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ERRORS IN GRAPHICAL
METHODS

by

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CHAPTER I ERRORS IN GRAPHICAL COMPUTATIONS

Part 1. The Error in Locating a Point.

The first error in graphical computation to be considered is the accuracy with which a point can be located with a sharp pencil on millimeter paper. Of course, this error will be different for each individual so each individual will have to determine his own accuracy in locating a point.

In this article my intention is only to suggest a method for determining this error.

Figure 1, column 1, I have estimated points .5 millimeters from the left hand line, while in columns 2 and 3 the points are estimated at .3 and .1 millimeters respectively from the left hand line. Figure 2 shows the points in Figure 1 magnified about nineteen times by means of a reflectoscope. Figure 1 was reflected on a screen and the points with the lines on both sides were marked off on a sheet of paper as shown in Figure 2. By measurements from Figure 2 I was able to determine the errors as tabulated on page 5.

Fig. 1

<i>Trials</i>	<i>.5 mm</i>	<i>.3 mm</i>	<i>.1 mm</i>
<i>1</i>	☐	☐	☐
<i>2</i>	☐	☐	☐
<i>3</i>	☐	☐	☐
<i>4</i>	☐	☐	☐
<i>5</i>	☐	☐	☐
<i>6</i>	☐	☐	☐
<i>7</i>	☐	☐	
<i>8</i>	☐	☐	
<i>9</i>	☐	<i>First column shows points located 5mm from line on left</i>	
		<i>Second " " " " 3mm " " " "</i>	
<i>10</i>	☐	<i>Third " " " " 1mm " " " "</i>	

Fig. 2

First column shows points located .5 m.m. from line on left

Second " " " " .3 m.m. " " " "

Third " " " " .1 m.m. " " " "

<u>Trials</u>	<u>.5 m.m.</u> <u>Errors</u>	<u>.3 m.m.</u> <u>Errors</u>	<u>.1 m.m.</u> <u>Errors</u>
1	4/36 " .11 " 1/36	.03 " 1/36	0 " 0 " 1/36
2	1/36 " .03 " 1/36	.03 " 1/36	3/36 " .08 "
3	1/36 " .03 " 1/36	.03 " 1/36	2/36 " .05 "
4	3/36 " .08 " 0	0 " 0	3/36 " .08 "
5	0/36 " .0 " 4/36	.11 " 4/36	1/36 " .03 "
6	2/36 " .05 " 1/36	.03 " 1/36	3/36 " .08 "
7	3/36 " .08 " 0	0 " 0	
8	1/36 " .03 " 1/36	.03 " 1/36	
9	2/36 " .05 "		
10	1/36 " .03 "		

Part 2. The Error in Drawing a Line Through Two Points.

Let P_1 and P_2 be two points at a distance d apart through which a line is to be drawn (fig. a). Let r be the maximum error in locating each point, then the interval of error is a circle of radius r . Let W be the maximum error at a distance l from P_1 . If $l > d$, the line that gives the maximum error must be drawn through the lower edge of one circle and the upper edge of the other. In this case

$$W = 2r/d (l - d/2) = (2l - d)/d r$$

If $l = d$, $W = r$; but, if $l < d$, the formula will not hold, since the maximum error in this case is given by a line drawn through the upper (or lower) edge of each point; hence $W = r$.

Fig. a.

We can see from the formula above that in order to obtain the greatest accuracy d should be chosen as large as possible.

Part 3. Errors in Constructions Due to the Uncertainty of the Intersection of Two Lines.

If two lines l_1, l_2 of width w intersect at an angle ϕ , the line l_3 drawn vertically through their intersection may be drawn anywhere within the interval t (Fig b).

Fig. b.

$$t = t_1 + t_2; \quad t_1 = w \cot \phi \quad \text{and} \quad t_2 = w \csc \phi \quad \text{hence} \quad t = w(\cot \phi + \csc \phi).$$

From this formula it is evident that the angle ϕ must be

given a value as near $\frac{\pi}{2}$ as the problem will permit for t to be made as small as possible.

Part 4. The Error in Multiplying Two Numbers Graphically by Method I*.

The method I of multiplying two numbers C_1 and C_2 , graphically is shown in (Fig. c)

If C_1 is to be multiplied by some number C_2

the length $OB = C_2$ is layed off on the y axis, $OA = 1$ is layed off on the x axis. Through A, draw a 45° line intersecting the y axis at D. Draw a line through A and B. If $QP_1 = C_1$, through P_1 draw a parallel line to the x axis intersecting AD at F; now through F draw a line parallel to the y axis intersecting the line through A and B at E. Through E draw a line parallel to the x axis intersecting QP_1 at P_2 . $QP_2 = C_1 C_2$.

By a theorem in plane geomertry $ER:FR = BD:DO$, or $ER = (FR \times BO) / DO = C_1 C_2$. Since $QP_2 = ER$, then $QP_2 = C_1 C_2$.

* This method is intended to be used in such problems as that of obtaining the graph of $C_2 f(x)$ from that of $f(x)$. C_1 then represents the value of $f(x)$ for some given abscissa.

The errors that we are to consider here are the ones due to the location of a point and to the uncertainty in the location of the point of intersection of two lines.

In Fig. 3, let ω be the interval of error in drawing the lines P_1F and AS . This interval of error is due to the error r in locating the points P , A and S (Part 1). Let t be the interval in which the vertical line EF must be drawn, then S will be the interval in which line EP_2 will be drawn.

$$\delta = \delta_1 + \delta_2$$

$$\delta_1 = t \tan \beta, \quad \text{for } C_1 > 1$$

and $\delta_2 = 2W \sec \beta$; but since $W =$

$$[(2-d)/d]r \text{ (Part 2)}, \quad \delta_2 = 2r [(2d-d)/d] \sec \beta$$

$$l = C_1 C_2 \csc \beta; \text{ but } \csc \beta = d/C_2 = \sqrt{C_2^2 + 1}/C_2$$

hence $l = C_1 \sqrt{C_2^2 + 1}$

$$\begin{aligned} \sec \beta &= \sec (\beta - \beta') = 1/(\cos \beta \cos \beta' + \sin \beta \sin \beta') \\ &= d^2 / (d + 2rC_2) \\ &= (C_2^2 + 1) / (\sqrt{C_2^2 + 1} + 2rC_2) \end{aligned}$$

Now δ_2 can be expressed in terms of C_1 and C_2 by substituting in the above equation which gives:

$$\begin{aligned} (1) \quad \delta_2 &= 2r (2C_1 - 1) (C_2^2 + 1) / (\sqrt{C_2^2 + 1} + 2rC_2) \\ &= 2r (2C_1 - 1) (C_2^2 + 1) / (\sqrt{C_2^2 + 1} + 2rC_2). \end{aligned}$$

$$\delta_1 = t \tan \beta,$$

$t = \omega (\cot \beta + \csc \beta)$ (part 3) and since $\beta = \frac{\pi}{2}$ in this problem, $t = \omega (1 + \sqrt{2}) = 2r(1 + \sqrt{2})$

$$\tan \beta = \tan (\beta - \beta') = (\tan \beta - \tan \beta') / (1 + \tan \beta \tan \beta')$$

$$= (C_2 - 2r/d) / (1 + 2rC_2/d) \quad (C_2 d - 2r) / (d + 2rC_2)$$

$$= (C_2 \sqrt{C_2^2 + 1^2} - 2r) / (\sqrt{C_2^2 + 1^2} + 2rC_2)$$

$$(2) \text{ hence } d' = 2r (1 + \sqrt{2}) (C_2 \sqrt{C_2^2 + 1^2} - 2r) / (\sqrt{C_2^2 + 1^2} + 2rC_2)$$

$$(3) \text{ therefore } d' = 2r \left[(1 + \sqrt{2}) (C_2 \sqrt{C_2^2 + 1^2} - 2r) / (\sqrt{C_2^2 + 1^2} + 2rC_2) + (2C_1 - 1)(C_2^2 + 1^2) / (\sqrt{C_2^2 + 1^2} + 2rC_2) \right]$$

Approximately this error may be written, since in general r will be small compared with C_2 , and unity.

$$d' \approx 2r \left[(1 + \sqrt{2})C_2 + (2C_1 - 1) \sqrt{C_2^2 + 1^2} \right]$$

The relative error will be

$$d' / C_2 C_1 \approx 2r \left[(1 + \sqrt{2}) / C_1 + (2 - 1/C_1) (\sqrt{1 + 1/C_2^2}) \right]$$

or approximately, if $C_1 > 1$ and $C_2 > 1$, $6r$. If $r = .1$ m.m. and as unity we take, for instance 2 cm this gives an error of 3 %. This is probably considerably in excess of the error actually made.

The above formula does not hold for $C_1 < 1$ since in this case $l < d$, hence $2W = 2r = \omega$ (Part 2). Also we can readily see that β_1 becomes β , Therefore.

$$d' = t \tan \beta = 2r (1 + \sqrt{2}) C_2$$

and

$$d_2 = \omega \csc \beta = 2r \sqrt{C_2^2 + 1^2}$$

(4) thus

$$d' = 2r \left[(1 + \sqrt{2})C_2 + \sqrt{C_2^2 + 1^2} \right]$$

Part 5. Error in Multiplying Two Numbers Graphically by Method II.

The method II of multiplying two numbers C_1 and C_2 , graphically is shown in Fig. d.

On the Y axis, lay off $OE=1$, $OP_1 \approx C_1$ and $OD \approx C_2$.

Through E draw a line cutting the X axis in some point B.

Join B and P_1 . Draw through D a parallel to BE cutting the X axis at A.

Fig. d.

Now through A draw a line parallel to BP_1 and intersecting the y axis at P_2 .

$$OP_2 \approx C_1 C_2$$

By a theorem in plane geometry

$$OP_2/OB = OA/OB$$

and $OA/OB = C_2$,

therefore $OP_2 \approx C_1 C_2$.

In this method, the errors due to the uncertainty of the position of a point and of the intersection of two lines are considered as in Method I.

In Fig. 4 $\delta \approx 2W \sec \beta$

but since $W = [(2l - d)/d] r$ (part 2),

(5) $\delta \approx 2r [(2l - d)/d] \sec \beta$

$$l \approx C_1 C_2 \csc \beta \approx C_2 d \text{ and } d = \sqrt{C_1^2 + a^2}$$

therefore $l \approx C_2 \sqrt{C_1^2 + a^2}$

(6) $\sec \beta_1 \approx \sec (\beta - \beta') = 1/(\cos \beta \cos \beta' + \sin \beta \sin \beta')$

Substituting in (6), the values of these trigonometric functions from the Fig. d, the formula will read:

$$\begin{aligned}\sec \beta_1 &= 1/(a/d + 2C_1 r/d^2) = d^2/(ad + 2C_1 r) \\ &= (C_1^2 + a^2)/(a\sqrt{C_1^2 + a^2} + 2C_1 r)\end{aligned}$$

Equation (5) can now be written

$$\begin{aligned}(7) \quad \delta &= 2r \left\{ \left[2C_2 \sqrt{C_1^2 + a^2} - \sqrt{C_1^2 + a^2} \right] / \sqrt{C_1^2 + a^2} \right\} \\ &\quad \left\{ (C_1^2 + a^2)/(a\sqrt{C_1^2 + a^2} + 2C_1 r) \right\} \\ &= 2r (2C_2 - 1)(C_1^2 + a^2)/(a\sqrt{C_1^2 + a^2} + 2C_1 r)\end{aligned}$$

or approximately this error may be written, since in general r will be small compared with C_1 and unity.

$$\delta \approx 2r (2C_2 - 1)\sqrt{C_1^2 + a^2}/a$$

or if $a=1$, the relative error becomes

$$\delta/C_1 C_2 = 2r (2 - 1/C_2) \sqrt{1 - 1/C_1^2}, \text{ which for large}$$

values of C_1 and C_2 is approximately $4r$ which is a better result than that given by method I.

CHAPTER II. ERRORS IN THE GRAPHICAL METHODS
IN THE INFINITESIMAL CALCULUS

Part I. Description of a Graphical Method for
Integration.

At this point in the discussion, it will be necessary that a brief description of a method of graphic integration be given before considering the errors.

Suppose a function, $y=f(x)$, given by a curve whose ordinates is y and whose abscissa is x . The problem is to find a curve whose ordinate $\bar{y}$ is an integral of the function $f(x)$,

$$\bar{y} = \int_a^x f(x) \cdot dx.$$

$\bar{y}$ measures the area between the ordinates at a and x , the curve, $y=f(x)$, and the x axis.

If $f(x)$ is a constant, the equation $y=f(x)=C$ is represented by a straight line parallel to the x axis and

$$\bar{y} = \int_a^x C dx = C(x-a)$$

$\bar{y}$ is the ordinate of a straight line intersecting the axis of x at the point $x=a$. The constant C is the slope of the line or in other words is the change of $\bar{y}$ for an increase in x equal to 1. Let P be the point on the axis of x for the value of $x=a$ and Q be the point where the line $x=C$ cuts the y axis. The desired line is parallel to PQ and cuts the x axis at the point $x=a$ (in the Fig. $a=0$).

But if $f(x)$ is a function represented by any curve, in order to get the curve

$$\bar{y} = \int_a^x f(x) dx$$

we substitute in the place of the function $y = f(x)$ a function that has different constant values in successive intervals and changing the constant values abruptly at the end of each interval. This step function is determined as follows: through points m_1, n_2, n_3 , etc., on the curve which represents $f(x)$ are drawn lines parallel to the axis of x , these vertical lines parallel to the y axis are drawn intersecting the horizontal lines at A_2, B_1 at B_2, C_1 etc.

The vertical lines $A_2 B_1$ is drawn so that the areas between the horizontal or stepping line and the given curve are equal on both sides of the vertical line $A_2 B_1$ and between

the points n_1 and n_2 . In a similar manner the vertical $B_2 C_1$ is drawn between the points n_2 and n_3 . This step line is integrated in the following manner. The horizontal lines projected on the axis of y are δ, β, γ , etc. From a point P on the axis of x at a distance of -1 from the origin, lines are drawn to points δ, β, γ , etc. From A_2 draw a line parallel to the line $P\delta$, and intersecting the vertical line $B_2 C_1$ in some point K_2 . From K_2 draw a line parallel to $P\beta$ and intersecting the vertical line $C_2 D_1$ in some point K_3 . Proceed in this way until the step function has been integrated. The broken lines thus formed is the integral curve of the step function. It intersects the vertical lines through the points n_1, n_2 , etc., at points numbered 1, 2, 3, etc. These points are called points of accuracy since the integrals of the function $f(x)$ and the step function will have the same value up to those points. It is readily seen that the broken line consists of a series of tangents of the integral curve

$$\bar{y} = \int_a^x f(x) dx$$

at the points of accuracy. The integral curve $\bar{y} = \int_a^x f(x) dx$ can now be drawn in by means of a curved ruler so as to touch at these points of accuracy. Since the accuracy in drawing the integral curve

$$\bar{y} = \int_a^x f(x) dx$$

depends first upon the accuracy with which the vertical lines of the steps are constructed, we will make the construction

of these lines our first problem.

Part 2. Method for Constructing Vertical Lines
between the Horizontal Steps in Graphical Integration.

Suppose that N_1 and N_2 are two points on the curve through which the horizontal lines of two consecutive steps are drawn (see Fig. h) and suppose that the arc between N_1 and N_2 is the arc of a parabola with the axis parallel to the axis of x . The position of the vertical line of the step between N_1 and N_2 can be found as follows. Through m , the center of the chord $N_1 N_2$, draw a line parallel to the axis of x . H is the point of intersection with the arc. The vertical line AB of the step intersects MH at a point K which is twice the distance from M as it is from H .

To show that this is true, it must be shown that the area $N_1 H N_2 D N_1$ is equal to the rectangle $A B N_2 D$. The area $N_1 H N_2 M N_1$ and the triangle $N_1 N_2 D$. The triangle $N_1 N_2 D$ is equal to the rectangle $C F N_2 D$. The area $N_1 H N_2 N_1$ is equal to two thirds of the area of the parallelogram $G E N_2 N_1$ and therefore equal to the rectangle

A B F C. Since the area $N_1 H N_2 D N_1$ is equal to the rectangle A B N_2 D, it follows that the areas on both sides of the vertical A B between the stepping line and the curve are equal. If the curve $N_1 N_2$ is supposed to be an arc of a parabola with its axis parallel to the axis of y, the constructions will be slightly altered. Through the center M of the chord $N_1 N_2$ (Fig. 1) draw the vertical line C F intersecting the curve at H.

On M H find K a point twice the distance from M as its distance from H and through it draw a parallel to $N_1 N_2$. It will intersect a horizontal through M at a point L whose distance from M is twice its distance from R. The vertical A B must pass through L. This is true since the area $N_1 H N_2 N_1$ is equal to $2/3$ the parallelogram G E $N_1 N_2$, which in turn is equal to the rectangle A B F C. Hence the area $N_1 N_2 D N_1$ is equal to the rectangle A B N_2 D, and therefore the areas on both sides of the vertical A B between the stepping lines and the curve are equal.

Now since the arc on any curve can be approximated to by an arc of a parabola with sufficient accuracy if the arc

of the curve be taken sufficiently small, we can use this method in determining the vertical lines. *

With the above methods we are able to construct the broken line which consists of lines tangent to the integral curve

$$\bar{y} = \int_a^x f(x) dx$$

at the points of accuracy. Now we will consider a method for checking the integral curve between any two successive points of accuracy.

Part 3. Method for Locating a Point on the Integral Curve $\bar{y} = \int_a^x f(x) dx$ Between Two Points of Accuracy.

Suppose P_1 and P_3 (Fig.j) be two points of accuracy and A the point of intersection of the tangent lines to the integral curve at P_1 and P_3 , and also suppose that the integral curve between P_1 and P_3 is the arc of a parabola with its axis parallel to the axis of y. A third point, P_2 , can be located on the curve in the following way.

Fig j

*This method may be conveniently used in some instances to locate the vertical lines but, generally, it would be better to locate the vertical lines with the eye and when necessary use this method as a check.

Draw the chord $P_1 P_3$ (Fig. j) and locate its middle point B. The mid point P_2 of the line A B is a point of the curve.

To show that this is true, let a and b be the coordinates of the point P_1 and $(a+h)$ and $(b+k)$ be the coordinates of the point P_3 . The equation of the parabola is $y = \frac{1}{2} x^2$. If the coordinates of the point P_2 satisfy the equation of the parabola, the above construction is true. The coordinates of the point B are $(\frac{a+h}{2}, \frac{b+k}{2})$ but since $k = \frac{1}{2}(2ah + h^2)$ and $b = \frac{1}{2}a^2$, the coordinates of B can be written $[\frac{a+h}{2}, \frac{1}{2}(2a^2 + 2ah + h^2)/2]$. The coordinates of the point A is found by solving the equation of the two tangent lines $P_1 A$ and $P_3 A$ simultaneously for x and y . The equation of $P_1 A$ is

$$y = 2 \frac{1}{2} ax - \frac{1}{2} a^2$$

and the equation of $P_3 A$ is

$$y = 2 \frac{1}{2} (a+h)x - \frac{1}{2} (a+h)^2.$$

Solving these two equations simultaneously for x and y , we find $x = a + h/2$ and $y = \frac{1}{2}(a^2 + ah)$ which are the coordinates of the point A. Since the abscissas of the points A and B are equal to $(a + h/2)$ the line A B is parallel to the axis of y and therefore the abscissa of the point P_2 on this line is equal to $(a + h/2)$. The ordinate of P_2 is equal to one half the sum of the ordinates of A and B, i.e.

$$\begin{aligned} & \frac{1}{2} \left[\gamma(2a^2 + 2ah + h^2)/2 + \gamma(a^2 + ah) \right] \\ &= \gamma(4a^2 + 4ah + h^2)/4 \end{aligned}$$

Now if these coordinates of P_2 be substituted in the equation $y = \gamma x^2$, we will get the equation

$$\gamma(4a^2 + 4ah + h^2)/4 = \gamma(4a^2 + 4ah + h^2)/4.$$

Thus the point P_2 is a point of the curve. ~~Since any part~~ of a curve can be approximated to by the arc of a parabola with sufficient accuracy if the ~~part~~ to be approximated is sufficiently small, the above method will be useful.

We will ~~not~~ consider a method of locating a point on the integral curve between two points of accuracy that is more general than the one considered above.

Part 40. Graphical Method of Finding a Point Between Two Points of Accuracy on a First, Second, or Third Degree Curve Touching the Tangent Lines at the Points of Accuracy.

Suppose P_1 and P_2 (Fig. k) are two points of accuracy whose coordinates are respectively $(-h, \gamma)$ and (h, γ') , and that the tangent lines $P_1 A$ and $P_2 A$ have the respective slopes of ρ and ρ' . Let us suppose that the curve passing through the points P_1 and P_2 and touching the tangent lines at these points is a third degree curve and represented by the equation

$$(1) \quad y = a + bx + cx^2 + dx^3$$

Where a, b, c, d are constant to be determined.

If the curve represented by Equation (1) passes

through P_1 and P_2 and touches the tangent lines at these points, the following conditions must be satisfied:

$$(2) \quad y' = a + bh + ch^2 + dh^3$$

$$(3) \quad y = a - bh + ch^2 - dh^3$$

$$(4) \quad \beta = b + 2ch + 3dh^2$$

$$(5) \quad \beta = b - 2ch + 3dh^2$$

Adding (2) and (3) and dividing by 2 we get

$$(6) \quad (y + y')/2 = a + ch^2$$

Subtracting (4) from (5) dividing by 4 and multiplying by h we get

$$(7) \quad h \frac{\beta - \beta'}{4} = -ch.$$

Now if we add (6) and (7) we get

$$(8) \quad a = (y + y')/2 + h (\beta - \beta')/4;$$

but a is the value of y in (1) when $x=0$, hence the ordinate a of the point P_3 on the curve mid way between P_1 and P_2 is

$$a = \frac{y + y'}{2} + h \frac{\beta - \beta'}{4}.$$

The graphical construction of the point P_3 is shown in (Fig.k). A vertical line is drawn mid way between P_1 and P_2 and is intersected at the point B by the chord $P_1 P_2$. Project P_2 on the vertical line OB at the point E. EF is equal to $h\beta'$. Let G be the projection of B on the vertical line CP_1 . Draw GK parallel to $P_1 A$, EH parallel to FG and HL parallel to GK. Thus $KL = h\beta'$ and $BK = h\beta$, hence $BL = h(\beta - \beta')$. Locate P_3 on the vertical line OB with its distance from L equal to three times its distance from B. $BP_3 = h/4 (\beta - \beta')$, therefore $OP_3 = OB + BP_3 = (y + y')/2 + h (\beta - \beta')/4 = a$. Thus P_3 is a point of the curve. (Fig. L) is the construction with only the necessary construction lines drawn.

The preceding argument and constructions are readily seen to hold for first and second degree curves also.

Since any ~~part~~ of a curve can be approximated to by a first, second or third degree curve with sufficient accuracy if the point to be approximated is sufficiently small, we can use this method for checking a curve or finding another point on it *.

* Methods of parts 3 + 4 may in some instances be useful in locating points on the integral curve but in general the integral curve should be located with the eye and if necessary checked with the above methods.

Part 5 . Extension of Kutta's Method for Solving Ordinary Differential Equations to the Solution of Partial Differential Equations of the First Order, i.e., of the Form $P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} - R = 0$.

The method starts from given initial conditions, i.e. $z = f(x)$ when $y = \bar{y}_0$ and we calculate the change in z corresponding to a certain small change in y . Let Δy be the change in y and Δz be the change in z so that $z + \Delta z$, $y + \Delta y$ are the coordinates of a curve on the surface satisfying the differential equation and passing through the initial curve. z is calculated in the following manner. We calculate in succession four values of $\Delta z_1, \Delta z_2, \Delta z_3, \Delta z_4$ as follows:

$$(1) \quad P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} - R = 0$$

is solved for $\frac{\partial z}{\partial y}$ which is written

$$\frac{\partial z}{\partial y} = \left[\frac{R - P \frac{\partial z}{\partial x}}{Q} \right]$$

$$z = \frac{\partial z}{\partial y} \Delta y = \left[\frac{R - P \frac{\partial z}{\partial x}}{Q} \right] \Delta y$$

$$\Delta z_1 = \left[\frac{\partial z}{\partial y} \right]_1 \Delta y = \left[\frac{R - P \frac{\partial z}{\partial x}}{Q} \right]_1 \Delta y \quad (z, y),$$

then

$$z_1 = z + \Delta z_1;$$

$$\Delta z_2 = \left[\frac{\partial z}{\partial y} \right]_2 \Delta y = \left[\frac{R - P \frac{\partial z}{\partial x}}{Q} \right]_2 \Delta y \quad (z + \Delta z_1/2, y + \Delta y/2),$$

then

$$z_2 = z + \Delta z_2;$$

$$\Delta z_3 = \left[\frac{\partial z}{\partial y} \right]_3 \Delta y = \left[\frac{R - P \frac{\partial z}{\partial x}}{Q} \right]_3 \Delta y \quad (z + \Delta z_2/2, y + \Delta y/2),$$

then

$$z_3 = z + \Delta z_3;$$

$$\Delta z_4 = \left. \frac{\partial z}{\partial y} \right|_4 \Delta y = \left[\frac{R - \frac{P}{Q} \frac{\partial z}{\partial x}}{Q} \right] \Delta y \quad (z + \Delta z_3, y + \Delta y),$$

then

$$z_4 = z + \Delta z_4.$$

We now form the arithmetical means

$$p = \frac{\Delta z_2 + \Delta z_3}{2} \quad \text{and} \quad q = \frac{\Delta z_1 + \Delta z_4}{2},$$

and find with a high degree of approximation as long as Δy is not too large

$$\Delta z = p + \frac{1}{3} (q - p).$$

The new values

$$\bar{y}_1 = \bar{y}_0 + \Delta y, \quad \bar{z}_1 = \bar{z}_0 + \Delta z$$

are then substituted for $\bar{y}_0$ and $\bar{z}_0$ and in the same way the coordinates of a third curve are calculated and so on.

This calculation may be performed graphically in a very convenient way, if we consider the plane of the paper the coordinate plane Ozx and the coordinate plane Oxy rotated about the horizontal axis Ox until its positive direction comes into coincidence with the negative direction of the plane Ozx . Projections on the positive side of the Oxy plane will fall on the paper below the axis of x and projections on the negative side of the Oxy plane will fall on the paper above the axis of x according to the rules of projective geometry.

Any Equation of the type (1) can be solved graphically by means of a straight edge and a pair of dividers

and if coordinate paper is used the dividers are not necessary.

It will be necessary for us to find $\frac{\partial z}{\partial x}$, i.e. the slope, of a curve at a point. To do this graphically draw a line through the origin parallel to the tangent line at the point. The line through the origin will intersect a vertical line through the point at a distance of unity on the axis of x . The length of this vertical leg is the slope, i.e. $\frac{\partial z}{\partial x}$, at the given point and can be read off from the axis of z .

To illustrate the method more clearly, I am solving several problems both numerically and graphically.

Suppose

$$x \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} - 1 = 0,$$

and that the initial conditions are

$$z = \bar{z}_0 = x \text{ when } y = \bar{y}_0 = 0.$$

Let $y = .2$. Substituting in the formulas on page 23,

$$\text{we get } \left. \frac{\partial z}{\partial y} \right|_1 = 1 - x \frac{\partial \bar{z}_0}{\partial x} = 1 - x,$$

$$\text{hence } \Delta z_1 = (1 - x) \cdot 2 = .2 - .2x$$

$$\text{and } z_1 = x + (.2 - .2x) = .2 + .8x.$$

$$* {}_0 z_1 = \bar{z}_0 + \frac{1}{2} \Delta z_1 = x + (.1 - .1x) = .1 + .9x,$$

$$\text{hence } \frac{\partial z_1}{\partial x} = .9.$$

* For convenience, throughout the rest of this paper I will let ${}_0 z_1$ be the line half way between the lines $\bar{z}_0$ and z_1 and ${}_0 z_2$ the line half way between $\bar{z}_0$ and z_2

$$\left. \frac{\partial z}{\partial y} \right|_2 = 1 - x \frac{\partial z_1}{\partial x} = 1 - .9x,$$

hence $\Delta z_2 = .2 - .18x$

and $z_2 = z_1 + \Delta z_2 = x + (.2 - .18x) = .2 + .82x.$

$${}_0z_2 = \bar{z}_0 + \frac{1}{2} \Delta z_2 = x + (.1 - .09x) = .1 + .91x.$$

since $\frac{\partial z_2}{\partial x} = .91,$

$$\left. \frac{\partial z}{\partial y} \right|_3 = 1 - x \frac{\partial z_2}{\partial x} = 1 - .91x,$$

hence $\Delta z_3 = .2 - .182x$

and $z_3 = \bar{z}_0 + \Delta z_3 = x + (.2 - .182x) = .2 + .818x.$

Since $\frac{\partial z_3}{\partial x} = .818,$

and $\left. \frac{\partial z}{\partial y} \right|_4 = 1 - x \frac{\partial z_3}{\partial x} = 1 - .818x,$

then $\Delta z_4 = .2 - .1636x$

and $z_4 = z_0 + \Delta z_4 = x + (.2 - .1636x) = .2 + .8364x$

$$q = \frac{1}{2} (\Delta z_1 + \Delta z_4) = \frac{1}{2} [(.2 - .2x) + (.2 - .1636x)] \\ = .2 - .1818x$$

$$p = \frac{1}{2} (\Delta z_2 + \Delta z_3) = \frac{1}{2} [(.2 - .18x) + (.2 - .182x)] \\ = .2 - .181x$$

$$\Delta z = (.2 - .181x) + \frac{1}{3} (-.0008x) = .2 - .1812\frac{2}{3}x$$

hence the new values are

$$\bar{y}_1 = \bar{y}_0 + \Delta y = .2, \quad \bar{z}_1 = \bar{z}_0 + \Delta z = .2 + .8187\frac{1}{3}x$$

At the point where $x = 1$ the approximate value for z by this method is

$$\bar{z}_1 = 1.0187\frac{1}{3}.$$

The accurate solution for the differential equation is

$$z = y + x e^{-y}.$$

Now at the point where $y = .2$ and $x = 1$

$$z = .2 + e^{-.2},$$

$$\text{hence } z = .2 + (1 - .2 + \frac{.04}{2} - \frac{.008}{3!} + \frac{.0016}{4!} - \frac{.00032}{5!} + \frac{.000064}{6!})$$

$$= 1.087307555$$

We find the numerical error to be

$$E_1 = \bar{z}_1 - z = .0000025777----$$

Now let us repeat the operation for a third line upon the surface with the line just found as initial conditions, i.e.

$$z = \bar{z}_1 + .2 + ax \quad \text{where } a = .8187\frac{1}{3}.$$

Now

$$y = \bar{y}_1 = .2;$$

and again let

$$y = .2.$$

Substituting in the formulas on page 23 as in the previous step, we get the following.

$$\text{Since } \frac{\partial \bar{z}_1}{\partial x} = a,$$

$$\frac{\partial z}{\partial y} = 1 - x \quad \frac{\partial \bar{z}_1}{\partial x} = 1 - ax$$

and

$$\Delta z_1 = .2 - .2ax,$$

$$\text{hence } z_1 = \bar{z}_1 + \Delta z_1 = (.2 + ax) + (.2 - .2ax) = .4 + .8ax.$$

$$\bar{1}z_1 = \bar{z}_1 + \frac{1}{2}\Delta z_1 = (.2 + ax) + (.1 - .1ax) = .3 + .9ax).$$

Since $\frac{\partial \bar{1}z_1}{\partial x} = .9a,$

$$\left[\frac{\partial z}{\partial y}\right]_2 = 1 - x \frac{\partial \bar{1}z_1}{\partial x} = 1 - .9ax$$

and $\Delta z_2 = .2 - .18ax$

hence $z_2 = \bar{z}_1 + \Delta z_2 = (.2 + ax) + (.2 - .18ax) = .4 + .82ax.$

$$\bar{1}z_2 = \bar{z}_1 + \frac{1}{2}\Delta z_2 = (.2 + ax) + (.1 - .09ax) = .3 + .91ax.$$

Since $\frac{\partial \bar{1}z_2}{\partial x} = .91ax,$

$$\left[\frac{\partial z}{\partial y}\right]_3 = 1 - x \frac{\partial \bar{1}z_2}{\partial x} = 1 - .91x$$

and $\Delta z_3 = .2 - .182ax;$

hence $z_3 = \bar{z}_1 + \Delta z_3 = (.2 + ax) + (.2 - .182ax) = .4 + .818ax.$

Since $\frac{\partial z_3}{\partial x} = .818ax,$

$$\left[\frac{\partial z}{\partial y}\right]_4 = 1 - x \frac{\partial z_3}{\partial x} = 1 - .818ax$$

and $\Delta z_4 = .2 - .1636ax$

hence $z_4 = \bar{z}_1 + \Delta z_4 = (.2 + ax) + (.2 - .1636ax) = .4 + .8364ax$

$$\begin{aligned} q &= \frac{1}{2} (\Delta z_1 + \Delta z_4) = \frac{1}{2} [(.2 - .2ax) + (.2 - .1636ax)] \\ &= .2 - .1818ax \end{aligned}$$

$$\begin{aligned} p &= \frac{1}{2} (\Delta z_2 + \Delta z_3) = \frac{1}{2} [(.2 - .18ax) + (.2 - .182ax)] \\ &= .2 - .181ax \end{aligned}$$

$$\text{and } \Delta z = (.2 - .181ax) + \frac{1}{3} (-.0008ax) = .2 - .1812\frac{2}{3}ax$$

hence the new values are

$$\begin{aligned}\bar{y}_2 &= \bar{y}_1 + \Delta y = .4, \quad \bar{z}_2 = \bar{z}_1 + \Delta z = .4 - .1812\frac{2}{3}ax \\ &= .4 + .8187\frac{1}{3}ax.\end{aligned}$$

At the point where $x = 1$ the approximate value for z is

$$\begin{aligned}\bar{z}_2 &= .4 + .8187\frac{1}{3}a \\ &= 1.07032427111\end{aligned}$$

Now at the point where $y = .4$ and $x = 1$

$$z = .4 + e^{-.4}$$

$$\begin{aligned}\text{hence } z &= .4 + (1 - .4 + \frac{.16}{2!} - \frac{.064}{3!} + \frac{.0256}{4!} - \frac{.01024}{5!} \\ &\quad + \frac{.004096}{6!} - \frac{.0016384}{7!}) \\ &= 1.07137953.\end{aligned}$$

We find the numerical error to be

$$E_2 = z - \bar{z}_2 = .00105525\text{-----} < \frac{1}{10} \%$$

The graphical solution is shown in Fig. 5.

We will now solve the same equation,

$$x \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} - 1 = 0,$$

With the new initial conditions:

$$z = \bar{z}_0 = x^2 \quad \text{when } y = \bar{y}_0 = 0.$$

Let $\Delta y = .2$; and again substitute in the formulas on page 20. We thus get the following:

$$\text{Since } \frac{\partial \bar{z}_0}{\partial x} = 2x,$$

$$\left[\frac{\partial z}{\partial y} \right]_1 = 1 - x \frac{\partial \bar{z}_0}{\partial x} = 1 - 2x^2$$

and

$$\Delta z_1 = .2 - .4x^2$$

Fig. 5

hence $z_1 = x^2 + \Delta z_1 = x^2 + (.2 - .4x^2) = .2 + .6x^2.$

${}_0z_1 = \bar{z}_0 + \frac{1}{2} \Delta z_1 = x^2 + (.1 - 2x^2) = .1 + .8x.$

Since $\frac{\partial {}_0z_1}{\partial x} = 1.6x,$

$\left[\frac{\partial z}{\partial y} \right]_2 = 1 - x \frac{\partial {}_0z_1}{\partial x} = 1 - 1.6x^2$

and $\Delta z_2 = .2 - .32x^2$

hence $z_2 = x^2 + \Delta z_2 = x^2 + (.2 - .32x^2) = .2 + .68x^2$

${}_0z_2 = \bar{z}_0 + \frac{1}{2} \Delta z_2 = x^2 + (.1 - .16x^2) = .1 + .84x^2.$

Since $\frac{\partial {}_0z_2}{\partial x} = 1.68x,$

$\left[\frac{\partial z}{\partial y} \right]_3 = 1 - x \frac{\partial {}_0z_2}{\partial x} = 1 - 1.68x^2$

and $\Delta z_3 = .2 - .336x^2$

hence $z_3 = z_0 + \Delta z_3 = x^2 + (.2 - .336x^2) = .2 + .664x^2.$

Since $\frac{\partial z_3}{\partial y} = 1.328x,$

$\left[\frac{\partial z}{\partial y} \right]_4 = 1 - x \frac{\partial z_3}{\partial x} = 1 - 1.328x^2$

and $\Delta z_4 = .2 - .2656x^2$

hence $z_4 = \bar{z}_0 + \Delta z_4 = x^2 + (.2 - .2656x^2) = .2 + .7344x^2$

$q = \frac{1}{2} (\Delta z_1 + \Delta z_4) = \frac{1}{2} [(.2 - .4x^2) + (.2 - .2656x^2)]$
 $= .2 - .3328x^2$

$p = \frac{1}{2} (\Delta z_2 + \Delta z_3) = \frac{1}{2} [(.2 - .32x^2) + (.2 - .336x^2)]$
 $= .2 - .328x^2$

Fig. 6

and $\Delta z = (.2 - .328x^2) + \frac{1}{3} (-.0048x^2) = .2 - .3296x^2$.

Hence the new values are

$$\bar{y}_1 = \bar{y}_0 + \Delta y = .2, \quad \bar{z}_1 = \bar{z}_0 + \Delta z = .2 + .6704x^2$$

These values are also arrived at graphically in (fig.6).

Our third problem is to solve the equation

$$P \frac{\partial z}{\partial y} + Q \frac{\partial z}{\partial x} - 1 = 0$$

Where only the contours of P and Q are given (Fig.7) and the initial conditions are

$$z = z_0 = \sin x \quad \text{when } y = 0.$$

Since this problem cannot be solved numerically, we can arrive at the solution only by a graphical method.

If we let $\Delta y = .2$, then $\bar{y}_1 = \bar{y}_0 + \Delta y = .2$.

For this value of $\bar{y}_1$ we find graphically (Fig. 7) the value of $\bar{z}_1$.

Hand-drawn graph on green grid paper. The vertical axis is labeled with values from 1.1 to 2.9. The horizontal axis is labeled with values from 1 to 6. Several curves are plotted, some labeled with mathematical expressions like $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$. Arrows indicate directions of flow or gradients. The graph is divided into two main sections by a horizontal line at $y=2$. The top section shows curves starting from the left and moving right, while the bottom section shows curves starting from the left and moving right, with some curves labeled with '1' and '2'.

when the initial conditions are

and P and Q are given by means of
contour lines.

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~~It is never to leave the room.~~
~~For the first time he checked out of the room.~~