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On the Motion of a Sphere On a
Rough Horizontal Plane.

The Billiard Ball Problem.

A thesis presented to the Faculty
of the University of Missouri for the
Degree of Master of Arts.

By

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Introduction.

The problem discussed in these pages is that of the motion of a billiard ball when struck by a horizontal cue. This is a special case of the motion of a sphere on a rough horizontal plane.

The first chapter considers the motion of the ball on a rough table, having no regard for the cause of the motion. In the second, the results are applied to the case where the stroke of the cue is horizontal. The third treats of the motion after impact with another ball, neglecting the friction between the balls, and the fourth takes up the case where such friction exists. The last chapter is a discussion of the motion which follows impact with the cushion.

Throughout the entire discussion the following terms and notation have been used:

The point of contact is the point where the ball touches the table.

The superior center of percussion of the ball is a point on the vertical passing through the center, at a distance above the center, equal to two-fifths of the radius.

The inferior center of percussion is a point on the vertical through the center, at a distance below the center, equal to two-fifths of the radius.

Direct rotation is rotation in the same sense as when the ball rolls without slipping, on the table.

Retrograde rotation is opposite

in sense to direct rotation.

The state of slipping is the state of the ball at the instant when the velocity of the point of contact is zero.

The final state of the ball is that in which we find the ball when it rolls on the table in a straight line, without slipping.

In this state, the velocity of rotation of the point of contact is equal and directly opposite to the velocity of translation of the center of the ball.

The final direction is the direction of motion of the ball when it is in the final state.

M' is mass of the cue.

W' the velocity of cue before it strikes the ball.

R the radius of the ball.

M the mass of the ball.

W , the initial velocity of the center of the ball—that which it takes after the stroke of the cue.

W_0 , the initial velocity of the center for the same velocity of the cue when the line of stroke passes through the center of the ball.

W , the velocity of the center of the ball at any instant.

ω the velocity of rotation of the superior center of percussion.

p, q, r , the three projections upon the axes of the velocity of rotation.

l the distance above the table of the point where the line of stroke meets the vertical plane taken through the center of the ball, perpendicular to the vertical plane of the stroke.

h the distance from the vertical plane of stroke to the center of the ball.

a the shortest distance from the
line of stroke to the center of the ball.
When the stroke of the cue is hori-
zontal, $a^2 = (R-l)^2 + r^2$.

Chapter I.

Let X and Y be the x and y components of the force produced by friction between the ball and the table. The point of contact is the point $(0, 0, -R)$, if the center of the ball be taken as origin. Equating moments about the center,

$$\left. \begin{aligned} \int dm \left(y \frac{d^2 x}{dt^2} - x \frac{d^2 y}{dt^2} \right) &= 0 \\ \int dm \left(x \frac{d^2 z}{dt^2} - z \frac{d^2 x}{dt^2} \right) &= RX \\ \int dm \left(z \frac{d^2 y}{dt^2} - y \frac{d^2 z}{dt^2} \right) &= -RY \end{aligned} \right\} I.$$

Call the angular velocity of rotation ω , and the angles made by the instantaneous axis with the fixed axes α, β, γ . Then say,

$$\beta = \omega \cos \alpha, \quad \gamma = \omega \cos \beta, \quad \gamma = \omega \cos \gamma.$$

Also, call K the moment of

inertia of the ball with respect to one of its diameters. Then ΣK will be the sum of the moments of the motion with respect to the axis of rotation. This sum, projected upon the coordinate planes will give K_r , K_q , K_p . Then,

$$\left. \begin{aligned} \int dm \left(y \frac{dx}{dt} - x \frac{dy}{dt} \right) &= K_r \\ \int dm \left(x \frac{dz}{dt} - z \frac{dx}{dt} \right) &= K_q \\ \int dm \left(z \frac{dy}{dt} - y \frac{dz}{dt} \right) &= K_p \end{aligned} \right\} \text{II.}$$

Or,

$$\left. \begin{aligned} \int dm \left(y \frac{d^2x}{dt^2} - x \frac{d^2y}{dt^2} \right) &= K \frac{dr}{dt} \\ \int dm \left(x \frac{d^2z}{dt^2} - z \frac{d^2x}{dt^2} \right) &= K \frac{dq}{dt} \\ \int dm \left(z \frac{d^2y}{dt^2} - y \frac{d^2z}{dt^2} \right) &= K \frac{dp}{dt} \end{aligned} \right\} \text{III.}$$

Then from I and III, we have,

$$\left. \begin{aligned} K \frac{dt}{dt} &= 0 \\ K \frac{dq}{dt} &= RX \\ K \frac{dp}{dt} &= -RY \end{aligned} \right\} \text{IV}$$

We can combine these equations with those for the motion of the center of gravity, which are,

$$\left. \begin{aligned} M \frac{dU}{dt} &= X \\ M \frac{dV}{dt} &= Y \end{aligned} \right\} \text{V}$$

where U and V are the components of the velocity of the center of gravity, and can get U, V, p, q, r where we know the forces X and Y .

Eliminating X and Y from III, V.

$$\left. \begin{aligned} K \frac{dt}{dt} &= 0 \\ K \frac{dq}{dt} &= RM \frac{dU}{dt} \\ K \frac{dp}{dt} &= -RM \frac{dV}{dt} \end{aligned} \right\} \text{VI.}$$

Putting for K its value $\frac{2}{3}MR^2$, these become,

$$\left. \begin{aligned} R \frac{dr}{dt} &= 0 \\ \frac{2}{3} \cdot R \frac{dq}{dt} &= \frac{dU}{dt} \\ -\frac{2}{3} \cdot R \frac{d\beta}{dt} &= \frac{dV}{dt} \end{aligned} \right\} \text{VI}$$

Integrating these equations and denoting by $U_1, V_1, \beta_1, q_1, r_1$ the initial values of U, V, β, q, r ,

$$\left. \begin{aligned} r &= r_1 \\ U - U_1 &= \frac{2}{3} \cdot R (q - q_1) \\ V - V_1 &= \frac{2}{3} \cdot R (\beta - \beta_1) \end{aligned} \right\} \text{VII}$$

These equations show that r will be constant, and that β and q change only with U and V . Thus the ball can change its axis of rotation only when it does not move uniformly, and reciprocally.

Let u and v be components of

of the velocity of the superior center of percussion. Then,

$$\left. \begin{aligned} \frac{2}{5} R \beta &= v \\ \frac{2}{5} R \alpha &= -u \end{aligned} \right\} \text{VII.}$$

Combining VII and VIII,

$$u - u_1 = U_1 - U$$

$$v - v_1 = V_1 - V, \text{ etc.}$$

$$\left. \begin{aligned} u + U &= u_1 + U_1 \\ v + V &= v_1 + V_1 \end{aligned} \right\} \text{IX.}$$

Hence,

The resultant of the velocity of the superior center of percussion and of the center of the ball is always constant during the motion.

Now, the ball will cease slipping and roll in a straight line

when the velocity of rotation of the highest point is equal to the velocity of the center. The components of the velocity of this point are, from VII,

$$\frac{1}{2}u, \frac{1}{2}v.$$

Then the ball will roll in a straight line when,

$$U = \frac{1}{2}u, \quad V = \frac{1}{2}v.$$

Substitute these values in IX,

$$\left. \begin{aligned} U &= \frac{2}{7}(U_1 + u_1) \\ V &= \frac{2}{7}(V_1 + v_1) \end{aligned} \right\} \text{X.}$$

These values of U and V are the velocities that the center of the ball will have when it begins rolling. Then we see that,

The final direction is independent of the nature of the friction; and

it is always the resultant of the velocity of the center of the ball and of the velocity of the superior center of percussion.

We will now consider the motion before this rolling stage is reached. We may neglect the friction of rolling, then very small. We will consider only the friction of slipping, and call its coefficient μ . Now Σ become,

$$\left. \begin{aligned} X &= -\mu mg \cos \theta \\ Y &= -\mu mg \sin \theta \end{aligned} \right\} \text{VII.}$$

where θ is the angle which the direction of slipping makes with the x axis.

The velocity of rotation of the point of contact has for components,

$$-\frac{3}{2}u, \quad -\frac{3}{2}v.$$

Then the absolute velocity will be,
 $u' = U - \frac{3}{2}u$ and $v' = V - \frac{3}{2}v$.

That is,

$$\left. \begin{aligned} \cos \theta &= \frac{u'}{\sqrt{u'^2 + v'^2}} \\ \sin \theta &= \frac{v'}{\sqrt{u'^2 + v'^2}} \end{aligned} \right\} \text{XII}$$

Substituting these values in XI,

$$\left. \begin{aligned} X &= -\mu g \frac{u'}{\sqrt{u'^2 + v'^2}} \\ Y &= -\mu g \frac{v'}{\sqrt{u'^2 + v'^2}} \end{aligned} \right\} \text{XIII}$$

Equations IX now become,

$$\left. \begin{aligned} \frac{dU}{dt} &= -\mu g \frac{u'}{\sqrt{u'^2 + v'^2}} \\ \frac{dV}{dt} &= -\mu g \frac{v'}{\sqrt{u'^2 + v'^2}} \end{aligned} \right\} \text{XIV}$$

Substitute in these the values of u and v given by II, and call U_2 and V_2 the final velocities found in X.

Fig. 1.

$$\left. \begin{aligned} \frac{dU}{dt} + \frac{1}{2} \mu g \left(\frac{U-U_2}{\sqrt{U^2+V^2}} \right) &= 0 \\ \frac{dV}{dt} + \frac{1}{2} \mu g \left(\frac{V-U_2}{\sqrt{U^2+V^2}} \right) &= 0 \end{aligned} \right\} \text{XV.}$$

Eliminate $\sqrt{U^2+V^2}$.

$$\frac{dU}{U-U_2} = \frac{dV}{V-U_2} \quad \text{XVI.}$$

Integrating and putting in the proper limits,

$$\frac{U-U_2}{U_1-U_2} = \frac{V-U_2}{V_1-U_2} \quad \text{XVII}$$

This linear relation shows that if OB (fig. I) represents the velocity at any point, B lies on a right line, O being a fixed point.
We may write XVIII,

$$\frac{U_1-U_2}{V_1-U_2} = \frac{U-U_2}{V-U_2} \quad \text{XVIII}$$

But from values of $\sin \theta$ and $\cos \theta$

given by XIII we see,

$$\frac{U - U_2}{V - V_2} = \tan \theta$$

Then $\tan \theta = \frac{U_1 - U_2}{V_1 - V_2}$ XIX.

That is, θ is constant throughout the motion, or

The direction of friction does not vary during the motion.

Consider again the equations

$$\frac{dU}{dt} = -mg \cos \theta$$

$$\frac{dV}{dt} = -mg \sin \theta$$

Integrating these and calling x and y the coordinates of the center of the ball at the end of a time t , and putting in the proper constants,

$$\left. \begin{aligned} \frac{dx}{dt} = U &= U_1 - mg \cos \theta \cdot t \\ \frac{dy}{dt} = V &= V_1 - mg \sin \theta \cdot t \end{aligned} \right\} \text{XX}$$

$$\left. \begin{aligned} x &= U_1 t - \frac{1}{2} mg t^2 \cos \theta \\ y &= V_1 t - \frac{1}{2} mg t^2 \sin \theta \end{aligned} \right\} \text{XXI}$$

These values of x and y show that the ball describes a parabola, for if we eliminate t , the resulting equation will be that of a parabola.

If we wish to get the path of the ball after it has reached its final state, we will make $U = U_2$, $V = V_2$, and from XXI,

$$\left. \begin{aligned} U_2 &= U_1 - mg \cos \theta \cdot t \\ V_2 &= V_1 - mg \sin \theta \cdot t \end{aligned} \right\} \text{XXII}$$

$$t = \frac{U_1 - U_2}{mg \cos \theta} = \frac{U_1 - U_2}{\frac{mg u_1}{\sqrt{u_1^2 + v_1^2}}}$$

$$\begin{aligned}
 t &= \frac{U_1 - \frac{5}{7}(U_1 + u_1)}{mg} \cdot \frac{\sqrt{u_1'^2 + v_1'^2}}{U - \frac{5}{2}u} \\
 &= \frac{\frac{2}{7}U_1 - \frac{5}{7}u_1}{U - \frac{5}{2}u} \cdot \frac{\sqrt{u_1'^2 + v_1'^2}}{mg} \\
 &= \frac{\frac{2}{7}(U_1 - \frac{5}{2}u_1)}{U - \frac{5}{2}u} \cdot \frac{\sqrt{u_1'^2 + v_1'^2}}{mg} \\
 t &= \frac{\frac{2}{7}\sqrt{u_1'^2 + v_1'^2}}{mg} \quad \underline{\underline{XXIII}}
 \end{aligned}$$

If we call x_2 and y_2 the coordinates of the point where the parabola ends, then

$$\left. \begin{aligned}
 x_2 &= v_1 t = \left(\frac{U_1 + U_2}{2}\right) \frac{2}{7} \frac{\sqrt{u_1'^2 + v_1'^2}}{mg} \\
 y_2 &= v_2 t = \left(\frac{V_1 + V_2}{2}\right) \frac{2}{7} \frac{\sqrt{u_1'^2 + v_1'^2}}{mg}
 \end{aligned} \right\} \underline{\underline{XXIV}}$$

We know $U_2 = \frac{5}{7}(U_1 + u_1)$
 $V_2 = \frac{5}{7}(V_1 + v_1)$

Substituting these,

$$\left. \begin{aligned} X_2 &= \frac{2}{7} \frac{\sqrt{u'^2 + v'^2}}{\mu g} \left[\frac{1}{2} U_1 + \frac{5}{7} U_1 + \frac{5}{7} u_1 \right] \\ Y_2 &= \frac{2}{7} \frac{\sqrt{u'^2 + v'^2}}{\mu g} \left[\frac{1}{2} V_1 + \frac{5}{7} V_1 + \frac{5}{7} v_1 \right] \\ X_2 &= \frac{12}{49} \frac{\sqrt{u'^2 + v'^2}}{\mu g} \left[U_1 + \frac{5}{12} u_1 \right] \\ Y_2 &= \frac{12}{49} \frac{\sqrt{u'^2 + v'^2}}{\mu g} \left[V_1 + \frac{5}{12} v_1 \right] \end{aligned} \right\} \text{XXXVII}$$

Now in figure I, let the coordinates of K and S be the projections along the axes of the velocity of the center of the ball. Then OK and OS will represent the initial and final velocities.

As found, XXXVII,

$$\frac{U - U_1}{U_1 - U_2} = \frac{V - V_1}{V_1 - V_2}$$

Fig. 2.

Fig. 3.

Then if the variable velocity be represented by a right line through the origin, its extremity must lie on the line joining K and S .

In figure 2, OB is a vector whose magnitude and direction represent the initial velocity of the center of the ball. OF is equal in magnitude and opposite in direction to OG which represents the initial velocity of rotation. Then BF is the effective velocity, and is the locus of B' as B moves about O during the motion. The final position of B' will be such as to make $BB' = \frac{1}{2} BF$.

In figure 3, let OF and OB be initial velocities of rotation and translation, and OB' the velocity

when slipping ceases. M is the middle point of BB' . Then it is evident that (x_2, y_2) of XII will lie on the line QM . We want to find this point L . Draw OP parallel to BT and equal to μg . Lay off $OL = BB'$ and through Q draw OQ parallel to PM . Then the intersection of OQ and QM , L , is the required point. Through L draw LV parallel to OB which represents the final direction. Then LV gives the path of the ball after it begins to roll.

This line LV passes through (x_2, y_2) and is parallel to the final velocity whose components are U_2 and V_2 .

Then the equation of LV is,

$$\frac{y - y_2}{x - x_2} = \frac{V_2}{U_2}$$

To find where LV meets FOG, make $x = 0$ and $y = y_2 - \frac{V_2}{U_2} x_2$.

Substituting for x_2 and y_2 ,

$$y = \frac{12}{49} \frac{\sqrt{u'^2 v'^2}}{\mu g} \left[(V_1 + \frac{5}{12} v_1) - \frac{V_2}{U_2} (U_1 + \frac{5}{12} u_1) \right]$$

For points on FOG, $u_1 = 0, v_1 = \omega$.

$$y = \frac{12}{49} \frac{\sqrt{u'^2 v'^2}}{\mu g} \left[U_1 + \frac{5}{12} u_1 \right]$$

$$y = -\frac{1}{7} \frac{\sqrt{u'^2 v'^2}}{\mu g} \omega$$

Then if LV be extended to meet OG, it will cut OG in a point at a distance $\frac{1}{7} \frac{\sqrt{u'^2 v'^2}}{\mu g}$ below O.

The final direction LV we found

$$\frac{y-y_2}{x-x_2} = \frac{V_2}{U_2}$$

The perpendicular upon UV from origin,

$$\pi = \frac{y_2 U_2 - x_2 V_2}{\sqrt{U_2^2 + V_2^2}} = \frac{W_2}{7\mu g} \left[\frac{U_2 V_1 - U_1 V_2}{\sqrt{U_2^2 + V_2^2}} \right]$$

In the triangle whose vertices are the points $(0, 0)$, (U_1, V_1) , (U_2, V_2) ,

$$2\Delta = \begin{vmatrix} U_1 & V_1 & 1 \\ U_2 & V_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \sqrt{U_1^2 + V_1^2} \sqrt{U_2^2 + V_2^2} \sin(\Phi - \Psi)$$

That is, $\sin(\Phi - \Psi) = \frac{U_1 V_2 - U_2 V_1}{W_2 W_1}$

$$\pi = \frac{W_2 W_1}{7\mu g} \sin(\Phi - \Psi)$$

This gives the length of the perpendicular in terms of the angles

which the initial and final velocities make with the x axis.

Suppose that ω , the velocity of rotation of the center of percussion is directed in the same sense as the velocity of translation W . Then the final velocity would become $\frac{2}{7}(W + \omega)$, and ~~XXV~~,

$$y_2 = \frac{12}{49} \frac{W\omega}{\mu g} (W + \frac{5}{12}\omega)$$

$$\text{Or, } y_2 = \frac{12}{49} [(W - \frac{5}{12}\omega)(W + \frac{5}{12}\omega)] \text{ XXVII}$$

If the ball is without rotation, $\omega = 0$,

$$y_2 = \pm \frac{24}{49} \frac{W^2}{2\mu g}$$

If without velocity of translation,

$$y_2 = \pm \frac{25}{49} \frac{\omega^2}{2\mu g}$$

At any time,

$$t = \pm \frac{W_1 - W}{ug},$$

$$\mp ugt = W_1 - \frac{dy}{dt}. \quad \text{Integrate.}$$

$$y = W_1 t \mp \frac{1}{2} ugt^2 \quad \text{XXIV}$$

Substitute for t .

$$y = \frac{1}{2ug} (W_1^2 - W^2) \quad \text{XXVIII}$$

$$W^2 = W_1^2 \mp 2ugy \quad \text{XXIX}$$

Then we see that the velocity as a function of the distance y is represented by the ordinates of a parabola. The equation of this parabola may be written,

$$W^2 = 2ug[Y \pm y], \quad \text{XXX.}$$

where $\frac{W_1^2}{2ug} = Y$. The vertex is at a distance Y from the starting point of the ball, and is before or behind the player according as the posi-

tive or negative sign be taken in XXX;
or according to whether $W_1 - \frac{1}{2}w_1 \geq 0$.

Whatever the value of w , it will be given by the part of the ordinate between one or the other of these parabolas and a line parallel to the x axis at a height from it $W_1 + w$, where w_1 is the value of w at the origin.

In order to simplify some of the formulas found, experiments have been made to determine W_0 , γ , μ , etc.

A ball was suspended by a thread two meters in length and given a blow with the cue. It was observed that for an ordinary blow the ball rose vertically 1.20 m. and for a hard blow, 2.50 m. Then since,

$$m \frac{d^2 z}{dt^2} = mg, \quad \frac{dz}{dt} = 2gz, \quad \text{Or,}$$

$$\left. \begin{array}{l} \frac{W_0}{2g} = 1.20 \\ \frac{W_0}{2g} = 2.50 \end{array} \right\} \begin{array}{l} W_0 = 4.86 \\ W_0 = 7.00 \end{array} \quad \text{XXXI}$$

To determine μ the ball was drawn over the table by a thread stretched by a weight. The motion of the ball was about uniform when $\mu = .25$. Then,

$$\frac{W_0}{2\mu g} = 4.80 \text{ or } 10.00$$

Thus, when the ball starts without rotation, the distance y_2 being given by

$$y_2 = \frac{24}{49} \frac{W_0^2}{2\mu g} \text{ or very nearly,}$$

$$y_2 = \frac{1}{2} \frac{W_0^2}{2\mu g} \text{ becomes}$$

$$y_2 = 2.40 \text{ or } y_2 = 5.00.$$

Chapter II.

Let us take for origin the center of the ball at the time of the stroke, for y axis a line through this point parallel to the vertical plane passing through the axis of the cue. The z axis is measured downward, as before, and the x axis is perpendicular to these two.

Equating moments about the origin,

$$\left. \begin{aligned} \frac{2}{3} MR^2 \dot{\phi}_1 &= MW_1 (l - R) \\ \frac{2}{3} MR^2 \dot{q}_1 &= 0 \\ \frac{2}{3} MR^2 \dot{r}_1 &= MW_1 h \end{aligned} \right\} \text{XXXII}$$

Dividing through by MR^2

$$\left. \begin{aligned} \frac{2}{3} R \dot{\phi}_1 &= W_1 \left(\frac{l - R}{R} \right) \\ \frac{2}{3} \dot{q}_1 &= 0 \\ \frac{2}{3} R \dot{r}_1 &= -W_1 \frac{h}{R} \end{aligned} \right\} \text{XXXIII}$$

Fig. 4.

The initial velocity of rotation of the superior center of percussion is directed along the y axis, since $\frac{1}{2} R q_1 = 0$ & if $R p_1$ is positive, that is if $l > R$, then ω_1 is in the same direction as the motion. We saw in the last chapter that,

$$W + \omega = W_1 + \omega_1$$

$$\text{Also } \frac{1}{2} R p_1 = \omega_1$$

$$\text{That is, } \omega_1 = W \left(\frac{l - R}{R} \right)$$

$$\text{and } W_1 + \omega_1 = W_1 \frac{l}{R}$$

Thus, at any instant,

$$W + \omega = W_1 \frac{l}{R} \quad \text{XXXIV}$$

In figure 4, ML represents W , the initial velocity. FT is this velocity when the ball has reached T . Now draw PHE parallel to MDK and at

a height from it $PM = W, \frac{l}{R}$. Then the distance from the parabola down to PE will represent w . When $l > R$, the point A falls below PE .

The final velocity is given by,

$$W = \frac{5}{7} (W_1 + w_1),$$

which now becomes,

$$W = \frac{5}{7} W_1 \frac{l}{R} \quad \text{XXXV}$$

Thus, W can never fall below PE , if the ball were struck so that

$$W = \frac{5}{7} W_1 \frac{l}{R},$$

we would no longer have a parabola, but a straight line.

In order to get MW , the distance traversed by the ball before reaching the final state, we take the value of y_2 found in the preceding chapter,

$$y_2 = \frac{W_1^2}{2\mu g} \left[1 - \left(\frac{5}{7} \cdot \frac{l}{R} \right)^2 \right],$$

or very nearly,

$$y_2 = \frac{W_1^2}{2\mu g} \left[1 - \frac{1}{2} \frac{l^2}{R^2} \right] \quad \text{XXXVII}$$

At the point where the center of percussion has no velocity, that is where the ball has ceased rolling backward and has not begun to roll forward, $W = W_1 \frac{l}{R}$. Calling this point y_0 ,

$$y_0 = \frac{W_1^2}{2\mu g} \left(1 - \frac{l^2}{R^2} \right)$$

Then y_0 is very nearly twice y . The ball will not pass through this state, and y_0 will not exist unless $l < R$.

Call ϕ the fraction of the vis viva lost in contact. Then,

$$(1-\theta)M'W'^2 = M'W_1'^2 + MW^2 + MR(p_1'^2 + q_1'^2 + r_1'^2)$$

now we know, from XXII,

$$M'W' = MW_1 + MW_1'$$

$$\frac{2}{3}MR\dot{\phi}_1^2 = MW_1 + MW_1'$$

$$\frac{2}{3}MR\dot{q}_1^2 = 0$$

$$\frac{2}{3}MR\dot{r}_1^2 = -MW_1 h$$

$$a^2 = h^2 + (l-R)^2$$

Substituting,

$$(1-\theta)M'W'^2 = M'W_1'^2 + MW_1^2 \left(1 + \frac{5}{2}\frac{a^2}{R^2}\right)$$

$$W_1 = W' \frac{1 + \sqrt{1-\theta - \theta \frac{M'}{M} \left(1 + \frac{5}{2}\frac{a^2}{R^2}\right)}}{1 + \frac{5}{2}\frac{a^2}{R^2} + \frac{M}{M'}}$$

$$W_1' = W' \frac{1 + \frac{5}{2}\frac{a^2}{R^2} - \frac{M}{M'} \sqrt{1-\theta - \theta \frac{M'}{M} \left(1 + \frac{5}{2}\frac{a^2}{R^2}\right)}}{1 + \frac{5}{2}\frac{a^2}{R^2} + \frac{M}{M'}}$$

In order that these equations hold, the cue and the ball must sepa-

rate after the stroke. That is $W_1 > W_1'$.
That is,

$$1 + \sqrt{1 - \theta - \theta \frac{M'}{M} \left(1 + \frac{5a^2}{2R^2}\right)} > 1 + \frac{5a^2}{2R^2} - \frac{M}{M'} \sqrt{1 - \theta - \theta \frac{M'}{M} \left(1 + \frac{5a^2}{2R^2}\right)}$$

That is,

$$\left(1 + \frac{M}{M'}\right) \sqrt{1 - \theta - \theta \frac{M'}{M} \left(1 + \frac{5a^2}{2R^2}\right)} > \frac{5a^2}{2R^2} \quad \text{XXXVIII}$$

Then in order that $\frac{a^2}{R^2}$ be a maximum, θ must be zero. Then the greater $\frac{a^2}{R^2}$ may be, the more has θ decreased. Suppose that instead of abandoning the cue to itself at the moment of the stroke the player should immediately withdraw it. We may introduce these conditions in our equations by supposing M' and θ very large. That is, the left side of the above inequality diminishes. Thus, in

order to be able to strike the ball as far as possible from the center without diminishing the rotation that the stroke tends to produce the cue should be held free in the hand, and not move forward after the stroke.

Now let us see what the influence of different manners of striking the ball will be upon the distance that the ball will have traversed before retrograde rotation begins.

We found that,

$$y_0 = \frac{W^2}{2ug} \left(1 - \frac{b^2}{R^2}\right). \text{ Subs. in XXXVII.}$$

$$y_0 = \frac{W^2}{2ug} \left(1 - \frac{b^2}{R^2}\right) \frac{[1 + \sqrt{1 - 6 - 8 \frac{M}{M'} (1 + \frac{3a^2}{2R^2})}]}{(1 + \frac{3a^2}{2R^2} + \frac{M}{M'})^2}$$

$$y_2 = \frac{W''}{2\mu g} \left(1 - \frac{25}{49} \frac{l^2}{R^2}\right) \frac{[1 + \sqrt{1 - 0 - 0 \frac{M}{M'} (1 + \frac{5}{2} \frac{a^2}{R^2})}]}{(1 + \frac{5}{2} \frac{a^2}{R^2} + \frac{M}{M'})^2}$$

We have said that in the case of the horizontal stroke,

$$\frac{a^2}{R^2} = \frac{h^2}{R^2} + (1 - \frac{l}{R})^2$$

If now we wish to examine the maximum values of y_0 and y_2 , we must put $h=0$. Now substitute for $\frac{a^2}{R^2}$ its value $(1 - \frac{l}{R})^2$

$$y_0 = \frac{W''}{2\mu g} \frac{(1 - \frac{l}{R}) [1 + \sqrt{1 - 0 - 0 \frac{M}{M'} (1 + \frac{5}{2} (1 - \frac{l}{R})^2)}]}{[1 + \frac{M}{M'} + \frac{5}{2} (1 - \frac{l}{R})^2]^2} \quad \left. \vphantom{\frac{W''}{2\mu g}} \right\} \text{XL}$$

$$y_2 = \frac{W'' [1 - (\frac{5}{7} \frac{l}{R})] [1 + \sqrt{1 - 0 - 0 \frac{M}{M'} (1 + \frac{5}{2} (1 - \frac{l}{R})^2)}]}{[1 + \frac{M}{M'} + \frac{5}{2} (1 - \frac{l}{R})^2]^2}$$

If the ball were struck at the center,

W₀ the velocity would be,

$$W_0 = W' \frac{1 + \sqrt{1 - \theta - \theta \frac{M'}{M}}}{1 + \frac{M}{M'}} \quad \text{Or}$$

$$W' = \frac{W_0 (1 + \frac{M}{M'})}{1 + \sqrt{1 - \theta - \theta \frac{M'}{M}}} \quad \text{XLI}$$

Substitute this value of W' in X.

$$y_0 = \frac{W_0^2}{2uq} \left(1 - \frac{l^2}{R^2}\right) \left(1 + \frac{M}{M'}\right)^2 \left[1 + \sqrt{1 - \theta - \theta \frac{M'}{M}} - \frac{M'}{M} \theta \left(1 + \frac{1}{2} \left(1 - \frac{l^2}{R^2}\right)\right)\right] \\ \left(1 + \sqrt{1 - \theta - \theta \frac{M'}{M}}\right)^2 \left(1 + \frac{M}{M'} + \frac{1}{2} \left(1 - \frac{l^2}{R^2}\right)^2\right) \quad \text{XLII}$$

$$y_2 = \frac{W_0^2}{2uq} \left(1 - \frac{25}{49} \frac{l^2}{R^2}\right) \left(1 + \frac{M}{M'}\right)^2 \left[1 + \sqrt{1 - \theta - \theta \frac{M'}{M}} - \frac{M'}{M} \theta \left(1 + \frac{1}{2} \left(1 - \frac{l^2}{R^2}\right)\right)\right] \\ \left(1 + \sqrt{1 - \theta - \theta \frac{M'}{M}}\right)^2 \left(1 + \frac{M}{M'} + \frac{1}{2} \left(1 - \frac{l^2}{R^2}\right)^2\right) \quad \text{XLIII}$$

Now these hold only when separation of cue and ball follow the stroke immediately. That is, when a does not exceed a certain value. Now we may ask, at what distance from the center should we strike

the ball in order to render y_0 and y_2 as great as possible.

Put $\frac{a}{R} = \bar{r} = 1 - \frac{b}{R}$. Then,

$$y_0 = \frac{(2\bar{r} - \bar{r}^2)(1 + \sqrt{1 - \beta\bar{r}^2})^2}{(a + b\bar{r})^2},$$

where α, β, a, b are numerical constants. Equating the first derivative of y_0 to zero, we get the value of $\bar{r}$ which makes y_0 a maximum. We usually consider $\frac{M'}{M} = 3$. By experiment it has been found that $\beta = .13$. Substituting these, we find the required value of $\bar{r}$ to be very nearly .25. Then in order to make the point of slipping as far as possible from the point of stroke, with a stroke whose intensity is already

determined, it is necessary to strike one-fourth of the radius above the center. In the same way, for the same conditions, we can find the maximum of y_z to be one-tenth

Now we will see how the stroke should be given to render a maximum the horizontal velocity of rotation of points on the horizontal equator. That is, the value of R_r , when $l = R$ and $a = h$. We have,

$$R_r = \frac{1}{2} W, \quad \frac{h}{R} = \frac{5h}{2R} \frac{(1 + \frac{M}{M'}) [1 + \sqrt{1 + \theta - \theta \frac{M'}{M} (1 + \frac{1}{2} \frac{h^2}{R^2})}]}{(1 + \frac{M}{M'}) + \frac{1}{2} \frac{h^2}{R^2} (1 + \sqrt{1 - \theta - \theta \frac{M'}{M}})}$$

Differentiating with respect to $\frac{h}{R}$, and putting in the same values as before for M , M' and θ , we find

$$\frac{h}{R} = .5, \text{ and } R_r = .75 W. \quad XLII$$

We might also wish to render a maximum the final velocity and at the same time the velocity when the ball reaches a state of slipping. These velocities are,

$$\sqrt{\frac{5}{7}} W, \frac{L}{R} \text{ and } W, \frac{L}{R}.$$

Consider $h = 0$, $a^2 = (l - R)^2$

$$W, \frac{L}{R} = \frac{(1 + \frac{a}{R}) W_0 (1 + \frac{M}{M'}) (1 + \sqrt{1 - \theta - \theta \frac{M}{M'} (1 + \frac{5}{2} \frac{a^2}{R^2})})}{(1 + \frac{M}{M'} + \frac{5}{2} \frac{a^2}{R^2}) (1 + \sqrt{1 - \theta - \theta \frac{M}{M'}})}$$

Differentiate with respect to $\frac{a}{R}$, equate to zero, and find the value of $\frac{a}{R}$ which will make $W, \frac{L}{R}$ a maximum is nearly .19. Thus it is necessary to strike the ball at about one-fifth of the radius below the center to render $W, \frac{L}{R}$ a maximum.

Fig 5.

Chapter III

We will now investigate what happens when a moving ball strikes one which is not in motion. The friction between the balls is very slight. It has been determined by experiment to be about .028, and for the present, we will neglect it.

Let ϕ be the angle which the tangent at the point of stroke makes with the direction of the motion before the stroke. Then ϕ varies from $\frac{\pi}{2}$ to 0, as the ball is struck directly or more or less to one side. After the stroke, since there is no friction between the balls, w is unchanged, but W becomes $W \sin \phi$. Thus if AD (fig. 5) represent the original velocity W , AB , its projection upon

the tangent at B, will be that velocity after the impact. HR represents. Call the angle of greater deviation of the final path ψ . Then,

$$\psi = \angle OBR.$$

$$\sin \psi = \frac{OB}{OR} = \frac{W}{W+2w}$$

From the way in which ϕ and ψ are measured on arcs of the circle,

$$\psi = \phi - \frac{\pi}{2} - \phi \quad \text{or,}$$

$$2\phi = \frac{\pi}{2} - \psi$$

$$\text{Then } \cos 2\phi = \sin \psi = \frac{W}{W+2w}$$

If at the moment of the stroke rotation were retrograde, HR would be measured in the same direction as HD . R would be inside the circle, and BR still parallel

to the final direction. Now we saw,

$$\cos 2\phi = \frac{W}{W+2w}$$

Then $w = W \cos^2 \phi$, and the condition that the ball recoil when the motion of rotation is retrograde is

$$W \cos^2 \phi < w \text{ or } \cos \phi < \sqrt{\frac{w}{W}}$$

As the ball moves, D and H remain fixed. If it has not reached the final state, H approaches D and moves away from H. If the ball has started with a retrograde motion, that is, if it has been struck by the cue above the center, the point H is found just below H, then moves up, and when it passes over H, the ball is in a state of

slipping, and then there is no curve described after impact.

When the ball is struck just when it has reached the state of slipping, $H H = \frac{2}{3} H D$, or $w = \frac{2}{3} W$, &

$$\sin \psi = \frac{W}{W + 2w} = \frac{5}{9}$$

$$\psi = 33^{\circ} 44'$$

As $\cos 2\phi = \sin \psi = \frac{5}{9}$, we may find $\phi = 62^{\circ} 54'$

Then if the ball of the player has just reached the final state, in order that its deviation from its original direction be a maximum, after impact, it should strike the other ball almost at its side.

We have found that the

shortest distance from H to the final path is.

$$\pi = \frac{W_a W_i}{7\mu g} \sin(\phi - \psi)$$

If in this we put $\sin \psi = \frac{5}{9}$,
 $w = \frac{2}{3}W$ and $W_a = W \cos \phi = \frac{\sqrt{7}}{3}W$,

$$\pi = \frac{1}{18} \cdot \frac{W^2}{2\mu g}$$

Also, we have seen that if W_1 be the velocity of the center when the impact takes place,

$$W = \frac{5}{7} \left(W_1 \frac{l}{R} \right)$$

Then

$$\pi = \frac{1}{36} \left(\frac{W_1^2 \frac{l^2}{R^2}}{2\mu g} \right) \text{ very nearly.}$$

If we take the greatest values of W_1 corresponding to $\frac{W_1^2}{2\mu g} = 2.50$, which is for a very strong stroke of the cue, and if we suppose

that $l=R$, $\pi = .27$.

This value of π which corresponds to the greatest deviation may be regarded, without sensible error, as the maximum value of the distance.

If impact takes place soon after the stroke of the cue, the ratio $\frac{w}{v}$ is equal to $\frac{R-l}{R}$. The maximum of this corresponds to the lowest point at which we may strike without allowing the cue to leave the ball after the stroke. We have found this point to correspond to,

$$\frac{R-l}{R} = .60 \text{ which gives } l = .40R.$$

We may now see what the perpendicular becomes when

we start with this value of h .

We have,

$$\pi = \frac{w \cdot W_a \sin \phi}{7ug}$$

The maximum of $\sin \phi$ for all values of ϕ is unity. To this corresponds very nearly the maximum of π , because of the very small variation of W_a near the point which corresponds to $\sin \phi = 1$.

For $h = .40R$, $w = .60W$ and in consequence $W_a = 2.15W$, we see that the maximum of π is very nearly

$$\pi = .36 \frac{W^2}{ug}$$

Also we have seen that when the stroke is given at a distance .60 above the center, we have,

$$W_1 = .5 W_0$$

where W_0 is the velocity the ball would take if the same blow of the cue were given at the center. Thus we would have for the maximum of π .

$$\pi = .09 \frac{W_0^2}{r \mu g}$$

For a very hard stroke of the cue, we have found that,

$$\frac{W_0}{r \mu g} = 10$$

Thus we have, very nearly,

$$\pi = .90$$

For an ordinary stroke of the cue, which corresponds to $\frac{W_0}{r \mu g} = 4.80$,

Whatever be the motion of the

other ball after the stroke, when we neglect friction between the balls, we do not need to determine. It is clear that it will always move in the direction of the normal at the point of stroke, and that the velocity of translation of its center will be $W \sin \phi$. Its velocity of rotation will be zero at first, and will become $\frac{1}{2} W \sin \phi$ when it has reached the final state.

Fig. 6

Chapter IV

In this chapter, we will suppose the balls are not perfectly elastic. Then the ball of the player will keep a part of its normal velocity after impact. If this velocity were $W \sin \phi$ before impact, after impact it becomes $\alpha W \sin \phi$. Then (fig. 6) AD is W , $AA' = \alpha W$, $BB' = \alpha W \sin \phi$. Then AB' is the resultant of $W \cos \phi$ and $\alpha W \cos \phi$. Thus, we see that if the balls are not perfectly elastic, the tendency will be to diminish the angle between AD and the final direction. Ordinarily, the friction between the balls may be neglected but in certain plays it may not.

so we will now take this into account, and later will use the formulas thus obtained, with slight modifications, for the contact of the ball with the cushion, in which case friction is never neglected. We may, however, neglect the friction between the ball and the surface of the table, since this can have only a vertical component, which is extremely small.

Let us consider first the effect of friction, in general, in the contact of two spheres. Take the XY plane parallel to the plane tangent to the balls at the point where impact occurs. Call Φ the force which is produced

in the impact by friction. Let $U, V, p, q, r, U', V', p', q', r'$ have meanings assigned to them as before, for the two balls. Also S and S' are the velocities of the two along the Z axis. The subscripts attached to anyone of these signifies that we consider that quantity at the moment of impact.

Now since the force developed by impact is imparted to the two balls equally but in opposite directions,

$$\left. \begin{aligned} M \frac{dU}{dt} &= -X = -M \frac{dU'}{dt} \\ M \frac{dV}{dt} &= -Y = -M \frac{dV'}{dt} \\ M \frac{dS}{dt} &= -Z = -M \frac{dS'}{dt} \end{aligned} \right\} \#$$

Integrating,

$$\left. \begin{aligned} U+U' &= U_1+U_1' \\ V+V' &= V_1+V_1' \\ S+S' &= S_1+S_1' \end{aligned} \right\} B$$

taking moments about the origin,

$$\left. \begin{aligned} \frac{2}{3}MR^2\frac{d\phi}{dt} &= 0 & \frac{2}{3}MR^2\frac{d\phi'}{dt} &= 0 \\ \frac{2}{3}MR^2\frac{d\theta}{dt} &= -RZ & \frac{2}{3}MR^2\frac{d\theta'}{dt} &= RZ \\ \frac{2}{3}MR^2\frac{d\psi}{dt} &= RY & \frac{2}{3}MR^2\frac{d\psi'}{dt} &= -RY \end{aligned} \right\} C$$

Integrating and combining,

$$\left. \begin{aligned} \phi+\phi' &= \phi_1+\phi_1' \\ \theta+\theta' &= \theta_1+\theta_1' \\ \psi+\psi' &= \psi_1+\psi_1' \end{aligned} \right\} D$$

Eliminating Z and Y between

C and H, and integrating,

$$\left. \begin{aligned} \frac{2}{3} R (p - p_1) &= 0 \\ \frac{2}{3} R (q - q_1) &= S - S_1 \\ \frac{2}{3} R (r - r_1) &= V - V_1 \end{aligned} \right\} E$$

$$\left. \begin{aligned} \frac{2}{3} R (p' - p'_1) &= 0 \\ \frac{2}{3} R (q' - q'_1) &= -(S' - S'_1) \\ \frac{2}{3} R (r' - r'_1) &= (V' - V'_1) \end{aligned} \right\} F$$

It should be noticed that, in the moment equations, we should take for V and V' only the components of friction, since the normal force is, at that instant, zero. V and V' will be Φ multiplied by the sine and cosine of the angle which the velocity

makes with these axes. Then they are given by:

$$Y = \frac{(Rr + V) - (Rr' + V')}{\sqrt{[(Rr + V) - (Rr' + V')]^2 + [(S - Rq) - (S' - Rq')]^2}} \phi$$

$$Z = \frac{(S - Rq) - (S' - Rq')}{\sqrt{[(Rr + V) - (Rr' + V')]^2 + [(S - Rq) - (S' - Rq')]^2}} \phi$$

Calling the radical ϕ and substituting in our first equations,

$$M \frac{dV}{dt} = -X$$

$$M \frac{dS}{dt} = \frac{S - S' - Rq + Rq'}{\phi} \phi$$

$$M \frac{dV}{dt} = \frac{V - V' - Rr + Rr'}{\phi} \phi$$

Combining the last two of these equations,

$$\frac{dV}{V - V' + Rr - Rr'} = \frac{dS}{S - S' - Rq + Rq'}$$

Now because we have the nine linear relations between the twelve unknown quantities involved, the denominator of dV will become a linear function V ; that of dS , a linear function of S .

Integrating, we see that this equation establishes a constant ratio between the denominators during impact. As these are proportional to the cosines of the angles that the friction makes with the axes, we may say,

Friction preserves a constant ratio during impact.

We will now consider this friction in the impact of a moving

ball with one which is at rest. We will always call W the velocity of the center of the ball in motion, and ϕ the angle which the direction of this velocity makes with the tangent at the point of impact.

Let us take for y axis the direction of the tangent at the point of impact, that is of the velocity $W \cos \phi$ of the center of the ball immediately after impact. If α be the angle that the velocity at the point of impact makes with the direction of $W \cos \phi$ will be determined by

$$\sin \alpha = \frac{Rq}{\sqrt{(R + W \cos \phi)^2 + Rq^2}} = - \frac{Rq}{W}$$

$$\cos \alpha = \frac{Rr + W \cos \phi}{\sqrt{(Rr + W \cos \phi)^2 + R^2 \dot{\phi}^2}} = \frac{Rr + W \cos \phi}{\omega}$$

Call f the ratio between the friction and momentum produced in impact. The effect of friction will be to produce at the center of the ball a horizontal velocity

$$-f W \sin \phi \cos \alpha,$$

opposite the the velocity $W \cos \phi$, and a vertical velocity

$$-f W \sin \phi \sin \alpha.$$

We need not consider the latter, as we see that only the horizontal component will be effective. The friction will produce, besides, moments of rotation, which will be,

will be,

$$-f W \sin \phi \cdot \cos \alpha \cdot R \\ + f W \sin \phi \cdot \sin \alpha \cdot R$$

The velocity $W \cos \phi$ will preserve its direction and become,

$$W \cos \phi - \mu \cos \phi \cos \alpha,$$

The quantities $\frac{2}{5} R \dot{\phi}$, $\frac{2}{5} R \dot{\theta}$, $\frac{2}{5} R \dot{\psi}$, will become after impact

$$\frac{2}{5} R \dot{\phi}$$

$$\frac{2}{5} R \dot{\theta} + \mu R W \sin \phi \sin \alpha$$

$$\frac{2}{5} R \dot{\psi} - \mu R W \sin \phi \cos \alpha$$

Thus, after impact, we have,

$$\frac{2}{5} R \dot{\phi}' = \frac{2}{5} R \dot{\phi}$$

$$\frac{2}{5} R \dot{\theta}' = \frac{2}{5} R \dot{\theta} + f W \sin \phi \sin \alpha$$

$$\frac{2}{5} R \dot{\psi}' = \frac{2}{5} R \dot{\psi} - f W \sin \phi \cos \alpha$$

$$W' = W \cos \phi - \mu W \sin \phi \cos \alpha$$

$$\text{Put } W \sin \phi = \beta$$

Then these equations become,

$$\frac{2}{3} R p' = \frac{2}{3} R p$$

$$\frac{2}{3} R q' = \frac{2}{3} R q + \beta \sin \alpha$$

$$\frac{2}{3} R r' = \frac{2}{3} R r - \beta \cos \alpha$$

designate by u and v the components of the velocity of the superior center of percussion.

$$\left. \begin{aligned} u &= \frac{2}{3} R q \\ v &= \frac{2}{3} R p \end{aligned} \right\}$$

$$u' = u - \beta \sin \alpha$$

$$v' = v$$

Substitute these and $R r = \frac{5}{2} W_1 \frac{h}{R}$,

$$\sin \alpha = \frac{\frac{5}{2} u}{\omega}$$

$$\cos \alpha = \frac{W \cos \phi - \frac{2}{3} W_1 \frac{h}{R}}{\omega}$$

$$\omega = \sqrt{\left(\frac{5}{2} u\right)^2 + \left(W \cos \phi - \frac{2}{3} W_1 \frac{h}{R}\right)^2}$$

With the values we now have for $\sin \alpha$, $\cos \alpha$, W' , u , v' , it is easy to construct all that pertains to the path of the ball after impact.

It remains now to examine how we shall construct $\sin \alpha$ and $\cos \alpha$. First, if impact takes place after a horizontal stroke of the cue, and before another impact, the velocity of rotation of the center of percussion, w , being in the direction of the velocity W , or HD (fig. 7), $\frac{1}{2}u$ will be the projection of $\frac{1}{2}w$. That is, $\frac{1}{2}u$ will be the projection of HF on DB or KB . Now $W \cos \phi$ is HB . If on HB we draw, starting at H , a distance HC equal to $W, \frac{h}{2R}$, then $CB = W \cos \phi - \frac{1}{2}W, \frac{h}{R}$. Now since

Fig. 8.

$$\left. \begin{aligned} \sin \alpha &= -\frac{PQ}{\omega} = \frac{\frac{1}{2}u}{\omega} \\ \cos \alpha &= \frac{W \cos \phi - \frac{5}{2}W \frac{h}{R}}{\omega} \end{aligned} \right\}$$

then if $\frac{1}{2}u$ be AB and $W \cos \phi - \frac{5}{2}W \frac{h}{R}$ be CB , the angle ACB must be the angle α .

In figure 8 we draw a circle on AD as diameter, where $AD' = AD + \frac{1}{2}AD$. Then $BG = \frac{1}{2}W \sin \phi$. This length BG will represent β . If now we make BG turn around B until it is parallel to AD , and through its intersection with AD' draw $B'B'$ parallel to DB , then the distance AB' represents the value of W after the impact.

$$W' = AB' = W \cos \phi - \beta \cos \alpha.$$

We will have the new velocity

by moving the point B to B' ; and making HH' thus approach HB along a perpendicular. Then $HH' = u$ is

$$u' = u - \beta \sin \alpha$$

We have now the elements HB' and HH' of the motion of the ball of the player after impact, considering the effect of friction produced between the balls during impact.

It should be noticed that in order to have the final direction $H'B'$, we might connect H with B'' , as it is easily seen, that HB'' is parallel to HB' . The final direction must always pass through H . It will pass through B or B' according as there is not or is friction. We see then

that friction tends to increase the angle that the final direction makes with FD when rotation is retrograde.

The other ball, before at rest will follow the normal at the point of impact if there be no friction. Let us see what if not friction between the two balls will have upon its path. Its velocity of translation will be represented by BD , and its velocity of rotation be zero, when there is no friction. Also because the friction on this ball is equal and directly opposite to that which is produced upon the ball of the player, we will have its final path by joining D

with B'' . This path is now directed from B'' to D . We see then that the direction of $B''D$ differs from that of BD by a very small angle, so that the effect of friction is very slight.

Chapter V

Let us see what will happen when the ball of the player strikes against the cushion, its center having a velocity W which makes an angle ϕ with the cushion, and the elements of motion of rotation being Rp , Rq , Rr , referred to the cushion as axis PY .

To obtain the elements of motion after impact, we will call ε a very small numerical fractional coefficient, such that the normal velocity given by the cushion after impact will be.

$$\varepsilon W \sin \phi$$

The momentum given to the

ball by the cushion during impact is

$$W \sin \phi (1 + \epsilon)$$

Call μ , the coefficient of friction during the impact. Then the momentum due to this friction is

$$\mu (1 + \epsilon) W \sin \phi$$

Using the same notation as before,

$$U' = -\epsilon W \sin \phi$$

$$V' = W \cos \phi - \mu W \sin \phi \cos \alpha$$

$$\frac{2}{3} R \dot{\phi}' = \frac{2}{3} R \dot{\phi}$$

$$\frac{2}{3} R \dot{q}' = \frac{2}{3} R \dot{q} + \mu' W \sin \phi \sin \alpha$$

$$\frac{2}{3} R \dot{r}' = \frac{2}{3} R \dot{r} + \mu W \sin \phi \cos \alpha$$

Substituting $u = -\frac{2}{3} R \dot{q}$, $v = \frac{2}{3} R \dot{\phi}$, we will have, ∇

$$\left. \begin{aligned} u' &= u - u' \sin \phi \sin \alpha \\ v' &= v \\ \frac{2}{5} R r' &= \frac{2}{5} R r - u' W \sin \phi \cos \alpha \end{aligned} \right\}$$

As before, put $\beta = u W \sin \phi$

$$\left. \begin{aligned} U' &= -\frac{2}{5} W \sin \phi \\ V' &= W \cos \phi - \beta \cos \alpha \\ u' &= u - \beta \sin \alpha \\ v' &= v \\ R r' &= R r - \frac{2}{5} \beta \cos \alpha \end{aligned} \right\}$$

If we have $U' = 0$, these equations become exactly those which give the motion after the impact of two perfectly elastic balls.

In figure 8, if ODH is the direction in which the ball is moving before it strikes the cushion, we may find OB , its direct-

ion after impact. Let fall DO perpendicular to the cushion, and lay off $OB = \epsilon OD$. Then OB will be the required path. We see that this is determined by its former direction and by the coefficient of elasticity.

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