GENERALIZATIONS OF THEOREMS
FROM
THE THEORY OF FUNCTIONS

197

by

Julia Spalding, B. S.

SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS

in the

GRADUATE SCHOOL

of the

UNIVERSITY OF MISSOURI

1915

INTRODUCTION.

Practically all of the text books and lectures on the Theory of Functions of a Complex Variable treat only those functions which have a derivative in every point of some given region. This derivative must be independent of the direction in which the increment of the variable is taken. Functions which have this property are called analytic functions of a complex variable. In some cases these are called simply functions, the non analytic functions not being considered functions at all. Professor Osgood in his "Lehrbuch der Funktionentheorie" is an exception to this custom. He often develops a general theorem and then gives a corresponding one for the analytic function, sometimes with added results, as a special case.

There exist, of course, many rather simple functions, which do not possess a derivative. In this paper the results of some of the important theorems concerning analytic functions will be extended to the non analytic ones. In some cases the extension can be made immediately, the analytic property not being an essential part of the hypothesis. In other cases the results in the case of the non analytic functions are different, though still very interesting. There are functions which are continuous and have partial derivatives in different directions, but which are not analytic. Results very similar to those for analytic functions can often be obtained for these. In all cases the object has been to reduce the hypothesis to a minimum. This occasionally results in much more difficult proofs but in a wider application of the theorem. In other cases where the analytic property is not assumed, other hypotheses must be added which restrict the application of the theorem.

In all work z will be taken as the complex variable, made up of the two real variables x and y , which are connected by the relation, $z = x + iy$. w will be the letter used for the function of z , consisting of the two real parts, u and v , so that $w = u + iv$. This will make u and v both real functions of the two real variables, x and y , so that

$$w = u + iv = f(z) = f(x + iy) = A(x, y) + iB(x, y).$$

--oOo--

CHAPTER I

THE EXISTENCE OF THE DERIVATIVE.

1. Usual Theory. Since the existence of a derivative is such an important property in the Theory of Functions of a Complex Variable, it will be well first to find just what are the necessary and sufficient conditions which a function must satisfy in order to have a derivative.

Given:

$$w = u + iv = A(x, y) + iB(x, y).$$

Building up the quotient function for the derivative in the same way as in elementary calculus, we have

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}.$$

In the usual theory this limit is considered only when it exists independently of the direction taken by the increment. We will occasionally consider the limit as a function of the angle between Δx and Δy , that is of the derivative $\frac{dw}{dz}$. At present, however, keeping to the ordinary meaning of the derivative, the limit must be the same whether the increment is real ($\Delta y = 0$) or pure imaginary ($\Delta x = 0$).

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta_x u + i \Delta_x v}{\Delta x} = \lim_{\Delta y \rightarrow 0} \frac{\Delta_y u + i \Delta_y v}{i \Delta y} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\Delta_x u}{\Delta x} + i \frac{\Delta_x v}{\Delta x} \right] = \lim_{\Delta y \rightarrow 0} \left[\frac{\Delta_y u}{i \Delta y} + \frac{\Delta_y v}{\Delta y} \right] \end{aligned}$$

Passing to the limit,

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating real and imaginary parts, we have:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned}$$

These equations, called the Cauchy Riemann Equations, form a necessary condition for the existence of a derivative in the usual sense.

To prove that these equations are sufficient to insure a derivative, let

$$w = u + iv = f(z) = A(x,y) + iB(x,y)$$

and

$$A_x = B_y$$

$$A_y = -B_x.$$

We must assume that these partial derivatives all exist. Building up the difference quotient in the usual way, we have:

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} \\ &= \frac{A(x + \Delta x, y + \Delta y) - A(x, y)}{\Delta x + i \Delta y} + i \frac{B(x + \Delta x, y + \Delta y) - B(x, y)}{\Delta x + i \Delta y}. \end{aligned}$$

The remaining work depends on the continuity of the partial derivatives A_x, A_y, B_x, B_y . At first assume that these derivatives are all continuous. Then it is well known⁽¹⁾ that

$$\begin{aligned} \Delta u &= A(x + \Delta x, y + \Delta y) - A(x, y) \\ &= A_x(x + \theta \Delta x, y + \theta \Delta y) \Delta x + A_y(x + \theta \Delta x, y + \theta \Delta y) \Delta y, \quad 0 < \theta < 1. \\ \Delta v &= B(x + \Delta x, y + \Delta y) - B(x, y) \\ &= B_x(x + \theta' \Delta x, y + \theta' \Delta y) \Delta x + B_y(x + \theta' \Delta x, y + \theta' \Delta y) \Delta y, \quad 0 < \theta' < 1 \end{aligned}$$

To simplify the notation, let

$$\begin{aligned} A(x + \theta \Delta x, y + \theta \Delta y) &= A^* \\ B(x + \theta' \Delta x, y + \theta' \Delta y) &= B^\# \end{aligned}$$

Then

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} \\ &= \lim_{\Delta z \rightarrow 0} \frac{A_x^* \Delta x + A_y^* \Delta y + i (B_x^\# \Delta x + B_y^\# \Delta y)}{\Delta x + i \Delta y}. \end{aligned}$$

Dividing numerator and denominator by Δx and passing to the limit,

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} = \frac{A_x + A_y \frac{dy}{dx} + i B_x + i B_y \frac{dy}{dx}}{1 + \frac{dy}{dx} i}.$$

Applying the Cauchy Riemann equations to this result,

--o--

- (1) If $f(x,y)$ is a function of the two independent variables x and y and if the partial derivatives f_x and f_y are continuous, then

$$f(x+h, y+k) - f(x, y) = h f_x(x + \theta h, y + \theta k) + k f_y(x + \theta h, y + \theta k), \quad 0 < \theta < 1.$$

Goursat "Mathematical Analysis", Vol.I, page 15.

$$\begin{aligned} \lim_{\Delta Z \rightarrow 0} \frac{\Delta w}{\Delta Z} &= \frac{A_x \left(1 + i \frac{dy}{dx}\right) + A_y \left(\frac{dy}{dx} - i\right)}{1 + i \frac{dy}{dx}} \\ &= \frac{(A_x - i A_y) \left(1 + i \frac{dy}{dx}\right)}{1 + i \frac{dy}{dx}} = A_x - i A_y. \end{aligned}$$

As a result we may state:

Theorem I. A function of a complex variable has a derivative or is analytic, if the Cauchy Riemann equations hold and all the partial derivatives exist and are continuous.

If no assumption concerning the continuity of the partial derivatives is made, the first part of the work goes as before.

$$\lim_{\Delta Z \rightarrow 0} \frac{\Delta w}{\Delta Z} = \lim_{\Delta Z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}.$$

Then by another form of the Law of the Mean,⁽²⁾

$$\begin{aligned} \Delta u &= A(x + \Delta x, y + \Delta y) - A(x, y) \\ &= A_x(x + \theta \Delta x, y + \Delta y) \Delta x + A_y(x, y + \theta' \Delta y) \Delta y, \quad 0 < \theta < 1, \quad 0 < \theta' < 1. \\ \Delta v &= B(x + \Delta x, y + \Delta y) - B(x, y) \\ &= B_x(x + \theta \Delta x, y + \Delta y) \Delta x + B_y(x, y + \theta' \Delta y) \Delta y, \quad 0 < \theta < 1, \quad 0 < \theta' < 1. \end{aligned}$$

In order for $A_x(x + \theta \Delta x, y + \Delta y)$ to approach $A_x(x, y)$ as Δx and Δy approach zero, A_x must be a continuous function of x and y . For the same reason B_x must be continuous. In order for $A_y(x, y + \theta' \Delta y)$ to approach $A_y(x, y)$, A_y must be a continuous function of y for each constant x . Similarly B_y must be continuous

--o--

(2) If $f(x, y)$ is a function of the two independent variables x and y , then $f(x+h, y+k) - f(x, y) = h f_x(x + \theta h, y + k) + k f_y(x, y + \theta' k)$
 $0 < \theta < 1, \quad 0 < \theta' < 1,$ whether the derivatives

are continuous or not.

Goursat, page 15.

in y for each x . If these conditions are fulfilled:

$$\begin{aligned} \lim_{\Delta z \neq 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} &= \frac{A_x + A_y \frac{dy}{dx} + i B_x + i B_y \frac{dy}{dx}}{1 + i \frac{dy}{dx}} \\ &= \frac{A_x + i B_x + i(B_y - i A_y) \frac{dy}{dx}}{1 + i \frac{dy}{dx}}. \end{aligned}$$

If the Cauchy Riemann equations are substituted, this becomes:

$$\frac{(A_x - i A_y)(1 + i \frac{dy}{dx})}{1 + i \frac{dy}{dx}} = A_x - i A_y.$$

Therefore:

Theorem II. A function $f(z)$ will possess a derivative independent of $\frac{dy}{dx}$, if

$$\begin{aligned} A_x &= B_y \\ A_y &= -B_x \end{aligned}$$

and A_x and B_x are continuous in x and y , and A_y and B_y are continuous in y for each constant x .

By a third form of the Law of the Mean⁽³⁾ which holds whether the derivatives are continuous or not,

$$\begin{aligned} \Delta u &= A(x + \Delta x, y + \Delta y) - A(x, y) \\ &= A_x(x + \theta \Delta x, y + \Delta y) \Delta x + A_y(x, y + \theta \Delta y) \Delta y, \quad 0 < \theta < 1. \end{aligned}$$

$$\begin{aligned} \Delta v &= B(x + \Delta x, y + \Delta y) - B(x, y) \\ &= B_x(x + \theta' \Delta x, y + \Delta y) \Delta x + B_y(x, y + \theta' \Delta y) \Delta y, \quad 0 < \theta' < 1. \end{aligned}$$

In this case also, A_x and B_y must be continuous functions of x and y , and A_y and B_x must be continuous in y for each constant x . This may therefore be regarded as a second proof of Theorem II, stated just above.

--o--

(3) If $f(x, y)$ is a function of the two independent variables, x and y ,

$$f(x + h, y + k) - f(x, y) = h f_x(x + \theta h, y + k) + k f_y(x, y + \theta k), \quad 0 < \theta < 1.$$

whether f_x and f_y are continuous or not.

Goursat, page 16^y - footnote.

2. Other Forms of the Sufficient Conditions. The work of forming the difference quotient may be performed differently, resulting in a slightly different statement for the sufficient conditions for the derivative.

As before let:

$$w = u + iv = f(z) = A(x,y) + iB(x,y)$$

where

$$A_x = B_y$$

$$A_y = -B_x.$$

Then:

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}.$$

In order to separate this quotient into real and imaginary parts, multiply both numerator and denominator by $\Delta x - i \Delta y$.

$$\begin{aligned} \frac{\Delta w}{\Delta z} &= \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} \cdot \frac{\Delta x - i \Delta y}{\Delta x - i \Delta y} \\ &= \frac{\Delta \bar{u} \cdot \Delta \bar{x} + \Delta \bar{v} \cdot \Delta \bar{y}}{\Delta \bar{x}^2 + \Delta \bar{y}^2} + i \frac{\Delta \bar{v} \cdot \Delta \bar{x} - \Delta \bar{u} \cdot \Delta \bar{y}}{\Delta \bar{x}^2 + \Delta \bar{y}^2}. \end{aligned}$$

In order for these terms to approach a limit as Δx and Δy approach zero, u and v must be functions which have total differentials at the point (x,y) .

$$\text{i.e. } \Delta u = A(x + \Delta x, y + \Delta y) - A(x, y) = A_x(x, y) \Delta x + A_y(x, y) \Delta y + \epsilon_1(\Delta x, \Delta y)$$

where ϵ_1 is an infinitesimal of higher order than Δx or Δy so that:

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\epsilon_1}{\Delta \bar{x}^2 + \Delta \bar{y}^2} = 0.$$

Also:

$$\Delta v = B(x + \Delta x, y + \Delta y) - B(x, y) = B_x(x, y) \Delta x + B_y(x, y) \Delta y + \epsilon_2(\Delta x, \Delta y)$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\epsilon_2}{\Delta \bar{x}^2 + \Delta \bar{y}^2} = 0.$$

Then:

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta \bar{u} \cdot \Delta \bar{x} + \Delta \bar{v} \cdot \Delta \bar{y}}{\Delta \bar{x}^2 + \Delta \bar{y}^2} =$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{A_x \Delta \bar{x}^2 + (A_y + B_x) \Delta \bar{x} \cdot \Delta \bar{y} + B_y \Delta \bar{y}^2}{\Delta \bar{x}^2 + \Delta \bar{y}^2} + \frac{\epsilon}{\Delta \bar{x}^2 + \Delta \bar{y}^2} \right]$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{A_x (\overline{\Delta x}^2 + \overline{\Delta y}^2)}{\overline{\Delta x}^2 + \overline{\Delta y}^2} + \frac{\epsilon}{\overline{\Delta x}^2 + \overline{\Delta y}^2} \right] = A_x.$$

Similarly

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\overline{\Delta v} \cdot \overline{\Delta x} - \overline{\Delta u} \cdot \overline{\Delta y}}{\overline{\Delta x}^2 + \overline{\Delta y}^2} =$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{B_x \overline{\Delta x}^2 + (B_y - A_x) \overline{\Delta x} \cdot \overline{\Delta y} - A_y \overline{\Delta y}^2}{\overline{\Delta x}^2 + \overline{\Delta y}^2} + \frac{\epsilon'}{\overline{\Delta x}^2 + \overline{\Delta y}^2} \right] =$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{B_x (\overline{\Delta x}^2 + \overline{\Delta y}^2)}{\overline{\Delta x}^2 + \overline{\Delta y}^2} + \frac{\epsilon'}{\overline{\Delta x}^2 + \overline{\Delta y}^2} \right] = B_x.$$

Therefore:

$$\lim_{\substack{\Delta z \rightarrow 0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} = A_x + i B_x.$$

Theorem III. If $w = u + i v$ is a function of the complex variable z such that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and that u and v are functions of x and y which have total differentials, then w will be an analytic function.

CHAPTER II

DEFINITE INTEGRALS.

3. Maximum Value Theorem. In the case of analytic functions a theorem, which is used in the proof of many important later theorems, states that the integral of an analytic function along any path is less than or equal to the maximum value of the function along that path multiplied by the length of the path. This theorem admits of immediate extension to non analytic functions.

Theorem IV. The integral of a function, which is single valued and continuous along a curve, is less than or equal to the maximum value of the function along that curve multiplied by the length of the curve.

Given: $f(z)$ a continuous function along a curve C , $|f(z)| \leq M$,
 L the length of C .

To prove:

$$\left| \int_C f(z) dz \right| \leq M \cdot L.$$

As usual in building up an integral by means of an infinite sum, divide the curve into n equal parts of length Δz by means of the points $z_1, z_2, \dots, z_n$.

Then:

$$\begin{aligned} \left| \int_C f(z) dz \right| &= \lim_{\Delta z \rightarrow 0} \left| f(z_1)\Delta z + f(z_2)\Delta z + \dots + f(z_n)\Delta z \right| \\ &\leq \lim_{\Delta z \rightarrow 0} \left[|f(z_1)\Delta z| + |f(z_2)\Delta z| + \dots + |f(z_n)\Delta z| \right] \\ &\leq \lim_{\Delta z \rightarrow 0} \left[|f(z_1)| \cdot |\Delta z| + |f(z_2)| \cdot |\Delta z| + \dots + |f(z_n)| \cdot |\Delta z| \right] \\ &\leq \lim_{\Delta z \rightarrow 0} M \left| \Delta z + \Delta z + \dots \right| \leq M \cdot L \\ \therefore \left| \int_C f(z) dz \right| &\leq M \cdot L. \end{aligned}$$

If $f(z)$ is limited and has a finite number of discontinuities along C , the curve may be divided into a finite number of parts, in each of which the function is continuous. Then the result proved above can be applied to each part.

Therefore:

Theorem IV'. The integral of a function, which is finite and has

at most a finite number of discontinuities along a curve, is less than or equal to the maximum value of the function along that curve, multiplied by the length of the curve.

4. Cauchy Integral Theorem. One of the theorems, on which much of the theory of functions of a complex variable is based, is the Cauchy Integral Theorem. This enables one to find the value of an analytic function from its value on the boundary of some given region. A similar theorem which holds for non analytic functions as well is developed by Osgood.⁽⁴⁾

Theorem V. If C is an arbitrary, single valued, regular curve and $\phi(t)$ a function, with t a variable point of C, then $F(z)$, the function defined by the equation

$$F(z) = \int_C \frac{\phi(t)}{t-z} dt,$$

is analytic in every point of the plane except on the curve C.

By definition, in order to prove a function analytic in any given region, it is necessary to prove that the derivative exists at every point of that region.

Let z_0 be any arbitrary point of the plane not on C and draw a curve T around z_0 , so that no point of T is nearer than η to C. Take a second point, $z_0 \neq z$, in T and build up the difference quotient.

$$\begin{aligned} \frac{F(z_0 + \Delta z) - F(z_0)}{\Delta z} &= \int_C \frac{1}{\Delta z} \left[\frac{1}{(t-z_0-\Delta z)} - \frac{1}{(t-z_0)} \right] \phi(t) dt \\ &= \int_C \frac{\phi(t)}{(t-z_0-\Delta z)(t-z_0)} dt. \end{aligned}$$

From the hypothesis, $(t-z_0-\Delta z)(t-z_0) > \eta^2$, so the denominator of this fraction cannot vanish.

$$\lim_{\Delta z \rightarrow 0} \int_C \frac{\phi(t)}{(t-z_0-\Delta z)(t-z_0)} dt = \int_C \frac{\phi(t)}{(t-z_0)^2} dt.$$

$$F'(z) = \int_C \frac{\phi(t)}{(t-z)^2} dt.$$

Since this derivative of $F(z)$ exists at every point of the plane except on C, $F(z)$ is analytic with the same exception.

As an example of the different results obtainable in the analytic case, consider the Cauchy Theorem⁽⁵⁾ itself. If $f(z)$

--o--

(4) Osgood "Lehrbuch der Funktionentheorie", page 241.

(5) Osgood, page 251.

is a function defined at every point in a closed region S , whose boundary is a regular curve C , and is analytic in the interior of S , then $f(z)$ at any interior point of S is given by the formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt.$$

This analytic case, in which the two functions are the same, may be regarded as a special case under the previous theorem.

5. Higher Derivatives. It was just proved that $F(z)$, defined by:

$$F(z) = \int_C \frac{\phi(t)}{t-z} dt,$$

was analytic, and in particular that

$$F'(z) = \int_C \frac{\phi(t)}{(t-z)^2} dt.$$

By building up the successive difference functions, it can be shown, under the same hypothesis, that:

$$F''(z) = 2! \int_C \frac{\phi(t)}{(t-z)^3} dt$$

$$F'''(z) = 3! \int_C \frac{\phi(t)}{(t-z)^4} dt$$

⋮

$$F^{[n]}(z) = n! \int_C \frac{\phi(t)}{(t-z)^{n+1}} dt.$$

$F(z)$ and its successive derivatives are analytic except on the curve C . $\phi(t)$ may or may not be analytic.

As a result of these statements, the following may be proved directly.

If $\phi(t)$ is continuous and bounded along a curve C , L the length of C , z a point of the plane such that $|t-z| > \eta$ and

$$F(z) = \int_C \frac{\phi(t)}{t-z} dt$$

then:

$$\left| F(z) \right| \leq \left| \int_C \frac{\phi(t)}{t-z} dt \right| \leq \frac{M}{\eta} \cdot L.$$

$$\begin{aligned}
|F'(z)| &\leq 1! \left| \int_C \frac{\phi(t)}{(t-z)^2} dt \right| \leq 1! \frac{M}{\eta^2} \cdot L \\
|F''(z)| &\leq 2! \left| \int_C \frac{\phi(t)}{(t-z)^3} dt \right| \leq 2! \frac{M}{\eta^3} \cdot L \\
&\vdots \\
|F^{[n]}(z)| &\leq n! \left| \int_C \frac{\phi(t)}{(t-z)^{n+1}} dt \right| \leq n! \frac{M}{\eta^{n+1}} \cdot L.
\end{aligned}$$

Therefore:

Theorem VI. If $F(z)$ is the function defined by the equation:

$$F(z) = \int_C \frac{\phi(t)}{t-z} dt, \text{ where } \phi(t) \text{ has}$$

a maximum value M on C , L is the length of C , and z is a point of the plane such that $|t-z| > \eta$, then the maximum value of any derivative is given by the formula:

$$F^{[n]}(z) \leq n! \frac{M}{\eta^{n+1}} \cdot L.$$

6. Liouville Theorems. As a result of the statements for analytic functions corresponding to those just proved, there is a series of theorems called the Liouville Theorems.⁽⁶⁾ These may be stated as follows:

A function which is analytic and bounded everywhere, even at the point at infinity, is a constant.

A function which is bounded and analytic everywhere with the exception of one pole of the first order is equal to

$$\frac{a_1}{z} + a_0.$$

A function which is bounded and analytic everywhere with the exception of one pole of the n^{th} order is equal to

$$\frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \dots + \frac{a_1}{z} + a_0.$$

Analogous theorems can be proved for non analytic functions by using the statements proved concerning the value of the successive derivatives.

Given:

(a) A function $\phi(t)$, continuous and finite all over the Z plane.

(b) A set of curves C , such that the length of each curve $l_c \leq k \rho_{\max}$, where $\rho_{\max}$ is the maximum distance of any point on that curve from an arbitrary point z_0 and k a constant.

It is necessary to define the curves, since in non analytic

--o--

(6) See Osgood, page 255.

functions the shape of the curve may change the value of the integral:

$$(c) \lim_{\rho_{\min.}} \frac{\rho_{\max.}}{\rho_{\min.}} = k \text{ where } |t - z_0| \geq \rho_{\min.}$$

$$(d) F(z) = \int_C \frac{\phi(t)}{t - z_0} dt.$$

Then

$$|F'(z_0)| \leq \frac{M \cdot l_c}{\rho_{\min.}^2} \leq \frac{M \cdot K \cdot \rho_{\max.}}{\rho_{\min.}^2} \leq \frac{M \cdot K \cdot k}{\rho_{\min.}}.$$

$$\lim_{\rho_{\min.} \rightarrow \infty} |F'(z_0)| \leq \lim_{\rho_{\min.} \rightarrow \infty} \frac{M \cdot K \cdot k}{\rho_{\min.}} = 0, \text{ since}$$

M, K, k are constants.

Since z_0 was any point of the plane,

$$F'(z) = 0$$

$$F(z) = \text{constant}.$$

Theorem VII. If $F(z)$ is the function defined by the equation,

$$F(z) = \int_C \frac{\phi(t)}{t - z_0} dt, \text{ where } \phi(t) \text{ is a function continuous and}$$

finite all over the z plane, and C is a set of curves, such that the length of each is less than or equal to a constant times the maximum distance of any point on that curve from z_0 , and the ratio of the maximum to the minimum distance is a constant, then $F(z)$ is a constant.

Given:

(a) $\phi(t)$ a function which is continuous and finite with the exception of one pole of the first order at z_0 .

(b) A set of curves C , such that $l_c \leq K \rho_{\max.}$

$$(c) \lim_{\rho_{\min.}} \frac{\rho_{\max.}}{\rho_{\min.}} = k.$$

If we set $\phi(t) = \frac{f(t)}{t - z_0}$
finite everywhere.

$f(t)$ will be continuous and

$$f(t) = \phi(t) (t - z_0)$$

$$|f(t)| \leq M |t - z_0| \leq M \rho_{\max.}$$

$$\begin{aligned}
 \text{If } F(z) &= \int_C \frac{f(t)}{t-z_0} dt \\
 |F''(z)| &\leq 2! \frac{M \rho_{\max} \cdot l_c}{\rho_{\min}^3} \leq 2! \frac{M \cdot K \cdot \rho_{\max}^2}{\rho_{\min}^3} \leq 2! \frac{M \cdot K \cdot k}{\rho_{\min}}. \\
 \lim_{\rho_{\min} \rightarrow \infty} |F''(z)| &\leq \lim_{\rho_{\min} \rightarrow \infty} 2! \frac{M \cdot K \cdot k^2}{\rho_{\min}} = 0. \\
 F'''(z) &= 0 \\
 F'(z) &= C_1 \\
 F(z) &= C_1 z + C_2.
 \end{aligned}$$

The general case is very similar.

Given:

(a) $\phi(t)$ continuous and finite except for a pole of the n^{th} order at z_0 .

$$(b) \quad l_c \leq K \cdot \rho_{\max}.$$

$$(c) \quad \lim_{\rho_{\min} \rightarrow \infty} \frac{\rho_{\max}}{\rho_{\min}} = k$$

$$\text{Let } \phi(t) = \frac{f(t)}{(t-z_0)^n}$$

then $f(t) = \phi(t)(t-z_0)^n$ and

$$|f(t)| \leq M |t-z_0|^n \leq M \rho_{\max}^n.$$

$$\text{If } F(z) = \int_C \frac{f(t)}{t-z_0} dt$$

$$F^{[n+1]}(z) \leq 2! \frac{M \cdot \rho_{\max}^n \cdot l_c}{\rho_{\min}^{n+2}} \leq 2! \frac{M \cdot \rho_{\max}^{n+1} \cdot K}{\rho_{\min}^{n+2}}$$

$$\leq 2! \frac{M \cdot K \cdot k}{\rho_{\min}}.$$

$$\lim_{\rho_{\min} \rightarrow \infty} F^{[n+1]}(z) = \lim_{\rho_{\min} \rightarrow \infty} 2! \frac{M \cdot K \cdot k}{\rho_{\min}} = 0.$$

$$F^{[n+1]}(z) = 0$$

$$F^{[n]}(z) = C_0$$

$$F^{[n-1]}(z) = C_0 z + C_1$$

$$\vdots$$

$$F(z) = C_0 z^n + C_1 z^{n-1} + \dots + C_{n-1} z + C_n.$$

Theorem VIII. If $F(z)$ is the function defined by the equation,

$$F(z) = \int_C \frac{f(t)}{t-z_0} dt$$
, in which $f(t) = \phi(t)(t-z_0)$, where $\phi(t)$
 is a function, which is continuous and finite all over the Z plane
 with the exception of one pole of the n^{th} order at z_0 , and C is a
 set of curves, such that the length of each is less than or equal
 to a constant times the maximum distance of any point on that curve
 from z_0 and the ratio of the maximum to the minimum distance is a
 constant, then

$$F(z) = C_0 z^n + C_1 z^{n-1} + \dots + C_{n-1} z + C_n.$$

CHAPTER III

NON ANALYTIC TRANSFORMATIONS.

7. Introduction. It is often convenient in the study of any kind of functions to make transformations from one plane to another in order to simplify either the form of the equation or the shape of the curve. Analytic transformations have some special properties which make them useful in mapping. For example, the transformation is conformal, that is the angles and infinitesimal elements of area are preserved.

There are many rather simple non analytic transformations which show interesting changes of form from the Z to the W plane. The angles, of course, are not preserved, but we can often find in just what proportions these angles are changed. At first a few of the simplest transformations will be studied, showing what some familiar curves in the Z plane become when transformed. Then some of the important theorems and developments in series will be transformed. Some of these will give simple and useful results, other such complicated ones that they are of little value.

8. A Special Example. A transformation, as simple as any analytic one, is that which stretches the plane in one direction but not in the other. An example of this is:

$$\begin{aligned}x &= u \\ y &= 2v.\end{aligned}$$

In the usual way,

$$w = f(z) = f(x + yi) = u + vi = x + \frac{y}{2} i.$$

To show that this is not analytic, apply the Cauchy Riemann equations.

$$\begin{aligned}\frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial y} &= \frac{1}{2} \\ \frac{\partial u}{\partial x} &\neq \frac{\partial v}{\partial y}\end{aligned}$$

The following general points of correspondence can be seen immediately.

The origins correspond in the two planes.

The x axis corresponds to the u axis, for

$$\begin{aligned}y &= 0 \\ v &= 0.\end{aligned}$$

The y axis corresponds to the v axis, for

$$\begin{aligned}x &= 0 \\u &= 0.\end{aligned}$$

The lines $x = \text{constant}$ correspond to $u = \text{same constant}$.

The lines $y = \text{constant}$ correspond to $v = \frac{1}{2}$ that constant.

All other straight lines go into straight lines with half the slope.

$$\begin{aligned}y &= ax + b & m_1 &= a \\v &= \frac{au}{2} + \frac{b}{2} & m_2 &= \frac{a}{2}.\end{aligned}$$

Circles in the Z plane become ellipses in the W with their major axes equal to the diameter of the circle and their minor axes half that diameter.

$$\begin{aligned}x^2 + y^2 &= r^2 \\u^2 + 4v^2 &= r^2.\end{aligned}$$

The figure will show these points of correspondence.

Since the transformation is not conformal, angles are not preserved. To find out what change is made in the angles, take any two straight lines in the Z plane and transform them to the W .

$$\begin{aligned}y &= ax + b \\y &= cx + d.\end{aligned}$$

If θ is the angle between these lines,

$$\tan \theta = \frac{c-a}{1+ac}.$$

In the W plane, the lines corresponding to these are:

$$v = \frac{a}{2} u + \frac{b}{2}$$

$$v = \frac{c}{2} u + \frac{d}{2}$$

If ω is the angle between these lines,

$$\tan \omega = \frac{2(c-a)}{4+ac}.$$

9. The Polar Transformation. The polar transformation, so often used in elementary work with real variables, is non analytic when applied to complex variables. Let:

$$\begin{aligned} x &= u \cos v \\ y &= u \sin v. \end{aligned}$$

Solving these for u and v ,

$$u = (x^2 + y^2)^{\frac{1}{2}}$$

$$v = \tan^{-1} \frac{y}{x}$$

Then:

$$\begin{aligned} z &= x + yi = u \cos v + i u \sin v \\ w &= u + vi = (x^2 + y^2)^{\frac{1}{2}} + i \tan^{-1} \frac{y}{x}. \end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}.$$

Therefore the transformation is not analytic.

The general points of correspondence are:

The X axis:

$$\begin{aligned} y = 0 & \text{ becomes} \\ u \sin v = 0 & \text{ and corresponds to} \\ u = 0 & \text{ and to the lines} \\ v = 0, \pi, 2\pi, \dots, n\pi. \end{aligned}$$

The y axis:

$$\begin{aligned} x = 0 & \text{ becomes} \\ u \cos v = 0 & \text{ and corresponds to} \\ u = 0 & \text{ and the lines} \\ v = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{(2n+1)\pi}{2}. \end{aligned}$$

The line $x = 1$
or $u \cos v = 1$ gives a series of parabolas extending to the right of the line $u = 1$, and to the left of $u = -1$.
The vertices are at $(1, 0)$ $(1, 2\pi)$ ----- and at $(-1, \pi)$, $(-1, 3\pi)$ ---

The line $y = 1$
or $u \sin v = 1$ gives a similar series of parabolas with vertices at $(1, \frac{\pi}{2})$, $(1, \frac{5\pi}{2})$, ----- and at $(-1, \frac{\pi}{2})$, $(-1, \frac{3\pi}{2})$ --

The lines $u = \text{constant}$ become circles in the W plane with the constant as radius and the center at the origin.

$$\begin{aligned} u &= c \\ (x^2 + y^2)^{\frac{1}{2}} &= c \\ x^2 + y^2 &= c^2. \end{aligned}$$

The lines $v = \text{constant}$ correspond to straight lines through the origin.

$$\begin{aligned} v &= c \\ \tan^{-1} \frac{y}{x} &= c \\ y &= x \tan c. \end{aligned}$$

The figure will show some of these points of correspondence.

Since the transformation is not analytic, angles will not be preserved: For example the line

$$y = x$$

makes an angle of $\frac{\pi}{4}$ with

$$\begin{aligned} y &= 0. \\ y &= x \quad \text{corresponds to} \\ u &= 0, \quad v = \frac{\pi}{4}, \frac{5\pi}{4}, \dots \\ y &= 0 \quad \text{corresponds to} \\ u &= 0 \end{aligned}$$

angle of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ with $v = 0, \pi, 2\pi, \dots$
 $v = 0$ coincides with $u = 0$ and makes an
 $v = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$ make angles of $\frac{\pi}{2}$
 with $u = 0$ and are parallel to $v = 0, \pi, 2\pi, \dots$

10. General Linear Transformation. The general linear transformation is sometimes analytic and sometimes not, depending on the values of the coefficients of the variables.
 Let:

$$\begin{aligned} x &= au + bv + c \\ y &= a'u + b'v + c'. \end{aligned}$$

Then:

$$u = \frac{\begin{vmatrix} x-c & b \\ y-c' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} = \frac{b'x - b'c - by + bc'}{ab' - a'b},$$

$$v = \frac{\begin{vmatrix} a & x-c \\ a' & y-c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} = \frac{ay - ac' - a'x + a'c}{ab' - a'b}.$$

In order for this transformation to be analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{b'}{ab' - a'b} = \frac{a}{ab' - a'b}$$

$$b' = a.$$

$$\frac{-b}{ab' - a'b} = -\frac{-a'}{ab' - a'b}$$

$$a' = -b.$$

Therefore the transformation

$$\begin{aligned} x &= au + bv + c \\ y &= -bu + av + c' \end{aligned}$$

is an analytic transformation.

11. An Example of the Use of Non Analytic Transformations.

Some of these non analytic transformations may be used to make analytic a function which was originally not so. To illustrate, take an analytic function of w .

$$\begin{aligned} F(w) &= (u + vi)^2 + 3(u + vi) - 7 \\ &= (u^2 - v^2 + 3u - 7) + i(2uv + 3v) \\ &= A + Bi \end{aligned}$$

This is analytic, for:

$$\begin{aligned} A_u &= B_v & 2u + 3 &= 2u + 3 \\ A_v &= -B_u & -2v &= -2v. \end{aligned}$$

If we perform the transformation:

$$\begin{aligned} u &= x \\ v &= \frac{1}{2}y, \end{aligned}$$

the result will be a non analytic function of x and y .

$$\begin{aligned} f(z) &= (x^2 - \frac{y^2}{4} + 3x - 7) + i(xy + \frac{3}{2}y) \\ A_x &\neq B_y & 2x + 3 &\neq x + \frac{3}{2} \\ A_y &\neq -B_x & -\frac{1}{2}y &\neq -y. \end{aligned}$$

If the function $f(z)$ had been given at first, it could have been made analytic by the transformation:

$$\begin{aligned} x &= u \\ y &= 2v. \end{aligned}$$

12. Classes of Non Analytic Functions. If we have two non analytic functions,

$$\begin{aligned} w &= f(z) \\ \bar{w} &= F(z), \end{aligned}$$

in general the quotients $\frac{\Delta w}{\Delta z}$ and $\frac{\Delta \bar{w}}{\Delta z}$ will be functions of $\frac{dy}{dx}$, that is of the direction taken by the increment of z .

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \phi(x, y, y')$$

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta \bar{w}}{\Delta z} = \bar{\phi}(x, y, y').$$

If the quotient $\frac{\phi(x, y, y')}{\bar{\phi}(x, y, y')} = \psi(x, y)$ is independent of y' , then $f(z)$ and $F(z)$ may be said to be non analytic functions of the same class.

To illustrate this idea let:

$$w = z(x + \frac{y}{2}i) + 5 = (2x + 5) + iy.$$

$$\bar{w} = (x + \frac{y}{2}i)^2 + 2(x + \frac{y}{2}i) - 1 = (x^2 - \frac{y^2}{4} + 2x - 1) + i(xy + y).$$

$$\text{Then, } \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \phi(x, y, y') = \frac{2 + y'i}{1 + y'i}$$

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta \bar{w}}{\Delta z} = \bar{\phi}(x, y, y') = \frac{2x + 2 - \frac{y}{2}y' + i[y + (x + 1)y']}{1 + y'i}$$

$$\frac{\bar{\phi}(x, y, y')}{\phi(x, y, y')} = x + 1 + \frac{y}{2}i = \psi(x, y).$$

In this case, since ψ is independent of y' , w and $\bar{w}$ might be called non analytic functions of the same class. We might say also that w is an analytic function of $\bar{w}$.

13. Development in Series. In general a development in series is impossible for a non analytic function, all these developments depending primarily on the existence of the derivative. It is possible, however, to develop some particular functions by the use of the non analytic transformations.

For example take the familiar Taylor's Series for e^w (7)

$$e^w = 1 + \frac{w}{1!} + \frac{w^2}{2} + \frac{w^3}{3} + \dots$$

--o--

(7) See Goursat, page 100. (9).

Separating this into real and imaginary parts,

$$\begin{aligned}
 e^w &= e^{u+vi} = 1 + \frac{u+vi}{1!} + \frac{(u+vi)^2}{2!} + \frac{(u+vi)^3}{3!} + \dots \\
 &= 1 + \frac{u}{1!} + \frac{u^2 - v^2}{2!} + \frac{u^3 - 3uv^2}{3!} + \dots \\
 &\quad + i \left[\frac{v}{1!} + \frac{2uv}{2!} + \frac{3u^2v - v^3}{3!} + \dots \right]
 \end{aligned}$$

Writing e^w in the trigonometric form:

$$\begin{aligned}
 e^w &= e^u \cdot e^{vi} = e^u (\cos v + i \sin v) \\
 &= e^u \cos v + i e^u \sin v.
 \end{aligned}$$

Making the polar transformation:

$$\begin{aligned}
 x &= u \cos v \\
 y &= u \sin v, \\
 u &= (x^2 + y^2)^{\frac{1}{2}} \\
 v &= \tan^{-1} \frac{y}{x}.
 \end{aligned}$$

$$\begin{aligned}
 e^w &= e^u \frac{x}{u} + i e^u \frac{y}{u} \\
 &= \frac{e^{(x^2+y^2)^{\frac{1}{2}}}}{(x^2+y^2)^{\frac{1}{2}}} \cdot x + i \frac{e^{(x^2+y^2)^{\frac{1}{2}}}}{(x^2+y^2)^{\frac{1}{2}}} \cdot y.
 \end{aligned}$$

Making the same transformation on the series,

$$\begin{aligned}
 e^{u+vi} &= 1 + \frac{(x^2+y^2)^{\frac{1}{2}}}{1!} + \frac{x^2+y^2 - (\tan^{-1} \frac{y}{x})^2}{2!} + \dots \\
 &\quad + i \left[\frac{\tan^{-1} \frac{y}{x}}{1!} + \frac{2(x^2+y^2)^{\frac{1}{2}} \tan^{-1} \frac{y}{x}}{2!} + \dots \right]
 \end{aligned}$$

These two statements for e^w must be equal to each other.

$$\begin{aligned}
 \frac{e^{(x^2+y^2)^{\frac{1}{2}}}}{(x^2+y^2)^{\frac{1}{2}}} (x+iy) &= 1 + \frac{(x^2+y^2)^{\frac{1}{2}}}{1!} + \frac{x^2+y^2 - (\tan^{-1} \frac{y}{x})^2}{2!} + \dots \\
 &\quad + i \left[\frac{\tan^{-1} \frac{y}{x}}{1!} + \frac{2(x^2+y^2)^{\frac{1}{2}} \tan^{-1} \frac{y}{x}}{2!} + \dots \right].
 \end{aligned}$$

This gives an example of a non analytic function developed in Taylor's Series.

14. The General Transformation. The transformations studied so far have been particular ones, known to be non analytic. It is interesting to take a general transformation, about which nothing is known, and show what certain theorems, for example Taylor's Series or the Cauchy Integral Theorem, become when subjected to this transformation.

Let:

$$\begin{aligned} u &= f(x, y) \\ v &= \phi(x, y). \end{aligned}$$

Suppose these equations solvable for x and y so that:

$$\begin{aligned} x &= F(u, v) \\ y &= \Phi(u, v). \end{aligned}$$

Develop an analytic function $A(z)$ in Taylor's Series ⁽⁸⁾ about the point a .

$$A(z) = A(a) + \frac{A'(a)}{1!} (z-a) + \frac{A''(a)}{2!} (z-a)^2 + \dots$$

Let the coordinates of a be (η, β) , then:

$$\begin{aligned} A(x+yi) &= A(\eta + \beta i) + \frac{A'(\eta + \beta i)}{1!} [(x+yi) - (\eta + \beta i)] \\ &+ \frac{A''(\eta + \beta i)}{2!} [(x+yi) - (\eta + \beta i)]^2 + \dots \end{aligned}$$

Substituting the values given above; the result is

$$\begin{aligned} A[F(u, v) + i \Phi(u, v)] &= A[f(\eta, \beta) + i \phi(\eta, \beta)] + \\ &\frac{A'[f(\eta, \beta) + i \phi(\eta, \beta)]}{1!} \left[\{F(u, v) + i \Phi(u, v)\} - \{f(\eta, \beta) + i \phi(\eta, \beta)\} \right] \\ &+ \frac{A''[f(\eta, \beta) + i \phi(\eta, \beta)]}{2!} \left[\{F(u, v) + i \Phi(u, v)\} - \{f(\eta, \beta) + i \phi(\eta, \beta)\} \right]^2 \\ &+ \dots \end{aligned}$$

The Laurent Ring ⁽⁹⁾ development may be formed from the same general transformation. Let $A(z)$ be an analytic function with a pole of the K^{th} order at $a(\eta, \beta)$.

--o--

(8) See Osgood, page 267 § 7.

(9) See Whittaker "Modern Analysis", page 61.

$$H(z) = \frac{B_k}{(z-a)^k} + \frac{B_{k-1}}{(z-a)^{k-1}} + \dots + \frac{B_1}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \dots$$

$$H(x+yi) = \frac{B_k}{[(x+yi) - (\eta + \beta i)]^k} + \frac{B_{k-1}}{[(x+yi) - (\eta + \beta i)]^{k-1}} + \dots + \frac{B_1}{[(x+yi) - (\eta + \beta i)]} + A_0 + A_1[(x+yi) - (\eta + \beta i)] + A_2[(x+yi) - (\eta + \beta i)]^2 + \dots$$

Using the general transformation,

$$H[\bar{F}(u,v) + i\bar{\Phi}(u,v)] = \frac{B_k}{\left[\{\bar{F}(u,v) + i\bar{\Phi}(u,v)\} - \{f(\eta, \beta) + i\phi(\eta, \beta)\}\right]^k} + \frac{B_{k-1}}{\left[\{\bar{F}(u,v) + i\bar{\Phi}(u,v)\} - \{f(\eta, \beta) + i\phi(\eta, \beta)\}\right]^{k-1}} + \dots + \frac{B_1}{\left[\{\bar{F}(u,v) + i\bar{\Phi}(u,v)\} - \{f(\eta, \beta) + i\phi(\eta, \beta)\}\right]} + A_0 + A_1\left[\{\bar{F}(u,v) + i\bar{\Phi}(u,v)\} - \{f(\eta, \beta) + i\phi(\eta, \beta)\}\right] + \dots$$

In order to find the integral of a function of z under a general transformation, let:

$$x = \bar{F}(u, v)$$

$$y = \bar{\Phi}(u, v).$$

Then

$$dx = \bar{F}_u(u, v) du + \bar{F}_v(u, v) dv$$

$$dy = \bar{\Phi}_u(u, v) du + \bar{\Phi}_v(u, v) dv.$$

$$dz = dx + i dy = \bar{F}_u(u, v) du + \bar{F}_v(u, v) dv + i [\bar{\Phi}_u(u, v) du + \bar{\Phi}_v(u, v) dv]$$

$$\int dz = z = x + i y.$$

By using the transformation above,

$$\int dz = \int (\bar{F}_u + i \bar{\Phi}_u) du + (\bar{F}_v + i \bar{\Phi}_v) dv$$

$$= \bar{F}(u, v) + i \bar{\Phi}(u, v).$$

$$\begin{aligned}
\int A(z) dz &= \int A(x+yi)(dx + i dy) \\
&= \int A[\bar{r}(u,v) + i\bar{\phi}(u,v)] \left[(\bar{r}_u(u,v) + i\bar{\phi}_u(u,v)) du + (\bar{r}_v(u,v) + i\bar{\phi}_v(u,v)) dv \right] \\
&= \int A[\bar{r}(u,v) + i\bar{\phi}(u,v)] [\bar{r}_u(u,v) + i\bar{\phi}_u(u,v)] du \\
&\quad + \int A[\bar{r}(u,v) + i\bar{\phi}(u,v)] [\bar{r}_v(u,v) + i\bar{\phi}_v(u,v)] dv.
\end{aligned}$$

15. Developments under Polar Transformation. The general Taylor's Series might be transformed by the polar transformation and its region of convergence found. As before:

$$\begin{aligned}
H(z) &= H(a) + \frac{H'(a)}{1!} (z-a) + \frac{H''(a)}{2!} (z-a)^2 + \dots \\
&= H(\eta + \beta i) + \frac{H'(\eta + \beta i)}{1!} [(x+yi) - (\eta + \beta i)] + \\
&\quad \frac{H''(\eta + \beta i)}{2!} [(x+yi) - (\eta + \beta i)]^2 + \dots
\end{aligned}$$

The polar transformation, as before, gives:

$$\begin{aligned}
x &= u \cos v \\
y &= u \sin v. \\
u &= (x^2 + y^2)^{\frac{1}{2}} \\
v &= \tan^{-1} \frac{y}{x}.
\end{aligned}$$

The point (u, v) corresponding to (η, β) is $\left[(\eta^2 + \beta^2)^{\frac{1}{2}}, \tan^{-1} \frac{\beta}{\eta} \right]$

Then:

$$\begin{aligned}
H(u \cos v + i u \sin v) &= H(u, + i v) + \\
&\frac{H'(u, + i v)}{1!} [(u \cos v + i u \sin v) - (u, + i v)] \\
&+ \frac{H''(u, + i v)}{2!} [(u \cos v + i u \sin v) - (u, + i v)]^2 + \dots
\end{aligned}$$

The original Taylor's Series in the z plane will converge in some circle of radius r about the origin:

$$x^2 + y^2 = r^2.$$

The corresponding series in the W plane will converge in a vertical strip of width $2r$, for the equation of the circle becomes

$$u = \pm r.$$

The strip will be infinite in extent but will be divided into rectangles, in each of which the function repeats itself. The complete rectangle will be of length $2r$ and of width 2π .

The preceding figure is similar to that for an elliptic integral: (10)

$$W = \int \frac{dx}{\sqrt{x^3 - px + q}},$$

in which the whole plane is divided into a network of rectangles, in each of which the function takes on all values taken on at all. Let e_1, e_2 , and e_3 be the roots of

$$x^3 - px + q = 0.$$

The half rectangle in the W plane has for its vertices the points corresponding to e_1, e_2, e_3 , and e_4 or infinity. This figure is reflected to form the complete rectangle.

(10) See Whittaker, Chap. XIV, page 322 ff.

In order to find the equation of this elliptic integral in terms of u and v , let x have the coordinates r and θ .

$$x = r + i\theta$$

$$r = (u^2 + v^2)^{\frac{1}{2}}$$

$$\theta = \tan^{-1} \frac{v}{u}.$$

To give the half rectangle the width π , let

$$W = \frac{2\pi}{\omega_2} \int_{e_1}^x \frac{dx}{\sqrt{x^3 - px + q}},$$

where ω_2 represents the original width of the rectangle.

$$dx = dr + i d\theta = \frac{u du + v dv}{(u^2 + v^2)^{\frac{1}{2}}} + i \frac{u dv - v du}{u^2 + v^2}$$

$$W = \frac{2\pi}{\omega_2} \int_{e_1}^{\left[(u^2+v^2)^{\frac{1}{2}} + i \tan^{-1} \frac{v}{u}\right]} \frac{\frac{u du + v dv}{(u^2 + v^2)^{\frac{1}{2}}} + i \frac{u dv - v du}{u^2 + v^2}}{\sqrt{\left\{(u^2+v^2)^{\frac{1}{2}} + i \tan^{-1} \frac{v}{u}\right\}^3 - p \left\{(u^2+v^2)^{\frac{1}{2}} + i \tan^{-1} \frac{v}{u}\right\} + q}}$$

$$= \frac{2\pi}{\omega_2} \int_{e_1}^{\left[(u^2+v^2)^{\frac{1}{2}} + i \tan^{-1} \frac{v}{u}\right]} \frac{\frac{u du + v dv}{(u^2 + v^2)^{\frac{1}{2}}} + i \frac{u dv - v du}{u^2 + v^2}}{\sqrt{(u^2+v^2)^{\frac{3}{2}} - (u^2+v^2)^{\frac{1}{2}} \left[3 \left(\tan^{-1} \frac{v}{u}\right)^2 - p\right] + q + i \left[\tan^{-1} \frac{v}{u} \left\{3(u^2+v^2) - p \left(\tan^{-1} \frac{v}{u}\right)\right\}\right]}}$$

UNIVERSITY OF MISSOURI
COLUMBIA

DEPARTMENT OF MATHEMATICS

May 18, 1915.

Dear Walter Miller,
Graduate Faculty.
University of Missouri.

Dear Sir:

I have examined the thesis of Miss Julia Spalding submitted in partial fulfillment of the requirements for the degree of Master of Arts, and it is my opinion that it meets the requirements which the thesis should, and ought to be accepted.

Very truly yours,
O. W. Kellogg

May 13, 1915.

Professor O. D. Kellogg,

Engineering Bldg.

Dear Sir:-

It is customary for the Graduate Committee to refer dissertations, submitted by candidates for the degree of Master of Arts, to some member of the Group who is not connected with the Department in which the candidate's work has been done. I am sending you herewith a dissertation which has been submitted by Miss Julia Spalding. I shall be greatly obliged if you will kindly examine the same at your earliest convenience and report to us for the Graduate Committee whether in your opinion the dissertation meets the general standard which has been established in this University for the Master's dissertation.

Very truly yours,

A handwritten signature in dark ink, appearing to read "Walter Miller". The signature is fluid and cursive, with a long horizontal stroke extending to the right.

Chairman, Graduate Committee.

UNIVERSITY OF MISSOURI
COLUMBIA

DEPARTMENT OF MATHEMATICS

Dean Walter Miller,
The Graduate School.

May 13/15-

Sir:- This note is written to accompanying a
thesis by Miss Julia Spalding on Generalizations of
norms from The Theory of Functions, submitted
partial fulfillment of the requirements for
Degree of Master of Arts.

Miss Spalding has prepared this thesis
under my direction; it has my full approval;
and I recommend that it be accepted.

Yours
W. R. Hedrick

MU Libraries
University of Missouri --Columbia

MU Theses and Dissertations (Pre-1920)

Local identifier Spalding1915

Capture information

Date captured	04212017
Scanner manufacturer	Zeutschel
Scanner model	OS 15000
Scanning system software	OmniScan v. 12.8 SR2 (2675)
Optical resolution	600 dpi
Color settings	grayscale, 8 bit and color, 24 bit
File types	tiff

Source information

Format	Book
Content type	Text [with images]
Source ID	010-100953942
Notes	<p>Pages typed and single-sided.</p> <p>Title page has perforated and ink property stamp.</p> <p>Call number written on page 1.</p> <p>Handwritten annotations throughout.</p> <p>Inside front cover has label pasted in.</p> <p>Purple ink property stamp on page 26.</p> <p>Equations are handwritten.</p> <p>Pocket pasted in at end.</p> <p>Inside back cover has barcode and call number.</p> <p>Two approval letters and university correspondence follow final text.</p> <p>38 blank pages at end of book.</p>

Derivatives - Access copy

Compression	Tiff compressed with LZW before conversion to pdf
Editing software	Adobe Photoshop CS5
Resolution	600 dpi
Color	Grayscale and color
File types	pdf
Notes	<p>Grayscale pages cropped, canvassed, and images brightened.</p> <p>Color pages cropped and canvassed.</p> <p>Blank pages removed.</p>