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THE DESIGN OF SAMPLED-DATA FEEDBACK SYSTEMS

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The Design of Sampled-Data Feedback Systems

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SAMPLED-DATA feedback control systems, or pulsed servomechanisms, are characterized by the distinguishing feature that at one or more points in the system signal information is in the form of a pulse train which can be represented as a sequence of numbers. One example in which the signal information inherently is represented by a sequence of numbers is provided by a closed-loop system containing a digital computer as one element. Sampled-data systems are also useful

when the energy available from a sensitive element is so small that it is imperative to avoid loading the output more than necessary.

The sampling process both simplifies and complicates the design of the control system. Because the loop is closed only during the sampling times, the system operates most of the time open-loop, and evaluation of the response is a simple open-loop analysis problem. The intermittency of the closing of the loop intro-

duces, however, stability problems nonexistent in the continuous system. The following discussion indicates certain basic ideas concerning the design of sampled-data systems, particularly the manner in which the simplification introduced by the sampling can be utilized in design.

The system to be considered is shown in Fig. 1. The distinguishing component is the sampler, which is represented as a switch alternately closed for T_1 seconds

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Fig. 1. A sampled-data feedback system

and opened for T_2 seconds. The sampling period T , assumed constant in this paper, is defined by the equation

$$T_1 + T_2 = T \quad (1)$$

A sampled-data system implies, by definition, that T_1 is small compared with both T_2 (or T) and the time constants of the remainder of the system.

For the purposes of illustration throughout this paper, only the single-loop unity-feedback configuration of Fig. 1 is considered. Here $G_2(s)$ is the transfer function of the controller elements and $G_1(s)$ describes the compensation network or control system. The product of $G_1(s)$ and $G_2(s)$ is denoted by $G(s)$. Furthermore, the primary emphasis of the following discussion is placed on the design to meet specifications phrased in terms of the desired characteristics of the step-function response, although the methods are readily extended to the consideration of other aperiodic inputs.

The continuous actuating signal is $e(t)$, while the actuating signal in sampled form is $e^*(t)$. The $e^*(t)$ has the form of a train of pulses with a width T_1 and a height proportional to the magnitude of $e(t)$ at the sampling instant. As long as the assumptions of the smallness of T_1 are valid, the analysis can be simplified by approximating the pulse train by a train of unit impulses with areas representing the sample amplitudes of $e(t)$. Substitution of the impulse train for the actual pulse train, with a correspondence of area to amplitude, tacitly assumes that T_1 equals unity. When the gain K of the open-loop part of the system is set on the basis of this assumption, the actual gain used in the physical system must be K/T_1 .

Simplicity of System Response

By its very nature a sampled-data system should be more easily analyzed than the corresponding continuous system.

Indeed, some of the most powerful methods for the analysis of continuous systems are based on sampled approximations of the continuous functions. The greater simplicity of sampled-data feedback systems results from the open-loop nature of the operation at all times except the sampling instants. The response of the over-all system of Fig. 1 is simply the sum of weighted impulse responses of g .

If the system shown in Fig. 1 is at rest and an arbitrary input function $r(t)$ is applied, the g network remains unexcited until the first sampling instant t_0 . As a result of the first sampling, g is excited by an impulse of area

$$e(t_0) = r(t_0) - c(t_0) \quad (2)$$

In the interval $t_0 < t < t_0 + T$ the output $c(t)$ is simply the constant $e(t_0)$ multiplied by $g(t)$, the impulse response of g . At the second sampling period another impulse of area $e(t_1)$ is applied to g . Since g is assumed linear superposition holds and the output $c(t)$ is the sum of the two impulse responses.

If this procedure is continued, the output $c(t)$ can be calculated through any desired number of sampling periods. The output during the n th interval is simply the sum of n weighted, delayed impulse responses, in contrast to the continuous system where the output is the convolution of the actuating signal and $g(t)$. If an aperiodic function such as a unit step or a unit ramp function is applied to the system, the nature of $c(t)$ can often be determined by following the output through only a few sampling periods. The main purpose of this paper is to indicate the manner in which this physical view of system performance can be utilized in design; indeed, a wide variety of design problems can be solved with only this physical reasoning and without the introduction of more powerful mathematical methods.

Of the more elaborate mathematical tools available for the analysis and design of sampled-data systems the most promising appears to be the z -transformation.¹ The main limitation of the z -transform is that the response is determined only at the sampling instants. Since there exists an infinite number of sampled-data systems which have the same response values at the sampling instants, other techniques must be used to determine the nature of the complete response. The significance of this ambiguity is illustrated by the four systems shown in Fig. 2. The step-function response of each of the four systems assumes exactly the same set of values at the sampling instants, but the curves between sampling instants are remarkably different, as shown in Fig. 3. Since all four responses go through the same points at the sampling instants, the four associated z -transforms are identical. This uncertainty introduced in the use of the z -transform is added to the ambiguity introduced when the pulse train is replaced by an impulse train: an ambiguity which arises because a true impulse is also represented as an impulse in the approximation. For example, a fifth system can be added to Figs. 2 and 3; i.e., a system which possesses a step-function response which is entirely an impulse train.

A large part of this limitation of the z -transformation method is removed by the techniques set forth in Appendix I. These methods allow determination of the value of the output at any desired sub-multiple of the sampling period. Actually, this extended z -transformation approach is quite similar to the impulse response approach proposed in this paper; the main difference is primarily a matter of emphasis. If the extended z -transformation method is used as a basis for design, the designer is apt to become so enamored with the mathematics that all problems are worked with transformation methods. In the design of sampled-data systems it is important that the basic simplicity of system operation be kept firmly in mind. This can be accomplished if the basic approach to design is through the summation of the weighted

Fig. 2 (left). Four systems with the same z -transform and a T of 1 second

Fig. 3 (right). Step-function responses of the systems of Fig. 2

impulse responses of g . The z -transform, the theory of difference equations, and other mathematical methods are introduced to extend the designer's insight, to suggest system designs which may not be intuitively apparent, and to facilitate those aspects of analysis where intuitive and physical reasoning breaks down.

Impulse Response Approach

ANALYSIS

The basic characteristics and advantages of the impulse response approach to system design are conveniently described by considering first the analysis problem. In particular, the following simple example illustrates that analysis based on the impulse response of the open-loop system is capable of presenting essentially complete information about the system output. If a plot of the continuous output is not required, the output at regular intervals can be determined. The work required in the impulse response approach depends only on the amount of information desired.

The characteristics of the analysis are apparent from the simple example of Fig. 1 with

$$G(s) = \frac{13.4}{s(s+10)} \quad \text{and} \quad T=0.1 \text{ second} \quad (3)$$

The response of g to a unit impulse is

$$g(t) = 1.34(1 - e^{-10t}) \quad (4)$$

It is assumed that the system is at rest when the unit step function is applied as the input $r(t)$.

The first sampling (assumed at $t=0$, just after the application of the step) occurs with the actuating signal equal to unity. Thus at $t=0$ a unit impulse is applied to g which responds in accordance with equation 4 until the next sampling

Fig. 4. Response of simple system

instant. This response is shown as $g_0(t)$ in Table I and Fig. 4. At the end of one sampling period the output has reached a value of 0.846, the error e_1 is 0.154, and an impulse of area 0.154 is applied to g . The corresponding component of the response is shown as $g_1(t)$ in Table I and Fig. 4. In Table I the $g_1(t)$ column is simply the $g_0(t)$ column multiplied by e_1 and shifted down by an amount corresponding to one sampling period. The output $c(t)$ between T and $2T$ seconds is

$$c(t) = g_0(t) + g_1(t) \quad (5)$$

and is also shown in Table I and Fig. 4. The remainder of the $c(t)$ curve is constructed in a similar manner, with the procedure followed for as many sampling periods as desired. Examination of the nature of $g_0(t)$ and the $c(t)$ curve of Fig. 4 shows that the output is settling down rapidly.

It should be pointed out here that the entries in Table I are simply the coefficients which are involved if the z -transform analysis is used with a subharmonic period of $T/5$. See Appendix I. The advantage of viewing the analysis as the superposition of impulse responses essentially lies in the possibility of utilizing the curve summation indicated in Fig. 4. In practice it is rarely necessary to calculate accurately more than the response

at the actual sampling points. A knowledge of the graphical nature of $g(t)$ enables the designer to sketch the approximate shape of the response between sampling points without elaborate calculations.

DESIGN

The foregoing analysis is readily extended to design. There are two basic design problems in the theory of feedback control systems: the adjustment of gain and the determination of appropriate values for circuit parameters. Both facets of design are conveniently illustrated in terms of the simple example used in the preceding section, with

$$G(s) = \frac{10K}{s(s+10)} \quad \text{and} \quad T=0.1 \text{ second} \quad (6)$$

A trial-and-error procedure suffices for the rapid adjustment of the gain K to meet any of the customary specifications on relative stability, rise time, etc. For example, if the desired overshoot is specified, a value of K is chosen and the $c(t)$ curve is developed until the first maximum is reached. If the overshoot is too large, a smaller value of K is chosen and the analysis repeated. With a little experience the proper value of K can be established with one or two attempts. With K set at 1.34 the example of the preceding section demonstrates the solution with the specified overshoot about 29 per cent.

Insight into the more general problem of fixing system parameters can be gained from a study of the effect of variation of aT on the response of a system with a sampling period T and

$$G(s) = \frac{aK}{s(s+a)} \quad (7)$$

If a maximum value of overshoot is specified (25 to 30 per cent in this paper), an appropriate value of K and the corresponding form of the response can be determined for any value of aT .

If aT has a value of 5, the $g(t)$ response has essentially reached steady state before the next sampling instant. Parts A, B, and C of Fig. 5 show the nature of $c(t)$

Table I. Response of Simple System

t	$e_0 = 1$ $g_0(t)$	$e_1 = 0.154$ $g_1(t)$	$e_2 = -0.29$ $g_2(t)$	$e_3 = -0.204$ $g_3(t)$	$c(t)$
0	0				0
0.02	0.242				0.242
0.04	0.442				0.442
0.06	0.605				0.605
0.08	0.738				0.738
0.10	0.846	0			0.846
0.12	0.935	0.037			0.972
0.14	1.01	0.068			1.078
0.16	1.07	0.093			1.163
0.18	1.12	0.114			1.234
0.20	1.16	0.130	0		1.29
0.22	1.19	0.144	-0.07		1.264
0.24	1.22	0.156	-0.129		1.247
0.26	1.24	0.165	-0.175		1.23
0.28	1.26	0.173	-0.214		1.219
0.30	1.27	0.179	-0.245	0	1.204
0.32	1.28	0.183	-0.271	-0.049	1.143
0.34	1.29	0.188	-0.293	-0.090	1.095
0.36	1.30	0.191	-0.311	-0.123	1.057
0.38	1.31	0.194	-0.325	-0.150	1.029
0.40	1.32	0.196	-0.336	-0.173	1.01

Fig. 5. Response of system with $aT=5$ as K is varied

Curve A— $K=0.5$
Curve B— $K=1$
Curve C— $K=1.25$

as K is varied. For most purposes the system of part B with K equal to unity would be preferred. The only advantage of the system of part C is that it has a slightly shorter rise time, but this is certainly outweighed by the oscillatory nature of the response. For this reason the value of gain is ordinarily never set higher than the magnitude to put the $c(t)$ curve through unity at the end of one sampling period.

The results of continuing this investigation for aT equal to 2.5, 1.5, 1.0, and 0.5 are shown in parts A, B, C, and D respectively of Fig. 6. As aT is decreased the magnitude of $c(t)$ after one period decreases and more sampling periods are required for the $c(t)$ curve to reach its maximum value. This is shown in Table II which also includes the value of $c(T)$ and K when K assumes the value placing the system on the verge of instability. The equations for $K_{\max}$ and $c(T)_{\max}$ are derived in Appendix II and are

$$K_{\max} = \frac{2(1 + e^{-aT})}{1 - e^{-aT}} \quad (8)$$

$$c(T)_{\max} = 2(1 + e^{-aT}) \quad (9)$$

SYNTHESIS

When more freedom is allowed the designer and more comprehensive problems are considered, it becomes necessary to shape the response characteristics to meet arbitrary specifications. In terms of the design of sampled-data systems, synthesis involves the reshaping of the impulse response $g(t)$.

This reshaping of a time function can be accomplished in a variety of ways ranging from almost pure synthesis to

simply extended analysis. Idealistically, the designer simply forces the system to behave in the desired manner. The unwanted poles and zeros of $G_2(s)$, the transfer function of the controller system, are canceled by zeros and poles of $G_1(s)$, and the critical frequencies reinserted are selected to yield a system which meets the performance specifications.

If the system required to effect such a straightforward compensation is undesirably complex, the impulse response can be shaped more indirectly. For example, $G_2(s)$ conventionally includes a pole at the origin: i.e., $G_2(s)$ can be written as $T(s)/s$. The impulse response $g_2(t)$ is then simply the step-function response of a system with the transfer function $T(s)$. Hence, all the literature and knowledge concerning the effects of various pole-zero configurations can be brought to bear on the problem of reshaping $g_2(t)$ or the step-function response for $T(s)$.

If the designer leans even more heavily away from synthesis and toward design by analysis, the poles of $G_2(s)$ can be modified one by one until the appropriate response is obtained. For example, a typical design problem might be to decrease the rise time and settling time while leaving the overshoot unchanged. The approach is illustrated with the specifications

$$G_2(s) = 500K/s(s+10)(s+50) \text{ with } T=0.05 \text{ second} \quad (10)$$

Since it is the pole at -10 which is essentially limiting the rise time, it is clear that $G_1(s)$ should effectively move this pole farther to the left in the s -plane. If $G_1(s)$ is

$$G_1(s) = \frac{5(s+10)}{s+50} \quad (11)$$

the pole is moved to -50 , and $G(s)$ becomes

$$G(s) = \frac{2,500K}{s(s+50)^2} \quad (12)$$

With K set to meet the overshoot requirements the curve labeled part B of Fig. 7 results, in contrast to the curve of part A for the response without compensation. More stringent compensation specifications can be met by moving the

Fig. 6. Response of system as aT is varied

Curve A— $aT=2.5$, $K=1.09$
Curve B— $aT=1.5$, $K=1.29$
Curve C— $aT=1.0$, $K=1.34$
Curve D— $aT=0.5$, $K=0.77$

pole at -10 farther to the left with a

$$G_1'(s) = \frac{10(s+10)}{s+100} \quad (13)$$

Part C of Fig. 7 shows the result after K is adjusted to make $c(T)$ equal to unity, in which case the overshoot is 17 per cent.

With a little experience the designer can examine $G_2(s)$ and T and in most cases see the nature of the needed $G_1(s)$. If the $g_2(t)$ is monotonic increasing, specific statements regarding the desired compensation can be made. If $g_2(t)$ essentially reaches steady state in one sampling period, little needs to be done in the way of compensation. If this is not true, the pole nearest the origin is predominant in determining the rise time and is the one to move first. If there are several poles near the origin, the movement of any one of these to the left decreases rise time. If the resulting system does not satisfy the specifications, additional poles must be moved. Although relations similar to equations 8 and 9 cannot be derived for the general monotonic increasing case, it can be stated that $K_{\max}$ and $c(T)_{\max}$ are both equal to or greater than 2. If a system is designed with $c(T)$ equal to or less than unity, the system is operating well below the limit of stability and each successive maximum value of $c(t)$ is lower than the preceding.

If $g_2(t)$ is oscillatory, general statements are more difficult. In certain cases the gain for absolute stability is lowered as a result of the oscillation, but in other cases, even though the gain for absolute stability is not affected, K has to be set at a lower value to meet the overshoot requirement and the entire response is more sluggish.

The curve of the desired $g(t)$ is one

Fig. 7. Response of system being synthesized

Table II. Design Summary

aT	Per Cent Overshoot	K	$K_{\max}$	$c(T)$	$c(T)_{\max}$	Periods to Maximum
5	0	1	2	1	2.03	1
2.5	8	1.09	2.36	1	2.16	2
1.5	22.5	1.29	3.15	1	2.45	2
1.0	29	1.34	4.32	0.846	2.74	2
0.5	26.9	0.77	8.18	0.303	3.22	5

which reaches the range of 80 to 100 per cent of its steady-state value in one sampling period and settles down to the steady state rapidly. In most cases an examination of $g_2(t)$ indicates the nature of the required compensation network.

Limitations of the Impulse Response Approach

When the z -transformation theory is used to determine stability the function $G(z)/[1+G(z)]$, where $G(z)$ is the z -transform of $G(s)$, is examined and the system becomes absolutely unstable when a pole or poles move out of the unit circle in the z -plane. If the first pole leaves the unit circle on the negative real axis (corresponding to system oscillation at one-half the sampling frequency, or oscillation basically because of the sampling), the impulse response approach can be used to determine the maximum gain. However, if the first pair of poles leave the unit circle at some point other than on the negative real axis, it is impossible to determine the maximum gain directly except in special cases. (This mode of instability can happen in a second-order system with a dead-time delay or a holding circuit). The only method left to the designer using the impulse response approach is to set the gain and follow the output long enough to determine system stability.

A similar situation occurs in attempting to find the steady-state output when the input to the system is a sinusoidal function. When the z -transformation is used the envelope of the output and the bandwidth can be found directly. If the impulse response approach is used and the sampling period is not a subharmonic of the sinusoidal period, the $c(t)$ curve never repeats, and it is difficult to recognize steady-state conditions. If the two methods of analysis are combined, the envelope can be found by z -transformation methods and the output between sampling instants by the impulse response approach, giving the complete picture of the output.

Conclusions

The present z -transformation theory suffers from the uncertainty surrounding the behavior of the system between sampling instants. This is partly removed by the procedure of finding the output at submultiples of the sampling period and is entirely removed by the impulse response approach. The z -transformation method gives a comprehensive picture of absolute stability, but this is not the prob-

Table III. z -Transform

Time Function	Laplace Transform	z -Transform
$u_0(t)$	1	1
$u_{-1}(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$e^{-\alpha T}$	$\frac{1}{s+\alpha}$	$\frac{z}{z-e^{-\alpha T}}$

lem faced by the engineer who is trying to meet such specifications as maximum overshoot. The impulse response approach can be used to set the gain to meet overshoot specification and an examination of $g_2(t)$ in most cases indicates the nature of the compensation network needed to meet the specifications. Thus the optimum design method for a large variety of sampled-data systems is a design on the basis of a shaping of the impulse response of the open-loop system. The z -transformation and Nyquist diagram are used to increase the designer's circumspection.

Appendix I. Output Between Sampling Instants

The sampled functions are described throughout this paper in terms of the corresponding z -transforms. The definition of the z -transform follows that used by Ragazzini and Zadeh.¹ The z -transform is obtained by substituting z for e^{sT} in the Laplace transform of the sequence of samples. As a consequence of this definition, the expansion of $C(z)$, in a power series in $1/z$, places in evidence the successive sample amplitudes of $c(t)$. $C(z)$ is obtained in closed form with the aid of Table III. (The Ragazzini-Zadeh article contains a more extensive list of transforms.) Conversion from $C(s)$ to $C(z)$ is accomplished by a partial fraction expansion of $C(s)$, followed by a term-by-term identification based on the table.

Thus the z -transformation can be used to evaluate rapidly the output of a sampled-data system at the sampling instants. In this basic application the z -transform $C(z)$ contains no information about the behavior of the actual $c(t)$ between sampling instants. The artifice of the introduction of a fictitious sampler permits the circumvention of this disadvantage and allows evaluation of the output at submultiples of the sampling period.

In a closed-loop system with the sampler following the error-measuring device, the actuating signal $e^*(t)$ is a train of impulses at the sampling frequency and is zero between sampling instants. The z -transform of the actuating signal can be expanded in the form

$$E(z) = \frac{R(z)}{1+G(z)} = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \dots \quad (14)$$

The time between successive values is T . As shown in Fig. 8, a fictitious sampler is added in tandem with the actual sampler. The period of the fictitious sampler is assumed a submultiple of the period T .

$$T' = \frac{T}{n} \quad (15)$$

In this example, if n is given a value of 2, the actuating signal at the output of the fictitious sampler is represented by

$$E'(z) = A_0 + \frac{0}{z} + \frac{A_1}{z^2} + \frac{0}{z^3} + \frac{A_2}{z^4} + \frac{0}{z^5} + \frac{A_3}{z^6} + \dots \quad (16)$$

where the time between successive values corresponds to T' and the A 's are the same as in equation 14. $E'(z)$ can be written as

$$E'(z) = \frac{R(z^2)}{1+G(z^2)} \quad (17)$$

and the z -transform of the output $C'(z)$ as

$$C'(z) = \frac{R(z^2)}{1+G(z^2)} G'(z) \quad (18)$$

in which $G'(z)$ is the z -transform of $G(s)$ with respect to T' and $R(z^2)$ and $G(z^2)$ are the z -transforms of $R(s)$ and $G(s)$ with respect to T , but with z replaced by z^2 .

$G(s)$ from equation 3 serves as an example

$$G(s) = \frac{13.4}{s(s+10)} = 1.34 \left(\frac{1}{s} - \frac{1}{s+10} \right) \quad (19)$$

The corresponding $G(z)$ is given from Table III

$$G(z) = 1.34 \left(\frac{z}{z-1} - \frac{z}{z-e^{-1}} \right) = \frac{0.846z}{z^2 - 1.368z + 0.368} \quad (20)$$

With the input a unit step function, $E(z)$ is

$$\frac{R(z)}{1+G(z)} = \frac{z^2 - 0.368z}{z^2 - 0.522z + 0.368} \quad (21)$$

and

$$E'(z) = E(z^2) = \frac{z^4 - 0.368z^2}{z^4 - 0.522z^2 + 0.368} \quad (22)$$

$G'(z)$ is given by

$$G'(z) = 1.34 \left(\frac{z}{z-1} - \frac{z}{z-e^{-0.5}} \right) = \frac{0.526z}{z^2 - 1.607z + 0.607} \quad (23)$$

The output is described by the z -transform

$$C'(z) = \frac{0.526z^5 - 0.194z^3}{z^6 - 1.607z^5 + 0.085z^4 + 0.838z^3 + 0.052z^2 - 0.592z + 0.223} \quad (24)$$

When equation 24 is expanded by

Fig. 8. Simple system with fictitious sampler

ordinary long division of the denominator into the numerator, the first few terms are

$$C'(z) = 0 + \frac{0.526}{z} + \frac{0.846}{z^2} + \frac{1.12}{z^3} + \frac{1.29}{z^4} + \dots \quad (25)$$

This result agrees with the curve of $c(t)$ in Fig. 4.

This procedure for determining the response between sampling instants is straightforward. The only difficulty arises in the evaluation of $G'(z)$ from the known $G(s)$ or $G(z)$. $p_{G'}$, a pole of $G'(z)$, is related to the corresponding pole p_G of $G(z)$ by the equation

$$p_{G'} = (p_G)^{1/n} \quad (26)$$

The residues of $G(z)$ and $G'(z)$ are equal in the corresponding poles. Unfortunately, however, there is no simple correlation between the numerator polynomials of $G(z)$ and $G'(z)$. Except in simple cases, the numerator of $G'(z)$ must be evaluated by summation of the terms in the partial fraction expansion.

Appendix II. Stability of Second-Order System

Equations 8 and 9 are derived in the following manner

$$G(s) = \frac{aK}{s(s+a)} = \frac{K}{s} - \frac{K}{s+a} \quad (27)$$

The z -transformation of equation 27 is

$$G(z) = \frac{Kz}{z-1} - \frac{Kz}{z-\epsilon^{-aT}} = \frac{K(1-\epsilon^{-aT})}{z^2 - (1+\epsilon^{-aT})z + \epsilon^{-aT}} \quad (28)$$

which gives

$$\frac{G(z)}{1+G(z)} = \frac{K(1-\epsilon^{-aT})z}{z^2 + K(1-\epsilon^{-aT}) - (1+\epsilon^{-aT})z + \epsilon^{-aT}} \quad (29)$$

As one step of the z -transformation the change of variable

$$z = \epsilon^{sT} \quad (30)$$

is made to map the left half of the s -plane to the interior of the unit circle in the z -plane. As K is increased the system becomes unstable when the poles of $G(z)/[1+G(z)]$ leave the unit circle. Thus stability requires that the zeros of

$$p(z) = z^2 + K(1-\epsilon^{-aT}) - (1+\epsilon^{-aT})z + \epsilon^{-aT} \quad (31)$$

lie within the unit circle. The system is stable if

$$p(0) < 1 \quad (32)$$

$$p(1) > 0 \quad (33)$$

$$p(-1) > 0 \quad (34)$$

Equation 32 gives

$$p(0) = \epsilon^{-aT} < 1 \quad (35)$$

which is true for all aT greater than zero and for all K . Equation 33 yields

$$p(1) = K(1-\epsilon^{-aT}) > 0 \quad (36)$$

which is true for all positive aT and for all K . Finally, equation 34 gives

$$p(-1) = -K(1-\epsilon^{-aT}) + 2(1+\epsilon^{-aT}) > 0 \quad (37)$$

from which $K_{\max}$ or equation 8 can be derived.

The inverse Laplace transform of equation 27 yields

$$g(t) = K(1-\epsilon^{-aT}) \quad (38)$$

When $K_{\max}$ is substituted and $g(T)$ evaluated, equation 9 results.

Equations 32, 33, and 34 are valid for the second-order system only. In the general case, specific procedures are available for establishing stability in terms of the z -transform.²

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