UM Libraries Depository

103304901011

This book is subject to
revised text revisions

This Thesis Has Been

MICROFILMED

Negative No. T- 220

Form 26

A VECTOR TREATMENT
of the
PROJECTIVE PROPERTIES OF PLANE CURVES

Ernest Franklin Canaday, A. B.
"

SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS

in the

GRADUATE SCHOOL

of the

UNIVERSITY OF MISSOURI

1916

A VECTOR TREATMENT
of the
PROJECTIVE PROPERTIES OF PLANE CURVES.

INTRODUCTION.

1. Notations. Suppose that we have given a plane curve C , and a point O , not lying in the plane of the curve. If we draw vectors from point O , one to every point of the given curve, we produce a conical surface, the elements of which are vectors. If we then cut this surface by any plane, not passing thru the point O , the intersection will be a new plane curve bearing certain fixed relations to the original curve C . In other words the two plane curves are projectively related.

Fig. 1.

In this discussion we will denote arc length of the curve C by (σ) , the curvature by $\frac{1}{\rho(\sigma)}$, and a vector from point O to curve C by $\phi(\sigma)$ abbreviated to ϕ .

2. First and second derivatives of the vector φ .

We will first find what $\varphi'(\sigma)$ represents geometrically.

Fig. 2.

First use any parameter (t). Then $\frac{\Delta \varphi}{\Delta t}$ is a vector in the di-

rection of the secant line. Passing to the limit the secant

line becomes the tangent. Therefore $\lim_{\Delta t \rightarrow 0} \frac{\Delta \varphi}{\Delta t}$ is a vector in

the direction of the tangent line.

Replacing parameter t by (σ),

$$\lim_{\sigma \rightarrow 0} \text{of length } \frac{\Delta \varphi}{\Delta \sigma} = 1.$$

Therefore $\varphi'(\sigma)$ is a unit tangent vector when arc length (σ)

is taken as parameter. If we multiply $\varphi'(\sigma)$ scalarwise by

itself, we have,

$$\varphi'(\sigma) \mid \varphi'(\sigma) = 1.$$

Deriving, we have,

$$\varphi'(\sigma) \mid \varphi''(\sigma) = 0.$$

φ'' is therefore perpendicular to the tangent and is in the

direction of the normal to the curve.

Since $\varphi'' = \lim_{\Delta \sigma \rightarrow 0} \frac{\varphi'(\sigma + \Delta \sigma) - \varphi'(\sigma)}{\Delta \sigma}$, φ'' is in the plane of

the curve C .

3. Length of second derivative of ϕ .

Fig. 3.

Draw the unit tangent at point σ and call it MP. At point $(\sigma + \Delta\sigma)$ draw the unit tangent $QN = \phi'(\sigma + \Delta\sigma)$. Call it QN. Move QN end point falls on parallel to itself until it ~~terminates in~~ terminates in point P. Then $MT = \Delta t$, the change in the tangent. Draw arc MT equal to $\Delta\theta$, the change in the direction of the tangent. We now have,

$$(3) \text{ Length } \phi''(\sigma) = \text{length of } \lim_{\Delta(\sigma) \rightarrow 0} \frac{\phi'(\sigma + \Delta\sigma) - \phi'(\sigma)}{\Delta\sigma},$$

$$(4) \text{ Length } \phi''(\sigma) = \lim_{\Delta\sigma \rightarrow 0} \frac{\text{length } \Delta t}{\Delta\sigma}, \text{ and since } \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = 1,$$

$$(5) \text{ Length } \phi''(\sigma) = \lim_{\Delta\sigma \rightarrow 0} \frac{\Delta\theta}{\text{length } \Delta t} \cdot \frac{\text{length } \Delta t}{\Delta\sigma},$$

$$(6) \text{ Length } \phi''(\sigma) = \lim_{\Delta\sigma \rightarrow 0} \frac{\Delta\theta}{\Delta\sigma},$$

$\lim_{\Delta\sigma \rightarrow 0} \frac{\Delta\theta}{\Delta\sigma}$ is by definition the curvature of C. Therefore the length

of ϕ'' is $\frac{1}{\rho(\sigma)}$, the curvature of C at the point of tangency of $\phi'(\sigma)$

4. Relations among the derivatives of ϕ .

Proof that ϕ' and all the other derivatives of $\phi(\sigma)$, lie in the plane of curve C.

First use any parameter such as t .

Having given the plane curve C and $\phi(t)$ a vector from the origin O to point P on the curve, draw $\phi'(t)$ the tangent vector. Then draw the tangent vector $\phi'(t + \Delta t)$ and move it parallel to itself until its ~~terminates in~~ end point coincides with point P. $\phi'(t)$ and $\phi'(t + \Delta t)$ being tangents to the plane curve C, are in the plane of the curve. It follows that
$$\frac{\phi'(t + \Delta t) - \phi'(t)}{\Delta t}$$
 whose limit is $\phi''(t)$ will also lie in the same plane as $\phi'(t)$.

In like manner it can be shown that $\phi'''(t)$, $\phi^{(4)}(t)$ ----- $\phi^{(n)}(t)$ all lie in the plane of C.

It follows that every vector in the plane of the curve can be expressed linearly in terms of any two derivatives of the vector ϕ , hence we have,

Theorem I. If a vector $\psi(t)$ determines a plane curve, ψ satisfies a differential equation of the form,

$$(7) \psi''(t) = S \psi'(t) + T \psi(t).$$

We can also prove the converse,

Theorem II. If a vector ψ satisfies an equation of the form,

$$(7) \psi'' + S \psi' + T \psi = 0,$$

then ψ defines a plane curve. The proof is as follows:

Let $h_1(t)$ and $h_2(t)$ be two independent scalar values for ψ , satisfying equation (7). Then every solution may be written,

$$(8) \psi' = A h_1(t) + B h_2(t) + C,$$

where A and B are independent of (t). Hence ψ may be written,

$$(9) \psi = A H_1(t) + B H_2(t) + C,$$

where A, B, and C are independent of (t).

We obtain VECTOR solutions by choosing A, B, and C as constant vectors. In any given vector solution, C is a constant vector. We therefore obtain the vector ψ by adding to the

constant vector C a (variable) linear combination of the vectors A and B .

The end of the vector ψ must therefore trace out a curve in the plane of the vectors A and B . This plane passes thru the end of the vector C .

PART I.

5. Differential equation satisfied by $\varphi(t)$.

As has already been stated $\varphi'(t)$, $\varphi''(t)$, and $\varphi'''(t)$ all lie in the same plane and hence $\varphi'''(t)$ can be expressed as a scalar times $\varphi''(t)$ plus a scalar times $\varphi'(t)$. That is to say, there exists a differential equation of the third order which is satisfied by the vector φ .

Since $\varphi''(\sigma)$ is equal in length to the curvature $\frac{1}{\rho(\sigma)}$, of curve C, and is in the direction of the normal, we may write it thus ;

$$(10) \quad \varphi''(\sigma) = \frac{1}{\rho(\sigma)} n,$$

n , being a unit normal vector.

Deriving equation (10) we obtain,

$$(11) \quad \varphi''' = \frac{1}{\rho} n' - \frac{\rho'}{\rho^2} n.$$

Since n is a vector of constant length, n' is perpendicular to n . That is to say, n' is in the direction of the tangent.

$$(12) \quad \text{Let } n' = m \varphi'(\sigma).$$

Since vectors n' and $\varphi'(\sigma)$ have the same direction we know,

$$(13) \quad n \mid \varphi'(\sigma) = 0.$$

Deriving (13) we have

$$(14) \quad n \left| \varphi''(\sigma) + n' \right| \varphi'(\sigma) = 0.$$

Multiplying equation (10) scalarwise by n gives us,

$$(15) \quad n \left| \varphi''(\sigma) = \frac{1}{\rho(\sigma)} (n \left| n), \text{ but } n \left| n = 1, \text{ then} \right.$$

$$(16) \quad n \left| \varphi''(\sigma) = \frac{1}{\rho(\sigma)}.$$

If we substitute this value for $n \left| \varphi''(\sigma)$ in equation (14)

$$(17) \quad n' \left| \varphi'(\sigma) = - \frac{1}{\rho(\sigma)}.$$

Again, multiply equation (12) scalarwise by $\varphi'(\sigma)$. We have,

$$(18) \quad n' \left| \varphi'(\sigma) = m \varphi'(\sigma) \left| \varphi'(\sigma) = m.$$

By equation (17) $n' \left| \varphi'(\sigma) = - \frac{1}{\rho(\sigma)}$, hence

$$(19) \quad m = - \frac{1}{\rho(\sigma)}, \quad \text{and therefore,}$$

$$(20) \quad n' = - \frac{1}{\rho(\sigma)} \varphi'(\sigma).$$

We now substitute in (11) the values of n and n' as given by equations (10) and (20).

$$(21) \quad \varphi'''(\sigma) = - \frac{1}{\rho^2} \varphi'(\sigma) - \frac{\rho'(\sigma)}{\rho(\sigma)} \varphi''(\sigma).$$

From now on, for the sake of brevity we shall omit (σ) with $\varphi(\sigma)$, $\rho(\sigma)$ and their derivatives. Equation (21) may then be written,

$$(22) \quad \varphi''' + \frac{\rho'}{\rho} \varphi'' + \frac{1}{\rho^2} \varphi' = 0.$$

This is the differential equation which φ must satisfy if it traces out the plane curve C.

Conversely, by Theorem II, if φ satisfies an equation of the form of equation (22), then φ defines a plane curve.

6. The general solution of equation (22).

One solution of $\varphi''' + \frac{\rho'}{\rho} \varphi'' + \frac{1}{\rho^2} \varphi' = 0$ is

$$(23) \quad \varphi' = \cos \int \frac{d\sigma}{\rho}, \quad \text{for then by deriving we have,}$$

$$(24) \quad \varphi'' = -\frac{1}{\rho^2} \sin \int \frac{d\sigma}{\rho},$$

$$(24) \quad \varphi''' = -\frac{1}{\rho^2} \cos \int \frac{d\sigma}{\rho} + \frac{\rho'}{\rho^2} \sin \int \frac{d\sigma}{\rho}.$$

Adding equations (23) plus $\frac{\rho'}{\rho}$ times (24) plus $\frac{1}{\rho^2}$ times (25),

$$(26) \quad \varphi''' + \frac{\rho'}{\rho} \varphi'' + \frac{1}{\rho^2} \varphi' = 0.$$

In like manner we may prove that another solution is

$$(28) \quad \varphi' = \sin \int \frac{d\sigma}{\rho}.$$

We may therefore write the general solution in the form,

$$(29) \quad \varphi' = A \cos \int \frac{d\sigma}{\rho} + B \sin \int \frac{d\sigma}{\rho},$$

where A and B are arbitrary constants.

Integrating equation (29) we obtain,

$$(30) \quad \varphi = A \int \left(\cos \int \frac{d\sigma}{\rho} \right) d\sigma + B \int \left(\sin \int \frac{d\sigma}{\rho} \right) d\sigma + C.$$

As we have seen on page five, if we wish ϕ to be a vector*

solution, we must choose A, B, and C to be constant vectors.

* If vector solutions are desired such that ϕ' will be of unit length, we must choose the vectors A and B of unit length and at right angles to each other. This is proved as follows.

From equation (29) we may obtain the length of ϕ' . Into this value substitute a and b for the scalar values of A and B, and α for the angle between A and B. We then have,

$$(31) \text{ Length of } \phi' = \sqrt{a^2 \cos^2 \left(\frac{d\sigma}{\rho} \right) + b^2 \sin^2 \left(\frac{d\sigma}{\rho} \right) + 2ab \cos \alpha \sin \left(\frac{d\sigma}{\rho} \right) \cos \left(\frac{d\sigma}{\rho} \right)}.$$

Therefore the quantity under the radical must equal unity. In order that this quantity may equal unity for any value of (σ) , its derivative taken as to (σ) must vanish. That is to say, if

$$(32) a^2 \cos^2 \left(\frac{d\sigma}{\rho} \right) + b^2 \sin^2 \left(\frac{d\sigma}{\rho} \right) + 2ab \cos \alpha \sin \left(\frac{d\sigma}{\rho} \right) \cos \left(\frac{d\sigma}{\rho} \right) = 1.$$

Taking the first derivative,

$$(32a) -\frac{2a^2}{\rho} \cos \left(\frac{d\sigma}{\rho} \right) \sin \left(\frac{d\sigma}{\rho} \right) + \frac{2b^2}{\rho} \sin \left(\frac{d\sigma}{\rho} \right) \cos \left(\frac{d\sigma}{\rho} \right) + \frac{2ab}{\rho} \cos \alpha \left(-\sin^2 \left(\frac{d\sigma}{\rho} \right) + \cos^2 \left(\frac{d\sigma}{\rho} \right) \right) = 0.$$

$$(32b) \frac{-b^2 + a^2}{\rho} \sin 2 \left(\frac{d\sigma}{\rho} \right) = 2ab \cos \alpha \cos 2 \left(\frac{d\sigma}{\rho} \right).$$

In order that this equation may hold, regardless of the value of (σ) , the coefficients of $\sin 2 \int \frac{d\sigma}{\rho}$ and $\cos 2 \int \frac{d\sigma}{\rho}$ must each equal to zero, that is to say,

$$a^2 - b^2 = 0 \quad \text{and} \quad \frac{2ab}{\rho} \cos \bar{a}\bar{b} = 0.$$

If $a^2 - b^2 = 0$ we see that a must equal b , that is, A must equal B in length. If $\frac{2ab}{\rho} \cos \bar{a}\bar{b} = 0$ one factor must equal zero.

Neither a nor b can be zero, for then ϕ would be a constant vector C and we would have no curve. We must then conclude

that $\cos \bar{a}\bar{b} = 0$, .. Then angle $\bar{a}\bar{b}$ is ninety degrees and vectors A and B are at right angles.

Substituting $|A| = |B|$ and $\cos \bar{a}\bar{b} = 0$ in equation (32) gives

$$(33) \quad |A| \cos^2 \int \frac{d\sigma}{\rho} + |A| \sin^2 \int \frac{d\sigma}{\rho} = 1.$$

$$(34) \quad |A| \left(\sin^2 \int \frac{d\sigma}{\rho} + \cos^2 \int \frac{d\sigma}{\rho} \right) = |A| = 1.$$

Hence $|A| = 1$, but $|A| = |B|$, therefore $|B| = 1$.

Vectors A and B have thus been shown to be equal in length,

namely unity, and to be perpendicular to each other.

7. Curve characterized by the General Solution.

We now show that the curve traced out by vector ϕ , as we will designate the general vector solution of the differential

equation, $\varphi''' + \frac{\rho'}{\rho^2} \varphi'' + \frac{1}{\rho^2} \varphi' = 0$

is projectively related to the original curve C , traced out by the vector φ .

In equation (30) $\Phi = A \int \cos \int \frac{d\sigma}{\rho} d\sigma + B \int \sin \int \frac{d\sigma}{\rho} d\sigma + C$,
we will use the abbreviations $c(\sigma) = \int \cos \int \frac{d\sigma}{\rho} d\sigma$ and
 $s(\sigma) = \int \sin \int \frac{d\sigma}{\rho} d\sigma$.

Choosing three mutually perpendicular unit vectors $e_1, e_2,$

and e_3 as reference vectors, we set up the figure thus :

Fig. 6.

$$\text{Let } (35) \quad A = a_1 e_1 + a_2 e_2 + a_3 e_3,$$

$$(36) \quad B = b_1 e_1 + b_2 e_2 + b_3 e_3,$$

$$(37) \quad C = c_1 e_1 + c_2 e_2 + c_3 e_3,$$

where the coefficients of e_1, e_2, e_3 , are constants.

We may then write equation (30) thus,

$$(38) \quad \phi = (a_1 e_1 + a_2 e_2 + a_3 e_3) c(\sigma) \\ + (b_1 e_1 + b_2 e_2 + b_3 e_3) s(\sigma) \\ + (c_1 e_1 + c_2 e_2 + c_3 e_3) \cdot$$

Rearranging , we obtain,

$$(39) \quad \phi = (a_1 c(\sigma) + b_1 s(\sigma) + c_1) e_1 \\ + (a_2 c(\sigma) + b_2 s(\sigma) + c_2) e_2 \\ + (a_3 c(\sigma) + b_3 s(\sigma) + c_3) e_3.$$

This may be written,

$$(40) \quad \phi = X e_1 + Y e_2 + Z e_3,$$

where we set

$$(41) \quad X = (a_1 c(\sigma) + b_1 s(\sigma) + c_1)$$

$$(42) \quad Y = (a_2 c(\sigma) + b_2 s(\sigma) + c_2)$$

$$(43) \quad Z = (a_3 c(\sigma) + b_3 s(\sigma) + c_3).$$

Now the vector ϕ , defining the original curve is a

solution of the differential equation: hence we may write,

$$(44) \quad \phi = (\bar{a}_1 c(\sigma) + \bar{b}_1 s(\sigma) + \bar{c}_1) e_1 \\ + (\bar{a}_2 c(\sigma) + \bar{b}_2 s(\sigma) + \bar{c}_2) e_2 \\ + (\bar{a}_3 c(\sigma) + \bar{b}_3 s(\sigma) + \bar{c}_3) e_3,$$

where $\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{a}_2, \bar{b}_2$, etc. are constants. Rearranging, we have

$$(45) \quad \varphi = x e_1 + y e_2 + z e_3, \quad \text{where}$$

$$(46) \quad x = \bar{a}_1 c(\sigma) + \bar{b}_1 s(\sigma) + \bar{c}_1,$$

$$(47) \quad y = \bar{a}_2 c(\sigma) + \bar{b}_2 s(\sigma) + \bar{c}_2,$$

$$(48) \quad z = \bar{a}_3 c(\sigma) + \bar{b}_3 s(\sigma) + \bar{c}_3,$$

The condition that equations (46), (47), (48) be consistent is:

$$(49) \quad \frac{\begin{vmatrix} \bar{a}_1 & \bar{b}_1 & x \\ \bar{a}_2 & \bar{b}_2 & y \\ \bar{a}_3 & \bar{b}_3 & z \end{vmatrix}}{(\bar{a} \quad \bar{b} \quad \bar{c})} = 1,$$

where $(\bar{a}, \bar{b}, \bar{c})$ represents the determinant of the quantities $\bar{a}_i, \bar{b}_i, \bar{c}_i$. This is the plane of the original curve.

Solving (46), (47) and (48) for $c(\sigma)$ and $s(\sigma)$ we obtain,

$$(50) \quad c(\sigma) = \frac{\begin{vmatrix} x & \bar{b}_1 & \bar{c}_1 \\ y & \bar{b}_2 & \bar{c}_2 \\ z & \bar{b}_3 & \bar{c}_3 \end{vmatrix}}{(\bar{a} \quad \bar{b} \quad \bar{c})}$$

$$(51) \quad s(\sigma) = \frac{\begin{vmatrix} \bar{a}_1 & x & \bar{c}_1 \\ \bar{a}_2 & y & \bar{c}_2 \\ \bar{a}_3 & z & \bar{c}_3 \end{vmatrix}}{(\bar{a} \quad \bar{b} \quad \bar{c})}$$

Substituting these values of $c(\sigma)$ and $s(\sigma)$ in (41), (42),

and (43), we have the following equations.

$$(52) \quad X = A_1 x + B_1 y + C_1 z + D_1,$$

$$(53) \quad Y = A_2 x + B_2 y + C_2 z + D_2,$$

$$(54) \quad Z = A_3 x + B_3 y + C_3 z + D_3,$$

$A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, D_3$, etc. are constant combinations of $a_1, b_1, c_1, a_2, b_2, c_2$, etc.

The above relations show that any curve defined by any vector φ , obtained by solving the differential equation $\varphi''' + \frac{\rho'}{\rho} \varphi'' + \frac{1}{\rho^2} \varphi' = 0$, is projectively related to any other curve defined in a similar way for the same equation.

Suppose,

$$(55) \quad \phi = X e_1 + Y e_2 + Z e_3, \quad \text{where}$$

$$(56) \quad X = a_1 x + b_1 y + c_1 z + d_1,$$

$$(57) \quad Y = a_2 x + b_2 y + c_2 z + d_2,$$

$$(58) \quad Z = a_3 x + b_3 y + c_3 z + d_3.$$

Where $\varphi = x e_1 + y e_2 + z e_3$, is the original vector then does ϕ satisfy the differential equation $\phi''' + \frac{\rho'}{\rho} \phi'' + \frac{1}{\rho^2} \phi' = 0$?

Substituting in the differential equation the values of $\varphi', \varphi'',$ and φ''' found by taking successive derivatives of the equation $\varphi = x e_1 + y e_2 + z e_3$, we obtain the equation,

$$(59) \left(x''' + \frac{\rho'}{\rho} x'' + \frac{1}{\rho^2} x' \right) e_1 + \left(y''' + \frac{\rho'}{\rho} y'' + \frac{1}{\rho^2} y' \right) e_2 + \left(z''' + \frac{\rho'}{\rho} z'' + \frac{1}{\rho^2} z' \right) e_3 = 0.$$

If a vector equals zero each of its components is equal to zero, therefore the coefficients of e_1 , e_2 , and e_3 are each equal to zero.

Substituting values of X , Y , and Z into equation (55), then taking the first, second and third derivatives and substituting the values thus found for ϕ' , ϕ'' and ϕ''' into the differential equation we have the following,

$$(60) \left(x''' + \frac{\rho'}{\rho} x'' + \frac{1}{\rho^2} x' \right) a + \left(y''' + \frac{\rho'}{\rho} y'' + \frac{1}{\rho^2} y' \right) b + \left(z''' + \frac{\rho'}{\rho} z'' + \frac{1}{\rho^2} z' \right) c = 0$$

We know this to be true because the coefficients of a , b , and c , being equal to the coefficients of e_1 , e_2 , and e_3 in (59), are each identically equal to zero.

Thus we have proved that ϕ satisfies the differential equation $\phi''' + \frac{\rho'}{\rho} \phi'' + \frac{1}{\rho^2} \phi' = 0$. or, in other words, the vector defining any curve projectively related to the original curve satisfies the same differential equation.

PART II.

TRANSFORMATIONS and INVARIANTS.

8. Section of the cone by a plane. If the vectors φ , from the origin to points of the curve C be drawn, we obtain a conical surface. We may then set up a one to one correspondence between the points of an arbitrary curve drawn on the cone, and the points on the curve C , by making those points on the two curves, which lie on the same element of the cone, correspond to each other.

Fig. 7.

If we denote by f , the vectors from the origin to points on this new curve C' , it is clear that vectors f and φ extending to corresponding points, have the same direction and hence either is a scalar times the other.

We may then write this relation thus,

$$(61) \quad \varphi = u f.$$

Taking successive derivatives we obtain,

$$(62) \quad \varphi' = u f' + u' f,$$

$$(63) \quad \varphi'' = u f'' + 2 u' f' + u'' f,$$

$$(64) \quad \varphi''' = u f''' + 3u'' f' + 3u' f'' + u'' f.$$

Substituting these values into equation (22) gives us

$$(65) \quad u f''' + (3u' + \frac{\rho'}{\rho} u) f'' + (3u'' + 2\frac{\rho'}{\rho} u' + \frac{1}{\rho^2} u) f' + (u''' + \frac{\rho'}{\rho} u'' + \frac{1}{\rho^2} u') f = 0.$$

$$(66) \quad \text{Set} \quad 3 \frac{u'}{u} + \frac{\rho'}{\rho} = 3 p \quad \text{and also}$$

$$(67) \quad (3 \frac{u''}{u} + 2 \frac{\rho'}{\rho} \frac{u'}{u} + \frac{1}{\rho^2}) = 3 q \quad \text{then}$$

$$(68) \quad f''' + 3p f'' + 3q f' + (u''' + \frac{\rho'}{\rho} u'' + \frac{1}{\rho^2} u') f = 0.$$

We now assume that the curve defined by vector f is a PLANE curve.

Since the vector is not in the plane of its derivatives, the coefficients of f in equation (68) must be zero. Otherwise we would have an equation of the form

$$P f''' + Q f'' + R f' + S f = 0,$$

and this equation can not exist unless vector f is in the plane of the vectors f' , f'' , and f''' . (See Theorem I.)

Setting the coefficient of f equal to zero gives us.

$$(69) \quad u''' + \frac{\rho'}{\rho} u'' + \frac{1}{\rho^2} u' = 0.$$

Thus we see that if $\phi = u f$, u must satisfy the same differential equation as the vector ϕ satisfies, when the vector f defines a plane curve.

9. Invariants. In equation (68) after setting the coefficient of f equal to zero, the remaining coefficients are called $3p$ and $3q$. The values of p and q depend on u , u'' , and u''' . There are however, combinations of p and q whose values are independent of u and its derivatives. These combinations are invariants of the differential equation $u''' + \frac{\rho'}{\rho}u'' + \frac{1}{\rho^2}u' = 0$ under the transformation $\phi = u f$.

Dividing equations (58) and (59) by 3, we have

$$(70) \quad p = \frac{u'}{u} + \frac{\rho'}{3\rho} \quad \text{or} \quad \frac{u'}{u} = p - \frac{\rho'}{3\rho},$$

$$(71) \quad q = \frac{u''}{u} + \frac{2}{3} \frac{\rho'}{\rho} \frac{u'}{u} + \frac{1}{3\rho^2}, \quad \text{or}$$

$$\frac{u''}{u} = q - \frac{2}{3} \frac{\rho'}{\rho} \frac{u'}{u} - \frac{1}{3\rho^2}.$$

Deriving equation (70) we have,

$$(72) \quad \frac{u u'' - (u')^2}{u^2} = p' - \frac{\rho \rho'' - (\rho')^2}{3\rho^2} = \frac{u''}{u} - \left(\frac{u'}{u}\right)^2.$$

Substituting from equations (70) and (71) into (72) the values

of $\frac{u''}{u}$ and $\frac{u'}{u}$, we have the following equation,

$$(73) \quad p' - \frac{\rho''}{3\rho} + \frac{1(\rho')^2}{3(\rho)^2} = q - \frac{2}{3} \frac{\rho'}{\rho} p + \frac{2}{9} \frac{(\rho')^2}{(\rho)^2} - \frac{1}{3\rho^2} - p^2 + \frac{2}{3} \frac{\rho'}{\rho} p - \frac{1}{9} \frac{9(\rho')^2}{(\rho)^2}.$$

Transposing and combining we have,

$$(74) \quad q - p^2 - p' = \frac{1}{3\rho^2} + \frac{2}{9} \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{3\rho}.$$

Here we have an invariant combination of p and q and their derivatives, for, as shown by the equation, the value of the left hand side depends on ρ and its derivatives and not on u . It is therefore the same for every value of u , we may choose.

Another invariant may be obtained as follows: rearranging equation (68) we have,

$$(75) \quad \frac{u''}{u} - \frac{(u')^2}{(u)^2} = p' - \frac{1}{3} \frac{\rho''}{\rho} + \frac{1}{3} \frac{(\rho')^2}{(\rho)^2},$$

Deriving (75) gives

$$(76) \quad \frac{u'''}{u} - 3 \frac{u''}{u} \frac{u'}{u} + 2 \frac{(u')^3}{(u)^3} = p'' - \frac{1}{3} \frac{\rho'''}{\rho} + \frac{1}{3} \frac{\rho''\rho'}{\rho^2} + \frac{2}{3} \frac{\rho'\rho''}{\rho^2} - \frac{2}{3} \frac{(\rho')^3}{(\rho)^3}.$$

From equation (69),

$$(77) \quad \frac{u'''}{u} = - \frac{\rho'}{\rho} \frac{u''}{u} - \frac{1}{\rho^2} \frac{u'}{u}.$$

Substituting the right hand side of (77) into (76) for $\frac{u'''}{u}$, and then substituting the values of $\frac{u'}{u}$ and $\frac{u''}{u}$ we have the following equation containing p , q and ρ and their derivatives.

$$\begin{aligned}
 (78) \quad & -\frac{\rho'}{\rho} q + \frac{2(\rho')^2}{3(\rho)^2} - \frac{2(\rho')^3}{9(\rho)^3} + 3\rho\rho' - \frac{p}{\rho^2} + \frac{\rho'}{3\rho^3} \\
 & + 2p^3 - 2p^2 \frac{\rho'}{\rho} + \frac{2}{3} p \frac{(\rho')^2}{(\rho)^2} - \frac{2}{27} \frac{(\rho')^3}{(\rho)^3} - 3p q \\
 & + 2p^2 \frac{\rho'}{\rho} - \frac{2}{3} p \frac{(\rho')^2}{(\rho)^2} + \frac{p}{\rho^2} + \frac{3q\rho'}{3\rho} - \frac{6}{9} p \frac{(\rho')^2}{(\rho)^2} + \frac{6}{27} \frac{(\rho')^3}{(\rho)^3} \\
 & - 3\rho\rho' = p'' - \frac{1(\rho'')}{3(\rho)} + \frac{\rho'\rho''}{\rho^2} - \frac{2}{3} \frac{(\rho')^3}{(\rho)^3}.
 \end{aligned}$$

This reduces to our second invariant:

$$(79) \quad 2p^3 - 3pq - p'' = \frac{\rho'\rho''}{\rho^2} - \frac{\rho'''}{3\rho} - \frac{\rho'}{3\rho^3} - \frac{16(\rho')^3}{27(\rho)^3}.$$

We find another invariant as follows. Substitute from

(70) and (71) into (76) the values of $\frac{u''}{u}$ and $\frac{u'}{u}$. Instead of

substituting the value of $\frac{u''}{u}$ from (73) as before, derive (71)

$$(80) \quad \frac{u'''}{u} - \frac{u''u'}{uu} = q' - \frac{2\rho'}{3\rho} \left(\frac{u''}{u} - \frac{(u')^2}{(u)^2} \right) - \frac{2u'}{3u} \left(\frac{\rho''}{\rho} - \frac{(\rho')^2}{(\rho)^2} \right) - \frac{2\rho'}{3\rho^3}$$

and substitute the value of $\frac{u'''}{u}$ from this equation into (72).

$$\begin{aligned}
 (81) \quad & q' - \frac{2\rho'}{3\rho} p - \frac{2\rho''}{3\rho} p + \frac{2(\rho')^2}{3(\rho)^2} p + \frac{4\rho'\rho''}{9\rho^2} - \frac{4(\rho')^3}{9(\rho)^3} + \frac{2\rho'}{3\rho^3} \\
 & - 2\left(p - \frac{\rho'}{3\rho}\right) \left[p' - \frac{\rho''}{3\rho} + \frac{1(\rho')^2}{3(\rho)^2} \right] = p'' - \frac{\rho'''}{3\rho} + \frac{\rho'\rho''}{3\rho^2} + \frac{2\rho'\rho''}{3\rho^2} - \frac{2\rho'}{3\rho}.
 \end{aligned}$$

Combining the terms this equation reduces to our third invariant equation.

PART III.

MISCELLANEOUS FORMULAS.

10. The osculating conic to the original curve C. In the following we find the equation of the osculating conic to ~~the~~ any curve ^{from} ~~in terms of~~ its vector representation. We first write the general equation of a conic.

$$(82a) \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + e = 0.$$

From this we will find the equation of the osculating conic, referred to the tangent and normal at the point of contact as axes

Deriving (82a) and dividing it by two we have the equation,

$$(83) \quad ax + hy + g + (hx + by + f) \frac{dy}{dx} = 0.$$

Transposing (82a) and dividing by f,

$$(84) \quad 2y = \frac{ax^2}{f} + \frac{2hxy}{f} + \frac{by^2}{f}. \quad \text{This may be written,}$$

$$(85) \quad 2y = ax^2 + 2hxy + by^2,$$

where a, h, and b are new constants equal respectively to the

old a, h and b each divided by f. Deriving (84) we have,

$$(86) \quad \frac{dy}{dx} = ax + hy + (hx + by) \frac{dy}{dx}.$$

$$(87) \quad \frac{dy}{dx^2} = a + 2h \frac{dy}{dx} + b \left(\frac{dy}{dx} \right)^2 + (hx + by) \frac{d^2y}{dx^2}.$$

$$(88) \quad \frac{d^2y}{dx^3} = 3h \frac{dy}{dx^2} + 3b \frac{dy}{dx} \frac{d^2y}{dx^2} + (hx + by) \frac{d^3y}{dx^3}.$$

$$(89) \quad \frac{d^4 y}{dx^4} = 4h \frac{d^3 y}{dx^3} + 4b \frac{dy}{dx} \frac{d^3 y}{dx^3} + 3b \left(\frac{dy}{dx} \right)^2 + (hx + by) \frac{d^4 y}{dx^4} \quad 25.$$

We then find the values of $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, and $\frac{d^3 y}{dx^3}$ at the origin by

putting the values $x = 0$ and $y = 0$ into the above equations.

$$(90) \quad \left[\frac{dy}{dx} \right] = 0,$$

$$(91) \quad \left[\frac{d^2 y}{dx^2} \right] = a,$$

$$(92) \quad \left[\frac{d^3 y}{dx^3} \right] = 3ah,$$

$$(93) \quad \left[\frac{d^4 y}{dx^4} \right] = 12 a h^2 + 3 a b.$$

We now compare these values of $\frac{dy}{dx}$ and its derivatives with those

values found from the vector representation

Fig. 8.

From here on l shall represent the piece of curve between points σ and N , and l will be used for the digit one.

By Taylor's theorem of expansion we have,

$$(94) \quad \varphi(\sigma_0 + l) - \varphi(\sigma_0) = \varphi'(\sigma_0)l + \varphi''(\sigma_0)\frac{l^2}{2} + \varphi'''(\sigma_0)\frac{l^3}{6} + \varphi^{(4)}(\sigma_0)\frac{l^4}{24} + \dots$$

Substituting the values of φ'' , φ''' , $\varphi^{(4)}$, etc into this equation,

$$(95) \quad \varphi(\sigma_0 + 1) - \varphi(\sigma_0) = \left(1 - \frac{1^3}{6} + \frac{\rho' 1^4}{8\rho^3} + \dots \right) \varphi'(\sigma_0) \\ + \frac{1}{\rho} \left[\frac{1^2}{2} - \frac{\rho'}{6\rho} + \frac{1}{24} \left\{ 2 \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{1}{\rho^2} \right\} 1^4 \right]. \quad \begin{matrix} I = 1 \\ 1 = l \end{matrix}$$

The values substituted were ,

$$(96) \quad \varphi'' = \frac{1}{\rho} n$$

$$(97) \quad \varphi''' = - \frac{\rho'}{\rho} \varphi'' - \frac{1}{\rho^2} \varphi'$$

$$(98) \quad \varphi^{(4)} = - \frac{\rho'}{\rho} \left(- \frac{\rho'}{\rho} \varphi'' - \frac{1}{\rho^2} \varphi' \right) - \left\{ \frac{\rho''}{\rho} - \frac{(\rho')^2}{(\rho)^2} \right\} \varphi'' - \frac{1}{\rho^2} \varphi' \\ + \frac{2\rho' \varphi'}{\rho^3} = \left\{ 2 \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{1}{\rho^2} \right\} \varphi'' + \frac{3\rho'}{\rho^3} \varphi'.$$

From equation (95) we see that the coordinates of point P are the two components of the vector $(\sigma_0 + 1)$, in the direction of the tangent and normal respectively.

The x and y components of $(\sigma_0 + 1)$ are

$$(99) \quad \bar{x} = 1 - \frac{1^3}{2} + \frac{\rho' 1^4}{8\rho^3} + \dots \text{Higher powers of } 1.$$

$$(100) \quad \bar{y} = \frac{1}{\rho} \left[\frac{1^2}{2} - \frac{\rho'}{6\rho} + \frac{1}{24} \left\{ 2 \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{1}{\rho^2} \right\} 1^4 + \dots \text{H.P. of } 1 \right]$$

$$(101) \quad \frac{dy}{dx} = \frac{\frac{I}{\rho} \left\{ 1 - \frac{\rho'}{2\rho} l^2 + \frac{I}{6} \left[2 \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{I}{\rho^2} l^3 + \dots \right] \right\}}{\left(1 - \frac{l^2}{2\rho^2} + \frac{\rho'}{2\rho^3} l^3 + \dots \right)}$$

$$(102) \quad \frac{dy}{dx} = \frac{I}{\rho} \left\{ 1 - \frac{\rho'}{2\rho} l^2 + \frac{I}{6} \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{I}{\rho^2} l^3 \right\} \left\{ I + \frac{l^2}{2\rho^2} - \frac{\rho' l^3}{2\rho^3} + \frac{l^4}{4\rho^4} \right\}$$

$$\text{Set } \left\{ \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{I}{\rho^2} \right\} = m$$

$$(103) \quad \frac{dy}{dx} = \frac{I}{\rho} \left\{ \left(1 - \frac{\rho'}{2\rho} l^2 + \frac{m}{6} l^3 + \frac{I}{2\rho^2} l^3 - \frac{3\rho'}{4\rho^3} l^4 + \dots \right) \right\}.$$

$$(104) \quad \frac{d^2y}{dx^2} = \frac{I}{\rho} \left\{ I - \frac{\rho'}{\rho} l + \left(\frac{m}{2} + \frac{3}{2\rho^2} \right) l^2 - \frac{3\rho'}{\rho^3} l^3 \right\} \left\{ I + \frac{l^2}{2\rho^2} - \frac{\rho' l^3}{2\rho^3} + \dots \right\}$$

$$(105) \quad \frac{d^2y}{dx^2} = \frac{I}{\rho} \left\{ I - \frac{\rho'}{\rho} l + \left(\frac{m}{2} + \frac{2}{\rho^2} \right) l^2 - 4 \frac{\rho'}{\rho^3} l^3 + \dots \right\}.$$

$$(106) \quad \frac{d^3y}{dx^3} = \frac{I}{\rho} \left\{ -\frac{\rho'}{\rho} + \left(m + \frac{4}{\rho^2} \right) l - 12 \frac{\rho'}{\rho^3} l^2 + \dots \right\} \frac{I}{\frac{dx}{dl}}.$$

$$(107) \quad \frac{d^3y}{dx^3} = \frac{I}{\rho} \left\{ -\frac{\rho'}{\rho} + \left(m + \frac{4}{\rho^2} \right) l - 12 \frac{\rho'}{\rho^3} l^2 - \frac{\rho' l^2}{2\rho^3} + \dots \right\}.$$

$$(108) \quad \frac{d^4y}{dx^4} = \frac{I}{\rho} \left\{ m + \frac{4}{\rho^2} \right\} + \text{terms in powers of } l.$$

Putting l equal to zero in the above equations and combining with

equations (90) ----(93) we obtain values for a , h , and b .

$$(109) \quad \frac{dy}{dx} = \frac{I}{\rho} \quad \text{Therefore } a = \frac{I}{\rho}$$

$$(110) \quad \frac{d^2y}{dx^2} = -\frac{\rho'}{\rho^2}, \quad a h = -\frac{\rho'}{3\rho^2}, \quad h = -\frac{\rho'}{3\rho}$$

$$(111) \quad \frac{d^3y}{dx^3} = 12 a h^2 + 3 a^2 b = \frac{I}{\rho} \left\{ 2 \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{\rho} - \frac{I}{\rho^2} \right\} + \frac{4}{\rho^3}$$

Substituting in (111) the values of a and h, we have,

$$(112) \quad \frac{12\rho'^2}{9\rho^3} + \frac{3b}{\rho^2} = \frac{2\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2} - \frac{I}{\rho^3} + \frac{4}{\rho^3}.$$

$$(113) \quad \frac{4\rho'^2}{3\rho} + 3b = \frac{2\rho'^2}{\rho} - \rho'' - \frac{I}{\rho} + \frac{4}{\rho}$$

$$(114) \quad b = \rho \left\{ \frac{2}{9} \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{3\rho} + \frac{I}{\rho^2} \right\}.$$

We may now substitute these values of a, b, and h into the equation of a conic in the general form and we will obtain the equation of the osculating conic to our original curve.

$$(115) \quad 2y = \frac{I}{\rho} x^2 - \frac{2\rho'}{3\rho} xy + \rho \left\{ \frac{2}{9} \frac{(\rho')^2}{(\rho)^2} - \frac{\rho''}{3\rho} + \frac{I}{\rho^2} \right\} y.$$

$$(82) \quad q' - 2pp' - p'' = \frac{7\rho\rho''}{9\rho^2} - \frac{4(\rho')^3}{9(\rho)^3} - \frac{2\rho'}{3\rho^3} - \frac{\rho'''}{3\rho}.$$

The left hand side of this equation is clearly an invariant.

This invariant, we observe, is the first derivative of the
of
invariant, equation (74). Other invariant expressions may be
found by taking further derivatives of equations (79) and (82).

11. Differential equation satisfied by the squared length of a vector from any point to a curve. If λ equals the squared length, then

$$(116) \quad \varphi | \varphi = \lambda.$$

Taking successive derivatives ,

$$(117) \quad 2\varphi | \varphi' = \lambda'$$

$$(118) \quad \varphi | \varphi' = \frac{\lambda'}{2}$$

$$(119) \quad \varphi | \varphi'' + \varphi' | \varphi' = \frac{\lambda''}{2}, \quad \text{but } \varphi' | \varphi' = I \quad \text{then,}$$

$$(120) \quad \varphi | \varphi'' + I = \frac{\lambda''}{2},$$

$$(121) \quad \varphi | \varphi''' = \frac{\lambda'''}{2}.$$

Multiplying equation (22) scalarwise by φ we have,

$$(122) \quad \varphi | \varphi''' + \frac{\rho'}{\rho} \varphi | \varphi'' + \frac{I}{\rho^2} \varphi | \varphi' = 0 \quad \text{then}$$

$$(123) \quad \lambda''' + \frac{\rho'}{\rho} \lambda'' + \frac{I}{\rho^2} \lambda' = 0$$

This is the differential equation satisfied by the squared length of vector φ from a point to the curve.

12. Length of a vector from any point in space to a conic.

Call the squared length λ . Our transformation was $\varphi = u f$.

Squaring both sides of the equation,

$$(124) \quad \varphi | \varphi = u^r f | f.$$

$$(125) \quad \lambda = \varphi | \varphi.$$

$$(126) \quad \ell = f | f, \quad \text{then} \quad \ell = \frac{\lambda}{u^2}$$

Any conic may be regarded as a projection of a circle. The vector to any conic may therefore be found, according to theorem one, by solving the differential equation,

$$(127) \quad \varphi'' + \frac{1}{\rho^2} \varphi' = 0$$

which is the form our general differential equation takes when we put ρ equal to a constant, as it is in a circle.

Equation (30) may then be integrated and we obtain expressions for λ and u .

$$(128) \quad \lambda = A \cos \frac{\sigma}{\rho} + B \sin \frac{\sigma}{\rho} + C,$$

$$(129) \quad u = A' \cos \frac{\sigma}{\rho} + B' \sin \frac{\sigma}{\rho} + C'.$$

Since $\ell = \frac{\lambda}{u^2}$, we have the following formula for the squared

length of a vector to a conic.

$$(130) \quad \ell = \frac{A \cos \frac{\sigma}{\rho} + B \sin \frac{\sigma}{\rho} + C}{\left(A' \cos \frac{\sigma}{\rho} + B' \sin \frac{\sigma}{\rho} + C' \right)^2}.$$

In like manner the formula for the length of a vector to any curve in general may be found. It will however be much more complicated.

Approved May, 15, 1916.

Louis Ingold

UNIVERSITY OF MISSOURI
COLUMBIA

LAWS OBSERVATORY
OFFICE OF THE DIRECTOR

May 19, 1916.

Dean Walter Miller,
211 Academic Hall.

My dear Dean Miller,

I have examined the thesis submitted by
Mr. Ernest F. Canaday, entitled A VECTOR TREATMENT OF THE
PROJECTIVE PROPERTIES OF PLANE CURVES. In my opinion it
meets the general standard which has been established in
this University for the Master's dissertation.

Very truly yours,

RECEIVED
OCT 20 1916
UNIV. OF MD.

VectorTreatmentPlaneCurvesSpecs.txt

MU Libraries
University of Missouri --Columbia

Digitalization Information Page

Local identifier VectorTreatmentPlaneCurves

Citation

AUTHOR	Canaday, Ernest Franklin.
TITLE	A vector treatment of the projective properties of plane curves.
IMPRINT	1916.

Capture information

Date captured	04/2014
Scanner manufacturer	Zeutschel
Scanner model	OS 15000
Scanning system software	Omni scan v. 12.4 SR4 (1947) 64-bit
Optical resolution	600 dpi
Color settings	24 bit color
File types	tiff
Notes	Some page curvature due to tight binding. 52 blank pages at end of book were not scanned.

Source information

Content type	text
Format	book
Source ID	103304901011
Notes	Inside front cover has two labels pasted in. Title page has perforated property stamp. Some pages have handwritten corrections and hand drawn figures. Pages 23 is inserted out of order and follows page 28. Purple ink property stamp on page 30. Signature follows page 30. Last page of text is correspondence to Dean Walter Miller. 52 blank pages at end of book. Pocket and date due slip pasted in at end. Inside back cover has stamp, barcode, and call number.

Derivatives - Access copy

Compression	Tiff: compression: 1
Editing software	Adobe Photoshop CS5
Editing characteristics	Pages cropped and resized.
Resolution	300 dpi
Color	gray scale / color
File types	pdf
Notes	Pages typed single-sided. Blank pages removed Page 1

VectorTreatmentPlaneCurvesSpecs.txt
from access copy.