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STUDY OF THE CONVERGENCE OF SERIES IN
CERTAIN ORTHOGONAL FUNCTIONS

by

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The thesis of Mrs. Ruben Axel Green
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Introduction.

The question of the accuracy with which a given function, say $f(x)$, can be expressed by means of a known function or in a series of known functions is a very important one. Such developements are continually arising in problems in applied mathematics such as in mathematical physics and astronomy. The value of any developement depends largely on the accuracy with which one can judge its error at any point.

Reviewing hastily some of the work that has been done along this line, we see that Weierstrass¹ was the first to give the proof for the following theorem:

If in the interval $a \leq x \leq b$ $f(x)$ is a single valued, defined, continuous and real function of the real variable x , and if δ is an arbitrarily chosen positive quantity, then there is an integral rational function $P(x)$ such that throughout the interval $a \leq x \leq b$, $|f(x) - P(x)| < \delta$.

It is clear that in general the degree of the polynomial $P(x)$ is dependent upon δ . As δ is chosen smaller and smaller the degree of $P(x)$ will in general increase. Lesbesgue² has considered the question of what the relation is between δ and the degree of the polynomial of lowest degree which satisfies the above inequality throughout the interval for any given function, $f(x)$.

For any polynomial $P(x)$ of the n th degree, the expression $|f(x) - P(x)|$ has an upper limit in the interval $a \leq x \leq b$. This upper limit is a function of the coefficients of $P(x)$. If we consider several polynomials of the n^{th} degree or lower, this function has a definite lower limit which for a given a , b , and

$f(x)$ depends only on n . Let us call this lower limit $\varphi(n)$.

It is in general positive. It becomes zero only when $f(x)$ is itself a polynomial of the n^{th} degree or lower. The theorem by Weierstrass states that for a continuous function $f(x)$, the $\lim_{n \rightarrow \infty} \varphi(n) = 0$. Introducing the notation $\varphi(n) = O(\psi(n))$ and $\varphi(n) = o(\psi(n))$, by which we shall mean that for n sufficiently large $|\varphi(n)| < A\psi(n)$, where A is a constant independent of n , or that $\lim_{n \rightarrow \infty} \varphi(n)/\psi(n) = 0$ respectively, we can state Weierstrass' result by saying that $\varphi(n) = o(1)$ for a continuous function. Lesbesgue² gives a result which says that in case a function satisfies a Lipschitz condition, i.e.

$$|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|,$$

where λ is a constant, $\varphi(n) = O(\sqrt{\frac{\log n}{n}})$.

Assuming practically the same conditions de la Vallee Poussin³ finds that $\varphi(n) = O(\frac{1}{\sqrt{n}})$. Later⁴ he finds that when $f(x)$ possesses a derivative of limited variation, $\varphi(n) = O(\frac{1}{n})$. Lesbesgue⁵ has considered the case where $f(x)$ satisfies a Lipschitz-Dini condition, i.e. $\lim_{\delta \rightarrow 0} \omega(\delta) \log \delta = 0$ where $\omega(\delta)$ represents the upper limit of the difference between the maximum and minimum values assumed by $f(x)$ in an interval of length δ . He finds that in such a case $\varphi(n) = o(\frac{1}{\log n})$.

More in line with this paper is the work done on approximations by means of trigonometric sums. There is a very close relation between trigonometric and polynomial approximations. From a result by Lesbesgue⁶ about the convergence of a Fourier's series one can get, assuming the Lipschitz condition, that $\varphi(n) = O(\frac{\log n}{n})$. Jackson⁷ gives a somewhat better result finding that $\varphi(n) = O(\frac{1}{n})$. He also shows that there exist func-

tions satisfying a Lipschitz condition for which $\varphi(n)$ is not $o(\frac{1}{n})$. In the case of trigonometric approximations we will write $\tau(n)$ in place of $\varphi(n)$ to distinguish the two cases. Lebesgue⁵ has found that in case a function satisfies a Lipschitz condition $\tau(n) = O(\frac{1}{n})$ and later⁶ he finds that $\tau(n) = O(\frac{\log n}{n})$. Jackson's result is that $\tau(n) = O(\frac{1}{n})$. In a later paper⁹ he proves the following theorem:

"Theorem I. There exists an absolute numerical constant K having the following property: If $f(x)$ is a real function of the real variable x , of period 2π , which everywhere satisfies the Lipschitz condition $|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|$, λ being a constant, then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the n^{th} order at the most, such that for all values of x $|f(x) - T_n(x)| \leq \frac{K\lambda}{n}$." He finds that $\frac{\pi}{2} < K \leq 3$. A generalization of this theorem is his Theorem III of the same paper. "

"For each positive integral value of k there exists a constant K_k having the following property: If $f(x)$ is a function of period 2π possessing a $(k-1)^{\text{th}}$ derivative which everywhere satisfies the Lipschitz condition $|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|$

λ being a constant, then there exist for every positive integral value of n a trigonometric sum $T_n(x)$, of the n^{th} order at most, such that for all values of x

$$|f(x) - T_n(x)| \leq \frac{K_k \lambda}{n^k} . "$$

Similar theorems are given for polynomial approximations.

Jackson has also given results for the degree of convergence of certain series. For instance in the case of a Fourier's series he gets, in the paper just quoted, the follow-

ing theorem:

"Theorem X. If $f(x)$ is a function of period 2π satisfying the Lipschitz condition, it is represented by the partial sum of its Fourier's series to terms of the n^{th} order, provided $n \geq 5$, with an error not exceeding $6\lambda \frac{\log n}{n}$."

In a later paper⁸ he has a generalization of this result, viz.,

"Theorem X. If $f(x)$ is a function of period 2π possessing a $(k - 1)^{\text{th}}$ derivative which everywhere satisfies the Lipschitz condition $|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leq \lambda |x_2 - x_1|$, where λ is a constant, then $f(x)$ is everywhere approximately represented by the partial sum of its Fourier's series to terms of the n^{th} order, $n \geq 5$, with an error not exceeding $36\lambda \frac{\log n}{n^k}$. If k is odd, the factor 36 may be replaced by 12."

For the case of the development of a function in series of Legendre's polynomials Jackson¹⁰ has also given a result. Denoting by $r_n(x)$ the difference $\sum_{m=0}^n a_m P_m(x) - f(x)$, where $P_m(x)$ are the polynomials of Legendre, and $a_m = \frac{2m+1}{2} \int_{-1}^1 f(x) P_m(x) dx$ he gets:

"Theorem II. If $f(x)$ is a function possessing a $(k - 1)^{\text{th}}$ derivative which satisfies a Lipschitz condition throughout the closed interval $(-1, 1)$, then, for this function $f(x)$,

$$|r_n(x)| = O\left(\frac{\log n}{n^k}\right).$$

Gronwall¹¹ gives a somewhat more general result in discussing the degree of convergence of a Laplace's series of which a Legendre's series is but a special case.

In this present paper we will develop some theorems concerning the degree of convergence of certain series, in

particular a Fourier's series, a Legendre's series, and a series of Bessel's functions, getting results which are indeed very much similar to the ones already mentioned but which are derived in a much simpler manner. Before proceeding directly to the proof of these theorems, however, we will give some theorems showing the conditions under which the Fourier coefficients in the developement of a function will converge to zero.

Chapter I.

1. Total variation finite. Let $f(x)$ be a real function of the real variable x and let us impose the condition that the total variation of $f(x)$ shall remain finite in an interval from $x = 0$ to $x = 2\pi$. This condition forces $f(x)$ to be bounded in the interval under consideration i.e., it is possible to find a positive quantity N such that $|f(x)| \leq N$ when $0 \leq x \leq 2\pi$.

We have then the Fourier developement:

$$(1) \quad f(x) = \frac{b_0}{2} + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \dots \\ + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots$$

$$\text{where} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx, \quad \text{and} \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx.$$

Since the absolute value of the $\sin kx$ and $\cos kx$ can never exceed unity we can study the degree of convergence of this developement by studying the degree of convergence of the coefficients, b_k and a_k . These coefficients may indeed approach zero without the series converging but we shall study this convergence to zero of the coefficients before imposing further conditions.

We can write:

$$(2) \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \left\{ \int_0^{\frac{\pi}{2k}} f(x) \cos kx \, dx + \int_{\frac{\pi}{2k}}^{\frac{3\pi}{2k}} f(x) \cos kx \, dx \right. \\ \left. + \int_{\frac{3\pi}{2k}}^{\frac{5\pi}{2k}} f(x) \cos kx \, dx + \dots + \int_{\frac{(2k-1)\pi}{2k}}^{\frac{(4k-1)\pi}{2k}} f(x) \cos kx \, dx + \int_{\frac{(4k-1)\pi}{2k}}^{2\pi} f(x) \cos kx \, dx \right\}$$

And similarly:

$$(3) \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \left\{ \int_0^{\frac{\pi}{2k}} f(x) \sin kx \, dx + \int_{\frac{\pi}{2k}}^{\frac{\pi}{k}} f(x) \sin kx \, dx \right. \\ \left. + \int_{\frac{\pi}{k}}^{\frac{3\pi}{2k}} f(x) \sin kx \, dx + \dots + \int_{\frac{(2k-1)\pi}{2k}}^{\frac{(4k-1)\pi}{2k}} f(x) \sin kx \, dx + \int_{\frac{(4k-1)\pi}{2k}}^{2\pi} f(x) \sin kx \, dx \right\}.$$

By the law of the mean we know that

$$\int_a^b f(x) \phi(x) \, dx = \mu \int_a^b \phi(x) \, dx,$$

where $m < \mu < M$, m and M being the minimum and maximum values assumed by the function, $f(x)$, in the interval $a \leq x \leq b$. There is no restriction on $f(x)$ other than that it be bounded.

Also we have that

$$\int_{\frac{(4k-m)\pi}{2k}}^{\frac{(4k-(m-1))\pi}{2k}} \cos kx \, dx = + \frac{1}{k} \text{ or } - \frac{1}{k}; \text{ and } \int_{\frac{(4k-m)\pi}{2k}}^{\frac{(4k-(m-1))\pi}{2k}} \sin kx \, dx = + \frac{1}{k} \text{ or } - \frac{1}{k},$$

where $n = 1, 2, 3, \dots, 4k$.

We will then have, after substitution in (2),

$$(4) \quad b_k = \frac{1}{\pi} \left\{ \frac{1}{k} (\mu_1 - \mu_2 - \mu_3 + \mu_4 + \mu_5 - \dots - \mu_{4k-2} - \mu_{4k-1} + \mu_{4k}) \right\},$$

where $\mu_1, \mu_2, \mu_3, \mu_4, \dots$ etc., lie between the minimum and maximum values assumed by $f(x)$ in the intervals

$$0 \leq x \leq \frac{\pi}{2k}, \quad \frac{\pi}{2k} \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{3\pi}{2k}, \quad \dots \text{ respectively.}$$

$$\text{Then } |b_k| \leq \left\{ \frac{1}{\pi} \left[\frac{1}{k} (|\mu_1 - \mu_2| + |\mu_4 - \mu_3| + |\mu_5 - \mu_6| + \dots + |\mu_{4k} - \mu_{4k-1}|) \right] \right\} \quad (5)$$

Similarly after substitution in (3) we have,

$$(6) \quad a_k = \frac{1}{\pi} \left\{ \frac{1}{k} (\mu_0 + \mu_1 - \mu_2 - \mu_3 + \mu_4 + \mu_5 - \dots + \mu_{4k-3} - \mu_{4k-2} - \mu_{4k-1}) \right\},$$

where $\mu_0, \mu_1, \mu_2, \mu_3, \dots$ etc., lie between the minimum and maximum values assumed by $f(x)$ in the intervals

$$0 \leq x \leq \frac{\pi}{2k}, \quad \frac{\pi}{2k} \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{3\pi}{2k}, \quad \dots \text{ respectively.}$$

$$\text{Then } |a_k| \leq \frac{1}{\pi} \left\{ \frac{1}{k} [|\mu_0 - \mu_{4k-1}| + |\mu_1 - \mu_2| + |\mu_4 - \mu_3| + \dots + |\mu_{4k-3} - \mu_{4k-2}|] \right\} \quad (7)$$

But in equation (5) each of the quantities $|\mu_1 - \mu_2|, |\mu_4 - \mu_3|, |\mu_5 - \mu_6|, \dots$ is evidently less than the total variation of $f(x)$ in the intervals

$$0 \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{2\pi}{k}, \quad \frac{2\pi}{k} \leq x \leq \frac{3\pi}{k}, \quad \dots \text{ respectively.}$$

For denoting the total variation by ω , we have $\omega_1 \geq M_1 - m_1$, where M_1 and m_1 are the maximum and minimum values of $f(x)$ in the interval $0 \leq x \leq \pi/k$. But $\mu_1 > m_1$ and $\mu_2 < M_1$, since the minimum of a function in an interval is always less than or at the most equal to the minimum of the function in a subinterval and the maximum of a function in an interval is always greater than or at the most equal to the maximum of the function in a subinterval. Then after substitution we will have $\omega_1 \geq \mu_2 - \mu_1$,

or we can write $\omega_1 \geq |\mu_1 - \mu_2|$. Similarly $\omega_2 \geq |\mu_4 - \mu_3|$, $\omega_3 \geq |\mu_5 - \mu_6|, \dots$

where $\omega_1, \omega_2, \omega_3, \dots$ represent the total variation of $f(x)$ in the intervals

$$0 \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{2\pi}{k}, \quad \frac{2\pi}{k} \leq x \leq \frac{3\pi}{k}, \quad \dots \text{ respectively.}$$

But the total variation, Ω , of the function $f(x)$ in

the interval $0 \leq x \leq 2\pi$ is

$$\Omega = \omega_1 + \omega_2 + \omega_3 + \dots + \omega_{2k}.$$

Thus we have

$$(8) \quad |b_k| \leq \frac{1}{\pi} \cdot \frac{1}{k} \Omega :$$

Following the same method for equation (7) we get

$$\omega_0 \geq |\mu_0 - \mu_{4k-1}|, \quad \omega_1 \geq |\mu_1 - \mu_2|, \quad \omega_2 \geq |\mu_4 - \mu_3|, \dots$$

where $\omega_1, \omega_2, \omega_3, \dots$ represent the total variation of $f(x)$ in the intervals,

$$\frac{\pi}{2k} \leq x \leq \frac{3\pi}{2k}, \quad \frac{3\pi}{2k} \leq x \leq \frac{5\pi}{2k}, \quad \frac{5\pi}{2k} \leq x \leq \frac{7\pi}{2k}, \dots \text{ respectively}$$

and ω_0 represents the total variation in the two intervals

$$0 \leq x \leq \frac{\pi}{2k} \quad \text{and} \quad \frac{(4k-1)\pi}{2k} \leq x \leq 2\pi.$$

Then we have as before:

$$(9) \quad |a_k| \leq \frac{1}{\pi} \cdot \frac{1}{k} \cdot \Omega.$$

Thus if Ω remains finite in the interval $0 \leq x \leq 2\pi$ the quantities $|b_k|$ and $|a_k|$ will be defined as well as finite and the

$$\lim_{k \rightarrow \infty} |b_k| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} |a_k| = 0.$$

Hence we can conclude:

If the total variation of a function remains finite the coefficients of its Fourier developement converge to zero.

2. Integrable in the Riemann sense. Now let us assume that the function, $f(x)$, is integrable in the Riemann sense.

Then
$$\int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x, \text{ where } m_i \leq f(\xi_i) \leq M_i.$$

We could have written equation (4) in the following form, considering $1/k = \Delta x$:

$$b_k = \frac{1}{\pi} \left\{ (\mu_1 \Delta x + \mu_4 \Delta x + \mu_5 \Delta x + \dots + \mu_{4k} \Delta x) - (\mu_2 \Delta x + \mu_3 \Delta x + \mu_6 \Delta x + \dots + \mu_{4k-1} \Delta x) \right\}.$$

And we have $m_i \leq \mu_i \leq M_i$.

$$(10) \quad \therefore \lim_{k \rightarrow \infty} b_k = \frac{1}{\pi} \left\{ \frac{1}{2} \int_0^{2\pi} f(x) dx - \frac{1}{2} \int_0^{2\pi} f(x) dx \right\} = 0.$$

Similarly equation (6) can be written in the form :

$$a_k = \frac{1}{\pi} \left\{ (\mu_0 \Delta x + \mu_1 \Delta x + \mu_4 \Delta x + \mu_5 \Delta x + \dots + \mu_{4k-3} \Delta x) - (\mu_2 \Delta x + \mu_3 \Delta x + \mu_6 \Delta x + \dots + \mu_{4k-1} \Delta x) \right\}.$$

$$(11) \quad \lim_{k \rightarrow \infty} a_k = \frac{1}{\pi} \left\{ \frac{1}{2} \int_0^{2\pi} f(x) dx - \frac{1}{2} \int_0^{2\pi} f(x) dx \right\} = 0.$$

Thus we can conclude:

If a function is integrable in the Riemann sense the coefficients of its Fourier developement converge to zero.

3. Generalized Lipschitz condition. Now let us assume that the function, $f(x)$, satisfies the following condition:

$$|\Delta f| < A |\Delta x|^a$$

where A is any constant and a satisfies the condition $0 < a < 1$.

This restriction on $f(x)$ is not nearly as great as that imposed by the so called Lipschitz condition which is that

$$|\Delta f| \leq A |\Delta x| .$$

Writing Δx as $\frac{1}{k}$ we will then have

$$|\Delta f| < A \left| \frac{1}{k^a} \right| .$$

From (5) we have, setting $|\mu_1 - \mu_2| \leq |\Delta f_1|$, $|\mu_4 - \mu_3| \leq |\Delta f_2|$

$$|kb_k| \leq \frac{1}{\pi} \left\{ \sum_i^k |\Delta f_i| \right\} .$$

Then $|k b_k| \leq \frac{kA}{\pi} \left| \frac{1}{k^a} \right| = \frac{C_1}{k^{a-1}}$, where C_1 is a constant.

Then $|b_k| = \frac{C_1}{k^a}$ and $\lim_{k \rightarrow \infty} |b_k| = 0$. (12)

Similarly: $|a_k| = \frac{C_2}{k^a}$ and $\lim_{k \rightarrow \infty} |a_k| = 0$. (13)

Hence:

If a function satisfies a generalized Lipschitz condition the coefficients of its Fourier developement converge to zero.

Chapter II.

1. Remainder in a Fourier's series. We will now proceed to the consideration of the accuracy of a Fourier developement taken for a finite number of terms and will develop an expression for the remainder of a Fourier's series after n terms.

Let us consider the case where the function of x , $f(x)$, is a periodic function of period 2π , having an infinite number of derivatives of the same period.

The Fourier developement of $f(x)$ will then be of the form;

$$(1) \quad \begin{aligned} f(x) = & \frac{b_0}{2} + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \dots \\ & + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots \end{aligned}$$

We will investigate the coefficients:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx .$$

Integrating the first by parts we have:

$$a_k = -\frac{1}{k\pi} \left[f(x) \cos kx \right]_0^{2\pi} + \frac{1}{k\pi} \int_0^{2\pi} f'(x) \cos kx \, dx .$$

k being an integer, we have $\left[f(x) \cos kx \right]_0^{2\pi} = 0$.

Repeating the integration by parts we get:

$$a_k = \frac{1}{\pi k^2} \left[f'(x) \sin kx \right]_0^{2\pi} - \frac{1}{\pi k^2} \int_0^{2\pi} f''(x) \sin kx \, dx .$$

Again: $\left[f'(x) \sin kx \right]_0^{2\pi} = 0$.

Repeating the integration again:

$$a_k = \frac{1}{\pi k^3} \left[f''(x) \cos kx \right]_0^{2\pi} - \frac{1}{\pi k^3} \int_0^{2\pi} f'''(x) \cos kx \, dx .$$

And again:
$$a_k = -\frac{1}{\pi k^4} \left[f'''(x) \cos kx \right]_0^{2\pi} - \frac{1}{\pi k^4} \int_0^{2\pi} f^{(4)}(x) \sin kx \, dx .$$

This process could be kept up indefinitely since

$$\left[f''(x) \cos kx \right]_0^{2\pi}, \quad \left[f'''(x) \cos kx \right]_0^{2\pi}, \quad \text{and similar factors}$$

all drop out, being equal to zero, and since $f(x)$ has an infinite number of derivatives by hypothesis.

Carrying through the same process for b_k we get:

$$b_k = \frac{1}{\pi k} \left[f(x) \sin kx \right]_0^{2\pi} - \frac{1}{\pi k} \int_0^{2\pi} f'(x) \sin kx \, dx .$$

But
$$\left[f(x) \sin kx \right]_0^{2\pi} = 0 .$$

Integrating again:

$$b_k = \frac{1}{\pi k^2} \left[f'(x) \cos kx \right]_0^{2\pi} - \frac{1}{\pi k^2} \int_0^{2\pi} f''(x) \cos kx \, dx .$$

Repeating again we have since
$$\left[f'(x) \cos kx \right]_0^{2\pi} = 0 ,$$

$$b_k = -\frac{1}{\pi k^3} \left[f''(x) \sin kx \right]_0^{2\pi} + \frac{1}{\pi k^3} \int_0^{2\pi} f'''(x) \sin kx \, dx .$$

And again:

$$b_k = -\frac{1}{\pi k^4} \left[f'''(x) \cos kx \right]_0^{2\pi} + \frac{1}{\pi k^4} \int_0^{2\pi} f^{(4)}(x) \cos kx \, dx ,$$

since
$$\left[f''(x) \sin kx \right]_0^{2\pi} \quad \text{and} \quad \left[f'''(x) \cos kx \right]_0^{2\pi} = 0 .$$

Then the absolute value of the n^{th} Fourier coefficient will be:

$$|a_n| \leq \frac{1}{\pi n^k} \int_0^{2\pi} |f^k(x)| |\cos nx| dx, \text{ when } k \text{ is odd or}$$

(2)

$$|a_n| \leq \frac{1}{\pi n^k} \int_0^{2\pi} |f^k(x)| |\sin nx| dx, \text{ when } k \text{ is even.}$$

Similarly:

$$|b_n| \leq \frac{1}{\pi n^k} \int_0^{2\pi} |f^k(x)| |\sin nx| dx, \text{ when } k \text{ is odd or}$$

(3)

$$|b_n| \leq \frac{1}{\pi n^k} \int_0^{2\pi} |f^k(x)| |\cos nx| dx, \text{ when } k \text{ is even.}$$

By the law of the mean we have:

$$|a_n| \leq \frac{1}{\pi n^k} |f^k(\bar{x})| \int_0^{2\pi} |\cos nx| dx, \text{ where } 0 < \bar{x} < 2\pi.$$

But
$$\int_0^{2\pi} |\cos nx| dx = \frac{1}{n} \cdot 4n = 4.$$

$$(4) \therefore |a_n| = \frac{4}{\pi n^k} |f^k(\bar{x})|. \text{ And similarly } |b_n| = \frac{4}{\pi n^k} |f^k(\bar{x})|.$$

Then in order to have a good representation of $f(x)$ by its Fourier development the remainder after n terms of the series must approach zero. From (1), since $|\sin kx| \leq 1$ and $|\cos kx| \leq 1$, we can write the remainder:

$$R_n \leq |a_n| + |b_n| + |a_{n-1}| + |b_{n-1}| + \dots$$

Substituting the values determined for $|a_n|$, $|b_n|$, etc.,

$$(5) \quad R_n \leq |f^k(\bar{x})| \frac{4}{\pi} \left\{ \frac{1}{n^k} + \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+1)^k} + \dots \right\}.$$

$$R_n \leq |f^k(\bar{x})| \frac{8}{\pi} \left\{ \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \dots \right\}.$$

$$\text{Let } F(n,k) = \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \dots \quad (6)$$

Then

$$(7) \quad R_n(x) = \theta \left[\frac{8}{\pi} |f^k(\bar{x})| F(n,k) \right], \text{ where } 0 < \theta < 1, \text{ or}$$

$$(8) \quad R_n(x) \leq \left[\frac{8}{\pi} M_k F(n,k) \right],$$

where M_k is the maximum value of $|f^k(\bar{x})|$.

We can then judge the character of our remainder by computing $|f^k(\bar{x})|$ for a number of values of k and noting what choice of k will make $R_n(x)$ the smallest. For this purpose we will tabulate $F(n,k)$ for a number of values of n and k .

2. Range of k^{th} derivative of $f(x)$. We shall at this point prove a rather striking theorem:

If a function of x , $f(x)$, is a periodic function possessing an infinite number of derivatives having the same period as $f(x)$, and if $f(x)$ cannot be developed in a finite Fourier's series, then after a certain point the maximum value of the k^{th} derivative of $f(x)$ will increase faster than an exponential function of the form n^k .

$$\text{Let } M_k = \text{maximum value of } |f^k(x)|.$$

Then from equations (4):

$$|a_n| \leq \frac{4}{\pi n^k} M_k \quad \text{and} \quad |b_n| \leq \frac{4}{\pi n^k} M_k.$$

Rewriting:

$$|a_n| \leq \frac{4}{\pi} \frac{M_k}{n^k} \quad \text{and} \quad |b_n| \leq \frac{4}{\pi} \frac{M_k}{n^k}.$$

We see at once that if $|a_n| = 0$ or $|b_n| = 0$ we would know nothing definite about the conduct of M_k . It might increase as fast as n^k or it might not, just as the case happened

to be.

But our hypothesis was that $|a_n| > 0$ or $|b_n| > 0$, or both, for an infinite number of values of n . This will then force M_k to increase faster than n^k increases for any value of n or otherwise the fraction M_k/n^k will approach zero and hence violate the hypothesis of our theorem, since if M_k/n^k approaches zero $|a_n|$ and $|b_n|$ must also approach zero, each being less than M_k/n^k .

Thus if we have a function that satisfies the conditions of our theorem and we assign integral values to $k, 1, 2, 3, \dots$ then after a certain value of k is reached, M_k will increase faster than an exponential function of the type n^k .

As a corollary to the above theorem we can state the following:

If M_k is bounded by a function of the type m^k , $f(x)$ is a trigonometric polynomial of order at most equal to m .

For the reasoning shows that then $a_n = 0$ and $b_n = 0$ for $n > m$ and the function would thus be a trigonometric polynomial of order at most equal m .

A simple example of such a function is the function

$$f(x) = \sin 2x. \quad \text{Then } f'(x) = 2 \cos 2x, \quad f''(x) = -4 \sin 2x,$$

$$f'''(x) = -2^3 \cos 2x, \quad f^{(4)}(x) = 2^4 \sin 2x, \text{ etc.,}$$

and thus we have $|f^{(k)}(x)| \leq 2^k$. Therefore $M_k = 2^k$, which verifies our corollary.

3. Table of values of $F(n,k)$. We will now tabulate the function $F(n,k)$ for values of n ranging from 1 to 20 and for k ranging from 1 to 20. In computing this table I have made use of some results given by Stieltjes¹².

n	F(n,2)	F(n,3)	F(n,4)	F(n,5)	F(n,6)
1	1.6449341	1.2020569	1.0823232	1.0369278	1.017343062
2	.6449341	.2020569	.0823232	.0369278	.017343062
3	.3949341	.0770569	.0198232	.0056778	.001718062
4	.2838229	.0400199	.0074775	.0015625	.000346320
5	.2213229	.0243949	.0035713	.0005860	.000102180
6	.1813229	.0163949	.0019713	.0002660	.000038180
7	.1535452	.0117652	.0011997	.0001374	.000016747
8	.1331371	.0088498	.0007832	.0000779	.000008248
9	.1175121	.0068967	.0005391	.0000473	.000004434
10	.1051664	.0055249	.0003866	.0000304	.000002553
11	.0951664	.0045249	.0002866	.0000204	.000001553
12	.0869019	.0037736	.0002183	.0000142	.000000989
13	.0799575	.0031949	.0001701	.0000121	.000000655
14	.0740403	.0027397	.0001351	.0000075	.000000448
15	.0689383	.0023753	.0001091	.0000056	.000000316
16	.0644938	.0020790	.0000893	.0000043	.000000229
17	.0605876	.0018349	.0000741	.0000034	.000000170
18	.0571274	.0016313	.0000621	.0000027	.000000129
19	.0540409	.0014600	.0000526	.0000021	.000000090
20	.0512709	.0013141	.0000449	.0000017	.000000069

n	F(n,7)	F(n,8)	F(n,9)	F(n,10)
1	1.008349277	1.00407735620	1.002008392826	1.000994575128
2	.008349277	.00407735620	.002008392826	.000994575128
3	.000536777	.00017110620	.000055267826	.000018012628
4	.000079530	.00001869047	.000004462563	.000001077541
5	.000018495	.00000343228	.000000648033	.000000123909
6	.000005695	.00000087228	.000000136033	.000000021509
7	.000002123	.00000027691	.000000036805	.000000004971
8	.000000909	.00000010363	.000000012051	.000000001435
9	.000000433	.00000004401	.000000004599	.000000000504
10	.000000224	.00000002078	.000000002018	.000000000218
11	.000000124	.00000001078	.000000001018	.000000000118
12	.000000073	.00000000612	.000000000955	.000000000080
13	.000000046	.00000000380	.000000000402	.000000000064
14	.000000031	.00000000258	.000000000308	.000000000057
15	.000000022	.00000000192	.000000000260	.000000000054
16	.000000017	.00000000152	.000000000234	.000000000053
17	.000000014	.00000000129	.000000000220	.000000000053
18	.000000012	.00000000115	.000000000212	
19	.000000010	.		
20	.000000009			

n	F(n,11)	F(n,12)	F(n,13)	F(n,14)
1	1.000494188604	1.0002460865533	1.0001227133475	1.0000612481350
2	.000494188604	.0002460865533	.0001227133475	.0000612481350
3	.000005907354	.0000019459283	.0000006430350	.0000002129788
4	.000000262325	.0000000642519	.0000000158096	.0000000039037
5	.000000023917	.0000000046499	.0000000009091	.0000000001786
6	.000000003437	.0000000005539	.0000000000899	.0000000000148
7	.000000000681	.0000000000945	.0000000000134	.0000000000021
8	.000000000175	.0000000000224	.0000000000031	.0000000000007
9	.000000000059	.0000000000081	.0000000000013	.0000000000005
10	.000000000027	.0000000000046	.0000000000010	
11	.000000000017	.0000000000036	.0000000000009	
12	.000000000014	.0000000000033		
13	.000000000012	.0000000000032		
14	.000000000012			
15	.000000000012			
16	.000000000011			

n	F(n,15)	F(n,16)	F(n,17)
1	1.0000305882363	1.000015282259408	1.000007637197637
2	.0000305882363	.000015282259408	.000007637197637
3	.0000000706582	.000000023470346	.000000007803106
4	.0000000009665	.000000000239773	.000000000059582
5	.0000000000353	.000000000006942	.000000000001375
6	.0000000000026	.000000000000379	.000000000000064
7	.0000000000005		
8	.0000000000003		

n	F(n,18)	F(n,19)	F(n,20)
1	1.00000381729326499	1.00000190821271655	1.000000953962033872
2	.00000381729326499	.00000190821271655	.000000953962033872
3	.00000000259599937	.00000000086408374	.000000000287717466
4	.00000000001482458	.00000000000369214	.000000000000920267
5	.00000000000027267	.00000000000005416	.0000000000000107728
6	.00000000000001052	.00000000000000173	.0000000000000002871

4. Comparison of results. As has already been mentioned in the introduction Jackson has found that in the case of a periodic function of period 2π whose $(k - 1)^{\text{th}}$ derivative satisfies a Lipschitz condition the remainder after n terms of its Fourier's series is:

$$R_n(x) \leq 36 \lambda \frac{\log n}{n^k}.$$

If k is odd the factor 36 can be replaced by 12.

We will compare this with the result which we have just derived, namely,

$$R_n(x) \leq \frac{8}{\pi} M_k F(n, k),$$

M_k being the maximum value assumed by the k^{th} derivative of $f(x)$ in the interval 0 to 2π .

We see at once that $8/\pi < 12$ and $M_k \leq \lambda$, λ being the constant of the Lipschitz condition. If we let $\frac{\log n}{n^k} = J(n, k)$, we have left to compare only $J(n, k)$ and $F(n, k)$. When $F(n, k) < J(n, k)$ our result gives a more accurate expression for R_n and when $F(n, k) > J(n, k)$ Jackson's result is the better.

Since we have already constructed a table of values for $F(n, k)$ we have but to make a corresponding table for $J(n, k)$ and compare the two. In order to compare the functions accurately for large values of n and k it would be necessary to compute $F(n, k)$ to a much greater number of decimal places than has been done in our table. However, in any particular problem in which the development of a function arises, the development, to be of any value, must be such that it will be necessary to use only a few terms of it. Thus it will not be necessary to consider large values of n . However it will be advantageous to

let k be as large as possible. In making the following comparison I have considered values of n from 3 to 8 and of k from 2 to 20 inclusive, this being the extent of k in our table of $F(n,k)$.

n

3	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 8$, $F < J$ for $k = 9$ to $k = 20$
4	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 6$, $F < J$ for $k = 7$ to $k = 20$
5	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 5$, $F < J$ for $k = 6$ to $k = 20$
6	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 5$, $F < J$ for $k = 6$ to $k = 20$
7	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 6$, $F < J$ for $k = 6$ to $k = 20$
8	$F(n,k) > J(n,k)$ for $k = 2$ to $k = 6$, $F < J$ for $k = 6$ to $k = 20$

As far as I can judge, as k is increased further $F(n,k)$ still remains less than $J(n,k)$. Since in general k will be chosen quite large we see by the above table that our result has some advantage over that of Jackson's for this range of n .

Chapter III.

1. Remainder in a Legendre's series. We will now consider the developement of a function of x , $f(x)$, in a series of Zonal Harmonics or what are usually called Legendre's polynomials, in the interval from $x = -1$ to $x = +1$.

We have the developement:

$$(1) \quad f(x) = A_0 P(x) + A_1 P_1(x) + A_2 P_2(x) + \dots + A_m P_m(x) + \dots$$

where

$$(2) \quad A_m = \frac{2m+1}{2} \int_{-1}^{+1} f(x) P_m(x) dx, \quad m = 1, 2, 3, \dots$$

If the series (1) converges it will be sufficient to study the convergence of the series of coefficients in investigating this development since we know that $|P_m(x)| \leq 1$.

In doing this we will use a method somewhat similar to the one used in the case of a Fourier development.

Integrating (2) by parts we will have:

$$(3) \quad A_m = \frac{2m+1}{2} \left\{ \left[\int_{-1}^x P_m(x) dx f(x) \right]_{-1}^{+1} - \int_{-1}^{+1} \left(\int_{-1}^x P_m(x) dx \right) f'(x) dx \right\}$$

From Legendre's equation,

$$\frac{d}{dx} (1 - x^2) \frac{dP_m(x)}{dx} + m(m+1) P_m(x) = 0,$$

we have

$$\int_{-1}^x P_m(x) dx = - \frac{1}{m(m+1)} (1 - x^2) \frac{dP_m(x)}{dx}, \quad m \neq 0.$$

Hence (3) becomes, the first term being equal to zero,

$$(4) \quad A_m = \frac{2m+1}{2m(m+1)} \int_{-1}^{+1} \frac{d}{dx} P_m(x) \left[(1-x^2) f'(x) \right] dx.$$

The work can be very much simplified by making the following substitutions: Let

$$\begin{aligned} F_0(x) &= f(x), & F_1(x) &= (1-x^2) \frac{d}{dx} f(x), \\ F_2(x) &= \frac{d}{dx} F_1(x), & F_3(x) &= (1-x^2) \frac{d}{dx} F_2(x), \\ F_4(x) &= \frac{d}{dx} F_3(x), & F_5(x) &= (1-x^2) \frac{d}{dx} F_4(x), \\ && \text{etc.} \end{aligned}$$

Equation (4) then becomes:

$$A_m = \frac{2m+1}{2m(m+1)} \int_{-1}^{+1} \frac{d}{dx} P_m(x) F_1(x) dx.$$

Integrating by parts again:

$$(5) \quad A_m = \frac{2m+1}{2m(m+1)} \left\{ \left[P_m(x) F_1(x) \right]_{-1}^{+1} - \int_{-1}^{+1} P_m(x) \frac{d}{dx} F_1(x) dx \right\}.$$

Again the first term drops out and we have:

$$A_m = - \frac{2m+1}{2m(m+1)} \int_{-1}^{+1} P_m(x) F_2(x) dx.$$

Proceeding as before we get:

$$\begin{aligned} A_m &= - \frac{2m+1}{2m(m+1)} \left\{ \left[\int_{-1}^x P_m(x) dx F_2(x) \right]_{-1}^{+1} - \int_{-1}^{+1} \left(\int_{-1}^x P_m(x) dx \right) \frac{d}{dx} F_2(x) dx \right\} \\ &= \frac{2m+1}{2m^2(m+1)} \int_{-1}^{+1} \frac{d}{dx} P_m(x) \left[(1-x^2) \frac{d}{dx} F_2(x) \right] dx. \end{aligned}$$

$$A_m = \frac{2m+1}{2m^2(m+1)^2} \int_{-1}^{+1} \frac{d}{dx} P_m(x) F_3(x) dx$$

In the same way we get:

$$A_m = - \frac{2m+1}{2m^2(m+1)^2} \int_{-1}^{+1} P_m(x) F_4(x) dx.$$

Hence after integration by parts $2k-1$ times we will have:

$$A_m = \frac{2m+1}{2m^k(m+1)^k} \int_{-1}^{+1} \frac{d}{dx} P_m(x) F_{2k-1}(x) dx, \quad k = 1, 2, 3, \dots$$

and after integration by parts $2k$ times,

$$A_m = - \frac{2m+1}{2m^k(m+1)^k} \int_{-1}^{+1} P_m(x) F_{2k}(x) dx, \quad k = 0, 1, 2, \dots$$

We have however that

$$F_1(x) = (1-x^2) f'(x), \quad F_2(x) = (1-x^2) f''(x) - 2x f'(x),$$

$$F_3(x) = (1-x^2)^2 f'''(x) - 4x(1-x^2) f''(x) - 2(1-x^2) f'(x).$$

And similarly:

$$F_k(x) = c_1(x) f^k(x) + c_2(x) f^{k-1}(x) + c_3(x) f^{k-2}(x) + \dots + c_k f'(x),$$

where $c_i(x)$ are integral rational functions of x .

Now suppose that the k th derivative of $f(x)$ is bounded, i.e.,

$$|f^k(x)| < N$$

where N is a positive quantity. Then $F_k(x)$ will also be

bounded. Throughout the discussion it must be remembered that we are considering only the values of x in the interval -1 to $+1$.

Call the maximum value of $|F_k(x)|$ in the interval, M_k . Then if k is an even number equal, say, to $2n$ we will have:

$$|A_m| \leq \frac{2m+1}{2m^{n(m+1)n}} M_k \int_{-1}^{+1} |P_m(x)| dx$$

or if k is an odd number, $k = 2n - 1$,

$$|A_m| \leq \frac{2m+1}{2m^{n(m+1)n}} M_k \int_{-1}^{+1} \left| \frac{d}{dx} P_m(x) \right| dx.$$

Thus we have an upper bound for $|A_m|$. We need only consider one of these namely the first. To determine the value

of the integral $\int_{-1}^{+1} |P_m(x)| dx$ we will use Schwarz's inequality

which states that

$$\left(\int_a^b f(x) \cdot \phi(x) dx \right)^2 \leq \int_a^b (f(x))^2 dx \cdot \int_a^b (\phi(x))^2 dx,$$

where $f(x)$ and $\phi(x)$ are two real and continuous functions.

Now if we set $f(x) = 1$ and $\phi(x) = |P_m(x)|$ we have:

$$\left(\int_{-1}^{+1} |P_m(x)| dx \right)^2 \leq 2 \int_{-1}^{+1} (P_m(x))^2 dx \text{ since } (|P_m(x)|)^2 = (P_m(x))^2.$$

But $\int_{-1}^{+1} (P_m(x))^2 dx = \frac{2}{2m+1}$. Hence $\int_{-1}^{+1} |P_m(x)| dx < \frac{2}{\sqrt{2m+1}}$.

Since $|P_m(x)| > (P_m(x))^2$ throughout the interval - 1 to +1, we can say that

$$\frac{2}{2m+1} < \int_{-1}^{+1} |P_m(x)| dx \leq \frac{2}{\sqrt{2m+1}}.$$

We have that the remainder after m terms of our series is

$$R_m \leq |A_m| + |A_{m+1}| + |A_{m+2}| + \dots + |A_{m+p}| + \dots$$

This will then become:

$$R_m \leq M_k \left\{ \frac{(2m+1)^{\frac{1}{2}}}{m^n(m+1)^n} + \frac{(2m+3)^{\frac{1}{2}}}{(m+1)^n(m+2)^n} + \frac{(2m+5)^{\frac{1}{2}}}{(m+2)^n(m+3)^n} + \dots \right\}.$$

where $2n$ is again equal to k .

The series

$$\frac{(2m+1)^{\frac{1}{2}}}{m^n(m+1)^n} + \frac{(2m+3)^{\frac{1}{2}}}{(m+1)^n(m+2)^n} + \frac{(2m+5)^{\frac{1}{2}}}{(m+2)^n(m+3)^n} + \dots$$

is a convergent series for values of $n \geq 1$ and $m \geq 1$. It converges very rapidly as n is taken larger and larger. Since it will be in general possible to assign a moderately large value to n it will only be necessary to compute a few terms of the series to obtain a result accurate enough for all purposes. M_k then being known or computed we will have an upper bound for the remainder after n terms of a Legendre's series.

Chapter IV.

1. Remainder in a series of Bessel's functions. Let us now consider the development of a function in a series of Bessel's functions of the zeroth order, between the limits 0 and 1.

We have:

$$f(x) = A_1 J_0(\mu_1 x) + A_2 J_0(\mu_2 x) + A_3 J_0(\mu_3 x) + \dots$$

where $\mu_1, \mu_2, \mu_3, \dots$ are the successive roots of the transcendental equation $J_0(\mu x) = 0$ and

$$A_k = \frac{2}{J_1^2(\mu_k)} \int_0^1 x f(x) J_0(\mu_k x) dx.$$

We will use the same method that we have used in discussing the other series.

Integrating A_k by parts we have:

$$A_k = \frac{2}{J_1^2(\mu_k)} \left\{ \left[\frac{x f(x) J_1(\mu_k x)}{\mu_k} \right]_0^1 - \frac{1}{\mu_k} \int_0^1 x f'(x) J_1(\mu_k x) dx \right\}.$$

We will at this point simplify our work a great deal by setting up the following auxiliary functions:

$$F_0(x) = f(x), \quad F_1(x) = \frac{f'(x)}{x}, \quad F_2(x) = \frac{F_1'(x)}{x},$$

$$\dots \quad F_{n-1}(x) = \frac{F_{n-2}'(x)}{x}, \quad F_n(x) = \frac{F_{n-1}'(x)}{x}.$$

It will be necessary to impose the following conditions upon these functions:

$$F_0(1) = F_1(1) = F_2(1) = \dots = F_{n-1}(1) = 0.$$

With these conditions we will then have:

$$\left[\frac{x f(x) J_1(\mu_k x)}{\mu_k} \right]_0^1 = 0 \text{ and consequently}$$

$$A_k = \frac{-2}{J_1^2(\mu_k)} \frac{1}{\mu_k} \int_0^1 x^2 F_1(x) J_1(\mu_k x) dx .$$

Integrating by parts again:

$$\begin{aligned} A_k &= \frac{-2}{J_1^2(\mu_k)} \frac{1}{\mu_k} \left\{ \left[\left(\int_0^x x^2 J_1(\mu_k x) dx \right) F_1(x) \right]_0^1 - \int_0^1 \left[\int_0^x x^2 J_1(\mu_k x) dx \right] F_1'(x) dx \right\} \\ &= \frac{-2}{J_1^2(\mu_k)} \frac{1}{\mu_k} \left\{ \left[\frac{x^2 J_2(\mu_k x) F_1(x)}{\mu_k} \right]_0^1 - \frac{1}{\mu_k} \int_0^1 x^2 J_2(\mu_k x) F_1'(x) dx \right\} . \\ &= \frac{2}{J_1^2(\mu_k)} \frac{1}{\mu_k^2} \int_0^1 x^2 J_2(\mu_k x) F_1'(x) dx \\ &= \frac{2}{J_1^2(\mu_k)} \frac{1}{\mu_k^2} \int_0^1 F_2(x) x^3 J_2(\mu_k x) dx . \end{aligned}$$

Repeating the integration by parts:

$$\begin{aligned} A_k &= \frac{2}{J_1^2(\mu_k)} \frac{1}{\mu_k^2} \left\{ \left[F_2(x) \int_0^x x^3 J_2(\mu_k x) dx \right]_0^1 - \int_0^1 \left(\int_0^x x^3 J_2(\mu_k x) dx \right) F_2'(x) dx \right\} \\ &= \frac{2}{J_1^2(\mu_k)} \frac{1}{\mu_k^2} \left\{ \left[\frac{F_2(x) x^3 J_3(\mu_k x)}{\mu_k} \right]_0^1 - \frac{1}{\mu_k} \int_0^1 F_2'(x) x^3 J_3(\mu_k x) dx \right\} . \\ &= \frac{-2}{J_1^2(\mu_k)} \frac{1}{\mu_k^3} \int_0^1 F_3(x) x^4 J_3(\mu_k x) dx . \end{aligned}$$

Repeating this process m times we will finally get:

$$A_k = \frac{(-1)^m 2}{J_1^2(\mu_k)} \frac{1}{\mu_k^m} \int_0^1 F_m(x) x^{m+1} J_m(\mu_k x) dx$$

By the law of the mean we can say:

$$A_k = \frac{(-1)^m 2}{J_1^2(\mu_k)} \frac{1}{\mu_k^m} F_m(\bar{x}) J_m(\mu_k \bar{x}) \int_0^1 x^{m+1} dx, \text{ where } 0 \leq x \leq 1.$$

$$\text{Then } A_k = \frac{2}{J_1^2(\mu_k)} \frac{1}{\mu_k^m} \frac{MF_m(x) \cdot MJ_m(\mu_k x)}{m+2}$$

where by $MF_m(x)$ and $MJ_m(\mu_k x)$ we mean the maximum values assumed by $F_m(x)$ and $J_m(\mu_k x)$ respectively in the interval $x = 0$ to $x = 1$.

We have however that $|J_m(x)| \leq 1$ for all values of x . For Whittaker's Modern Analysis, article 153, gives us an expression for the Bessel functions in the form of an integral. The following formula is developed:

$$J_m(x) = \frac{1}{\pi} \int_0^\pi \cos(m\theta - x \sin\theta) d\theta.$$

But since the value of the integrand can never exceed unity we can conclude that $|J_m(x)| \leq 1$ for all values of x . It can also be shown that the first maximum of $J_m(x)$ is the largest one.

Then in considering the convergence of the series

$$f(x) = A_1 J_0(\mu_1 x) + A_2 J_0(\mu_2 x) + A_3 J_0(\mu_3 x) + \dots$$

we can consider instead the series

$$f(x) \leq |A_1| + |A_2| + |A_3| + \dots + |A_n| + \dots$$

Hence we have:

$$R_n(x) \leq |A_n| + |A_{n-1}| + |A_{n-2}| + \dots$$

$$R_n(x) \leq \frac{2}{m+2} MF_m(x) \left\{ \frac{MJ_m(\mu_n x)}{\mu_n^m J_1^2(\mu_n)} + \frac{MJ_m(\mu_{n+1} x)}{\mu_{n+1}^m J_1^2(\mu_{n+1})} + \frac{MJ_m(\mu_{n+2} x)}{\mu_{n+2}^m J_1^2(\mu_{n+2})} + \dots \right\}$$

Or we could write:

$$R_n(x) \leq \frac{2}{m+2} MF_m(x) \cdot MJ_m(\mu_n x) \left\{ \frac{1}{\mu_n^m J_1^2(\mu_n)} + \frac{1}{\mu_{n+1}^m J_1^2(\mu_{n+1})} + \dots \right\}$$

Since $\mu_n, \mu_{n+1}, \mu_{n+2}, \dots$ are successive roots of the equation $J_0(x) = 0$, $J_1^2(\mu_n)$, $J_1^2(\mu_{n+1})$, $J_1^2(\mu_{n+2})$, $\dots$ are all greater than zero.

If then the series

$$\frac{1}{\mu_n^m J_1^2(\mu_n)} + \frac{1}{\mu_{n+1}^m J_1^2(\mu_{n+1})} + \frac{1}{\mu_{n+2}^m J_1^2(\mu_{n+2})} + \dots$$

is a convergent series it will be possible to compute an upper bound for the values of $R_n(x)$.

We will now proceed to show that the above series is convergent for values of $m > 2$, and to obtain an upper limit for its value.

We have the following asymptotic formula¹⁵:

$$(1) \quad J_0(x) = \frac{2 \cos(\pi/4 - x)}{\sqrt{2\pi x}} + \theta 2 \sqrt{\frac{2}{\pi}} g^2 e^{-1} x^{-5/2}$$

where $-1 < \theta < 1$ and $g < 3.5$.

Then $x J_0(x) = \frac{2x^{\frac{1}{2}} \cos(\pi/4 - x)}{\sqrt{2\pi}} + \theta 2\sqrt{\frac{2}{\pi}} g^2 e^{-1} x^{-\frac{3}{2}}$

and $x J_0(x) = \frac{d}{dx}(x J_1(x))$ by a well known formula.

Integrating between two indefinite limits, a and λ we will have:

$$\int_a^\lambda x J_0(x) dx = \lambda J_1(\lambda) - a J_1(a).$$

And then

$$(2) \quad \lambda J_1(\lambda) - a J_1(a) = \frac{2}{\sqrt{2\pi}} \int_a^\lambda x^{\frac{1}{2}} \cos(\pi/4 - x) dx - \left[\theta 4\sqrt{\frac{2}{\pi}} g^2 e^{-1} x^{-\frac{1}{2}} \right]_a^\lambda$$

Let us call the second member of the right hand side of this equation R . R then tends toward zero as λ and a increase.

If now we could choose a as a root of $J_1(x) = 0$ we would have an expression for $\lambda J_1(\lambda)$. However if we choose a not equal to such a root it amounts only to adding a constant to $\lambda J_1(\lambda)$.

Integrating the first member of the right hand side of equation (2) by parts we get, leaving off for the moment the constant term:

$$\begin{aligned} \int x^{\frac{1}{2}} \cos(\pi/4 - x) dx &= -\sin(\frac{\pi}{4} - x) x^{\frac{1}{2}} + \frac{1}{2} \int x^{-\frac{1}{2}} \sin(\frac{\pi}{4} - x) dx \\ &= -\sin(\frac{\pi}{4} - x) x^{\frac{1}{2}} + \frac{1}{2} \left\{ \cos(\frac{\pi}{4} - x) x^{-\frac{1}{2}} + \frac{1}{2} \int x^{-\frac{3}{2}} \cos(\frac{\pi}{4} - x) dx \right\} \\ (3) \quad &= -\sin(\frac{\pi}{4} - x) x^{\frac{1}{2}} + \frac{1}{2} \cos(\frac{\pi}{4} - x) x^{-\frac{1}{2}} + \frac{1}{4} \int x^{-\frac{3}{2}} \cos(\frac{\pi}{4} - x) dx \end{aligned}$$

But in the last term of the above equation we have,

by the law of the mean,

$$\begin{aligned} \int_a^\lambda x^{-\frac{3}{2}} \cos\left(\frac{\pi}{4} - x\right) dx &= \cos\left(\frac{\pi}{4} - \bar{x}\right) \int_a^\lambda x^{-\frac{3}{2}} dx \\ &= -2 \cos\left(\frac{\pi}{4} - \bar{x}\right) x^{-\frac{1}{2}}, \end{aligned}$$

where $\bar{x}$ lies between a and λ . This term is of the order $x^{-\frac{1}{2}}$ and can be included in the R of equation (2). The same is true of the second term of equation (3).

Hence, we will now have:

$$\lambda J_1(\lambda) - a J_1(a) = \frac{2}{\sqrt{2\pi}} \left\{ -\sin\left(\frac{\pi}{4} - x\right) x^{\frac{1}{2}} \Big|_a^\lambda + R_1 \right\}$$

$$\text{where } R_1 = \left[-\theta 4\sqrt{\frac{2}{\pi}} g^2 e^{-1} x^{-\frac{1}{2}} + \frac{1}{2} \cos\left(\frac{\pi}{4} - x\right) x^{-\frac{1}{2}} - \frac{1}{2} \cos\left(\frac{\pi}{4} - \bar{x}\right) x^{-\frac{1}{2}} \right]_a^\lambda$$

Then

$$\lambda J_1(\lambda) - a J_1(a) + \frac{2}{\sqrt{2\pi}} \left\{ \sin\left(\lambda - \frac{\pi}{4}\right) \lambda^{\frac{1}{2}} - \sin\left(a - \frac{\pi}{4}\right) a^{\frac{1}{2}} \right\} + R_1$$

and

$$\begin{aligned} \lambda J_1(\lambda) &= \frac{2}{\sqrt{2\pi}} \sin\left(\lambda - \frac{\pi}{4}\right) \lambda^{\frac{1}{2}} - R_1(\lambda) \\ &= a J_1(a) - \frac{2}{\sqrt{2\pi}} \sin\left(a - \frac{\pi}{4}\right) a^{\frac{1}{2}} - R_1(a). \end{aligned}$$

Hence if a is a constant

$$\lambda J_1(\lambda) - \frac{2}{\sqrt{2\pi}} \sin\left(\lambda - \frac{\pi}{4}\right) \lambda^{\frac{1}{2}} - R_1(\lambda) = K, \text{ a constant.}$$

We will now prove that this constant is zero.

It is a well known fact that in any range 2π of the

variable, $J_1(\lambda)$ has at least one root and the same is true of $\lambda J_1(\lambda)$. The function $\frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \lambda^{\frac{1}{2}}$ also has roots

lying at intervals of π apart. Hence the two curves must cross. Suppose they cross at $\lambda = \lambda^*$. For such a value, $R_1(\lambda^*) = -K$.

But such an interval can be found so that λ^* is as big as you please and since $R_1(\lambda)$ is of order $\frac{1}{\sqrt{\lambda}}$, $R_1(\lambda^*)$ can be made as small as we please.

Hence $R_1(\lambda^*) = 0$ and consequently $K = 0$.

Thus we have:

$$\lambda J_1(\lambda) = \frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \lambda^{\frac{1}{2}} + R_1(\lambda), \text{ where } |R_1(\lambda)| < \frac{C}{\sqrt{\lambda}}$$

C being a constant.

Now the maxima of this function are not less in absolute value than $\frac{2}{\sqrt{2\pi}} \sqrt{\mu_i} - \frac{C}{\sqrt{\mu_i}}$, where μ_i are the roots of $J_0(x) = 0$.

The general term of the series which we are investigating can be written:

$$\frac{1}{\mu_i^{m-2} (\mu_i J_1(\mu_i))^2}.$$

This now becomes:

$$\frac{1}{\mu_i^{m-2} (\mu_i J_1(\mu_i))^2} < \frac{1}{\mu_i^{m-2} (\frac{4}{2\pi} \mu_i - \frac{4C}{\sqrt{2\pi}} - \frac{C^2}{\mu_i})} < \frac{1}{\mu_i^{m-2} (\frac{4}{2\pi} \mu_i - \frac{4C}{\sqrt{2\pi}})}$$

Or if $\frac{2\mu_i}{2\pi} > \frac{4C}{\sqrt{2\pi}}$ our general term is less than $\frac{\pi}{\mu_i^{m-1}}$

But no roots of $J_0(x) = 0$ are closer spaced than $\frac{\pi}{2}$. Hence $\mu_i > \pi i/2$ and the i^{th} term of our series becomes less than $\frac{\pi}{(\frac{\pi i}{2})^{m-1}}$ or $\frac{2^{m-1}}{\pi^{m-2}} \frac{1}{i^{m-1}}$, $i = 1, 2, 3, \dots$

Hence our series converges and converges at least as

fast as $N \sum_{n=1}^{\infty} \frac{1}{n^{m-1}}$, where N is a constant. It is of course clear that we will have to choose $m > 2$.

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