TRANSCENDENTALISM OF CURVES

AND OF NUMBERS

by

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PART 1

TRANSCENDENTALISM OF CURVES

Lindemann's proof that the number e can not be the root of an algebraic equation with algebraic coefficients and algebraic exponents furnishes us with an example of a curve, namely $y = e^x$ which has only one algebraic point, that is, only one point each of whose coordinates is algebraic. We will divide points on a curve into three classes with respect to the character of their coordinates: (1) totally algebraic, in which the x and y coordinates are both algebraic numbers, (2) totally transcendental, in which the x and y coordinates are both transcendental, (3) mixed, in which one coordinate is algebraic and the other transcendental. With reference to the so called transcendentalism of the exponential curve Klein* says, "Dieser Verlauf der exponential Kurve ist gewiss eine höchst merkwürdige Tatsache." Klein was evidently thinking of the enormous amount of algebraic points in the plane, and it was his first thought, just as it is ours, that a curve would have difficulty in avoiding them. The transcendentalism Klein spoke of is by no means "merkwürdig". There are uncountable hosts of curves with like peculiarities. Let us consider first the algebraic ones. Before we begin our discussion, however, let us state a few facts about transcendental and algebraic numbers which will make what we shall say clearer.

1) A finite number of rational operations on an algebraic number yields an algebraic number.

2) The product of a finite number of algebraic numbers is an algebraic number.

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* F. Klein, "Elementar Mathematik von höheren Standpunkte aus".

- 3) The sum of a finite number of algebraic numbers is algebraic.
- 4) The imaginary number $\sqrt{-1}$ is algebraic.
- 5) If a number is transcendental, any integral power of the number is transcendental.
- 6) The sum of an algebraic number and a transcendental number is transcendental.
- 7) The product of an algebraic number and a transcendental number is transcendental.

All of these results follow directly from the definitions of algebraic and transcendental numbers.

1. ALGEBRAIC CURVES.

The curves,

$$F(x,y)=t,$$

where F is a rational function with algebraic coefficients, and t is any transcendental number has no algebraic point. x and y can not both be algebraic numbers, because in that case $F(x,y)$ would be algebraic and our equation would not be satisfied. Since t can take on an uncountable number of values, the above equation furnishes us with an uncountable number of curves with no algebraic point.

As an example consider the straight line,

$$y=mx+b$$

Four cases may arise:

- 1) m transcendental and b algebraic.

$$y-b=mx.$$

In this case x and y cannot both be algebraic except when $x=0$.

- 2) m algebraic and b transcendental.

$$y-mx=b$$

In this case x and y cannot both be algebraic.

3) m algebraic and b algebraic.

When this occurs x and y are either both algebraic or both transcendental.

4) m transcendental and b transcendental.

Of this case we can say nothing.

Hence a straight line has no algebraic point if it has a transcendental slope or a transcendental y -intercept, provided it does not have both.

As a related example, consider the circle,

$$(x-a)^2 + (y-b)^2 = r^2.$$

1) If a and b are algebraic and r transcendental, the circle has no algebraic point.

2) If either a or b is transcendental, but not both, and r algebraic, the circle has no algebraic point.

Hence a circle with a transcendental radius has no algebraic point provided its center is at an algebraic point. A circle with an algebraic radius has no algebraic point provided its center is at a mixed point.

2. TRIGONOMETRIC CURVES*

Theorem: In the equation,

$$y = \sin x$$

x and y cannot both be algebraic, if we except the values $(0,0)$.

Using the exponential definitions of sine and cosine, we have,

$$e^{i\theta} = \cos \theta + i \sin \theta = \sqrt{1 - \sin^2 \theta} + i \sin \theta.$$

*For these equations we are indebted to G. N. Bauer and H. L. Slobin, "Some Transcendental Curves and Numbers", *Rendiconti del Circolo Matematico di Palermo*, Tomo xxxvi, 1913.

If θ is algebraic, ie is algebraic. Hence the right hand member of the equation cannot represent an algebraic number and it follows that $\sin \theta$ is transcendental. The sine curve has only one algebraic point, the point $(0,0)$. Similar results hold for all the other direct and inverse trigonometric functions. In general,

$$\phi_1(x,y) = \phi_2(x,y) \sin \phi_3(x,y),$$

where ϕ_1, ϕ_2, ϕ_3 are rational functions, all the constants involved being algebraic, has no algebraic points except a finite number satisfying either of the two pairs of equations,

$$\begin{cases} \phi_1(x,y)=0 \\ \phi_2(x,y)=0 \end{cases} \quad \begin{cases} \phi_1(x,y)=0 \\ \phi_3(x,y)=0 \end{cases}$$

The sine may be replaced by any of the other direct or inverse trigonometric functions with suitable changes in the resulting pairs of equations.

3. HYPERBOLIC CURVES*

Using the exponential definitions for $\sinh$, $\cosh$, etc., we have,

$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \text{etc.}$$

If θ is algebraic (not 0), the right hand member of the equation is a transcendental number. Hence if θ is algebraic $\cosh \theta$ cannot be algebraic. In general, just as above, the curves,

$$\phi_1(x,y) = \phi_2(x,y) \tanh \phi_3(x,y)$$

have only a finite number of algebraic points satisfying either of the two pairs of equations,

$$\begin{cases} \phi_1(x,y)=0 \\ \phi_2(x,y)=0 \end{cases} \quad \begin{cases} \phi_1(x,y)=0 \\ \phi_3(x,y)=0 \end{cases}$$

*See Bauer, Slobin article referred to on page 3.

Similar results may be obtained for all the other direct and inverse hyperbolic functions.

4. EXPONENTIAL CURVES.

The curves,

$$\phi_1(x,y)^{\phi_2(x,y)} + \phi_3(x,y) = 0,$$

have only a finite number of algebraic points corresponding to the solutions of the equations,

$$\begin{cases} \phi_1(x,y)=0 \\ \phi_3(x,y)=0 \end{cases} \quad \begin{cases} \phi_2(x,y)=0 \\ \phi_1(x,y) + \phi_3(x,y)=0 \end{cases}$$

Similarly, in,

$$\phi_1(x,y) \log \phi_2(x,y) + \phi_3(x,y) = 0,$$

there are no algebraic points except a finite number satisfying either of the two pairs of equations,

$$\begin{cases} \phi_1(x,y)=0 \\ \phi_3(x,y)=0 \end{cases} \quad \begin{cases} \phi_2(x,y)=1 \\ \phi_3(x,y) \neq 0. \end{cases}$$

The results we have obtained justify, we trust, the statement that we made in the beginning. Klein's so called transcendentalism is by no means remarkable. We may define as many curves as we please with no algebraic points. These curves may be algebraic ones of the simplest sort, or they may be trigonometric or hyperbolic curves of the utmost complexity. Straight lines, circles, sine curves, sinh curves, etc., uncountable numbers of them, wind their way thru the plane without passing thru a single algebraic point, or at most a finite number of them.

Other Results Concerning the Character of Points on Curves.

We have just considered curves with no algebraic points, that is curves with two general types of points, transcendental and mixed. Such considerations lead us to others. Are there any curves all of whose points are algebraic? Are there any curves all of whose points are transcendental? Do there exist continuous curves all of whose points belong to a single type? We propose to discuss these questions.

Are there any curves all of whose points are algebraic? Any curve which contains algebraic points exclusively cannot be continuous, for the number of points in the plane each of whose coordinates is algebraic is countable. Any function defined for only a countable number of points cannot be continuous.

We have just seen that every continuous curve has either mixed points or transcendental points, or both. Roughly it would seem as if every such curve would have purely transcendental points. Let us examine this.

Theorem:

Given a function $F(x)$, defined and continuous at every point in an interval a, b , and suppose $F(\xi) - F(\eta)$ is not equal to zero for ξ and η , two distinct values in the interval, then $F(x)$ has an uncountable number of transcendental points in the interval.

Since $F(x)$ is continuous in the interval a, b and $F(\xi)$ is not equal to $F(\eta)$, the function is defined at every point in the interval. There is an uncountable number of points of the curve in this interval. Evidently some of the points must be transcendental, because in the interval a, b there are only a countable number

of purely algebraic and mixed points belonging to $F(x)$.

We have assumed more than was necessary for our hypothesis. Continuity is not necessary. All that is essential is that $F(x)$ be defined for an uncountable number of points in any interval, and in that interval takes on the same value at most a countable number of times. Hence we will restate our theorem:

If a function $F(x)$, is defined for an uncountable number of points in an interval a, b , and if $F(x_1) - F(x_2)$ is not equal to zero for more than a countable number of x 's in the interval, then $F(x)$ has an uncountable number of transcendental points in the interval considered.

The conditions of our theorem are sufficient to insure transcendental points to our curve, but not necessary. We will give examples of curves which violate these conditions and do not have transcendental points, and also examples of those which violate them and do have transcendental points.

The function,

$$\begin{cases} F(x) = 0 & \text{when } x \text{ is irrational or zero.} \\ F(x) = \frac{1}{q} & \text{when } x \text{ is of the form } \frac{p}{q} \end{cases}$$

has no transcendental point. In any interval it has only a countable number of discontinuities, but in this interval the ordinates are equal for more than a countable number of abscissa's.

The straight lines,

$$x = \alpha$$

$$y = \beta$$

where α and β are transcendental numbers, violate the conditions of our theorem. However, these functions have transcendental points

The question, can any continuous curve be composed of purely transcendental points is easily disposed of. No such curve can be continuous within any interval α, β , because within this interval there will be a countable number of abscissas (the algebraic) for which the function is not defined.

We have noticed that there are no continuous curves with solely algebraic points, none with solely transcendental points. Our theorem of page (7) enables us to conclude that no continuous curves except parallels to the axes avoid transcendental points. Hence there exist no curves, with the possible exceptions of those parallel to the axes, which contain mixed points and no others. We need not make the exception, however, for it is quite easy to see that the parallels to the axes, namely, $x = \alpha$, $y = \beta$ where α and β are either algebraic or transcendental cannot contain solely mixed points. There exists no continuous curve which contains points of a single type. We do not have examples of curves with algebraic points exclusively, or transcendental points exclusively, or mixed points exclusively. Every curve must have at least two distinct types of points. Every continuous curve belongs to one of the four classes of curves:

1). Curves with purely algebraic and purely transcendental points. The simplest example of such a curve is $y=x$, or in general,

$$y=x+\alpha, \quad y=(x-\alpha)^n$$

where α is any algebraic number, and n any integer.

2). Curves with algebraic points and mixed points.

The only continuous curves in this class are the parallels to the axes,

$$x=a,$$

$$y=b,$$

where a and b are algebraic numbers.

3). Curves with transcendental points and mixed points. Such curves we have already discussed at length.

Effect of Transformation of Coordinates on the Character of Points.

1. TRANSLATION.

$$x=x'+a$$

$$y=y'+b$$

We will divide this into three cases:

1). a and b both algebraic.

In this case algebraic points are transformed into algebraic points, transcendental points into transcendental points, and mixed points into mixed points.

Hence the type of points of a curve $F(x)$ remain unchanged when the origin is transferred to any algebraic point.

2). a and b both transcendental.

In this case algebraic points are transformed into transcendental points, transcendental points into any of the three types, and mixed points into mixed points or transcendental points.

In this case a curve with no transcendental point becomes a

curve with no algebraic point.

3). a algebraic, b transcendental, or vice versa.

In this case algebraic points go into mixed points, transcendental points into transcendental points or mixed points, and mixed points into any of the three types.

Under this transformation a curve with no mixed points goes into a curve with no algebraic points.

2. ROTATION.

$$x = x' \cos \theta - y' \sin \theta.$$

$$y = x' \sin \theta + y' \cos \theta.$$

Consider the two cases,

1). θ algebraic, not zero.

In this case we can make no general statement as to the effect of the transformation on points.

(2). $\sin \theta$ algebraic.

In this case, algebraic points go into algebraic points, transcendental points into any of the three types, and mixed points into transcendental points.

3. CHANGE FROM RECTANGULAR TO POLAR COORDINATES.

$$\theta = \tan^{-1} \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

Such a transformation affects points as follows:

1). An algebraic point in rectangular coordinates becomes a mixed point in polar coordinates (excepting points for which $y=0$).

2). Of transcendental points we can make no general statement.

3). A mixed point becomes a mixed point or a transcendental point.

Like Considerations for Surfaces.

All of the results which we have obtained for a space of two dimensions may be extended to a space of three dimensions.

1. SURFACES WITH NO ALGEBRAIC POINTS.

All the curves in two dimensions which have no algebraic points become cylindrical surfaces in three dimensions with no algebraic points. This furnishes us with an uncountable class of transcendental surfaces. Let us consider the more general surfaces with no algebraic points.

1). Algebraic Surfaces.

The surfaces,

$$F(x, y, z) = t,$$

where t is any transcendental number, and F is a rational function, all the constants involved being algebraic, has no algebraic point.

As a related example, not a special case of the above, consider the conicoids,

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1,$$

for which we obtain the following results:

If a is transcendental (all the other constants being algebraic) the families of surfaces defined above have no algebraic points except those along the intersection of the surface with $x=0$.

If b is transcendental and all the other constants algebraic, the surfaces have no algebraic points except those along the intersections with $y=0$.

Similarly for c transcendental. Concerning the other cases we can draw no general conclusions.

Similar results may be obtained for the planes,

$$ax+by+cz=d.$$

2). Exponential Surfaces.

The surfaces,

$$\psi_1(x,y,z) e^{\psi_2(x,y,z)} + \psi_3(x,y,z) = 0$$

where the ψ_i are rational functions involving algebraic numbers only, has no algebraic points except a finite number satisfying either of the two pairs of equations,

$$\begin{cases} \psi_1(x,y,z)=0 \\ \psi_2(x,y,z)=0 \end{cases} \quad \begin{cases} \psi_2(x,y,z)=0 \\ \psi_1(x,y,z) + \psi_3(x,y,z)=0 \end{cases}$$

This result is merely an extension of the result obtained in two dimensional space.

3). Trigonometric and Hyperbolic Surfaces.

The results of the previous discussion concerning two dimensional space may be generalized here in a manner analogous to the one used above.

PART II

TRANSCENDENTALISM OF NUMBERS.

The numbers corresponding to points in the continuum are divided into two classes, algebraic and transcendental. A number α is said to be an algebraic number of rank m if it satisfies an irreducible algebraic equation of the form,

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_{m-1} x + a_m = 0$$

where the coefficients and exponents are integers. All numbers which do not satisfy such equations are classed as transcendental. It has been shown by G. Cantor* that there are two kinds of infinite sets of objects: (1) countable sets, whose individual members can be enumerated so that to each member a definite place can be assigned in the system, (2) uncountable sets, for which this is not possible. The algebraic numbers are countable. The transcendental numbers are uncountable. Hence the class of transcendental numbers is incomparably greater than the class of algebraic numbers.

1. LIOUVILLE NUMBERS.

The existence of transcendental numbers was unknown until 1844 when Liouville, (Compte Rendus) showed that there was a larger class beyond the algebraic. Liouville's Theorem is as follows:

If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ be a sequence of rational fractions in their lowest terms defining an algebraic number ϵ of rank m , then for every constituent $\frac{p}{q}$ from and after an assignable stage, we have,

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{q^{m+1}} \quad (1)$$

Liouville's inequality furnishes us with a criterion for judging numbers. If a given number α violates (1), it is transcendental, but it does not necessarily follow that all transcendental numbers do violate it.

The Liouville numbers are of the form,

$$\frac{1}{10^{L_1}} + \frac{1}{10^{L_2}} + \frac{1}{10^{L_3}} + \dots + \frac{1}{10^{L_m}} + \dots = .1100010000\dots \quad (2)$$

or the same form with 1 replaced by any other algebraic number.

Consider the general series,

$$\eta_f = \frac{1}{10^{f(1)}} + \frac{1}{10^{f(2)}} + \dots + \frac{1}{10^{f(m)}} + \dots \quad (3)$$

where $f(m)$ is an increasing integral function. Let us find the condition that this represent a transcendental number. Since $f(m)$ is a steadily increasing integer, it follows that,

$$\left| \eta_f - \frac{I}{10^{f(m)}} \right| \leq \frac{1}{10^{f(m+1)-1}}$$

for all values of m , $\frac{I}{10^{f(m)}}$ being the sum of the first m terms of series(3). The following condition is sufficient to make η_f transcendental,

$$\left| \eta_f - \frac{I}{10^{f(m)}} \right| \leq \frac{1}{10^{f(m+1)-1}} \leq \frac{1}{10^{n f(m)}}, \quad n, \text{ an integer}$$

that is,

$$\frac{1}{10^{f(m+1)-1}} \leq \frac{1}{10^{n f(m)}}$$

$$\frac{1}{10^{f(m+1)-1-nf(m)}} \leq 1$$

Hence, $f(m+1)-1-nf(m)$ is positive or zero, that is,

$$\frac{f(m+1)-1}{f(m)} \geq n. \quad n^{(1)}, m^{(2)}$$

which is equivalent to,

$$\lim_{m \rightarrow \infty} \frac{f(m+1)}{f(m)} = \infty$$

To sum up our results in a theorem:

The infinite series,

$$\frac{1}{10^{f(1)}} + \frac{1}{10^{f(2)}} + \dots + \frac{1}{10^{f(m)}} + \dots$$

where $f(m)$ is an integral function of a positive integer represents a transcendental number, provided $\lim_{m \rightarrow \infty} \frac{f(m+1)}{f(m)} = \infty$.

We can now write down Liouville numbers ad libitum,

$$\frac{1}{10^{1^1}} + \frac{1}{10^{2^2}} + \frac{1}{10^{3^3}} + \dots + \frac{1}{10^{m^m}} + \dots$$

$$\frac{1}{10^{a^a}} + \frac{1}{10^{(2a)^{2a}}} + \frac{1}{10^{(3a)^{3a}}} + \dots + \frac{1}{10^{(ma)^{ma}}} + \dots$$

where a is any integer.

$$\frac{1}{10^{(11)^a}} + \frac{1}{10^{(12)^a}} + \dots + \frac{1}{10^{(1m)^a}} + \dots$$

etc.

All the numbers we have considered can easily be transferred to other scales than the scale of ten.

2. OTHER TRANSCENDENTAL NUMBERS.

The best known single transcendental numbers are e , the base of the Napierian Logarithm system and π , the ratio of the circumference of a circle and its diameter. Hermite, *Compte Rendus*, vol.77 (1873) first established the transcendency of e . Nine years later Lindemann, *Mathematische Annalen*, vol.20(1882) proved the transcendency of π and thus once and for all proved the impossibility of the squaring of the circle, one of the famous problems of antiquity. Hermite proved that e can not be the root of any algebraic equation whose coefficients and exponents are integers. Lindemann showed that e can not be the root of an equation whose exponents are any algebraic numbers whatever. Lindemann's theorem enables us to set up a large class of transcendental numbers.

The numbers e^{α} , where α may take on all the algebraic numbers of the continuum is transcendental.

Furthermore, any algebraic function of e with algebraic coefficients and algebraic exponents represents a transcendental number.

Our discussion of the transcendental points on curves shows that we have a countable number of transcendental numbers gained by substituting any algebraic number (with the few exceptions we noted) in any of the series representing direct or inverse circular or hyperbolic functions.

If a number t is transcendental, t plus any algebraic number whatever is transcendental.

If a number t is transcendental, t multiplied by any algebraic number (excepting zero) is transcendental.

PART III

AN INVESTIGATION OF THE HILBERT PROOF OF THE TRANSCENDENTALISM OF e .

As we have already explained, in 1844 Hermite first showed that e is not algebraic by proving that it cannot satisfy an algebraic equation of degree m . Is it not possible to prove the same thing by showing that e cannot satisfy the Liouville inequality? The following results show the efforts of the writer attacking the problem. This Chapter may be regarded as a preliminary report on an investigation of rather far reaching character which the writer hopes to carry further after this year.

The simplest proof we know of the transcendentalism of e is the Hilbert proof. We will first point out the similarities between this proof and the proposed Liouville proof. In our discussion we will follow the Hilbert proof as given in Goursat-Hedrick, "Mathematical Analysis", Ginn and Company, 1904.

By partial integration there results the equality,

$$\int_a^b e^{\omega x} f(x) dx = \left\{ e^{\omega x} \left[\frac{f(x)}{\omega} - \frac{f'(x)}{\omega^2} + \frac{f''(x)}{\omega^3} + \dots + (-1)^n \frac{f^{(n)}(x)}{\omega^{n+1}} \right] \right\}_a^b \quad (1)$$

Setting $a = 0$ and $\omega = -1$, it becomes,

$$\int_0^b e^{-x} f(x) dx = - \left[e^{-x} F(x) \right]_0^b, \quad (2)$$

where,

$$F(x) = f(x) + f'(x) + f''(x) + \dots + f^{(n)}(x) \quad (3)$$

And this again may be written in the form,

$$F(b) = e^b F(0) - e^b \int_0^b e^{-x} f(x) dx. \quad (4)$$

Hilbert's next step is to suppose, for the purpose of a Reductio ad Absurdum, that e satisfies an equation of the form,

$$c_0 + c_1 e + c_2 e^2 + c_3 e^3 + \dots + c_m e^m = 0 \quad (5)$$

Then setting $b=0,1,2,3,\dots,m$ successively in (4) and adding the results obtained after multiplying them respectively by $c_0, c_1, c_2, \dots, c_m$, he obtains,

$$c_0 F(0) + c_1 F(1) + c_2 F(2) + \dots + c_m F(m) + \sum_{i=0}^m c_i e^i \int_0^i e^{-x} f(x) dx = 0 \quad (6)$$

Hilbert proceeds to show that such a relation is impossible if the polynomial $f(x)$, which is up to the present arbitrary, be properly chosen. His choice is as follows:

$$f(x) = \frac{1}{p-1} x^{p-1} \left\{ (x-1)(x-2)\dots(x-m) \right\}^p,$$

where p is a prime number greater than m . The polynomial is of degree $mp + p - 1$, and all the coefficients of its successive derivatives past the p^{th} are integral multiples of p , since the product of p successive integers is divisible by $p!$. Moreover, $f(x)$, together with its first $(p-1)$ derivatives, vanishes for $x=1,2,3,\dots,m$, and it follows that $F(1), F(2), \dots, F(m)$ are all integral multiples of p . It only remains to calculate $F(0)$ i.e.

$$F(0) = f(0) + f'(0) + \dots + f^{(p-1)}(0) + f^{(p)}(0) + f^{(p+1)}(0) + \dots$$

In the first place, $f(0) = f'(0) = \dots = f^{(p-2)}(0) = 0$, while $f^{(p)}(0), f^{(p+1)}(0)$ are all integral multiples of p , as we have just shown. To find $f^{(p-1)}(0)$ multiply the coefficient of x^{p-1} in $f(x)$ by $\frac{p-1}{p}$, which gives $\pm (1 \cdot 2 \cdot 3 \dots m)^p$. Hence the sum,

$$c_0 F(0) + c_1 F(1) + \dots + c_m F(m).$$

is equal to an integral multiple of p added to,

$$\pm c_0 (1 \cdot 2 \cdot 3 \dots m)^p$$

If p is taken greater than m or c_0 , the above number cannot be divisible by p ; hence the first portion of the sum (6) will be an integer different from zero.

The last step of the proof is to show that,

$$\sum_0^m c_i e^{i \int_0^i e^{-x} f(x) dx}$$

can be made smaller than any preassigned quantity by taking p sufficiently large. As x varies from 0 to 1 each factor of $f(x)$ is less than m ; hence,

$$|f(x)| < \frac{1}{p-1} m^{mp+p-1}$$

$$\left| \int_0^i f(x) e^{-x} dx \right| < \frac{1}{p-1} m^{mp+p-1} \int_0^i e^{-x} dx < \frac{1}{p-1} m^{mp+p-1},$$

from which it follows that,

$$\left| \sum_0^m c_i e^{i \int_0^i e^{-x} f(x) dx} \right| < \frac{M m^{mp+p-1}}{p-1} e^m = \phi(p),$$

where M is an upper limit of $|c_0| + |c_1| + \dots + |c_m|$.

As p increases indefinitely the function $\phi(p)$ approaches zero, for it is the general term of a convergent series. It follows that a prime number p can be chosen so large that (6) is impossible. Such is Hilbert's method of showing that e is transcendental.

Now let us indicate the proposed Liouville method and point out the similarity of it to the previous method. We will follow the Hilbert proof to (4). Then for the purpose of simplification let us assign to ξ the value 1, so that we have,

$$F(1) = e F(0) - e \int_0^1 e^{-x} f(x) dx,$$

$$\left| e - \frac{F(1)}{F(0)} \right| = \left| \frac{e \int_0^1 e^{-x} f(x) dx}{F(0)} \right|$$

Proving that e violates the Liouville Theorem amounts to proving that,

$$\left| e - \frac{F(1)}{F(0)} \right| = \left| \frac{e \int_0^1 e^{-x} f(x) dx}{F(0)} \right| \leq \frac{1}{\{F(0)\}^{m+1}}, \text{ for every } m^{\textcircled{0}}.$$

i.e. that,

$$\left| e \int_0^1 e^{-x} f(x) dx \right| \left[F(0) \right]^m \leq 1, \quad m^{\textcircled{0}}.$$

Five conditions must be satisfied in order to conduct the Liouville proof, namely,

- 1) $f(x)$ must be a polynomial.
- 2) $F(1)$ and $F(0)$ must be integers(not zero).
- 3) $F(1)$ must be prime to $F(0)$.
- 4) $\left| e \int_0^1 e^{-x} f(x) dx \right|$ must be made as small as we please.

$$3) \left| e \int_0^1 e^{-x} f(x) dx \right| [F(0)]^m \leq 1 \text{ for every } m^{\textcircled{1}}.$$

Let us see how many of these conditions the Hilbert proof makes use of. In the first place, 1) is essential to both theorems. Hilbert chose his polynomial so that $F(1)$ and $F(0)$ are both integers. Furthermore, $F(0)$ cannot be zero, since it equals an integral multiple of $p \pm (1 \cdot 2 \cdot 3 \cdots m)^p$, where p is a prime number larger than m . The Hilbert proof makes $\sum_0^m c_i e^i \int_0^1 e^{-x} f(x) dx$, which corresponds to our $\left[e \int_0^1 e^{-x} f(x) dx \right]$ less than any preassigned quantity. Hence the Hilbert proof makes use of conditions, (1), (2), and (4). To prove e not algebraic by the Liouville method we must choose an $f(x)$ which exhibits two peculiarities in addition to those which the Hilbert $f(x)$ possesses. That the Hilbert proof and the proposed Liouville proof are very close together is quite evident from the above.

Having noticed the similarities between the two proofs, let us consider the methods which the writer used. In the first place it seems barely possible that a shortened form of the function which is so potent for Hilbert, may serve our purposes. Let us try,

$$f(x) = \frac{x^{p-1}}{1^{p-1}} (x-1)^p.$$

Evidently for this function,
$$e \int_0^1 e^{-x} f(x) dx < \frac{K}{1^{p-1}}$$

$$F(0) = \frac{1}{1^{p-1}} \left\{ \frac{1^{2p-1}}{1^{p-1}} - \binom{p}{1} \frac{1^{2p-2}}{1^{p-1}} + \binom{p}{2} \frac{1^{2p-3}}{1^{p-1}} - \cdots + p \frac{1^p}{1^{p-1}} - \frac{1^{p-1}}{1^{p-1}} \right\}$$

$$F(1) = \frac{1}{1^{p-1}} \left\{ [1 + (2p-1) + (2p-1)(2p-2) + \dots + 1^{2p-1}] - \binom{p}{1} [1 + (2p-2) + \dots + 1^{2p-2}] \right. \\ \left. + \dots - [1 + (p-1) + \dots + 1^{p-1}] \right\}$$

The obstacle here lies in the fact that $F(0)$ cannot be controlled, and in fact, approaches infinity faster than p (as can be seen by grouping the above terms in twos). Hence there is no possibility of proving,

$$\frac{1}{1^{p-1}} \left\{ F(0) \right\}^m \leq 1$$

with the function we assumed. In addition, $F(1)$ approaches infinity more rapidly than p . Hence there is no possibility of gaining anything by putting our equality in the form,

$$\left| e^{-1} - \frac{F(0)}{F(1)} \right| = \left| \frac{\int_0^1 e^{-x} f(x) dx}{F(1)} \right|$$

and showing that $\left[\int_0^1 e^{-x} f(x) dx \right] [F(1)]^m \leq 1$

Thus attempt number one proves futile.

As a second attempt let us try to determine $f(x)$ so that $F(0) = p$, and $e \int_0^1 e^{-x} f(x) dx < \frac{K}{1^p}$ (suggested by the fact that $\frac{p^m}{1^p} \leq 1$)

⁽¹⁾ ⁽²⁾ Since $f(x)$ is a polynomial, it has the form,

$$\frac{1}{1^p} [a_0 x^p + a_1 x^{p-1} + a_2 x^{p-2} + \dots + a_{p-2} x^2 + a_{p-1} x]$$

For this function,

$$F(0) = \frac{1}{1^p} \left[a_0 1^p + a_1 1^{p-1} + a_2 1^{p-2} + \dots + a_{p-2} 1^2 + a_{p-1} 1 \right]$$

$$= a_0 + \frac{a_1}{p} + \frac{a_2}{p(p-1)} + \dots + \frac{1}{1^p} a_{p-1}$$

Now choose the arbitrary constants, $a_0, a_1, a_2, \dots, a_{p-1}$, in the following manner: $a_0 = 1$, $a_1 = p$, $a_2 = p(p-1)$ $a_{p-1} = \frac{1^p}{1!}$

This choice makes $F(0) = p$. Let us examine $F(1)$.

$$F(0) = \frac{1}{1^p} \left\{ \left[1 + p + p(p-1) + \dots + 1^p \right] + p \left[1 + (p-1) + (p-1)(p-2) + \dots + 1^{p-1} \right] \right.$$

$$\left. + p(p-1) \left[1 + (p-2) + \dots + 1^{p-2} \right] + \dots + \frac{1^p}{1!} \left[1 + \frac{1}{1!} \right] \right\}$$

$$= \left[1 + \frac{1}{1!} + \frac{1}{1^2} + \dots + \frac{1}{1^p} \right] + \left[1 + \frac{1}{1!} + \frac{1}{1^2} + \dots + \frac{1}{1^{p-1}} \right] + \dots + \left[1 + \frac{1}{1!} \right]$$

The difficulty here lies in proving $F(1)$ an integer.

The same general problem which we have been discussing may be studied from the standpoint of another proof of the transcendentalism of e . Veblen Lennes in, "An Introduction to Infinitesimal Analysis", John Wiley and Sons, 1907, gives a proof of the transcendency of e which suggested to us the investigation we have begun. The Veblen Lennes proof is too long to reproduce. We will refer the reader to Veblen Lennes, page 19. In the course of the proof we have the equality,

$$n_p e^k = p W_{kp} + r_{kp}$$

where W_{kp} is an integer or zero, n_p an integer, not zero and not divisible by p , and r_{kp} an expression which can be made as small as we please by choosing p sufficiently large. Dividing by n_p

and transposing, we have,

$$\left| e^k - \frac{p W_{kp}}{n_p} \right| = \left| \frac{n_{kp}}{n_p} \right|$$

This last equality bears a striking resemblance to the reversed form of the Liouville inequality,

$$\left| e - \frac{p}{q} \right| \leq \frac{1}{q^{m+1}}$$

The fraction, $\frac{p W_{kp}}{n_p}$ corresponds to $\frac{p}{q}$. Both numerator and denominator are integers which are relatively prime provided W and n are prime to each other. The expression, $\frac{n_{kp}}{n_p}$ can be made as small as we please, and so corresponds to the right hand member of the Liouville inequality, $\frac{1}{q^{m+1}}$. Proving that e violates the Liouville theorem, in this case, amounts to proving that,

$$\left| e^k - \frac{p W_{kp}}{n_p} \right| = \left| \frac{n_{kp}}{n_p} \right| \leq \frac{1}{n_p^{m+1}}$$

for every integral m , that is, that,

$$\left| n_{kp} \right| [n_p] \leq 1, \quad m^{(1)}, p^{(2)}$$

We found this extremely difficult to handle on account of the complexity of n_p

As we said in the beginning, this Chapter is a preliminary report of an investigation which the writer hopes to carry on later. We have not been able to prove that e violates the Liouville inequality. Perhaps it does do so, perhaps it does not. Perhaps e is one of an uncountable class of numbers which fail to violate Liouville's Theorem. At the present time we do not know.

Columbia, Missouri, May 15, 1916.

To the Dean of the Graduate Faculty

The thesis of Miss Jewell C. Hughes,

"Transcendentalism of Curves and of Numbers",

is approved, as fulfilling part of the requirements

for the Master's degree.

W. D. A. Westfall

UNIVERSITY OF MISSOURI
COLUMBIA

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OFFICE OF THE DIRECTOR

May 19, 1916.

Dean Walter Miller,
211 Academic Hall.

My dear Dean Miller,

I have examined the thesis submitted by
Miss Jewell C. Hughes, entitled TRANSCENDENTALISM OF CURVES
AND OF NUMBERS. In my opinion it meets the general standard
which has been established in this University for the Master's
dissertation.

Very truly yours,

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