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THE OSCILLATION OF CERTAIN SETS OF
ORTHOGONAL FUNCTIONS

by

Charles Albert Epperson, A. B., B. S. (in Ed.)

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I. INTRODUCTION.

Definitions. A set of functions $\varphi_1(x), \varphi_2(x), \dots$ is said to be orthogonal on the interval (a, b) if $\int_a^b \varphi_i(x) \varphi_j(x) dx = 0$, ($i \neq j$), and if $\int_a^b \varphi_i^2(x) dx \neq 0$. It is said to be normed if $\int_a^b \varphi_i^2(x) dx = 1$. Evidently the functions of an orthogonal set may be normed by dividing each by a suitable constant. In the following pages we shall speak of approximating to a function $f(x)$ by a linear polynomial $c_1 \varphi_1(x) + c_2 \varphi_2(x) \dots c_n \varphi_n(x)$ in the functions $\varphi_i(x)$. This approximation will invariably be understood to be in the same sense that the integral of the square of the error is a minimum, that is, $\int_a^b [f(x) - c_1 \varphi_1(x) \dots - c_n \varphi_n(x)]^2 dx$ is less than for any other system of coefficients $c_1, c_2, \dots c_n$. It is well known that upon this assumption the c_i are given by the formulae $c_i = \int_a^b f(x) \varphi_i(x) dx / \int_a^b \varphi_i^2(x) dx$

In the classic memoirs of Sturm and Liouville (Liouville, Journal de Mathematiques: Vol. I-II, 1836), two classes of theorems are found concerning sets of orthogonal functions. The first deal with the number of sign-changes in φ_n , and the second with the number of sign-changes in a polynomial $c_1 \varphi_1(x) + \dots c_n \varphi_n(x)$. The functions of the orthogonal sets with which they deal are solutions of a differential equation containing a single parameter.

The question arises as to whether these two classes of theorems are consequences of the mere orthogonality of the function sets, or whether the differential equation is a necessary condition for them. We shall find that orthogonality alone is not a sufficient condition for the oscillation theorems in question; but that with the addition of the hypothesis of the non-vanishing of a certain set of determinants

it becomes so.

The first chapter will be given to this subject. The later ones will be occupied with the verification for certain special sets of orthogonal functions of the oscillation theorems of Sturm.

The first chapter contains some results apparently new.

Chapter I

Theorems on general sets of orthogonal functions
and their oscillation properties.

In sets of orthogonal functions $\varphi_0, \varphi_1, \varphi_2 \dots \varphi_n$, as defined in the introduction, it is frequently found that φ_{i+1} has one more sign-change than φ_i in the interval¹⁾ for which the functions are orthogonal. That this is not a consequence of orthogonality alone is seen by the following example, since in it φ_0 does not vanish, φ_1 vanishes once, and φ_2 also vanishes but once. The example is:

$$\begin{aligned}
 \varphi_0 &= 1 & 0 \leq x \leq 1/3 \\
 \varphi_1 &= 10 + 27(x - 1/3) & 1/3 \leq x \leq 1/2 \\
 &= -5 + 18x & 1/2 \leq x \leq 2/3 \\
 &= 13 - 18x & 2/3 \leq x \leq 1 \\
 &= 1 & \\
 \varphi_2 &= 1 & 0 \leq x \leq 1/3 \\
 &= -5 + 18x & 1/3 \leq x \leq 1/2 \\
 &= 13 - 18x & 1/2 \leq x \leq 2/3 \\
 &= 1 - 27(x - 2/3) & 2/3 \leq x \leq 1
 \end{aligned}$$

and the interval of orthogonality is $(0,1)$. Our investigation shows, however, that a sufficient condition for the sign-changes observed ³ is the non vanishing of the determinants:

$$\begin{vmatrix}
 \varphi_0(x_0) & \varphi_1(x_0) & \varphi_2(x_0) & \dots & \varphi_n(x_0) \\
 \varphi_0(x_1) & \varphi_1(x_1) & & & \varphi_n(x_1) \\
 \vdots & & & & \\
 \varphi_0(x_n) & \varphi_1(x_n) & & & \varphi_n(x_n)
 \end{vmatrix} \quad \begin{matrix} n = (1, 2, 3, 4 \dots) \\ x_i \neq x_j \end{matrix}$$

1) Interval will be taken with out exception to mean open interval.

A set of orthogonal functions satisfying these conditions will be said to have the property "D".

Let us notice also that the non-vanishing of this determinant is the necessary condition for the possibility of making a linear combination of $n + 1$ functions pass through $n + 1$ points, at least one of whose ordinates is not zero. Thus the condition for interpolation is the condition for the observed sign changes.

As a result of this we further find that the difference between a function $f(x)$ and its approximation by a polynomial, $\Phi_n(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) + \dots + c_n \varphi_n(x)$ in functions having the property "D", changes signs at least $n + 1$ times in a given interval, so that the approximating curve $\Phi_n(x)$ crosses the curve approximated to at least $n + 1$ times. The proofs of these statements form the subject matter of this chapter.

Theorem I. In a set of orthogonal functions, $\varphi_0, \varphi_1, \dots, \varphi_n$ having the property "D", φ_n has not more than n sign-changes in the interval of orthogonality.

For, by hypothesis, the determinant

$$\begin{vmatrix} \varphi_0(x_0), \varphi_1(x_0), & \dots & \varphi_n(x_0) \\ \varphi_0(x_1), \varphi_1(x_1) & \dots & \varphi_n(x_1) \\ \vdots & & \\ \varphi_0(x_n), \varphi_1(x_n) & \dots & \varphi_n(x_n) \end{vmatrix} \quad x_i \neq x_j$$

does not vanish. If φ_n has $n+1$ sign-changes, set $x_0, x_1, \dots, x_n$ respectively equal to the $n + 1$ values of x at which φ_n changes sign. Each element of the last column is thus seen to be zero and the determinant vanishes, which contradicts our hypothesis.

Theorem II. If $\Phi_n(x)$ is a linear combination of

continuous orthogonal functions, as $\Phi_n(x) = c_0 \varphi_0 + c_1 \varphi_1 + \dots + c_n \varphi_n$
and if the set $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n$ has the property "D", then Φ_n
has not more than n zeros in the interval of orthogonality.

If $\Phi_n(x)$ has $n+1$ zeros at all the points at which it vanishes, $x_0, x_1, x_2, \dots, x_n$. Substitute these points in the equation for $\Phi_n(x)$. We then have $n+1$ homogeneous equations in the $n+1$ coefficients c_j . Hence the determinant

$$\begin{vmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \dots & \varphi_n(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \dots & \varphi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_n) & \varphi_1(x_n) & \dots & \varphi_n(x_n) \end{vmatrix} \quad x_i \neq x_j$$

must vanish, which contradicts the assumption that the determinant is different from zero.

Theorem III. If Φ_n , defined as in Theorem II, has n roots in the interval of orthogonality, it changes sign at each.

To show that Φ_n actually changes sign and does not vanish without changing signs at any of its n roots, we will suppose that at one point, say r_2 , is a root of even order. For simplicity suppose that $n=3$. Then $\Phi_3(x)$ can have only three roots in our given interval by Theorem II.

Suppose that r_2 is the double root,

and that Φ_3 is positive between r_1 and r_2 and ^{between} r_2 and r_3 . Let us now take a

function $\Phi_2 = c_0 \varphi_0 + c_1 \varphi_1 + c_2 \varphi_2$

and make Φ_2 vanish at r_1 and r_3 and equal to -1 at r_2 . Suppose

now that the lesser of the two

maxima of Φ_3 between r_1 and r_3 is b ,

and that the minimum value of Φ_2

is $-p$. Multiply $\bar{\Phi}_2$ by $b/2p$. Call the resulting function $h\bar{\Phi}_2$. $h\bar{\Phi}_2$ is negative between r_1 and r_3 and less than or equal in absolute value to $b/2$ everywhere in that interval. Now $\bar{\Phi}_3$ is greater than or equal to b in both the intervals (r_1, r_2) and (r_2, r_3) and is zero at r_2 . Adding $\bar{\Phi}_3$ and $h\bar{\Phi}_2$ we get a function $\bar{\Phi}_3$, which vanishes at r_1 , between r_1 and r_2 , between r_2 and r_3 , and at r_3 . But $\bar{\Phi}_3$ is of the form $\bar{\Phi}_3 = k_0\varphi_0 + k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3$ and by Theorem II cannot vanish more than three times in the given interval. Hence r_2 must be a change of sign and not a double root.

Our proof is for the case $n = 3$, and shows the impossibility of one vanishing point without change of sign. It applies however in all essentials to any n and to showing the impossibility of any number of vanishing points without sign-change.

Theorem IV. If any function is orthogonal to the set $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{n-1}$, having the property "D", it changes sign at least n times in the interval of orthogonality.

Suppose φ_k has less than n sign-changes, say $k < n$. Consider $\bar{\Phi}_k(x) = c_0\varphi_0 + c_1\varphi_1 + \dots + c_k\varphi_k$. By Theorem II and the hypothesis that the set $\varphi_0, \varphi_1, \dots, \varphi_k$ has the property "D", we can make $\bar{\Phi}_k$ change signs at exactly the k points at which φ_k changes sign and in addition can make it have one value different from zero in common with φ_k . The product $\bar{\Phi}_k\varphi_k$ is then everywhere positive in the interval of orthogonality and its integral taken over that interval cannot then be zero. This contradicts our orthogonality hypothesis.

Corollary I. φ_n changes sign exactly n times in the interval of orthogonality.

By Theorem I, φ_n does not change signs more than n times.

By Theorem IV φ_n must change signs more than $n - 1$ times.
Hence φ_n must change signs n times.

Theorem V. If $f(x)$ is approximated to by the continuous function $A_n(x) = c_0 \varphi_0 + c_1 \varphi_1 + c_2 \varphi_2 + \dots + c_n \varphi_n$, where the set $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n$ has the property "D", the remainder $E(x) = f(x) - A_n(x)$ is orthogonal to $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n$, hence by Theorem IV crosses $f(x)$ at least $n + 1$ times in the interval of orthogonality.

Call out interval of orthogonality (a, b) . As stated in the introduction the approximation considered is that one in which the coefficients are the Fourier coefficients.

$$A_n(x) = \alpha_0 \varphi_0(x) + \alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x)$$

where
$$\alpha_m = \frac{\int_a^b f(x) \varphi_m(x) dx}{\int_a^b \varphi_m^2(x) dx}$$

Now

$$E(x) = f(x) - \alpha_0 \varphi_0(x) - \alpha_1 \varphi_1(x) - \dots - \alpha_n \varphi_n(x).$$

Multiplying this by $\varphi_m(x)$, ($m \leq n$), and integrating over the interval a to b , we have

$$\begin{aligned} \int_a^b E(x) \varphi_m(x) dx &= \int_a^b f(x) \varphi_m(x) dx - \alpha_0 \int_a^b \varphi_0(x) \varphi_m(x) dx - \alpha_1 \int_a^b \varphi_1(x) \varphi_m(x) dx \\ &\quad - \dots - \alpha_m \int_a^b \varphi_m^2(x) dx - \dots \\ &\quad - \int_a^b \varphi_m(x) \varphi_m(x) dx \\ &= \int_a^b f(x) \varphi_m(x) dx - \alpha_m \int_a^b \varphi_m^2(x) dx \\ &= (\alpha_m - \alpha_m) \int_a^b \varphi_m^2(x) dx \\ &= 0 \end{aligned}$$

Hence $E(x)$ is orthogonal to $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)$, and hence changes sign by Theorem IV, $n+1$ times at least.

Theorem VI. If the coefficients, c_m and c_r are neither zero, the function $\psi(x) = \sum_{m=1}^r a_j \varphi_j(x)$ changes sign at least m and at most r times in the interval (a, b) .

$\psi(x)$ is orthogonal to any function $\varphi_{m-1}(x)$, hence

by Theorem IV must vanish at least m times. It may also be considered a part of $\Phi_n(x)$ in which all the coefficients up to c_n are zero, hence by Theorem II cannot vanish more than r times.

Chapter II

Legendre Polynomials

The Legendre Polynomials, $P_0, P_1, \dots$, are polynomial solutions of the differential equation

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_m}{dx} \right] + m(m+1)P_m = 0$$

with the constant factor so chosen as to make $P_m(1) = 1$. It is well known that $P_m(x) = \frac{1}{2^m m!} \frac{d^m (x^2 - 1)^m}{dx^m}$. We shall verify for them the oscillation theorems of Sturm and prove that they have the property "D".

Theorem I. The zeros of P_m are all simple.

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m (x^2 - 1)^m}{dx^m}$$

Now $(x^2 - 1)^m$ has a root of order m at $+1$ and a root of order m at -1 and no other roots. $d(x^2 - 1)^m/dx$ has a zero of orders $m - 1$ at $+1$, and a zero of order $m - 1$ at -1 , and one zero which separates the zeros of $(x^2 - 1)^m$, and no other zeros. $d^2(x^2 - 1)^m/dx^2$ has a zero of order $m - 2$ at $+1$ and a zero of order $m - 2$ at -1 and two zeros which separate the zeros of $d(x^2 - 1)^m/dx$, and no other zeros. Keeping this up we see that $d^m(x^2 - 1)^m/dx^m$ has no zero at $+1$ or -1 but has m zeros which separate the zeros of $d^{m-1}(x^2 - 1)^m/dx^{m-1}$, and no other zeros. Following the work we see that these m zeros are all simple. But the zeros of $P_m(x)$ are the zeros of $d^m(x^2 - 1)^m/dx^m$ and hence are all simple.

Theorem II. The zeros of $P_m(x)$ lie between successive zeros of $P_{m-1}(x)$ within the interval $(-1, +1)$.

Consider
$$P_m(x) = \frac{1}{2^m m!} \frac{d^m (x^2 - 1)^m}{dx^m}$$

$$= k \frac{d^{m-1} \left[d(x^2 - 1)^m/dx \right]}{dx^{m-1}} \quad k = 1/2^m m!$$

$$\begin{aligned}
&= k_1 \frac{d^{m-1}}{dx^{m-1}} \left[(x^2 - 1)^{m-1} (2x) \right] \\
&= k_1 \left[2x \frac{d^{m-1}}{dx^{m-1}} (x^2 - 1)^{m-1} + 2(m-1) \frac{d^{m-2}}{dx^{m-2}} (x^2 - 1)^{m-1} \right] \\
(A) \quad &= k_1 2x P_{m-1}(x) + k_2 \int_{-1}^x P_{m-1}(x) dx
\end{aligned}$$

Let us now suppose that r_{i-1} and r_i are two successive zeros of P_{m-1} . The zeros of P_{m-1} are all simple and lie between the successive zeros of $\int_{-1}^x P_{m-1} dx$ from Theorem I. It then follows that the curve $\int_{-1}^x P_{m-1} dx$ changes sign between r_{i-1} and r_i once and only once. Then from equation (A), P_m changes sign once between r_{i-1} and r_i . This accounts for the $m-2$ zeros of P_m each of which lies between successive zeros of P_{m-1} . To show that a zero of P_m lies between -1 and r_1 and one between r_{m-1} and $+1$ is then easy. Considering again equation A), the integral of P_{m-1} is zero at -1 . If P_{m-1} is positive, at -1 $2x P_{m-1}$ is negative, and the slope of the curve $\int_{-1}^x P_{m-1} dx$ is positive. Hence $\int_{-1}^x P_{m-1} dx$ is positive from -1 and attains a maximum at r_1 . But $2x P_{m-1}$ is negative from -1 to r_1 . Hence their sum $P_m(x)$ which is negative at -1 and positive at r_1 is zero between -1 and r_1 .

The same argument holds if P_{m-1} is negative at -1 , and exactly the same situation exists between r_{m-1} and $+1$. Hence two successive zeros of P_m include a zero of P_{m-1} , and the theorem is proven.

It is well known that the Legendre polynomials $P_0, P_1, P_2, \dots$ are orthogonal in the interval $(+1, -1)$, and that P_n vanishes n times. We can also state the theorem:

Theorem III. The Legendre Polynomials have the Property "D".

$$1) \quad \frac{d^m}{dx^m} (U \cdot V) = U^m V'' + m U^{m-1} V' + \dots$$

For consider the determinant

$$\Delta = \begin{vmatrix} P_0(x_0), P_1(x_0) & P_n(x_0) \\ P_0(x_1), P_1(x_1) & P_n(x_1) \\ \vdots & \vdots \\ P_0(x_n), P_1(x_n) & P_n(x_n) \end{vmatrix} \quad x_i \neq x_j$$

$P_j(x)$ is a polynomial of the j -th degree in x . Hence by a proper combination of columns the determinant Δ can be reduced to the determinant

$$\Delta_1 = K \begin{vmatrix} 1 & x_0 & x_0^2 & x_0^3 & \dots & x_0^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^n \end{vmatrix}$$

whose value is known to be

$(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n)(x_1 - x_2) \dots (x_1 - x_n) \dots (x_{n-1} - x_n)$. Since $x_i \neq x_j$, none of the factors can be zero, hence the determinant cannot vanish.

If we consider a linear combination of Legendre Polynomials as $A_n(x) = c_0 P_0 + c_1 P_1 + c_2 P_2 + \dots + c_n P_n$ we see immediately that A_n may have n simple roots but that it cannot have more than n roots, since A is a polynomial of the n -th degree in x . Further, that if A_n has n different roots, they are all simple, it follows from the fact that A_n is a polynomial of the n -th degree.

We can now state

Theorem IV. If $f(x)$ is approximated to by a linear combination of Legendre Polynomials, $A_n(x) = c_0 P_0 + c_1 P_1 + \dots + c_n P_n$ the remainder $E(x) = f(x) - A_n(x)$ is orthogonal to $P_0, P_1, P_2, \dots, P_n$, and $A_n(x)$ crosses $f(x)$ at least $n+1$ times in the interval $(-1, +1)$.

As stated in the introduction consider the approximation as the one in which the coefficients in $A_n(x)$ are the Fourier Coefficients, that is,

$$c_m = \frac{2m+1}{2} \int_{-1}^{+1} f(x) P_m(x) dx$$

$$\left(\text{since } \int_{-1}^{+1} [P_m(x)]^2 dx = \frac{2}{2m+1} \right)$$

$$\text{Now } E(x) = f(x) - c_0 P_0(x) - c_1 P_1(x) - \dots - c_n P_n(x)$$

Multiplying this through by $P_m(x)$ and integrating we have

$$\begin{aligned} \int_{-1}^{+1} E(x) P_m(x) dx &= \int_{-1}^{+1} f(x) P_m(x) dx - c_0 \int_{-1}^{+1} P_0(x) P_m(x) dx - \dots \\ &\quad - c_m \int_{-1}^{+1} [P_m(x)]^2 dx - \dots - \int_{-1}^{+1} P_n(x) P_m(x) dx \\ &= \int_{-1}^{+1} f(x) P_m(x) dx - c_m \int_{-1}^{+1} [P_m(x)]^2 dx \\ &= \left[\frac{2m+1}{2} \int_{-1}^{+1} f(x) P_m(x) dx - c_m \right] \frac{2}{2m+1} \\ &= (c_m - c_m) \frac{2}{2m+1} = 0 \end{aligned}$$

Hence $E(x)$ is orthogonal to $P_0, P_1, P_2, \dots, P_n$, and by Theorem IV of Chapter I must vanish at least $n+1$ times in the given interval. Hence $A_n(x)$ must cross $f(x)$ at least $n+1$ times.

Theorem V. If neither of the constants c_m or c_n is zero, the function $\psi(x) = \sum_{m=0}^n c_m P_m(x)$ changes sign at most k times and at least m times in the interval $(-1, +1)$.

$\psi(x)$ cannot change sign more than k times since it is a polynomial of the k -th degree in x . To show that it cannot change sign less than m times, consider the function $\psi_1(x) = \sum_{i=0}^{n-1} c_i P_i(x)$. This is orthogonal to $\psi(x)$. Since it is a polynomial of the $(m-1)$ -th degree it can have but $m-1$ changes of sign. Hence, since $\psi(x)$ is orthogonal to $\psi_1(x)$, it must by Theorem IV of Chapter I change sign at least m times in the given interval.

Chapter III. Section I.

Trigonometric Polynomials

If we are considering a sine or cosine polynomial as,

$$A'_n(x) = b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx, \quad \text{or}$$

$$A''_n(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx, \quad \text{from the}$$

two preceding chapters we would expect $A'_n(x)$ to change sign not more than $n - 1$ times and $A''(x)$ to change sign not more than n times in the interval $0 < x < \pi$. If, however, we have the trigonometric polynomial

$$A_n(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx$$

and are considering the interval $0 < x < 2\pi$, which is the interval of orthogonality, we would expect $A_n(x)$ to have not more than $2n$ roots in this interval. That this is the case will now be shown.

The function $A_n(x)$ can be written in the form $P_n \cos x + \sin x P_{n-1}(\cos x)$, where $P_n(\cos x)$ means a polynomial of the n th degree in $\cos x$. That this is true can be easily shown by induction. $A_n(x)$ can be made to have $2n$ changes of sign in the given interval by proper choice of coefficients. To show that it cannot have more than $2n$ changes of sign in this interval, consider the product

$$(1) \quad P_n(\cos x) + \sin x P_{n-1}(\cos x) \quad \text{by}$$

$$(2) \quad P_n(\cos x) - \sin x P_{n-1}(\cos x) \quad \text{which is}$$

$$P_n^2(\cos x) - (1 - \cos^2 x) P_{n-1}^2(\cos x)$$

This is a polynomial of the $2n$ -th degree in $\cos x$, hence cannot have more than $4n$ changes of sign in the given interval. To see how these changes of sign are distributed between the

two factors, note that for every point r at which (1) changes sign there is a point $2\pi - r$ for which (2) changes sign. That is, the changes of sign of the product (1) by (2) are equally distributed between (1) and (2). Hence A_n cannot have more than $2n$ changes of sign in the interval $0 < x < 2\pi$.

We have thus proven

Theorem I. The trigonometric polynomial

$$\begin{aligned} A_n(x) = & \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx \\ & b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx \end{aligned}$$

cannot have more than $2n$ roots in the interval $(0, 2\pi)$ for any choice of coefficients.

Theorem II. If $A_n(x)$ has $2n$ zeros in the interval $(0, 2\pi)$ it changes sign at each.

Note that $A_n(x)$ is a periodic function of period 2π . Now suppose that one of the $2n$ zeros were a double root. Then at least two zeros are double roots to make $A_n(x)$ periodic. The derivative of $A_n(x)$ must have a zero between each zero of $A_n(x)$ and also a zero at each of the double roots, or in all $2n + 1$ zeros. But $\frac{d}{dx} A_n(x)$ is identical with $A_n(x)$ except for the constant coefficient, hence by Theorem I cannot have more than $2n$ zeros. Hence $A_n(x)$ cannot have any double roots.

Theorem III. The functions, $\frac{a_0}{2}, \cos x, \cos 2x, \cos 3x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx$, have the property "D" if the interval $(0, 2\pi)$ be considered.

To prove this let us write the determinant in the form:

$$\begin{vmatrix}
 \cos nx_0, & \cos(n-1)x_0, & \dots, & \frac{a_0}{2}, & \sin x_0, & \dots, & \sin nx_0 \\
 \cos nx_1, & & & & & & \\
 \cos nx_2, & & & & & & \\
 \vdots & & & & & & \\
 \cos nx_{2n}, & & & & & & \sin nx_{2n}
 \end{vmatrix}$$

This determinant is equal to the determinant:

$$\frac{a_0}{2} \begin{vmatrix}
 \frac{e^{inx_0} + e^{-inx_0}}{2}, & \frac{e^{i(n-1)x_0} + e^{-i(n-1)x_0}}{2}, & \dots, & 1, & \frac{e^{ix_0} - e^{-ix_0}}{2i}, & \dots, & \frac{e^{inx_0} - e^{-inx_0}}{2i} \\
 \frac{e^{inx_1} + e^{-inx_1}}{2}, & & & & & & \\
 \vdots & & & & & & \\
 \frac{e^{inx_{2n}} + e^{-inx_{2n}}}{2}, & & & & & &
 \end{vmatrix}$$

By addition of proper columns, this determinant reduces to

the following determinant:

$$\frac{K \cdot 1}{i^n} \begin{vmatrix}
 e^{inx_0}, & \dots, & e^{ix_0}, & 1, & \frac{e^{ix_0} - e^{-ix_0}}{2}, & \dots, & \frac{e^{inx_0} - e^{-inx_0}}{2} \\
 e^{inx_1}, & & & & & & \\
 e^{inx_2}, & & & & & & \\
 \vdots & & & & & & \\
 e^{inx_{2n}}, & & & & & &
 \end{vmatrix}$$

This determinant is in turn equal to the following determinant:

$$\frac{K}{2^n i^n} \begin{vmatrix}
 e^{inx_0}, & e^{i(n-1)x_0}, & \dots, & e^{ix_0}, & 1, & e^{-ix_0}, & \dots, & e^{-inx_0} \\
 e^{inx_1}, & & & & & & & \\
 e^{inx_2}, & & & & & & & \\
 \vdots & & & & & & & \\
 e^{inx_{2n}}, & & & & & & &
 \end{vmatrix}$$

Multiplying each row of this determinant by e^{inx} , it is equal to the determinant:

$$\frac{K}{2^{n+1}} \cdot \frac{1}{e^{i n(x_0 + x_1 + \dots + x_n)}} \begin{vmatrix} e^{i 2 n x_0} & e^{i (2 n - 1) x_0} & \dots & e^{i x_0} & 1 \\ e^{i 2 n x_1} & e^{i (2 n - 1) x_1} & \dots & e^{i x_1} & 1 \\ e^{i 2 n x_2} & & & & \\ \vdots & & & & \\ e^{i 2 n x_n} & & & e^{i x_n} & 1 \end{vmatrix}$$

Now the value of this determinant is known to be

$$(e^{i x_0} - e^{i x_1})(e^{i x_1} - e^{i x_2}) \dots (e^{i x_{n-1}} - e^{i x_n})(e^{i x_n} - e^{i x_0}) \dots$$

$$(e^{i x_0} - e^{i x_n})(e^{i x_2} - e^{i x_3}) \dots (e^{i x_{n-1}} - e^{i x_n}). \quad \text{But as } x_i \neq x_j,$$

the value of each factor is different from zero, hence the determinant is different from zero.')

We can now state

Theorem IV. If $f(x)$ is a periodic function approximated to by the trigonometric polynomials

$$\begin{aligned} A_n(x) = & a_0/2 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx \\ & + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx. \end{aligned}$$

the remainder, $E(x) = f(x) - A_n(x)$ is orthogonal to $a_0/2$, $\cos jx$, $\sin jx$, ($1 \leq j \leq n$), and $A_n(x)$ crosses $f(x)$ at least $2n + 2$ times in the interval $(0, 2\pi)$.

Choosing the coefficients A_n as the Fourier coefficients

$$A_n \text{ then becomes } \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx$$

$$+ b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx, \text{ where } \int_0^{2\pi} f(x) \cos kx$$

$$\alpha_k = \frac{\int_0^{2\pi} f(x) \cos kx \, dx}{\int_0^{2\pi} \cos^2 kx \, dx} \quad \text{and} \quad \beta_k = \frac{\int_0^{2\pi} f(x) \sin kx \, dx}{\int_0^{2\pi} \sin^2 kx \, dx}$$

$$\text{Now } E(x) = f(x) - \frac{a_0}{2} - a_1 \cos x - \dots - a_n \cos nx$$

$$- b_1 \sin x - \dots - b_n \sin nx$$

')

It might be noted here that this has been proved only for odd order determinants.

Integrating in this manner,

$$\begin{aligned} \int_0^{2\pi} E(x) \cos nx dx &= \int_0^{2\pi} f(x) \cos nx dx - \frac{\alpha}{2} \int_0^{2\pi} \cos nx dx - \dots \\ &\quad - \alpha_1 \int_0^{2\pi} \cos^2 nx dx - \beta_1 \int_0^{2\pi} \sin x \cos nx dx \\ &\quad - \beta_2 \int_0^{2\pi} \sin nx \cos nx dx. \end{aligned}$$

This reduces to

$$\begin{aligned} \int_0^{2\pi} E(x) \cos nx dx &= \int_0^{2\pi} f(x) \cos nx dx - \alpha_1 \int_0^{2\pi} \cos^2 nx dx \\ &= (\alpha_n - \alpha_n) \int_0^{2\pi} \cos^2 nx dx \\ &= 0 \end{aligned}$$

Hence $E(x)$ is orthogonal to

$$a_0/2, \cos jx, \sin jx, \quad 1 \leq j \leq n$$

If $E(x)$ does not have more than $2n$ changes of sign in the interval $(0, 2\pi)$ by applying Theorems III, I, and II of this chapter in order, we can ~~change~~, as in Theorem IV, Chapter I, contradict the orthogonality just proven. Hence $A_n(x)$ must cross $f(x)$ at least $2n + 2$ times.

Theorem V. If the constants a_n and b_n are not both zero, and also a_k and b_k are not both zero, the function

$$\psi(x) = \sum_{n=1}^k (a_n \cos nx + b_n \sin nx) \quad 1 \leq n \leq k$$

vanishes and changes sign at at least $2m$ and at most $2k$ distinct points of the interval $0 < x < 2\pi$

That the function cannot vanish more than $2k$ times, and that if it vanishes $2k$ times each zero represents a change of sign, I have just shown in the preceding theorems. To show that it cannot vanish less than $2m$ times consider the function

$$\psi_1(x) = \frac{a_0}{2} + \sum_{n=1}^{m-1} (a_n \cos nx + b_n \sin nx) \quad 1 \leq n \leq m-1$$

This function is orthogonal to our function $\psi(x)$. As in Theorem I we can show that $\psi_1(x)$ can have $2m - 2$ changes of sign in our interval. Hence $\psi(x)$ by Theorem IV Chapter I

to be always orthogonal to $\psi_1(x)$ must change sign more than $2m - 2$ times or the orthogonality can be contradicted. Hence $\psi(x)$, being periodic, must change signs at least $2m$ times. (This Theorem is stated and proved by Bôcher Annals of Mathematics, April 1906, but from an entirely different point of view.)

Section II

Sine Polynomials

If we consider the sine polynomial

$$A'_n(x) = b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx, \text{ and}$$

and the interval $(0, \pi)$ as the interval of orthogonality, we may draw conclusions similar to those of Section I. Note that $\sin nx$ has only $n - 1$ zeros in this interval, hence we should expect that $A'_n(x)$ could not change signs more than $n - 1$ times in this interval. It can of course be made to vanish $n - 1$ times in the interval. To prove that it cannot vanish more than $n - 1$ times we must first prove

Theorem I. The functions $\sin x$, $\sin 2x$, $\sin 3x$, ..., $\sin nx$, have the property "D" if the interval considered be $(0, \pi)$

The determinant involved in this case is

$$\begin{vmatrix} \sin x_1 & \sin 2x_1 & \sin 3x_1 & \dots & \sin nx_1 \\ \sin x_2 & \sin 2x_2 & \sin 3x_2 & \dots & \sin nx_2 \\ \sin x_3 & \sin 2x_3 & \sin 3x_3 & \dots & \sin nx_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin x_n & \sin 2x_n & \sin 3x_n & \dots & \sin nx_n \end{vmatrix}$$

This determinant can be written in the form

$$\begin{vmatrix} \sin x_1, & \sin x_1 P_1(\cos x_1), & \sin x_1 P_2(\cos x_1) & \dots & \sin x_1 P_{n-1}(\cos x_1) \\ \sin x_2, & & & & \\ \sin x_3, & & & & \\ & & & & \\ \sin x_n, & & & & \sin x_n P_{n-1}(\cos x_n) \end{vmatrix} \quad 19$$

where we understand by $P_m(\cos x)$ a polynomial of the m -th degree in $\cos x$. By a proper combination of columns the determinant is easily reduced to the following form:

$$\sin x_1, \dots, \sin x_n \begin{vmatrix} 1, & 2 \cos x_1, & 4 \cos^2 x_1, & \dots & 2^{n-1} \cos^{n-1} x_1 \\ & 2 \cos x_2 & & & \\ & & & & \\ & & & & \\ & & & & 2^{n-1} \cos^{n-1} x_n \end{vmatrix}$$

By removing the constants from each column to the outside of the determinant we have it in the form

$$(K) \sin x_1, \dots, \sin x_n \begin{vmatrix} 1, & \cos x_1, & \cos^2 x_1, & \dots & \cos^{n-1} x_1 \\ & \cos x_2 & & & \\ & & & & \\ & & & & \\ & & & & \cos^{n-1} x_n \end{vmatrix}$$

The value of the determinant is in this way seen to be $(K \sin x_1 \sin x_2 \dots \sin x_n)(\cos x_1 - \cos x_2)(\cos x_1 - \cos x_3) \dots (\cos x_1 - \cos x_n)(\cos x_2 - \cos x_3) \dots (\cos x_{n-1} - \cos x_n)$. Now as $x_i \neq x_j$, no factor can vanish in our interval $0 < x < \pi$. Hence, the determinant cannot vanish, which was to be proven.

We can now prove

Theorem II. $A'_n(x)$ cannot vanish more than $n - 1$ times in the interval $(0, \pi)$ for any choice of coefficients.

The proof follows exactly the proof of Theorem II Chapter I

Theorem III. If $A_n'(x)$ has $n - 1$ roots in the interval $(0, \pi)$ it changes sign at each.

The method of proof is the same as that of Theorem III Chapter I.

Theorem IV. If any function is orthogonal to $\sin x$, $\sin 2x$, ... $\sin nx$, in the interval $(0, \pi)$, it changes sign at least n times in that interval.

Compare Theorem IV Chapter I.

Theorem V. If the function $f(x)$ is approximated to by the sine polynomial

$$A_n'(x) = b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx$$

the remainder $E(x) = f(x) - A_n'(x)$ is orthogonal to $\sin x$, $\sin 2x$, ... $\sin nx$, and $A_n'(x)$ crosses $f(x)$ at least n times in the interval of orthogonality.

That $E(x)$ is orthogonal to $\sin x$, $\sin 2x$, ..., $\sin nx$, in the interval $(0, \pi)$ is seen by comparing this example with Theorem IV Section I of this chapter. That $E(x)$ vanishes at least n times in the interval $(0, \pi)$ follows from Theorem IV of this section.

Section III

Cosine Polynomials

A cosine polynomial should show properties similar to those of a sine polynomial if the same interval of orthogonality be considered. The cosine polynomial

$$A_n''(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx.$$

can vanish n times in the interval $(0, \pi)$, and from the analogy to the sine polynomial we should expect that it could not vanish more than n times in that interval. We

find that this is the case.

We will now state theorems exactly analogous to those of Section II, leaving the proofs to the reader, as they are merely repetitions of the proofs of that section.

Theorem I. The functions $a_0/2, \cos x, \cos 2x, \cos 3x, \dots, \cos nx$, have the property "D" if the interval considered be $(0, \pi)$.

Theorem II. $A_n''(x)$ cannot vanish more than n times in the interval $(0, \pi)$ for any choice of coefficients.

Theorem III. If $A_n''(x)$ has n roots in the interval $(0, \pi)$ it changes sign at each.

Theorem IV. If any function is orthogonal to $a_0/2, \cos x, \cos 2x, \dots, \cos nx$, in the interval $(0, \pi)$ it changes sign at least $n + 1$ times in that interval.

Theorem V. If the function $f(x)$ is approximated to ~~##~~ by the cosine polynomial

$$A_n''(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx$$

the remainder

$$E(x) = f(x) - A_n''(x)$$

is orthogonal to $a_0/2, \cos x, \dots, \cos nx$, and $A_n''(x)$ crosses $f(x)$ at least $n + 1$ times in the interval $(0, \pi)$.

Chapter IV.

Bessel FunctionsSection I. Bessel Functions of Order Zero.

A Bessel Function of the n -th order is a solution of the differential equation

$$\frac{d^2 J_n}{dx^2} + \frac{1}{x} \frac{dJ_n}{dx} + \left(1 - \frac{n^2}{x^2}\right) J_n = 0$$

which is finite for $x = 0$. The Bessel Function of zero order is a solution of the equation

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x} \frac{dJ_0}{dx} + J_0 = 0$$

We find in all such cases of approximation by means of a linear combination of orthogonal Bessel Functions that the necessary condition for interpolation is satisfied and that the approximating function crosses the function approximated to the number of times we should expect.

If we have the Bessel Functions $J_0(\mu_1 x)$, $J_0(\mu_2 x)$, ... $J_0(\mu_n x)$, where $\mu_1, \mu_2, \dots, \mu_n$ are the roots of $J_0(x)$ taken in order of magnitude, $J_0(\mu_1 x)$ does not vanish in the interval $(0,1)$ ^{positive}, $J_0(\mu_2 x)$ vanishes once in that interval, $J_0(\mu_3 x)$ vanishes twice, $J_0(\mu_n x)$ vanishes $n - 1$ times. The functions $J_0(\mu_i x)$ and $J_0(\mu_j x)$, ($i \neq j$) themselves are not orthogonal in the interval but the function $\sqrt{x} J_0(\mu_i x)$ and $\sqrt{x} J_0(\mu_j x)$ are orthogonal in that interval. (Byerly, Chapter VII)

If we have a linear combination of these functions as

$$\Phi_n(x) = c_1 J_0(\mu_1 x) + c_2 J_0(\mu_2 x) + \dots + c_n J_0(\mu_n x)$$

we see that $\Phi_n(x)$ could have $n - 1$ zeros in the interval $(0,1)$. That it cannot have more than $n - 1$ zeros is then to be expected. We will thus state

Theorem I. $\bar{\Phi}_n(x)$ as just defined cannot have more than $n - 1$ zeros in the interval $(0,1)$ for any choice of coefficients.

To prove this, consider the original differential equation of which $J_0(x)$ is the solution, namely

$$[x J_0'(x)]' + x J_0(x) = 0$$

which by a transformation becomes

$$\frac{d}{dx} \left[x \frac{d}{dx} J_0(\mu x) \right] + \mu^2 x J_0(\mu x) = 0$$

Now we will adopt the notation

$$\varphi_1(x) = J_0(\mu_1 x)$$

$$\varphi_2(x) = J_0(\mu_2 x)$$

$$\vdots$$

Where $\mu_1, \mu_2, \mu_3, \dots, \mu_n$ are the positive zeros of $J_0(x)$ in order of magnitude. From the nature of the Bessel Functions

$$\varphi_i(1) > 0 \text{ and } \varphi_i(0) = 1 \quad (i = 1, 2, 3, \dots)$$

$\bar{\Phi}_n(x)$ now becomes

$$\bar{\Phi}_n(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \dots + c_n \varphi_n(x)$$

We now want to prove that $\bar{\Phi}_n(x)$ has no more zeros than $\varphi_n(x)$ in the interior of the interval $(0,1)$. Let k be the number of zeros of $\bar{\Phi}_n(x)$ in the interval $(0,1)$. Then

$\bar{\Phi}_n'(x)$ has $k - 1 + 1$ or more zeros in that interval since $\bar{\Phi}_n(x)$ vanishes at $x = 1$. Also $x \bar{\Phi}_n'(x)$ has k or more zeros in the interval, as the x factor does not remove any of the zeros. Then $[x \bar{\Phi}_n'(x)]'$ has $k-1+1$ or more zeros in the interval as $x \bar{\Phi}_n'(x)$ vanishes at $x = 0$. As $[x \bar{\Phi}_n'(x)]'$ has k or more zeros, then from the identity

$$[x \varphi_i'(x)]' + x \mu_i^2 \varphi_i(x) \equiv 0,$$

the expression

$$-x [\mu_1^2 c_1 \varphi_1(x) + \mu_2^2 c_2 \varphi_2(x) + \dots + \mu_n^2 c_n \varphi_n(x)]$$

has k or more zeros in the given interval. Then

$${}_{2\lambda}\Phi(x) = c_1 \left(\frac{\mu_1}{\mu_n}\right)^{2\lambda} \varphi_1(x) + \left(\frac{\mu_2}{\mu_n}\right)^{2\lambda} c_2 \varphi_2(x) + \dots + \left(\frac{\mu_{n-1}}{\mu_n}\right)^{2\lambda} c_{n-1} \varphi_{n-1}(x) + \varphi_n(x)$$

has k or more zeros as the division by constants does not affect the number of zeros. Repeating the argument by substituting in the differential equation

${}_{2\lambda}\Phi(x) = \left(\frac{\mu_1}{\mu_n}\right)^{2\lambda} c_1 \varphi_1(x) + \dots + \left(\frac{\mu_{n-1}}{\mu_n}\right)^{2\lambda} c_{n-1} \varphi_{n-1}(x) + \varphi_n(x)$ has k or more zeros in our interval. That is ${}_{2\lambda}\Phi$ has an equal or greater number of zeros than $\Phi_n(x)$ in the interval considered ~~than~~ (x) . Now let

$${}_{2\lambda}\Phi(x) - \varphi_n(x) \equiv E(x)$$

and let λ become so great that the maximum of the absolute value of any term of $E(x)$ at any point is less than ϵ/n , where ϵ is to be chosen later, i.e.,

$$\left| \left(\frac{\mu_i}{\mu_n}\right)^{2\lambda} c_i \varphi_i(x) \right| < \epsilon/n \quad i = (1, 2, 3, \dots, n-1).$$

$E(x)$ is then everywhere less than ϵ . Consider now the derivative $\varphi_n'(x)$ at the points $x_1, x_2, x_3, \dots, x_{n-1}$ at which $\varphi_n(x)$ vanishes and choose that derivative at x_i of minimum absolute value, say of value η . Choose an interval about x_i of width $\delta_i/2$ on either side of x_i such that nowhere in the interval δ_i is the slope of $\varphi_n(x) < \eta/2$

Now $E(x)$ being a continuous function, we will choose ϵ to satisfy the two conditions, first, that the maximum absolute slope of $E(x) < \eta/2$, and second, that $\epsilon < \eta \cdot \delta_i/2$. Now call the sum of the intervals $\delta_1 + \delta_2 + \delta_3 + \dots + \delta_{n-1}$ about the zeros of $\varphi_n(x)$

M , and call the remainder of the interval $(0,1)$, M . It is now clear, that in M the sum of $\varphi_n(x) + E(x)$ cannot have any zeros. Also in N the slope of the curve $_{\lambda}\bar{\Phi}$ cannot be zero. Hence in any of the intervals, J , the curve cannot have more than one simple and no double roots. It now follows that the zeros of $_{\lambda}\bar{\Phi}$ are simple and equal in number to the zeros of $\varphi_n(x)$. Since $\varphi_n(x)$ has only $n-1$ zeros, $_{\lambda}\bar{\Phi}(x)$ cannot have more than $n-1$ zeros. But the number of zeros of $_{\lambda}\bar{\Phi}(x)$ is greater than or equal to the number of zeros of $\bar{\Phi}_n(x)$. Hence $\bar{\Phi}_n(x)$ has not more than $n-1$ zeros in the given interval, and the proof is complete.

§ Theorem II. If $\bar{\Phi}_n(x)$ has $n-1$ zeros in the interval $(0,1)$ it changes sign at each.

To show this suppose $\bar{\Phi}_n(x)$ has the $n-1$ roots and one is a double root. Then $\frac{d}{dx} \bar{\Phi}_n(x)$ would have $n-2+2$ or more

zeros in the interval $(0,1)$. This follows from Rolle's

Theorem, noting the fact that $\bar{\Phi}_n(x) = 0$ at $x = 1$.

But the derivative of $\bar{\Phi}_n(x)$ is a linear combination of

$-J_{\nu}(\mu x)$, (μ_s are the μ_s of J_0), call $\psi_n(x)$. If we

multiply $\psi_n(x)$ by x we do not introduce any new zeros in

the interval $(0,1)$. Differentiate $x \psi_n(x)$ and we have

a function which in the interval $(0,1)$ has $n-1+1$ or

more zeros, since $x \psi_n(x) = 0$ at $x = 0$. But $\frac{d}{dx} [x \psi_n(x)]$

is of the form $x \bar{\Phi}_n(x)$ which by Theorem I cannot have more

than $n-1$ zeros in the given interval. The theorem is

proven.

Theorem III. The Bessel Functions $J_0(\mu_1 x)$, $J_0(\mu_2 x)$, — $J_0(\mu x)$ have the property "D" when the interval considered is $(0,1)$.

The determinant involved is

$$\begin{vmatrix} J_0(\mu_1 x) & J_0(\mu_2 x) & \dots & J_0(\mu_n x) \\ J_0(\mu_1 x) & J_0(\mu_2 x) & \dots & J_0(\mu_n x) \\ J_0(\mu_1 x) & J_0(\mu_2 x) & & \\ J_0(\mu_1 x) & J_0(\mu_2 x) & \dots & J_0(\mu_n x) \end{vmatrix}$$

If this determinant vanishes we can find n points at which a function

$$\Phi_n(x) = c_1 J_0(\mu_1 x) + c_2 J_0(\mu_2 x) + \dots + c_n J_0(\mu_n x)$$

vanishes. But from Theorem I this is impossible. Hence the determinant must be different from zero for $x_1 < x_2 < \dots < x_n$.

Theorem IV. If any function is orthogonal to $J_0(\mu_1 x), J_0(\mu_2 x), \dots, J_0(\mu_n x)$, it changes sign at least n times in the interval $(0,1)$.

Compare Theorem IV Chapter I.

Theorem V. If $f(x)$ is approximated to by the linear combination of Bessel Functions

$\Phi_n(x) = c_1 J_0(\mu_1 x) + c_2 J_0(\mu_2 x) + \dots + c_n J_0(\mu_n x)$
the remainder $E(x) = f(x) - \Phi_n(x)$ is orthogonal to $x J_0(\mu_j x)$ ($1 \leq j \leq n$) and $\Phi_n(x)$ crosses $f(x)$ at least n times in the interval $(0,1)$.

As in the preceding theorems consider the approximation as that one in which the coefficients are the Fourier Coefficients.

$$\Phi_n(x) = \alpha_1 J_0(\mu_1 x) + \alpha_2 J_0(\mu_2 x) + \dots + \alpha_n J_0(\mu_n x)$$

where

$$\alpha_j = \frac{\int_0^1 x f(x) J_0(\mu_j x) dx}{\int_0^1 x (J_0(\mu_j x))^2 dx}$$

Now $E(x) = f(x) - \alpha_1 J_0(\mu_1 x) - \alpha_2 J_0(\mu_2 x) - \dots - \alpha_n J_0(\mu_n x)$

Multiplying this through by $x J_0(\mu_j x)$ and integrating between

the limits (0,1) we have,

$$\begin{aligned}
 \int_0^1 x E(x) J_0(\mu_j x) dx &= \int_0^1 x f(x) J_0(\mu_j x) dx - \alpha_1 \int_0^1 x J_0(\mu_1 x) J_0(\mu_j x) dx \\
 &\quad \dots - \alpha_j \int_0^1 x (J_0(\mu_j x))^2 dx - \dots \\
 &\quad - \alpha_n \int_0^1 x J_0(\mu_n x) J_0(\mu_j x) dx \\
 &= \int_0^1 x f(x) J_0(\mu_j x) dx - \alpha_j \int_0^1 x (J_0(\mu_j x))^2 dx \\
 &= (\alpha_j - \alpha_j) \int_0^1 x (J_0(\mu_j x))^2 dx \\
 &= 0
 \end{aligned}$$

Hence $f(x)$ is orthogonal to $x \Phi_n(x)$. That $E(x)$ must change sign n times in the interval (0,1) follows from Theorem IV, since $x E(x)$ is orthogonal to $J(\mu_j x)$ ($1 \leq j \leq n$)

Section II. Bessel Functions of Order m .

Upon investigation we have found that the theorems stated for Bessel Functions of order zero hold also for the Bessel Functions of higher order. These Theorems will be stated and the proof given only when the parallel is not so close as to be obvious.

The functions $\sqrt{x} J_m(\mu_1 x)$, $\sqrt{x} J_m(\mu_2 x)$, [where $\mu_1, \mu_2 \dots$ are the roots of $J_m(x) = 0$], are orthogonal in the interval (0,1).¹⁾ $J_m(\mu_1 x)$ does not vanish in that interval, $J_m(\mu_2 x)$ vanishes once, $J_m(\mu_3 x)$ vanishes twice, $J_m(\mu_n x)$ vanishes $n-1$ times, where the μ 's are taken in order of magnitude.

A linear combination of these functions

$$\Phi_n(x) = c_1 J_m(\mu_1 x) + c_2 J_m(\mu_2 x) + \dots + c_n J_m(\mu_n x)$$

can vanish then, $n-1$ times in the interval. Our first proposition is

1) Gray and Matthews, A Treatise on Bessel Functions, (1895) Chapter VI.

Theorem I. $\bar{\Phi}_n(x)$ as just defined cannot have more than $n-1$ zeros in the interval $(0,1)$ for any choice of coefficients.

To prove this theorem, I wish to show that

$$\bar{\Phi}_n(x) = c_1 J_n(\mu_1 x) + c_2 J_n(\mu_2 x) + \dots + c_n J_n(\mu_n x)$$

has no more zeros in the interval $(0,1)$ than $J_n(\mu_n x)$.

Suppose $\bar{\Phi}_n(x)$ has k zeros in the interval $(0,1)$. Then $x^m \bar{\Phi}_n(x)$ has also k zeros in that interval and a zero at each end. It follows that $[x^m \bar{\Phi}_n]'$ has $k-1+2 = k+1$ or more zeros in that interval. But (Byerly, Fourier Series, Chapter VII)

$$[x^m \bar{\Phi}_n]' = x^m [\mu_1 c_1 J_{n-1}(\mu_1 x) + \dots + \mu_n c_n J_{n-1}(\mu_n x)]$$

Write this last expression in the form

$$[x^m \bar{\Phi}_n]' = \mu_n x^m \left[\frac{\mu_1}{\mu_n} c_1 J_{n-1}(\mu_1 x) + \frac{\mu_2}{\mu_n} c_2 J_{n-1}(\mu_2 x) + \dots + J_{n-1}(\mu_n x) \right]$$

(The μ 's are those of J_n , c_n is set equal to 1) Now

the expression $x^{-(2m-1)} [x^m \bar{\Phi}_n(x)]'$ has $k+1$ or more zeros in

that interval. Then $\{x^{-(2m-1)} [x^m \bar{\Phi}_n(x)]'\}'$ has k or more

zeros in the given interval. But

$$\begin{aligned} \{x^{-(2m-1)} [x^m \bar{\Phi}_n(x)]'\}' &= -\mu_n x^{-(2m-1)} \left[\frac{\mu_1^2}{\mu_n} c_1 J_{n-2}(\mu_1 x) + \frac{\mu_2^2}{\mu_n} c_2 J_{n-2}(\mu_2 x) + \dots \right. \\ &\quad \left. + \mu_n J_{n-2}(\mu_n x) \right]. \end{aligned}$$

Then

$$\Delta_1(x) = \frac{\{x^{-(2m-1)} [x^m \bar{\Phi}_n(x)]'\}'}{-\mu_n^2 x} = \left[\left(\frac{\mu_1}{\mu_n} \right)^2 c_1 J_{n-2}(\mu_1 x) + \left(\frac{\mu_2}{\mu_n} \right)^2 c_2 J_{n-2}(\mu_2 x) + \dots + J_{n-2}(\mu_n x) \right]$$

has k or more zeros in the given interval. By carrying out the same process on $\Delta_1(x)$ we see that $\Delta_2(x)$ has k or more zeros in our interval. Likewise

$$\Delta_{2\lambda}(x) = \left[\left(\frac{\mu_1}{\mu_n} \right)^{2\lambda} c_1 J_{n-2\lambda}(\mu_1 x) + \left(\frac{\mu_2}{\mu_n} \right)^{2\lambda} c_2 J_{n-2\lambda}(\mu_2 x) + \dots + J_{n-2\lambda}(\mu_n x) \right]$$

has k or more zeros in our given interval.

Now as in Theorem I of Section I $\Delta_{\lambda\lambda}(x)$ cannot have more than $n-1$ zeros in the given interval for any choice of the constants. As the number of zeros of $\Delta_{\lambda\lambda}$ is greater than or equal to the number of zeros of $\bar{\Phi}_n(x)$, $\bar{\Phi}_n(x)$ cannot have more than $n-1$ zeros in the given interval for any choice of the coefficients. This proves our theorem.

Theorem II. If $\bar{\Phi}_n(x)$ has $n-1$ roots in the interval $(0,1)$ it changes sign at each.

This follows from the method of proof of Theorem I, since upon two differentiations the function returns to the original form.

Theorem III. The Bessel Functions $J_n(\mu, x)$, $J_m(\mu_1 x)$, ..., $J_m(\mu_n x)$ have the property "D" when the interval considered is $(0,1)$.

Compare Theorem III Section I

Theorem IV. If any function is orthogonal to $J_n(\mu, x)$, $J_m(\mu_1 x)$, ..., $J_m(\mu_n x)$ it changes sign at least n times in the interval $(0,1)$.

Compare Theorem IV Chapter I.

Theorem V. If $f(x)$ is approximated to by the linear combination of Bessel Functions

$\bar{\Phi}_n(x) = c_1 J_n(\mu_1 x) + c_2 J_n(\mu_2 x) + \dots + c_n J_n(\mu_n x)$
the remainder $E(x) = f(x) - \bar{\Phi}_n(x)$ is orthogonal to
 $x J_n(\mu_j x)$, $(1 \leq j \leq n)$ and $\bar{\Phi}_n(x)$ crosses $f(x)$ at
least n times in the interval $(0,1)$.

Compare Theorem V Section I.

Chapter V.

A General Differential Equation
and some of its Solutions.

In this chapter we shall discuss certain solutions of the general differential equation

$$(1) \quad \frac{d}{dx} \left[p(x) \frac{dV}{dx} \right] + \lambda r(x) V = 0$$

in which V is continuous, and $p(x)$ and $r(x)$ are each continuous and greater than zero in the interval (a, b) .

We will take as boundary conditions that

$$(2) \quad \begin{aligned} V(a) &= k V(b) & k &= p(b)/p(a) \\ V'(a) &= V'(b) \end{aligned}$$

For convenience we will discuss first the case in which $V(a) = k V(b) \neq 0$. and show that the results obtained hold in all essentials when $V(a) = k V(b) = 0$.

If $V = V_i$ and $V = V_j$ are two values of V which satisfy the differential equation for different values of λ , by substituting then in equation (1) we have the two following equations

$$(3) \quad \frac{d}{dx} \left[p(x) \frac{dV_i}{dx} \right] + \lambda_i r(x) V_i = 0$$

$$(4) \quad \frac{d}{dx} \left[p(x) \frac{dV_j}{dx} \right] + \lambda_j r(x) V_j = 0$$

Multiplying (3) by V_j and (4) by V_i and subtracting, we have

$$V_j (pV_i')' - V_i (pV_j')' = -(\lambda_i - \lambda_j) r(x) V_i V_j$$

Taking the integral between the limits a and b we have

$$\int_a^b \left[V_j (pV_i')' - V_i (pV_j')' \right] dx = -(\lambda_i - \lambda_j) \int_a^b r(x) V_i V_j dx$$

Integrating the left hand side by parts we have

$$\begin{aligned} \left[V_j (p V_i') \right]_a^b - \int_a^b p V_j' V_i' dx - V_i \left[p V_j' \right]_a^b + \int_a^b p V_i' V_j' dx \\ = -(\lambda_i - \lambda_j) \int_a^b r(x) V_i V_j dx \end{aligned}$$

This is then equal to the equation

$$(5) \quad p \left[V_j V_i' - V_j' V_i \right]_a^b = -(\lambda_i - \lambda_j) \int_a^b r(x) V_i V_j dx$$

Writing out the left hand side we have,

$$\begin{aligned} p(a) \left[\frac{p(b)}{p(a)} V_j(b) V_i'(b) - \frac{p(b)}{p(a)} V_j'(b) V_i(b) \right. \\ \left. - V_j(a) V_i'(a) + V_j'(a) V_i(a) \right] = -(\lambda_i - \lambda_j) \int_a^b r(x) V_i V_j dx \end{aligned}$$

If V_i and V_j satisfy the boundary conditions, we will see that

$$(6) \quad -(\lambda_i - \lambda_j) \int_a^b r(x) V_i V_j dx = 0$$

This equation shows orthogonality between $\sqrt{r} V_i$ and $\sqrt{r} V_j$ and since $r(x)$ does not vanish in the interval a to b , either V_i or V_j must vanish and change sign in that interval. Further since V_i and V_j both satisfy the boundary conditions, which are those for periodicity, the number of changes of sign is always even. Further, no one of the roots of any V is a double root. For writing (1) in the form

$$(7) \quad p \frac{d^2 V}{dx^2} + \frac{dp}{dx} \frac{dV}{dx} + \lambda r(x) V = 0$$

we see that if dV/dx and V were simultaneously zero, that $d^2 V/dx^2$ and all higher derivatives are zero at the same time. This would mean that V was identically zero. Hence V can have no double roots.

Let us now consider the distribution of the roots of V for different values of λ . If we assume that in equation (5) $\lambda_i > \lambda_j$, we can show that V_i must vanish between successive roots of V_j . For, change the limits of integration to ξ_1 and ξ_2 , where ξ_1 and ξ_2 are successive roots of V_j and assume both V_i and V_j positive between ξ_1

and ξ_2 . Carrying out the integration we have

$$(8) \quad p(\xi_2) V_i(\xi_2) V_j'(\xi_2) - p(\xi_1) V_i(\xi_1) V_j'(\xi_1) = (\lambda_i - \lambda_j) \int_{\xi_1}^{\xi_2} r(x) V_i V_j dx$$

Examining the result we see that the left hand member of the equation is negative while the right hand member is positive.

Assuming V_i negative, $\xi_1 < x < \xi_2$, we get the same result.

Thus V_i must change sign between consecutive roots of V_j .

This fact together with our boundary conditions tells us

that the number of roots of V_i in the interval (a, b) is greater than or equal to the number of roots of V_j .

I now wish to show that λ_i can be so chosen that the number of roots of V_i is greater than the number of roots of V_j and that there are an infinite number of values of λ for which the boundary conditions are satisfied.

To do this we must first note that $V(\lambda, x)$ is uniformly continuous in λ . For, there is always a solution V_i of the differential equation (1) such that $V_i(a) = 0$, $V_i'(a) = 1$. Substituting these in our equation we have

$$(9) \quad \frac{d}{dx} [p V_i'(\lambda)] + \lambda r(x) V_i(\lambda) = 0, \quad \text{and letting } V_i(\lambda')$$

correspond to $V_i(\lambda + d\lambda)$ we have

$$(10) \quad \frac{d}{dx} [p V_i'(\lambda')] + \lambda' r(x) V_i(\lambda') = 0$$

From these equations we can get

$$(11) \quad \frac{d}{dx} p [V_i'(\lambda') - V_i'(\lambda)] + r(x) [\lambda' V_i(\lambda') - \lambda V_i(\lambda)] = 0$$

or

$$(12) \quad \frac{d}{dx} p(\Delta V_i)' + r(x) [\Delta \lambda V_i(\lambda') + \lambda \Delta V_i(\lambda)] = 0$$

Let $\Delta V_i = \omega$ and we have

$$(13) \quad \frac{d}{dx} [p \omega'] + \lambda r(x) \omega = - V_i(\lambda') r \Delta \lambda$$

with the condition that $\omega(a) = 0$, $\omega'(a) = 0$. If we again make the substitution $\omega' = z$ where $\omega = \int_a^x z dx$, we have

$$p z = - \int_a^x [V_i(\lambda') r(x) \Delta \lambda + \lambda r(x) \omega] dx$$

If now we replace in these equations the variables by greater finite functions, as

z by $\bar{z}$, $\bar{\omega}$ by $\int_a^x \bar{z} dx$, $\bar{\omega}'$ by $\bar{z}$, $\frac{rV_1(\lambda')}{p}$ by M , where M

is constant, and $-\frac{\lambda r(x)}{p}$ by R , we have

$$\bar{z} = - \int_a^x (M\Delta\lambda - R\bar{\omega}) dx$$

and our differential equation becomes

$$(14) \quad \bar{\omega}'' - R\bar{\omega} = -M\Delta\lambda = k \text{ (constant)}$$

Differentiating,

$$\begin{aligned} \bar{\omega}''' - R\bar{\omega}' &= 0 \\ \bar{\omega} &= c_1 e^{\sqrt{R}(x-a)} + c_2 e^{-\sqrt{R}(x-a)} + c_3 \\ &= c_1 e^{\sqrt{R}(x-a)} + c_2 e^{-\sqrt{R}(x-a)} - \frac{M\Delta\lambda}{R} \end{aligned}$$

By boundary conditions,

$$0 = c_1 + c_2 - \frac{M\Delta\lambda}{R} \quad \text{or} \quad c_1 = c_2 = \frac{\Delta\lambda M}{2R}$$

or $\bar{\omega} = \Delta\lambda$ (multiplied by a finite quantity). Therefore in the limit $\bar{\omega}$ approaches 0 with $\Delta\lambda$. Thus we see that

$$\begin{aligned} |V(\lambda + h, x) - V(\lambda, x)| &< \epsilon \\ |h| &< \delta \text{ independently of } x. \end{aligned}$$

Now since V is uniformly continuous in λ with respect to x ,

$(pV')' = -\lambda V' r$ also is uniformly continuous in λ . Also

$$pV' = -\lambda \int_a^x V r(x) dx + p(a)$$

is uniformly continuous in λ . Hence as $p > 0$, V' is uniformly continuous in λ . Now from the continuity in x for constant λ and from the uniform continuity in λ , we may argue continuity in both variables.

Let us now consider the problem of the periodic boundary conditions. For every λ there are by the existence theorems two solutions, $S_1(\lambda, x)$ and $S_2(\lambda, x)$ satisfying the

conditions

$$(15) \quad \begin{aligned} S_1(\lambda, a) &= 0 & S_1'(\lambda, a) &= 1 \\ S_2(\lambda, a) &= 1 & S_2'(\lambda, a) &= 0 \end{aligned}$$

These solutions are uniformly continuous in λ and x in any interval. The boundary conditions to be considered are

$$I \quad p(a) V(\lambda, a) = p(b) V(\lambda, b) \quad \text{and}$$

$$II \quad V'(\lambda, a) = V'(\lambda, b)$$

Let us show there is always a non-identically vanishing solution satisfying I. The general solution of the differential equation is

$$V(\lambda, x) = c_1 S_1(\lambda, x) + c_2 S_2(\lambda, x).$$

Then we must have

$$c_1 [p(b) S_1(\lambda, b) - p(a) S_1(\lambda, a)] + c_2 [p(b) S_2(\lambda, b) - p(a) S_2(\lambda, a)] = 0$$

Let us choose

$$(16) \quad \begin{aligned} c_1 &= [p(b) S_2(\lambda, b) - p(a) S_2(\lambda, a)] \\ c_2 &= -[p(b) S_1(\lambda, b) - p(a) S_1(\lambda, a)] \end{aligned}$$

Thus c_1 and c_2 are continuous in λ , and so $V(\lambda, x)$ will be uniformly continuous in λ and x .

Remark: Neither S_1 nor S_2 are identically zero in x for any λ as is shown by their initial values. Hence $V(\lambda, x)$ can be identically zero only in case $c_1 = c_2 = 0$.

$S_1(\lambda, x)$ is a Sturm solution of the differential equation, and as $S_1(\lambda, a) = 0$, $c_2 = -p(b) S_1(\lambda, b)$ and vanishes only for isolated values of λ . (Liouville's Journal) If c_2 is also zero, $S_2(\lambda, x)$ satisfies the condition I and hence so does any solution of the differential equation. Let us see whether we can then satisfy the second condition. To this end we must have

$$k_1 S_1'(\lambda, a) + k_2 S_2'(\lambda, a) = k_1 S_1'(\lambda, b) + k_2 S_2'(\lambda, b)$$

or

$$k_1 [S_1'(\lambda, a) - S_1'(\lambda, b)] + k_2 [S_2'(\lambda, a) - S_2'(\lambda, b)] = 0$$

These equations can always be solved for the non-vanishing k_1 and k_2 unless either S_1 or S_2 satisfy II. Hence in any case there is a solution corresponding to the boundary conditions I and II in case c_1 and c_2 both vanish.

Summing up we can say the function $V(\lambda, x)$

$$V(\lambda, x) = c_1 S_1(\lambda, x) + c_2 S_2(\lambda, x)$$

with the c_1 and c_2 defined by equations (16) vanishes identically in x for no other values of λ than the Sturm values'). If it vanishes identically for any of these, there is a non-identically vanishing solution of the boundary conditions I and II.

Let us now consider the above V , satisfying I identically in λ . For $\lambda = 0$ it is

$$V(0, x) = [p(b) - p(a)] \int_a^x \frac{p(a)}{p(x)} dx - p(b) \int_a^b \frac{p(a)}{p(x)} dx$$

So that

$$V(b) = - p(a) \int_a^b \frac{p(a)}{p(x)} dx$$

$$V(a) = - p(b) \int_a^b \frac{p(a)}{p(x)} dx$$

Also

$$V(0, x) = - p(b) \int_x^b \frac{p(a)}{p(x)} dx - p(a) \int_a^x \frac{p(a)}{p(x)} dx$$

which shows that $V(0, x)$ is always negative. For $V'(0, x)$ we have

$$V'(0, x) = [p(b) - p(a)] \frac{p(a)}{p(x)}$$

1) The Sturm solutions are those solutions for isolated values of λ such that $S(a, \lambda) = S(b, \lambda) = 0$

$$F(0) = V'(0, b) - V'(0, a) = [p(b) - p(a)] p(a) \left[\frac{1}{p(b)} - \frac{1}{p(a)} \right] \\ - \frac{[p(b) - p(a)]^2}{p(b)}$$

This is negative unless $p(b) = p(a)$, in which case $V'(0, x) = 0$ and $V(0, x) = \text{constant}$. Thus if $p(b) = p(a)$, $\lambda = 0$ is a characteristic value for the boundary conditions I and II and corresponds to a non-vanishing solution. There can be no non-vanishing solutions for other λ as the solutions considered are orthogonal.

Let us now consider the case for $p(b) \neq p(a)$. Here $F(0) < 0$.

From

$$p(x) \begin{vmatrix} V(\mu, x) & V(\lambda, x) \\ V'(\mu, x) & V'(\lambda, x) \end{vmatrix}_b^a = (\mu - \lambda) \int_a^b V(\mu, x) V(\lambda, x) r(x) dx$$

or since $p(b)V(\mu, b) = p(a)V(\mu, a)$ identically in μ

$$(17) \quad p(a) \begin{vmatrix} V(\mu, a) & V(\lambda, a) \\ F(\mu) & F(\lambda) \end{vmatrix} = (\mu - \lambda) \int_a^b V(\mu, x) V(\lambda, x) r(x) dx$$

we deduce for $\lambda = 0$

$$(18) \quad p(a) \begin{vmatrix} V(\mu, a) & V(0, a) \\ F(\mu) & F(0) \end{vmatrix} = \mu \int_a^b V(\mu, x) V(0, x) r(x) dx$$

which we will consider for negative μ . $V(\mu, x)$ is always negative, since if it vanished, the boundary conditions would compel it to vanish twice, and hence it would contain between its roots a root of $V(0, x)$. As $V(0, x)$ is always negative this is impossible. Hence we have

$$\mu(a)V(\mu, a)F(0) - p(a)V(0, a)F(\mu) < 0$$

and transposing the first term, $F(\mu) < 0$. Hence so long as μ is negative there is no solution satisfying the boundary conditions. Let us now consider the case of positive μ . Consider the class C , of values for which

$V(\mu x)$ has no roots and $\mathcal{G}_2$ for which it has roots. The cut defines a number $\bar{\lambda}$. Then $V(\bar{\lambda})$ has a root, say $\bar{x}$ because of continuity. This root cannot be in the interior of the interval, for all the roots are simple and $V(\bar{\lambda})$ is nowhere positive. Hence $V(\bar{\lambda})$ has a root at both extremities, and we have a Sturm solution. Hence, either $V(\bar{\lambda}) \equiv 0$, in which case as before remarked the conditions I and II can be satisfied, or $V(\bar{\lambda}) \not\equiv 0$.

If $V(\bar{\lambda}) \equiv 0$, the solution, $S(\bar{\lambda}, x)$ of the boundary problem I and II cannot have any interior roots, since if it had such roots it would have to have two because of periodicity and by continuity there would be a function $S(\bar{\lambda} - \epsilon, x)$ for small positive ϵ which would also have two interior roots. In the interval between the roots all solutions of the differential equation, for say $\bar{\lambda} - \frac{\epsilon}{2}$ would have a root, and hence in particular $V(\bar{\lambda} - \frac{\epsilon}{2}, x)$ contrary to the supposition that $V(\bar{\lambda}, x)$ had no solutions for $\lambda < \bar{\lambda}$. Thus if $V(\bar{\lambda}) \equiv 0$, the corresponding solution of the problem I and II does not vanish.

If $V(\bar{\lambda}) \not\equiv 0$, as it is nowhere positive in the interval and vanishes at the end points, $V'(\bar{\lambda}, b) > 0$, $V'(\bar{\lambda}, a) < 0$ and hence $F(\bar{\lambda}) > 0$, $F(\bar{\lambda}) < 0$, $F(\lambda)$ which is continuous has vanished between 0 and $\bar{\lambda}$, say at λ' . Then $V(\lambda', x)$ is a non-vanishing solution of the boundary problem I and II.

In any case therefore there is a non-vanishing solution of the boundary problem I and II. Definitely this solution is negative.

We shall now show the existence of a second solution. By (17) with $\lambda = \lambda'$,

$p(a)$

$$p(a) \begin{vmatrix} V(\mu a) & V(\lambda'_1 a) \\ F(\mu) & 0 \end{vmatrix} = (\mu - \lambda'_1) \int_a^b V(\mu x) V(\lambda'_1 x) r(x) dx$$

and hence so long as the integral (which for $\mu = \lambda'_1$ is positive) keeps its sign, and $\mu > \lambda'_1$,

$$p(a) V(\lambda'_1 a) F(\mu) < 0 \quad \text{and as} \quad V(\lambda'_1 a) < 0 \\ F(\mu) > 0$$

Consider the function $p(a) V(\mu a) = p(b) V(\mu b) = G(\mu)$.

I wish first to show that it changes sign an infinite number of times and then that between any two of its sign-changes $F(\mu)$ has a root.

Let us suppose that $\bar{\mu}$ is a value greater than the last value of μ for which $V(\mu a)$ is zero. To fix ideas, suppose $V(\mu a) > 0$ for $\mu > \bar{\mu}$. Let $2r$ be the number of roots of $V(\bar{\mu} x)$. Then we can find a $\bar{\bar{\mu}}$ such that $V(\bar{\bar{\mu}} x)$ has more than $2r$ roots. Divide all numbers into two classes; G_1 containing values of μ such that the corresponding $V(\mu x)$ has no more than $2r$ roots, and G_2 such that $V(\mu x)$ has more than $2r$ roots. Call the number defined μ^* , $\bar{\mu} < \mu^* < \bar{\bar{\mu}}$. If the sequence $\epsilon_1, \epsilon_2, \dots$ approximates 0, the functions $V(\mu^* + \epsilon_1), V(\mu^* + \epsilon_2), \dots$, each has at least $2(r+1)$ roots. The system of the first roots of each function will have at least one limit point r_1 , and so will the second, r_2 , and so on to r_{2n+2} . These all lie in the interior, because of continuity, since $V(\mu a) > 0$ and $V(\mu b) > 0$. But by continuity, they are all roots of $V(\mu^* x)$. Hence $V(\mu^* x)$ has $2r+2$ or more roots in the interior of the interval, while $V(\mu^* - \epsilon, x)$ has no more than $2r$ roots. $V(\mu^* x)$ must therefore either

(1) alternate in signs at each root, or

(2) have the same signs in two successive intervals between roots.

The first is impossible because there would then be a value δ such that there would be a value of x in each interval for which $|V(\mu^* x)| > \delta$. In this case we could choose ϵ so small that for all x

$$|V(\mu^{*- \epsilon}, x) - V(\mu^* x)| < \frac{\delta}{2}, \quad \text{and hence } V(\mu^{*- \epsilon}, x)$$

change signs in each root interval of $V(\mu^* x)$, and hence has $2r+2$ roots which is contrary to the definition of μ^* .

The second is impossible, as it would give a double root, which would contradict $V(\mu^* b) > 0$. Hence

$$p(a) V(\mu a) = p(b) V(\mu a) = G(\mu)$$

cannot be constantly different from zero from any point a it must therefore vanish infinitely often.

Now by (17)

$$\begin{vmatrix} G(\mu) & G(\lambda) \\ F(\mu) & F(\lambda) \end{vmatrix} = (\mu - \lambda) \int_a^b V(\mu x) r(x) \, dx \, V(\lambda, x) \, dx$$

Let λ be one of the roots of G . Then

$$G(\mu) F(\lambda) = (\mu - \lambda) \int_a^b V(\mu x) V(\lambda, x) r(x) \, dx$$

If $V(\lambda, x) \equiv 0$, there is a solution of the boundary problem I, II. (Compare Remark) Otherwise the integral is positive for small $(\mu - \lambda)$. Hence $G(\mu)F(\lambda)$ changes signs as μ passes through λ ; hence $F'(\lambda)$ is different from zero, and if

$$(1) \quad G(\mu) \text{ changes from } - \text{ to } +, \quad F(\lambda) > 0$$

$$(2) \quad G(\mu) \text{ changes from } + \text{ to } -, \quad F(\lambda) < 0$$

Thus if we have two successive roots of $G(\mu)$ (and there are an infinite number), either there is a solution of the boundary problem I, II for one of the two roots, or else at

one $G(\mu)$ changes from + to - and at the other from - to +, and hence $F(\lambda)$ has opposite signs, and must have vanished in between.

In every other interval $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots$, including the endpoints, there must be a root λ' . ~~Hence each solution has exactly two more solutions than the preceding.~~

Thus there are in any case an infinite number of solutions of the boundary problem I, II. The characteristic values $\lambda', \lambda'', \lambda''', \dots$, lie either in the interior or at an endpoint of every root interval for the Sturm case $\lambda, \lambda_2, \dots$.

We have thus seen that the differential equation (1) has a solution $V(\lambda, x)$ which for infinitely many values of λ satisfies the boundary conditions. We have seen also that there is no solution satisfying the boundary conditions for negative λ . We can also show that λ cannot be imaginary. For, assume that $\lambda = a + ib$, where a , and b , are real. Then $V = P + iQ$. Substituting these values in equation (1) we have

$$(19) \quad \frac{d}{dx} \left[p \frac{d(P+iQ)}{dx} \right] + (a + ib) r(x) (P+iQ) = 0$$

From this we get,

$$(20) \quad \frac{d}{dx} \left[p \frac{dP}{dx} \right] + r(x) (aP - bQ) = 0$$

$$(21) \quad \frac{d}{dx} \left[p \frac{dQ}{dx} \right] + r(x) (bP + aQ) = 0$$

Multiplying (20) by Q and (21) by P and subtracting we have

$$(22) \quad Q \frac{d}{dx} \left[p \frac{dP}{dx} \right] - P \frac{d}{dx} \left[p \frac{dQ}{dx} \right] = b r(x) (P^2 + Q^2)$$

Taking the integral between a and b we have

$$(23) \quad \int_a^b \left[Q (pP')' - P (pQ')' \right] dx = b \int_a^b r(x) (P^2 + Q^2) dx$$

Integrating by parts we have

$$(24) \quad p \left[QP' - Q'P \right]_a^b = b \int_a^b r(x) (P^2 + Q^2) dx$$

But by our original boundary conditions the left hand side of equation (24) is zero. Hence as the quantities under the integral are both positive, (24) can be satisfied only when b_1 is zero, which is when λ is real.

Let us now build up a linear combination of the functions V of the form $U = A_0 V_0 + A_1 V_1 + \dots + A_n V_n$ where V_j ($j \leq n$) is a solution of our differential equation (1) satisfying the boundary conditions and such that V_0 does not vanish in our interval, V_1 vanishes twice, V_2 four times and so on. V_n then vanishes $2n$ times. The A 's are any constants. Let us now inquire the number of times U can vanish in this interval, for any choice of the A 's. It can of course vanish $2n$ times by choosing all the A 's except A_n as zero.

I wish to show that U cannot vanish for more than $2n$ times for any choice of coefficients. To do this substitute V_0 and V_n in our differential equation (1) and we have.

$$(25) \quad \frac{d}{dx} \left[p \frac{dV_n}{dx} \right] + \lambda_n r(x) V_n = 0$$

$$(26) \quad \frac{d}{dx} \left[p \frac{dV_0}{dx} \right] + \lambda_0 r(x) V_0 = 0$$

From these two equations we can obtain the equation

(Compare 24)

$$\begin{aligned} (\lambda_0 - \lambda_n) \int_a^x r(x) V_0 V_n dx &= p \left[V_0 \frac{dV_n}{dx} - V_n \frac{dV_0}{dx} \right]_a^x \\ &= p V_0^2 \frac{d}{dx} \left[\frac{V_n}{V_0} \right] - p V_n^2 \frac{d}{dx} \left[\frac{V_0}{V_n} \right]_a^x \\ &= p V^2 \frac{d}{dx} \left[\frac{V_n}{V_0} \right] - C \quad (\text{constant}) \end{aligned}$$

Now choose m , $0 \leq m \leq n$, and multiply by the coefficients $A_0, A_1, \dots, A_n$ and adding, after setting

$$(\lambda_0 - \lambda_m) A_m = a_m, \quad a_0 = 0$$

we have

$$\begin{aligned}
 (27) \quad \int_a^b r(x) V_0 (a_0 V_0 + a_1 V_1 + \dots + a_n V_n) dx \\
 = p V_0^2 \frac{d}{dx} \left[\frac{U}{V_0} \right] - C_1 \\
 = p V_0^2 \frac{d\psi}{dx} - C_1, \quad \frac{U}{V_0} = \psi
 \end{aligned}$$

Suppose that $U = 0$ has μ zeros in the interval a to b .

($\mu = 2k$ from our boundary conditions) Then $\frac{U}{V_0}$ has also μ

zeros in that interval, since V does not vanish. If

$U = 0$ has μ zeros in the interval, $\frac{dU}{dx}$ has also μ or more

zeros from Rolle's Theorem and the boundary conditions, viz.

$$U(a) = k U(b), \quad U'(a) = U'(b).$$

Also ψ has μ zeros and as it satisfies the boundary

conditions, $\frac{d\psi}{dx}$ has μ or more zeros in the interval. Now

differentiating (27) we have

$$(28) \quad r(x) V_0 (a_0 V_0 + a_1 V_1 + \dots + a_n V_n) = \frac{d}{dx} \left[p V_0^2 \frac{d\psi}{dx} \right] +$$

$p V_0^2 \frac{d\psi}{dx}$, as we have just seen, has μ or more zeros in our

interval. Hence by Rolle's Theorem, $\frac{d}{dx} \left[p V_0^2 \frac{d\psi}{dx} \right]$ has $\mu - 1$

or more zeros in that interval.

But $\frac{d}{dx} \left[p V_0^2 \frac{d\psi}{dx} \right]$ equals $r(x) V_0 (a_0 V_0 + a_1 V_1 + \dots + a_n V_n)$ which is the product of two factors, one of which does not

vanish and the other of which satisfies our original boundary

conditions. Hence the factor $a_0 V_0 + a_1 V_1 + \dots + a_n V_n$

must have an even number of roots, and as it has $\mu - 1$ or

more roots, it must have μ or more roots in the given interval.

Hence we see that if

$$U = A_0 V_0 + A_1 V_1 + \dots + A_n V_n = 0$$

has μ zeros in the interval a to b that

(29) $(\lambda_0 - \lambda_0)A_0V_0 + (\lambda_1 - \lambda_0)A_1V_1 + \dots + (\lambda_n - \lambda_0)A_nV_n = 0$
 has μ or more zeros in that interval.

Repeating the same argument i times we see that
 (30) $(\lambda_0 - \lambda_0)^i A_0V_0 + (\lambda_1 - \lambda_0)^i A_1V_1 + \dots + (\lambda_n - \lambda_0)^i A_nV_n = 0$
 has μ or more zeros in the same interval.

By reasoning exactly analogous to that used in the Bessel Function proofs, Chapter IV, i. e., by writing (30) in the form

$$(31) \left(\frac{\lambda_0 - \lambda_0}{\lambda_n - \lambda_0} \right)^i A_0V_0 + \dots + \left(\frac{\lambda_{n-1} - \lambda_0}{\lambda_n - \lambda_0} \right)^i A_{n-1}V_{n-1} + A_nV_n = 0,$$

Since in the first part of the proof we have shown that $\lambda_n > \lambda_{n-1}$, we can show that (31) cannot have more than $2n$ zeros in our given interval. Since (31) has the same number of or more zeros than U , U cannot have more than $2n$ zeros in our given interval. We have thus proven

Theorem I. If $U(x)$ is a linear combination of the functions, $V_0, V_1, V_2, V_3, \dots, V_n$ satisfying the differential equation

$$\frac{d}{dx} \left[p \frac{dV}{dx} \right] + \lambda r(x) V = 0$$

and the boundary conditions as defined on page 30, in which the subscripts taken in order correspond to the different values of λ taken in order of magnitude, then for any choice of coefficients, $U = 0$ cannot vanish a greater number of times than V_n in the interval of orthogonality.

Theorem II. If $U = 0$ has $2n$ zeros in the interval (a, b) , it changes sign at each.

In the proof of Theorem I, we showed that two differentiations of U does not decrease its number of zeros in the interval of orthogonality. If $U = 0$ has one double root, it must have two double roots to satisfy the conditions of periodicity

and an even number of roots. Two differentiations will then throw a function which has at least $2n+1$ roots. But on differentiating twice in the manner used in the proof of Theorem I, we get a function $U''(x)$ which is of the same form as $U(x)$ and by Theorem I cannot have more than $2n$ zeros. Hence our theorem is proven.

If we now consider the special case of our boundary conditions, namely

$$V(a) = k V(b) = 0$$

$$V'(a) = V'(b)$$

we see that V satisfying these conditions must change signs an odd number of times in our interval of orthogonality, but that otherwise our statements will stand as presented.

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