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CONDITIONALLY CONVERGENT VECTOR SERIES

by

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INTRODUCTION. FUNDAMENTAL PROPERTIES OF SERIES.

I. Series of Real Numbers. In this paper we propose to study the behavior of series of complex numbers, or of vectors in two dimensions; and to generalize this study to the case of vectors in n dimensions. The particular properties to be studied are described on page 7. We shall first state a few well-known properties of series, both of real numbers and of complex numbers.

Consider any finite series of constant terms;

$$(I) \quad a_0 + a_1 + a_2 + \dots + a_n + \dots$$

A sequence can be built from this series as follows:

$$s_0 = a_0$$

$$s_1 = a_0 + a_1$$

$$s_2 = a_0 + a_1 + a_2$$

$$\dots$$

$$s_n = a_0 + a_1 + \dots + a_n$$

$$\dots$$

We say the series (I) converges if the limit of s_n , as n becomes infinite, exists. Then the value of $\lim_{n \rightarrow \infty} s_n$ is called the sum of the series.

Cauchy's necessary and sufficient condition that series (I) converge states: ')

If for a preassigned positive quantity ϵ there exists a number N such that $|a_n + a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \epsilon$ for $n \geq N$ and for any positive value of p , then the series (I) converges.

Regarding the series from the sequence standpoint, Cauchy's test becomes: ^① Given $\epsilon > 0$, ^② a number N exists, ^③ such ^④ that $|s_{n+p} - s_n| < \epsilon$, whenever $n \geq N$, for any p .

We say series (I) converges absolutely if the series composed of the absolute values of the terms of series (I), taken in the same order, converges. In symbols, the series (I) converges if

$$\textcircled{1} \epsilon, \textcircled{2} N, |a_n| + |a_{n+1}| + \dots + |a_{n+p}| < \epsilon, \textcircled{3} n \geq N, \textcircled{4} p$$

Not all series converge absolutely. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots + \frac{1}{2n+1} - \frac{1}{2n+2} + \dots$$

converges to the sum $\log_e 2$, but the series of absolute values diverges and the sum of the first n terms of that series becomes infinite as n becomes infinite. We say that a series converges conditionally if it converges and if the series of absolute values diverges.

The following series converges absolutely:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \dots + \left(-\frac{1}{2}\right)^n + \dots = \frac{2}{3}$$

for the series of absolute values also converges:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \dots + \left(\frac{1}{2}\right)^n + \dots = 2.$$

The order of the terms in such a series can be changed without affecting its convergence and the sum of the rearranged series will then be the same as that of the original series.')

Such a statement is not valid for conditionally convergent series. For example in the series mentioned above:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots + \frac{1}{2n+1} - \frac{1}{2n+2} \dots,$$

we may rearrange the terms by putting two negative terms after each positive term:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{4} \dots + \frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} \dots$$

The sum of the new series is the limit of the expression

$$\sum_{n=0}^{m-1} \left(\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} \right)$$

as m becomes infinite. It can be seen from the identity

1) Goursat-Hedrick: Mathematical Analysis, Vol. I, p.392.

$$\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} = \frac{1}{2} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right)$$

that the sum of the new series is just half the sum of the original one.

Furthermore this same series can be rearranged so as to converge toward any real number whatever. This is true for all conditionally convergent series of real terms. For let s_n denote the sum of the first n terms of such a series, s_p the sum of the positive terms occurring among the first n terms and s'_{n-p} the sum of the absolute values of the corresponding negative terms. Then $s_n = s_p - s'_{n-p}$. As n becomes infinite, each of the sums s_p and s'_{n-p} must increase indefinitely; for otherwise the series would diverge, or else converge absolutely. By the Cauchy test, since the series converges, the general term a_n approaches zero as a limit as n becomes infinite. If it is required to rearrange the series so as to converge to the sum K , we should proceed as follows: take positive terms from the given series in the order in which they occur in it until their sum exceeds K . Then add to these in the order in which they occur in the series negative terms until the total sum is less than K . Next add positive terms, beginning where we left off, until the total sum exceeds K . Then add negative terms as before; and so on forever. We see that s_n is alternately greater than and less than K and that it differs from K by a quantity which approaches zero as a limit; for, since the series converges, the limit of a_n as n becomes infinite is zero, where a_n denotes the general term of the series.

It is this property of conditional convergence which I wish to investigate for series of complex terms and for other series.

2. Series of Complex Numbers. Let us now turn to series of complex terms. Let

$$\text{II.} \quad \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_n + \dots$$

be such a series, where $\alpha_k = a_k + i b_k$. Such a series can be decomposed into two series, namely the one consisting of the real components of the α_k :

$$\text{III.} \quad a_0 + a_1 + a_2 + \dots + a_n + \dots ;$$

and the other of the coefficients of the imaginary components:

$$\text{IV.} \quad b_0 + b_1 + b_2 + \dots + b_n + \dots$$

If series III converges, say to A , and if series IV converges, say to B , then we define series II as being convergent to the sum $A + iB$. The convergence of series II may also be defined in the following manner, which is equivalent to the definition just given: if the sum of the first n terms of series II approaches a limit K as n becomes infinite, then the series is said to converge to the sum K .

Series II may be regarded as a series of vectors. The real part of each term then becomes the x component of the vector representing the term and the coefficient of the imaginary part becomes the y component. Each partial sum $S_n = \alpha_0 + \alpha_1 + \dots + \alpha_n$ is represented as a vector starting at the origin and terminating in the point (A_n, B_n) .

Since $|\alpha_n + \alpha_{n+1} + \dots + \alpha_{n+p}| \leq |\alpha_n| + |\alpha_{n+1}| + \dots + |\alpha_{n+p}|$, we can apply Cauchy's necessary and sufficient condition for the convergence of a series of complex numbers to series II, namely that corresponding to any preassigned number ϵ a positive integer N should exist such that the difference $|\overline{A_{n+p} + iB_{n+p}} - \overline{A_n + iB_n}|$ is less in absolute value than ϵ for any $n > N$ and any pos-

itive integer p . Geometrically this means that if a circle of radius ϵ be placed about (A_n, B_n) as center, then the

terminal point of each vector representing $A_{n+p} + iB_{n+p}$ lies in this circle. A square of side 2ϵ might have been used in place of the circle of radius ϵ .

When the series of absolute values of the terms of II V. $\sqrt{a_0^2 + b_0^2} + \sqrt{a_1^2 + b_1^2} + \dots + \sqrt{a_n^2 + b_n^2} + \dots$

converges, each of the series III and IV evidently converges absolutely, for $|a_n| \leq \sqrt{a_n^2 + b_n^2}$ and $|b_n| \leq \sqrt{a_n^2 + b_n^2}$. In this case the series II is said to be absolutely convergent. Conversely, if each of the series III and IV converges absolutely, the series II converges absolutely, for $\sqrt{a_n^2 + b_n^2} \leq |a_n| + |b_n|$.

As in the case of a series of real terms, we may change the arrangement of the terms of an absolutely convergent series without altering its sum. ')

What happens when we rearrange the terms of one that does not converge absolutely? By analogy to the case of a real series, whose partial sums are represented in a one dimensional space, one might be led to believe that a conditionally convergent series of complex terms, whose partial sums are represented in a space of two dimensions, could be rearranged so that the series as rearranged would converge toward any preassigned complex number, that is to any preassigned point in the complex plane. That this is not the case appears from the following example,-

$$I \quad (1 + i) - \left(\frac{1}{2} + i\frac{1}{2}\right) + \left(\frac{1}{3} - i\frac{1}{4}\right) - \dots + (-1)^n \left[\frac{1}{n} - i\left(\frac{1}{2}\right)^{n-1}\right] + \dots$$

Upon examining the two series formed of the real parts of

') Goursat-Hedrick: Mathematical Analysis, Vol. I, p. 355-356.

the successive terms of I and of the coefficients of the imaginary parts

$$\text{II.} \quad 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \dots - \left(\frac{1}{2}\right)^n - \dots,$$

$$\text{III.} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n+1} \frac{1}{n} + \dots,$$

we see that the sum of III is $\log_e 2$. Since II is conditionally convergent and I is absolutely convergent, the terms of III can be rearranged so that the sum of the series is any real number K. Geometrically speaking this means that the vector series can be arranged so that the polygon representing it converges toward any desired point on the real axis; but it cannot be made to converge toward any other point.

Consider next the series

$$\text{IV.} \quad (1 + i) e^{i\frac{\pi}{4}} - \left(\frac{1}{2} + i\frac{1}{2}\right) e^{i\frac{\pi}{4}} + \dots + (-1)^n \left[\frac{1}{n} - i\left(\frac{1}{2}\right)^{n-1}\right] e^{i\frac{\pi}{4}} + \dots$$

From a vector standpoint each term of this series is obtained by rotating the corresponding term in series I through an angle of 45° . Therefore the sum of IV is the same as the sum of I rotated through an angle of 45° '); and if we rearrange the terms of IV we can make it converge toward any point on the straight line $x = iy$; but it cannot be made to converge toward any point not on that line.

There are indeed series of complex terms that can be rearranged so as to converge toward any preassigned complex number. Such an one is:

$$(1 + i) - \left(\frac{1}{2} + i\frac{1}{3}\right) + \left(\frac{1}{3} + i\frac{1}{5}\right) - \left(\frac{1}{4} + i\frac{1}{7}\right) + \dots \\ \dots + \left(\frac{1}{2n+1} + i\frac{1}{4n+1}\right) - \left(\frac{1}{2n+2} + i\frac{1}{4n+3}\right) + \dots$$

The sum of the above series as it stands is $\log 2 + \frac{\pi}{4}$.

If we rearrange this series by putting two negative terms after each positive term, -

$$\left[(1 + i) - \left(\frac{1}{2} + i\frac{1}{3}\right) - \left(\frac{1}{4} + i\frac{1}{7}\right)\right] + \left[\left(\frac{1}{3} + i\frac{1}{5}\right) - \left(\frac{1}{6} + i\frac{1}{11}\right) - \left(\frac{1}{8} + i\frac{1}{15}\right)\right] + \dots \\ \dots + \left[\left(\frac{1}{2n+1} + i\frac{1}{4n+1}\right) - \left(\frac{1}{4n+2} + i\frac{1}{8n+3}\right) - \left(\frac{1}{4n+4} + i\frac{1}{8n+7}\right)\right] + \dots,$$

*) For proof, see p. 13.

we get as its sum $\frac{1}{2} \log_e 2 + i \lim_{n \rightarrow \infty} \sum_{n=0}^{2n} \frac{52n^2 + 6n - 19}{(n+4)(8n+3)(8n+7)}$.

By making still a third arrangement of the terms of the above series wherein we put two positive terms after each negative term,-

$$\left[\left(-\frac{1}{2} - i\frac{1}{3} \right) + \left(1 + i1 \right) + \left(\frac{1}{3} + i\frac{1}{5} \right) \right] + \left[\left(-\frac{1}{4} - i\frac{1}{7} \right) + \left(\frac{1}{5} + i\frac{1}{9} \right) + \left(\frac{1}{7} + i\frac{1}{13} \right) \right] + \dots$$

$$\dots + \left[\left(-\frac{1}{2n+2} - i\frac{1}{4n+3} \right) + \left(\frac{1}{4n+1} + i\frac{1}{8n+11} \right) + \left(\frac{1}{4n+3} + i\frac{1}{8n+5} \right) \right] + \dots,$$

we get the sum $\lim_{n \rightarrow \infty} \sum_{n=0}^{2n} \frac{8n+5}{(2n+2)(4n+1)(4n+3)} + i \lim_{n \rightarrow \infty} \sum_{n=0}^{2n} \frac{24n+13}{(4n+3)(8n+11)(8n+5)}$.

The three sum points for the three different arrangements of the terms do not lie on the same straight line. According to the main theorem to be proved below, this is a sufficient condition that the series can be rearranged so as to converge toward any point in the complex plane. ')

Thus we see that there are at least two general classes of conditionally convergent series of complex terms:

I. Those whose sums resulting from all possible rearrangements of the terms correspond to all the points on a fixed straight line and to those points only.

II. Those whose sums correspond to all the points in the complex plane.

This leads to the question: is there any other class? For example, can a series be found whose sums resulting from all possible rearrangements of the terms correspond to all the points on two fixed straight lines and to those points only; or to all the points in a portion of the plane and to those points only?

This problem has recently been investigated by E. Steinitz of Germany. In the "Journal für die reine und

') Compare p. 23.

angewandte Mathematik", Band 143, Heft 1 (July, 1913), under the title, "Bedingt konvergente Reihen und konvexe Systeme", Steinitz has proved that there exist only the two classes of conditionally convergent series of complex terms mentioned above.

The same problem was attacked by P. Lévy in the "Nouvelles Annales de Mathématiques", Vol. V, 1905, p. 506-511, in an article entitled: "Sur les Séries Semi-Convergentes"; but his work was not detailed and was lacking in rigor.

This paper treats of the problem from an entirely different standpoint.

Chapter I. The Fundamental Theorem.

We shall prove the following theorem which essentially disposes of the entire question:

Theorem I. If the series of complex terms

$$a_0 + a_1 + a_2 + a_3 + \dots + a_n + \dots,$$

where $a_k = x_k + iy_k$, converges to a point $P (A + iB)$,

and if this series can be rearranged in the form

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n + \dots,$$

so that the new series converges to another point $Q (U + iV)$,

then the series can be arranged so that it will converge to

any preassigned point R on the straight line joining P and Q .

Stated in symbolic form the hypothesis is:

- (1) $a_0 + a_1 + a_2 + \dots + a_n + \dots = A + iB$
- (2) $\overset{①}{\epsilon}, \overset{②}{K}, |a_k + a_{k+1} + \dots + a_{k+p}| < \overset{③}{\epsilon}, k > K, \overset{④}{p} > 0$
- (3) $\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_n + \dots = U + iV$
- (4) $\overset{①}{\epsilon}, \overset{②}{J}, |\alpha_j + \alpha_{j+1} + \dots + \alpha_{j+q}| < \overset{③}{\epsilon}, j > J, \overset{④}{q} > 0$

Furthermore every α is an a and every a is an α .

The proof of the theorem is as follows. Take

$$(5) \quad a_0 + a_1 + a_2 + \dots + a_{k_1} = A_1 + iB_1; \quad k \geq K.$$

Fig. 2. Geometrically the left hand side of equation (5)

is represented by the addition of vectors whose

summand terminates in the point $(A_1 + iB_1)$, which lies

inside a square of side 2ϵ about $(A + iB)$ as center.

Since every a is an α , all the terms of (5) can be found

among a finite number of terms of series (2) taken in order

beginning with the first and ending say with α_c . Take $j_1 \geq c$

and $j_1 \geq J$; then

$$(6) \quad \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_{j_1} = U_1 + iV_1$$

Geometrically the left hand side of (6) is represented by the addition of vectors whose summand
Fig. 3 terminates in the point $(U_1 + iV_1)$, which lies inside of a square of side 2ϵ about $(U + iV)$.

Subtract (5) from (6). Call the remaining terms α' ; then

$$(7) \quad \alpha'_1 + \alpha'_2 + \alpha'_3 + \dots + \alpha'_{z_1} = (U_1 + iV_1) - (A_1 + iB_1) = D_1.$$

Adding these terms to those of series (5), we obtain

$$(8) \quad a_0 + a_1 + \dots + a_{k_1} + (\alpha'_1 + \alpha'_2 + \dots + \alpha'_{z_1}) = U_1 + iV_1.$$

Geometrically (8) expresses the fact that the α' terms,

Fig. 4 regarded as vectors, proceed from the point $(A_1 + iB_1)$ to the point $(U_1 + iV_1)$ inside the square of side 2ϵ about $(U + iV)$.

All the terms of (8) can be found among the first k_2 terms of series (1), where $k_2 > k_1 \geq K$:

$$(9) \quad a_0 + a_1 + a_2 + \dots + a_{k_2} = A_2 + iB_2$$

Subtracting (8) from (9) and calling the remaining terms a' , we obtain a series

$$(10) \quad a'_1 + a'_2 + \dots + a'_{\ell_1} = -D_2,$$

where $D_2 = D_1 \pm \delta_1$, $\delta_1 < \epsilon$. This is true of the magnitude of D_2 because by taking the terms in series (1) past a_{k_1} out to a_{k_2} , we have, by the Cauchy test,

$$|(\alpha'_1 + \alpha'_2 + \dots + \alpha'_{z_1}) + (a'_1 + a'_2 + \dots + a'_{\ell_1})| < \epsilon$$

Since the sum of the terms in the first parenthesis is D_1 , the sum of those in the second parenthesis must be $-D_1 \pm \delta_1$, where $\delta_1 < \epsilon$. Geometrically the a' terms are represented by small vectors of magnitude less than

Fig. 5. or equal to ϵ which proceed from the point $(U_1 + iV_1)$ and advance to the point $(A_2 + iB_2)$

which lies inside the square of side 2ϵ about $(A + iB)$.

All the terms of (9) can be found among the first j_2 terms of series (2), where $j_2 > j_1 \geq J$:

$$(11) \quad \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_{j_2}' = U_2 + iV_2.$$

Subtracting (9) from (11) and calling the remaining terms α'' , we obtain the series

$$(12) \quad \alpha_1'' + \alpha_2'' + \alpha_3'' + \dots + \alpha_{j_2}'' = (U_2 + iV_2) - (A_2 + iB_2) = D_3.$$

Call $|Q - P| = 2\epsilon = d$. By Cauchy's test it follows that $|D_3| \geq d$, and $|D_3| \leq d + 4\epsilon$.

Geometrically the α'' terms are represented by vectors of magnitude not greater than ϵ which proceed from $(A_2 + iB_2)$

to $(U_2 + iV_2)$, where $(U_2 + iV_2)$ is a point
Fig. 6. inside the square of side 2ϵ about $(U + iV)$.

We could proceed in this manner forever and obtain an infinite number of groups of α^i terms such that each group would be represented graphically as an addition of vectors starting at a point inside the square of side 2ϵ about $(A + iB)$ and proceeding to a point inside the square of side 2ϵ about $(U + iV)$. Likewise we would obtain an infinite number of groups of α^i terms such that each group would be represented graphically as an addition of vectors starting at a point inside the square of side 2ϵ about $(U + iV)$ and proceeding to a point inside the square of side 2ϵ about $(A + iB)$. Furthermore let us bear in mind that the length of each of the α^i and α^i vectors is not greater than ϵ .

It is easy to show that series (1) can be rearranged so that it will converge toward $U + iV$. Take (8) :

$$(8) \quad (\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_{k_1}) + (\alpha_1' + \alpha_2' + \dots + \alpha_{j_1}') = U_1 + iV_1.$$

Add the rest of series (2) past α_{j_1}' to (8) :

$$(13) \quad \alpha_0 + \alpha_1 + \dots + \alpha_{k_1} + \alpha_1' + \alpha_2' + \dots + \alpha_{j_1}' + \alpha_{j_1+1}' + \dots + \alpha_{j_1+n}' + \dots$$

Since (8) contains only a finite number of terms, whose sum is $U_1 + iV_1$, where $(U_1 + iV_1)$ is a point inside the square of side 2ϵ about $(U + iV)$, and since the terms past α_j in series (2) when added in series (2) at the point $(U_1 + iV_1)$ do not vectorially go outside the square of side 2ϵ about $(U + iV)$, then certainly their addition to (8) will bring the same result, - namely convergence to $U + iV$.

It remains to be proved that series (1) can be rearranged so that it will converge toward any preassigned point on the straight line joining P and Q.

As a first step we shall translate and rotate the axes so that the line joining P and Q will coincide with the x axis. This will not affect the convergence of the series, because of the following Lemmas.

Lemma I. If the series

$$(x_0 + iy_0) + (x_1 + iy_1) + \dots + (x_n + iy_n) \dots = X + iY$$

is convergent to the point $(X + iY)$ and if the x axis be translated so that the new axis system has for its origin the point $(0, K)$ of the old axis system, then the series will converge toward the point $[X, i(Y + k)]$

Set up a sequence from the series (1):

$$(x_0 + iy_0) = A_0 + iB_0$$

$$(x_0 + iy_0) + (x_1 + iy_1) = A_1 + iB_1$$

$$\dots \dots \dots$$

$$(x_0 + iy_0) + (x_1 + iy_1) + \dots + (x_n + iy_n) = A_n + iB_n$$

$$\dots \dots \dots$$

The sequence $A_j + iB_j$ approaches $X + iY$ as a limit as j becomes infinite.

After the translation of axes the sequence becomes:

$$[A_0 + i(B_0 + k)], [A_1 + i(B_1 + k)], \dots, [A_n + i(B_n + k)], \dots$$

Hence this sequence will approach the limit
Fig. 7. of the former sequence plus the constant k.

Changing the last sequence back into a series,
 its sum is $X + i(Y + k)$:

$$[\lambda_0 + i(q_0 + k)] + [\lambda_1 + i(q_1 + k)] + \dots + [\lambda_n + i(q_n + k)] + \dots = [X + i(Y + k)].$$

By similar reasoning it could be shown that the other
 axis or both axes could be translated without affecting the
 convergence of the series.

Lemma II. If the axes be rotated through an angle θ ,
then the above series converges to the point $[X(\cos \theta + \sin \theta),$
 $Y(\cos \theta - \sin \theta)i]$, which is the point $(X + iY)$ rotated
through the angle θ .

Consider the sequence set up above,-

$$\text{I. } A_0 + iB_0, A_1 + iB_1, \dots, A_n + iB_n, \dots$$

whose limit as n becomes infinite is $X + iY$. After
 rotating the axes this sequence becomes

$$\begin{aligned} \text{II. } & A_0(\cos \theta + \sin \theta) + iB_0(\cos \theta - \sin \theta), \dots \\ & \dots, A_n(\cos \theta + \sin \theta) + iB_n(\cos \theta - \sin \theta), \dots \end{aligned}$$

Consider the sequence of A's in I:

$$\text{III. } A_0, A_1, A_2, \dots, A_n \xrightarrow{n \rightarrow \infty} X$$

After rotating the axes this sequence becomes

$$\begin{aligned} \text{IV. } & A_0(\cos \theta + \sin \theta), A_1(\cos \theta + \sin \theta), \dots, \\ & \dots A_n(\cos \theta + \sin \theta), \dots \end{aligned}$$

Since each term of IV results from multiplying each term
 of III by a constant, $(\cos \theta + \sin \theta)$, the limit of sequence

IV must be the limit of sequence III multiplied

Fig. 8. by $(\cos \theta + \sin \theta)$. Similarly the limit of
 the B sequence of II can be shown to be

$iY(\cos \theta - \sin \theta)$. Changing sequence II back into a series
 we get as its sum $X(\cos \theta + \sin \theta) + iY(\cos \theta - \sin \theta)$.

We are now at liberty to make the change of axes referred to above. Produce the upper and lower sides of the squares about P and Q until they meet, thus forming a rectangle of width 2ϵ and length $d + 4\epsilon$. Extend this rectangle 2ϵ units to the left.

Fig. 9.

We pointed out above that any one group of the α^i terms, if regarded as vectors, would advance from a point inside the square about P to a point inside the square about Q. In so doing, the vectors might roam considerably in the vertical direction. However we shall now proceed to prove that in any event this group of α^i terms can be rearranged so that they will never advance outside the rectangle.

Let us remember that the magnitude of each α^i vector is never greater than ϵ . Suppose we are dealing with the α^k terms where $\alpha_n^k = x_n + iy_n$. The absolute value of the sum of the y components of these vectors is never greater than ϵ , because even if the terms do wander in the vertical direction, they always go from a point inside the square of side 2ϵ about $(A + iB)$ to a point inside the square of side 2ϵ about $(U + iV)$.

Let us rearrange the α^k terms, and suppose that at some stage before we get to the last term we arrive at a point $(x' + iy')$, where y' is positive and less than or equal to ϵ . Suppose that the next vector we add takes us to $(x'' + iy'')$, where y'' is greater than or equal to ϵ . Since we have

$$|y_0 + y_1 + \dots + y_{z_k}| \leq \epsilon$$

there must be a vector or vectors with negative y components

that will bring us to a point whose ordinate is greater than zero and less than or equal to ϵ . But we can add this vector or vectors at the point (x',iy') , - which will still leave us inside the rectangle, - and then add the vector with positive y component that previously gave trouble. Finally it is evident that the last vector cannot take us to a point outside the rectangle, for this would give

$$|y_0 + y_1 + \dots + y_{z_k}| > \epsilon.$$

A similar argument holds in case the vectors are supposed to wander below the rectangle.

Corollary I. If a point R is selected between P and Q, it is evident from the foregoing proof that we can proceed with α' vectors from a point inside the square of side two ϵ about P to a point inside a square of side 2ϵ about R, without going outside the rectangle.

Corollary II. By using α' terms it can be shown by the same manner of proof as employed above that we can proceed from a point inside the square of side 2ϵ about Q to a point inside the square of side 2ϵ about P or about any point R between P and Q in the interval P,Q without ever going outside the rectangle.

Corollary III. If the point R is distant to the right of Q by $d + 2\epsilon$ units, then it is possible to select two groups of α' terms that can be arranged in such a manner that they all lie inside the rectangle of width 2ϵ referring to P and R.

From equation (5) we have

$$(5) \quad a_0 + a_1 + a_2 + \dots + a_{K_1} = A_1 + iB_1.$$

It was indicated above that the point $(A + iB)$ lies inside the square of side 2ϵ about P , and I have shown that the α' terms can be added to (5) in such a manner that vectorially they proceed from $(A + iB)$ to a point $(U + iV)$ inside the square about Q without going outside the rectangle relating to P and Q . From now on let us think of each group of α' and a' terms as being rearranged so that its terms regarded as vectors never depart from the rectangle of width 2ϵ and length $d + 6\epsilon$.

When the α' terms are added to (5), we have

$$a_0 + a_1 + a_2 + \dots + a_k + (\alpha'_1 + \alpha'_2 + \dots + \alpha'_{z_1}) = U + iV.$$

Let us add to this equation the α'' terms:

$$a_0 + a_1 + \dots + a_k + (\alpha'_1 + \alpha'_2 + \dots + \alpha'_{z_1}) + (\alpha''_1 + \alpha''_2 + \dots + \alpha''_{z_2}).$$

The α'' vectors might wander outside the long rectangle relating to P and R , as in figure 11. If they do, the

vertical variation could never take us to a

Fig. 11. point whose ordinate is greater in absolute value than 3ϵ . To remedy this, it is only necessary to go back and choose K and J so that the α' and α'' terms will be less in absolute value than $\frac{\epsilon}{4}$.

This will enable us to proceed from a point

Fig. 12. inside the square of side 2ϵ about P to a point inside the square of side 2ϵ about R without leaving the rectangle.

By extension it is easy to show that R may be chosen $n(d + 2\epsilon)$ units to the right of P . By using a' terms R could be chosen $n(d + 2\epsilon)$ units to the left of P .

Lastly R could be chosen as any point between $(n + 1)(d + 2\epsilon)$ and $n(d + 2\epsilon)$ units to the right of P (or left of P).

Thus we see that if ϵ is chosen however small, and if any point R on the straight line joining P and Q is selected, then a group of terms of series (2) can be found which will take us from a point inside a square of side 2ϵ about P to a point within a square of side 2ϵ about R without departing from the sides of the rectangle relating to P and R ; or, if you please, a group of terms of series (1) can be found that will take us from a point inside a square of side 2ϵ about R to a point within a square of side 2ϵ about P without departing from the rectangle relating to R and P .

We shall need one further restriction on the arrangement of the terms in rectangles. It might be possible that as we proceed to the right with α' vectors, there would be a succession of these terms which would go to the left a distance greater than 6ϵ . To avoid this we shall proceed as follows:

About P and R as centers place squares of side 6ϵ , and form a rectangle by joining the upper and lower sides of these squares. Beginning at the left draw vertical lines at intervals of 6ϵ and number the square blocks thus formed. Then beginning at the left draw vertical lines at intervals of 3ϵ , thus dividing each block into a right and left half. Next put squares of side 2ϵ about P and R and form a rectangle by joining the upper and lower sides of these squares.

It is now our purpose to arrange the α' vectors so that the tip of no vector shall be outside the larger of

the two rectangles, and so that if vector K terminates in block N, then no vector past the Kth will lie wholly or in part in block (N - 1).

To do this, take first those α^v vectors having positive x and y components and arrange these terms in order according to the magnitude of the ratio $\frac{x}{y}$. If at any stage one of these vectors, α_k , has as origin a point in the right half of block N of the rectangle of width 2ϵ and terminates in a point above this rectangle, we shall add in its place a vector having a positive x component and a negative y component, if such a vector exists. If no such vector exists, then there must be one or more vectors having negative x components and negative y components and such that the ratio $\frac{|x|}{|y|}$ is less than that of α_k and such that the sum of the absolute values of their y components is at least equal to the y component of α_k . We should add these vectors in place of α_k .

If α_k has as origin a point in the left half of block N of the rectangle of width 2ϵ and terminates in a point above this rectangle, we shall proceed in the same manner as above, except that if a vector or vectors with negative x components have to be used and they lead to a point in block (N - 1), then instead of adding them in place of α_k , we shall add them in place of the first vector terminating in block N.

After all vectors with positive y components have been added, then add vectors with negative y components in order of the magnitude of the ratio $\frac{x}{|y|}$. Treat the addition of these vectors in the same manner as was used above in the case of vectors with positive x components. Finally add

any vectors with negative x components that may be left after the addition of those with positive x components.

The plan of arranging the vectors just described we shall call the "block arrangement". In order to make the block arrangement in a rectangle of width 2ϵ , we could select α'_i terms which have an absolute value less than or equal to $\frac{\epsilon}{3}$.

To return to the proof proper, let any point R on the straight line joining P and Q be selected. Suppose R is to the right of P . Then K and J of (3) and (4) can be so chosen that the sum

$$(14) \quad (\alpha'_1 + \alpha'_2 + \dots + \alpha'_{2_1}) + (\alpha''_1 + \alpha''_2 + \dots + \alpha''_{2_2}) + \dots + (\alpha'_1 + \dots + \alpha'_{f_1})$$

will take us from a point inside the square of side 2 about P to a point inside the square of side 2ϵ about R without going outside of the rectangle relating to P and R . Call the group of terms in (14) ω .

By (5) we have, - $a_0 + a_1 + \dots + a_{K_1} = A_1 + iB_1$, where $(A_1 + iB_1)$ is a point inside the square of side 2ϵ about P . Add (14) to (5), -

$$(15) \quad (a_0 + a_1 + \dots + a_{K_1}) + \omega = M_1 + iN_1,$$

where $(M_1 + iN_1)$ is a point inside the square of side 2ϵ about R . All the terms of (15) can be found among the first x_1 terms of series (1), -

$$(16) \quad a_0 + a_1 + a_2 + \dots + a_{x_1} = A_{x_1} + iB_{x_1}.$$

Subtract (15) from (16) and call the remainder

$$(17) \quad (\alpha'_1 + \alpha'_2 + \dots + \alpha'_{f_1}) + (\alpha''_1 + \dots + \alpha''_{f_2}) + \dots + (\alpha'_1 + \dots + \alpha'_{f_m})$$

These terms take us from $(M_1 + iN_1)$ without going outside the rectangle relating to P and R . Let us give them the block

arrangement, using a rectangle of width 4ϵ and call the whole group Γ_i . It remains to be shown that the a_i and the rest of the α_i terms can be added to (15) thus forming a new arrangement of series (1) so that for any preassigned number θ , however small, there will come a stage in the addition of a_i and α_i terms to (15) when the summand regarded as a vector always has its terminal point inside a square of side 2θ about R .

Because of the convergence of series (1), $|N_i| \leq \epsilon$. If $|N_i| = \epsilon$, we shall go back and choose K and J so that $|N_i| < \epsilon$. Suppose N is positive. Call $\epsilon_i - N_i = \delta_i$. Take $\beta_i < \frac{\delta_i}{2}$; $\beta_i < \frac{\epsilon}{64}$. Call the distance $|R - P| = 1$. Call $|A_{x_i} - M_i| = \xi_i$. Take the index i so large that consecutive groups of α_i terms can be found, each of which is less in absolute value than β_i . Choose such a number of these terms that the sum of their components in the direction of the x axis is less in absolute value than ξ_i , and greater than $\xi_i - \beta_i$, and moreover so that they can be arranged in such a manner that vectorially they never depart from the sides of a rectangle of width $2\beta_i$. Using a rectangle of width $4\beta_i$ let us give these terms the block arrangement by drawing vertical lines at intervals of 4ϵ , and call them collectively Δ_i .

Next add to (15) the first block of the terms of Γ_i , then the first block of Δ_i , then the second block of Γ_i , and so on until all the terms of Γ_i and Δ_i are used. During this procedure the terminal point of any summand will never have an ordinate greater than 6ϵ , nor will its abscissa be greater or less than the abscissa of R by more than 6ϵ ; and when the last term of Γ_i and Δ_i is added, the terminal

point of the summand will be inside the square of side 2ϵ about R , say $(M_1 + iN_1)$. Our new series thus far is

$$(18) \quad a_0 + a_1 + \dots + a_{k_1} + \omega + (\overline{\Gamma}_1 + \Delta_1) = M_1 + iN_1$$

Let us take the terms of series (1) in order from the first until the first term of Δ_1 is reached, say to a_{x_2} :

$$(19) \quad a_0 + a_1 + \dots + a_{k_1} + a_{k_1+1} + \dots + a_{x_2} = A_{x_2} + iB_{x_2}.$$

Subtract (18) from (19), call the remaining terms Φ , and add them to (18) in the same order as they occur in series (1):

$$(20) \quad a_0 + a_1 + \dots + a_{k_1} + \omega + (\overline{\Gamma}_1 + \Delta_1) + \Phi = M_2 + iN_2.$$

While these terms are being added, the terminal point of no summand will be exterior to a square of side 3ϵ about R and the point $(M_3 + iN_3)$ will lie inside a square of side $\frac{\epsilon}{16}$ about R .

The proof from here on is parallel to that already given. All the terms of (20) can be found among the first x_3 terms of series (1),-

$$(21) \quad a_0 + a_1 + \dots + a_{x_1} + a_{x_1+1} + \dots + a_{x_2} + \dots + a_{x_3} = A_{x_3} + iB_{x_3}.$$

Subtract (20) from (21), give the remaining terms a block arrangement in a rectangle of width $\frac{\epsilon}{16}$, and call them collectively $\overline{\Gamma}_2$. Call $\frac{\epsilon}{32} - |N_3| = \delta_3$. Take $\beta_1 < \frac{\delta_2}{2}$; $\beta_2 < \frac{\epsilon}{2^{12}}$. Call $|A_{x_3} - M_3| = \xi_2$. Take the index i so large that consecutive groups of α^i terms can be found, each of which is less in absolute value than β_1 . Choose such a number of these terms that the sum of their components in the direction of the x axis is less in absolute value than ξ_2 and greater than $\xi_2 - \beta_2$, and moreover so that they can be arranged in such a manner that vectorially they never depart from the sides of a rectangle of width $2\beta_2$. Using a rectangle of width $4\beta_2$ let us give these terms the block arrangement by drawing vertical lines at intervals of $\frac{\epsilon}{16}$.

Add to (20) the first block of the terms of Γ_2 , then the first block of Δ_2 , and so on until all the terms of Γ_2 and Δ_2 are used. During this procedure the terminal point of no summand will have an ordinate greater than $\frac{3\epsilon}{16}$ in absolute value, nor will its abscissa be greater or less than the abscissa of R by more than $\frac{3\epsilon}{16}$; and when the last term of Γ_2 and Δ_2 is added, the terminal point of the summand will be inside a square of side $\frac{3\epsilon}{32}$ about R , - say $(M_4 + iN_4)$.

At this stage our new series is

$$(22) \quad a_0 + a_1 + \dots + a_{k_1} + \omega + (\Gamma_1 + \Delta_1) + \Phi_1 + (\Gamma_2 + \Delta_2) = M_4 + iN_4.$$

Let us take the terms of series (1) in order from the first until the first term of Δ_2 is reached, - say to x_4 :

$$(23) \quad a_0 + a_1 + a_2 + \dots + a_{x_4} = A_{x_4} + iB_{x_4}.$$

Subtract (22) from (23), call the remaining terms Φ_2 and add them to (22) in the same order as they occur in series (1). While these terms are being added, the terminal point of no summand will be exterior to a square of side $\frac{3\epsilon}{32}$ about R , and the point $(M + iN)$ will be interior to a square of side $\frac{2\epsilon}{2^n}$ about R .

By continuing this process we get for the rearranged series,

$$(24) \quad (a_0 + a_1 + \dots + a_{k_1}) + \omega + (\Gamma_1 + \Delta_1) + \Phi_1 + (\Gamma_2 + \Delta_2) + \Phi_2 \\ + (\Gamma_3 + \Delta_3) + \dots + (\Gamma_n + \Delta_n) + \Phi_n + \dots$$

Chapter II. Final Theorem for Two Dimensions.

In Chapter I it was proved that if series (1) has two different sum points, P and Q , corresponding to two different arrangements of its terms, then it can be arranged so as to converge toward any point on the straight line joining P and Q . Suppose now that this series for three different arrangements of its terms has as corresponding sums three sum points not on the same straight line, and that these sums are $(A + iB)$, $(U + iV)$, and $(R + iS)$. It is easy to show that the terms can be arranged in such a manner that the series will converge toward any point whatever in the complex plane. Suppose you choose the point $(P + iQ)$. Select that one of the three points $(A + iB)$, $(U + iV)$ and $(R + iS)$ that is at the greatest distance from $(P + iQ)$,— say $(R + iS)$. Draw the straight line determined by $(R + iS)$ and $(P + iQ)$. It intersects the straight line joining $(U + iV)$ and $(A + iB)$ in some point $(G + iH)$. By the theorem proved in Chapter I, series (1) can be rearranged so as to converge toward $(G + iH)$. Since $(G + iH)$ and $(P + iQ)$ lie on the same straight line, it must be possible, according to the theorem, to rearrange the terms of series (1) so that the resulting series will converge toward the sum $(P + iQ)$.

As has already been indicated in this paper, the terms of a complex series can be regarded as vectors. The vectorial representation of the first n terms of a series is a vector polygon. We say that such a polygon converges absolutely in the direction of a certain straight line if the

series made up of the components of the individual vectors in the direction of the straight line converges absolutely.

We shall define a countably infinite set of vectors as being absolutely convergent if it converges absolutely in every direction. This is equivalent to saying that it is absolutely convergent if its components converge absolutely in two directions. The following Lemma will establish the equivalence.

Lemma I. Let (1) $d_0 + d_1 + d_2 + \dots + d_n + \dots$
represent a vector polygon with an infinite number of sides.
Call its components in the direction of the straight line
(L), $x + iky = 0$.

(2) $a_0 + a_1 + a_2 + \dots + a_n + \dots$,

and its components in the direction of the straight line
(M), $x + iky = 0$.

(3) $b_0 + b_1 + b_2 + \dots + b_n + \dots$.

If both the series (2) and (3) converge absolutely, then
(1) converges absolutely in the direction of any straight
line in the plane.

Suppose that the theorem were not true and that the components of (1) in the direction of the straight line N, $x + iky = 0$ gives a conditionally convergent series,-

(4) $c_0 + c_1 + c_2 + \dots + c_n + \dots$.

Further more suppose that the components of series (1) in the direction perpendicular to L, $x - iky = 0$ give an absolutely convergent series,-

(5) $d_0 + d_1 + d_2 + \dots + d_n + \dots$.

If both series (3) and (5) converge absolutely,
Fig. 14. then (1) must converge absolutely. (Definition given at the first of the paper.) The

projection of any term of (1) on N is less than or equal to that term. Hence the projection of (1) on N must give an absolutely convergent series. This is contrary to our supposition. Hence if (5) is absolutely convergent, the Lemma is proved. Suppose (5) is conditionally convergent. Then (5) can be arranged so as to converge toward the sum corresponding to any point D_K on the straight line perpendicular to L , -

$$(6) \quad d'_0 + d'_1 + d'_2 + \dots + d'_n + \dots = D_K$$

$$(7) \quad |d'_n + d'_{n+1} + \dots + d'_{n+p}| < \overset{(1)}{\epsilon}, \quad \overset{(3)}{n} > \overset{(2)}{N}, \quad \overset{(4)}{p}$$

Since series (2) converges absolutely, then for any arrangement of its terms its sum is always the same. Call it A .

$$(8) \quad a_0 + a_1 + a_2 + \dots + a_n + \dots = A.$$

Let us denote by θ the angle between the lines L and M .

Then the projection of series (1) on M gives a series whose sum is $A \cos \theta + i D_K \sin \theta$. Since D_K takes on more than one value, the series must be conditionally convergent. But this is contrary to the hypothesis. Thus the Lemma is proved and we can draw the following conclusions:

(1). If series (1) converges absolutely in two directions, it has only one sum for all possible arrangements of its terms.

(2). If series (1) converges absolutely in only one direction, then its sum points lie on a straight line perpendicular to the line of absolute convergence.

(3). If (1) is conditionally convergent in the directions of two straight lines that are perpendicular to each other, then its sum points either lie on a straight line or

else they fill the whole plane.

(4). If (1) is conditionally convergent in every direction, then it can be arranged so as to converge toward any point in the complex plane.

Chapter III. Vector Series in Three Dimensions.

Let us now consider a series whose individual terms have components in the directions of the three axes of a rectangular coordinate system:

$$(1) \quad \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_n + \dots$$

where $\alpha_k = x_k e_1 + y_k e_2 + z_k e_3$. The coefficients of the e_i terms of (1) give the series

$$(2) \quad x_0 + x_1 + x_2 + \dots + x_n + \dots$$

Likewise from the coefficients of the e_2 and e_3 terms we have the following series :

$$(3) \quad y_0 + y_1 + y_2 + \dots + y_n + \dots ;$$

$$(4) \quad z_0 + z_1 + z_2 + \dots + z_n + \dots$$

Suppose series (2) converges to the sum X , (3) to the sum Y , and (4) to the sum Z ; then we define (1) to be convergent and to have the sum $X e_1 + Y e_2 + Z e_3$.

Taking

$$s_0 = \alpha_0$$

$$s_1 = \alpha_0 + \alpha_1$$

$$\dots$$

$$s_n = \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_n$$

$$\dots$$

we may regard the partial sums s as points in a three dimensional space, and if the limit of s exists as n becomes infinite, then we say (1) is convergent.

If the series of absolute values of (2), (3) and (4) respectively converge,-

$$(5) \quad |x_0| + |x_1| + |x_2| + \dots + |x_n| + \dots = K_1 ;$$

$$(6) \quad |y_0| + |y_1| + |y_2| + \dots + |y_n| + \dots = K_2 ;$$

$$(7) \quad |z_0| + |z_1| + |z_2| + \dots + |z_n| + \dots = K_3 ;$$

then we define series (1) as being absolutely convergent toward the sum $K_1\epsilon_1 + K_2\epsilon_2 + K_3\epsilon_3$, -

$$(8) \quad |\alpha_0| + |\alpha_1| + |\alpha_2| + \dots + |\alpha_n| + \dots = K_1\epsilon_1 + K_2\epsilon_2 + K_3\epsilon_3 .$$

Since $|\alpha_k| = \sqrt{x_k^2 + y_k^2 + z_k^2}$, series (8) may be written in the form:

$$(9) \quad \sqrt{x_0^2 + y_0^2 + z_0^2} + \sqrt{x_1^2 + y_1^2 + z_1^2} + \dots + \sqrt{x_n^2 + y_n^2 + z_n^2} + \dots .$$

Since $|x_k| \leq \sqrt{x_k^2 + y_k^2 + z_k^2}$, $|y_k| \leq \sqrt{x_k^2 + y_k^2 + z_k^2}$, $|z_k| \leq \sqrt{x_k^2 + y_k^2 + z_k^2}$, it is evident that if series (9) converges, then series (5), (6) and (7) converge. Hence either the convergence of (9) or the convergence of (5), (6) and (7) is a necessary and sufficient condition for the absolute convergence of (1).

Since $|\alpha_n + \alpha_{n+1} + \dots + \alpha_{n+p}| \leq |\alpha_n| + \dots + |\alpha_{n+p}|$, we may apply Cauchy's necessary and sufficient condition for the convergence of a series to (1); that is, if for a preassigned positive quantity ϵ there exists a number N such that $|\alpha_n + \alpha_{n+1} + \dots + \alpha_{n+p}| \leq \epsilon$ for $n > N$ and for any positive value of p , then series (1) converges. If we regard the terms as vectors, this test states that the vector polygon converges to a point if for some small preassigned quantity ϵ , we can take N so large that if a sphere of radius ϵ be placed about the terminal point of any vector past the N th, then the remaining vectors will remain entirely inside this sphere. We could use a cube of side 2ϵ in place of the sphere.

We shall define series (1) as being absolutely convergent in the direction of a certain straight line if the components of (1) in the direction of this straight line give an

absolutely convergent series. The following series converges absolutely in the direction of the x axis, but conditionally in the directions of the y axis and the z axis:

$$(\epsilon_1 + \epsilon_2 + \epsilon_3) - \left(\frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + \frac{1}{2}\epsilon_3\right) - \left(\frac{1}{4}\epsilon_1 - \frac{1}{2}\epsilon_2 - \frac{1}{3}\epsilon_3\right) - \dots \\ \dots - \left(\frac{1}{2^{2n}}\epsilon_1 + \frac{1}{2^{n+1}}\epsilon_2 - \frac{1}{2^{n+1}}\epsilon_3\right) - \left(\frac{1}{2^{2n+1}}\epsilon_1 + \frac{1}{2^{n+2}}\epsilon_2 + \frac{1}{2^{n+2}}\epsilon_3\right) + \dots$$

The following series converges absolutely in the direction of the straight line in the xz plane making an angle of 45° with the x axis, and yet it converges conditionally in the directions of the three axes:

$$(\rho^{\epsilon_3 \frac{\pi}{4}} \epsilon_1 + \epsilon_2 + \epsilon_3) - \left(\frac{1}{2}\rho^{\epsilon_3 \frac{\pi}{4}} \epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{2}\epsilon_3\right) - \dots \\ \dots - \left(\frac{1}{2^{2n}}\rho^{\epsilon_3 \frac{\pi}{4}} \epsilon_1 - \frac{1}{4^{n+1}}\epsilon_2 - \frac{1}{2^{n+1}}\epsilon_3\right) - \left(\frac{1}{2^{2n+1}}\rho^{\epsilon_3 \frac{\pi}{4}} + \frac{1}{4^{n+3}}\epsilon_2 + \frac{1}{2^{n+2}}\epsilon_3\right) + \dots,$$

where $\rho^{\epsilon_3 \frac{\pi}{4}}$ means a rotation of 45° in the direction of the z axis. The following series converges absolutely in the directions of both the x axis and the y axis, but converges conditionally in the direction of the z axis:

$$(\epsilon_1 + \epsilon_2 + \epsilon_3) - \left(\frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + \frac{1}{2}\epsilon_3\right) - \left(\frac{1}{4}\epsilon_1 + \frac{1}{4}\epsilon_2 - \frac{1}{3}\epsilon_3\right) - \dots \\ \dots - \left(\frac{1}{2^{2n}}\epsilon_1 + \frac{1}{2^{2n}}\epsilon_2 - \frac{1}{2^{n+1}}\epsilon_3\right) - \left(\frac{1}{2^{2n+1}}\epsilon_1 + \frac{1}{2^{2n+1}}\epsilon_2 + \frac{1}{2^{n+2}}\epsilon_3\right) + \dots$$

By the following Lemma the above series converges absolutely in the direction of any straight line in the xy plane or of any straight line parallel to the xy plane. Such a series is said to be absolutely convergent over the xy plane.

Lemma I. If a series of three dimensional vectors converges absolutely in the direction of two intersecting straight lines X and Y , then it converges absolutely in the direction of any line in the plane of X and Y .

Designate by Φ the plane of X and Y , by R the straight line in Φ perpendicular to X , and by Z the straight line perpendicular to X and Y at their point of intersection. Take any line in Φ making an angle θ

with X . The projection of any term d_k of the original series on K is equal to the sum of the projections on K of the three terms given by projecting d_k on the three lines X , R and Z . But the projection of d_k on Z is zero. Thus the problem becomes a two dimensional one and we can apply Lemma I of Chapter II.

In any case where we speak of a series being absolutely convergent over a plane, if the plane about which we are speaking passes through the origin, we shall call it "the" plane of absolute convergence; any plane parallel to the plane of absolute convergence we shall call a plane of absolute convergence.

The following series converges absolutely over the plane determined by the y axis and the line in the zx plane that makes an angle of 45° with the x axis, but converges conditionally in the direction of the z axis:

$$\begin{aligned} & (\lambda \epsilon_1 + \epsilon_2 + \epsilon_3) - \left(\frac{1}{2} \lambda \epsilon_1 + \frac{1}{2} \epsilon_2 + \frac{1}{2} \epsilon_3\right) - \left(\frac{1}{4} \lambda \epsilon_1 + \frac{1}{4} \epsilon_2 - \frac{1}{3} \epsilon_3\right) - \dots \\ & \dots - \left(\frac{1}{2^{2n}} \lambda \epsilon_1 + \frac{1}{2^{2n}} \epsilon_2 - \frac{1}{2^{2n+1}} \epsilon_3\right) - \left(\frac{1}{2^{2n+1}} \lambda \epsilon_1 + \frac{1}{2^{2n+1}} \epsilon_2 + \frac{1}{2^{2n+2}} \epsilon_3\right) - \dots, \end{aligned}$$

where $\lambda = 2^{\epsilon_3 \frac{1}{2}}$. I say that this series converges absolutely only over the plane Φ , or a plane parallel to Φ . Suppose it converges absolutely over any line making an angle with the plane Φ . Then it would follow that the series converges absolutely over any plane and hence in the direction of the z axis. But this is contrary to the hypothesis.

A series could be set up that would converge conditionally in the directions of the three axes and yet be absolutely convergent over a plane. Such a series is one that would converge absolutely only over a fixed plane making an angle with each of the three coordinate axes.

If a series converges absolutely over two coordinate planes, - say the xz and yz planes, - then it converges absolutely over the yz plane; for according to the hypothesis the series converges absolutely in the direction of the z axis and of the y axis. This forces absolute convergence over the yz plane.

The main theorem of this paper, which was proved for vectors in two dimensions, can be proved in a similar manner for vectors in three dimensions. We should first prove the following rather particular case in three dimensions:

Suppose

$$(1) \quad \underline{q_0 + q_1 + q_2 + \dots + q_n + \dots = \epsilon_1}$$

where $q_k = x_k \epsilon_1 + y_k \epsilon_2 + z_k \epsilon_3$, and that series (1) can be rearranged so as to give the following series:

$$(2) \quad \underline{a_0 + a_1 + a_2 + \dots + q + \dots = 2 \epsilon_1},$$

where every q is an a and every a is an q . Then it follows that (1) can be rearranged so as to converge toward $\lambda \epsilon_1$, where λ is any arbitrary real number, positive or negative.

Once this theorem is proved we can generalize it to cover all cases in three dimensions by proper translation and rotation of the axes. In brief the theorem generalized would state that if series (1) can be rearranged so as to converge to two points, then it can be rearranged so as to converge toward any point on the straight line determined by the two given points. If for three different arrangements of the terms, the three corresponding sum points do not lie on the same straight line, then the sum points of

series (1) for all possible arrangements of the terms
comprise at least all points in the plane of the three given
points. Finally, if (1) has at least four sum points not
in the same plane, then any point in the three dimensional
space is a sum point.

corresponding to any point on the straight line joining P
and P .

If in addition to the above arrangements of series (1),
 we also have:

$$c_0 + c_1 + \dots + c_n + \dots = P_3$$

$$d_0 + d_1 + \dots + d_n + \dots = P_4$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$\beta_0 + \beta_1 + \dots + \beta_n + \dots = P_k$$

where every c is an α and every α is a c ; every d is
 an α and every α is a d ; every β is an α and
 every α is an β ; then if $P_1, P_2, \dots, P_k$ do not
 lie in a configuration of less than K dimensions, then
series (1) can be made to converge toward any point in the
configuration of K dimensions containing $P_1, P_2, \dots, P_k$.

Chapter V. Curvilinear Vector Systems.

Consider the representation of the components of a vector OP on the three axes of a spherical coordinate system, where ρ represents the distance from P to the origin, θ represents the longitude and ϕ the latitude. Suppose we have a series of such vectors,-

$$(1) \quad (a_0\rho + b_0\theta + c_0\phi) + \dots + (a_n\rho + b_n\theta + c_n\phi) + \dots$$

If the three series made up respectively of the a 's, b 's and c 's all converge,-

$$(2) \quad a_0 + a_1 + a_2 + \dots + a_n + \dots = A,$$

$$(3) \quad b_0 + b_1 + b_2 + \dots + b_n + \dots = B,$$

$$(4) \quad c_0 + c_1 + c_2 + \dots + c_n + \dots = C,$$

then series (1) is said to converge to the sum $A\rho + B\theta + C\phi$.

If series (2), (3) and (4) converge absolutely, then series (1) converges absolutely. Suppose that (3) and (4) converge absolutely and that (2) converges conditionally.

It is evident that the sum points of (1) in this case lie on the straight line $\theta = c$, $\phi = k$, where c and k are constants. If (2) and (3) converge absolutely and (4) conditionally, the sum points of (1) lie on a meridian circle, given by the equations $\rho = c_1$, $\theta = c_2$. If (2) and (4) converge absolutely and (3) conditionally, the sum points of (1) lie on a circle of latitude given by the equations $\rho = c_1$, $\phi = c_2$.

If (2) converges absolutely and (3) and (4) converge conditionally, the sum points of (1) lie either on the whole surface of a sphere of constant radius or else on a spiral that winds an infinite number of times about the two poles of

of this sphere. This is the loxodromic spiral given by the equation: $\theta \tan \lambda = \log \tan \left(\frac{\phi}{2} + \frac{\pi}{4} \right) + C$. This curve cuts the meridian circles at the constant angle λ .

If (3) converges absolutely and (2) and (4) conditionally, the sum points of (1) lie either in a plane or on a spiral in this plane given by the equation $\rho = \lambda \phi$. This curve cuts the circles of latitude at the constant angle λ .

If (4) converges absolutely and (2) and (3) conditionally, the sum points of (1) lie either on the whole surface of a cone with vertex at the origin or else they lie on a spiral on this cone given by the equation $\rho = \lambda \theta$. This curve cuts the meridians at the constant angle λ .

If (2), (3) and (4) all converge conditionally, then the sum points of (1) either fill the whole of the three space, or else they lie on a surface given by one linear equation or on a curve given by two linear equations.

Suppose now that we have a general coordinate system with three axes ρ , θ , and ϕ , and that series (1) is composed of terms whose components are represented graphically as projections along the three axes. We can make these generalizations:

I. If series (2), (3) and (4) all converge absolutely, series (1) has only one sum point for all possible arrangements of its terms.

II. If (2) and (3) converge absolutely and (4) conditionally, the sum points of (1) lie on the configuration $\rho = c_1$, $\theta = c_2$, where c_1 and c_2 are constants.

III. If (3) and (4) converge absolutely and (2) converges conditionally, the sum points of (1) lie on the

configuration $\Theta = c_1, \varnothing = c_2$.

IV. If (2) and (4) converge absolutely and (3) converges conditionally, the sum points of (1) lie on the configuration $\rho = c_1, \varnothing = c_2$.

V. If (2) converges absolutely and (3) and (4) conditionally, the sum points of (1) lie either on the surface $\rho = c_1$ or on a curve on this surface given by two linear equations.

VI. If (3) converges absolutely and (2) and (4) conditionally, the sum points of (1) lie either on the surface $\Theta = c_1$ or on a curve on this surface given by two linear equations.

VII. If (4) converges absolutely and (2) and (3) conditionally, the sum points of (1) lie either on the surface $\varnothing = c_1$, or else on a curve on this surface given by two linear equations.

VIII. If (2), (3) and (4) all converge conditionally, the sum points of (1) either fill the whole of the three space or they lie on a surface given by one linear equation or on a curve given by two linear equations.

Chapter VI. Steinitz's Proof.

In his solution of the problem of possible classes of conditionally convergent vector series, Steinitz takes as hypothesis the following:

Consider the series

$$(1) \quad \gamma_1 + \gamma_2 + \gamma_3 + \dots + \gamma_n + \dots,$$

where $\gamma_k = x_k + i y_k$, $|\gamma_k| \neq 0$, and the following condition is satisfied. Call the terms belonging to (1) Γ . Set up partial sums of the members of Γ in all possible ways. Call this set of partial sums Γ_* . Suppose the projections of Γ_* on every straight line through the origin extend to $+\infty$ and $-\infty$.

From this he concludes that if σ is any point in the complex plane, then series (1) can be arranged in such a manner that its sum will be σ .

In outline his proof is as follows: Suppose we have a finite or an infinite set of points in a plane, - $A_1, A_2, A_3, \dots, A_n, \dots$. Consider the set of points (P) satisfying these conditions: (1) Every A is a P; (2) if P_1 and P_2 are two points of (P), then every point of $P_1 P_2$ belongs to (P). We call such a set of points convex. The smallest convex set containing (A) we shall denote by $\overline{(P)}$.

The main theorems that Steinitz uses in his proof are these:

Theorem I. If A is an infinite system of complex

numbers, whose absolute value is never greater than the positive number m , then for every number γ in $\overline{A}_\kappa$ there is at least one number in A_κ such that $|\gamma - \sigma| \leq 2m$. In our problem $\overline{A}_\kappa$ embraces all points in the complex plane. Hence if σ is any point, then in Γ_κ there is at least one partial sum π such that $|\pi - \sigma| \leq 2m$.

Theorem II. If a finite number of complex numbers, whose absolute values are less than or equal to m , has as sum a number σ whose absolute value is less than or equal to qm , where q is any positive number, then the given numbers can be assigned such an arrangement $d_1, d_2, \dots, d_p$, that the absolute values of the sums

$$\sigma_k = d_1 + d_2 + d_3 + \dots + d_k \quad (k = 1, \dots, p).$$

are always less than or equal to $(4 + q)m$.

It is clear that if we leave out of Γ a finite number of elements, the partial sums of the remaining system B are such that their projections on any straight line through the origin extend to $+\infty$ and $-\infty$. Hence B embraces all the points in the plane and if σ is any point, q the upper limit of the absolute values of the numbers of B , then there is a partial sum π of the system B which satisfies the condition $|\sigma - \pi| \leq 2q$. We can further stipulate that π contains a given element β out of B . For we can pick out of the remaining elements of B a partial sum π' such that $|\sigma - \beta - \pi'| \leq 2q$. Then the partial sum $\pi = \beta + \pi'$ satisfies the stipulated condition.

The following is a brief of the main steps which Steinitz takes in proving the theorem proper:

Let us designate by g_k ($k = 0, 1, 2, \dots$ ad inf.)

the upper limit of the set of numbers $|Y_k|, |Y_{k+1}|, \dots, |Y_{k+p}|, \dots$.

Then $g_k > 0$, $g_k > g_{k+1}$; $\lim g_k = 0$, as $k \rightarrow \infty$. Among the partial sums of T_* there is one which satisfies the condition $|\pi - \sigma| \leq 2g_k$. Let us designate by π_0 the first partial sum that has this property and by T_1 the system which remains when the elements of π_0 are left out of T .

The upper limit of the absolute values of the elements of T_1 is then less than or equal to g_1 . Hence we can pick π so that $|\pi_0 + \pi - \sigma| \leq 2g_1$. Proceeding in this manner we would get

$$|\pi_0 + \pi_1 + \pi - \sigma| \leq 2g_2$$

$$\dots \dots \dots$$

$$|\pi_0 + \pi_1 + \dots + \pi_k - \sigma| \leq 2g_k \quad [k = 0, 1, \dots]$$

Since

$$\pi_k = (\pi_0 + \pi_1 + \dots + \pi_k - \sigma) - (\pi_0 + \dots + \pi_{k-1} - \sigma),$$

it follows that

$$\pi_k \leq 2(g_k + g_{k-1}) \leq 4g_{k-1}.$$

From Theorem II it follows that the terms of π_k can be given an arrangement $Y_{k1}, Y_{k2}, \dots, Y_{kp_k}$, such that

$$|Y_{k1} + Y_{k2} + \dots + Y_{kl}| \leq 8g_{k-1} \quad (l = 1, \dots, p_k).$$

From the above method of choosing the partial sums π and arranging their terms, it is clear that we shall set up a series

$$Y_{01} + Y_{02} + \dots + Y_{0p_0} + Y_{11} + \dots + Y_{1p_1} + \dots + Y_{k1} + \dots + Y_{kp_k} + \dots$$

such that if S_{kl} denotes the sum of the terms up to and including Y_{kl} , then for $k > 1$, $|S_{kl} - \sigma| \leq 10g_{k-1}$, and since the limit of g_{k-1} as k becomes infinite is zero, it follows that the series converges toward the sum σ .

The same type of proof for vectors in n dimensions shows that if the partial sums have components from $-\infty$ to $+\infty$ on every line of an r dimensional space lying in the n dimensional space, then the terms can be arranged so that the sum σ is at any point of the r dimensional space; but no arrangement can make the sum lie outside of this r dimensional space unless the $(r+1)$ dimensional space determined by the previous r dimensional space and the new point enjoys the same properties we assumed for the given r dimensional space.

Thus Steinitz accounts for all possible cases of conditionally convergent vector series in a space of a finite number of dimensions.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 12

Figure 13

Figure 14

[illegible]

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