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THE ACCELERATION OF GRAVITY
An Absolute Determination with a Kater's
Pendulum for Columbia, Missouri.

by

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Galileo Galilei* (1564-1642) was the first to question the assertion of Aristotle that bodies fall with velocities proportional to their weights. He dramatically refuted this Aristotelian doctrine by dropping from the Tower of Pisa two iron balls of very unequal weight. This demonstration failed to convince a part of the faculty of the University of Pisa who tried to explain the phenomenon as an optical illusion or as caused by the intervention of the Devil. Galileo soon found untenable his first hypothesis that there were equal additions of velocity as the object fell equal distances and he held instead that there were equal additions to the velocity in equal intervals of time. He showed that the distance passed over would be proportional to the square of the time of falling and proved this conclusion experimentally by means of the inclined plane. To measure intervals of time he weighed the water which flowed from an orifice while the ball was rolling down the plane. Galileo observed also that, as nearly as he could judge, using his pulse as a chronometer, a pendulum made oscillations in equal intervals of time regardless of the amplitude; he showed that the period of a pendulum was proportional to the square root of its length and suggested the pendulum as a means of measuring the passage of time.

Mersenne, in 1644, made the first determination of the pendulum which makes a complete vibration in two seconds. In 1657, Huggens followed out the suggestion of Galileo and constructed

* Poynting and Thomson: Text-Book of Physics; Properties of Matter, Chap. II.

a pendulum clock. It was then possible to determine the length of the seconds pendulum with a degree of accuracy since the clock could be rated by star observations. Huygens showed in the case of the physical pendulum (in which the parts are rigidly bound together, as opposed to the ideal pendulum in which a particle is supported by a weightless thread) that for each axis of suspension, there is another axis, the axis of oscillation, whose distance from the axis of suspension is the length of the ideal pendulum swinging with the same period. The axes of oscillation and suspension, he showed also, are interchangeable. Galileo's conclusion that the period of a pendulum was independent of the amplitude of the arc through which it was swung, was shown to be untrue by D. Bernoulli in 1747. He worked out a formula for correcting the observed period to that for an arc infinitely small.

At Paris in 1792 Borda and Cassini, in connection with the investigation as to whether the seconds pendulum would be a suitable standard of length, swung the pendulum which is called Borda's Pendulum. It consisted of a platinum bob, 1.5 inches in diameter, suspended from a knife edge by a fine iron wire about twelve feet long. The knife edge and the wire holder were so constructed that their period was the same as that of the pendulum; this permitted their moment of inertia to be disregarded. In comparing the period of the pendulum with the period of a pendulum clock, a method of coincidences was used.

Prony in 1800 proposed that instead of the simple

pendulum of Borda, a compound pendulum be used in which the axes of suspension and oscillation could be interchanged. Eleven years later, Bohnenberger made the same suggestion, but it was not until ¹⁸¹⁷~~1817~~ that Captain Kater* hit upon the same plan independently and constructed what is known as "Kater's Convertible Pendulum". It consisted of a brass bar 170cm. long, 4cm. wide and 0.3cm. thick. Two knife edges, which pointed inward, were firmly fastened to the bar 100cm. apart and a weight of about one kilogram was fastened above one knife edge. If the periods of the pendulum swung about either knife edge are equal, it can be shown that the length of the corresponding simple pendulum is the distance between the knife edges (page 12) On the bar between the knife edges were two adjustable weights, ~~and~~ a large one for getting the periods nearly equal, and ~~then~~ another, much smaller, for finer adjustment. These weights he adjusted until the difference in the period of the pendulum as it swung about one knife edge and the period as it swung about the other was less than the observed error in the period as it swung about one of the knife edges. (See figure 1.) Kater used a method of "coincidences" in determining the period. Fastened to the bob of the clock was a circle of white paper which was just covered by the black tail of the pendulum when both were at rest. A slit in the focal plane of the telescope used in making the observations was of the same width as the image of tail of the pendulum. A coincidence occurred when the images

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*Phil. Trans., 1818, 1819.

Figure 1

Kater's
Pendulum

Figure 2

Simple Pendulum

Figure 3

Compound Pendulum

of the pendulum and the bob crossed the slit in the telescope without any of the white spot being visible. The distance between the knife edges was measured carefully and corrections were made for amplitude, bouyancy of the air and temperature.

Following the suggestions made by Bessel, Repsold constructed in 1860 a pendulum which was symmetrical in shape but unsymmetrical in the distribution of its mass. The symmetrical shape made unnecessary a correction for a different air effect for the light end and the heavy end. The knife edges were interchangeable; this eliminated the effect due to their possible dissimilarity. The design of the pendulum was such that the periods about either knife edge were nearly equal. It can be shown*~~that~~ if the periods are not exactly equal, that the expression for gravity is given in terms of the sum of the square of the periods and the distance between the knife edges plus a correction term involving the difference in the squares of the periods and the difference of the distances from the center of gravity to the knife edges. The latter needs to be known only approximately if the difference between the periods is small.

There are several other methods for the absolute determination of gravity, among which the best known are: the Atwoods machine, the inclined plane, the Morin machine, the falling fork. All of these methods are valuable rather as means of demonstrating uniform acceleration, than for accurate quantitative work.

* Poynting and Thomson; loc. cit.

The Atwoods machine is a piece of apparatus for decreasing the apparent velocity of a falling body. It consists of a wheel over which passes a thread which has unequal weights attached to either end. If P is the larger weight and p the less, it can be shown that the acceleration is;

$$a = g \frac{P}{P+p}$$

For accurate results, corrections must be made for the friction of the bearings of the wheel, the angular momentum given the wheel, the weight of the thread. An exact measurement of time is difficult.

The inclined plane is another device for decreasing the velocity of the falling body. If l is the length of the plane, ~~plane~~ and h the height, the acceleration is given;

$$a = g \frac{h}{l}$$

Correction must be made for friction, the angular momentum of the moving mass (a ball or a carriage with wheels). There is difficulty in measuring time exactly, here also.

The Morin machine consists of a revolving vertical cylinder upon which a freely falling body traces a path. The falling fork traces a path, usually on a plate of smoked glass. Both of these pieces of apparatus work with a large amount of friction.

The most accurate absolute determinations of gravity have been made with pendulums of the Kater or Repeold type. The best absolute determinations of gravity have been made at Potsdam where the value is 981.274. On account of the care necessary in and the difficulty of making absolute determinations,

methods of relative determination are used instead, in obtaining data concerning the local variation of gravity. The value at Washington, D. C., referred to Potsdam is 980.111; to this value the relative determinations in the United States are referred.

The comparison work of the United States Coast and Geodetic Survey* has been done principally with half seconds pendulums which are not adjustable or reversible. These pendulums, three in number have the planes attached to the *pendulum* bar and the knife edges to the supports, since the knife edges are quite liable to injury and, if on the pendulum, could not be repaired without changing its period. The pendulums are swung in a partially exhausted receptacle. The periods are determined by a coincidence method in which a break circuit chronometer actuates a shutter, this gives a flash which is reflected from a mirror on the head of the pendulum into the observing telescope. The swaying of the support is measured by an interferometer**. For these pendulums the corrections for amplitude, temperature, pressure and swaying of the support, were all determined empirically from the pendulums themselves.

Another method of relative determination is that of R. Trelfall and J.A. Pollock^o who published in 1899 an account of their quartz thread gravity balance. The balance consists

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* U.S.C.G.S Report, 1891, App. No. 1. 1894, App. No. 15

** U.S.C.G.S Report, 1910, App. No. 6

^o Phil. Trans., 1899

of a horizontal quartz thread, to the middle of which is fastened a piece of brass wire. The wire is fastened a little off center and the magnitude of the acceleration is measured by the amount of twist it is necessary to apply to the quartz thread to make the wire horizontal. The apparatus is calibrated by making readings at two places where absolute measurements of gravity have been made. A number of corrections are necessary among which the most important are for temperature and for the change in the elasticity of the quartz thread. The investigators calculate for the apparatus an error of less than one part in 200,000.

The boiling point-barometer method for relative determination has been used by Hecker* and others, chiefly for gravity determinations at sea where other methods are not available. The principle upon which the method is based is that the height of the mercurial column of the barometer is a function of the weight of the mercury as well as the ^{of} pressure of the atmosphere. The boiling point of pure water is dependent upon atmospheric pressure and this makes it possible to determine the effect of gravity upon the barometer column. Hecker has claimed a higher degree of accuracy for his determinations than those** who have used the same method are willing to allow him. The interpretation which he placed upon his results as to the density of the earth beneath certain parts of the ocean has not been accepted.

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* Hecker: Bestimmung der Schwerkraft auf dem Schwarzen Meere.

** Bauer, in Am. Jour. Sci. No. 181: On Gravity Determinations at Sea.

The mathematical theory of Kater's Pendulum will now be considered, together with that of the necessary corrections.

If in the case of the ideal pendulum, (figure 2), m is the mass of the particle, l the length of the massless thread, θ the angle the thread makes with the vertical, then the tangential component of the force due to gravity is $-mg \sin \theta$. But $F = ma$ where a is the linear acceleration. Since s (the space passed over) is given by $s = l\theta$, $a = \frac{d^2 s}{dt^2} = \frac{d^2 (l\theta)}{dt^2} = l \frac{d^2 \theta}{dt^2}$. The equation on motion is then,

$$m l \frac{d^2 \theta}{dt^2} = -mg \sin \theta$$

or,

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin \theta$$

Since the difference between $\sin \theta$ and θ , for small values of θ , is small compared with unity, for an approximate solution, θ may be substituted for $\sin \theta$:

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{l}$$

To solve this differential equation, the solution $\theta = e^{mt}$ is assumed. To determine m , the solution is substituted in the differential equation, giving

$$m^2 e^{mt} = -\frac{g}{l} e^{mt}$$

Hence; $m = \pm i\sqrt{\frac{g}{l}}$

The particular solutions will be $Ae^{i\sqrt{\frac{g}{l}}t}$, $Be^{-i\sqrt{\frac{g}{l}}t}$

These solutions may be put in the form

$$(A+B)(e^{i\sqrt{\frac{g}{l}}t} + e^{-i\sqrt{\frac{g}{l}}t}) = (A+B) \cos \sqrt{\frac{g}{l}}t$$

$$i(A+B)(e^{i\sqrt{\frac{g}{l}}t} - e^{-i\sqrt{\frac{g}{l}}t}) = i(A-B) \sin \sqrt{\frac{g}{l}}t$$

The complete solution is, then: $\theta = E \cos \sqrt{\frac{g}{l}}t + F \sin \sqrt{\frac{g}{l}}t$

$$A+B = E$$

$$i(A-B) = F$$

There is no loss in generality if the following values for the constants of integration be assumed:

$$E = C \cos \alpha$$

$$F = C \sin \alpha$$

The general solution becomes:

$$\mathcal{D} = C \cos \alpha \cos \sqrt{\frac{g}{L}} t + C \sin \alpha \sin \sqrt{\frac{g}{L}} t = C \cos \left(\sqrt{\frac{g}{L}} t - \alpha \right)$$

If $\sqrt{\frac{g}{L}} t$ be increased by 2π , $\mathcal{D}$ will remain unchanged.

The function then is periodic and the period* is

$$T = 2\pi \sqrt{\frac{L}{g}}$$

Since in reality $\mathcal{D}$ is not equal to $\sin \mathcal{D}$, it is necessary

to solve the exact equation,**

$$\frac{d^2 \mathcal{D}}{dt^2} + \frac{g}{L} \sin \mathcal{D} = 0$$

Multiply by $\frac{d\mathcal{D}}{dt}$

$$\frac{d\mathcal{D}}{dt} \cdot \frac{d^2 \mathcal{D}}{dt^2} + \frac{g}{L} \frac{d\mathcal{D}}{dt} \sin \mathcal{D} = 0$$

from which is gotten

$$\frac{1}{2} \frac{d}{dt} \left(\frac{d\mathcal{D}}{dt} \right)^2 - \frac{g}{L} \frac{d(\cos \mathcal{D})}{dt} = 0$$

Integrating

$$\frac{1}{2} \left(\frac{d\mathcal{D}}{dt} \right)^2 - \frac{g}{L} \cos \mathcal{D} = h$$

If for $\mathcal{D} = \mathcal{D}_0$, $v_0 = \left(\frac{d\mathcal{D}}{dt} \right)_0$, then $\left(\frac{v_0^2}{L^2} \right) = 2 \left(\frac{g}{L} \cos \mathcal{D}_0 + h \right)$

Substitute this value of h ,

$$\left(\frac{d\mathcal{D}}{dt} \right)^2 - \frac{v_0^2}{L^2} = \frac{2g}{L} (\cos \mathcal{D} - \cos \mathcal{D}_0)$$

Solve for $\frac{d\mathcal{D}}{dt}$

$$\frac{d\mathcal{D}}{dt} = \pm \frac{1}{L} \sqrt{v_0^2 + 2gL(\cos \mathcal{D} - \cos \mathcal{D}_0)}$$

Integrating

$$t = \pm \int_{\mathcal{D}_0}^{\mathcal{D}} \frac{L d\mathcal{D}}{\sqrt{v_0^2 + 2gL(\cos \mathcal{D} - \cos \mathcal{D}_0)}}$$

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* By period is meant the time required for a complete swing; that is, the interval of time required for the position and direction of swing to become identical.

** Webster, Dynamics, Sect. 22.

Since $\cos \frac{\theta}{2} = 1 - 2 \sin^2 \frac{\theta}{4}$

$$t = \pm \int_{\theta_0}^{\theta} \frac{L d\theta}{\sqrt{v_0^2 + 4gl \sin^2 \frac{\theta_0}{2} - 4gl \sin^2 \frac{\theta}{2}}}$$

Substitute $\frac{v_0^2 + 4gl \sin^2 \frac{\theta_0}{2}}{4gl} = k^2$ where v_0 and θ_0 are quantities such that $k^2 < 1$

The equation becomes

$$t = \pm \sqrt{\frac{L}{g}} \int_{\theta_0}^{\theta} \frac{d\theta}{2\sqrt{k^2 - \sin^2 \frac{\theta}{2}}}$$

Making a change of variable

$$\begin{aligned} k \sin \phi &= \sin \frac{\theta}{2} \\ k \cos \phi \frac{d\phi}{dt} &= \cos \frac{\theta}{2} \frac{d\theta}{dt} \end{aligned}$$

$$t = \pm \sqrt{\frac{L}{g}} \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{k^2 - \sin^2 \frac{\theta}{2}}} = \pm \sqrt{\frac{L}{g}} \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

Using Legendre's notation for the elliptic integral: $F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$

$$t = \pm \sqrt{\frac{L}{g}} [F(\phi, k) - F(\phi_0, k)]$$

The upper sign is used if the pendulum is rising at θ_0 . At the lowest point of the swing, $\theta = 0$, $\phi = 0$

$$t_m = \mp \sqrt{\frac{L}{g}} F(\phi_0, k)$$

The upper sign is taken if the pendulum is rising at θ_0 . Subtracting,

$$t_m - t = \sqrt{\frac{L}{g}} F(\phi, k)$$

If the particle passes through the lowest point and rises until $\frac{d\theta}{dt} = 0$

$\frac{d\theta}{dt} = 0$, that is, when $\sin^2 \frac{\theta}{2} = k^2$, at the time t_h

$$\sin \frac{\theta}{2} = -k; \sin \phi = -1 \quad \phi = -\frac{\pi}{2} \quad \text{and}$$

$$t_m - t_h = \sqrt{\frac{L}{g}} F(-\frac{\pi}{2}, k) \quad \text{or,}$$

$$t_h - t_m = \sqrt{\frac{L}{g}} F(\frac{\pi}{2}, k)$$

But

$$\sqrt{\frac{L}{g}} F(\frac{\pi}{2}, k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

It is evident that the period of a complete oscillation is $4(t_h - t_m)$

$$T = 4\sqrt{\frac{L}{g}} K(k) = 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

Expanding by the binomial theorem;

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \left[1 + \frac{1}{2} k^2 \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \phi + \dots \right] d\phi$$

Since $\int_0^{\frac{\pi}{2}} \sin^{2n} \phi \, d\phi = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}$

(Peirce; Tables 483)

Integrating termwise

$$K(k) = \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots \right]$$

If α be the maximum for ϕ , then $\psi = \frac{\pi}{2}$, $k = \sin \frac{\alpha}{2}$

$$T = 2\pi \sqrt{\frac{L}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{\alpha}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4 \frac{\alpha}{2} + \cdots \right]$$

For small angles* terms after the second may be dropped

$$T = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{4} \sin^2 \frac{\alpha}{2} \right)$$

If T_0 is the period of the pendulum for infinitely small arcs, then

$$T_0 = T_c \left(1 + \frac{1}{4} \sin^2 \frac{\alpha}{2} \right)$$

where T_0 is the observed value of the period.

$$T_c = T_0 \left(1 - \frac{1}{4} \sin^2 \frac{\alpha}{2} \right)$$

Since $\sin^2 \frac{\alpha}{2}$ is a small quantity T_0 may be replaced by

where T_m is approximately the value of T_0 . In the following

table, d is the mean of the half amplitudes in millimeters of the

pendulum at the beginning and end of the observation, $C = \frac{1}{4} \sin^2 \frac{d}{2200} T_m = 2.0110$

Table I.		
d	C	$C \times T_m$
14.00	.0000405	.000081
14.25		84
14.50		87
14.75		90
15.00	.0000465	93
15.25		96
15.50		99
15.75		102
16.00	.0000540	1'05

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* In the observations recorded on page , the half arc is less than 1°

$$\sin^2 \frac{\alpha}{2} = 0.00007615$$

$$\sin^4 \frac{\alpha}{2} = 0.000\ 000\ 006 \quad \text{for } \alpha = 1^\circ$$

If instead of the ideal pendulum, the physical (compound) pendulum is considered, the constants involved in the equation of motion are different. If h is the distance from S , (figure 3), the point of suspension, to G , the center of gravity, the expression for the work is

$$W = Mgh(1 - \cos \theta)$$

and the expression for the kinetic energy

$$\pi = \frac{1}{2} K \omega^2 = \frac{1}{2} K \left(\frac{d\theta}{dt} \right)^2$$

where K is the moment of inertia about S . Substituting in Lagrange's Equations of Motion* in generalized coordinates;

$$\frac{d}{dt} \left[\frac{\partial (\pi - W)}{\partial \dot{\theta}} \right] - \frac{\partial (\pi - W)}{\partial \theta} = 0$$

The equation of motion is;

$$K \frac{d^2 \theta}{dt^2} + Mgh \sin \theta = 0$$

This is the form of the equation of the ideal pendulum, if $L = \frac{K}{Mh}$

The period is given by $T = 2\pi \sqrt{\frac{K}{Mgh}}$

To show that the axes of oscillation and suspension are interchangeable, let K be written in the form $K = K_0 + Mh^2$ ** where K_0 is the moment of inertia about an axis through the center of gravity parallel to the axis of suspension. If r

is the radius of gyration, by definition, $K_0 = Mr^2$. Since $L = \frac{K}{Mh}$

$$L = \frac{M(r^2 + h^2)}{Mgh} \quad \text{or,} \quad L = \frac{r^2}{h} + h$$

If a point O beyond the center of gravity be chosen so that its distance $h' = L - h$ Substituting this in $L = \frac{r^2}{h} + h$ it is found that $r^2 = hh'$. If now O be made the axis of suspension, the length of the corresponding simple pendulum

will be: $L' = \frac{r^2}{h'} + h'$ Substituting this in the equation $r^2 = hh'$

* Webster; Dynamics, Sect. 36. ** loc. cit. Sect. 70

* Webster Dynamics

it is found that $L' = h + h' = L$ That is, the axes of suspension and oscillation are interchangeable. When the weights of the Kater's pendulum are so adjusted that the period about one knife edge is equal to the period about the other, one knife edge is the axis of oscillation of the other as an axis of suspension; the distance between the knife edges is then the length of the equivalent ideal pendulum.

Again, if the expressions for the periods of the pendulum about the two knife edges

$$T = 2\pi \sqrt{\frac{K_0 + Mh^2}{Mgh}}$$

$$T' = 2\pi \sqrt{\frac{K_0 + Mh'^2}{Mgh'}}$$

be squared, subtracted and the expression solved for g ,

$$g = \frac{4\pi^2(h^2 - h'^2)}{T^2h - T'^2h'}$$

When $T = T'$ and h is not equal to h' ,

$$g = 4\pi^2 \frac{h + h'}{T^2} = \frac{4\pi^2 L}{T^2}$$

If $h = h'$, the expression for g is indeterminate.

It has been assumed thus far that there was no damping of the motion of the pendulum. If now the damping^{*} due to the air be assumed proportional to the velocity, the equation of motion is

$$\frac{d^2\theta}{dt^2} + k \frac{d\theta}{dt} + h^2 \sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + k \frac{d\theta}{dt} + h^2 \theta = 0$$

or,

To solve this differential equation, a solution of the form $\theta = e^{bt}$ is assumed. Substituting to determine b , it is found that

$$b^2 + kb + h^2 = 0 \quad b = \frac{-k \pm \sqrt{k^2 - 4h^2}}{2}$$

Since k^2 is small compared to $4h^2$, b will be imaginary and the

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* Webster; Dynamics, Sect. 43

solution derived as on page 9, is;

$$D = e^{-\frac{Kt}{2}} A \cos\left(\frac{1}{2}\sqrt{4h^2 - K^2}t - \alpha\right)$$

This represents a simple harmonic motion damped by the factor $e^{-\frac{Kt}{2}}$

Maximum elongations (the measure of the period) occur when $\frac{dD}{dt} = 0$

that is, when,

$$\tan\left[\frac{1}{2}\sqrt{4h^2 - K^2}t - \alpha\right] = -\frac{K}{\sqrt{4h^2 - K^2}}$$

Accordingly the period will be

$$T = \frac{4\pi}{\sqrt{4h^2 - K^2}}$$

The ratio of two successive maximum elongations on the same side

of the middle point will be given by

$$\frac{D_1}{D_2} = \frac{e^{-\frac{Kt}{2}} (\cos \frac{1}{2}\sqrt{4h^2 - K^2}t - \alpha)}{e^{-\frac{K(t+T)}{2}} (\cos \frac{1}{2}\sqrt{4h^2 - K^2}(t+T) - \alpha)}$$

Since the cosine factors are equal, $\frac{D_1}{D_2} = e^{\frac{KT}{2}}$

From which

$$\delta = \log_e \frac{D_1}{D_2} = \frac{1}{2}KT = \frac{2\pi}{\sqrt{\frac{4h^2}{K^2} - 1}}$$

If there were no damping, the period would be $T_0 = \frac{2\pi}{h}$

Hence the observed value may be written as $T_0 = T_0 \left(1 + \frac{\delta^2}{4\pi^2}\right)^{\frac{1}{2}}$

Solving for T_0 and expanding by the binomial theorem;

$$T_0 = T_0 \left(1 + \frac{\delta^2}{4\pi^2}\right)^{-\frac{1}{2}} = T_0 \left(1 - \frac{\delta^2}{8\pi^2} + \frac{3\delta^4}{128\pi^4} + \dots\right)$$

When the damping is small, the terms after the second may be

dropped. If

$$\text{If } r = \frac{D_0}{D_1} = \frac{D_1}{D_2}, \quad r^2 = \frac{D_0}{D_2} \quad \text{and} \quad r = \left(\frac{D_0}{D_n}\right)^{\frac{1}{n}}$$

$$\delta = \log_e r = \frac{1}{n} \log_e \frac{D_0}{D_n}$$

The observations given on page 27 show that for 1080 swings,

the amplitude decreased from 40mm. to 20mm. Hence

$$\delta = \frac{1}{n} \log_e r = \frac{\log_e 2}{1080} = \frac{.693}{1080} \quad \frac{\delta^2}{8\pi^2} = .000000005$$

But the first significant figure in this result is ~~less~~ farther

(Page 25)

out than the last figure for the observed period, and the effect

of the damping can be disregarded.

Let the effect of a change in temperature upon the period of the pendulum be considered. If the length of the pendulum at t_0 is l , the length at t_n is $l[1 + \alpha(t_n - t_0)]$, where

α is the coefficient of linear expansion. If the period at t_0 is $T_0 = 2\pi\sqrt{\frac{l}{g}}$ the period for an increase of n degrees is;

$$T_n = 2\pi\sqrt{\frac{l(1+\alpha n)}{g}} = 2\pi\sqrt{\frac{l}{g}} (1+n\alpha)^{\frac{1}{2}} = T_0(1+n\alpha)^{\frac{1}{2}}$$

From which $T_0 = T_n(1+n\alpha)^{-\frac{1}{2}}$. Expanding and dropping terms after

the second $T_0 = T_n(1 - \frac{n\alpha}{2})$ Taking an approximate

value T_m in the second term, the expression becomes $T_0 = T_n - \frac{n\alpha}{2} T_m$

In the following table $t_0 = 20^\circ$, the coefficient of linear expansion

of brass, $\alpha = 0.0000187$

$T_m = 2.0110$

Table II.	
t°	$n\alpha T_m$
20	.0000000
21	.0000288
22	376
23	564
24	940
25	1128

The determination of the period resolves itself into the problem of counting a number of swings and measuring the time required to make them. Since counting the swings is rather tedious and since the observer will make errors from fatigue, methods of estimating the number of swings have been devised. Of these the most important are "middle elongations" and "coincidences".

The method of middle elongations is as follows.

The instant at which the pendulum passes its lowest point is recorded for 10 consecutive passages. After a short interval, ten more passages are recorded. To find the period, the means of

the first and tenth passages, 2nd and 9th, 3rd and 8th, 4th and 7th, 5th and 6th are taken. Their average gives the instant of maximum elongation between the 5th and 6th passages. The difference between two "middle elongations" will give the interval required for an integral number of swings. If this be divided by the approximate value of the period gotten from the ten observations of the passages, the number of swings made will be indicated as the nearest integer. If the interval between middle elongations be divided by this number, the period is gotten more closely. The personal equation of the observer is a large factor in this method and for this reason is usually discarded in exact measurements for the method of coincidences.

The method of coincidences compares a pendulum whose period is unknown with a pendulum whose period is known or with a clock. Let the case of a chronometer causing a flash at intervals of two seconds and a pendulum whose period is a little greater be considered. As the pendulum swings, there will be times at which the pendulum will cross its lowest point as the chronometer marks the second. The clock and the pendulum will then be in coincidence. The pendulum approaches coincidence alternately from the right and from the left. Between two coincidences from the same direction, there will be n complete swings which will require $(2n + 2)$ seconds for their completion.

Then the period of the pendulum is $T = \frac{2n+2}{n} = 2\left(1 + \frac{1}{n}\right)$

If there is an error of e seconds in judging the exact time of coincidence, the error for the whole interval will be not

greater than $2e$. Then $T_e = 2\left(1 + \frac{1}{n+|e|}\right) = 2\left(1 + \frac{1}{n} + \left|\frac{e}{n^2}\right| - \dots\right)$

where terms after the third do not need to be considered.

The error is given $E = T_0 - T = 2 \left| \frac{e}{n^2} \right|$

In the case of m intervals instead of one, e would remain the

same while n would be multiplied by m and the error, $E = 2 \left| \frac{e}{m^2 n^2} \right|$

In this determination $e = 2$, $n = 180$, $m = 6$, so that for a

single interval $E = \frac{2 \times 2}{(180)^2} = \frac{1}{8100}$ and for m intervals $E = \frac{1}{291600}$

The theory of Kater's Convertible pendulum having been discussed, the procedure in this determination and the data obtained will be considered.

The pendulum was swung in Room 7 of the Physics laboratory of the University of Missouri. Room 7 is in the basement on the north side of the building and has double doors and windows. The temperature was fairly constant, varying never more than half a degree between 8 A.M. and 5 P. M.

The pendulum support was screwed to a wooden bracket which was in turn fastened to a board held in place on an inside wall by four four-inch screws set in wooden plugs in holes drilled in the brick wall. The pendulum was suspended eight feet from the floor and floor and was swung eight inches from the wall.

The auxilliary apparatus was set up as follows. (Figure 5). A chronometer operated a make circuit relay which in turn operated three other relays. Of these, one was connected with the electromagnet of the chronograph pen, the second with the electromagnet which controlled the speed of the chronograph, the third, with a break circuit, with the primary of an induction coil. The secondary of the induction coil was connected with the spark gap behind the pendulum. The terminals of the spark gap were of

Figure 44

B, the box containing the spark gap.

z, z, the zinc electrodes

g, ground glass

m, metal plate

P, the pendulum

Figure 5

C, Chronometer

C.C. Chronograph Control

P, Chronograph pen

R, Relay

V, Variable resistance

I.C. Induction Coil

S, Spark Gap

K, Key

L, Leyden jar

zinc and were placed two or three millimeters apart. A Leyden jar was connected across the terminals of the spark gap. The spark was made as the circuit was broken.

In front of the spark gap (figure 4) was a piece of ground glass to diffuse the light and a metal plate which had in it three holes, each one millimeter in diameter. The two outside holes were in a horizontal line and the middle one slightly above it. The distance between the outer edges of the outside holes was enough greater than the diameter of the circular opening in the pendulum that when the pendulum was at rest, the edges of the outside holes of the plate coincided with the edges of the hole in the pendulum when viewed through the observing telescope.

To record a coincidence, the supplementary key which closed the pen circuit was depressed as soon as the spark was visible through all three holes and released as soon as a swing was made for which the spark was seen through two holes only. The use of the holes instead of an opening the size of that in the pendulum was of advantage in that the observer did not have to form a judgment of the shape of a comparatively large surface of light, but only of the number of spots of light which were visible. The key was held down each time two seconds longer than there were three spots visible, but this was taken into account in reading the chronograph sheet. The time that these spots were visible was from four to six seconds at the beginning of an observation and from eight to twelve at its close. The midpoint of this interval was taken as the instant of coincidence.

In beginning an observation, the pendulum was let

down on the planes of the support by means of the supplementary arms. It was placed each time as nearly as possible in the same position with respect to these arms (and consequently to the planes.) Any vibration in the plane of the knife edges was stopped and the pendulum set swinging by blowing on a piece of paper held against the pendulum. Pushing or pulling on the pendulum with an object of any kind had a tendency to cause vibrations in planes other than that perpendicular to the knife edges. The amplitude was measured by a paper scale attached to either end of the pendulum, 110cm. from the most distant knife edges. The initial double amplitude was from $43/1100$ to $37/1100$ radians.

The pendulum is of brass, with steel knife edges. The bar of the pendulum is not straight, the center being about 3mm. out of line; it becomes somewhat straighter when suspended. The knife edges are somewhat uneven. The one nearest the large mass lacks 15' of being perpendicular to the bar of the pendulum. To measure the distance between the knife edges, the pendulum was clamped to a rigid iron bar. The microscopes of the comparator were firmly clamped to the bed and after being focused on the standard meter were not disturbed in any way. The cross hairs of the microscopes were set on the lines of the meter and the distance between the zeros of the microscope worked out. The pendulum was put in place of the meter and blocked up until the knife edges were in focus, and the distance between the knife edges measured with reference to the distance between the zero points of the microscopes. Four different points were taken on the knife edges, the same distances from the bar at each end of the

pendulum; the distances between these four pairs of points measured, the pendulum was then turned over and the same four points again taken. The points were half and a quarter of an inch from the bar. Means of readings for eight positions of each point gave at 17.8° . (The thermometer was the same that was used when observing the period of the pendulum);

99.982	99.983	99.983	99.985	Mean	99.983
--------	--------	--------	--------	------	--------

Accordingly at 20° the length is 99.987 centimeters.

To get the periods about the knife edges equal to each other, rough determinations of the period were made as the large weight was shifted the length of the pendulum, the smaller weight remaining fixed. Readings of the period were taken every ten centimeters of the length, beginning at the heavy end. These values of the period were plotted as ordinates and the positions of the weight as abscissas. (Figure 6. The red curve indicates the period with the heavy end up) The curves cross in two places. This shows roughly the position of the large weight for equality of the periods.

Placing the large weight in the position indicated by the graph, the small weight was moved through a range of positions. It appeared from the plot of the periods against positions (figure 7) that the two curves would not cross.* The position of the large weight was changed slightly and another set of readings taken for different positions of the

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* A full discussion of this point is to be found in A Study of the Reversible Pendulum by John C. Shedd and James A. Birchby in The Physical Review, XXV, pp274-293; XXXIV, pp 110-124.

small weight. This time the two curves (figure 8) intersected in two places. The one of these that was on the part of the bar which bore a scale was taken and the small weight was shifted until the difference in period as the pendulum was swung about either knife edge was less than the errors in observing the period.

Table III shows the data recorded for a single swinging of the pendulum. Table IV gives the data for the swings ~~are~~ used in the computation of g . The period is given in sidereal seconds.

Swing #1. June 29 Heavy end down.

Table III.						
No. of Coinc.	Time	Interval of 6 Coin	Mean for 1 int. in seconds	T in seconds	temp	Amp. in mm
1R	5°11'44"				23.9°	37
1 L	114 48					
2R	17 48					
L	20 51					
3	23 52					
	26 55					
7	48 8	36'24"				
	51 12	24				
8	54 12	24				
	57 17	26				
9	6 00 18	26				
	3 22	27	364.22	2.011044	24.0°	20

The rate of the ^hronometer was computed by Mr. C. C. Wylie, assistant at the Lows Observatory. The rate was +0.3 seconds ^{found} Correcting the period on page 25, $T = 2.010808$. Reducing to mean solar time by the ratio, $365.25/366.25$, $T = 2.005412$. If this be substituted in the formula given on page 14;

$$g = \frac{4\pi^2 L}{T^2} = 981.508 \text{ centimeters per second per second.}$$

($1 = 99.987$. See page 22).

Table IV.

Date	No.	Coin. Int. in secs.	Mean temp.	Amplitude		Period Uncor't	Corrections		Period Corrected
				In'l	Final		Arc	temp.	
June 23	1 Down	364.22	24.0°	37	20	2.011044	.000087	.000075	2.010882
	2 Up	365.53	24.0	38	20	1002	87	75	840
	3 Down	366.17	24.0	39	21	0984	102	75	807
	4 Up	365.58	24.0	39	21	1002	93	75	834
June 24	5 Down	365.83	24.1	38	22	0994	96	76	824
	6 Up	366.61	24.2	40	21	0971	96	79	796
	7 Down	365.80	24.2	40	23	0995	102	77	814
	8 Up	365.95	24.1	41	21	0991	99	77	815
	9 Down	364.61	24.1	40	23	1031	102	77	852
	10 Up	367.80	24.0	38	19	0935	84	75	776
July 2	11 Down	364.53	25.4	40	23	0911	102	122	787
	12 Up	362.28	25.5	41	22	1125	99	124	902

Mean period, Heavy end down 2.010812

Heavy end up 2.010818

Mean 2.010815

The Smithsonian Physical Tables give the values of g , for St. Louis and Kansas City.

	Latitude	Elevation	g
St. Louis	38°38'	505'	980.000
Kansas City	39°06'	912'	979.989
Columbia, Mo.	38° 57'	757'	*****

These values are from the comparison work of the U. S. Coast and Geodetic Survey and are based on the value of 980.111 at Washington.

Using Table 84 of The Smithsonian Physical Tables (Fifth Revised Edition), the value of g at Columbia should be 980.006.

The apparent error in the results of the determination at Columbia are probably due to bending of the pendulum bar and to the imperfections of the knife edges. It was observed that as the position of the pendulum was changed with reference to the planes, that there was a change in period which amounted to one part in a thousand.

The following list of apparatus used is given for reference:

Kater's Convertible Pendulum and support by the Societe Genevoise
Break Circuit Sidereal Time Chronometer No. 383 by Wm. Bond & Son,
Boston. Loaned by The Laws Observatory, University of Missouri.
Standard Meter of Invar, No. 65, by the Societe Genevoise
Laboratory thermometer, No. 55.

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