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IRROTATIONAL FLOW OVER SPILLWAYS OF FINITE HEIGHT

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IRROTATIONAL FLOW OVER SPILLWAYS OF FINITE HEIGHT

By John J. Cassidy,¹ M. ASCE

INTRODUCTION

Flow over a well-designed spillway generally occurs in an abruptly contracting fashion. Such abruptly contracting flows are understood to be approximately irrotational in nature.² Hence, it would seem logical to conclude that if potential theory could be used for the determination of free-surface profiles, discharge coefficients, and pressure distributions, the results would approximate flow over the prototype structure. Two conditions make such a solution particularly difficult for the case of flow over a spillway of finite height and given shape.

1. The elevation of the total-head line is unknown at the outset; and
2. The velocity at every point on the free surface is a function of the elevation of the point and the location of the total-head line.

Aside from academic interest in overcoming these difficulties, the development of a method of solution is of practical interest. Theoretically, at least, complete control can be maintained over all pertinent variables in an analytic solution, which is, in general, not possible in an experimental study. Comparison of the results of experimental and irrotational-flow studies would make possible a quantitative estimate of the roles of boundary roughness, viscosity, and surface tension.

Apparently, A. Lauck³ performed the first solution for free-surface, irrotational flow in a gravitational field. By the simultaneous solution of two integral equations, one derived from the Cauchy integral theorem and one from the Bernoulli equation, Lauck was able to obtain the discharge coefficient and

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¹ Assoc. Prof., Dept. of Civ. Engrg., Univ. of Missouri, Columbia, Mo.

² Rouse, Hunter, "Elementary Mechanics of Fluids," John Wiley and Sons, New York, N. Y., 1946.

³ Lauck, A., "Ueberfall über ein Wehr," *Zeitschrift für Angewandte Mathematik und Mechanik*, Akademie-Verlag GMBH, Berlin, Germany, Vol. 5, 1925.

coordinates for the free-surface profile produced by flow over an infinitely-high weir.

Only a relatively few solutions have been reported since Lauck's 1925 paper and most of these have been described by G. Birkhoff⁴ and by H. Rouse⁴ and S. Garg⁴ in discussions of Birkhoff's paper. No further account of those methods is warranted herein except to note that until 1961, when the writer first began this investigation, no solutions had been obtained for free-surface, ir-rotational flows over curved shapes of finite height in a gravity field.

Recently, however, two important papers have appeared. T. S. Strelkoff⁵ was able to obtain free-surface profiles and discharge coefficients for flow over weirs of finite height, and G. Watters and R. L. Street⁶ have shown free-surface profiles obtained for certain cases of flow over steps and curved humps. Both obtained their solutions by solving integral equations. In Strelkoff's case, the integral equation was derived by means of conformal mapping and the use of distributed singularities; he was able to formulate an algorithm for correction of an initially assumed total head. Watters and Street formulated an integral equation by expressing the complex velocity as an infinite series; they included the Froude number of the downstream flow as an independent variable. However, the shapes of curved flow obstructions apparently cannot be specified independently in either Strelkoff's method or that of Watters and Street.

Because no applicable methods of solution seemed to be available when the writer began this investigation, the study reported herein was conducted for the purpose of developing an analytic method for the determination of discharge coefficients, pressure distributions, and free-surface profiles produced by ir-rotational flow over a finite-height spillway of specified shape in a gravity field.

Notation.—The symbols used in this paper are defined where they first appear, and are arranged alphabetically in the Appendix.

THE PHYSICAL PICTURE

A dimensional analysis of the variables involved in flow over a given spillway shape is of considerable aid to understanding. The variables that determine the free-surface profile in such a flow (see Fig. 1) are related as

$$y = \phi(x, h, h_D, w, k, \gamma, \rho, \mu, \sigma, V_0) \dots \dots \dots (1)$$

in which y is the vertical coordinate, x is the horizontal coordinate, h is the head on the spillway, h_D describes the spillway shape, w is the spillway height, k is a boundary roughness characteristic, γ , ρ , μ , and σ , are the specific weight, density, dynamic viscosity, and surface energy of the fluid respectively, and V_0 is the approach velocity.

⁴ Birkhoff, G., "Calculation of Potential Flows with Free Streamlines," *Journal of the Hydraulics Division, ASCE*, Vol. 87, No. HY6, Proc. Paper 2977, November, 1961; with discussion by Hunter Rouse, Vol. 88, No. HY2, Proc. Paper 3087, March, 1962; and S. Garg, Vol. 88, No. HY4, Proc. Paper 3209, July, 1962.

⁵ Strelkoff, T. S., "Solution of Highly Curvilinear Gravity Flows," *Journal of the Engineering Mechanics Division, ASCE*, Vol. 90, No. EM3, Proc. Paper 3950, June, 1964.

⁶ Watters, G., and Street, R. L., "Two-Dimensional Flow Over Sills in Open Channels," *Journal of the Hydraulics Division, ASCE*, Vol. 90, No. HY4, Proc. Paper 3967, July, 1964.

The variable h_D is assumed to fully specify the spillway shape. It is obvious that most complex shapes could not be completely specified by one such variable. However, the standard spillway profile is an excellent example of a complex shape that can be so specified (in that case h_D would be the design head). Application of the π theorem to Eq. 1 produces

$$\frac{y}{h_D} = f\left(\frac{x}{h_D}, \frac{h}{h_D}, \frac{w}{h_D}, \frac{k}{h_D}, \frac{V_o h_D \rho}{\mu}, \frac{V_o}{\sqrt{\frac{\sigma}{\rho h_D}}}, \frac{V_o}{\sqrt{g h_D}}\right) \dots (2)$$

At the present state of mathematical science it is impossible to determine the exact functional relationship indicated in Eq. 2. Indeed, an experimental investigation involving all the dimensionless parameters in Eq. 2 would not be practicable if only because of time requirements. Theoretical consideration of each of the parameters provides the means for simplification of the problem.

FIG. 1.—FLOW OVER SPILLWAY IN PHYSICAL PLANE

The Reynolds number $V_o h_D \rho / \mu$ and the relative roughness k/h_D play similar roles and can be considered simultaneously. Viscosity requires that the velocity of the fluid in contact with the boundary be zero, and, hence, a boundary layer exists at the solid boundary. The free surface is displaced outward a distance equal to the displacement thickness. Because the resulting velocity profile places the high-velocity fluid in an advantageous position to pass over the spillway, the discharge coefficient would tend to be increased above the inviscid case. However, viscosity also produces energy dissipation and a resulting slope in the total-head line. A single total-head measurement well upstream, thus, will not indicate the total-head at the spillway. As a result of the energy dissipation, the discharge coefficient would tend to be decreased. Roughness acts to increase both the displacement thickness and energy dissipation rate. Thus, viscosity and the roughness each produces two separate effects that tend to be mutually compensating. However, the degree of compensation is unknown.

The effects of surface energy, as embodied in the Weber number $V_o / \sqrt{\sigma / \rho h_D}$, can safely be neglected provided the investigation is limited to large scales where pressure differences across curved interfaces are small.

The role of the Froude number $V_0/\sqrt{g h_D}$ is dependent on whether the approaching flow is rapid or tranquil. For upstream Froude numbers less than unity, h/h_D and the Froude number cannot be expressed independently because the discharge coefficient, and, hence, the Froude number as well, will be a unique function of the spillway shape and the head. A solution for the free-surface profile under such conditions must then simultaneously yield a value for the discharge coefficient. If the upstream Froude number is greater than unity the control section is upstream from the spillway and the discharge coefficient becomes an independent variable.

If viscosity, surface energy, and boundary roughness are neglected and the investigation is confined to the case of subcritical approaching flow, Eq. 2 becomes

$$f\left(\frac{y}{h_D}, \frac{x}{h_D}, \frac{w}{h_D}, \frac{h}{h_D}\right) = 0 \quad \dots\dots\dots (3)$$

Surface profiles and discharge coefficients obtained by investigation of the functional relationship indicated in Eq. 3 could well be expected to be valid for flow at large Reynolds numbers over smooth boundaries of large spillways.

MATHEMATICAL ANALYSIS

If it is assumed that the entire flow field is irrotational, Eq. 3 is applicable, and the classical methods of potential theory may be used in the analytical investigation of the flow pattern.

The region between the solid boundary and the free surface, as shown in Fig. 1, can be mapped into the complex-potential plane shown in Fig. 2, in which ϕ and ψ are the velocity potential and the stream function, respectively. For any given spillway, h/h_D , w/h_D , and the configuration of the solid boundary are known. Because $y = y(x)$ is known on the solid boundary, the angle of inclination of the velocity vector can be determined for any point except the stagnation point B.

The complex velocity can be written as

$$\frac{dW}{dz} = V e^{-i\theta} \quad \dots\dots\dots (4)$$

in which W and $z = x + iy$ refer to the complex-potential and the physical planes, respectively, V = the magnitude of the velocity vector, and θ = its inclination. Then the natural logarithm of dW/dz is

$$\ln\left(\frac{dW}{dz}\right) = \ln(V) - i\theta \quad \dots\dots\dots (5)$$

Because the geometry is somewhat complicated in the physical plane, it is desirable to work in the complex-potential plane which is always rectangular. A solution to the problem requires that V , θ , x , and y be known as functions of ϕ and ψ . Because $\ln(V)$ and θ are harmonic everywhere except where $V = 0$, $\ln(V)$ and θ must satisfy the Laplace equation,

$$\frac{\partial^2 \ln(V)}{\partial \psi^2} + \frac{\partial^2 \ln(V)}{\partial \phi^2} = 0 \quad \dots\dots\dots (6)$$

and

$$\frac{\partial^2 \theta}{\partial \psi^2} + \frac{\partial^2 \theta}{\partial \phi^2} = 0 \dots\dots\dots (7)$$

A connection between $\ln(V)$ and θ is provided by the Cauchy-Riemann equations

$$\frac{\partial \ln(V)}{\partial \phi} = -\frac{\partial \theta}{\partial \psi} \dots\dots\dots (8)$$

and

$$\frac{\partial \ln(V)}{\partial \psi} = \frac{\partial \theta}{\partial \phi} \dots\dots\dots (9)$$

Equations 6, 7, 8, and 9 provide necessary relationships between the magnitude of the velocity vector and its inclination in the W-plane, but a connection between the complex-potential plane and the physical plane is also required. This connection can be formulated after consideration of the ex-

FIG. 2.—COMPLEX-POTENTIAL PLANE

pressions for the stream function and the velocity potential. Because both are functions of y and x , their total derivatives may be written as

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \dots\dots\dots (10)$$

and

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \dots\dots\dots (11)$$

Substituting the horizontal velocity component $u = -\partial \psi / \partial y = \partial \phi / \partial x$ and the vertical velocity component $v = \partial \phi / \partial y = \partial \psi / \partial x$ into Eqs. 10 and 11 produces

$$d\psi = -v dx + u dy \dots\dots\dots (12a)$$

and

$$d\phi = u dx + v dy \dots\dots\dots (12b)$$

Eqs. 12 and 13, when solved simultaneously, yield

$$dx = \frac{u d\phi - v d\psi}{u^2 + v^2} \dots\dots\dots (13a)$$

and

$$dy = \frac{v d\phi + u d\psi}{u^2 + v^2} \dots\dots\dots (13b)$$

From the definition of the velocity vector, $u = V \cos \theta$ and $v = V \sin \theta$. Thus

$$dx = \frac{\cos \theta}{V} d\phi - \frac{\sin \theta}{V} d\psi \dots\dots\dots (14)$$

and
$$dy = \frac{\sin \theta}{V} d\phi + \frac{\cos \theta}{V} d\psi \dots\dots\dots (15)$$

The following derivatives, valid anywhere in the flow field,

$$\frac{dx}{d\phi} = \frac{\cos \theta}{V} - \frac{\sin \theta}{V} \frac{d\psi}{d\phi} \dots\dots\dots (16)$$

$$\frac{dy}{d\phi} = \frac{\sin \theta}{V} + \frac{\cos \theta}{V} \frac{d\psi}{d\phi} \dots\dots\dots (17)$$

$$\frac{dx}{d\psi} = \frac{\cos \theta}{V} \frac{d\phi}{d\psi} - \frac{\sin \theta}{V} \dots\dots\dots (18)$$

and
$$\frac{dy}{d\psi} = \frac{\sin \theta}{V} \frac{d\phi}{d\psi} + \frac{\cos \theta}{V} \dots\dots\dots (19)$$

follow directly from Eqs. 14 and 15.

Finally, the following form of the Bernoulli equation

$$\frac{V^2}{2g} + y = H \dots\dots\dots (20)$$

must be satisfied along the free surface where the pressure, p , is equal to zero. Eqs. 6 through 9 and Eqs. 16 through 20 appear to represent a system of nine equations in seven unknowns. However, only two of Eqs. 6 through 9 are independent because both Eq. 6 and Eq. 7 can be formed by cross-differentiation and addition of Eqs. 8 and 9. Thus, the system is in reality one of seven simultaneous equations involving seven unknowns. To obtain an analytical solution to the problem of irrotational flow over a spillway, this system of equations must be solved according to the proper boundary conditions. Certain mathematical difficulties are to be expected because real solutions to the system of equations need not necessarily exist for all values of the total head, H .

PROCEDURE FOR NUMERICAL SOLUTION

Numerical procedures and an IBM 7070 digital computer were used to obtain solutions to the foregoing system of equations for three different spillway shapes. To make the solutions as general as possible, dimensionless variables in the following forms were used throughout the analysis and subsequent representation of the results. Thus,

$$\frac{V}{\sqrt{2g h_D}}, \frac{\psi}{\sqrt{2g h_D^{3/2}}}, \frac{\phi}{\sqrt{2g h_D^{3/2}}}, \frac{p}{\gamma h_D}, \frac{y}{h_D}, \frac{x}{h_D}, \frac{H}{h_D}, \frac{w}{h_D}, \frac{h}{h_D}$$

However, for purposes of simplicity, only the numerators of these dimensionless variables have been used herein to represent the corresponding

dimensionless forms. The following finite difference equation, derived by W. G. Bickley,⁷ was substituted for Eq. 7 as

$$20 \theta_0 = 4(\theta_1 + \theta_2 + \theta_3 + \theta_4) + (\theta_5 + \theta_6 + \theta_7 + \theta_8) \dots \quad (21)$$

Terms of the order m^6 have been neglected in Eq. 21. The finite-difference mesh spacing m and the subscripts are defined in Fig. 3. Eqs. 8 and 9 were written in integral form as

$$\Delta \ln(V) = \int_{\phi_1}^{\phi_2} 2 - \frac{\partial \theta}{\partial \psi} d\phi \dots \quad (22)$$

and

$$\Delta \ln(V) = \int_{\psi_1}^{\psi_2} 2 \frac{\partial \theta}{\partial \phi} d\psi \dots \quad (23)$$

in which $\Delta \ln(V)$ = the change in $\ln(V)$ over the interval of integration.

Eqs. 16, 17, 18, and 19 were expressed in the integral forms

$$\Delta x = \int_{\psi_1}^{\psi_2} 2 - \frac{\sin \theta}{V} d\psi \dots \quad (24)$$

$$\Delta x = \int_{\phi_1}^{\phi_2} 2 \frac{\cos \theta}{V} d\phi \dots \quad (25)$$

$$\Delta y = \int_{\psi_1}^{\psi_2} 2 \frac{\cos \theta}{V} d\psi \dots \quad (26)$$

and

$$\Delta y = \int_{\phi_1}^{\phi_2} 2 \frac{\sin \theta}{V} d\phi \dots \quad (27)$$

valid only along lines of constant ϕ or ψ .

As was noted previously, a singularity exists at the stagnation point on the upstream side of the spillway. At this point $\ln(V)$ becomes infinite and θ becomes undefined. This difficulty was avoided by using the mapping function $W = z^2$ to represent flow in the corner near the stagnation point.

The numerical procedures were programmed for solution in an iterative fashion. Initially, values of θ and $\ln(V)$ as functions of ϕ and ψ were assumed for all boundaries. Realistic assumptions were made in agreement with a rough flow net drawn for an assumed free surface. To compute initial values for $\ln(V)$, it was necessary to assume an initial value for the total head and, in the course of solution, it was then necessary to improve on the assumed values for both the free-surface coordinates and the total head.

Improvement of the values of H , $x(\phi, \psi)$ and $y(\phi, \psi)$ was accomplished in the following iterative fashion:

⁷ Bickley, W. G., "Finite-Difference Formulae for the Square Lattice," *Quarterly Journal of Mechanics and Applied Mathematics*, London, England, Vol. 1, Part 1, 1948.

1. A $\theta(\phi, \psi)$ field was computed in agreement with the assumed values on the boundary by successively applying Eq. 21 to each interior point in the W-plane (a relaxation process).

2. Beginning with values of $\ln(V(\phi, \psi))$ on the free surface, as determined from velocities obtained by application of the Bernoulli equation, a $\ln(V(\phi, \psi))$ field was determined in the W-plane through numerical integration of Eqs. 22 and 23.

3. Values for $y(\phi, \psi)$ or $x(\phi, \psi)$ were determined for the free surface and the solid boundary using Eqs. 24, 25, 26, and 27. If these values differed from the previously used $y(\phi, \psi)$ and $x(\phi, \psi)$ by less than 1.0%, the solution was assumed to be complete.

FIG. 3.—FINITE-DIFFERENCE MESH

4. If the values of $x(\phi, \psi)$ or $y(\phi, \psi)$ computed in Step 3 differed too greatly from the previously used values, new values of $\theta(\phi, \psi)$ were computed for all boundaries in agreement with the new $x(\phi, \psi)$ and $y(\phi, \psi)$. Observation of the behavior of the free-surface coordinates provided an algorithm for the correction of the approximated total head. This correction is explained in greater detail in the following text. After proper adjustment of the total head, values of $\ln(V(\phi, \psi))$ were computed along the free surface in accordance with the new $y(\phi, \psi)$ and H .

5. The programmed solution was then cycled back to Step 1 for another iteration. This procedure was continued until $x(\phi, \psi)$, $y(\phi, \psi)$, and H remained constant throughout successive iterations.

As was stated earlier, it was necessary to assume an initial value for the total head. By observing the behavior of the coordinates for the free surface

profile, it was possible to determine whether the assumed total head was smaller or larger than the correct value. Fig. 4 illustrates the behavior of the free-surface profile with assumed values of the discharge coefficient. When the discharge coefficient was too large (this results in a total head which is also too large), the computed free-surface ordinates became progressively smaller with each succeeding iteration. An opposite trend occurred when the assumed discharge coefficient was too small. When a satisfactory value for the discharge coefficient was assumed, the computed free-surface profile approached a limiting position.

FIG. 4.—BEHAVIOR OF NUMERICALLY-DETERMINED FREE-SURFACE PROFILE WHEN C_D WAS IN ERROR

Three separate spillway shapes were analyzed in the foregoing manner. The first shape corresponded to that of a standard spillway, whereas the second shape was proportioned according to the nappe produced by flow over a weir subject to a negative pressure head $\Delta/p\gamma$ beneath the nappe equal to one-half the head on the weir. The third shape was formed by a vertical upstream face, a circular crest, and a plane downstream slope. Each shape was analyzed for flow at several heads. The resulting free-surface profiles and pressure distributions are shown in Figs. 5, 6, and 7 and the final values for the discharge coefficients are shown in Fig. 8.

Accuracy of the numerical procedures was estimated by solving one problem at successively reduced finite-difference intervals (increased number of

FIG. 5.—FREE-SURFACE PROFILES AND PRESSURE DISTRIBUTIONS FOR STANDARD SPILLWAY ($\Delta p/\gamma h_D = 0.0$)

FIG. 6.—FREE-SURFACE PROFILES AND PRESSURE DISTRIBUTIONS FOR SPILLWAY DESIGNED FOR $4p/\gamma h_D = -0.5$

FIG. 7.—FREE-SURFACE PROFILES AND PRESSURE DISTRIBUTIONS FOR SPILLWAY WITH CIRCULAR CREST

streamlines). For the standard spillway shape with flow at design head, nineteen streamlines were required to produce x and y coordinates that were within 1.0% of the apparent limiting value. The accuracy of the assumed discharge coefficient also affected the inherent accuracy in the computed co-

FIG. 8.—DISCHARGE COEFFICIENTS FOR THREE SPILLWAY SHAPES

ordinates. Discharge coefficients within 1.0% of their apparent limiting values were found to produce the above noted accuracy in coordinates.

Computed free-surface coordinates were also plotted in the fashion shown in Figs. 10, 11, and 12. Observation of these plots indicates an apparent linear relationship between y and h for particular values of x. This apparent linear

property was used to extrapolate free-surface ordinates for particular values of h and x after free-surface profiles had been obtained for two values of h . It should be noted that Rouse⁸ has shown a similar apparent linear property for the coordinates of free-surface profiles produced by flow over sharp-crested weirs.

RESULTS

Free-Surface Profiles.—Two different sets of figures are used to show the computed free-surface coordinates. Fig. 5, 6, and 7 show x versus y

FIG. 9.—MINIMUM PRESSURES FOR THREE SPILLWAY SHAPES

plotted as functions of h whereas Figs. 10, 11, and 12 show y versus h plotted as functions of x . The diameter, D , was used to form the dimensionless parameters describing flow over the circular crest because it was the independent geometric variable. Figs. 5, 6, and 7 provide physical clarity whereas Figs. 10, 11, and 12 illustrate an apparent functional relationship.

⁸ "Engineering Hydraulics," Edited by Hunter Rouse, John Wiley and Sons Inc., New York, N. Y., 1950.

Experimental results obtained by H. Rouse and L. Reid⁹ and the writer¹⁰ have been plotted on Fig. 5 for the purpose of comparison. The close agreement of the results of experiment and theory indicate that the effects of

FIG. 10.—FREE-SURFACE COORDINATES FOR STANDARD SPILLWAY ($\Delta p/\gamma h_D = 0.0$)

viscosity and surface tension are small with regard to the location of the free-surface profile.

⁹ Rouse, H., and Reid, L., "Model Research on Spillway Crests," *Civil Engineering*, Vol. 5, No. 1, January, 1935.

¹⁰ Cassidy, J. J., "Spillway Discharge at Other Than Design Head," thesis presented to the Univ. of Iowa, at Iowa City, Iowa, in June, 1964, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Because of complexity, nothing has been said with regard to the uniqueness of the solutions obtained using the method reported herein. However, the close comparison of experimental and theoretical results serves to indicate that the solutions obtained mathematically were those of interest.

FIG. 11.—FREE-SURFACE COORDINATES FOR SPILLWAY DESIGNED FOR $\Delta p/\gamma h_D = -0.5$

Pressure Distributions.—Once the velocity field has been determined, the Bernoulli equation was used to compute dimensionless pressures on the solid boundaries. The resulting pressure distributions are shown in Figs. 5, 6, and 7. For the two spillway shapes patterned after weir nappes, points of minimum pressure are seen to occur at points of maximum boundary curvature. However, for the circular crest, the point of minimum pressure is located at the downstream point of tangency. Close agreement between experiment and theory

is shown in Figs. 5 and 6, but no experimental values were available for comparison with the theoretical results obtained for the circular crest.

Minimum pressures occurring on the face of the spillways are plotted as functions of head in Fig. 9. The pressure magnitudes are seen to decrease rapidly with increasing head. Vapor pressure would be the lower limit of the negative pressures because cavitation would then occur. Another possible lower

FIG. 12.—FREE-SURFACE COORDINATES FOR SPILLWAY WITH CIRCULAR CREST

limit would be that at which separation of the nappe occurs. However, separation is a boundary-layer phenomenon and cannot be predicted from the consideration of pressure or piezometric head alone. It should be noted that separation on a standard spillway, according to O. Dillman,¹¹ will occur at a head equal to approximately 3.2 h_D .

¹¹ Dillman, O., "Untersuchungen an Ueberfallen," *Mitteilungen die Hydraulischen Instituts der T. H. Munchen*, Oldenbourg, Munich, Germany, No. 7, 1933.

Discharge Coefficients.—Fig. 8 shows experimental and analytical values of C as a function of head for each spillway. The analytically determined values are seen to be consistently lower than experimental values indicating that viscosity and surface tension produce a net decrease in the discharge coefficient.

Method of Solution.—A chief disadvantage associated with the method of solution described herein is that of practicability. Large computer storage capacity is a necessity because of the size of the $\ln(V)$ and θ arrays that must be stored. Approximately 6 hr of operational time were required on the IBM-7070 to obtain a solution for the standard spillway shape. These time and storage requirements necessarily eliminate the use of small computers for all but the simplest of problems.

The major attribute of the method is that of directness; all independent variables are specified in the physical plane making it possible to obtain free-surface profiles, discharge coefficients, and pressure distributions for flow over a particular boundary configuration.

CONCLUSIONS

In this study, a method was developed for the numerical solution of steady, irrotational, free-surface flow over a continuous lower boundary in a gravitational field. Pressure distributions, velocity distributions, free-surface coordinates, and discharge coefficients can be obtained therewith when all independent variables are specified in the physical plane. Because the method is one of successive approximation, it is a definite improvement over methods wherein the surface is located by trial-and-error procedures. For cases of spillways of finite height and tranquil approaching flow, the total head is initially unknown and cannot be obtained in a direct manner. Through observation of the behavior of the free-surface profile during the solution, an algorithm can be obtained for correcting assumed values of total head. Large computer-storage and long computer-time requirements are distinct disadvantages associated with application of the method. Neither of these disadvantages is as important for solution of problems involving super-critical approaching flow, for which the total head is an independent variable.

As a direct result of the experimental and analytical studies performed, the following conclusions can be drawn with regard to spillway flow characteristics:

1. Free-surface profiles obtained by potential theory and by experiment agree well and indicate that viscosity has a negligible influence on the location of the free surface. However, computed discharge coefficients for irrotational flow are larger than experimentally measured values by a small but noticeable quantity.

2. The location of the points of minimum pressure produced by flow over a spillway at a particular head is dependent on the boundary configuration. For a spillway shaped after weir-nappe profiles, the point of minimum pressure (and hence the point of incipient cavitation) occurs at the point of maximum curvature. For the circular crest used in this study, the minimum pressure occurred at the downstream point of tangency.

3. The curves for minimum pressure on the spillway surface developed in this study can be used to design spillways that will operate without cavitation

through the entire range of expected heads. Such a design would be more rational than one based on a design head chosen as some fraction of the maximum expected head.

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The study described herein was conducted while the writer was a graduate student in the Department of Mechanics and Hydraulics at the University of Iowa, Iowa City, Iowa. Computer time was donated by the University and all experimental work was performed in the laboratory of the Iowa Institute of Hydraulic Research. The writer is indebted to his advisor, Hunter Rouse, for both the original suggestion of the study and for his advice and encouragement as the study progressed.

APPENDIX. --NOTATION

The following symbols have been adopted for use in this paper:

- $C = q / \frac{2}{3} \sqrt{2g} h^{3/2}$ = discharge coefficient;
 D = diameter of the circular crest;
 g = gravitational acceleration;
 H = total head;
 h = head on the spillway;
 h_D = design head for the standard spillway or a parameter assumed to describe spillway shape;
 k = boundary-roughness scale;
 m = finite difference interval;
 p = pressure;
 $q = V_O(h+w)$ = discharge per unit width;
 u = velocity component in the x-direction;
 V = complex velocity;
 V_O = velocity of approach
 v = velocity component in the y-direction;
 W = complex potential;
 w = spillway height;
 x, y = cartesian coordinates;
 z = complex variable;
 γ = unit weight;
 Δp = difference in pressure between the lower and upper surfaces of weir nappe;
 θ = angle of inclination of the velocity vector;
 μ = dynamic viscosity;
 ρ = density;
 σ = surface energy;
 ϕ = velocity potential; and
 ψ = stream function.

KEY WORDS: coefficients; engineering mechanics; hydraulics; laminar flow; pressure; spillways; water flow

ABSTRACT: Potential theory is used to analyze free-surface, irrotational flow over spillways of finite height. Numerical values were obtained for free-surface coordinates, pressures on the surface of the spillway, and discharge coefficients for three separate spillway shapes. The spillways considered included (a) one corresponding to the shape of the nappe produced by flow over a ventilated, sharp-crested weir, (b) one corresponding to the shape of the nappe produced by flow over a ventilated, sharp-crested weir with negative pressure head beneath the nappe equal to one-half the head on the weir, and (c) one having vertical upstream face, a cylindrical crest, and a downstream tangent inclined 30° from the vertical. Because flows of several heads were analyzed for each spillway, certain conclusions were drawn relative to spillway operation under high heads.

REFERENCE: Cassidy, John J., "Irrotational Flow Over Spillways of Finite Height," Journal of the Engineering Mechanics Division, ASCE, Vol. 91, No. EM6, Proc. Paper 4591, December, 1965, pp. 155-173.

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