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A paper on Minimum Surfaces
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Minimum Surfaces

Introduction.

In this dissertation I propose to give some of the theory and develop some of the important Formulae upon which Minimum Surfaces are based.

In order to proceed with the development of Minimum Surfaces, it will be necessary to devote a brief space to the consideration of some of the general and elementary notions of Surface Theory.

What is the Mathematical meaning of a Surface?

It may be defined in either of two ways. First, we may determine the surface from its equations, that is, from the relation that exists among the coordinates of any point on the surface, and we may suppose that these coordinates can be expressed by means of two independent variables which we call u and v .

The second way of defining a surface is more general than the first.

If we take for u and v , two of the three rectangular coordinates, the expression for the third is precisely the equation of the surface, solved with respect to that coordinates.

Tangent-Plane to a Surface

Let the surface be given by the equation

$$z = f(x, y)$$

and let (x, y, z) be the coordinates of a point P of that surface.

Fig. 1.

If we draw the tangents to all the curves on the surface which pass through the point P , and these tangents lie in a plane, we speak of this plane as the tangent-plane to the surface at the point P . Suppose any curve C drawn on the surface and passing through the point P . Fig. 1.

The coordinates of a point of that curve are functions of a parameter u which satisfy the relation $z = f(x, y)$.

The equations of the tangent Pt to the curve C are then,

$$(1) \quad \frac{X-x}{dx} = \frac{Y-y}{dy} = \frac{Z-z}{dz}$$

Since x, y, z , being functions of a variable u , satisfy identically the relation $z = f(x, y)$.

on applying the rule for differentiation of compound functions,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

using the notation,

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q,$$

$$\text{or } \frac{\partial q}{\partial x} = p, \quad \frac{\partial z}{\partial x} = q.$$

we may write the above equation thus,

$$(2) \quad dz = p dx + q dy.$$

Replacing dz, dx, dy , by the proportional quantities $X-x, Y-y, Z-z$, from Equations (1)

we have,

$$(3) \quad Z - z = p(X - x) + q(Y - y)$$

which is the equation of the tangent plane at the point (x, y, z) .

Normal.

The normal to a surface at the point (x, y, z) is the perpendicular to the tangent plane and has for its equations,

$$(4) \quad \frac{X - x}{p} = \frac{Y - y}{q} = \frac{Z - z}{-1}$$

The tangent to a curve c at a point P is the limit of a straight line joining the point P to a point P' which is near P . It is not necessary to believe that the tangent plane to a surface at the point P is the limit of a plane passing through P and two points P' and P'' very near P on the surface. The plane $PP'P''$ will take various limiting positions accord.

ing to the manner in which P' and P'' approach P

If P' and P'' approach P in such a way that the straight-lines PP' and PP'' approach two distinct-tangents to the surface, the plane $PP'P''$ approaches the tangent-plane, if it exists, for that limiting position contains the two tangents to the surface.

But if the points P' and P'' approach P in such a way that both follow the same curve drawn on the surface through P , the plane $PP'P''$ approaches the osculating plane to the curve and not the tangent-plane to the surface.

Osculating Plane to a Curve.

Let P be a point on the curve C and P' a point near P , and Pt and $P't$, the tangents at these two points

The osculating plane at P is the limit towards which the plane passing through Pt and parallel to P_1t , approaches when P_1 approaches P .

The osculating plane is nearer to the curve than any other tangent plane. Without going through the development necessary to arrive at the osculating plane we will give its equations*,

$$(5) (X-x)(y'z''-z'y'') + (y-y')(z'x''-x'z'')(z-z')(x'y''-y'x'') = 0$$

where the curve is given by the following equations,

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$

$$\text{where } \begin{matrix} x' = f'(t) & y' = g'(t) & z' = h'(t) \\ x'' = f''(t), & y'' = g''(t), & z'' = h''(t). \end{matrix}$$

* See Humbert's Cours d'Analyse, Art 386, page 390.

Element of an arc on a surface
 A curve drawn on the surface, given by the equations,

$$x = f(u, v)$$

$$y = g(u, v)$$

$$z = h(u, v),$$

is defined by the relation between u and v , $v = \phi(u)$. The direction of its tangent at point (u, v) is determined by value of the ratio $\frac{dv}{du}$ or $\phi'(u)$ deduced from the equation of the curve.

$$x' = x_u + x_v \lambda$$

$$y' = y_u + y_v \lambda$$

$$z' = z_u + z_v \lambda,$$

where λ is $\frac{dv}{du}$.

These equations give the parametric directions of the tangents in terms of function of $\frac{dv}{du}$.

At a point (u, v) a tangent direction to the surface is defined by the ratio $\frac{dv}{du}$, and reciprocally, that direction

is limit of that straight line which joins the point (u, v) to the point $(u + du, v + dv)$.

The square of the element of length on the surface formed for the arguments, $u, v, u + du, v + dv$ is,

$$\left(\frac{ds}{du}\right)^2 = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$$

$$= (x_u + x_v \lambda)^2 + (y_u + y_v \lambda)^2 + (z_u + z_v \lambda)^2$$

that is,

$$\left(\frac{ds}{du}\right)^2 = E + 2F\lambda + G\lambda^2$$

where,

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = x_u^2 + y_u^2 + z_u^2$$

$$F = \left(\frac{\partial x}{\partial u} \frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u} \frac{\partial y}{\partial v}\right) + \left(\frac{\partial z}{\partial u} \frac{\partial z}{\partial v}\right) = x_u x_v + y_u y_v + z_u z_v$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = x_v^2 + y_v^2 + z_v^2$$

$$(6) \therefore ds^2 = E du^2 + 2F du dv + G dv^2.$$

This may be written in another form,

$$(7) ds^2 = A^2 du^2 + 2AC \cos \theta du dv + C^2 dv^2$$

where $A = \sqrt{E}$, $C = \sqrt{G}$, $\cos \theta = \frac{F}{\sqrt{EG}}$.

The formulae for E, F, G show that $A du$ is the arc of the curve $v = \text{const.}$, and $C dv$ is the arc of the curve $u = \text{const.}$ θ is the angle at which the curves cut at point considered. One has then, $F = 0$, when the two systems of coordinates lines are orthogonal.

That is, for an orthogonal system

$$(8) ds^2 = E du^2 + G dv^2$$

If $E = G, F = 0$,

$$(9) ds^2 = E (du^2 + dv^2),$$

which is an isothermal system.

What is the angle between the Normal to the Surface and the principal Normal to the curve?

Let the surface be given by the equations,

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases}$$

and the curve on the surface by the relation $v = \phi(u)$, then its equations are,

$$\begin{cases} x = f(u, \phi(u)) \\ y = g(u, \phi(u)) \\ z = h(u, \phi(u)). \end{cases}$$

We saw that $\cos \theta = \frac{F}{EG} = \frac{x_u x_v + y_u y_v + z_u z_v}{\sqrt{x_u^2 + y_u^2 + z_u^2} \sqrt{x_v^2 + y_v^2 + z_v^2}}$.
[Page 8.]

We need to know three numbers for which the direction cosines are proportional.

The direction cosines of the principal normal to the surface are proportional to the following quantities,

$$f = \frac{y_u z_v - y_v z_u}{H}, \quad g = \frac{z_u x_v - x_u z_v}{H},$$

$$h = \frac{x_u y_v - y_u x_v}{H},$$

where $H = EG - F^2$

Let us find now the direction cosines of P_n , the intersection of the osculating

plane with normal plane, (the principal normal to the curve).

$$(X-x) = \frac{r^2}{x'^2+y'^2+z'^2} \left[(x'^2+y'^2+z'^2)x'' - (x''x' + y'y'' + z'z'')x' \right]$$

$$(Y-y) = \frac{r^2}{x'^2+y'^2+z'^2} \left[(x'^2+y'^2+z'^2)y'' - (x'x'' + y'y'' + z'z'')y' \right]$$

$$(Z-z) = \frac{r^2}{x'^2+y'^2+z'^2} \left[(x'^2+y'^2+z'^2)z'' - (x'x'' + y'y'' + z'z'')z' \right]$$

The direction cosines α, β, γ , are proportional to these quantities,

$$\alpha : \beta : \gamma :: (X-x) : (Y-y) : (Z-z).$$

By quite a long calculation we arrive at the following result for $\cos \theta$.

$$(10) \cos \theta = \frac{r}{H} \frac{\begin{vmatrix} x'' & y'' & z'' \\ y'' & y'' & y'' \\ z'' & z'' & z'' \end{vmatrix}}{x'^2+y'^2+z'^2}$$

where $H = E G - F^2$,

$$\begin{cases} x' = x_u + x_v \lambda \\ y' = y_u + y_v \lambda \\ z' = z_u + z_v \lambda \end{cases}, \text{ where } \lambda = \frac{\partial v}{\partial u},$$

$$\begin{cases} x'' = x_{uu} + 2\lambda x_{uv} + \lambda^2 x_{vv} \\ y'' = y_{uu} + 2\lambda y_{uv} + \lambda^2 y_{vv} \\ z'' = z_{uu} + 2\lambda z_{uv} + \lambda^2 z_{vv} \end{cases},$$

and

$$x'^2 + y'^2 + z'^2 = E + 2F\lambda + G\lambda^2.$$

whence $\cos \theta$ takes the following form,

$$\cos \theta = \frac{r}{H[E + 2F\lambda + G\lambda^2]} \cdot \begin{vmatrix} x_{uu} + 2\lambda x_{uv} + \lambda^2 x_{vv} & x_u & x_v \\ y_{uu} + 2\lambda y_{uv} + \lambda^2 y_{vv} & y_u & y_v \\ z_{uu} + 2\lambda z_{uv} + \lambda^2 z_{vv} & z_u & z_v \end{vmatrix}$$

$$= \frac{r}{E + 2F\lambda + G\lambda^2} \left[\frac{1}{H} \begin{vmatrix} x_{uu} & x_u & x_v \\ y_{uu} & y_u & y_v \\ z_{uu} & z_u & z_v \end{vmatrix} + \frac{2\lambda}{H} \begin{vmatrix} x_{uv} & x_u & x_v \\ y_{uv} & y_u & y_v \\ z_{uv} & z_u & z_v \end{vmatrix} + \frac{\lambda^2}{H} \begin{vmatrix} x_{vv} & x_u & x_v \\ y_{vv} & y_u & y_v \\ z_{vv} & z_u & z_v \end{vmatrix} \right]$$

whence :

$$\cos \theta = \frac{r}{E + 2F\lambda + G\lambda^2} [L + 2M\lambda + N\lambda^2]$$

where,

$$(11) \quad L = \frac{1}{H} \begin{vmatrix} x_{uu} & x_u & x_v \\ y_{uu} & y_u & y_v \\ z_{uu} & z_u & z_v \end{vmatrix},$$

$$(12) \quad M = \frac{1}{H} \begin{vmatrix} x_{uv} & x_u & x_v \\ y_{uv} & y_u & y_v \\ z_{uv} & z_u & z_v \end{vmatrix},$$

$$(13) \quad N = \frac{1}{H} \begin{vmatrix} x_{vv} & x_u & x_v \\ y_{vv} & y_u & y_v \\ z_{vv} & z_u & z_v \end{vmatrix}.$$

$$(14) \quad \cos \theta = \frac{r (L + 2M\lambda + N\lambda^2)}{(E + 2F\lambda + G\lambda^2)},$$

whence,

$$(15) \quad r = \frac{E + 2F\lambda + G\lambda^2}{L + 2M\lambda + N\lambda^2} \cos \theta.$$

If P and λ are fixed, $r = \text{const.} \cos \theta$
 θ may still vary, that is, we may
 turn the osculating plane about
 the line PT the tangent to the curve.

As θ changes r changes. We
 may ask when is r a maximum?

Answer, when $\cos \theta = 1$, $\theta = 0$, that is
 when the normal to the surface
 lies in the osculating plane.

Normal Sections have largest
 radius of curvature. In the r
 for a normal section P and λ
 are kept constant, and θ varies,
 we can therefore replace any curve
 on the surface by a plane curve.

To find the maximum radius
 of curvature set $\frac{dr}{d\lambda} = 0$.
 we will get a maximum when
 when,

$$(16) (FN - GM)\lambda^2 + (EN - GL)\lambda - (EM - FL) = 0$$

Let the roots of this equation be λ_1, λ_2 ,

then,

$$(7), \quad \begin{cases} \lambda_1 + \lambda_2 = \frac{EN - GL}{FN - GM}, \\ \lambda_1 \lambda_2 = \frac{EM - FL}{FN - GM}. \end{cases}$$

To determine r_1, r_2 from the equations,

$$r_1 = \frac{E + 2F\lambda_1 + G\lambda_1^2}{L + 2M\lambda_1 + N\lambda_1^2},$$

$$r_2 = \frac{E + 2F\lambda_2 + G\lambda_2^2}{L + 2M\lambda_2 + N\lambda_2^2},$$

form an equation whose roots are the functions,

$$r_1 = \frac{F + G\lambda_1}{M + N\lambda_1},$$

$$r_2 = \frac{F + G\lambda_2}{M + N\lambda_2}.$$

By some easy calculations,

we get,

$$(18) \begin{cases} r_1 + r_2 = \frac{GL - 2FM + EN}{LN - M^2}, \\ r_1 r_2 = \frac{EG - F^2}{LN - M^2} \end{cases}$$

One calls the total curvature the quantity $\frac{1}{r_1 r_2}$, whence,

$$(19) \quad \frac{1}{r_1 r_2} = \frac{LN - M^2}{EG - F^2}.$$

The mean curvature of a surface at a point is the quantity $\frac{1}{r_1} + \frac{1}{r_2}$, whence,

$$(20) \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{GL - 2FM + EN}{EG - F^2}.$$

The surfaces with mean curvature zero, where $\frac{1}{r_1} + \frac{1}{r_2} = 0$, or $r_1 = -r_2$,

are the minimum surfaces.

- Indicatrix -

The Indicatrix is a conic which is the best approximation to section of surface by the normal plane. Its equation is,

$$f_{xx} (x - x_0)^2 + 2f_{xy} (x - x_0)(y - y_0) + f_{yy} (y - y_0)^2 = 0$$

Its discriminant is,

$$f_{xx} f_{yy} - f_{xy}^2$$

and according as it is equal to, less than, or greater than zero, the curve takes the form of a parabola, an hyperbola or an ellipse. In other words the Indicatrix is an ellipse, an hyperbola, or a parabola, according as the total curvature $\frac{1}{r_1 r_2}$ of

the surface at the point considered is positive, negative, or zero. The axes of the conic give the Principal directions of the surface at the point. Its Asymptotic directions are the

the asymptotic directions of the surface at the point in question, they coincide with the tangents at point of contact, with the intersection of the surface by the tangent plane.

Theorem —

The curvature at a point P of a line drawn on the surface is the same of any other curve on the surface which has at P the same tangent and same osculating plane as the first curve.

The curvature at a point P of a line drawn on the surface is the same as that of the section determined in surface by the osculating plane of the line at the point P .

Principal Tangents.

Fig. (4)

we have given two values of λ , (λ_1, λ_2) such that the directions of the tangents depend on these values of λ , we call these the principal tangents and the angle between them is $\frac{\pi}{2}$.

The radius of curvature at a point P of any plane section passing through P is the projection of the radius of curvature of the corresponding normal section. This is the theorem of Meunier.

From the curvature of a Normal Section,
$$r = \frac{E + 2F\lambda + G\lambda^2}{L + 2M\lambda + N\lambda^2},$$

where $\lambda = \frac{\partial u}{\partial v},$

We may determine at any point of the surface, the principal radii of curvature, the principal directions or tangents, and the parabolic points.

The previous equation for the radius of curvature may be written,

$$(21) (E - rL) + 2(F - rM)\lambda + (G - rN)\lambda^2 = 0$$

In finding a maximum or minimum of r we expressed the condition that the above Equation regarded as an equation of the second degree in λ should have equal roots. One has then,

$$(22) (F - rM)^2 - (E - rL)(G - rN) = 0,$$

an equation of the second degree in r , from which the values of r_1, r_2 may be found, or the principal radii of curvature.

Principal Directions.

The values of λ which correspond to the principal directions are those

for which R is a maximum and a minimum.

When the preceding equation in λ has a double root, that root corresponds to a principal direction. It satisfies the two equations,

$$(23) \begin{cases} (E - R L) + (F - R M) \lambda = 0 \\ (F - R M) + (G - R N) \lambda = 0 \end{cases}$$

eliminating R ,

$$(24) \frac{E + F \lambda}{F + G \lambda} = \frac{L + M \lambda}{M + N \lambda}$$

that is to say,

$$(25) \begin{vmatrix} E & F & G \\ L & M & N \\ \left(\frac{du}{dv}\right)^2 \frac{du}{dv} & 1 & \end{vmatrix} = 0 \quad \text{where } \lambda = \frac{du}{dv}$$

This equation gives the two values of λ which correspond to the principal directions.

Asymptotic Directions.

The normal sections drawn through the Asymptotes of the Indicatrix at a point P have their radii of curvature infinite and reciprocally. One obtains the asymptotic directions by writing for r an infinite value in the equation

$$r = \frac{E + 2F\lambda + G\lambda^2}{L + 2M\lambda + N\lambda^2}.$$

This will be effected by setting,

$$(26). \quad L + 2M\lambda + N\lambda^2 = 0$$

Parabolic Points.

These are the points where one of the principal radii of curvature is infinite, then the equations in r ,

$$(F - rM)^2 - (E - rL)(G - rN) = 0$$

for one of the points must have an infinite root; or $M^2 - LN = 0$,

this is a relation between u and v which defines on the surface a parabolic line all of whose points are parabolic.

Ombilic

An ombilic is a point where the indicatrix is a circle. The principal directions, the axes of the indicatrix is undetermined, that is to say in the equation

$$\begin{vmatrix} E & F & G \\ L & M & N \\ du^2 - 2dudv + dv^2 \end{vmatrix} = 0$$

the coefficients of du^2 , $dudv$, dv^2 are zero, which gives

$$(27) \quad \frac{E}{L} = \frac{F}{M} = \frac{G}{N},$$

now we have two equations between u and v , which determines on the surface a limited number of points which are ombilics.

Applications

Let a surface be given in the form $z = f(x, y)$, where we suppose $u = x$, $v = y$, then

$$E = 1 + \left(\frac{\partial z}{\partial x}\right)^2, F = \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}, G = 1 + \left(\frac{\partial z}{\partial y}\right)^2$$

Replace $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by p and q and the square of the element-length may be written

$$(28) ds^2 = (1 + p^2) dx^2 + 2pq dx dy + (1 + q^2) dy^2$$

$$E = 1 + p^2, F = pq, G = 1 + q^2.$$

The equations for the principal radii of curvature,

$$*(F - \rho M)^2 - (E - \rho L)(G - \rho N) = 0$$

becomes,

$$(29) \left(pq - \frac{\rho s}{\sqrt{1 + p^2 + q^2}} \right)^2 - \left(1 + p^2 - \frac{\rho r}{\sqrt{1 + p^2 + q^2}} \right) \left(1 + q^2 - \frac{\rho t}{\sqrt{1 + p^2 + q^2}} \right) = 0$$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$,

* we will use ρ to denote the radius of curvature to avoid confusing it with the symbol

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$M = \frac{s}{\sqrt{1+p^2+q^2}}, \quad N = \frac{t}{\sqrt{1+p^2+q^2}},$$

$$L = \frac{r}{\sqrt{1+p^2+q^2}}.$$

The equation (25) of the principal directions becomes,

$$\begin{vmatrix} 1+p^2 & pq & 1+q^2 \\ r & s & t \\ dy^2 & -dx dy & dx^2 \end{vmatrix} = 0$$

Equation (26) of the Asymptotic directions becomes

$$r dx^2 + 2s dx dy + t dy^2 = 0.$$

The condition for parabolic points

$$M^2 - LN = 0,$$

reduces to,

$$s^2 - rt = 0,$$

and the equation of the umbilics

$$\frac{E}{L} = \frac{F}{M} = \frac{G}{N} \text{ takes the form}$$

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t}.$$

Equation (29) may be written in the following form,

$$(30) \left\{ (s^2 - rt) \rho^2 + [(1 + p^2)t - 2pq s + (1 + q^2)r] \rho \sqrt{1 + p^2 + q^2} + (1 + p^2 + q^2) \right\} = 0$$

The roots of the equation in ρ are the principal radii of curvature ρ_1, ρ_2

The total curvature

$$\frac{1}{\rho_1 \rho_2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}$$

Expressing that equation (30) has equal roots and of contrary sign one has for the differential equation of those surfaces

$$(31) (1 + p^2)t - 2pq s + (1 + q^2)r = 0$$

This is the characteristic equation of the minimum surfaces.

Lines of Curvature

Lines of curvature on a surface are those lines which are tangents at every point to one of the principal directions at that point.

Through every point on the surface there will be two lines of curvature which are orthogonal to each other.

Let the surface be given in the form $z = f(x, y)$.

We have seen that the equations of the principal tangents or directions was given in the form,

$$\begin{vmatrix} E & F & G \\ L & M & N \\ dx^2 - dxdy & dy^2 \end{vmatrix} = 0$$

It may be put in the form

$$A(x, y) \left(\frac{dy}{dx} \right)^2 + B(x, y) \frac{dy}{dx} + C(x, y) = 0$$

where A, B, C are known functions of (x, y) .

Solving this equation for $\left(\frac{dy}{dx} \right)$ we get

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \begin{cases} F_1(x, y) \\ F_2(x, y) \end{cases}$$

$$\left\{ \frac{dy}{dx} = F_1(x, y) \right.$$

$$\left. \frac{dy}{dx} = F_2(x, y) \right.$$

which will give the solutions of the foregoing differential equation. Thus through every point on the surface we will get two lines of curvature.

As there are in every point two principal tangents always real, there are two real lines of curvature passing through every point on the surface. That is the lines of curvature form two real sets and further the curves of one set cut at right angles all the curves of the other set and the principal tangents at that point are rectangular.

The normals to a surface along a line of curvature envelop a curve, that is to say form a developable surface.

For the surface $z = f(x, y)$, the equations of the normal to the surface at a point (x, y, z) are

$$(X-x) + p(Z-z) = 0$$

$$(Y-y) + q(Z-z) = 0$$

The condition that the straight lines admit of an envelope is

$$* \frac{dp}{dx} \frac{d}{dx} (y + qz) = \frac{dq}{dx} \frac{d}{dx} (x + pz)$$

which after reductions becomes,

$$\frac{dp}{dx} \left(\frac{dy}{dx} + q \frac{dz}{dx} \right) = \frac{dq}{dx} \left(1 + p \frac{dz}{dx} \right)$$

$$\text{where } \frac{dp}{dx} = r + s \frac{dy}{dx}, \quad \frac{dq}{dx} = s + t \frac{dy}{dx},$$

$$\frac{dz}{dx} = p + q \frac{dy}{dx}.$$

* See Humbert's Cours d'Analyse.

$$(r + s \frac{dy}{dx}) (\frac{dy}{dx} (1 + q^2) + pq) = s + t \frac{dy}{dx} (1 + p^2 + pq \frac{dy}{dx})$$

$$\text{Let } \frac{dy}{dx} = y'$$

$$(r + sy') (y' (1 + q^2) + pq) = (s + ty') (1 + p^2 + pqy')$$

which may be verified to be the differential equation of the lines of curvature,

$$\frac{pqy' - (1 + p^2)}{r - sy'} + \frac{pq + (1 + q^2)y'}{ty' + s} = 0$$

Asymptotic Lines.

Asymptotic lines are those lines drawn on a surface which touch at every point one of the asymptotic directions - at that point. These may be found by integrating the equation of the asymptotic directions,

$$L + 2M\lambda + N\lambda^2 = 0$$

where L, M, N are known functions of u and v and $\lambda = \frac{dv}{du}$

There are evidently two asymptotic lines passing through every point on the surface that is the lines form two sets, but the asymptotic lines through a point are not necessarily real; they are real if the indicatrix in the point is an hyperbola, imaginary if the indicatrix is an ellipse.

If the asymptotic lines stand at right angles what kind of a surface do we get?

The angle between the asymptotic lines is 90° . We have for the differential equation of the asymptotic lines

$$L du^2 + 2M du dv + N dv^2 = 0$$

Then the indicatrix is rectangular Hyperbola.

$$\frac{du}{dv} = \frac{-M \pm \sqrt{M^2 - LN}}{L}$$

Asymptotes of indicatrix must stand at right angles, whence $r_1 = -r_2$, which gives the Minimum Surfaces.

Chapter I

The beginning of the theory of Minimum Surfaces dates back to the celebrated treatise of Lagrange (1760-'61) in which the calculus of Variations are also explained.

Let us consider a closed curve C and a surface S , bounded by this curve C . This surface will be called a Minimum Surface if it has the smallest-area in comparison with all the surfaces limited by the curve C .

Let the equation of the surface be taken in the form $z = f(x, y)$.

The area of the surface S is expressed by the double integral

$$\iint \sqrt{1 + p^2 + q^2} \, dx \, dy.$$

From a problem in the calculus of Variations, we get the condition that this integral should be a minimum to be,

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right) - \frac{\partial F}{\partial z} = 0$$

where $f = \sqrt{1+p^2+q^2}$

$$\text{hence } \frac{\partial f}{\partial z} = 0$$

and the condition becomes

$$(1) \frac{\partial}{\partial x} \left(\frac{p}{\sqrt{1+p^2+q^2}} \right) + \frac{\partial}{\partial y} \left(\frac{q}{\sqrt{1+p^2+q^2}} \right) = 0$$

A minimum Surface passing through a given contour satisfies this differential equation.

Through simple calculations equation (1) reduces to

$$(2). (1+q^2)x - 2pqz + (1+p^2)y = 0$$

which we have seen before characterizes the Minimum Surface.

This is a necessary condition for a Minimum Surface. Any function satisfying this differential equation will give a Minimum Surface.

Maurier in his "Memoire sur la courbure des Surfaces," (1776), was the first to point out the geometrical meaning of this equation (2).

He pointed out that the characteristic quality of the surface S which satisfies the foregoing differential equation was to have in every point, equal but opposite in sign principal radii of curvature. All surfaces fulfilling this requirement are called Minimum Surfaces.

Mensurier also noted after a result already given by Monge, that the equation,

$$(3) \quad q^2 r - 2 p q s + p^2 t = 0$$

is that of surfaces generated by a straight-line parallel to the (x, y) plane. Comparing (3) with (2) one has

$$(4) \quad r + t = 0,$$

which if it exists gives the minimum surfaces generated by a straight line parallel to a plane. The integration of (3) and (4), lead to the gauche helicoid with plane director.

or the screw surface.

Meusnier also determined the minimum surface of revolution and demonstrated that it is generated by the rotation of a catenary about its base.

The minimum surface of revolution or catenoid will be obtained direct by taking the equation of the surface in the form

$$z = f(x^2 + y^2)$$

and substituting in equation (2)

$$(1+q^2)r - 2pq\delta + (1+p^2)t = 0.$$

We will take for our surface

$$z = f(\sqrt{x^2 + y^2})$$

and proceed to compute the values for p, q, r, δ, t .

$$p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} f'$$

$$q = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} f'$$

36.

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{x^2}{x^2+y^2} f''$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{xy}{x^2+y^2} f''$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{y^2}{x^2+y^2} f''$$

Making the substitutions the differential equation becomes,

$$\begin{aligned} & \left[1 + \frac{x^2}{x^2+y^2} f'^2\right] \left[\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{y^2}{x^2+y^2} f''\right] \\ & - \frac{2xy}{x^2+y^2} f'^2 \left[-\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{xy}{x^2+y^2} f''\right] \\ & + \left[1 + \frac{y^2}{x^2+y^2} f'^2\right] \left[\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} f' + \frac{x^2}{x^2+y^2} f''\right] = 0 \end{aligned}$$

$$\frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} f' + f'' + \frac{(x^2+y^2)^2}{(x^2+y^2)^{\frac{5}{2}}} f'^3 = 0$$

$$(5) f' + \sqrt{x^2+y^2} f'' + f'^3 = 0$$

$$\text{Let } \sqrt{x^2+y^2} = w$$

$$\frac{\partial f}{\partial w} + w \frac{\partial^2 f}{\partial w^2} + \left(\frac{\partial f}{\partial w}\right)^3 = 0$$

$$\text{Let- } \frac{\partial f}{\partial w} = w$$

$$v + w \frac{dv}{dw} + v^3 = 0$$

$$(6) \quad \frac{dw}{w} + \frac{dv}{v+v^3} = 0$$

$$\log w + \frac{1}{2} \log \frac{v^2}{1+v^2} = c_2$$

$$v = \frac{c}{\sqrt{w^2 - c^2}}$$

$$\text{but } v = \frac{\partial f}{\partial w}$$

$$\therefore \text{whence } f = \int \frac{c dw}{\sqrt{w^2 - c^2}}$$

(7) $f = \log(w + \sqrt{w^2 - c^2})$,
 which is the form of the
 Calenary curve, and will give
 the minimum surface of revolution
 by revolving it about its base.

The screw surface will be obtained directly by taking the surface in the form $z = f\left(\frac{y}{x}\right)$

$$p = \frac{\partial z}{\partial y} = \frac{1}{x} f'\left(\frac{y}{x}\right)$$

$$q = \frac{\partial z}{\partial x} = -\frac{y}{x^2} f'\left(\frac{y}{x}\right)$$

$$r = \frac{\partial^2 z}{\partial x^2} = -\frac{2y}{x^3} f'\left(\frac{y}{x}\right) + \frac{y^2}{x^4} f''\left(\frac{y}{x}\right)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x^2} f'\left(\frac{y}{x}\right) + \frac{1}{x} \left(-\frac{y}{x^2}\right) f''\left(\frac{y}{x}\right)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{1}{x^2} f''\left(\frac{y}{x}\right)$$

whence $(1+q^2)r - 2pq s + (1+p^2)t = 0$
becomes

$$\frac{y}{x^3} \left[1 + \left(\frac{1}{x^2} f'^2\right)\right] \left[2f' + \frac{y}{x} f''\right] +$$

$$\frac{2y}{x^3} f'^2 \left[-f' - \frac{y}{x} f''\right] + \left[1 + \frac{y^2}{x^4} f'^2\right] \left[\frac{1}{x^2} f''\right] = 0$$

$$y(y^2 + f'^2)(2xf' + yf'') + 2yf'^2(-xf' - yf'') + (x^4 + y^2 f'^2) f'' = 0$$

or,

$$(8) \quad x^4 f'' + 2x^3 y' f' + x^2 y^2 f'' = 0$$

a differential equation which can be solved.

$$(9) \quad f'' + 2 \frac{y}{x} f' + \frac{y^2}{x^2} f'' = 0$$

make the substitution $\frac{y}{x} = w$, then (9) takes the form,

$$\frac{\partial^2 f}{\partial w^2} + 2w \frac{\partial f}{\partial w} + w^2 \frac{\partial^2 f}{\partial w^2} = 0$$

make still further the substitution $\frac{\partial f}{\partial w} = v$,

$$(1+w^2) \frac{\partial v}{\partial w} + 2wv = 0$$

$$\frac{\partial v}{v} + \frac{2w}{1+w^2} dw = 0$$

$$\log v + \log(1+w^2) = c$$

$$v(1+w^2) = K$$

$$\text{but } v = \frac{\partial f}{\partial w}$$

$$df = \frac{K dw}{1+w^2},$$

$$f = h + K \tan^{-1} w$$

$$\therefore f = h + K \tan^{-1} \frac{y}{x}$$

thus the equation of the sur-

face is given in the form $z = f(x, y)$
 we get the screw surface $z = A + B \tan^{-1} \frac{y}{x}$
 satisfying the differential equation
 of the minimum surfaces
 and when the surface is given
 by the equations

$$\begin{cases} x = r \cos \omega \\ y = r \sin \omega \\ z = a \omega \end{cases}$$

$$\tan^{-1} \frac{y}{x} = \tan^{-1} \tan \omega = \omega.$$

Whence $z = A + B \omega$.

Monge in 1784 gave a complete solution to equation (2) the characteristic equation of the minimum surfaces. In Crell's journal, there appeared in 1834 a work by Scherk which added important results to the theory of minimum surfaces and contained the first examples of minimum

deduced from the integrals of Monge and Legendre.

The Translation surface due to Scherk,

$$z = \frac{1}{a} [\log \cos(ax) - \log \cos(ay)]$$

may be obtained directly when the surface is taken in the form

$$z = f(x) + \phi(y)$$

$$p = \frac{\partial z}{\partial x} = f'(x), \quad q = \frac{\partial z}{\partial y} = \phi'(y), \quad r = \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x), \quad t = \frac{\partial^2 z}{\partial y^2} = \phi''(y),$$

then equation (2) becomes,

$$(1 + \phi'^2 y) f'(x) + (1 + f'^2 x) \phi''(y) = 0$$

$$\frac{\phi'' y}{1 + \phi'^2 y} = - \frac{f''(x)}{1 + f'^2 x}$$

In the solution $z = f(x) + \phi(y)$, $f(x)$ is a function of x alone; $\phi(y)$ a function of y alone. Thus in the above equation the variables are easily separated

and the right hand side is a function of x only, and the left member is a function of y only, whence we can equate each member to a constant.

$$\frac{\phi''(y)}{1+\phi'(y)^2} = a, \text{ and } \frac{f''(x)}{1+f'(x)^2} = -a$$

Integrating

$$\frac{1}{a} \tan^{-1} \phi'(y) = y,$$

$$\phi'(y) = \tan ay,$$

$$\phi(y) = -\frac{1}{a} \log \cos ay$$

By similar process,

$$f(x) = \frac{1}{a} \log \cos ax$$

Therefore for the complete solution we get-

$$(11) \quad z = \frac{1}{a} [\log \cos(ax) - \log \cos(ay)]$$

which is the translation surface.

In 1853-'60, Bonnet in his paper in the "Comptes rendus de l'Acad." indicated a new system of formulae relative to the general theory of surfaces.

The variables chosen by Bonnet are the following, given a surface (S) and a point (P) on that surface, the tangent plane is determined by its spherical representation p on the sphere of radius unity and by its distance to the centre of that sphere, with regard to the point p of the sphere, it is determined by its longitude φ and its latitude θ .

Bonnet's papers in the "Comptes rendus" gave a decided progress to the theory of Minimum Surfaces. The integral equation is given a new form from which can be obtained all the real sur-

faces and an unlimited number of algebraic surfaces. The complete solution of the following problem is also given.

"To determine the minimum surface passing through any curve and admitting in each point of that curve a given tangent-plane."

In 1866 the important work of Weierstrass appeared. He employs the formulae of Monge and treats them in a new way and obtains elegant results. He gave the way of finding all the minimum surfaces real and algebraic.* He treats the problem of Plateau and obtains important results. This remarkable problem is also treated by Riemann and is discussed fully in the collected works of Schwarz.

* See page

Surfaces of Revolution

Let us consider the surfaces of revolution and suppose that the z -axis is taken as the axis of the surfaces.

The Surface has for its equation

$$z = f(\sqrt{x^2 + y^2})$$

This can be represented parametrically by

$$(12) \quad \begin{cases} x = r \cos \omega \\ y = r \sin \omega \\ z = f(r) \end{cases}$$

r and ω being the polar coordinates of the projection of a point (x, y, z) on the plane of (x, y) .

The square of the element of length,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\omega^2 + f'(r)^2 dr^2 \end{aligned}$$

$$(13) \quad ds^2 = dr^2 [1 + f'(r)^2] + r^2 d\omega^2$$

Where $E = 1 + f'(r)^2$, $F = 0$, $G = r^2$.

The term in $dr dw$ vanishes, the curves $r = \text{const.}$ are the parallels, the curves $w = \text{const.}$ are the meridians, and they cut each other at right-angles.

— To find all the Minimum Surfaces of Revolution. —

The equations of a surface of revolution are,

$$\begin{cases} x = r \cos w \\ y = r \sin w \\ z = f(r) \end{cases}$$

The values of E, F, G , are $1 + f'^2(r)$, $0, r^2$ respectively; the proportional values of L, M, N , are $r f''(r)$, $0, r^2 f(r)$.

The Minimum Surface is given by

$$(14) \quad 2FM - EN - GL = 0$$

$$(1 + f'^2) r - 2 f g' r + (1 + g'^2) r = 0$$

[page 26 - (31)]

It takes this form,

$$(15) [1 + f'(r)^2] r^2 f'(r) + r^3 f''(r) = 0,$$

a differential equation where the unknown $f(r)$ may be found.

Separating the variables,

$$\frac{f''}{f'(1+f'^2)} = -\frac{1}{r},$$

Integrating—

$$\log f' - \frac{1}{2} \log(1 + f'^2) = -\log r + \log c$$

that is,
$$\frac{f'}{(1+f'^2)^{\frac{1}{2}}} = \frac{c}{r},$$

whence

$$(16) f' = c \log(r + \sqrt{r^2 - c^2}) + c'$$

The meridian in $\rho \times \phi$ plane is the curve $z = f(x)$

that is $z = c \log(x + \sqrt{x^2 - c^2}) + c'$

The value of the constant c' does not influence the form of the Sur-

face of revolution. Let $-c' = -c \log c$
 then the meridian takes the form,

$$(17) \quad z = c \log \frac{x + \sqrt{x^2 - c^2}}{c}.$$

$$x + \sqrt{x^2 - c^2} = c e^{\frac{z}{c}}$$

$$(18) \quad x = \frac{c}{2} (e^{\frac{z}{c}} + e^{-\frac{z}{c}}),$$

This curve is a catenary whose base is the z axis.

$\therefore$ The Minimum Surface of Revolution is generated by a catenary turning around its base.

As the catenary is the only curve whose radius of curvature is equal and of contrary sign to its normal the catenoid is the only surface of revolution for which the principal radii of curvature are equal but of contrary sign. By our previous definition of Minimum Surfaces this is a necessary condition for a Minimum Surface.

Let us briefly consider the catenary.

If an inelastic string of uniform density be allowed to hang freely from two fixed points the curve which it assumes is called the catenary. Its intrinsic equation is $s = a \tan \phi$.

Its equation in cartesian coordinates can be easily obtained. On the vertical through V take $OV = a$, draw Ox in a horizontal direction and assume Ox, Oy as the coordinate axes.

Let $PN = y$, $ON = x$

$$\frac{dy}{ds} = \tan \phi$$

$$\frac{dy}{ds} = \sin \phi$$

$$\frac{dx}{ds} = \cos \phi$$

$$\therefore \frac{dy}{d\phi} = \frac{dy}{ds} \cdot \frac{ds}{d\phi} = a \frac{\sin \phi}{\cos \phi}$$

$$\frac{dy}{d\phi} = \frac{a}{\cos \phi}$$

$y = a \sec \phi$, $x = a \log (\sec \phi + \tan \phi)$
no constant is added to either
integral, since $y = a$, $x = 0$, when
 $\phi = 0$.

$$\sec \phi + \tan \phi = e^{\frac{x}{a}}$$

$$\sec \phi - \tan \phi = \frac{1}{\sec \phi + \tan \phi} = e^{-\frac{x}{a}}$$

Hence we have

$$2 \sec \phi = e^{\frac{x}{a}} + e^{-\frac{x}{a}}$$

$$(19) \quad y = \frac{a}{2} (e^{\frac{x}{a}} + e^{-\frac{x}{a}})$$

$$(20) \quad s = \frac{a}{2} (e^{\frac{x}{a}} - e^{-\frac{x}{a}})$$

$$(21) \quad 2 \tan \phi = e^{\frac{x}{a}} - e^{-\frac{x}{a}}$$

$$y^2 - s^2 = a^2$$

$$s^2 = y^2 - a^2$$

$$(22) \quad \frac{y}{a} = \cosh \frac{x}{a}, \quad \frac{s}{a} = \sinh \frac{x}{a}$$

In order to plot the curve with most ease, put it in the logarithmic form.

Take the equation,

$$(23) \quad y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$$

$$\frac{2y}{a} = e^{\frac{x}{a}} + e^{-\frac{x}{a}}$$

$$\frac{2y}{a} e^{\frac{x}{a}} = e^{\frac{2x}{a}} + 1$$

$$e^{\frac{2x}{a}} - \frac{2y}{a} e^{\frac{x}{a}} + \frac{y^2}{a^2} = \frac{y^2 - a^2}{a^2}$$

$$e^{\frac{x}{a}} - \frac{y}{a} = \pm \frac{\sqrt{y^2 - a^2}}{a}$$

$$(24) \quad e^{\frac{x}{a}} = \frac{y}{a} \pm \frac{\sqrt{y^2 - a^2}}{a}$$

$$(25) \quad x = a \log_e \left[y \pm \sqrt{y^2 - a^2} \right]$$

To find the radius of curvature of the catenary,

$$y = \frac{a}{2} (e^{\frac{x}{a}} + e^{-\frac{x}{a}})$$

$$\frac{dy}{dx} = \frac{1}{2} (e^{\frac{x}{a}} - e^{-\frac{x}{a}})$$

$$\frac{d^2y}{dx^2} = \frac{y}{a^2}$$

whence we find the radius of curvature to be $-\frac{y^2}{a}$.

The length (n) of the normal upon the x -axis is $n = y\sqrt{1+y'^2}$ which is equal to $\frac{y^2}{a}$.

Hence the radius of curvature is equal to the part of the normal intercepted by the axis of x but measured in the opposite direction.

Let us now find the lines of curvature for a surface of revolution. The surface we have taken in the form

$$\begin{cases} x = r \cos w \\ y = r \sin w \\ z = f(r) \end{cases}$$

The differential equation of the lines of curvature,

$$\begin{vmatrix} L & M & N \\ E & F & G \\ \left(\frac{dw}{dr}\right)^2 - \frac{dw}{dr} & 1 & \end{vmatrix} = 0, \quad [\text{page 27.}]$$

becomes

$$(26) \quad \begin{vmatrix} 1 + f'(r)^2 & 0 & r^2 \\ r f'(r) & 0 & r^2 f'(r) \\ dw^2 - dr dw & dr^2 & \end{vmatrix} = 0$$

or,

$$r^2 dr dw \{ f'(r) [1 + f'(r)^2] - r f''(r) \} = 0$$

In general the coefficient of $dr dw$ is not zero then the differential equation reduces itself to $dr dw = 0$

which gives for the lines of curvature, the meridians $w = \text{const.}$ and the parallels $r = \text{const.}$

Asymptotic Lines.

The differential equation of the asymptotic lines we have seen is

$$L + 2M\lambda + N\lambda^2 = 0. \quad [\text{page 30}]$$

where λ is in this case $\frac{\partial w}{\partial r}$.

The proportional values of L, M, N have been found to be $rf''(r), 0, r^2f'(r)$. Thus the equation takes this form,

$$(28) \quad rf''(r) + r^2f'(r)\left(\frac{\partial w}{\partial r}\right)^2 = 0$$

Solve for $\frac{\partial w}{\partial r}$,

$$\frac{\partial w}{\partial r} = \pm \sqrt{\frac{-f'(r)}{rf''(r)}}.$$

$$(29) \quad w + \text{const.} = \int \pm \sqrt{\frac{-f'(r)}{rf''(r)}} dr,$$

which is the equation of the asymptotic lines for the surface of revolution. If the indicatrix is an ellipse the asymptotic lines will not exist.

- Helicoid -

The general helicoid is the surface generated by curves in the xy plane $y = f(x)$ which turn around Oz and is pushed up the z -axis. One can represent the helicoid parametrically by the same equations as the surface of revolution, by only adding to z a term aw , proportional to the angle of rotation. Its equations are $x = r \cos w$, $y = r \sin w$, $z = f(r) + aw$.

Calculating the values of E F G we find them to be respectively, $1 + f'(r)^2$, $af(r)$, $a^2 + r^2$.

$$ds^2 = [1 + f'(r)^2] dr^2 + 2drdw af(r) + dw^2(a^2 + r^2)$$

Let us compute the values of L, M, N

$$(30) L = \frac{1}{H} \begin{vmatrix} 0 & 0 & f''r \\ \cos w & \sin w & f(r) \\ r \sin w & r \cos w & a \end{vmatrix} = \frac{r}{H} f''(r),$$

$$(31) M = \frac{1}{H} \begin{vmatrix} -\sin w & \cos w & 0 \\ \cos w & \sin w & f(r) \\ -r \cos w & r \sin w & a \end{vmatrix} = \frac{a}{H}$$

$$(32) N = \frac{1}{H} \begin{vmatrix} -r \cos w & -r \sin w & 0 \\ \cos w & \sin w & f(r) \\ -r \sin w & r \cos w & a \end{vmatrix} = \frac{1}{H} r^2 f'(r),$$

Substituting these values in the differential equation of the asymptotic lines,

$$L + 2M \frac{dw}{dr} + N \left(\frac{\partial w}{\partial r} \right)^2 = 0$$

we get -

$$(33) r f''(r) + 2a \frac{dw}{dr} + r^2 f'(r) \left(\frac{\partial w}{\partial r} \right)^2 = 0$$

Solve for $\frac{dw}{dr}$ and integrate,

$$(34) w + \text{const} = \int a \pm \sqrt{a^2 - r^3 f'(r) f(r)} \frac{dr}{r^2 f'(r)},$$

The Equation of the Asymptotic lines, solved on the assumption that $f(r)$ is not zero

- lines of curvature for the helicoid. The differential equation of the lines of curvature becomes

$$(35) \quad \begin{vmatrix} 1 + f'(r)^2 & a f'(r) & a^2 + r^2 \\ r f'' & w & r^2 f' \\ \left(\frac{\partial w}{\partial r}\right)^2 & -\frac{\partial w}{\partial r} & 1 \end{vmatrix} = 0$$

The minimum screw surface, a special case of the helicoid is obtained by taking for the generating curve in the z or plane a normal to Oz . Let us take for the generating line $z=0$, then $f(r)=0$, and our surface has for its equations,

$$x = r \cos w$$

$$y = r \sin w$$

$$z = a w$$

For this surface $L=0$, $N=0$, $M=\frac{a}{H}$.

$E=1, F=0, G=a^2+r^2$,
whence we have for the square of
the element of length

$$ds^2 = dr^2 + (r^2 + a^2) dw^2.$$

The term in $dr dw$ vanishes, then
the curves $w = \text{const.}$ which are the
rectilinear generators, and the
curves $r = \text{const.}$, which are the cir-
cular helices are orthogonal.

Asymptotic lines.

Substituting in the differential equa-
tion of the Asymptotic lines, the values
of L, M, N , it becomes

$$2adr dw = 0,$$

which we have seen above has
two solutions, giving the generators
 $r = \text{const.}$ and the helices $w = \text{const.}$
These two systems then furnish the
Asymptotic lines.

Lines of Curvature. — Substituting
in the differential Equation of the
lines of curvature,

$$\begin{vmatrix} E & F & G \\ L & M & N \\ \left(\frac{dw}{dr}\right)^2 - \frac{dw}{dr} & 1 & \end{vmatrix} = 0$$

the values of E, F, G, L, M, N it reduces to the following,

$$(36) \begin{vmatrix} 1 & 0 & a^2 + r^2 \\ 0 & -a & 0 \\ dw^2 - drdw & dr^2 \end{vmatrix} = 0$$

whence,

$$(37) \quad dr^2 = (r^2 + a^2) dw^2$$

Separating the variables,

$$\frac{dr}{\sqrt{r^2 + a^2}} = \pm dw$$

Integrating we get -

$$\log(r + \sqrt{r^2 + a^2}) = \pm w + \log c$$

$$(r + \sqrt{r^2 + a^2}) = c e^{\pm w}$$

whence,

$$(38) \quad 2r = c e^{\pm w} - \frac{a^2}{c} e^{\mp w}.$$

By giving different values to w and through a proper choice of the arbitrary const c , the lines of curvature may be easily computed.

Let us develop some of the properties of the circular helix, which we saw was an asymptotic line on the screw surface. The coordinates of a point of the curve are given by the equations,

$$(38) \quad x = r \cos w, \quad y = r \sin w, \quad z = aw$$

whence,

$$x' = -r \sin w, \quad y' = r \cos w, \quad z' = a$$

$$ds = \sqrt{r^2 + a^2} dw$$

integrating

$$s = \sqrt{a^2 + r^2} w$$

$$s^2 = a^2 w^2 + r^2 w^2$$

The direction cosines of the tangent are

$$\alpha = \frac{dx}{ds} = -\frac{r \sin w}{\sqrt{r^2 + a^2}}$$

$$\beta = \frac{dy}{ds} = \frac{r \cos w}{\sqrt{r^2 + a^2}}$$

$$\gamma = \frac{dz}{ds} = \frac{a}{\sqrt{r^2 + a^2}}$$

The direction cosine γ being a const. the tangent of the helix makes a constant angle with the generators.

Radius of curvature —

Take the formulae for radius of curvature of a curve,

$$\frac{1}{R^2} = \left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 + \left(\frac{d\gamma}{ds}\right)^2$$

$$\frac{d\alpha}{ds} = -\frac{r \cos w}{r^2 + a^2}, \quad \frac{d\beta}{ds} = -\frac{r \sin w}{r^2 + a^2},$$

$$\frac{d\gamma}{ds} = 0, \quad \text{whence}$$

$$\frac{1}{R^2} = \frac{r^2}{(r^2 + a^2)^2}, \quad R = \frac{a^2 + r^2}{r}.$$

$$\alpha' = -\cos w, \quad \beta' = -\sin w, \quad \gamma' = 0$$

The cosines of the principal normal MN' are equal and of contrary sign to those of the radius OP .

Torsion.

$$\frac{1}{T^2} + \frac{1}{R^2} = \left(\frac{d\alpha'}{ds}\right)^2 + \left(\frac{d\beta'}{ds}\right)^2 + \left(\frac{d\gamma'}{ds}\right)^2$$

$$\frac{1}{T^2} + \frac{1}{R^2} = \frac{1}{r^2 + a^2}$$

$$(41) \quad \frac{1}{T^2} = \frac{a^2}{(r^2 + a^2)^2}, \quad T = \frac{r^2 + a^2}{a}.$$

Thus in a circular helix the radius of curvature and torsion are constant.

Osculating Plane — The equation of the osculating plane to the helix at the point x, y, z can be put in the form,

(42) $A(X-x) + B(Y-y) + C(Z-z) = 0$,
where the coefficients A, B, C have these values,

$$A = yz'' - z'y'' = ra \sin w$$

$$B = z'x'' - x'z'' = -ra \cos w$$

$$C = xy'' - y'x'' = a^2.$$

(42) takes this form

$$(X-x)ra \sin w - (Y-y)ra \cos w + (Z-z)a^2 = 0$$

Replace $r \sin w$, $r \cos w$ by the coordinates x, y at the point of contact then

$$(X-x)ry - (Y-y)rx + (Z-z)r^2 = 0$$

Such is the equation of the osculating plane to the helix at the point (x, y, z)

We are now in a position to discuss the relation between the Minimum Surface of revolution, the Catenoid, and the Minimum screw surface. We will first discuss the general case.

Theorem of Bour — "Every helicoid is applicable on a surface of revolution."

For two surfaces to be applicable upon each other, it is necessary that the corresponding arcs or elements of length be equal.

Let the helicoid S , be given by the equations, $x = r \cos w$, $y = r \sin w$, $z = f(r) + aw$.

The square of the element of length is

$$(43) ds^2 = [1 + f'(r)^2] dr^2 + 2af(r) dr dw + (r^2 + a^2) dw^2$$

where $E = 1 + f'(r)^2$, $F = af(r)$, $G = r^2 + a^2$.

Suppose the Surface of revolution S is defined by the equations,

$$(44) \begin{cases} x = u \cos \theta \\ y = u \sin \theta \\ z = \phi u \end{cases}$$

$$(45) \quad ds_1^2 = (r^2 + a^2) \left[dw^2 + \frac{a f'(r)}{r^2 + a^2} dr \right]^2 + \frac{a^2 + r^2 (1 + f'^2)}{r^2 + a^2}$$

we will have to establish a correspondence between $\mathcal{D}$ and $\mathcal{S}$, defined by the equations,

$$(46) \quad \begin{cases} \theta = w + a \int \frac{f'(r)}{r^2 + a^2} dr, & u = \sqrt{r^2 + a^2} \\ r = \sqrt{u^2 - a^2}, & dr = \frac{u du}{\sqrt{u^2 - a^2}} \end{cases}$$

Replace r and dr by their values and (45) becomes,

$$(47) \quad ds_1^2 = u^2 d\theta^2 + du^2 (a^2 + u^2 - a^2) \left[1 + f'^2 \sqrt{u^2 - a^2} \right] \\ = u^2 d\theta^2 + du^2 \left[\frac{u^2 + (u^2 - a^2) f'^2 \sqrt{u^2 - a^2}}{u^2 - a^2} \right].$$

$ds_1^2 \equiv ds^2$, if one is able to choose the unknown function $\phi(u)$ in such a way to verify the relation,

$$(48) \quad 1 + \phi'^2(u) = \frac{u^2 + (u^2 - a^2) f'^2 \sqrt{u^2 - a^2}}{u^2 - a^2}$$

$$\phi'^2(u) = \frac{a^2 + (u^2 - a^2) f'^2 \sqrt{u^2 - a^2}}{u^2 - a^2}.$$

It is sufficient to take

$$(49) \quad \phi(u) = \int \frac{\sqrt{a^2 + (u^2 - a^2) f'^2 \sqrt{u^2 - a^2}}}{u^2 - a^2}.$$

The theorem is then established. The surface of revolution is generated by revolving the curve $z = \varphi(x)$ around the z -axis.

The particular case is the one that interests us. For the minimum screw surface one has $f(x) = 0$. It is then applicable on the surface of revolution, around the z -axis which has for its meridian in the zox plane the curve $z = \varphi(x)$.

The function $\varphi(x)$ is defined thus

$$(50) \quad \varphi(x) = \int \frac{a^2}{(x^2 - a^2)} dx$$

That curve is $z = a \log(x + \sqrt{x^2 - a^2}) + c$, a catenary which has oz for its base. Then the screw surface is applicable on a surface of revolution generated by a catenary turning around its base.

If we write the two surfaces parametrically,

$$\begin{cases} (S_1); & x = r \cos \omega, y = r \sin \omega, z = a\omega \\ (S); & x = u \cos \theta, y = u \sin \theta, z = a \log(u + \sqrt{u^2 - a^2}). \end{cases}$$

the point (ω, r) of the helicoid corresponds to the point (θ, u) of the surface of revolution and are connected by the relations,

$$5) \quad \begin{cases} \theta = \omega \\ u = \sqrt{r^2 + a^2} \end{cases}$$

Then the rectilinear generators $\omega = \text{const.}$ of the screw surface surface are applicable on the meridians $\theta = \text{const.}$ of the surface of revolution and the helices $r = \text{const.}$ of the screw surface are applicable on the parallels $u = \text{const.}$ of the surface of revolution.

We have shown before that the meridians and parallels of the surface of revolution were its lines of curvature, and the generators and helices of the screw surface were its asymptotic lines, therefore the asymptotic lines of the screw surface go over into the lines of curvature of the surface of revolution.

Chapter II

Formulae of Weierstrass.

These formulae establish the relation which exists between the theory of functions of an imaginary variable and that of minimum surfaces since they show to every function of an imaginary argument - correspond a real minimum surface.

Let u and v be an isothermal system on the sphere and the square of the element-length becomes,

$$(1) \quad ds'^2 = \frac{1}{\kappa} (du^2 + dv^2)$$

The coordinates x, y, z , of a point on the sphere as functions of u and v are known. The coordinates

∴ a corresponding point on the minimum surface are determined from these Equations

$$(2) \begin{cases} dx = r_2 \left(\frac{\partial X}{\partial u} du - \frac{\partial X}{\partial v} dv \right) \\ dy = r_2 \left(\frac{\partial Y}{\partial u} du - \frac{\partial Y}{\partial v} dv \right) \\ dz = r_2 \left(\frac{\partial Z}{\partial u} du - \frac{\partial Z}{\partial v} dv \right) \end{cases}$$

The element of length of the surface δ is

$$(3) \quad ds^2 = r_2^2 (du^2 + dv^2)$$

The principal radii of curvature of δ are r_1, r_2 ; $r_1 = -r_2$.

Draw on the sphere the meridians and parallels and place

$$(4) \begin{cases} X = \sin \theta \cos \omega \\ Y = \sin \theta \sin \omega \\ Z = \cos \theta \end{cases}$$

Let us introduce the complex variable (τ) on the sphere

$$\tau = \cot \frac{\theta}{2} e^{i\omega}$$

and its conjugate

$$(\tau_0) = \cot \frac{\theta}{2} e^{-i\omega}$$

X, Y, Z can be expressed in terms of τ and τ_0 .

$$\begin{aligned} X &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \omega \\ &= 2 \frac{\cot \frac{\theta}{2}}{\csc^2 \frac{\theta}{2}} \cos \omega \end{aligned}$$

$$\cos \omega = \frac{e^{i\omega} + e^{-i\omega}}{2}$$

$$\begin{aligned} \tau + \tau_0 &= \cot \frac{\theta}{2} (e^{i\omega} + e^{-i\omega}) \\ &= 2 \cot \frac{\theta}{2} \cos \omega \end{aligned}$$

$$\tau \tau_0 = \cot^2 \frac{\theta}{2}$$

$$\tau \tau_0 + 1 = \cot^2 \frac{\theta}{2} + 1$$

$$\therefore X = \frac{\tau + \tau_0}{\tau \tau_0 + 1}$$

$$\begin{aligned} Y &= \sin \theta \sin \omega \\ &= \frac{2 \cot \frac{\theta}{2}}{\cot^2 \frac{\theta}{2} + 1} \sin \omega \end{aligned}$$

$$\begin{aligned}\tau - \tau_0 &= \cot \frac{\theta}{2} (e^{i\omega} - e^{-i\omega}) \\ &= \frac{1}{i} \cot \frac{\theta}{2} \sin \omega\end{aligned}$$

$$\therefore y = \frac{1}{i} \frac{(\tau - \tau_0)}{(\tau \tau_0 + 1)}$$

$$Z = \cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = (\cot^2 \frac{\theta}{2} - 1) \sin^2 \frac{\theta}{2}$$

$$\therefore Z = \frac{\tau \tau_0 - 1}{\tau \tau_0 + 1}$$

$$\text{since } \left\{ \begin{array}{l} \tau \tau_0 - 1 = \cot^2 \frac{\theta}{2} - 1 \\ \tau \tau_0 + 1 = \csc^2 \frac{\theta}{2} \end{array} \right\}$$

For the square of the linear element- ds' we have

$$(5) \quad ds'^2 = \frac{4 d\tau d\tau_0}{(\tau \tau_0 + 1)^2}$$

Take the complex variable

$$\sigma = u + iv,$$

as a function of τ , σ_0 a function of its conjugate τ_0 .

Equation (1) may be written

$$ds'^2 = \frac{1}{4} \frac{d\sigma}{d\tau} \frac{d\sigma_0}{d\tau_0} d\tau d\tau_0,$$

which gives with equation (5)

$$h_2 = \frac{(\tau \tau_0 + 1)^2}{4} \frac{\partial \sigma}{\partial \tau} \frac{\partial \sigma_0}{\partial \tau_0}.$$

Equation (3) becomes,

$$(6) \quad ds^2 = \frac{(\tau \tau_0 + 1)^2}{4} \left(\frac{d\sigma}{\partial \tau} \right)^2 \left(\frac{\partial \sigma_0}{\partial \tau_0} \right)^2 d\tau d\tau_0.$$

We will substitute in equations (2) (τ and τ_0), by taking into consideration that for an arbitrary function $\phi(u, v)$ as function of (τ and τ_0) the following relations hold,

$$(7) \quad \begin{cases} \frac{d\sigma}{\partial \tau} \frac{\partial \sigma_0}{\partial \tau_0} \left(\frac{1}{2} \frac{\partial \phi}{\partial u} + \frac{i}{2} \frac{\partial \phi}{\partial v} \right) = \frac{d\sigma}{\partial \tau} \cdot \frac{\partial \phi}{\partial \tau_0} \\ \frac{d\sigma}{\partial \tau} \frac{\partial \sigma_0}{\partial \tau_0} \left(\frac{1}{2} \frac{\partial \phi}{\partial u} - \frac{i}{2} \frac{\partial \phi}{\partial v} \right) = \frac{\partial \sigma_0}{\partial \tau_0} \cdot \frac{\partial \phi}{\partial \tau}; \end{cases}$$

and we obtain

$$(8) \quad \begin{cases} dx = \frac{1}{4} (1 - \tau^2) \left(\frac{d\sigma}{\partial \tau} \right)^2 d\tau + \frac{1}{4} (1 - \tau_0^2) \left(\frac{\partial \sigma_0}{\partial \tau_0} \right)^2 d\tau_0 \\ dy = \frac{1}{4} (1 + \tau^2) \left(\frac{d\sigma}{\partial \tau} \right)^2 d\tau - \frac{i}{4} (1 + \tau_0^2) \left(\frac{d\sigma_0}{\partial \tau_0} \right)^2 d\tau_0 \\ dz = \frac{\tau}{2} \left(\frac{d\sigma}{\partial \tau} \right)^2 d\tau + \frac{\tau_0}{2} \left(\frac{d\sigma_0}{\partial \tau_0} \right)^2 d\tau_0 \end{cases}$$

$$\begin{aligned}\frac{1}{2} \left(\frac{d\sigma}{d\tau} \right) &= F(\tau) = F(\xi + i\eta) \\ &= F_1(\xi, \eta) + i F_2(\xi, \eta)\end{aligned}$$

$$\begin{aligned}dx &= \frac{1}{2} \left\{ 1 - (\xi + i\eta)^2 (F_1 + i F_2) (d\xi + i d\eta) \right\} \\ &\quad + \frac{1}{2} \left\{ 1 - (\xi - i\eta)^2 (F_1 - i F_2) (d\xi - i d\eta) \right\} \\ &= \frac{1}{2} \left\{ 1 - (\xi^2 - \eta^2) - 2i\xi\eta (F_1 + i F_2) (d\xi + i d\eta) \right\} \\ &\quad + \frac{1}{2} \left\{ 1 - (\xi^2 - \eta^2) + 2i\xi\eta (F_1 - i F_2) (d\xi - i d\eta) \right\}\end{aligned}$$

$$\begin{aligned}dx &= \frac{1}{2} d\xi \left[2F_1 - 2(\xi^2 - \eta^2) F_1 + 4\xi\eta F_2 \right] \\ &\quad + \frac{1}{2} d\eta \left[-2F_2 + 2(\xi^2 - \eta^2) F_2 + 4\xi\eta F_1 \right]\end{aligned}$$

whence,

$$\begin{aligned}x &= \int [1 - (\xi^2 - \eta^2)] F_1 d\xi + \int 2\xi\eta F_2 d\xi \\ &\quad + \int -[1 - (\xi^2 - \eta^2)] F_2 d\eta + \int 2\xi\eta F_1 d\eta\end{aligned}$$

Substituting back, we get for the value of x —

$$x = \Re \int (1 - \tau^2) F(\tau) d\tau$$

$\Re$ being used to indicate that we take the real part of the function. By similar substitution and reduction we can compute the value of y and z .

Thus equations (8) become

$$(9) \quad \begin{cases} x = \Re \int (1 - \tau^2) F(\tau) d\tau \\ y = \Re \int \tau (1 + \tau^2) F(\tau) d\tau \\ z = \Re \int 2\tau F(\tau) d\tau. \end{cases}$$

Such are the formulae of Weierstrass. The square of the linear element and the (positive) principal radius of curvature are given by the following formulae,

$$(10) \quad ds^2 = (\tau\tau_0 + 1)^2 F(\tau) F_0(\tau_0) d\tau d\tau_0.$$

$$(11) \quad r_2 = \frac{(\tau\tau_0 + 1)^2 \sqrt{F(\tau) F_0(\tau_0)}}{2}$$

The lines of curvature, (u, v) are given by the integral $v = \int \sqrt{2F(\tau)} d\tau$, by equating the real and the coefficient of the imaginary part to a const.

Their equations are then,

$$(12) \begin{cases} \Re \int \sqrt{2 F(\tau)} d\tau = \text{constant} \\ \Re \int i \sqrt{2 F(\tau)} d\tau = \text{constant} \end{cases}$$

It follows that the real sheets of the minimum surface are represented by equations (9)

The two real variables on which depend the position of the point are the real part and the imaginary part of the argument τ .

The equations,

$$(13) \begin{cases} x = \Re [(1-\tau^2) \phi''(\tau) + 2\tau \phi'(\tau) - 2\phi(\tau)] \\ y = \Re i [(1+\tau^2) \phi''(\tau) - 2\tau \phi'(\tau) + 2\phi(\tau)] \\ z = \Re [2\tau \phi''(\tau) - 2\phi'(\tau)] \end{cases}$$

represent a real surface and for all functions $\phi(\tau)$ satisfying the relation

$$(14) \quad \phi'''(\tau) = F(\tau),$$

the differentials of the coordinates deduced from formulae (3) are

identical with those which are obtained by means of formulae (9).

If $q(z)$ is an algebraic function of (z) then (13) define an algebraic minimum surface.

Let $w = u + iv = f(x + iy)$, a function of the complex variable $(x + iy)$, if between (x, y) and the real part u of w an algebraic relation exists, so is w an algebraic function of $x + iy$.

In the neighborhood of a point $\begin{Bmatrix} x=0 \\ y=0 \end{Bmatrix}$ let w be expanded in Taylor's series.

$$w = a_0 + ib_0 + (a_1 + ib_1)(x + iy) + (a_2 + ib_2)$$

$$(x + iy)^2 + (a_3 + ib_3)(x + iy)^3 + \dots$$

where a, b are real constants and r is the radius of the circle of convergence.

Whence

$$(15) \quad u = a_0 + \frac{1}{2}(a_1 + ib_1)(x + iy) + \frac{1}{2}(a_2 + ib_2)(x + iy)^2 + \dots \\ + \frac{1}{2}(a_1 - ib_1)(x - iy) + \frac{1}{2}(a_2 - ib_2)(x - iy)^2 + \dots$$

we have

$$(16) \quad Q(u, x, y) = 0,$$

where Q is a rational function of (u, x, y) . Insert for u the value obtained from series (15), and develop according to powers of x and y . The coefficient of each single member is zero and remains zero, when in place of x and y we put $\bar{x}, \bar{y}$ provided the development remains convergent, which will hold if the absolute of $\bar{x}$ and $\bar{y}$ remains smaller than $\frac{r}{2}$.

Set then,

$$\bar{x} = \frac{x + iy}{2}, \quad \bar{y} = \frac{x - iy}{2i}$$

whence (15) becomes,

$$\bar{u} = a_0 + \frac{1}{2}(w - w_0),$$

where $w_0 = a_0 + ib$, thus equations (16) becomes an algebraic relation between w and $x + iy$.

In order that equations (12) may define an algebraic minimum surface there must exist between the quantities

$$\frac{x}{1-z} = \frac{z+z_0}{2}, \quad \frac{y}{1-z} = \frac{z-z_0}{2i}$$

and each of the quantities x, y, z an algebraic relation.

Then each of the three functions

$$f_1(z) = (1-z^2) \varphi''(z) + 2z \varphi'(z) - 2\varphi(z)$$

$$f_2(z) = i(1+z^2) \varphi''(z) - 2iz \varphi'(z) + 2i\varphi(z)$$

$$f_3(z) = 2z \varphi''(z) - 2\varphi'(z).$$

and also

$$\varphi(z) = \frac{1}{4} (z^2+1) f_1(z) - \frac{i}{4} (z^2+1) f_2(z) - \frac{1}{2} z f_3(z)$$

is an algebraic function of (z) .

We have the following proposition,
 "All Algebraic Minimum Surfaces can be obtained from equations (13), if in these equations for $\varphi(z)$ an algebraic function of (z) is inserted."

Chapter III.

Conformal Representation of Minimum Surfaces.

From Weierstrass's formulae we get the square of the element of length to be,

$$(1) \quad ds^2 = (1 + \tau \bar{\tau}_0)^2 F(\tau) \bar{F}_0(\bar{\tau}_0) d\tau d\bar{\tau}_0.$$

we compare this with the linear element of the spherical representation,

$$(2) \quad d\sigma^2 = dc^2 + dc'^2 + dc''^2 = \frac{4d\tau d\bar{\tau}_0}{(1 + \tau \bar{\tau}_0)^2}$$

Comparing these formulae leads to a proposition by Bonnet—

"The spherical representation of a minimum surface gives a conformal representation a geographical outline on the surface of the sphere."

It is easy to demonstrate that quality which characterizes minimum surfaces.

- Every surface other than the sphere for which the angles of two curves is equal to that of their spherical representation

is necessarily a minimum surface

Let us consider a curve C passing through a point P on the surface, and let PT be the tangent at the point P . This curve C has for its spherical representation a curve c , which passes through a point p , the image of P . The tangent to the curve c at the point p , is a straight line perpendicular to the conjugate tangent of PT . It follows that if two curves passing through a point P and at that point having the tangents PT , PT' , the angle of their spherical representation in point p is equal to that of the tangents PT_1 , PT'_1 , conjugates of PT and PT' .

The indicatrix of the sought surface is then a curve such that the angle of any two of its diameters is equal to that of the conjugate diameters.

This property belongs to a circle and to an equilateral hyperbola. In case the indicatrix is a circle we get an umbilic at that point, but the sphere is the only surface, every point of which is an umbilic. In case the indicatrix is an equilateral hyperbola the surface sought is a minimum surface.

Suppose we have given an orthogonal system (3) drawn on a sphere of radius (1). The problem is to find all the surfaces (Σ) such that, if one gets their spherical representation on the sphere the curves of the surface which correspond to those of spherical orthogonal system (5) form a system (5') equally orthogonal.

The problem is susceptible of a double solution. The system 5' being formed from the lines of curvature

of the surface, the question is reduced to the following—

To find the surfaces whose lines of curvature admit for a spherical representation the curves of a given orthogonal system."

Minding published a work in 1852 which threw some light on meridians and parallels applied to surfaces. The meridians are the curves for which the normal to the surface is parallel to a fixed vertical plane.

The parallels are the curves for which the tangent-plane makes a constant angle with the horizontal plane, or in the spherical representation of the surface the meridians and parallels of a surface correspond to the meridians and parallels of a sphere. Minding proposes to find all the surfaces for

which the meridians and parallels form an orthogonal system. Two distinct classes of surfaces satisfying the conditions are obtained.

The one class are the Monge Surfaces, the other are the minimum surfaces. It may be added that for all the minimum surfaces the network formed by the meridians and parallels is always isothermal for it corresponds to an isothermal system on the sphere.

The parallels are defined by $c'' = \text{const.}$ or, $\tau\tau_0 = \text{constant};$

the meridians by the equation $\frac{c}{c'} = \text{const.}$ or $\frac{\tau}{\tau_0} = \text{const.}$

If one replaces τ by $re^{i\omega}$, τ_0 by $re^{-i\omega}$ the square of the element of length has this expression,

$$(3) \quad ds^2 = (1+r^2) F(re^{i\omega}) F_1(re^{-i\omega}) (dr^2 + r^2 d\omega^2)$$

The curves $r = \text{const.}$ are the parallels and the curves $\omega = \text{const.}$ are the meridians of the surface and

the expression for the element of length shows that the family constitute an isothermal system.

In particular if one sets,

$$F(\tau) = A\tau^m, \quad F_0(\tau) = A_0\tau_0^m$$

the element of length becomes

$$(4) \quad ds^2 = AA_0(1+r^2)r^{2m}(dr^2 + r^2dw^2).$$

The surface is applicable on a surface of revolution, the meridians and parallels corresponding respectively on the two surfaces.

Conformal Representation due to Riemann.

Let us make correspond to a point (τ, τ_0) on the surface, a point on the plane whose rectangular coordinates are (u, v) determined by the formulae

$$(5) \quad \begin{cases} u + iv = \int \sqrt{2F(\tau)} d\tau \\ u - iv = \int \sqrt{2F_0(\tau_0)} d\tau_0. \end{cases}$$

The element of length becomes,

$$6 \begin{cases} ds^2 = \frac{1}{2} (1 + \tau \tau_0)^2 \sqrt{P(\tau) F_0(\tau_0)} (du^2 + dv^2) \\ = R (du^2 + dv^2) \end{cases}$$

R being the principal radius of curvature. The element of length of the spherical representation has the same expression,

$$(7) \quad d\sigma^2 = \frac{1}{R} (du^2 + dv^2)$$

The representation of the Minimum surface on the plane constitutes a geographical outline of the surface, on which the lines of curvature are represented by the straight-lines $u = \text{const.}$ and $v = \text{const.}$ parallel to the axes and the asymptotic lines by the straight lines

$$(8) \begin{cases} u + v = \text{const.} \\ u - v = \text{const.} \end{cases}$$

Parallels to the bisectors of the axes

one obtains other propositions by introducing in the formulae the two functions,

$$\alpha = u + iv$$

$$\beta = u - iv$$

one has then, $\tau = f(\alpha) = A$

$$\tau_0 = \varphi(\beta) = B$$

$$F(\tau) = \frac{1}{2A'^2}, \quad F_0(\tau_0) = \frac{1}{2B'^2}$$

A' and B' are the derivatives of the functions A and B .

ds and $d\sigma$, the elements of length are expressed as follows,

$$(9) \quad \begin{cases} ds^2 = \frac{(1 + AB)^2}{4A'B'} d\alpha d\beta \\ d\sigma^2 = \frac{4A'B' d\alpha d\beta}{(1 + AB)^2} \end{cases}$$

Using Weierstrass's formulae and setting in the new values of τ, τ_0 , $F(\tau), F_0(\tau_0)$, we get the following equations—

$$(10) \begin{cases} x = \int \frac{1-A^2}{4A'} d\alpha + \int \frac{1-B^2}{4B'} d\beta \\ y = i \int \frac{1+A^2}{4A'} d\alpha - i \int \frac{1+B^2}{4B'} d\beta \\ z = \int \frac{A}{2A'} d\alpha + \int \frac{B}{2B'} d\beta. \end{cases}$$

which are the forms given by Enneper.

Problem - To find all the minimum surfaces having for spherical representation, the isothermal system composed of two families of circles. The more general of the systems in which the circles of each family pass through two distinct points, correspond with convenient choice of the axes to the following values of A and B .

$$(11) \left\{ A = \sqrt{\frac{1-k}{1+k}} \tan \frac{\alpha}{2}, \quad B = \sqrt{\frac{1-k}{1+k}} \tan \frac{\beta}{2}, \right.$$

k being a real constant less than 1.

The preceding formulae then become

$$(12) \begin{cases} x = \frac{1}{2\sqrt{1-h^2}} [h(\alpha+\beta) + \sin \alpha + \sin \beta] \\ y = \frac{i}{2\sqrt{1-h^2}} [\alpha - \beta + h(\sin \alpha - \sin \beta)] \\ z = -\frac{1}{2} [\cos \alpha + \cos \beta]. \end{cases}$$

The element of length of the surface is determined by the formulae

$$ds^2 = \frac{1}{1-h^2} \left[\cos \frac{\alpha-\beta}{2} + h \cos \frac{\alpha+\beta}{2} \right]^2 d\alpha d\beta$$

set,

$$\alpha = \lambda + i\mu$$

$$\beta = \lambda - i\mu$$

where λ and μ are the parameters of the lines of curvature.

Multiplying by $\sqrt{1-h^2}$ we may express the coordinates as follows,

$$(13) \begin{cases} x = h\lambda + \sin \lambda \cos i\mu \\ y = -\mu + h \sin i\mu \cos \lambda \\ z = \sqrt{1-h^2} \cos \lambda \cos i\mu. \end{cases}$$

The element of length then has the form,

$$(14) \quad ds^2 = (\cos i\mu + h \cos \lambda)^2 (d\lambda^2 + d\mu^2).$$

These formulae were given by Bonnet. They give the catenoid. The section by the plane $x-y$ is composed of an unlimited number of catenaries all equal having their bases parallel to the axis of y . The catenaries are the plane lines of curvature to the surface. The lines of curvature $\mu = \text{const.}$ are the curves analogous to the cycloid and their form may be obtained by projecting a cycloid along on a plane parallel to the base of that curve. They are defined in the plane by

the two equations of the form

$$(15). \begin{cases} x = h\lambda + a \sin \lambda \\ y = \sqrt{a^2 - h^2} \cos \lambda \end{cases}$$

where λ is the variable parameter. On any surface the plane lines of curvature are characterized by having for a spherical representation a small circle of the sphere.

For obtaining all the minimum surfaces with plane lines of curvature in two systems it will suffice to associate with the preceding surface that which has for spherical representation of the lines of curvature the particular system of circles in which the circles of each family are tangent at ^{the} same point and have for their stereographic projection two systems of rectangular straight lines. The corresponding surface which is a limiting case

of the preceding was studied by Enneper. In that case it will be necessary to put $A=4$, $B=\beta$, or what is the same thing, as replacing in the formulae of Weierstrass $F(\tau)$, $F_0(\tau_0)$ by some constant.

Take the constant $= 3$

$$F(\tau) = 3, \quad F_0(\tau_0) = 3.$$

$$(16) \quad \begin{cases} x = \Re(3\tau - \tau^3) \\ y = \Re i(3\tau + \tau^3) \\ z = \Re 3\tau^2 \end{cases}$$

$$\text{set } \tau = \alpha - i\beta$$

α and β are the parameters of the lines of curvature.

By developing the expression of the coordinates one has,

$$(17) \quad \begin{cases} x = 3\alpha + 3\alpha\beta^2 - \alpha^3 \\ y = 3\beta + 3\alpha^2\beta - \beta^3 \\ z = 3\alpha^2 - 3\beta^2 \end{cases}$$

The surface is algebraic. In the geographical outline on the plane the plane sections are represented

by curves of third degree, each of which has a common point, consequently the surface is of the ninth degree.

The planes of the lines of curvature have respectively for their equations -

$$(18) \begin{cases} x + \alpha z - 3\alpha - 2\alpha^3 = 0 \\ y - \beta z - 3\beta - 2\beta^3 = 0 \end{cases}$$

The calculation of the square of the element of length gives

$$(19) ds^2 = 9(1 + \alpha^2 + \beta^2) [d\alpha^2 + d\beta^2].$$

putting $\beta = \text{constant}$,

$$(20) ds = 3(1 + \alpha^2 + \beta^2) d\alpha,$$

integrating,

$$s = 3(1 + \beta^2)\alpha + \alpha^3$$

The equation of the tangent plane at a point (α, β) takes the form,

$$(21) 2\alpha x - 2\beta y + (\alpha^2 + \beta^2 - 1)z + 3\beta^2 - 3\alpha^2 + \beta^4 - \alpha^4 = 0$$

It may be written thus,

$$(12) \quad (x-4\alpha)^2 + y^2 + (z-2\alpha^2+1)^2 - (y-4\beta)^2 - x^2 - (z+2\beta^2-1)^2 = 0$$

The equations,

$$(23) \quad \begin{cases} x = 4\alpha & , & y = 0 & , & z = 2\alpha^2 - 1 \\ x = 0 & , & y = -4\beta & , & z = -2\beta^2 + 1 \end{cases}$$

define parabolas which are confocal to each other.

"The Enneper surface" is the envelope of planes drawn, passed perpendicular to the middle of the chords which join the points of one curve to the points of the other. The normal planes to the two parabolas at the extremities of one of the chords are the planes of the lines of curvature which pass through the point of contact of the surface and to the plane perpendicular to the middle of the chord. The surface is also constructed by planes and by points.

Chapter IV.

Minimum Developable.

We may state this theorem - if every line upon a surface is a line of curvature, the surface is a plane or sphere and conversely.

We have seen that a line of curvature on a surface may be defined as any curve such that the normals to the surface along that curve form a developable surface.

Let the surface be given in the form $z = f(x, y)$

The tangent-plane at any point P has the equation,

$$p(X-x) + q(Y-y) - (Z-z) = 0$$

[(3) page 4]

The normal to the surface is given by the equation,

$$\frac{X-x}{p} = \frac{Y-y}{q} = \frac{Z-z}{-1} \quad [(4) \text{ page 4 }]$$

The condition for a surface to be a minimum surface is

$$(1+p^2)t - 2pq^2s + (1+q^2)r = 0$$

The condition for a developable surface is

$$(rt - s^2) = 0$$

Therefore the condition for the surface to be a minimum developable is

$$(1) \quad 1 + p^2 + q^2 = 0$$

The previous equations show that the normal line lies in the tangent plane and is a generator of the minimum developable.

Any line on the minimum developable is a line of curvature, for the normals to the surface form the minimum developable. The differential equation of the lines of curvature is given by the equation,

$$\frac{pqy' - (1+p^2)}{r - sy'} + \frac{pq + (1+q^2)y'}{ty' + s} = 0$$

$$pqty'^2 - (1+p^2)ty' - s(1+p^2) + pqy's + pqr - s(1+q^2)y'^2 - pqsy' + r(1+q^2)y' = 0$$

reducing we get,

$$(2) \quad [pqr - s(1+p^2)] + [r(1+q^2) - t(1+p^2)]y' + [pqt - s(1+q^2)]y'^2 = 0$$

All lines on the surface are lines of curvature when,

$$(3) \quad pqr - s(1+p^2) = 0$$

$$(4) \quad r(1+q^2) - t(1+p^2) = 0$$

$$(5) \quad pqt - s(1+q^2) = 0$$

These conditions are satisfied if the surface is a plane, a sphere or minimum developable. The relations (3), (4), (5), lead to one of these surfaces.

1) If we let $r = s = t = 0$ the above equations are satisfied and by integration we obtain a plane

$$z = ax + by + c.$$

2) Suppose $r \neq 0$, $s \neq 0$, $t \neq 0$ then equations (3), (4), (5) may be written

$$\frac{pq}{s} = \frac{1+p^2}{r} = \frac{1+q^2}{t} = \psi$$

where ψ is an unknown function of (x, y)

$$(6) \quad p q = s \psi$$

$$(7) \quad 1 + p^2 = r \psi$$

$$(8) \quad 1 + q^2 = t \psi$$

These equations being satisfied for all points on the surface we have,

$$(9) \quad r q + p s = \psi \frac{\partial s}{\partial x} + s \frac{\partial \psi}{\partial x}$$

$$(10) \quad 2 p s = \psi \frac{\partial r}{\partial y} + r \frac{\partial \psi}{\partial y},$$

which are obtained by differentiating (6) with regards to x and (7) with regards to y .

$$\text{Since } \frac{\partial s}{\partial x} = \frac{\partial r}{\partial y},$$

we have from (9) and (10)

$$(11) \quad r q - p s = s \frac{\partial \psi}{\partial x} - r \frac{\partial \psi}{\partial y}.$$

Similarly differentiating (6) with respect to y , and $(1 + q^2) = t \psi$ with respect to x and subtracting one from the other we have,

$$(12) \quad -3q + p t = -t \frac{\partial \psi}{\partial x} + s \frac{\partial \psi}{\partial y}.$$

from equations (11) and (12) we get;

$$(13) \quad p(rt - s^2) = -(rt - s^2) \frac{\partial \psi}{\partial x}$$

$$(14) \quad q(rt - s^2) = -(rt - s^2) \frac{\partial \psi}{\partial y}$$

whence either $rt - s^2 = 0$,

$$\text{or, } \frac{\partial \psi}{\partial x} = -p, \quad \frac{\partial \psi}{\partial y} = -q.$$

The condition $(rt - s^2) = 0$ is the condition that a surface should be developable. we have also $1 + p^2 + q^2 = 0$, which shows that the developable is also a minimum developable.

The equations $\frac{\partial \psi}{\partial x} = -p, \quad \frac{\partial \psi}{\partial y} = -q,$

give $\psi = c - z$

This value substituted in the equations

$$\begin{cases} pq = s\psi \\ 1 + p^2 = r\psi \\ 1 + q^2 = t\psi \end{cases}$$

gives relations from which we

get these equations

$$(15) \frac{\partial}{\partial x} (pz - cp) = \frac{\partial}{\partial y} (qz - cq) = -1$$

$$(16) \frac{\partial}{\partial y} (pz - cp) = \frac{\partial}{\partial x} (qz - cq) = 0$$

whence,
$$\begin{cases} pz - cp = a - x \\ qz - cq = b - y. \end{cases}$$

Therefore,

$$(17) (x-a)^2 + (y-b)^2 + (z-c)^2 = d^2,$$

which is the equation of a sphere.

Whence - Equations (3), (4), (5) lead to a plane, a sphere or minimum developable.

A bending or twisting of the Minimum Surface, in such a way that the minimum surface remains constant.

Let S and S' be two applicable minimum surfaces; in corresponding points the curvature of both surfaces and the absolute value of the principal radii of curvature are respectively equal to each other. It follows that the spherical representation of S and S' are congruent or symmetrical.

One of the two surfaces, say S' can be brought into such a position in space that the spherical picture of both surfaces cover each other and corresponding points of S and S' are determined through the same value of (ξ) .

Let $F(\xi)$, $f(\xi)$ be the corresponding value of the function F in Weierstrass formulae.

Then the element-length of both surfaces must be equal to each other.

$$ds^2 = (\tau\tau_0 + 1)^2 F(\tau) F_0(\tau_0) d\tau d\tau_0$$

=

$$ds'^2 = (\tau\tau_0 + 1)^2 f(\tau) f_0(\tau_0) d\tau d\tau_0$$

whence

$$(18) F(\tau) F_0(\tau_0) = f(\tau) f_0(\tau_0)$$

$$\left| \frac{f(\tau)}{F(\tau)} \right| = 1$$

$$(19) f(\tau) = e^{i\alpha} F(\tau).$$

where α is a real constant.

The element of length,

$$ds^2 = (\tau\tau_0 + 1)^2 F(\tau) F_0(\tau_0) d\tau d\tau_0$$

remains unchanged when $F(\tau)$ is replaced by $e^{i\alpha} F(\tau)$ whose value depends on the real constant α . Thus we have this proposition - The most-general bending of Minimum Surface by which the Minimum Surface remains constant is given when $F(\tau)$ is replaced

by $e^{i\alpha} F(\tau)$ in Weierstrass' formulae
 where α is a real arbitrary constant.
Associated Minimum Surfaces.

Let us consider the minimum surface defined by the general formulae of Monge.

$$(20) \begin{cases} x = A(t) + A_1(\tau) \\ y = B(t) + B_1(\tau) \\ z = C(t) + C_1(\tau) \end{cases}$$

where the functions A, B, C, A_1, B_1, C_1 satisfy respectively the two conditions

$$(21) \begin{cases} dA^2 + dB^2 + dC^2 = 0 \\ dA_1^2 + dB_1^2 + dC_1^2 = 0 \end{cases}$$

The element of length of that surface is given by the formulae

$$ds^2 = 2(dA dA_1 + dB dB_1 + dC dC_1)$$

This expression is the same for all the minimum surfaces defined by the formulae.

$$(22) \begin{cases} x = e^{i\alpha} A(t) + e^{-i\alpha} A_1(\tau) \\ y = e^{i\alpha} B(t) + e^{-i\alpha} B_1(\tau) \\ z = e^{i\alpha} C(t) + e^{-i\alpha} C_1(\tau) \end{cases}$$

where α is an arbitrary constant. All the surfaces are applicable on each other. The corresponding points of any two are determined by the same value of (t) and (τ) .

At these points the tangent-planes to the two surfaces are parallel. We speak of equations (22) as defining a family of associated surfaces. For $\alpha \neq 0$ we find the surfaces defined by equations (20). Especially interesting is the case where $\alpha = \pi$, the surface is defined by the formulae,

$$(23) \begin{cases} x_0 = i [A(t) - A_1(\tau)] \\ y_0 = i [B(t) - B_1(\tau)] \\ z_0 = i [C(t) - C_1(\tau)] \end{cases}$$

which we call Associated Minimum Surfaces.

When one uses Weierstrass' formula,

$$\begin{cases} x = R \int (1 - \tau^2) F(\tau) d\tau, \\ y = R \int i(1 + \tau^2) F(\tau) d\tau \\ z = R \int 2\tau F(\tau) d\tau \end{cases},$$

the associated minimum surfaces are obtained by replacing $F(\tau)$ by $F(\tau)e^{i\alpha}$, and in particular the conjugate surface is obtained by replacing $F(\tau)$ by $iF(\tau)$.

$$(24) \begin{cases} x_0 = R \int i(1 - \tau^2) F(\tau) d\tau, \\ y_0 = -R \int (1 + \tau^2) F(\tau) d\tau \\ z_0 = R \int 2i\tau F(\tau) d\tau \end{cases}$$

We denote the coordinates of a point of the associated surface, corresponding to a value of α in the following way,

$$(25) \begin{cases} X = x \cos \alpha + x_0 \sin \alpha \\ Y = y \cos \alpha + y_0 \sin \alpha \\ Z = z \cos \alpha + z_0 \sin \alpha \end{cases}$$

The square of the element of length of that surface ($dx^2 + dy^2 + dz^2$), being

independent of x , one deduces the relations,

$$26 \quad \begin{cases} dx^2 + dy^2 + dz^2 = dx_0^2 + dy_0^2 + dz_0^2 \\ dx dx_0 + dy dy_0 + dz dz_0 = 0 \end{cases}$$

Not only is a minimum surface and its conjugate applicable upon each other and have, as any two associated surfaces the property that their tangent planes in corresponding points are parallel but also the tangents to the two corresponding curves are always perpendicular.

Then we say that the corresponding elements of the surfaces are orthogonal.

To determine the minimum Sur-
faces which are applicable on a
Rotation Surface.

Let S be a minimum surface
applicable on the Rotation surface.
It permits of a constant bending
by which the spherical picture of
 S remains congruent and ex-
periences only a small turning
about the diameter of the sphere.

By a finite bending of the sur-
face S , its spherical picture is
turned about the z -axis. A turn-
ing about the axis is effected by
replacing (τ) by $e^{i\alpha}\tau$, where α is
the amplitude of the turning,
whence

$$ds^2 = (\tau\tau_0 + 1)^2 F(\tau) F_0(\tau_0) d\tau d\tau_0.$$

$$|F(\tau)| = |F(\tau e^{i\alpha})|.$$

$$F(\tau e^{i\alpha}) = e^{i\beta} F(\tau),$$

where β signifies a real constant,

whence,

$$(27). \quad \frac{\tau e^{i\alpha} F'(\tau e^{i\alpha})}{F(\tau e^{i\alpha})} = \tau \frac{F'(\tau)}{F(\tau)}.$$

$$(28). \quad \frac{F'(\tau)}{F(\tau)} = \frac{\kappa}{\tau}, \quad F(\tau) = c\tau^{\kappa},$$

where c and κ are constants, the second of which is real.

We have the proposition— "Those ^{minimum} surfaces which are applicable on the Rotation surface are obtained from Weierstrass' formulae

by setting

$$(29) \quad F(\tau) = c\tau^{\kappa},$$

where κ is a real arbitrary constant, and c a complex constant.

The Minimum Dcrew Surface.

Making use of the substitution

(29) just-found,

$$\sigma = \int \sqrt{2F(\tau)} d\tau$$

becomes, $\sigma = \sqrt{2c} \int \tau^{\frac{\kappa}{2}} d\tau,$

which gives with the exception of the case where $\kappa = -2,$

$$(30) \quad \sigma = u + iv = \sqrt{2C} \frac{z^{\frac{\kappa}{2} + 1}}{\frac{\kappa}{2} + 1},$$

and for the square of the element of length,

$$ds^2 = f(u^2 + v^2)(du^2 + dv^2).$$

Let us consider now, that

$$(31) \quad \begin{cases} u \cos \frac{\alpha}{2} - v \sin \frac{\alpha}{2} = \text{const.}, \\ u \sin \frac{\alpha}{2} + v \cos \frac{\alpha}{2} = \text{const.}, \end{cases}$$

are the lines of curvature of the associated surface, while through the substitution

$$(32) \quad \begin{cases} u' = u \cos \frac{\alpha}{2} - v \sin \frac{\alpha}{2} \\ v' = u \sin \frac{\alpha}{2} + v \cos \frac{\alpha}{2} \end{cases}$$

The square of the element of length remains unchanged, so we see that the surfaces in question are identical in form with their associated surfaces.

In case where $\kappa = -2$, Weierstrass'

formulae become,

$$(33) \begin{cases} x = \sqrt{r} \left(c \left(\frac{1}{z} + \tau \right) \right) \\ y = \sqrt{r} \left(i c \left(\frac{1}{z} - \tau \right) \right) \\ z = -\sqrt{r} \left(2 c \log(\tau) \right). \end{cases}$$

Then when (τ) is replaced by $\tau e^{2\alpha}$, z is increased by a constant, and x and y go over into

$$(34) \begin{cases} x \cos \alpha - y \sin \alpha \\ x \sin \alpha + y \cos \alpha \end{cases}$$

So it is clear that these surfaces are screw surfaces whose axis is the z -axis.

The minimum screw surface is obtained from Weierstrass' formulae by replacing $F(\tau)$ by $\left(\frac{c}{2\tau}\right)$.

To get the equations for the minimum screw surface in explicit form set-

$$c = \frac{m}{2} e^{i\theta}, \quad \tau = e^{-v+i\omega}$$

$$(35) \quad \begin{cases} x = m (\cos \beta \cosh v \cos w + \sin \beta \sinh v \sin w) \\ y = m (\cos \beta \cosh v \sin w - \sin \beta \sinh v \cos w) \\ z = m (v \cos \beta + w \sin \beta). \end{cases}$$

when $\beta = 0$, the corresponding surface is a catenoid, and its conjugate which corresponds to $\beta = \frac{\pi}{2}$,

$$(36) \quad z = m \tan^{-1} \frac{y}{x},$$

The minimum screw ruled surface.

On page 106 we have found the substitution $F(\tau) = c\tau^\kappa$. we considered the case where $\kappa = -2$. Let us see the form of surface when $\kappa = 1$.

$$F(\tau) = c\tau^\kappa$$

$$\varphi'''(\tau) = c(\tau)$$

Making the substitutions Weierstrass' formulae become,

$$\begin{cases} x = R\left[\frac{c}{4}(2\tau^2 - \tau^4)\right] \\ y = \sqrt{R}\left[\frac{c}{4}(i\tau^2 + i\tau^4)\right] \\ z = \sqrt[3]{R}\frac{2}{3}(c\tau^3) \end{cases}$$

$$\text{Let } \tau = u + iv$$

and these equations become

$$\begin{cases} x = \frac{c}{4} [2(u^2 - v^2) - (u^2 - v^2)^2 + 4u^3v] \\ y = \frac{c}{4} [4uv^3 - 2uv - 4u^3v] \\ z = \frac{2}{3} c (u^3 - 3uv^2). \end{cases}$$

These give x, y, z in terms of u and v and from these the form of the surface can be determined.

Formulae of Schwarz

Take the equations of the Minimum Surface in Monge's form,

$$\begin{cases} x = A(t) + A_1(\tau) \\ y = B(t) + B_1(\tau) \\ z = C(t) + C_1(\tau) \end{cases}$$

where the functions $A, B, C; A_1, B_1, C_1$ satisfy the relations,

$$dA^2 + dB^2 + dC^2 = 0$$

$$dA_1^2 + dB_1^2 + dC_1^2 = 0$$

In order that the surface contain a given curve C it is necessary that x, y, z designate the coordinates of any point of that curve, which are the known functions of a parameter λ . One can for determining t and τ the functions of λ satisfying the equations.

$$x = A + A_1, \quad y = B + B_1, \quad z = C + C_1.$$

If X, Y, Z designate the direction cosines of the Normal to the surface

at the point (x, y, z) of the curve C it is necessary to express that the tangent plane contain the tangents to two minimum curves of the surface which pass through the point and which gives the equations

$$X dA + Y dB + Z dC = 0$$

$$X dA_1 + Y dB_1 + Z dC_1 = 0$$

These in connection with the previous equations give all the conditions of the problem.

Differentiating equations ()

$$\begin{cases} dx = dA + dA_1 \\ dy = dB + dB_1 \\ dz = dC + dC_1 \end{cases}$$

whence,

$$X dx + Y dy + Z dz = 0,$$

which expresses that the tangent plane contains the tangents to the curve C , and which being verified by the given values of X, Y, Z - equations () (), reduce to each other.

There are then six equations in general for determining the differentials dA , dB , dC ; dA_1 , dB_1 , dC_1 .

From we get the values of dA , dB , dC , and set them in equations (), and there results

$$dx dA + dy dB + dz dC = \frac{ds^2}{2},$$

s being the arc of the curve C .

That equation taken in connection with the relations (), () gives the values of dA , dB , dC .

Form the determinant.

$$\Delta = \begin{vmatrix} x & y & z \\ dx & dy & dz \\ dA & dB & dC \end{vmatrix}$$

$$\Delta^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & ds^2 & \frac{ds^2}{2} \\ 0 & \frac{ds^2}{2} & 0 \end{vmatrix} = -\frac{ds^4}{4}$$

One has then,

$$\begin{aligned} \Delta &= (y dz - z dy) dA + (z dx - x dz) dB \\ &\quad + (x dy - y dx) dC \\ &= i \frac{ds^2}{2} \end{aligned}$$

We have a system of three equations

$$\begin{cases} X dA + Y dB + Z dC = 0 \\ dx dA + dy dB + dz dC = \frac{ds^2}{2} \\ \Delta = (Y dz - Z dy) dA + (Z dx - X dz) dB \\ \quad + (X dy - Y dx) dC \\ \quad = \pm i \frac{ds^2}{2} \end{cases}$$

all of which are of the first-degree. The values of dA , dB , and dC are easily calculated and are found to be

$$\begin{cases} dA = \frac{dx}{2} \pm \frac{i}{2} (Y dz - Z dy) \\ dB = \frac{dy}{2} \pm \frac{i}{2} (Z dx - X dz) \\ dC = \frac{dz}{2} \pm \frac{i}{2} (X dy - Y dx) \end{cases}$$

and from the relations,

$$\begin{cases} dx = dA + dA, \\ dy = dB + d\bar{B}, \\ dz = dC + d\bar{C}, \end{cases}$$

we get:

$$\begin{cases} dA_1 = \frac{dx}{2} + \frac{i}{2} (y dz - z dy) \\ dB_1 = \frac{dy}{2} + \frac{i}{2} (z dx - x dz) \\ dC_1 = \frac{dz}{2} + \frac{i}{2} (x dy - y dx) \end{cases}$$

Such are the formulae of Schwarz, which give the solution to the problem. — To construct the Minimum surface which goes through a given curve C and all along the curve has a given normal.

Chapter V.

"The Problem of Plateau"

The determination of the minimum surface passing through a given contour composed of straight lines or of planes, which cut the surface normally.

A closed boundary given and the problem is to determine the minimum surface, perfectly continuous, or subject to discontinuities of a known nature passing through the contour.

Plateau solved this problem physically by plunging the contour into a solution of glycerine. He achieved many noted results. Especially interesting is the case where the boundary is formed by straight lines, and that where some of the portions of the boundary are replaced by the planes which cut the surface normally.

The principal results achieved in this branch of the science are due to Riemann, Weierstrass and Schwarz.

We will cite some of the boundaries studied by Riemann, without going into a study of them.

(1) The contour is composed of two straight lines which do not lie in the same plane.

Supposing that the infinite sector extending between the two straight lines does not have discontinuities one gets the gauche helicoid with the plane director.

(2). Three straight lines two of which intersect, the third being parallel to the plane of the other two.

(3). Any gauche quadrilateral.

Riemann formed a linear equation of the second order, on which depends the solution of the problem.

He considered the particular case where the four sides of the quadrilateral are also edges of a regular tetrahedron.

Let us consider a piece of a minimum surface A bounded by a closed curve C and its spherical picture B , which is the conformal representation of the piece of surface A . We take further that the piece of surface B is spread out on the sphere.

For the determination of the minimum surface S we will introduce the complex variable,

$$\sigma = \int \sqrt{2 F(\tau)} d\tau,$$

whose real part u , and the imaginary part v each set equal to a constant give the lines of curvature on S .

First we will assume the function σ , within the piece of surface

A is finite, constant and single valued. So we get through the expression for the value of σ in the complex plane a new conformal representation B' of the piece of surface A . The piece of surface B on the sphere and the piece of surface B' in the plane are conformally represented. If the proposition is known according to which the points of both pieces of surfaces can be associated with each other, so is the surface B known. Then the Weierstrass function $F(\tau) = \frac{1}{2} \left(\frac{d\sigma}{d\tau} \right)^2$

is known.

The problem of Plateau contains also this proposition - To conformally represent the spherical polygon on the plane polygon B .

From the six edges of a regular tetrahedron we will suppose that the two opposite edges AD and BC taken away and we will consider the piece of minimum surface Σ which is bounded by the quadrangle $ABCD$.

The tangent-planes in the four corners are the sides of the tetrahedron.

It can be shown that the spherical picture of Σ is a spherical quadrilateral $A'B'C'D'$. The sum of those angles is 120° .

To obtain such a quadrilateral on the sphere we need to inscribe in the same a cube. Then the four corners of the cube surface are the four sought corners of the quadrilateral.

Now we see that the bisecting plane of the (BC) cut the surface Σ perpendicularly and divide it into two symmetrical sectors ABC and DBC . These have the spherical triangles $A'B'C'$, $D'B'C'$, for their spherical representation. We wish to determine the smallest sector ABC whose spherical picture is $A'B'C'$, in which the angle $A' = 120^\circ$, $B' = 60^\circ$, $C' = 60^\circ$.

The plane picture of this same sector in the complex σ -plane is a right-angle triangle abc whose hypotenuse bc is parallel to one of the coordinate axes in the σ -plane.

In order to find for the surface the corresponding Weierstrass function $F(z)$ we must express σ as a function of z so that the spherical triangle $A'B'C'$ will be

conformally represented by
the plane triangle (a, b, c)

We let fall from A' the $\perp A'H'$, so that in this way we divide the triangle $A'B'C'$ into two symmetrical triangles $A'B'H'$ and $A'C'H'$, and divide again each of these into symmetrical triangles by the lines $H'G'$ and $F'H'$. We make an analogous construction in the plane triangle (a, b, c). It is clear that the spherical triangle $B'H'F'$ with the angles of $60^\circ, 45^\circ, 90^\circ$ is to be conformal with the plane triangle (b, h, f) whose angles are $45^\circ, 45^\circ$ and 90° .

In order to get this representation we will picture both triangles $(B' F' H')$ and $(b f h)$ conformally on the plane of the complex variable z , so that to the boundary of each triangle corresponds the real axis. We can take arbitrarily those three points on the real axis in the z -plane which correspond to the corners, we will so choose them that

$$\begin{array}{ll} \text{in } B', b & z = 0 \\ \text{in } F', f & z = 1 \\ \text{in } H', h & z = \infty. \end{array}$$

Determination of $F(\tau)$ for the Schwarz Minimum Surface.

For the minimum surface of Schwarz we have the right-angle triangle $b f h$ in the o -plane whose hypotenuse $b h$ is parallel to one of the axes, say the real, to represent conformally on the positive half of the z -plane.
 in b : $z = 0$, in f : $z = 1$, in h : $z = \infty$.

The Schwarz-Chistoffel formula for the conformal representation of a polygon on the plane gives

$$\left(\frac{d\sigma}{dz}\right)^2 = \frac{C}{z^{\frac{3}{2}}(z-1)}$$

$$C = i A^2 \quad (A \text{ real})$$

$$\left(\frac{d\sigma}{dz}\right)^2 = \frac{i A^2}{z^{\frac{3}{2}}(z-1)}, \quad \sigma = A e^{\frac{i\pi}{4}} \int \frac{dz}{z^{\frac{3}{4}}(z-1)^{\frac{1}{2}}}$$

when $z = t^{-2}$, $\sigma = 2A\sqrt{i} \int \frac{dt}{\sqrt{t^3-1}}$,
 the integration of which leads to the Lemniscate function.

The Weierstrass Function $F(\tau)$ for the minimum surface of Schwarz is given from the equation,

$$F(\tau) = \frac{1}{2} \left(\frac{d\sigma}{d\tau} \right)^2 = \frac{iA^2}{2} \cdot \frac{1}{z^{\frac{3}{2}}(z-1)} \left(\frac{dz}{d\tau} \right)^3$$

Therefore the value of $F(\tau)$ can be found to be,

$$F(\tau) = \frac{1}{\sqrt{1-14\tau^4+\tau^8}}.$$

The Schwarz minimum surface is given through the following equations,

$$\begin{cases} x = R \int \frac{1-\tau^2}{\sqrt{1-14\tau^4+\tau^8}} d\tau \\ y = R \int \frac{i(1+\tau^2)}{\sqrt{1-14\tau^4+\tau^8}} d\tau, \\ z = R \int \frac{2\tau d\tau}{\sqrt{1-14\tau^4+\tau^8}}. \end{cases}$$

- The minimum surface which goes through the four edges of a regular tetrahedron, which arises from the conformal representation of the surface of a cube on a sphere.

Let ξ, η be the rectangular coordinates of a point in the plane, X, Y, Z the rectangular coordinates of a point in space.

$$X = \frac{2\xi}{\xi^2 + \eta^2 + 1}, \quad Y = \frac{2\eta}{\xi^2 + \eta^2 + 1},$$

$$Z = \frac{\xi^2 + \eta^2 - 1}{\xi^2 + \eta^2 + 1}$$

These equations represent a surface.

The Integral,

$$\oint + g i = \int_0^{\xi + \eta i} \frac{d(\xi + \eta i)}{\sqrt{1 - 14(\xi + \eta i)^4 + (\xi + \eta i)^8}}$$

represents the plane ξ, η in smallest part on a net which

arises when the surface of cube is stretched out on the plane p, q and which covers this plane 12 fold.

Through the substitution

$$\left\{ \frac{1 + 2\sqrt{3} (\xi + \eta i)^2 - (\xi + \eta i)^4}{1 - 2\sqrt{3} (\xi + \eta i)^2 - (\xi + \eta i)^4} \right\} = \mu^2$$

this integral is changed into

$$p + q i = \frac{1}{2\sqrt[4]{27}} \int \frac{d\mu}{\sqrt{\mu(\mu^2 - 1)}},$$

so this representation is dependent on the elliptic function and the lemniscate function. The conformal representation on the sphere is such that all the edges of the cube, the diagonals and middle lines of its faces picture great-circles and to every point of the surface of the cube only one point of the sphere corresponds.

Similar representation of the surface of the sphere on the plane is determined through the equation of the form,

$$dX^2 + dy^2 + dZ^2 = \frac{dp^2 + dq^2}{\rho}$$

From Weingarten, a definite minimum surface is determined through the equations,

$$\begin{cases} dx = \rho \left(\frac{\partial X}{\partial p} dp - \frac{\partial X}{\partial q} dq \right) \\ dy = \rho \left(\frac{\partial y}{\partial p} dp - \frac{\partial y}{\partial q} dq \right) \\ dz = \rho \left(\frac{\partial Z}{\partial p} dp - \frac{\partial Z}{\partial q} dq \right). \end{cases}$$

Let,

$\sqrt{1 - 14(\xi + \eta i)^4 + (\xi + \eta i)^8} = r(\cos \varphi + i \sin \varphi)$,
where r , $\cos \varphi$, $\sin \varphi$ are algebraic functions of ξ and η .

one finds for the principal radius of curvature the equation,

$$\rho = \frac{(\xi^2 + \eta^2 + 1)^2}{4r^2}$$

and for the differential coordinates of a point, the equations,

$$\begin{cases} dx = \frac{1}{2r^2} \left[(1 - \xi^2 + \eta^2) (\cos 2\varphi d\xi + \sin 2\varphi d\eta) - 2\xi\eta (\sin 2\varphi d\xi - \cos 2\varphi d\eta) \right] \\ dy = \frac{1}{2r^2} \left[(1 + \xi^2 - \eta^2) (\sin 2\varphi d\xi - \cos 2\varphi d\eta) - 2\xi\eta (\cos 2\varphi d\xi + \sin 2\varphi d\eta) \right] \\ dz = \frac{1}{2r^2} \left[2\xi (\cos 2\varphi d\xi + \sin 2\varphi d\eta) + 2\eta (\sin 2\varphi d\xi - \cos 2\varphi d\eta) \right] \end{cases}$$

The drawing represents a minimum surface hanging from four edges of a regular tetrahedron. A wire frame was constructed in the form of a regular tetrahedron from which two opposite sides were taken away, then the frame was plunged into a soap suds and glycerine solution and the minimum surface thus represented resulted.

The surface goes through the middle point of the tetrahedron from which stand the Normal IK to the surface, while the other two EF and GH lie entirely on the surface.

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