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On Finite Groups,
with Special Reference to
Klein's Ikosaeder.

A Thesis submitted to the
Faculty of the University of Missouri
for the degree of Master of Arts,
May 1, 1904.

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special reference to Klein's
Icosahedron.

Introduction.

In speaking of the icosahedron and other regular solids in the following work we shall include not only the space construction but also the sphere surface upon which the corners, edges and faces of the solids may be represented by means of projection from the middle point of the sphere. Of course many general forms might be found but we shall confine ourselves to the regular solids, that is, the regular tetrahedron, octahedron, cube, dodecahedron and icosahedron.

For our purpose let us construct a model containing all

of these solids. A large icosahedron may be made, the middle points of its faces will be the corners of a dodecahedron. If, now, any three perpendicular diagonals of the icosahedron are inserted, each of their end points will be corners of the octahedron. The intersection of these diagonals will be the center of a cube, the middle points of whose faces correspond to the corners of the octahedron. Finally a single tetrahedron may be inserted in the cube, the corners of the tetrahedron coinciding with alternate corners of the cube.

Let us consider the group from the geometric standpoint. The sum of all the revolutions necessary to carry a regular solid over into itself is a group. To illustrate this we shall use the solids above mentioned and also the dihedron. We may

think of the dihedron as a regular solid having its plane faces doubled but not enclosing any space, e. g., any polygon. Besides the figures themselves there are also their revolutions and reflections, that is, the elementary geometrical operations which carry the figures over into themselves. For our purpose each figure will be used in connection with its polar figure; for example, the icosahedron and dodecahedron, the octahedron and cube, the tetrahedron and its conjugate tetrahedron, the dihedron and the poles of the sphere.

Suppose we have a finite number of operations $(a, b, c, \dots k)$. Suppose we perform any one a . Now perform any other one, b . If the result, ab , is the same as some other operation h of the set we say that $(a, b, c, \dots k)$ form a group.

We must impose the condition that there be an identical operation, i.e., one which produces no change in any way; and the associative law must hold, i.e., $a(bc) = (ab)c$. By operation we mean any action whatever upon any object. This may be a substitution, a permutation of letters, turning a body about an axis or pivot, etc. A group may be represented on any solid which looks the same from two or more positions. For example, if we revolve an equilateral triangle through $\pm \frac{2\pi}{3}$ or $\pm \frac{4\pi}{3}$ about its center as a pivot, each vertex is carried into some other one, while the triangle as a whole remains unchanged. If we first make the revolution through $\frac{2\pi}{3}$, then through $\frac{4\pi}{3}$, the result is the same as not moving the triangle at all, or turning it entirely around

through 2π . If the triangle is turned through $\frac{2\pi}{3}$, then through $-\frac{2\pi}{3}$, the result produces no change whatever. Likewise if the triangle is revolved about one of its vertical diagonals as an axis, through an angle π , the faces of the triangle are reversed while the triangle as a whole remains unchanged. If the revolution through π about one of these axes is performed, then the revolution through $\frac{2\pi}{3}$, the result is the same as if the triangle had been turned through π about a certain other one of the axes.

As has been said the product of any two operations of a set composing a group produces another definite operation of the same set. In the case of the regular solids this is true of the revolutions but not of the reflections. Hence the rev-

solutions form a group but the reflections do not, for it will be shown later that the product of two reflections is a revolution, not a reflection. If, however, we consider the reflections in connection with certain revolutions they will form a group called the extended group.

We shall speak now only of finite groups, i.e., groups containing only a finite number of operations, each producing more than an infinitesimal change in any object. The number of distinct operations contained in any finite group is called the order of the group. The least possible number of repetitions of any operation necessary to bring it back to the identical operation is called the period of the operation. Any operation of a finite group has a period. One group H is a sub-

group of another, G , if all the operations of H are contained in G . The order of a subgroup is a divisor of the order of a group. For suppose N is the order of G and n the order of its subgroup H . Let $1, T_1, T_2, \dots, T_{n-1}$ be the n operations of H ; and let S_1 be any operation of G not contained in H . Then the operations $S_1, T_1 S_1, T_2 S_1, \dots, T_{n-1} S_1$ are all distinct from each other and from the operations of H . For if

$$T_p S_1 = T_q S_1,$$

then

$$T_p = T_q,$$

contrary to supposition; and if

$$T_p = T_q S_1,$$

$$S_1 = T_q^{-1} T_p,$$

and S_1 would be contained among the operations of H .

If the $2n$ operations thus obtained do not exhaust all the operations of G , let S_2 be any operation of G not contained among them. Then

it may be shown by repeating the previous reasoning, that the n operations $S_2, T_1 S_2, T_2 S_2, \dots, T_{n-1} S_2$ are all different from each other and from the previous $2n$ operations. If the group G is still not exhausted we may repeat this process and finally arrange the N operations of G thus:

$$\begin{array}{l}
 1, T_1, T_2, \dots, T_{n-1}; \\
 S_1, T_1 S_1, T_2 S_1, \dots, T_{n-1} S_1; \\
 S_2, T_1 S_2, T_2 S_2, \dots, T_{n-1} S_2; \\
 \dots \dots \dots \\
 S_{m-1}, T_1 S_{m-1}, T_2 S_{m-1}, \dots, T_{n-1} S_{m-1}.
 \end{array}$$

Hence $N = mn$, and n is a factor of N .

A subgroup which consists of the different powers of a single operation is called a cyclic subgroup, the order of the group being equal to the period of the given operation. When N , the order of the group, is a

prime p , G can have no subgroup except one of order unity, which is the identity. Every operation S of the group, except the identity, is of period p , and the group consists of the operations $1, S, S^2, \dots, S^{p-1}$. A group whose order is prime is therefore cyclic.

If we have given two operations S and T , we mean by their product ST that the operation S is first performed and then upon this the operation T is performed. In general

$$ST \neq TS.$$

If they are equal S and T are said to be permutable, and the group to which they belong is called an Abelian group. Consider also the operation S^{-1} (the inverse of S). If

$$S^{-1}TS = T$$

and S and T are not permutable,

T is different from T' ; and we say that T' is obtained from T by transformation and call T and T' conjugate operations.

A special case may arise when the original group and the transformed are the same. Any operation S of a group G which is identical with all its conjugate operations is called a self-conjugate operation. If T , an operation of the subgroup H , and its transformed operation $T' = S^{-1}TS$ are identical, whatever operation S is of G , then we call H a self-conjugate subgroup of G . Every group then contains at least two self-conjugate sub-groups, the group itself and a subgroup composed of the identical operation alone. If a group possesses no other self-conjugate subgroups it is called simple, otherwise it is called composite.

Let us define now that relation between two groups which is called isomorphism. Two groups are said to be isomorphic if their operations S and S' are performed in such a way that to every S_i there corresponds an S'_i , and to every S'_i there corresponds an S_i , so that every $S_i S_j$ corresponds to an $S'_i S'_j$. The groups are said to be simply isomorphic if the above correspondence is one-to-one. If, however, this correspondence has a multiple meaning the groups are said to be multiply isomorphic. In this case the order of one group is less than that of the other, so that to every operation S there corresponds a single operation S' , but to every S' there may be more than one S corresponding. The groups are called doubly isomorphic if this ratio is 2:1.

Chapter I.

The Regular Solids and the Group Theory.

Before this we have considered only the groups composed of the revolutions necessary to carry one of the solids mentioned above into itself. Let us now consider the rotation group formed by the repetition of a single periodic rotation. By such a group there usually remain fixed two points of the sphere, called the poles. If there are n rotations in all the group is composed of the n rotations through the angles $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{2(n-1)\pi}{n}$ about a fixed axis through the two poles. The operations of this group are all permutable with each other. Then each operation is conjugate only

with itself, so far as the subgroup which may be formed from single operations is concerned. The existence of such a group depends upon n , the order of the group. If n is a prime number the subgroup does not exist, for its order must be a factor of n . If however n is composite, to every factor of n there corresponds one and only one subgroup whose order is equal to that factor. This may be illustrated very simply by means of the rectangular array used in proving that the order of a subgroup is a divisor of the order of the group. In the case given N , the order of the group G , is equal to mn , where n is the order of the subgroup $H = \{1, T_1, T_2, \dots, T_{n-1}\}$ and m the order of another subgroup $K = \{1, S_1, S_2, \dots, S_{m-1}\}$, each being entirely distinct from

the other.

A self-conjugate subgroup of G not contained in a larger self-conjugate subgroup of G is called a maximal self-conjugate subgroup. Suppose G has a maximal self-conjugate subgroup H , H has a maximal self-conjugate subgroup K , etc. Such a series of groups terminating with the identity G_1 , in which each group is a maximal self-conjugate subgroup of the preceding, is called a composition series. Then we may analyse a group, i.e., resolve it into its various parts, if we can find its composition series.

This cyclic group composed of n rotations is simply isomorphic with the permutations on n letters, $(a_0, a_1, \dots, a_{n-1})$. The permutations of these revolutions may be represented geometrically. For we have only to

construct the n points which are found by our revolutions beginning with a certain given point a , and these points are themselves permuted by these revolutions.

In the dihedral group let us consider the great circle of the sphere which contains the n corner points or vertices of the dihedron, as the equator, having already determined both poles belonging to it. It is easily seen that if these poles are kept fixed the dihedron is carried into itself by the cyclic group of n revolutions already discussed. But this does not include all the dihedron revolutions. Mark the middle point of each edge of the dihedron. We shall now have also the diagonals which contain the end points and middle points of the dihedron. Call these diagonals sub-axes.

There are of course n of these. If n is odd the subaxis contains an end point and a middle point, if n is even the subaxis contains either two end points or two middle points. In either case the dihedron is unaltered if it is turned about its subaxis through the angle π . Therefore besides the n cyclic revolutions there are n revolutions of period 2. Hence the cyclic group is a subgroup of the whole dihedral group. If the dihedron is an equilateral triangle its cyclic group is simply isomorphic with the cube roots of unity.

We shall consider as conjugate such geometric figures as arise from one another by any operations of the given group. Suppose all the figures conjugate to a certain one have been constructed. These

are then T_k operations of one group which leave unchanged any single one of the figures thus constructed. Then the T_k operations form a self-conjugate subgroup of the given group. In the dihedron the revolutions about the main axis which turn through $\frac{2k\pi}{n}$ or $-\frac{2k\pi}{n}$ are conjugate; while those about the sub-axis, when n is odd are all conjugate, but when n is even they are divided into two classes.

This group is simply isomorphic with certain permutations of letters. For, corresponding to the n end points in the revolutions about the main axis will be the series $a_0, a_1, \dots, a_{n-1}$, as in the permutations of the cyclic group, taking a_v into $a_{v+k} \pmod{n}$. Also by the turnings about each sub-axis running through the point a_0 , a_v will go into a_{n-v} . Then from these two operations taken to-

gether will be formed the metacyclic group represented by the transformation $v' \equiv \pm v + k \pmod{n}$, which is simply isomorphic with the dihedral group.

In the dihedral group we considered $n > 2$. Now take $n = 2$. There will be an infinite number of ways of choosing this figure as there will be an infinite number of great circles passing through the two end points of our dihedron. Choose therefore the fixed great circle as the equator. The main axis of the figure will then be perpendicular to the two sub-axes, the three forming a set of orthogonal axes, and as before the group will consist of $2n = 4$ revolutions. If we consider these as coordinate axes the point (x, y, z) will be changed by these revolutions into the points $(x, -y, -z)$, $(-x, y, -z)$, and $(-x, -y, z)$.

This group contains, besides the identity, only operations of period 2, and one of these operations must be about the main axis, the other two about the sub-axis. Call this group the triaxial group to distinguish it from the dihedral group. All of its operations are permutable. It contains a certain subgroup of two rotations by which one of the three axes remains fixed, and then the identity.

All the revolutions which take the tetrahedron into itself take also the conjugate tetrahedron into itself. The eight corners of the tetrahedra taken together will be the corners of a cube. If then we mark on the sphere the six points corresponding to the middle points of the faces of the cube we shall have the corners of the octahedron. Then the

tetrahedron and octahedron groups are closely related. Consider the orthogonal axes as diagonals of the octahedron.

The tetrahedron group has 12 operations. There are four conjugate tetrahedron points, and each remains unchanged by three revolutions. There are then 8 operations of period 3. There are also three conjugate operations of period 2, the revolutions about the three perpendicular octahedron diagonals. These with the identity form the triaxial group, a self-conjugate subgroup of the tetrahedron group, since the diagonals remain unchanged by all the operations of the triaxial group, and only by them. The triaxial group may be again divided into the simple cyclic subgroup and the

identity.

Consider the four diagonals of the cube through the middle point of the sphere. They are permuted by the tetrahedron revolutions. By no revolutions of the tetrahedron, except the identity, do the four remain unchanged at the same time. There are also no two tetrahedron revolutions which produce the same permutation of the four diagonals. Then the tetrahedron group is simply isomorphic with the permutations of the four diagonals of the cube. The following arrangement of the four diagonals corresponds to the revolutions of the self-conjugate triaxial subgroup:-

1, 2, 3, 4;

2, 1, 4, 3;

3, 4, 1, 2;

4, 3, 2, 1.

If we include all the tetrahedron revolutions the other eight may be obtained by cyclic permutations of three of the four diagonals. These permutations of the four diagonals are called even permutations.

In the octahedron group we have the same figure as in the tetrahedron group. Let us also mark on the sphere twelve points corresponding to the middle points of the edges of the octahedron, and construct the six diagonals, each of which contains two points. Call these diagonals the cross lines of the figure.

The octahedron group contains the tetrahedron group as a self-conjugate subgroup. For the eight corners of the cube are obtained only in a certain manner from the tetrahedron and conjugate tetrahedron, and the latter remain

unchanged only by the twelve tetrahedron operations. Then the octahedron group consists of twenty-four operations in all, including those of both the tetrahedron and conjugate tetrahedron. There are six conjugate revolutions through π about the six cross lines of the figure, six revolutions through $\pm \frac{\pi}{2}$ of period 4 about the three octahedron diagonals. The last operations are also conjugate, for the four revolutions by which one of the diagonals remains fixed form a self-conjugate subgroup to a dihedral group of eight operations. The two revolutions of period 3 about one of the diagonals, and all the revolutions of period 3 as well, are conjugate; for each diagonal of the cube has become the main axis of a dihedral group of six revolutions. The opera-

tions of period 2 are divided into two classes, when the octahedron diagonal or the cross line remains fixed. Then the octahedron group contains as self-conjugate subgroups the tetrahedron group, the triaxial group, etc.

The four diagonals of the cube are permuted in 24 ways by the 24 operations of the octahedron group. The octahedron group is simply isomorphic with all the permutations on four letters.

The group of the icosahedron, so far as the dihedron, tetrahedron and octahedron are concerned, is simple, that is, has no self-conjugate subgroup besides itself and unity. We have already marked off on the sphere the twelve corners of the icosahedron, corresponding to the middle points of

the edges of the octahedron. Now mark the twenty corners of the dodecahedron as the middle points of the faces of the icosahedron, and also the thirty points corresponding to the middle points of the edges of the icosahedron. Call the six diameters which may be drawn between the twelve corners the diagonals of the icosahedron. Likewise between the twenty end points of the dodecahedron there will be ten diagonals of it, and finally fifteen cross lines between the thirty points as the middle points of the edges of the icosahedron.

The number of revolutions of the icosahedron is 60. Each of the twelve end points of the icosahedron remain unchanged by five revolutions. We have then, corresponding to the six icosahedron diagonals, four revolutions of period 5, i.e., alto-

gether 24 revolutions of this sort. In the same way about each of the ten diagonals of the dodecahedron there are two revolutions of period 3, making in all twenty revolutions, and 15 revolutions of period 2 about the 15 cross lines. Adding the identity we have

$$24 + 20 + 15 + 1 = 60$$

revolutions of the icosahedron by which it remains unaltered.

The fifteen operations of period 2, and the twenty operations of period 3 are conjugate; for the 15 cross lines and the ten diagonals of the dodecahedron are conjugate, and it does not matter whether we turn through the angle $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ about one of these diagonals, as both its end points are conjugate. The conjugate revolutions of period 5 may be divided into two classes,

those which turn about one of the icosahedron diagonals through the angle $\pm \frac{2\pi}{5}$ and those which turn through $\pm \frac{4\pi}{5}$.

There are several cyclic subgroups in the icosahedron; 15 of the kind for $n=2$, 10 for $n=3$, 6 for $n=5$. We also know that cyclic groups with the same value for n are conjugate. If we had a self-conjugate subgroup it must contain either all or none of the operations of period $n=2$ (since they are conjugate), likewise it must contain either all or none for $n=3$, or $n=5$. But the groups $n=2, 3, 5$ have respectively 15, 20, 24 operations different from the identity. Choose three numbers η, η', η'' between 0 and 1 so that the given self-conjugate subgroup contains the number of operations

$$1 + 15\eta + 20\eta' + 24\eta''.$$

But this number must be a factor of 60. Then either $\eta = \eta' = \eta'' = 0$, which says that the subgroup is the identity, or $\eta = \eta' = \eta'' = 1$, which says that the subgroup is the main group itself. The icosahedron group is therefore simple.

There are other subgroups besides the cyclic. There are six conjugate dihedral groups of order 5, and ten conjugate dihedral groups of order 3. The former have the diagonals of the icosahedron as main axes, the latter the diagonals of the dodecahedron; the subaxes are among the 15 cross lines. Similarly we might think that corresponding to the 15 cross lines there were 15 dihedral groups with $n=2$, i.e., triaxial groups. But in the triaxial group the main axis and the two sub-axes must form a rectangular set. There are then only

five triaxial groups conjugate among themselves. These correspond to the five rectangular sets of three into which the crosslines may be divided, one of these three being chosen as the diagonals of the octahedron. Each of these triplets will remain unchanged not only by the revolutions of the triaxial group, but also by 12 of the icosahedron revolutions. Then these revolutions form a tetrahedron group. For the eight corners of the cube belonging to the single triplet under consideration are all within the 20 corners of the dodecahedron. There are then in the icosahedron group the 8 rotations of period 3, which, taken with the original revolutions of the triaxial group, form a tetrahedron group. Therefore the five tetrahedron groups thus obtained are conjugate.

Consider the isomorphism of the icosahedron group as related to these five regular triplets. By each revolution of Period 5 these triplets will be permuted in a certain cyclic series; by each revolution of period 3 two of the triplets will remain unchanged and only the other three permuted in cycles; by each revolution of period 2 one of the triplets remains unchanged while the other four are permuted in pairs. Therefore the group of 60 icosahedron rotations is simply isomorphic with the group of 60 even permutations of five things.

We must now consider the symmetric planes of our model, on each side of which the model looks the same, and the sphere surface cut by them.

In the dihedron besides the plane of the equator we may have n other symmetric planes, each running through the main axis and a single subaxis. The sphere will be divided by these $n+1$ planes into $4n$ congruent isosceles triangles, having two angles equal to $\frac{\pi}{2}$ and one equal to $\frac{\pi}{n}$. At each end point and the middle point of each side of the dihedron four equal angles of such triangles will meet, and at each of the two poles there will be $2n$ equal angles.

In the tetrahedron there are six symmetric planes, each passing through an edge and perpendicular to the opposite edge. Each of the four equilateral triangles lying in these planes is divided into six congruent and symmetric triangles by three of the symmetric

planes on account of its three altitudes. If we carry this by central projection on to the sphere we have 24 of these congruent and symmetric triangles, each having the angles $\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, which have equal angles meeting in the corners of both the tetrahedron and its conjugate tetrahedron, and four equal angles in the corners of the octahedron.

In the octahedron we have, besides the symmetric planes of the tetrahedron, three others which contain two of the three octahedron diagonals. By these nine planes the surface of the octahedron made up of eight equilateral triangles may be easily divided just as the surface of the tetrahedron was divided. If we go to the surface of the sphere by central projection we find 48 congruent and symmetric triangles with the

angles $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, six of which meet in the corners of the cube, eight in the corners of the octahedron, and four in the end points of the cross lines.

In the icosahedron we have as symmetric planes the fifteen planes containing two of the six icosahedron diagonals. These divide the twenty equilateral triangles on the surface of the icosahedron in a similar manner to the case given above. We then have on the sphere 120 congruent and symmetric triangles, whose angles are $\frac{\pi}{3}$, $\frac{\pi}{5}$, $\frac{\pi}{2}$, six of which meet in the corners of the dodecahedron, ten in the corners of the icosahedron, and four in the end points of the cross lines. It may be observed from this that in each case the number of triangles is double the order of the corresponding group.

For the cyclic group we may also construct certain planes as symmetric planes. These will be the n planes running through the poles, which go over into each other by the revolutions of the group. They will divide the sphere into $2n$ congruent lunes whose angle is $\frac{\pi}{n}$, and each of which extends from one pole to the other.

We shall consider next the point groups which are obtained by turning a certain point of the sphere through the N revolutions of the group. In order to understand this idea better let us think of the sphere as divided into certain shaded and unshaded regions. Then by the revolutions of the single group each shaded region will be carried once and only once into each other shaded region, and likewise each unshaded region

will be carried into each unshaded region. That is, the number N of revolutions corresponds to one half the number of distinct regions. If now we have given a certain point on the sphere which may lie either in the shaded or the unshaded region, by means of this division into regions we can find the $(N-1)$ new positions which the point will take on when operated upon by the $(N-1)$ operations of the group different from identity; we have simply to mark the $(N-1)$ points having positions in their respective regions corresponding to the position of the beginning point in its region. There are in general n distinct types of the point group; they fall together only in case the original point lies in the corner of its region. If s shaded (and of course n unshaded) regions meet in

a certain corner, the point will remain unchanged by operations of the group, and there will be in all N different positions. These special point groups are no others than those which we have already discussed.

We shall consider the fundamental region of a group of point transformations such a division of the surface as contains one and only one of the given point groups. The edge points of such a region are arranged in pairs on account of the transformation of the group. We shall use as our fundamental region a shaded and its corresponding unshaded region. If we let a point move over a certain fundamental region the given point group will move over the entire sphere surface.

If we consider the reflections on the symmetric planes of our model we shall have another group called the extended group. A single different, shaded or unshaded, region is the fundamental region of the extended group; the extended group contains $2N$ operations. A combination of the revolutions already considered with the reflections in a symmetric plane is necessary in order to make each shaded region go into each unshaded region. In other words the sphere must be so divided that either a revolution or a revolution and reflection carry one region into a certain other one. The edge points of these fundamental regions are defined in such a way as to remain unchanged by a certain operation of the extended group, viz., the reflection in the sym-

metric plane. We shall obtain the extended group if we unite the original rotation group with the reflections in the symmetric plane in which the corresponding end point is found. Therefore there are particular groups of only N points which arise by the application of the extended group out of the edge points of the fundamental region, and at the same time the general point groups in the sense of the preceding paragraph. According to this there are under the latter point groups only those which remain unchanged by the additional operations of the extended group. Of course there are still, corresponding to the corners of the fundamental region, special point groups of $\frac{N}{2}$ points.

Without doubt the original group is a self-conjugate subgroup

of the extended group. Besides this the extended octahedron and icosahedron groups, as well as the dihedral group when n is even, contain a self-conjugate subgroup of only two operations. This arises from the double application of a certain transformation which carries each point of the sphere across to one diametrically opposite.

Heretofore we have considered the group as given and found its different operations out of which the group is formed by means of repetitions and combinations.

Let us consider the icosahedron group. Since each fundamental region of a group is formed from each other one by one operation of the group we may name the different fundamental regions according to the operations by which

they are formed from a certain one which we shall call 1, the beginning region. Think of one of the icosahedron diagonals as vertical. As the first fundamental region use one of the five isosceles triangles with the angles $\frac{2\pi}{5}$, $\frac{\pi}{3}$, $\frac{\pi}{3}$ grouped around the corner of the icosahedron. Such a triangle is one of the fundamental regions of the icosahedron group by the conditions for obtaining a fundamental region. These five isosceles triangles we shall call the first pentagon of the dodecahedron belonging to the icosahedron. The side of the triangles which is at the same time the side of the pentagon, is called the base.

We shall call the operation of rotating the figure through the angle $\frac{2\pi}{5}$ about its vertical diagonal

5. Then the five fundamental regions will be given by the series $1, S, S^2, S^3, S^4$; i.e., each region is designated by S^μ , $\mu = 0, 1, 2, 3, 4$.

Take now the operation T of period 2. It is the rotation through the angle π about a certain cross line of the icosahedron, one of whose end points is the middle point of the base of 1. By means of T our five regions S^μ may be changed into the regions $S^{\mu T}$, and give another pentagon. Also upon each of these we may perform the operation S^ν , producing the regions $S^{\mu T S^\nu}$. Therefore there are the fundamental regions of each of the five pentagons given by $S^{\mu T S^\nu}$, ($\mu, \nu = 0, 1, 2, 3, 4$).

We may now have a third operation U of period 2, whose axis is one of the horizontal cross lines perpendicular to the axis of T .

By means of U the six pentagons already given by the combinations of S and T , are taken over into the six pentagons diametrically opposite. Then the thirty fundamental regions still lacking are given by $US^\mu, US^\nu TS^\mu$, ($\mu, \nu = 0, 1, 2, 3, 4$). Going from the fundamental regions back to the rotations we have the 60 rotations of the icosahedron group given by $S^\mu, S^\mu TS^\nu, US^\mu, US^\nu TS^\mu$, ($\mu, \nu = 0, 1, 2, 3, 4$). The rotations S^μ, US^μ form the dihedral groups $n = 5$ of the icosahedron about its vertical diagonal; and the revolutions T, U, UT , together with the identity, form one of the five triaxial groups of the icosahedron. By a study of the model of the icosahedron we may see all the operations of any of its subgroups, as well as those of the other figures contained inside the icosahedron. For instance the

operations of the tetrahedron are as follows:

$$1, T, U, UT, S^3TS, STS^3, S^2TS^4, S^4TS^2, \\ US^3TS, USTS^3, US^4TS^2, US^2TS^4,$$

each of which is an operation of the icosahedron.

We may also obtain U by a combination of S and T . Perform upon T the operation S^2TS^3 and we have TS^2TS^3 one of the pentagons of the lower half of our model. But this same region has already been marked US^3T .

$$\therefore TS^2TS^3 = US^3T,$$

where U is the unknown. Multiply right-handedly by T and then by S^2 , and we have, since $T^2 = 1$ and $S^5 = 1$,

$$TS^2TS^3TS^2 = U.$$

Each of the other rotation groups, the cyclic, dihedral, triaxial, tetrahedron and octahedron may be generated in exactly the same way.

Chapter II.

Stereographic Projection.

Before going any farther with our discussion of the icosahedron group, let us consider some of the ideas of stereographic projection that we may better understand the results arising from certain operations of each of the groups already considered.

Obviously the simplest possible functions of a complex variable involve the variable rationally and only to the first degree. We may write these linear expressions in the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are any complex constants. To every value of z there will be a corresponding value of w . Hence if any values of z be represented by

a set of points in the complex plane, the corresponding values of w will be represented by another set of points in the plane. The geometrical relation between the two systems will be the same as the analytic.

Let us begin with the simple case where $w = az + \beta$. The point w corresponding to any z is found by adding to the vector z the vector β . We may then regard the entire plane as being translated from the position z to the position w through the vector distance β .

Now take the case $w = az$, where $a = a_1 + a_2 i$ or $= \rho(\cos \varphi + i \sin \varphi)$. First suppose a real; i.e. $\varphi = 0$. Then

each vector is multiplied by ρ , its position remaining unchanged. We have the expansion of the entire plane outward from the origin, ρ being the ratio of expansion. If $\rho = 1$ the corresponding operation in the plane will be a rotation of the plane about the origin through the angle ϕ . Every circle about the origin remains unchanged as a whole, and each ray is converted into another at the angular distance ϕ from its original position. If ρ and ϕ are any quantities whatever we have a combination of expansion and rotation. The lines of motion must be curves which have the property that as a point moves along one of them its radius vector, in turning through a constant angle ϕ , is multiplied in length by a constant ρ . These curves are

the logarithmic spirals $r = Ce^{k\varphi}$.

It is clear then that translation will change only the position of the figure, leaving the shape, size and direction of its parts unaltered. Rotation leaves the shape and size unchanged. Expansion leaves the shape unchanged but changes the size equally in all directions. Hence every integral linear transformation, being only a combination of these, changes every figure in the plane into a similar one. Every straight line goes into a straight line, every circle into a circle. Every angle becomes an equal angle since corresponding curves are similar. If from any point z we draw small vectors in all directions and consider the transformed figure consisting of small vectors drawn from the corresponding

point w , the ratio of two corresponding small vectors, called the ratio of similarity, will be constant on all sides of the given points.

Let us now consider the case where

$$w = \frac{1}{z} = \bar{z}' = \frac{1}{r(\cos \theta + i \sin \theta)} = \frac{1}{r}(\cos \theta - i \sin \theta) \\ = \frac{1}{r}[\cos(-\theta) + i \sin(-\theta)].$$

Let the coordinates of the point z be (r, θ) , and on this radius vector take (r', θ') such that $r' = r$, $\theta' = \theta$. The complex number corresponding to the second point is

$$z' = r'(\cos \theta + i \sin \theta).$$

Choose a third point (r'', θ'') symmetrical to the second with respect to the real axis, and we have $r'' = r' = r$, $\theta'' = -\theta' = -\theta$, and this is the point w .

Therefore by the operation given by $w = \frac{1}{z}$, every point is replaced by another on the same radius vec-

tor through the origin as the first point, and so that $r' = r^{-1}$. This is called inversion. Then the entire figure is reflected on the real axis as if it were a mirror. A positive rotation is reflected into a negative, and z is reflected into its conjugate $\bar{z}$. By inversion the unit circle remains unchanged for $r' = r^{-1} = 1$. Every point within the unit circle is changed to a point outside, and vice-versa. A straight line through the origin remains unchanged as a whole, but every point at infinity corresponds to the single point at the origin.

Let us consider some other transformations.

$$z = x + iy = r(\cos \varphi + i \sin \varphi)$$

$$r = \sqrt{x^2 + y^2}, \quad x = r \cos \varphi, \quad y = r \sin \varphi.$$

$$\begin{aligned} \text{But } z' &= \frac{1}{r}(\cos \varphi + i \sin \varphi) = \frac{r}{r^2}(\cos \varphi + i \sin \varphi) \\ &= \frac{x + iy}{x^2 + y^2}. \end{aligned}$$

Hence $x' = \frac{x}{x^2 + y^2}$, $y' = \frac{y}{x^2 + y^2}$,
 $x = \frac{x'}{x'^2 + y'^2}$, $y = \frac{y'}{x'^2 + y'^2}$.

Examine the straight line not passing through the origin,

$$Ax + By + C = 0 \quad \text{where } C \neq 0.$$

By transformation this becomes

$$Ax' + By' + C(x'^2 + y'^2) = 0$$

which is a circle through the origin, with its tangent at the origin

$$Ax + By = 0,$$

parallel to the original line.

Now examine the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

On inversion this becomes

$$\frac{x'^2 + y'^2}{(x'^2 + y'^2)^2} + \frac{2gx' + 2fy'}{x'^2 + y'^2} + c = 0$$

$$c(x'^2 + y'^2) + 2gx' + 2fy' + 1 = 0$$

which is a circle except for $c = 0$ when it becomes a straight line.

Hence a straight line goes into a circle through the ori-

gin, then the line goes into a line. A circle goes into a circle unless it passes through the origin when it becomes a straight line. A straight line may be considered a circle of infinite radius with its center at infinity. Then geometric inversion converts a circle into a circle. Also by inversion an angle is unchanged in amount but reversed in direction.

By various combinations of the operations already considered we may obtain parabolic, elliptic, hyperbolic and double spiral motion. (See Cole: "The Linear Functions of a Complex Variable", *Annals of Mathematics*, June, 1890.)

Thus far we have represented complex quantities by points in a plane. The same thing may be done on the sphere, preserving per-

fect continuity in the geometrical representation by means of stereographic projection. Suppose the complex number represented in the plane in the usual way. Hold the plane horizontal and tangent to the sphere at the origin. Call the point of tangency the south pole. Join each point of the plane to the north pole of the sphere. The point in which the line joining these two points cuts the sphere is called the corresponding point to the given

point. There is a point on the sphere corresponding to every complex number represented in the plane, and vice-versa. Evidently the distribution of the complex numbers on the sphere reproduces the continuity of the analytic system.

Let us draw various figures in the plane in order to determine the corresponding figures on the sphere. The bundle of rays through the origin becomes meridian circles, and the system of circles about the origin becomes parallels of latitude. One of the circles becomes the equator, and by properly choosing the radius of the sphere this may be made the unit circle. Then every circle through the origin within the unit circle will become a parallel of latitude in the southern hemisphere, and every circle with-

out becomes a parallel of latitude in the northern hemisphere. The Points at an infinite distance in the plane go into the north pole. Every straight line and every circle in the plane goes into a circle on the sphere which passes through the north pole in the case of the straight line. (See ^{plate I} figure p. 52). Every angle in the plane is equal to the corresponding angle on the sphere. For if A is any point in the plane (fig. ^{plate I} p. 55) and S its corresponding point on the sphere, and ST the intersection of the tangent plane at S with the plane POA , we have

$$\beta = \frac{1}{2} \text{arc } SP, \quad \alpha = \frac{1}{2} \text{arc } OP - \frac{1}{2} \text{arc } OS = \frac{1}{2} \text{arc } SP.$$

$$\therefore \alpha = \beta.$$

The tangent plane at S and the complex plane therefore make the same angle with the line of projection PA . If, now, any two straight lines in the plane be drawn through A these will become

circles through S on the sphere. The tangents to these circles will be the intersections of the tangent plane at S with the two planes of projection. But because of the symmetry of the complex plane and the tangent plane with respect to the line PA , it is clear that the angle between the two tangents at S is equal to the angle between the given straight lines in the complex plane. Hence the angles are unchanged by stereographic projection.

Let us consider the analytic theory of the relation between the representation of the complex number on the plane and on the sphere. As the origin of our coordinate system in space take the origin in the plane. Let ξ be the real axis and η the imaginary axis in the plane. The third axis ζ is perpendicular to the plane, with the positive direction upward. Choose the radius of the sphere $\frac{1}{2}$. The equation of the sphere is then

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}$$

$$\text{or } \xi^2 + \eta^2 + \zeta(\zeta - 1) = 0.$$

If x and y are the coordinates of any point in the plane and ξ, η, ζ those of the corresponding point on the sphere we have, by similar triangles,

$$\frac{\xi^2 + \eta^2}{1 - \xi} = \frac{x^2 + y^2}{x^2 + y^2 + 1},$$

$$\frac{\xi}{\eta} = \frac{x}{y}.$$

From these and the equation of the

sphere we obtain

$$\xi = \frac{x^2 + y^2}{x^2 + y^2 + 1}, \quad \eta = \frac{x}{y} \xi.$$

Solving for ξ, η, ζ we have

$$\xi = \frac{x}{x^2 + y^2 + 1}, \quad \eta = \frac{y}{x^2 + y^2 + 1}, \quad \zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Also $x = \frac{\xi}{1 - \xi}, \quad y = \frac{\eta}{1 - \xi}, \quad x^2 + y^2 = \frac{\xi}{1 - \xi}.$

Since on the equator of the sphere $\xi = \frac{1}{2}$, we have for the circle in the plane corresponding to the equator

$$x^2 + y^2 = 1,$$

which is the unit circle.

We may show analytically that every straight line and every circle becomes a circle on the sphere. If the equation of the straight line in the plane is

$$ax + by + c = 0,$$

and the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we have, by substituting the preceding values,

$$a\xi + b\eta + c(1-\xi) = 0$$

and
$$\xi + 2g\xi + 2f\eta + c(1-\xi) = 0$$

as the equations of the corresponding curves on the sphere. Both of these represent planes which of course cut the sphere in circles.

Let us consider now the analytic expressions in terms of the space coordinates ξ, η, ζ of the transformations of the spherical surface corresponding to the linear transformation of the complex variable. If x and y, ξ, η , and ζ be the coordinates of two corresponding points in the plane and on the sphere, and if x' and y', ξ', η' and ζ' are the coordinates of the transformed points, we have

$$x = \frac{\xi}{1-\xi}, \quad y = \frac{\eta}{1-\xi}, \quad x^2 + y^2 = \frac{\xi}{1-\xi},$$

$$x' = \frac{\xi'}{1-\xi'}, \quad y' = \frac{\eta'}{1-\xi'}, \quad x'^2 + y'^2 = \frac{\xi'}{1-\xi'}.$$

Let us find ξ', η' and ζ' in terms of ξ, η, ζ . Begin with the logarithmic

spiral motion in the plane given by

$$z' = \alpha z + \beta,$$

where $\alpha = a_1 + a_2 i$ and $\beta = b_1 + b_2 i$.

Then

$$x' + iy' = (a_1 + a_2 i)(x + yi) + b_1 + b_2 i.$$

$$\therefore x' = a_1 x - a_2 y + b_1,$$

$$y' = a_1 y + a_2 x + b_2,$$

$$x'^2 + y'^2 = (a_1^2 + a_2^2)(x^2 + y^2) + 2(a_1 b_1 + a_2 b_2)x + 2(a_1 b_2 - a_2 b_1)y + b_1^2 + b_2^2.$$

Hence from the equations already obtained,

$$\frac{\xi'}{1-\xi'} = \frac{a_1 \xi - a_2 \eta + b_1(1-\xi)}{1-\xi},$$

$$\frac{\eta'}{1-\xi'} = \frac{a_2 \xi + a_1 \eta + b_2(1-\xi)}{1-\xi},$$

$$\begin{aligned} \frac{\xi'}{1-\xi'} &= \frac{(a_1^2 + a_2^2)\xi + 2(a_1 b_1 + a_2 b_2)\xi + 2(a_1 b_2 - a_2 b_1)\eta + (b_1^2 + b_2^2)(1-\xi)}{1-\xi} \\ &= \frac{(a_1^2 + a_2^2 - b_1^2 - b_2^2)\xi + 2(a_1 b_1 + a_2 b_2)\xi + 2(a_1 b_2 - a_2 b_1)\eta + b_1^2 + b_2^2}{1-\xi}. \end{aligned}$$

If we denote the numerator on the right hand side of the third equation by F we have

$$\frac{\xi'}{1-\xi'} = \frac{F}{1-\xi}, \quad \xi' = \frac{F}{1-\xi}(1-\xi'),$$

$$\xi' = \frac{F}{1-\xi+F}, \quad 1-\xi' = \frac{1-\xi}{1-\xi+F}.$$

$$\xi' = \frac{a_1 \xi - a_2 \eta + b_1 (1-\xi)}{1-\xi} \cdot (1-\xi')$$

$$\therefore \xi' = \frac{a_1 \xi - a_2 \eta + b_1 (1-\xi)}{1-\xi+F}$$

$$\eta' = \frac{a_2 \xi + a_1 \eta + b_2 (1-\xi)}{1-\xi+F}.$$

The coordinates ξ' , η' and ξ' are then linear functions of ξ , η and ξ , and the denominators in the expressions just given are all equal.

Consider the case

$$z' = 1/z, \text{ or } x' + iy' = \frac{x - iy}{x^2 + y^2}.$$

$$\therefore x' = \frac{x}{x^2 + y^2}, \quad y' = \frac{-y}{x^2 + y^2},$$

$$x'^2 + y'^2 = \frac{1}{x^2 + y^2}.$$

$$\frac{\xi'}{1-\xi'} = \frac{\xi}{\xi},$$

$$\frac{\eta'}{1-\xi'} = -\frac{\eta}{\xi},$$

$$\frac{\xi'}{1-\xi'} = \frac{1-\xi}{\xi}.$$

Hence $\xi' = 1 - \xi$, $\therefore \xi' = \xi$, $\eta' = -\eta$.

These are again linear equations.

All the other linear transformations of a complex variable were obtained from combinations of those just considered. We may then conclude that "every transformation of the spherical surface which arises from a linear transformation of the complex variable, is, when analytically expressed in terms of the coordinates ξ, η, ζ , itself a linear transformation of these coordinates, in which the denominators in the equations of transformation are all equal."

Chapter III.

Linear Substitutions of the Complex Variable and the Regular Solids.

With the aid of these ideas on stereographic projection we may now proceed to the discussion of the icosahedron and other groups. For the representation of the values of the complex variable $z = x + iy$, we shall use the same sphere upon which we have already studied our revolution and point groups and marked our fundamental regions.

Let us take the system of points defined by the equation $f(z) = 0$ and consider the conditions under which such point groups would go over into themselves, either by turnings or reflections of the sphere. We

have as our fundamental theorem that every revolution of the $(x + iy)$ sphere about its middle point will be represented by a linear substitution of z ;

$$(1) \quad w = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

The z on the original sphere and the w obtained by this transformation are related in a one-to-one manner. Moreover we may have another transformation obtained by a reflection,

$$(2) \quad w = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}$$

where $\bar{z}$ is the conjugate imaginary $(x - iy)$ of z . Our equation $f(z) = 0$ remains unchanged by a group of linear substitutions (1), or by an extended group which contains, besides the substitutions (1), a corresponding number of substitutions (2).

On account of the many results arising from the equations $f(z) = 0$

let us consider them from the analytical standpoint. These variables are homogeneous. If we give z the value $\frac{z_1}{z_2}$ the substitution (1) (and likewise the substitution (2)) gives two operations:

$$(3) \quad \begin{cases} z_1' = \alpha z_1 + \beta z_2, \\ z_2' = \gamma z_1 + \delta z_2, \end{cases}$$

where the absolute value of the substitution determinant $(\alpha\delta - \beta\gamma)$ is some particular quantity. In place of $f(z) = 0$ or $f(\frac{z_1}{z_2}) = 0$, since we may multiply by any power of z_2 , we may consider the form $f(z_1, z_2) = 0$. This form always has the same degree as the given point group, since the occurrence of the point $z = \infty$ is indicated by a factor z_2 of f . By the substitution (3) f does not need to remain absolutely unchanged; it may vary by a factor, and we shall

now try to determine this factor. We shall find the invariant theory of binary forms in Algebra very helpful in complicated cases for determining various forms of f from a certain one. The result of the discussion is that for each each group of linear substitutions (1) corresponding to our rotation groups, there will be found a certain rational function

$$(4) \quad Z = R(z),$$

which represents the different point groups belonging to our group, if it is put equal to a certain constant.

Let us consider those linear transformations of $(x+iy)$ which correspond to the revolutions of the sphere about its middle point. Let the equation of our sphere, referred to a rectangular coordinate system, be

$$(5) \quad \xi^2 + \eta^2 + \zeta^2 = 1.$$

If we take $x+iy$, i.e., the $\xi\eta$ -plane,

as the equator, and project it stereographically on to the sphere from the pole $\xi = 0, \eta = 0, \zeta = 1$, we obtain the formulae (as already shown):

$$(6) \quad x = \frac{\xi}{1-\xi}, \quad y = \frac{\eta}{1-\xi}, \quad x+iy = \frac{\xi+i\eta}{1-\xi},$$

$$(7) \quad \xi = \frac{2x}{1+x^2+y^2}, \quad \eta = \frac{2y}{1+x^2+y^2}, \quad \zeta = \frac{-1+x^2+y^2}{1+x^2+y^2}.$$

Since we are considering those linear substitutions of z which correspond to the revolutions of the sphere, we must study the diametral points of the sphere, as one pair of them is unchanged by every revolution. In (6) substitute for ξ, η, ζ their negatives and we have the diametral point

$$x'-iy' = \frac{-\xi+i\eta}{1+\xi}.$$

Multiply this by the value of $(x+iy)$ in (6) and comparing with (5), we obtain

$$(x+iy)(x'-iy') = -1,$$

or if we put

$$x+iy = re^{i\theta},$$

$$x' + iy' = \frac{1}{x} e^{i(\varphi + \pi)}.$$

Hence the absolute values of the arguments of diametral points are reciprocal, while their amplitudes differ by π .

Let us take next the case where the revolution is through the angle α about the axis perpendicular to the equator plane, and if one looks at this from the point at infinity, above the equator plane, the revolution is opposite to the hands of a clock. A point which had the argument z now has the argument z' , corresponding to z in the same way that $(\xi' + i\eta')$ corresponded with $(\xi + i\eta)$ when the equator $(\xi\eta)$ plane was revolved in the given manner, for the denominator $(1 - \xi)$ in the formulae is unchanged by this revolution. We have in the $\xi\eta$ -plane for this revolution

$$\xi' = \xi \cos \alpha - \eta \sin \alpha,$$

$$\eta' = \xi \sin \alpha + \eta \cos \alpha,$$

or $\xi' + i\eta' = (\cos \alpha + i \sin \alpha)(\xi + i\eta),$

which may be written

(10) $z' = e^{i\alpha} z.$

If now we consider the revolution through the angle α by which the points (ξ, η, ξ) and $(-\xi, -\eta, -\xi)$ on the sphere are unchanged and by which the first point plays the same rôle that infinity did above, so that if we look from (ξ, η, ξ) outside the revolution is opposite the hands of a clock, we must replace z in (10) by such a linear function of z as will finally change into (ξ, η, ξ) and $(-\xi, -\eta, -\xi)$. The most general form of this function is

$$C \frac{z + \frac{\xi + i\eta}{1 + \xi}}{z - \frac{\xi + i\eta}{1 - \xi}}.$$

If we substitute this value in (10) we have

$$\frac{z' + \frac{\xi + i\eta}{1+\xi}}{z' - \frac{\xi + i\eta}{1-\xi}} = e^{i\alpha} \cdot \frac{z + \frac{\xi + i\eta}{1+\xi}}{z - \frac{\xi + i\eta}{1-\xi}},$$

or more simply

$$(11) \quad e^{-\frac{i\alpha}{2}} \cdot \frac{z'(1+\xi) + (\xi + i\eta)}{z'(1-\xi) - (\xi + i\eta)} = e^{\frac{i\alpha}{2}} \cdot \frac{z(1+\xi) + (\xi + i\eta)}{z(1-\xi) - (\xi + i\eta)}.$$

This is also a general formula for a certain revolution.

To get the value of z' make the following substitutions:

$$(12) \quad \xi \sin \frac{\alpha}{2} = a, \quad \eta \sin \frac{\alpha}{2} = b, \quad \xi \cos \frac{\alpha}{2} = c, \quad \cos \frac{\alpha}{2} = d,$$

Hence also,

$$(13) \quad a^2 + b^2 + c^2 + d^2 = 1.$$

Then we have the simple form:

$$(14) \quad z' = \frac{(d+ic)z - (b-ia)}{(b+ia)z + (d-ic)}.$$

This may be written

$$z' = \frac{Az + B}{Cz + D},$$

whence we obtain

$$(15) \quad \cos \frac{\alpha}{2} = \frac{A + D}{\sqrt{AD - BC}}.$$

We may separate formula (4) into two homogeneous linear substitutions:

$$(6) \quad \begin{cases} z_1' = (d + ic)z_1 - (b - ia)z_2, \\ z_2' = (b + ia)z_1 + (d - ic)z_2. \end{cases}$$

Here the a, b, c, d of (12) denote real quantities satisfying the equation

$$a^2 + b^2 + c^2 + d^2 = 1.$$

In order to extend the use of our substitution formulae let S be a substitution:

$$S \equiv \begin{cases} z_1' = (d + ic)z_1 - (b - ia)z_2, \\ z_2' = (b + ia)z_1 + (d - ic)z_2; \end{cases}$$

and T another substitution:

$$T \equiv \begin{cases} z_1'' = (d' + ic')z_1' - (b' - ia')z_2', \\ z_2'' = (b' + ia')z_1' + (d' - ic')z_2'. \end{cases}$$

We may then obtain the product TS by eliminating z_1' and z_2' .

$$TS \equiv \begin{cases} z_1'' = (d'' + ic'')z_1 - (b'' - ia'')z_2, \\ z_2'' = (b'' + ia'')z_1 + (d'' - ic'')z_2, \end{cases}$$

where

$$(17) \quad \begin{cases} a'' = (ad' + a'd) - (bc' - b'c) \\ b'' = (bd' + b'd) - (ca' - c'a) \\ c'' = (cd' + c'd) - (ab' - a'b) \\ d'' = -aa' - bb' - cc' + dd'. \end{cases}$$

This product TS is used in the same sense as TS in the revolutions of the icosahedron; first T is performed, then S upon this.

The case of the cyclic and dihedral groups is so simple that we can write the formulae down at once. In both these groups we shall let the poles be at $z=0$ and $z=\infty$. Then for the revolutions of the cyclic group

$$a=b=0, \quad c = \sin \frac{\alpha}{2}, \quad d = \cos \frac{\alpha}{2}, \quad \alpha = \frac{2K\pi}{n},$$

and for the $2n$ homogeneous substitutions of the cyclic group,

$$(18) \quad z_1' = e^{\frac{iK\pi}{n}} z_1, \quad z_2' = e^{-\frac{iK\pi}{n}} z_2, \\ (K=0, 1, \dots, (2n-1)).$$

In the dihedral group let

we choose one of the subaxes to coincide with the ξ axis of our space coordinate system, and the points on the sphere, $z=1$, $z=-1$, fixed. For the first revolution,

$$(19) \quad z_1' = \mp i z_2, \quad z_2' = \mp i z_1.$$

Combining with (18) we obtain for the $4n$ homogeneous substitutions of the dihedral group:

$$(20) \quad \begin{cases} z_1' = e^{\frac{iK\pi}{n}} z_1, & z_2' = e^{-\frac{iK\pi}{n}} z_2, \\ z_1' = i e^{-\frac{iK\pi}{n}} z_2, & z_2' = i e^{\frac{iK\pi}{n}} z_1, \end{cases} \\ (K = 0, 1, \dots, (2n-1)).$$

In the same way we have for the right substitutions of the triaxial group:

$$(21) \quad \begin{cases} z_1' = i^K z_1, & z_2' = (-i)^K z_2; \\ z_1' = -(-i)^K z_2, & z_2' = i^K z_1; \end{cases} \\ (K = 0, 1, 2, 3).$$

With the tetrahedron and octahedron we may take two different positions of the coordinates. First we use as the three perpendicular coordinate

axes ξ, η, ξ , the diagonals of the octahedron. In the second case revolve the coordinate system thus obtained through 45° about the ξ axis, so that the $\xi\xi$ -plane will coincide with one of the symmetric planes of the tetrahedron.

Let us begin with the first position. Of course we may simply write down the formulae (21) of the triaxial group. As before, the tetrahedron and the octahedron group we shall form next the homogeneous substitutions ~~groups~~ which correspond to the revolutions of period 3 about one of the diagonals of the cube. Two diametrically opposite corners of the cube have the coordinates

$$\xi = \eta = \xi = \pm \frac{1}{\sqrt{3}},$$

and since

$$\cos \frac{\pi}{3} = \frac{1}{2} = -\cos \frac{2\pi}{3},$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = \sin \frac{2\pi}{3},$$

we have for the homogeneous substitu-

tutions by which these two corners remain fixed

$$a = b = c = \pm d = \frac{1}{2}.$$

Hence we find the two substitutions

$$z_1' = \frac{(\pm 1 + i)z_1 - (1 - i)z_2}{2},$$

$$z_2' = \frac{(1 + i)z_1 + (\pm 1 - i)z_2}{2}.$$

If we combine these with the substitutions of (21) we obtain the following pairs of linear expressions as the 24 homogeneous tetrahedron substitutions:

$$(22) \quad \left\{ \begin{array}{l} z_1' = i^\kappa z_1, \quad z_2' = (-i)^\kappa z_2; \\ z_1' = -(-i)^\kappa z_2, \quad z_2' = i^\kappa z_1; \\ z_1' = i^\kappa \frac{(\pm 1 + i)z_1 - (1 - i)z_2}{2}, \\ z_2' = (-i)^\kappa \frac{(1 + i)z_1 + (\pm 1 - i)z_2}{2}; \\ z_1' = -(-i)^\kappa \frac{(1 + i)z_1 + (\pm 1 - i)z_2}{2}, \\ z_2' = i^\kappa \frac{(\pm 1 + i)z_1 - (1 - i)z_2}{2}; \end{array} \right. \quad (\kappa = 0, 1, 2, 3).$$

The octahedron group is obtained by adding the revolution

through $\frac{\pi}{2}$ about one of the three coordinate axes, say the ξ -axis. For one of the two corresponding homogeneous substitutions we have

$$(23) \quad \eta_1' = \frac{1+i}{\sqrt{2}} \eta_1, \quad \eta_2' = \frac{1-i}{\sqrt{2}} \eta_2.$$

Twenty four of the homogeneous octahedron substitutions are given in table (2.2). The others are obtained by multiplying each η_1' of (2.2) by $\frac{1+i}{2} \eta_1$, and each η_2' of (2.2) by $\frac{1-i}{\sqrt{2}} \eta_2$, making in all forty-eight substitutions.

For the second position of the coordinate system it is necessary, in order to obtain the substitution formulae from (2.2), (2.3), etc., only to find the coordinate transformation which carries the first position into the second. By such a transformation the original $\frac{\eta_1}{\eta_2}$ will be changed into $\frac{1+i}{\sqrt{2}} \frac{\eta_1}{\eta_2}$, and also $\frac{\eta_1'}{\eta_2'}$ into $\frac{1+i}{\sqrt{2}} \frac{\eta_1'}{\eta_2'}$. If, then, we wish to obtain the substitution formulae which correspond to this new

position of the coordinate axes, in the expressions for z_i' in (22) we must leave z_1 unchanged, and substitute $\frac{1-i}{\sqrt{2}} z_2$ for z_2 , and for z_2' we must leave z_2 unchanged and substitute $\frac{1+i}{\sqrt{2}} z_1$ for z_1 .

For the icosahedron group we shall arrange the position of the coordinate system in such a way that each revolution through $\frac{2\pi}{5}$ is in the positive direction about the ξ -axis, while the cross line about which was the revolution U through π , coincides with the η -axis. Then we have corresponding to the operations S and U :

$$(24) \quad \begin{cases} S = \begin{cases} z_1' = \pm \epsilon^3 z_1, \\ z_2' = \pm \epsilon^2 z_2; \end{cases} \\ U = \begin{cases} z_1' = \mp z_2, \\ z_2' = \pm z_1, \end{cases} \end{cases}$$

which is a form for the dihedral substitutions where $\epsilon = \frac{2i\pi}{5}$.

For the notation T two cases

may arise, according to the position of our coordinate system: the axis of T , which is the ξ -axis, in the ξ - η -plane may extend through the first and third quadrants, or through the second and fourth. Let us use the latter. If γ is the angle which this axis makes with $O\xi$ one of its end points will be given by the coordinates

$$\xi = -\sin \gamma, \quad \eta = 0, \quad \zeta = \cos \gamma;$$

whence $a = \mp \sin \gamma, \quad b = 0, \quad c = \pm \cos \gamma, \quad d = 0$.

We have still the angle γ to determine. Comparing the formulae (24) and those for Scaled T (17) we find

$$a' = b' = 0, \quad c' = \pm \sin \frac{\pi}{5}, \quad d' = \pm \cos \frac{\pi}{5}.$$

Also for the operation TS

$$\begin{aligned} d'' &= -aa' - bb' - cc' + dd' \\ &= \pm \cos \gamma \cdot \sin \frac{\pi}{5}. \end{aligned}$$

Since TS is of period 3 d'' must correspond with $\pm \cos \frac{\pi}{3} = \pm \frac{1}{2}$. Hence, choosing $\cos \gamma$ positive,

$$\cos \gamma \cdot \sin \frac{\pi}{5} = \frac{1}{2},$$

$$\cos \gamma = \frac{1}{2 \sin \frac{\pi}{5}}.$$

But

$$(\epsilon^2 - \epsilon^3)(\epsilon^4 - \epsilon) = \epsilon + \epsilon^4 - \epsilon^2 - \epsilon^3 = \sqrt{5}.$$

Then

$$\cos \gamma = \frac{1}{i(\epsilon^3 - \epsilon^2)} = \frac{\epsilon - \epsilon^4}{i\sqrt{5}},$$

and $\sin \gamma = \frac{\epsilon^2 - \epsilon^3}{i\sqrt{5}}.$

The values for a, b, c, d may now be found. Substituting in (6) we have for the homogeneous substitutions corresponding to the revolution T :

$$(26) \quad \begin{cases} \sqrt{5} z'_1 = \mp(\epsilon - \epsilon^4) z_1 \pm (\epsilon^2 - \epsilon^3) z_2, \\ \sqrt{5} z'_2 = \pm(\epsilon^2 - \epsilon^3) z_1 \pm (\epsilon - \epsilon^4) z_2. \end{cases}$$

From (24) and (26) we may form the icosahedron substitutions corresponding to the revolutions S^μ , US^μ , $S^\nu T S^\mu$, $US^\nu T S^\mu$, ($\mu, \nu = 0, 1, 2, 3, 4$).

The 120 homogeneous icosahedron substitutions are:

$$(27) \quad \begin{cases} S^\mu \equiv \begin{cases} z'_1 = \pm \epsilon^{3\mu} z_1, \\ z'_2 = \pm \epsilon^{2\mu} z_2; \end{cases} \\ US^\mu \equiv \begin{cases} z'_1 = \mp \epsilon^{2\mu} z_2, \\ z'_2 = \pm \epsilon^{3\mu} z_1; \end{cases} \end{cases}$$

$$(27) \begin{cases} S^\nu T S^\mu \equiv \begin{cases} \sqrt{5} z_1' = \pm \epsilon^{3\nu} \{ -(\epsilon - \epsilon^4) \epsilon^{3\mu} z_1 + (\epsilon^2 - \epsilon^3) \epsilon^{2\mu} z_2 \}, \\ \sqrt{5} z_2' = \pm \epsilon^{2\nu} \{ +(\epsilon^2 - \epsilon^3) \epsilon^{3\mu} z_1 + (\epsilon - \epsilon^4) \epsilon^{2\mu} z_2 \}; \end{cases} \\ US^\nu T S^\mu \equiv \begin{cases} \sqrt{5} z_1' = \mp \epsilon^{2\nu} \{ +(\epsilon^2 - \epsilon^3) \epsilon^{3\mu} z_1 + (\epsilon - \epsilon^4) \epsilon^{2\mu} z_2 \}, \\ \sqrt{5} z_2' = \pm \epsilon^{3\nu} \{ -(\epsilon - \epsilon^4) \epsilon^{3\mu} z_1 + (\epsilon^2 - \epsilon^3) \epsilon^{2\mu} z_2 \}. \end{cases} \end{cases}$$

By formula (15)

$$\cos \frac{\alpha}{2} = \frac{A + D}{\sqrt{AD - BC}}.$$

For the angle α of the revolution $S^\nu T S^\mu$ we have

$$\cos \frac{\alpha}{2} = \mp \frac{(\epsilon - \epsilon^4)(\epsilon^{3\mu+3\nu} - \epsilon^{2\mu+2\nu})}{2\sqrt{5}},$$

and likewise for $US^\nu T S^\mu$,

$$\cos \frac{\alpha}{2} = \mp \frac{(\epsilon^2 - \epsilon^3)(\epsilon^{3\mu+2\nu} - \epsilon^{2\mu+3\nu})}{2\sqrt{5}}.$$

Hence we have the period 2 in $S^\nu T S^\mu$ if $\mu + \nu \equiv 0$, and in $US^\nu T S^\mu$ if $3\mu + 2\nu \equiv 0 \pmod{5}$. We have the period 3 in $S^\nu T S^\mu$ if $\mu + \nu \equiv \pm 1$, in $US^\nu T S^\mu$ if $3\mu + 2\nu \equiv \pm 1 \pmod{5}$. In the 20 other cases $S^\nu T S^\mu$ as well as $US^\nu T S^\mu$ has the period 5. Also all US^μ have the

period 2, and all S^n with the exception of $S^0(1)$ have the period 5.

It is easy to pass from the homogeneous to the non-homogeneous substitutions if we allow the values of the substitution determinant, which up to this time have remained fixed, to vary, substituting z for $\frac{z_1}{z_2}$. For the non-homogeneous substitutions we have:

(1) in the cyclic group,
 (28) $z' = e^{\frac{2ik\pi}{n}} z, \quad (k = 0, 1, 2, \dots, (n-1));$

(2) in the dihedral group,
 (29) $\begin{cases} z' = e^{\frac{2ik\pi}{n}} z, \\ z' = -\frac{e^{-\frac{2ik\pi}{n}}}{z} \end{cases}, \quad (k = 0, 1, 2, \dots, (n-1));$

(3) in the tetrahedron group with the first position of the coordinate system,

(30a) $z' = \pm z, \pm \frac{1}{z}, \pm i \frac{z+1}{z-1}, \pm i \frac{z-1}{z+1},$
 $\pm \frac{z+i}{z-i}, \pm \frac{z-i}{z+i},$

or with the second position,

$$(30b) \quad z' = \pm z, \pm \frac{i}{z}, \pm \frac{(1+i)z + \sqrt{2}}{\sqrt{2}z - (1-i)}, \pm \frac{\sqrt{2}z - (1-i)}{(1+i)z + \sqrt{2}}, \\ \pm \frac{(1-i)z + \sqrt{2}}{\sqrt{2}z - (1+i)}, \pm \frac{\sqrt{2}z - (1+i)}{(1-i)z + \sqrt{2}};$$

(4) in the octahedron group with the same positions,

$$(31a) \quad z' = i^k z, \frac{i^k}{z}, i^k \cdot \frac{z+1}{z-1}, i^k \cdot \frac{z-1}{z+1}, \\ i^k \cdot \frac{z+i}{z-i}, i^k \cdot \frac{z-i}{z+i},$$

or

$$(31b) \quad z' = i^k z, \frac{i^k}{z}, i^k \cdot \frac{(1+i)z + \sqrt{2}}{\sqrt{2}z - (1-i)}, i^k \cdot \frac{\sqrt{2}z - (1-i)}{(1+i)z + \sqrt{2}}, \\ i^k \cdot \frac{(1-i)z + \sqrt{2}}{\sqrt{2}z - (1+i)}, i^k \cdot \frac{\sqrt{2}z - (1+i)}{(1-i)z + \sqrt{2}}, (\kappa = 0, 1, 2, 3);$$

(5) in the icosahedron group,

$$(32) \quad z' = \epsilon^\mu z, -\frac{\epsilon^{4\mu}}{z}, \epsilon^\nu \frac{-(\epsilon - \epsilon^4)\epsilon^\mu z + (\epsilon^2 - \epsilon^3)}{(\epsilon^2 - \epsilon^3)\epsilon^\mu z + (\epsilon - \epsilon^4)}, \\ -\epsilon^{4\nu} \frac{(\epsilon^2 - \epsilon^3)\epsilon^\mu z + (\epsilon - \epsilon^4)}{-(\epsilon - \epsilon^4)\epsilon^\mu z + (\epsilon^2 - \epsilon^3)}, \\ (\epsilon = e^{\frac{2i\pi}{5}}; \mu, \nu = 0, 1, 2, 3, 4).$$

These formulae correspond to the extended group. For instance in (30a) the $\xi\xi$ -plane of our coordinate system corresponds to a sym-

metric plane in the tetrahedron. We may now form the extended group by a combination of the reflections on these symmetric planes with the operations of the original group. These reflections are given by the formula,

$$z' = \bar{z},$$

where $\bar{z}$ is the conjugate of the imaginary z . Hence we may obtain formulae for the operation of the extended group if we set $z = \bar{z}$ in the formulae (28) to (32).

When we were working directly from our model we discussed the relation existing between each of the groups. Let us attempt to find an analogy in the case of the substitutions, and consider when the homogeneous substitution group and the rotation group are simply isomorphic. For a group of N rotations we have found $2N$ substitutions. We

ask whether, among these $2N$ substitutions, we cannot find N that form a group which is simply isomorphic with the rotation group, or whether an isomorphism exists such that we may obtain another value of the substitution determinant, which we have heretofore considered equal to $+1$.

Let us begin with the cyclic group. Consider a revolution through $\frac{2\pi}{n}$ by which a certain point (ξ, η, ζ) of the sphere remains fixed. For the original linear substitution (6):

$$z_1' = (d + ic) z_1 - (b - ia) z_2,$$

$$z_2' = (b + ia) z_1 + (d - ic) z_2,$$

the parameters are given by

$$a = \pm \xi \sin \frac{\pi}{n}, \quad b = \pm \eta \sin \frac{\pi}{n},$$

$$c = \pm \xi \sin \frac{\pi}{n}, \quad d = \pm \cos \frac{\pi}{n}.$$

Since we are going to take the substitution determinant equal to ρ^2 , we shall write in place of these

$$(34) \quad a_1 = \rho \xi \sin \frac{\pi}{n}, \quad b_1 = \rho \eta \sin \frac{\pi}{n},$$

$$c_1 = \rho \xi \sin \frac{\pi}{n}, \quad d_1 = \rho \cos \frac{\pi}{n}.$$

We obtain then as the k^{th} repetition of our substitution, as in (17),

$$a_k = \rho^k \xi \sin \frac{k\pi}{n}, \quad b_k = \rho^k \eta \sin \frac{k\pi}{n},$$

$$c_k = \rho^k \xi \sin \frac{k\pi}{n}, \quad d_k = \rho^k \cos \frac{k\pi}{n}.$$

Since we wish simple isomorphism with the rotation group the n^{th} repetition of our substitution must be the identity, also,

$$a_n = b_n = c_n = 0, \quad d_n = 1.$$

Then we must have

$$\rho^n = -1.$$

Hence we shall have simple isomorphism between the substitution and the rotation groups when and only when we have, in (34), ρ as the n^{th} root of -1 . However in the value of the substitution determinant two cases may arise. If n is odd $\rho = -1$, and the determinant still is $+1$. But if n is even the value $+1$ of the determinant is in

admissible. For instance if $n=2$ and the determinant equals -1 , the value of ρ equals $\pm i$.

Consider the dihedral group. We have the rotation S^{μ} ($S^n = 1$), to which will correspond substitutions of the determinant $\rho^{2\mu}$ where $\rho^n = -1$. We have also the rotation TS^{μ} of period 2. That we may have simple isomorphism the substitution which corresponds to T must be paired with the determinant (-1) . If now we multiply these two we have, for TS^{μ} , a substitution of the determinant $-\rho^{2\mu}$. But this must equal -1 since TS^{μ} has the period 2. Hence we have

$$\rho^n = -1, \rho^{2\mu} = +1, (\mu = 0, 1, \dots, (n-1)).$$

This is possible only when n is odd. Therefore in the dihedral group the desired simple isomorphism can be obtained only when n is odd, never if n is even.

In the case of the tetrahedron, octahedron and icosahedron simple isomorphism between the rotation group and the substitution group is impossible, since each contains as subgroup at least a dihedral group of even n , namely a triaxial group.

Chapter IV.

Other Graphical Methods of Representing a Group.

Up to this time we have made use only of those groups closely connected with the regular solids. As has been said a group may be represented on any body which looks the same from two or more positions. We have already made use of the case of the equilateral triangle. We might use equally well the isosceles triangle or the square or, in fact, any of the other plane figures which are at all regular, also the crystals and many of the prisms. We have considered the groups represented on a regular sphere and shall soon discuss those on a sphere having one or two holes in it. We might also use a sphere with three

holes in it as well as many other conformations having some degree of regularity about them. If we think of the outside surface or skin as flexible and loose from the surface proper, we may imagine it twisted or stretched in such a way as to bring into coincidence corresponding figures which have been described upon it. The motion is not necessarily a rotation as in the cases considered before this.

Suppose the group which we are going to consider is represented graphically on a plane. As before, the fundamental regions will be divided into shaded and unshaded parts. There will be corresponding sides upon the boundary as well as the interior, and when these are brought into coincidence by bending and stretching the figure, it will be no longer a plane but

some sort of continued surface, usually multiply connected. If N is the order of the group the continuous unbounded surface will be divided into $2N$ polygons, alternately shaded and unshaded. The correspondence between the operations of the group and the unshaded polygons on the surface is given by a rule that a single positive rotation of the white polygon Σ round one of its corners A_n leads to the white polygon $S_n \Sigma$.

Let us choose as an example the group defined by

$$S_1^4 = 1, \quad S_2^4 = 1, \quad S_3^4 = 1,$$

$$S_1 S_2 S_3 = 1, \quad S_2^{-1} S_1 S_2 S_1 = 1.$$

The plane figure for this group is given ^{in Plate III} on the next page, and it is easy to see that opposite sides of the boundary correspond. The operations of this group are not permutable. It is of order 8 with a single operation of order 2:

$$S_1^2 = S_2^2 = S_3^2 ;$$

This corresponds to a displacement of the triangles among themselves, in which all the six corners remain fixed. If the corresponding sides of the boundary are brought into coincidence the continuous surface formed will be a double-holed anchor ring, or sphere with two holes through it. Each half of the surface is divided into triangles in a manner exactly similar to the other half. The operation of order 2 replaces each triangle of one half by the corresponding triangle of the other, an operation which evidently leaves the six corners unchanged.

The number of sides of the plane figure C which represents the group, may be greatly modified by replacing individual polygons on the boundary by equivalent ones. However if corresponding sides of the boundary

are counted as a single side, and corresponding corners as a single corner, the number of corners, sides and polygons will remain constant no matter how the figure is modified. If A be the number of corners, and E the number of sides of C , $2N$ being the number of polygons, then the connectivity, $2p+1$, of the closed surface is given by the equation

$$2p = 2 + E + 2N - A.$$

Although p is independent of the form of C the connectivity of the surface is not in general independent of the generating operations of the group which determine the form of C . There is however a lower limit to p for any given group, whatever be its generating operations. This value of p we shall call the genus or deficiency of the group.

If N is the order of the group generated by n operations the

surface will be divided into $2N$ polygons of n sides each. Let $A_1, A_2, \dots, A_n$ be the angular points of one of these polygons; and suppose that on the surface there are C_1 corners in the set to which A_1 belongs, C_2 to which A_2 belongs, etc. Round each corner A_r there are $2m_r$ polygons, and each polygon has one and only one corner of the set to which A_r belongs. Hence

$$C_r m_r = N$$

$$\text{and } C_1 + C_2 + \dots + C_n = N \sum_i \frac{1}{m_i}$$

Also each side belongs to two and only to two polygons, so that the number of sides is N_n . Substituting these values for A and E in the formula for the connectivity we have

$$2(p-1) = N(n-2 - \sum_i \frac{1}{m_i}).$$

Let us consider first the case where $p=0$. The equation then becomes

$$2(1 - \frac{1}{N}) = \sum_i (1 - \frac{1}{m_i})$$

where m must be 2 or 3. First, let $n=2$. The only possible solution then is $N=m_1=m_2=n$, n being any integer. The corresponding group is a cyclic group of order n . Secondly, let $n=3$. Then one at least of the three numbers m_1, m_2, m_3 must equal 2, as otherwise the right hand side of the equation would not be less than 2. Set $m_1=2$. If now both m_2 and m_3 were greater than 3, the right hand side would still not be less than 2; we may therefore choose m_2 either 2 or 3. If $m_1=m_2=2$, the equation becomes

$$\frac{2}{N} = \frac{1}{m_3},$$

putting $m_3=n$, $N=2n$, where n is any integer. If $m_1=2$ and $m_2=3$, the equation is

$$\frac{2}{N} + \frac{1}{6} = \frac{1}{m_3}.$$

This has three solutions for N and m_3 in positive integers:

$$m_3=3, N=12;$$

$$m_3 = 4, N = 24;$$

$$m_3 = 5, N = 60.$$

We may therefore tabulate the results for the case $\phi = 0$ thus:

	m_1	m_2	m_3	N
I	n	n		n
II	2	2	n	$2n$
III	2	3	3	12
IV	2	3	4	24
V	2	3	5	60

These five distinct types represented graphically are given on the following pages. It will be seen that the first is the regular cyclic group, the second the dihedral group, the third the tetrahedron group, the fourth the octahedron group, the fifth the icosahedron group. It is from such plane figures as these that the model for our former discussion, as represented on the sphere, was obtained by stereographic projection.

In terms of their generating operations the five types of group of

I. Cyclic Group.

II. Dihedral Group.

III. Tetrahedron Group.

IV. Octahedron Group.

V. Icosahedron Group.

genus zero are given by the relations:

- I. $S_1^n = 1, S_2^n = 1, S_1 S_2 = 1;$
 II. $S_1^2 = 1, S_2^2 = 1, S_3^n = 1, S_1 S_2 S_3 = 1;$
 III. $S_1^2 = 1, S_2^3 = 1, S_3^3 = 1, S_1 S_2 S_3 = 1;$
 IV. $S_1^2 = 1, S_2^3 = 1, S_3^4 = 1, S_1 S_2 S_3 = 1;$
 V. $S_1^2 = 1, S_2^3 = 1, S_3^5 = 1, S_1 S_2 S_3 = 1.$

Let us now consider the case for $\ell = 1$. Here the order of the group disappears from the equation and we have

$$2 = \sum_{i=1}^n (1 - \frac{1}{m_i}),$$

and n must be either 3 or 4. If $n = 4$ the equation becomes

$$2 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4},$$

and $m_1 = m_2 = m_3 = m_4 = 2$.

If $n = 3$ the equation becomes

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3},$$

which has three solutions:

$$m_1 = 3, m_2 = 3, m_3 = 3;$$

$$m_1 = 2, m_2 = 4, m_3 = 4;$$

$$m_1 = 2, m_2 = 3, m_3 = 6.$$

Most of these may be rep-

represented as the elliptic functions are.

Let us take one example where $n=3$,

$$m_1 = m_2 = m_3 = 3.$$

We have then the group generated by S_1, S_2, S_3 where

$$S_1^3 = 1, S_2^3 = 1, S_3^3 = 1, S_1 S_2 S_3 = 1.$$

The generating operations are rotations through $\frac{2\pi}{3}$ about the angles of an equilateral triangle; and every operation of the group is either a translation or a rotation. If the plane figure given ^{in plate VII} below is transferred to the corresponding continuous surface, it is easy to see that that surface will be a simple anchor ring. For we may imagine the sides of this parallelogram brought into coincidence, forming a cylinder, then the ends of the cylinder joined to form a ring, in such a way that corresponding points, as indicated by the letters, are joined up and corresponding sides of the triangles

connected thus preserving perfect continuity with respect to both points and joined fundamental regions.

The group of genus 2 was discussed above. It is useless to discuss the other forms mentioned as they would each be similar to the example

given. We need note only that there are four distinct types of group of genus one and three of genus two. Of course the regular division of a continuous surface into fundamental regions of $2N$ black and white polygons is only one of many such methods for representing a group graphically, e.g., Cayley's color groups.

Chapter V.

Solution of Algebraic Equations.

Thus far all of our work concerning groups has been purely from the standpoint of the group itself, but the entire group theory is very closely connected with the solution of algebraic equations, as was shown by Galois. Let us restate the definition of a group:

Suppose we have given a set of operations $a, b, c, \dots, k$ performed on the same set of objects.

1. The operations are all distinct, no two producing the same change in any possible application.

2. The result of performing any number of the operations $a, b, c, \dots, k$ in succession is another definite operation of the same set, depending

only on the operations and the order in which they are carried out.

3. There is an operation inverse to every given operation. That is, a being any operation of the set, there is another operation a^{-1} such that aa^{-1} produces no change in the objects, $aa^{-1} = 1$.

A set of operations satisfying these conditions is called a group."

If this definition is modified to suit the particular kind of group which we are going to use, the substitution group, we must substitute everywhere for the words, operations $a, b, c, \dots, k$, substitutions $S_1, S_2, \dots, S_m$. The number m of distinct substitutions is called the order of the group, and the number n of letters operated on is called its degree. In fact the same definitions hold here as before. All the $n!$ substitutions on n letters form a group called the symmetric group.

A cycle of two letters is called a transposition, and every substitution can be expressed as the product of transpositions in various ways. If the various decompositions of a substitution into transpositions all contain an even number of transpositions, the substitution is said to be even. The totality of all the even substitutions on n letters is called the alternating group on n letters, whose order is $\frac{1}{2} \cdot n!$

If N is the order of a group G and P the order of a subgroup H , then $\nu = \frac{N}{P}$ is called the index of H under G . It is clear that all the substitutions on $x_1, x_2, \dots, x_n$ which leave unaltered a rational function $\varphi(x_1, x_2, \dots, x_n)$ form a group. (See Dickson: "Introduction to the Theory of Algebraic Equations", § 21.) The inverse is also true, that being given a group G of sub-

stitutions on $x_1, x_2, \dots, x_n$ we can construct a rational function $\varphi(x_1, x_2, \dots, x_n)$ belonging to G . We now find that if ψ is a rational function of $x_1, x_2, \dots, x_n$ belonging to a subgroup H of index ν under G , then ψ is ν -valued under G . The ν distinct functions are called the conjugate values of ψ .

We now come to an important step in this part of the work and have the following theorem:

"The ρ distinct values which a rational function $\varphi(x_1, x_2, \dots, x_n)$ takes when operated on by all $n!$ substitutions are the roots of an equation of degree ρ whose coefficients are rational functions of the elementary symmetric functions

$$C_1 = x_1 + x_2 + \dots + x_n, C_2 = x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n, \dots, C_n = x_1 x_2 \dots x_n."$$

Let the ρ distinct values which φ assumes be $\varphi_1 \equiv \varphi, \varphi_2, \varphi_3, \dots, \varphi_\rho$. These are the roots of an equation

$$(y - \varphi_1)(y - \varphi_2) \cdots (y - \varphi_n) = 0$$

whose coefficients are clearly symmetric functions of $x_1, x_2, \dots, x_n$ and rational functions of the c 's. Also any substitution on the x 's merely interchanges the φ 's. The equation having $\varphi_1, \varphi_2, \dots, \varphi_n$ as roots is called the resolvent equation for φ .

We may also state Lagrange's theorem: "If a rational function $\varphi(x_1, x_2, \dots, x_n)$ remains unaltered by all the substitutions which leave another rational function $\psi(x_1, x_2, \dots, x_n)$ unaltered, then φ is a rational function of ψ and $c_1, c_2, \dots, c_n$."

With the help of these theorems let us consider the solutions of the cubic and the quartic equations. The general cubic equation may be written

$$(1) \quad x^3 - c_1 x^2 + c_2 x - c_3 = 0.$$

Setting $x = y + \frac{1}{3}c_1$, (1) takes the form

$$(2) \quad y^3 + \phi y + q = 0,$$

if we make the following substitutions:

$$(3) \quad \phi = c_2 - \frac{1}{3} c_1^2, \quad q = -c_3 + \frac{1}{3} c_1 c_2 - \frac{2}{27} c_1^3.$$

The cubic (2) is called the reduced cubic equation. According to Cardan's solution we make the substitution

$$(4) \quad y = z - \frac{\phi}{3z},$$

and (2) becomes

$$z^3 - \frac{\phi^3}{27z^3} + q = 0,$$

and we have

$$(5) \quad z^6 + qz^3 - \frac{\phi^3}{27} = 0.$$

Solving this as a quadratic in z^3 we have

$$z^3 = -\frac{1}{2}q \pm \sqrt{R}$$

where $R \equiv \frac{1}{4}q^2 + \frac{1}{27}\phi^3$.

Let one of the cube roots of $-\frac{1}{2}q + \sqrt{R}$ be $\sqrt[3]{-\frac{1}{2}q + \sqrt{R}}$. The others are

$$\omega \sqrt[3]{-\frac{1}{2}q + \sqrt{R}}, \text{ and } \omega^2 \sqrt[3]{-\frac{1}{2}q + \sqrt{R}},$$

where ω is an imaginary cube root of unity. The three cube roots of unity are roots of the equation

$$v^3 - 1 = 0, \text{ or } (v-1)(v^2 + v + 1) = 0.$$

The roots of

$$v^2 + v + 1 = 0$$

are $-\frac{1}{2} + \frac{1}{2}\sqrt{-3} \equiv \omega$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3} \equiv \omega^2$.

Hence $\omega^2 + \omega + 1 = 0$, $\omega^3 = 1$.

Since

$$\left(-\frac{1}{2}q + \sqrt{R}\right)\left(-\frac{1}{2}q - \sqrt{R}\right) = \frac{1}{4}q^2 - R = -\frac{1}{27}p^3,$$

one cube root $\sqrt[3]{-\frac{1}{2}q - \sqrt{R}}$ may be chosen so that

$$\begin{aligned}\sqrt[3]{-\frac{1}{2}q + \sqrt{R}} \cdot \sqrt[3]{-\frac{1}{2}q - \sqrt{R}} &= -\frac{1}{3}p, \\ \omega \sqrt[3]{-\frac{1}{2}q + \sqrt{R}} \cdot \omega^2 \sqrt[3]{-\frac{1}{2}q - \sqrt{R}} &= -\frac{1}{3}p, \\ \omega^2 \sqrt[3]{-\frac{1}{2}q + \sqrt{R}} \cdot \omega \sqrt[3]{-\frac{1}{2}q - \sqrt{R}} &= -\frac{1}{3}p.\end{aligned}$$

Hence the six roots of (5) may be separated into pairs, the product of two in any pair being $-\frac{1}{3}p$. The root paired with z is then $-\frac{p}{3z}$, and their sum $z - \frac{p}{3z}$ is a root y of the cubic (2). That is, the six roots of (5) lead to only three roots of the cubic (2). We thus obtain Cardan's formulae for y_1, y_2, y_3 , the roots of (2):

$$(7) \quad \begin{cases} y_1 = \sqrt[3]{-\frac{1}{2}q + \sqrt{R}} + \sqrt[3]{-\frac{1}{2}q - \sqrt{R}}, \\ y_2 = \omega \sqrt[3]{-\frac{1}{2}q + \sqrt{R}} + \omega^2 \sqrt[3]{-\frac{1}{2}q - \sqrt{R}}, \\ y_3 = \omega^2 \sqrt[3]{-\frac{1}{2}q + \sqrt{R}} + \omega \sqrt[3]{-\frac{1}{2}q - \sqrt{R}}. \end{cases}$$

Multiplying these equations by $1, \omega^2, \omega$ and adding we get

$$\sqrt[3]{\frac{1}{2}q + \sqrt{R}} = \frac{1}{3}(y_1 + \omega^2 y_2 + \omega y_3).$$

Similarly, multiplying by $1, \omega, \omega^2$,

$$\sqrt[3]{\frac{1}{2}q - \sqrt{R}} = \frac{1}{3}(y_1 + \omega y_2 + \omega^2 y_3).$$

Cubing and subtracting, we have

$$\begin{aligned} R &= \frac{1}{54} \{ (y_1 + \omega^2 y_2 + \omega y_3)^3 - (y_1 + \omega y_2 + \omega^2 y_3)^3 \} \\ &= \frac{\sqrt{-3}}{18} (y_1 - y_2)(y_2 - y_3)(y_3 - y_1). \end{aligned}$$

The function

$$(y_1 - y_2)^2 (y_2 - y_3)^2 (y_3 - y_1)^2 = -27q^2 - 4p^3$$

is called the discriminant of the cubic (2). Therefore the roots of the general cubic (1) are

$$x_1 = y_1 + \frac{1}{3}c_1,$$

$$x_2 = y_2 + \frac{1}{3}c_1,$$

$$x_3 = y_3 + \frac{1}{3}c_1.$$

$$x_1 - x_2 = y_1 - y_2, \quad x_2 - x_3 = y_2 - y_3, \quad x_3 - x_1 = y_3 - y_1,$$

$$\begin{aligned} (8) \quad (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) &= (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \\ &= \frac{18}{\sqrt{-3}} \sqrt{R} = -6\sqrt{-3} \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}. \end{aligned}$$

Aside from the factors $\frac{1}{3}$ the roots of (1) are

$$\psi_1 = x_1 + \omega x_2 + \omega^2 x_3,$$

$$\psi_4 = x_1 + \omega x_3 + \omega^2 x_2,$$

$$\psi_2 = \omega^2 \psi_1 = x_2 + \omega x_3 + \omega^2 x_1,$$

$$\psi_5 = \omega^2 \psi_4 = x_3 + \omega x_2 + \omega^2 x_1,$$

$$\psi_3 = \omega \psi_1 = x_3 + \omega x_1 + \omega^2 x_2,$$

$$\psi_6 = \omega \psi_4 = x_2 + \omega x_1 + \omega^2 x_3,$$

which differ only in permutations of x_1, x_2, x_3 ; and ψ_1 is called a six-valued function.

The determination of the roots y_1, y_2, y_3 of the reduced cubic

$$y^3 + py + q = 0,$$

depends upon the chain of resolvent equations:

$$\xi^2 = \frac{q^2}{4} + \frac{p^3}{27}, \text{ where } \xi = \frac{\sqrt{-3}}{18} (y_1 - y_2)(y_2 - y_3)(y_3 - y_1);$$

$$z^3 = -\frac{q}{2} + \xi, \text{ where } z = \frac{1}{3} (y_1 + \omega y_2 + \omega^2 y_3);$$

$$y_1 = z - \frac{p}{3z}, \quad y_2 = \omega z - \frac{\omega^2 p}{3z}, \quad y_3 = \omega^2 z - \frac{\omega p}{3z}.$$

Of course the elementary symmetric functions belong to the symmetric group G_6 on y_1, y_2, y_3 . Solving a quadratic resolvent equation we find the two-valued function in ξ which belongs to the subgroup $G_3 = \{1, (123), (132)\}$ of G_6 . Solving

a cubic resolvent equation we then find the six-valued function z , belonging to the subgroup G_1 of G_3 . Also y_1, y_2, y_3 are rational functions of z, p, q which belong to the respective groups

$G_2' = \{1, (23)\}$, $G_2'' = \{1, (13)\}$, $G_2''' = \{1, (12)\}$, each containing G_1 . We may then arrange the following scheme:

$$\begin{array}{ccccccc}
 G_6 : & p, q & & & & & \\
 \downarrow 2 & & & & & & \\
 G_3 : & \xi & G_2' : & y_1 & G_2'' : & y_2 & G_2''' : & y_3 \\
 \downarrow 3 & & \downarrow 1 & & \downarrow & & \downarrow & \\
 G_1 : & z & G_1 : & z & G_1 : & z & G_1 : & z.
 \end{array}$$

By the same method for the general cubic

$$x^3 - c_1 x^2 + c_2 x - c_3 = 0$$

we find $\Delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ belong to the subgroup G_3 . To G_3 belongs also ψ , where

$$\psi = x_1 + \omega x_2 + \omega^2 x_3,$$

and for the general scheme we have:

$$\begin{array}{l}
 G_6: c_1, c_2, c_3 \\
 \begin{array}{l} 2 \\ | \end{array} G_3: (x_1 + \omega x_2 + \omega^2 x_3)^3 \\
 \begin{array}{l} 3 \\ | \end{array} G_1: (x_1 + \omega x_2 + \omega^2 x_3).
 \end{array}$$

The group of the cubic equation corresponds to the dihedral group of an equilateral triangle which we have already discussed. G_6 corresponds to the total dihedral group itself, G_3 to the cyclic subgroup of revolutions about the main axis of the dihedron through an angle of $\pm \frac{2\pi}{3}$ or $\pm \frac{4\pi}{3}$. G_2', G_2'', G_2''' correspond to the subgroups made up of the revolutions through π about each of the subaxes of the dihedron. Finally G_1 corresponds to the identity.

It is not necessary to carry out fully the work for the solution of the general equation of the fourth degree,

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

We shall simply state some of the results obtained by means of Ferrari's

and Lagrange's solutions (See Dickson: "Introduction to the Theory of Algebraic Equations", §4). We find

$$t = x_1 + x_2 - x_3 - x_4,$$

and the roots of the resolvent cubic equation to be

$$y_1 = x_1 x_2 + x_3 x_4, \quad y_2 = x_1 x_3 + x_2 x_4, \quad y_3 = x_1 x_4 + x_2 x_3.$$

We also have for the discriminant

$$\Delta = (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

We may form a 4! valued function

$$V = \sqrt{\psi} = (x_1 - x_2) + i(x_3 - x_4).$$

Our scheme for the equation of the fourth degree becomes:

$$\begin{array}{l} G_{24} : a, b, c, d \\ 3 \mid \\ G_8 : x_1 x_2 + x_3 x_4 \\ 2 \mid \\ H_4 : x_1 + x_2 - x_3 - x_4 \\ 2 \mid \\ G_2 : (x_1 - x_2 + i x_3 - i x_4)^2 \\ 2 \mid \\ G_1 : (x_1 - x_2 + i x_3 - i x_4) \end{array}$$

where

$$G_2 = \{ 1, (12)(34) \}$$

$$H_4 = \{1, (12), (34), (12)(34)\}$$

$$G_8 = \{1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}.$$

Or we may have the following scheme:

$$\begin{array}{l} G_{24} : a, b, c, d \\ 2 \mid \\ G_{12} : \Delta = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\ 3 \mid \\ G_4 : \varphi = y_1 + \omega y_2 + \omega^2 y_3 \\ 2 \mid \\ G_2 : \lambda = \varphi_1 \div (x_1 + x_2 - x_3 - x_4) \\ 2 \mid \\ G_1 : \psi = x_1 - x_2 + i x_3 - i x_4. \end{array}$$

where

$$G_{12} = \{1, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$$

and $G_4 = \{1, (12)(34), (13)(24), (14)(23)\}.$

The group of the quartic equation corresponds to the octahedron group or to the extended tetrahedron group, which are really the same. The first scheme given corresponds to the extended tetrahedron group; G_{24} corresponding to the entire group, G_8 to the revolutions of period 3 about the axis through each vertex of the tetrahedron

and its opposite face, H_4 to the triaxial subgroup of the tetrahedron, G_2 to the cyclic subgroup of the triaxial group, and finally G_1 to the identity. The second scheme given corresponds to the octahedron group; G_1 corresponding to the entire group, G_{12}^{24} to either the tetrahedron or the conjugate tetrahedron group, G_4 to the triaxial subgroup of the tetrahedron group, G_2 to the cyclic subgroup of the triaxial group, and G_1 to the identity.

The subgroup to which belong the conjugates of ψ , belonging to the subgroup H under G are called conjugate subgroups. In case they are all equal H is called a self-conjugate subgroup.

The plan for the solution of the general equation of degree n is to find a chain of binomial resolvent equations of prime degree, such

that a root of each is expressible as a rational function of the roots of the given equation. It is clear that the necessary condition for this is the existence of a series of composition whose factors of composition are all prime. This condition is satisfied if $n=3$ or if $n=4$, as has been shown, but if $n \geq 5$ the condition is not satisfied since one of the factors of composition, $\frac{1}{2}n!$, is not prime.

A domain of rationality is composed of a system of quantities derived from certain constants or variables by a finite number of additions, subtractions, multiplications and divisions, together with those constants. "An integral rational function whose coefficients belong to a domain R is said to be reducible in R if it can be decomposed into integral rational factors of lower degree whose coefficients likewise belong

to R ; irreducible in R if no such decomposition is possible." The $n!$ values of the function $V_1 = m_1 x_1 + m_2 x_2 + \dots + m_n x_n$, to which we have already referred, are the roots of an equation

$$F(V) = (V - V_1)(V - V_2) \dots (V - V_{n!}) = 0$$

whose coefficients belong to the domain R . Let $F_0(V)$ be the irreducible factor of $F(V)$. Then

$$F_0(V) = 0$$

is an irreducible equation called the Galois' resolvent of the equation

$$f(x) \equiv x^n - c_1 x^{n-1} - \dots - (-1)^n c_n = 0.$$

If the roots of the Galois' resolvent

$$V_1, V_a, V_b, \dots, V_l$$

are obtained from V_1 by the substitutions $1, a, b, \dots, l$, these substitutions form a group called the group of the equation $f(x) = 0$ for the domain R . This group possesses the following properties:

A. Every rational function $\varphi(x_1, x_2, \dots, x_n)$ of the roots which remains unaltered

by all the substitutions of G lies in the domain R .

B. Every rational function $\varphi(x_1, x_2, \dots, x_n)$ of the roots which equals a quantity in R remains unaltered by all the substitutions of G .

By means of these properties it is possible to actually determine the group G of an equation.

A group of substitutions on n letters is transitive if its substitutions change any one member into every other one; otherwise it is intransitive. The group of an irreducible equation is transitive; that of a reducible equation is intransitive.

By adjoining various quantities to the domain of rationality a group is reduced to various subgroups. If H is a maximal self-conjugate subgroup of G , the group of the resolvent is simple, and its equation

is called a regular simple equation. It is clear then that the solution of any equation can be reduced to the solution of a chain of simple regular equations, each of prime degree, solvable by radicals. Therefore if the group of an equation has a series of composition for which the factors of composition are all prime numbers, the equation is solvable by radicals, that is, by the extraction of roots of known quantities. We have also that a binomial equation of prime degree p ,

$$x^p - A = 0,$$

can be solved by means of a chain of Abelian equations of prime degree, where by Abelian equation we mean an irreducible equation whose roots are rational functions of each other.

We find finally through Galois the necessary and sufficient condition that an equation be solva-

ble by radicals is that its group have a series of composition in which the factors of composition are all prime. We may also state Abel's theorem: "The solution of an algebraically solvable equation can always be performed by a chain of binomial equations of prime degrees whose roots are rationally expressible in terms of the roots of the given equation and of certain roots of unity."

It is needless to carry on any further work with the cubic or quartic equations as a glance at the work already done will show that successive resolvents are found, the group of each one being obtained from the preceding by adjoining various quantities to the different domains. It is also unnecessary to attempt any work with the equation of the fifth or higher degree as the factors of the

series of composition are not all prime. It might be well to note that the extended group of the tetrahedron already considered bears the same relation to the group of the quartic equation that the extended group of the icosahedron does to the group of the quintic. The number of operations of the former equals $n!$, the number of substitutions on the n roots of the equation. There is also a similar correspondence between the subgroups of each.

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