THE UNIVERSITY OF MISSOURI

ENGINEERING REPRINT SERIES

Bulletin

Reprint Number 1

• • •
Engineering Experiment Station
Columbia, Missouri

RESPONSE OF CIRCUITS TO STEADY-STATE PULSES

D. L. WAIDELICH ✧ *Professor of Electrical Engineering*
Reprinted From IRE Vol. 37. No. 12
December, 1949

THE UNIVERSITY OF MISSOURI BULLETIN

VOLUME 52, NUMBER 8 ENGINEERING EXPERIMENT STATION REPRINT SERIES NO. 1

Published by the University of Missouri at Room 102, Building T-3, Columbia, Missouri. Entered as second-class matter, January 2, 1914, at the postoffice at Columbia, Missouri, under the Act of Congress of August 24, 1912. Issued three times a month. --300

MARCH 10, 1951

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Reprinted from the PROCEEDINGS OF THE I.R.E.
VOL. 37, NO. 12, DECEMBER, 1949

PRINTED IN THE U.S.A.

Response of Circuits to Steady-State Pulses*

D. L. WAIDELICH†, SENIOR MEMBER, IRE

Summary—A method of calculating the steady-state response of circuits to repeated pulses is given using the method of the steady-state operational calculus. A short table of transforms which have been found useful in these calculations is also presented. The response of several basic circuits to these pulses is obtained and shown as calculated curves, and the calculated curves are then compared with curves obtained experimentally. These curves have been found to be very useful in adjusting circuits to be used with pulses. Several other possible applications are discussed.

INTRODUCTION

MUCH WORK has been done lately on the manner in which electrical circuits affect the shape of pulses.^{1,2,3} MacLachlan⁴ and others^{5,6,7} have studied the response of circuits to the impulsive type of pulse, i.e., a pulse in which the energy has been transferred to the circuit before the circuit has had time to respond. Very little work on the response of circuits to steady-state pulses of the impulsive type has been done, however, and it is the purpose of this paper to examine this type of response.

ANALYSIS

When considering the impulsive type of pulse, the main requirement to be satisfied is that the duration of the pulse be much shorter than any of the time constants or natural periods of oscillation of the circuits used. For example, a one-microsecond pulse width should be satisfactory for a circuit whose smallest time constant is of the order of ten microseconds, and whose highest natural frequency is of the order of 100 kc. If the above is satisfied, the shape of the particular pulse used should have very little effect on the results. The area E is a satisfactory measure of the strength of a pulse, and a unit pulse will be defined as one with unit area.

The pulse voltage wave form of Fig. 1(a) has the steady-state direct transform⁸

$$S(e) = \frac{E}{ap} (1 - e^{-pa}). \quad (1)$$

* Decimal classification: R141.3. Original manuscript received by the Institute, February 24, 1949; revised manuscript received, September 26, 1949.

† University of Missouri, Columbia, Mo.

¹ E. Frank, "Pulsed Linear Networks," McGraw-Hill Book Company, New York, N. Y., 1945.

² E. C. Cherry, "Pulse response," *Jour. I.E.E.*, pt. III, vol. 92, pp. 183-196; September, 1945.

³ G. N. Glasoe, and J. V. Lebacqz, "Pulse Generators," McGraw-Hill Book Company, New York, N. Y., 1948.

⁴ MacLachlan, "Operational form of $f(t)$ for a finite interval with application to impulses," *Phil. Mag.*, vol. 26, Ser. 7/ pp. 695-704; November, 1938.

⁵ M. F. Gardner, and J. L. Barnes, "Transients in Linear Systems," John Wiley and Sons, Inc., New York, N. Y., pp. 252-263; 1942.

⁶ G. A. Campbell and R. M. Foster, "Fourier Integrals for Practical Applications," D. Van Nostrand Co., New York, N. Y., pp. 15-19; 1942.

⁷ J. C. Jaeger, "Switching problems and instantaneous impulses," *Phil. Mag.*, vol. 36, Ser. 7, pp. 644-651; September, 1945.

⁸ D. L. Waidelich, "Steady-state operational calculus," *PROC. I.R.E.*, vol. 34, pp. 78-83; February, 1946.

For very small values of a , the transform approaches the impulsive form

$$S(e) = E. \quad (2)$$

In applying the repeated pulses to circuits, a table of steady-state transforms was found very useful, and a short table of this kind is presented in Table I. The symbol Im_1 indicates an impulse of the first order and would be that of Fig. 1(a) with $E=1$ and a approaching zero. Similarly Im_2 indicates an impulse of the second order and is that of Fig. 1(b) with $E=1$ and a approaching zero. Figs. 1(c) and (d) show impulses of the third and fourth order, i.e., Im_3 and Im_4 .

Fig. 1—Wave forms of the voltage pulses of orders (a) one, (b) two, (c) three, (d) four.

RESPONSE OF CIRCUITS

In determining the response of circuits to repeated pulses, the first circuit to be considered is that of a resistance R and a capacitance C in series. If the repeated pulses have the direct transform of (2), the direct transform of the current i flowing is

$$S(i) = \frac{E}{R} \frac{p}{p + \frac{1}{CR}}, \quad (3)$$

and that of the voltages across the resistance R and the capacitance C are

TABLE I
STEADY-STATE TRANSFORM TABLE

Direct Transform $S[f(t)] = F(p)$	Inverse Transform $S^{-1}[F(p)] = f(t)$
1. 1	Im_1
2. p	Im_2
3. p^n	$Im_{(n+1)}$
4. $\frac{1}{p+a}$	$\frac{\epsilon^{-at}}{1 - \epsilon^{-aT}}$
5. $\frac{p}{p+a}$	$-a \frac{\epsilon^{-at}}{1 - \epsilon^{-aT}} + Im_1$
6. $\frac{1}{p^2 + a^2}$	$\frac{T}{2m} \frac{\cos m \left(\frac{t}{T} - \frac{1}{2} \right)}{\sin (m/2)}$ where $m = aT$
7. $\frac{p}{p^2 + a^2}$	$-\frac{\sin m \left(\frac{t}{T} - \frac{1}{2} \right)}{2 \sin (m/2)}$ where $m = aT$
8. $\frac{p^2}{p^2 + a^2}$	$-\frac{m}{2T} \frac{\cos m \left(\frac{t}{T} - \frac{1}{2} \right)}{\sin (m/2)} + Im_1$ where $m = aT$
9. $\frac{1}{(p+a)^2}$	$\frac{\epsilon^{-a(t-T)} [t - \epsilon^{-aT}(t-T)]}{2(\cosh aT - 1)}$
10. $\frac{p}{(p+a)^2}$	$\frac{\epsilon^{-a(t-T)}}{2(\cosh aT - 1)} [(1-at)(1 - \epsilon^{-aT}) - aT\epsilon^{-aT}]$
11. $\frac{p^2}{(p+a)^2}$	$\frac{\epsilon^{-a(t-T)}}{2(\cosh aT - 1)} [a(at-2)(1 - \epsilon^{-aT}) + a^2T\epsilon^{-aT}] + Im_1$
12. $\frac{1}{(p+\alpha_1)(p+\alpha_2)} = \frac{1}{(p+a)^2 - b^2} = \frac{1}{(p+a)^2 + \beta^2}$ $\alpha_1 = a + b = a + j\beta$ $\alpha_2 = a - b = a - j\beta$ $b = j\beta \neq 0$	$\frac{1}{(\alpha_2 - \alpha_1)} \left[\frac{\epsilon^{-\alpha_1 t}}{1 - \epsilon^{-\alpha_1 T}} - \frac{\epsilon^{-\alpha_2 t}}{1 - \epsilon^{-\alpha_2 T}} \right]$ $= \frac{\epsilon^{-a(t-T)}}{2b} \left[\frac{\sinh bt - \epsilon^{-aT} \sinh b(t-T)}{\cosh aT - \cosh bT} \right]$ $= \frac{\epsilon^{-a(t-T)}}{2\beta} \left[\frac{\sin \beta t - \epsilon^{-aT} \sin \beta(t-T)}{\cosh aT - \cos \beta T} \right]$
13. $\frac{p}{(p+\alpha_1)(p+\alpha_2)} = \frac{p}{(p+a)^2 - b^2} = \frac{p}{(p+a)^2 + \beta^2}$ $\alpha_1 = a + b = a + j\beta$ $\alpha_2 = a - b = a - j\beta$ $b = j\beta \neq 0$ when $b > a$, $\phi = \tanh^{-1} \frac{a}{b}$ when $a > b$, $\psi = \tanh^{-1} \frac{b}{a}$ $\sigma = \tanh^{-1} \frac{a}{\beta}$	$\frac{1}{(\alpha_2 - \alpha_1)} \left[\frac{-\alpha_1 \epsilon^{-\alpha_1 t}}{1 - \epsilon^{-\alpha_1 T}} + \frac{\alpha_2 \epsilon^{-\alpha_2 t}}{1 - \epsilon^{-\alpha_2 T}} \right]$ $= \frac{\epsilon^{-a(t-T)} \sqrt{b^2 - a^2}}{2b} \left\{ \frac{\cosh (bt - \phi) - \epsilon^{-aT} \cosh [b(t-T) - \phi]}{\cosh aT - \cosh bT} \right\}$ $= \frac{\epsilon^{-a(t-T)} \sqrt{a^2 - b^2}}{2b} \left\{ \frac{-\sinh (bt - \psi) + \epsilon^{-aT} \sinh [b(t-T) - \psi]}{\cosh aT - \cosh bT} \right\}$ $= \frac{\epsilon^{-a(t-T)} \sqrt{a^2 + \beta^2}}{2\beta} \left\{ \frac{\cos (\beta t + \sigma) - \epsilon^{-aT} \cos [\beta(t-T) + \sigma]}{\cosh aT - \cosh bT} \right\}$

TABLE I—(continued)

Direct Transform $S[f(t)] = F(p)$	Inverse Transform $S^{-1}[F(p)] = f(t)$
14. $\frac{p^2}{(p + \alpha_1)(p + \alpha_2)} = \frac{p^2}{(p + a) - b^2} = \frac{p^2}{(p + a)^2 + \beta^2}$ $\alpha_1 = a + b = a + j\beta$ $\alpha_2 = a - b = a - j\beta$ $b = j\beta \neq 0$ $\rho = \tanh^{-1} \left(\frac{2ab}{a^2 + b^2} \right)$ $\delta = \tan^{-1} \left(\frac{2a\beta}{a^2 - \beta^2} \right)$	$\frac{1}{(\alpha_2 - \alpha_1)} \left[\frac{\alpha_1^2 e^{-\alpha_1 t}}{1 - e^{-\alpha_1 T}} - \frac{\alpha_2^2 e^{-\alpha_2 t}}{1 - e^{-\alpha_2 T}} \right] + Im_1$ $= \frac{\epsilon^{-a(t-T)} a^2 - b^2 }{2b} \left\{ \frac{\sinh(bt - \rho) - \epsilon^{-aT} \sinh[b(t - T) - \rho]}{\cosh aT - \cosh bT} \right\} + Im_1$ $= \frac{\epsilon^{-a(t-T)} (a^2 + \beta^2)}{2\beta} \left\{ \frac{\sin(\beta t - \delta) - \epsilon^{-aT} \sin[\beta(t - T) - \delta]}{\cosh aT - \cos \beta T} \right\} + Im_1$

$$S(e_R) = E \frac{p}{p + \frac{1}{CR}} \quad (4)$$

and

$$S(e_C) = \frac{E}{RC} \frac{1}{p + \frac{1}{CR}} \quad (5)$$

By the use of transforms 4 and 5 of Table I and putting $\theta = (RC/T)$ and $\tau = (t/T)$, where $0 < \tau < 1$,

$$i = \frac{E}{R} \left[-\frac{\epsilon^{-\tau/\theta}}{\theta T(1 - \epsilon^{-1/\theta})} + Im_1 \right], \quad (6)$$

$$e_R = E \left[-\frac{\epsilon^{-\tau/\theta}}{\theta T(1 - \epsilon^{-1/\theta})} + Im_1 \right], \quad (7)$$

and

$$e_C = \frac{E}{RC} \frac{\epsilon^{-\tau/\theta}}{1 - \epsilon^{-1/\theta}} \quad (8)$$

The current i and the voltage across the resistance e_R have the same shape, and this shape is shown in Fig. 2 for various values of θ . The parameter θ is equal to the time constant of the circuit divided by the period of the applied pulses, so a larger time constant in the circuit

Fig. 2—Calculated wave forms for e_R in a series RC circuit.

is indicated by a larger value of θ for a given pulse period. The pulses of Fig. 2 are shown with a finite height which is as they would appear in a practical case. For low values of θ , the circuit approaches a differentiating circuit, and the wave form of Fig. 2 approaches the derivative of the impulse wave form which is an impulse of the second order. Experimental confirmation of the wave form for $\theta = 1$ is shown in Fig. 3, but only the tops of the pulses are visible in the photograph. The wave

Fig. 3—Experimental wave form for e_R ($\theta = 1.0$) in a series RC circuit.

forms of the voltage e_C across the capacitance are shown in Fig. 4, and it should be noted that no pulses appear in this case. For large values of θ , the circuit approaches an integrating circuit, and the wave form of Fig. 4 ap-

Fig. 4—Calculated wave forms for e_C in a series RC circuit.

proaches the integral of the pulse wave minus its average value, resulting in the saw-tooth wave shown. Again, the experimental wave form for $\theta = 1.0$ is shown in Fig. 5.

A number of other circuits have responses which are the same as those of a series resistance-capacitance cir-

Fig. 5—Experimental wave form for $e_C(\theta = 1.0)$ in a series RC circuit.

cuit. Among these are series resistance-inductance, parallel resistance-capacitance, and parallel resistance-inductance circuits. The applied pulses for the parallel circuits are current pulses.

Another circuit that is very commonly used is that of an inductance in series with a capacitance. The resistance is assumed zero, although it will be considered different from zero later on. The direct transforms of the current i , the voltage e_L across the inductance, and the voltage e_C across the capacitance are

$$S(i) = \frac{E}{L} \frac{p}{p^2 + \frac{1}{LC}} \quad (9)$$

$$S(e_L) = E \frac{p^2}{p^2 + \frac{1}{LC}} \quad (10)$$

and

$$S(e_C) = \frac{E}{LC} \frac{1}{p^2 + \frac{1}{LC}} \quad (11)$$

By the use of transforms 6, 7, and 8 of Table I and putting $m = (T/\sqrt{LC})$ and $\tau = (t/T)$ where $0 < \tau < 1$,

$$i = - \frac{E}{2L} \frac{\sin m(\tau - 1/2)}{\sin(m/2)}, \quad (12)$$

$$e_L = E \left[- \frac{m \cos m(\tau - 1/2)}{2T \sin(m/2)} + Im_1 \right], \quad (13)$$

and

$$e_C = \frac{ET \cos m(\tau - 1/2)}{2m \sin(m/2)}. \quad (14)$$

The current i is shown in Fig. 6 for various values of m , and it should be noticed that $(m/2\pi)$ is the number of cycles of the natural frequency of oscillation which occur during one period of the applied pulses. When $m = 2n\pi$ where n is a positive integer, the circuit is resonant at the n th harmonic of the applied pulse wave, and since the resistance in the circuit is assumed zero,

Fig. 6—Calculated wave forms for the current in a series LC circuit.

the response becomes infinitely large. When m becomes very small, the current wave form is the integral of the applied pulse wave form and is similar to that of Fig. 4 for the case of θ very large. The corresponding wave forms for the inductance voltage e_L are shown in Fig. 7,

Fig. 7—Calculated wave forms for e_L in a series LC circuit.

Fig. 8—Experimental wave form for $e_L(m = 3\pi)$ in a series LC circuit.

and the experimental wave for $m=3\pi$ is given in Fig. 8. The effect of resistance in the circuit may be noticed in that the peaks of the natural oscillation of the circuit voltage decrease in height with time. Similar results for the capacitive voltage e_C are shown in Fig. 9. For very low values of m the capacitive voltage approaches the second integral of the applied pulse voltage, which is a parabola. The paralleled inductance and capacitance circuit with a current-pulse wave form applied to it has wave forms exactly similar to those discussed above for the series inductance-capacitance circuit.

Fig. 9—Calculated wave forms for e_C in a series LC circuit.

When the resistance is not negligibly small in a series resistance-inductance-capacitance circuit, the response differs largely, depending upon whether the resistance is small or large. From transforms 12, 13, and 14 of Table I and with

$$\tau = (t/T), \quad 0 < \tau < 1,$$

$$\alpha_1 = \left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 - \frac{1}{LC}} \right] T, \quad \text{and}$$

$$\alpha_2 = \left[\frac{R}{2L} - \sqrt{\left(\frac{R}{2L} \right)^2 - \frac{1}{LC}} \right] T,$$

$$e_R = Ri = \frac{RE}{L(\alpha_2 - \alpha_1)} \left[\frac{-\alpha_1 \epsilon^{-\alpha_1 \tau}}{1 - \epsilon^{-\alpha_1}} + \frac{\alpha_2 \epsilon^{-\alpha_2 \tau}}{1 - \epsilon^{-\alpha_2}} \right], \quad (15)$$

$$e_L = \frac{E}{T(\alpha_2 - \alpha_1)} \left[\frac{\alpha_1^2 \epsilon^{-\alpha_1 \tau}}{1 - \epsilon^{-\alpha_1}} - \frac{\alpha_2^2 \epsilon^{-\alpha_2 \tau}}{1 - \epsilon^{-\alpha_2}} \right] + \frac{EIm_1}{T}, \quad (16)$$

$$e_C = \frac{ET}{LC(\alpha_2 - \alpha_1)} \left[\frac{\epsilon^{-\alpha_1 \tau}}{1 - \epsilon^{-\alpha_1}} - \frac{\epsilon^{-\alpha_2 \tau}}{1 - \epsilon^{-\alpha_2}} \right]. \quad (17)$$

This is the nonoscillatory case ($R > 2\sqrt{L/C}$), and a typical set of response curves are shown in Fig. 10 for

Fig. 10—Calculated wave forms for e_R , e_L , and e_C in a series RLC circuit with $R > 2\sqrt{L/C}$.

the parameters $\alpha_1 = 2.0$ and $\alpha_2 = 1.0$. The shapes of the curves for e_R and e_L resemble those of a series resistance-inductance circuit, as evidenced in Figs. 2 and 4. The voltage e_C across the capacitance, on the other hand, resembles the capacitive voltage of a series inductance-capacitance circuit as given in Fig. 9.

The oscillatory case ($R < 2\sqrt{L/C}$) is obtained also from transforms 12, 13, and 14 of Table I. If $\tau = (t/T)$, $0 < \tau < 1$,

$$m = T \sqrt{\frac{1}{LC} - \left(\frac{R}{2L} \right)^2}, \quad n = (TR/2L),$$

$$\sigma = \tan^{-1}(n/m), \quad \text{and} \quad \delta = \tan^{-1} \left(\frac{2mn}{n^2 - m^2} \right),$$

$$e_R = Ri = \frac{RE \epsilon^{-n(\tau-1)} \sqrt{m^2 + n^2}}{L2m(\cosh n - \cos m)} \left\{ \cos(m\tau + \sigma) - \epsilon^{-n} \sin[m(\tau - 1) + \sigma] \right\}, \quad (18)$$

$$e_L = \frac{E \epsilon^{-n(\tau-1)} (m^2 + n^2)}{2mT(\cosh n - \cos m)} \left\{ \sin(m\tau - \delta) - \epsilon^{-n} \sin[m(\tau - 1) - \delta] \right\} + \frac{EIm_1}{T}, \quad (19)$$

$$e_C = \frac{ET \epsilon^{-n(\tau-1)}}{2LCm(\cosh n - \cos m)} \left\{ \sin m\tau - \epsilon^{-n} \sin m(\tau - 1) \right\}. \quad (20)$$

A typical set of response curves for the case $m=3\pi$, $n=1.0$ is shown in Fig. 11. It should be noticed that the curves are very similar to those of Figs. 6, 7, and 9, except that the effect of the additional resistance is to damp out the oscillations. The responses of many other circuits may be obtained by using linear combinations of the transforms given in Table I.

Fig. 11—Calculated wave forms for e_R , e_L , and e_C in a series RLC circuit with $R < 2\sqrt{L/C}$.

In applying the above wave forms to the testing of amplifiers,^{9,10} the pulse-generator output is applied to

⁹ L. B. Arguimbau, "Network testing with square waves," *Gen. Rad. Exp.*, vol. 14, pp. 1-6; December, 1939.

¹⁰ D. L. Waideich, "Steady-state testing with saw-tooth waves," *Proc. I.R.E.*, vol. 32, pp. 339-348; June, 1944.

the amplifier, and the output of the amplifier is viewed on an oscilloscope screen. The frequency of the pulse generator may be varied to determine the width of the pass band of the amplifier and the type of distortion present at the edges of the pass band. For example, a simple resistance-capacitance coupled amplifier at low frequencies acts much like a series resistance-capacitance circuit with the output voltage appearing across the resistance. The resulting wave forms are similar then to those of Fig. 2, and the approximate lower half-power frequency would be that when the wave form for $\theta = (1/2\pi)$ of Fig. 12(a) would appear. The upper half-power frequency may be obtained in a similar manner, and the wave form to be used is that of Fig. 12(b) for $\theta = 2\pi$. Resonance occurring in the amplifier may be found by noticing for what frequencies of the pulse generator the output voltage becomes very large. The output voltage wave forms should be similar to those of Fig. 9. When the sinusoidal or nearly sinusoidal wave form is that for $m = 2\pi$, the frequency of the pulse generator is the same as the resonant frequency of the amplifier.

It has been shown¹⁰ that if the steady-state response of an amplifier is known to a saw-tooth wave of period T , the steady-state response to any nonsinusoidal wave of the same period T may be calculated. The same is true if a pulse wave is used. For example, if $e_0(t)$ is the steady-state output response voltage of a pulse of period T applied to an amplifier, and if $e(t)$ is any other non-

Fig. 12—Calculated wave forms for (a) the lower, (b) the upper half-power frequencies.

sinusoidal wave of period T , then the steady-state response e_s of the amplifier to $e(t)$ is

$$e_s = \int_0^T e(t - \tau) e_0(\tau) d\tau, \quad (21)$$

or another equivalent form is

$$e_s = \int_{t-T}^t e(\tau) e_0(t - \tau) d\tau. \quad (22)$$

If the equations for e and e_0 are known, it is possible to integrate (21) or (22). If equations are not known for either e or e_0 or both, a numerical solution is still possible as outlined in the previous reference.¹⁰

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