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VARIABLE-MESH DIFFERENCE EQUATION FOR THE STREAM FUNCTION IN AXIALLY SYMMETRIC FLOW

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Variable-Mesh Difference Equation for the Stream Function in Axially Symmetric Flow

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A finite difference equation is developed for the stream function in cylindrical coordinates with axial symmetry which is applicable to an irregular mesh having different length and radial dimensions. In addition, the length and radial dimensions may be varied, and the mesh made finer in any interior region. The equation also takes into account an irregular boundary.

THE stream function in cylindrical coordinates for the case of axial symmetry is

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (1)$$

A five-point mesh crossing an irregular boundary is used and is shown in Fig. 1. The mesh under consideration has spacing of h units in the z direction and k units in the r direction, and α and β are the ratios of the distance to the boundary divided by the mesh distance. If the function $\psi(z, r)$ is expanded in a Taylor's series in the r direction, dropping the argument for the derivatives, the following equations result:

$$\psi(z, r + k) = \psi(z, r) + k \frac{\partial \psi}{\partial r} + \frac{k^2}{2!} \frac{\partial^2 \psi}{\partial r^2} + \frac{k^3}{3!} \frac{\partial^3 \psi}{\partial r^3} + \frac{k^4}{4!} \frac{\partial^4 \psi}{\partial r^4} \dots \quad (2)$$

$$\psi(z, r - \alpha k) = \psi(z, r) - \alpha k \frac{\partial \psi}{\partial r} + \frac{\alpha^2 k^2}{2!} \frac{\partial^2 \psi}{\partial r^2} - \frac{\alpha^3 k^3}{3!} \frac{\partial^3 \psi}{\partial r^3} + \frac{\alpha^4 k^4}{4!} \frac{\partial^4 \psi}{\partial r^4} + \dots \quad (3)$$

If Eq. (2) is multiplied by α and the result added to Eq. (3),

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{2\psi(z, r + k)}{k^2(1 + \alpha)} + \frac{2\psi(z, r - \alpha k)}{k^2\alpha(1 + \alpha)} - \frac{2\psi(z, r)}{k^2\alpha^2} + (\alpha^2 - 1)0k + 0k^2 \quad (4)$$

where $0k$ and $0k^2$ are terms of the order of k and k^2 , respectively.

If Eq. (2) is multiplied by α^2 and the result subtracted from Eq. (3), the following is obtained when dividing by r :

$$\frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{\alpha \psi(z, r + k)}{rk(1 + \alpha)} - \frac{\psi(z, r - \alpha k)}{rk\alpha(1 + \alpha)} + \frac{(1 - \alpha)\psi(z, r)}{rk\alpha} + 0k^2 \quad (5)$$

In a similar way,

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{2\psi(z - h, r)}{h^2(1 + \beta)} + \frac{2\psi(z + \beta h, r)}{h^2\beta(1 + \beta)} - \frac{2\psi(z, r)}{h^2\beta^2} + (\beta^2 - 1)0h + 0h^2 \quad (6)$$

Equations (4-6) can be substituted in Eq. (1), and the result is a difference form of the stream function for the point $\psi(z, r)$ in terms of the four surrounding points:

$$\begin{aligned} & \psi(z, r + k) \left(\frac{2}{1 + \alpha} \right) \left(1 - \frac{\alpha k}{2r} \right) + \\ & \psi(z, r - \alpha k) \left(\frac{2}{1 + \alpha} \right) \left(\frac{1}{\alpha} \right) \left(1 + \frac{k}{2r} \right) + \\ & \psi(z + \beta h, r) \left[\frac{2\lambda^2}{\beta(1 + \beta)} \right] + \psi(z - h, r) \left(\frac{2\lambda^2}{1 + \beta} \right) - \\ & 2\psi(z, r) \left[\frac{\lambda^2}{\beta} + \frac{1}{\alpha} - \frac{k(1 - \alpha)}{2\alpha r} \right] + (1 - \beta^2)0h - \\ & (1 - \alpha^2)0k + 0h^2 + 0k^2 = 0 \quad (7) \\ & \lambda^2 = k^2/h^2, 0 < \alpha \leq 1, 0 < \beta \leq 1 \end{aligned}$$

Equation (7) is valid for any mesh point near a boundary. It also applies to an interior point where a change in mesh size is introduced. This feature is particularly valuable when evaluating the stream function near an abruptly changing boundary. For example, evaluating Eq. (7) near a change

Fig. 1 Five-point mesh used in difference equation.

in mesh size in the z direction corresponds to a vertical boundary in Fig. 1 through the point $\psi(z + h, r)$, and β becomes the ratio of the mesh sizes. The mesh need not be square nor regular, that is, k and h need not be equal nor do they always have to be constant.

For an interior point in a mesh where $h = k = \text{const}$, Eq. (7) reduces to the familiar form (e.g., see Salvadori and Baron¹)

$$\psi(z, r + k) \left(1 - \frac{k}{2r} \right) + \psi(z, r - k) \left(1 + \frac{k}{2r} \right) + \psi(z - h, r) + \psi(z + h, r) - 4\psi(z, r) = 0 \quad (8)$$

The error involved is of the order of $(1 - \beta^2)h$ in Eq. (7). Care must be exercised to ensure that β does not approach zero. In constructing the net it is necessary to make $(1 - \beta^2) \rightarrow h$ if the whole term is to be of the order of h^2 . The same argument holds for α . If the net is made fine enough, $h, k \ll 1$, and the characteristic dimension of the body under consideration is unity, the terms $0h^2$ and $0k^2$ tend to zero.

Also, it is assumed when using Taylor's expansion that all derivatives are bounded. This is, of course, not true at the stagnation point of a body of revolution for example. However, if the value of the mesh point at the stagnation point

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is defined as part of the boundary, the unboundedness problem will be avoided for interior points near the singularity.

The stability problem may be discussed along the lines presented by Forsythe and Wasow.² The term $1 - (\alpha k / 2r)$ will be positive if $\alpha < (2r/k)$. If this latter condition is met and if α, β, k, h, r are all positive and finite, Eq. (7) merely represents $\psi(z, r)$ as a weighted average of four surrounding points. Since the boundary is specified and finite, and since all derivatives are bounded in the region under consideration, all interior points must be finite: $0 \leq m \leq \psi(z, r) \leq M \leq \infty$. Thus, the difference equation should be stable throughout the region interior to the boundary.

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¹ Salvadori, M. G. and Baron, M. L., *Numerical Methods in Engineering* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961), p. 227.

² Forsythe, G. E. and Wasow, W. R., *Finite-Difference Methods for Partial Differential Equations* (John Wiley and Sons, Inc., New York, 1960), pp. 283-288.

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