above.

The foregoing shows us that the integral

$$I = \int_a^A f(x, t) dx$$

admits at the point $t = t_0$ of a derivative, represented by the

integral
$$\int_a^A \frac{\partial f}{\partial t_0} dx.$$

This important result was established only by supposing a and A finite, and $\frac{\partial f}{\partial t}$ continuous with respect to x and t , as long as x remains in the interval from a to A , and t in the interval from $t_0 - \delta$ to $t_0 + \delta$ (δ being chosen as small as we wish). But we can extend the demonstration to other cases.

The derivative of I at the point $t = t_0$ is by definition equal to

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{h} \int_a^A [f(x, t_0 + h) - f(x, t_0)] dx \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_a^A dx \int_{t_0}^{t_0 + h} \frac{\partial f}{\partial t} dt \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{t_0}^{t_0 + h} dt \int_a^A \frac{\partial f}{\partial t} dx = \int_a^A \frac{\partial f}{\partial t} dx. \end{aligned}$$

The theorem will be established for all cases in which the three transformations we have just made are allowable.

In the first place suppose the function $\frac{\partial f}{\partial t}$ is discontinuous with respect to t in a finite number of points only, between t_0 and $t_0 + h$, and that f remains continuous in these points. If therefore we can determine a quantity δ such that $f(x, t)$ be a continuous function of t in the whole rectangle $(a, A, t_0 - \delta, t_0 + \delta)$, and that for every value of x included between a and A , $\frac{\partial f}{\partial t}$ has only a limited number of discontinuities between $t_0 - \delta$ and $t_0 + \delta$, this transformation will be permitted when $|h|$ is less than δ .

If a and A are finite the reversal of the two integrations will be permitted under the conditions named above. (I, A, E)

The latter condition (E) will certainly be satisfied, if,

in the rectangle, f is a continuous function of the two variables x and t ; since its continuity will be uniform.

If one of the limits a, A , for example A , be replaced by ∞ , we can again reverse the integrations, if this operation be permitted in the interval from a to any number whatever A , and if, further, the integral

$$\int_A^\infty \frac{\partial f}{\partial t} dx$$

be determinate and have a modulus less than $\varepsilon_A g(t)$, $g(t)$ being integrable between $t_0 - \delta$ and $t_0 + \delta$, and ε_A approaching zero uniformly in this interval when A approaches

∞ . Finally, if the preceding conditions are satisfied, the integral $\int_a^A \frac{\partial f}{\partial t} dx$ will be continuous (even if the field be infinite). The latter transformation will therefore always be permitted.

Let us consider a case of the double integral, for example,

$$S_E f(x, y, t) dx dy.$$

We break up the field of integration (at first supposed limited) into two parts D and d of which the former is measurable and such that $\frac{\partial f}{\partial t}$ remains continuous when x, y describe the field D , and t the interval from $t_0 - \delta$ to $t_0 + \delta$. In the field D , we can derive under the sign of integration; we shall have, accordingly,

$$\frac{d}{dt_0} S_E = \frac{d}{dt_0} S_D + \frac{d}{dt_0} S_d = \int_D \frac{\partial f}{\partial t_0} dx dy + \lim S_d \left[\frac{f(x, y, t_0 + h) - f(x, y, t_0)}{h} \right] dx dy.$$

We make δ decrease indefinitely and suppose — that which will generally happen — that we can at the same time make the region D increase progressively at the expense of the region d , in such a way that the area of the latter will approach zero. The integral $\int_D \frac{\partial f}{\partial t_0} dx dy$ will approach $\int_E \frac{\partial f}{\partial t_0} dx dy$.

To see whether the formula is applicable, we shall have to discuss the second term and assure ourselves that it approaches zero. If, for example, we can show that, when d and $|h|$ are become sufficiently small, we always have

$$\left| \frac{f(x, y, t_0 + h) - f(x, y, t_0)}{h} \right| < g(x, y)$$

g designating a positive function whose integral in E is finite; this term will have its modulus less than $\int_d g(x, y) dx dy$, which approaches zero with d .

We suppose, now, that E is infinite, but that the formula of derivation is applicable in the whole limited region contained in E . We break up E into two regions D and d , of which the former is limited and contains all the points of E whose distance from the origin is not greater than a given number R . We have

$$\frac{d}{dt_0} \int_E = \int_D \frac{\partial f}{\partial t_0} dx dy + \lim_{h \rightarrow 0} \int_d \left[\frac{f(x, y, t_0 + h) - f(x, y, t_0)}{h} \right] dx dy$$

Make R approach ∞ . The first term approaches $S_E \frac{\partial f}{\partial t_0} dx dy$, and the formula will be applicable if the second term has zero for its limit, and unless if, for R sufficiently great and $|h|$ sufficiently small, we shall always have

$$\left| \frac{f(x, y, t_0+h) - f(x, y, t_0)}{h} \right| < g(x, y),$$

$g(x, y)$ admitting of a finite integral in E .

As an example take

$$\int_0^\infty \frac{e^{-\alpha x} \sin x}{x} dx = V(\alpha) \quad (\alpha > 0)$$

This integral is a continuous function of α , likewise for α approaching zero for positive values.

To obtain $V(\alpha)$, we first get the integral

$$\frac{dV}{d\alpha} = - \int_0^\infty e^{-\alpha x} \sin x \cdot dx.$$

By twice integrating by parts, we get

$$\int_0^\infty e^{-\alpha x} \sin x dx = \frac{1}{\alpha} \int_0^\infty e^{-\alpha x} \cos x dx,$$

$$\text{then } \int_0^\infty e^{-\alpha x} \cos x dx = \frac{1}{\alpha} - \frac{1}{\alpha} \int_0^\infty e^{-\alpha x} \sin x dx,$$

$$\text{whence } \frac{dV}{d\alpha} = - \frac{1}{1+\alpha^2}. \text{ Therefore } V = \tan^{-1} \frac{1}{\alpha}.$$

The theorem established for $\alpha > 0$ will hold good for $\alpha = 0$; hence we find

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

The question of differentiation under the Sign of integration can also be reduced to the question of differentiating a series term by term. From this point of view it is a question of the Equality of two double limits.

Take
$$\int_0^{\infty} f(x, y) dy,$$

where $f(x, y)$, $\frac{\partial f(x, y)}{\partial x}$ are continuous functions of the two independent variables x, y throughout the region $a \leq x \leq b$, $y \geq 0$. The question is whether

$$\frac{d}{dx} \int_0^{\infty} f(x, y) dx = \int_0^{\infty} \frac{\partial f(x, y)}{\partial x} dy$$

is a true equation; and it may be reduced to the question of differentiating term by term the series

$$\int_0^{\infty} f(x, y) dy = \int_0^1 f(x, y) dy + \int_1^2 f(x, y) dy + \dots$$

Here as in No. 2 two cases corresponding to case A and B arise, and we will restrict ourselves to the case that $\int_0^{\infty} f(x, y) dy$ has a derivative. Then this derivative $\frac{d}{dx} \int_0^{\infty} f(x, y) dy$ is the first member of the above equation. On the other hand, the term-by-term derivative of the series:

$$\frac{d}{dx} \int_0^1 f(x, y) dy + \frac{d}{dx} \int_1^2 f(x, y) dy + \dots$$

is equal to $\lim_{n \rightarrow \infty} \left[\frac{d}{dx} \int_0^n f(x, y) dy \right]$. This however has

~~then~~ for its value $\lim_{n \rightarrow \infty} \left[\int_0^n \frac{\partial f(x, y)}{\partial x} dy \right]$, differentia-
 tion, being here allowable because $\frac{\partial f(x, y)}{\partial x}$ is a continuous
 function of the two independent variables x, y , and
 the limits of integration are both finite. But this
 last expression is equal to

$$\int_0^\infty \frac{\partial f(x, y)}{\partial x} dy,$$

which is the second member of the Equation
 above; thus the main question has been reduc-
 ed to that of differentiating the Series

$$\int_0^\infty f(x, y) dy = \int_0^1 f(x, y) dy + \int_1^2 f(x, y) dy + \dots$$

term by term.

An example of a case in which differentia-
 tion under the sign of integration is not allowa-
 ble is the following:

$$f(x, y) = x^3 e^{-x^2 y}$$

$$\int_0^\infty x^3 e^{-x^2 y} dy = x$$

for all values of x .

$$\text{Hence } \frac{d}{dx} \int_0^\infty x^3 e^{-x^2 y} dy = 1$$

$$\text{But } \int_0^\infty \frac{\partial}{\partial x} (x^3 e^{-x^2 y}) dy = \int_0^\infty (3x^2 - 2x^4 y) e^{-x^2 y} dy,$$

and when $x=0$, the value of this expression is 0.

Conversely, the problem of differentiating a series term by term can, at least in the case that the derivatives of the terms are continuous functions, be reduced to that of differentiating

$$\int_0^{\infty} f(x, y) dy$$

under the sign of integration, $f(x, y)$ and $\frac{\partial f(x, y)}{\partial x}$ being continuous functions of the two independent variables x, y . For by (B)

$$u_i(x) = \int_{i-1}^i f(x, y) dy,$$

where $f(x, y)$ is continuous; and if $u'_i(x)$, $i = 1, 2, 3, \dots$, is a continuous function of x ,

then $\frac{\partial f(x, y)}{\partial x}$ is a continuous function of x, y

regarded as independent variables, and hence

$$u'_i(x) = \int_{i-1}^i \frac{\partial f(x, y)}{\partial x} dy.$$

The main question is whether, if $f(x)$ has a derivative,

$$f'(x) = u'_1(x) + u'_2(x) + \dots$$

is a true equation. The right hand side can

be written in the form

$$\begin{aligned} \lim_{u \rightarrow \infty} \left[\frac{d}{dx} \int_0^u f(x, y) dy \right] &= \lim_{u \rightarrow \infty} \left[\int_0^u \frac{\partial f(x, y)}{\partial x} dy \right] \\ &= \int_0^\infty \frac{\partial f(x, y)}{\partial x} dy, \end{aligned}$$

and thus the question is reduced to that of whether

$$\frac{d}{dx} \int_0^\infty f(x, y) dy = \int_0^\infty \frac{\partial f(x, y)}{\partial x} dy$$

is a true equation.

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May 1, 1899.

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This thesis is never to leave this room.

Neither is it to be checked out overnight.