**ON THE GLOBAL GAN-GROSS-PRASAD CONJECTURE
FOR GENERAL SPIN GROUPS**

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by
Melissa Emory
Dr. Shuichiro Takeda, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

ON THE GLOBAL GAN-GROSS-PRASAD CONJECTURE
FOR GENERAL SPIN GROUPS

presented by Melissa Emory,

a candidate for the degree of Doctor of Philosophy of Mathematics,

and hereby certify that in their opinion it is worthy of acceptance.

Professor Shuichiro Takeda

Professor William D. Banks

Professor Calin Chindris

Professor Dana Weston

Professor David Retzloff

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On the global Gan-Gross-Prasad conjecture for general spin groups

Melissa Emory

Dr. Shuichiro Takeda, Dissertation Supervisor

ABSTRACT

In the 1990s, Benedict Gross and Dipendra Prasad formulated an intriguing conjecture connected with restriction laws for automorphic representations of a particular group. More recently, Gan, Gross, and Prasad extended this conjecture, now known as the Gan-Gross-Prasad Conjecture, to the remaining classical groups. Roughly speaking, they conjectured the non-vanishing of a certain period integral is equivalent to the non-vanishing of the central value of a certain L -function. Ichino and Ikeda refined the conjecture to give an explicit relationship between this central value of a L -function and the period integral. We propose a similar conjecture for a non-classical group, the general spin group, and prove one case.

Chapter 1

Introduction

In the 1990s, Benedict Gross and Dipendra Prasad [GP92, GP94] formulated an intriguing conjecture regarding the restriction of representations, also known as branching laws, of special orthogonal groups. Restriction problems are one of the most natural problems regarding representations, present from the early days of representation theory. In general, the question is how a representation of a group decomposes when restricted to a subgroup. The Gross-Prasad conjecture describes these restrictions in terms of number theoretic invariants attached to Langlands parameters.

In 2012, Wee Teck Gan, Benedict Gross and Dipendra Prasad [GGP12] extended this conjecture, now known as the *Gan-Gross-Prasad (GGP) conjecture*, to the remaining classical groups. Roughly speaking, they conjecture that there exists a non-trivial period integral precisely when the central value of the tensor product L -function $L(1/2, \pi_{n+1} \times \pi_n)$ is non-zero. Here π_{n+1} and π_n are cuspidal automorphic representations of the groups $\mathrm{SO}(V_{n+1})$ and $\mathrm{SO}(V_n)$, where V_{n+1} and V_n are quadratic spaces of dimensions $n+1$ and n , respectively, such that $V_n \subseteq V_{n+1}$.

Atsushi Ichino and Tamotsu Ikeda [II10] refined the conjecture for the special orthogonal groups to give an explicit relationship between this central value of a L -function and the period integral and proved the first few cases. An analogous

conjecture was formulated for unitary groups by R. Neal Harris [Har14]. The purpose of this thesis is to formulate a similar conjecture for a non-classical group known as the general spin group (GSpin), and we prove one case.

1.1 The Gan-Gross-Prasad Conjecture

Here we recall the original global Gross-Prasad conjecture and then discuss the Ichino-Ikeda refinement. We then give an analogous global conjecture for general spin groups.

1.1.1 The Original Global Gross-Prasad Conjecture

Let $V_n \subset V_{n+1}$ be quadratic spaces of respective dimensions n and $n + 1$. Assume n is at least two and V_n is not isomorphic to the hyperbolic plane. Consider the algebraic groups $\mathrm{SO}(V_n) \subset \mathrm{SO}(V_{n+1})$ defined over a number field F . Let $\mathbb{A}_F$ be the ring of adeles defined over F . Denote $G_i := \mathrm{SO}(V_i)$ and let $\pi_n \cong \otimes_v \pi_{n,v}$ and $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$ be irreducible tempered cuspidal automorphic representations of $G_n(\mathbb{A}_F)$ and $G_{n+1}(\mathbb{A}_F)$, respectively. The following is the original global conjecture proposed by Gross and Prasad.

Conjecture 1.1 (Original Global Gross-Prasad Conjecture [GP92]). *Assume that for every place v of F , $\mathrm{Hom}_{G_n(F_v)}(\pi_{n+1,v} \otimes \bar{\pi}_{n,v}, \mathbb{C}) \neq \{0\}$. Then there exists vectors $\phi_i \in \pi_i$ such that*

$$\int_{G_n(F) \backslash G_n(\mathbb{A}_F)} \phi_{n+1}(g_n) \overline{\phi_n(g_n)} dg_n \neq 0$$

if and only if the tensor product L -function $L(1/2, \pi_{n+1} \times \pi_n)$ does not vanish.

Ginzburg-Jiang-Rallis have shown the forward implication, under some assumptions ([GJR05]).

1.1.2 The Ichino-Ikeda Refinement

Now we discuss the refined conjecture for special orthogonal groups proposed by Ichino-Ikeda [II10]. Let $\phi_i \in V_{\pi_{n+1}}$ and $f_i \in V_{\pi_n}$ and consider the $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$ -invariant functional

$$\mathcal{P} : (V_{\pi_{n+1}} \boxtimes \overline{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \overline{V}_{\pi_n}) \rightarrow \mathbb{C},$$

defined by

$$\mathcal{P}(\phi_1, \phi_2 : f_1, f_2) = \left(\int_{G_n(F) \backslash G_n(\mathbb{A}_F)} \phi_1(g) f_1(g) dg \right) \left(\int_{G_n(F) \backslash G_n(\mathbb{A}_F)} \overline{\phi_2(g) f_2(g)} dg \right).$$

If $\phi_1 = \phi_2$, we call it ϕ and similarly for f . Then $\mathcal{P}(\phi_1, \phi_2 : f_1, f_2) = \mathcal{P}(\phi, f)$, which we call the global period integral.

Next, we develop another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$ -invariant functional, but now from local integrals. For each place v of F , let $G_{i,v} = G_i(F_v)$. Fix local pairings

$$\mathcal{B}_{\pi_{i,v}} : \pi_{i,v} \otimes \overline{\pi}_{i,v} \rightarrow \mathbb{C}$$

such that

$$\mathcal{B}_{\pi_i} = \prod_v \mathcal{B}_{\pi_{i,v}}.$$

Here the $\mathcal{B}_{\pi_i}$ are the Peterson pairings

$$\begin{aligned} \mathcal{B}_{\pi_n}(f_1, f_2) &= \int_{G_n(F) \backslash G_n(\mathbb{A}_F)} f_1(g) \overline{f_2(g)} dg \\ \mathcal{B}_{\pi_{n+1}}(\phi_1, \phi_2) &= \int_{G_{n+1}(F) \backslash G_{n+1}(\mathbb{A}_F)} \phi_1(g) \overline{\phi_2(g)} dg \end{aligned}$$

and the dg are Tamagawa measures on $G_i(\mathbb{A}_F)$.

For each v , we consider the following $G_{n,v} \times G_{n,v}$ - invariant functional

$$\mathcal{P}_v^{\natural} : (\pi_{n+1,v} \boxtimes \overline{\pi}_{n+1,v}) \otimes (\pi_{n,v} \boxtimes \overline{\pi}_{n,v}) \rightarrow \mathbb{C}$$

defined by

$$\mathcal{P}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) = \int_{G_{n,v}} \mathcal{B}_{\pi_{n+1,v}}(\pi_{n+1,v}(g_{n,v})\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\pi_{n,v}}(\pi_{n,v}(g_{n,v})f_{1,v}, f_{2,v}) dg_{n,v}.$$

The $dg_{n,v}$ are local Haar measures chosen so that $\prod_v dg_{n,v} = dg_n$. As before, set $\mathcal{P}_v^\natural(\phi_v, \phi_v; f_v, f_v) = \mathcal{P}_v^\natural(\phi_v, f_v)$. Ichino-Ikeda prove that if $\pi_{i,v}$ is tempered, $P_v^\natural(\phi_v, f_v)$ converges absolutely. Moreover, when $P_v^\natural(\phi_v, f_v)$ does converge, they show

$$P_v^\natural(\phi_v, f_v) = \Delta_{G_{n+1,v}} \frac{L_v(1/2, \pi_{n,v} \times \pi_{n+1,v})}{L_v(1, \pi_{n,v}, \text{Ad}) L_v(1, \pi_{n+1,v}, \text{Ad})}$$

where ϕ_v, f_v are unramified and satisfy conditions (U1)-(U6) in [II10]. Here,

$$\Delta_{G_{n+1,v}} = \begin{cases} \zeta_v(2)\zeta_v(4) \cdots \zeta_v(2l) & \text{if } \dim V_{n+1} = 2l + 1, \\ \zeta_v(2)\zeta_v(4) \cdots \zeta_v(2l-2) \cdot L_v(l, \mu) & \text{if } \dim V_{n+1} = 2l \end{cases}$$

and $\mu = \mu_{V_i}$ is the Hecke character associated with the discriminant of the quadratic form associated to V_i . The $L(s, \pi_{n,v} \times \pi_{n+1,v})$ are the tensor product L -functions and the $L(s, \pi_{i,v}, \text{Ad})$ are the adjoint L -functions. Note that $\Delta_{G_{n+1}} = L(M_i^\vee(1), 0)$, where $M_i^\vee(1)$ is the twisted dual of the motive M_i associated to G_i by Gross [Gro97]. We normalize by setting

$$\mathcal{P}_v(\phi_v, f_v) = \Delta_{G_{n+1}}^{-1} \frac{L_v(1, \pi_{n,v}, \text{Ad}) L_v(1, \pi_{n+1,v}, \text{Ad})}{L_v(1/2, \pi_{n,v} \times \pi_{n+1,v})} \prod_v \mathcal{P}_v^\natural(\phi_v, f_v).$$

Finally, consider another $G_{n,v}(\mathbb{A}_F) \times G_{n,v}(\mathbb{A}_F)$ -invariant functional,

$$\prod_v \mathcal{P}_v : (V_{\pi_{n+1}} \boxtimes \overline{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \overline{V}_{\pi_n}) \rightarrow \mathbb{C}.$$

The refined global Gross-Prasad conjecture of Ichino-Ikeda gives the explicit constant of proportionality between $\mathcal{P}$ and $\prod_v \mathcal{P}_v$.

Conjecture 1.2 (Ichino-Ikeda Refinement of the global Gross-Prasad conjecture [II10]).

$$\mathcal{P}(\phi, f) = \frac{\Delta_{G_{n+1}}}{2^\beta} \frac{L(1/2, \pi_n \times \pi_{n+1})}{L(1, \pi_n, \text{Ad})L(1, \pi_{n+1}, \text{Ad})} \prod_v \mathcal{P}_v(\phi_v, f_v)$$

where 2^β is the product of cardinalities of the component groups attached to the L -packets for π_i .

The conjecture was proven unconditionally for $n = 2$ [Wal85] and $n = 3$ [Ich08]. The case for $n = 4$ was handled assuming that π_5 is a θ lift of a representation of SO_4 in [GI11].

Our goal is to provide a similar conjecture for general spin groups. For the general spin groups there exists a non-trivial center, so we adjust $\mathcal{P}$ from above as follows. Denote $G_i = \text{GSpin}(V_i)$ and let A_i be the identity component of the center of G_i . Let ω_{π_i} be the central characters of π_i restricted to A_i and we assume ω_{π_i} are such that

$$\omega^2 = \omega_{\pi_{n+2}} \omega_{\pi_{n+1}}.$$

Let $\phi_i \in V_{\pi_{n+1}}$ and $f_i \in V_{\pi_n}$ and consider the $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$ -invariant functional

$$\mathcal{P} : (V_{\pi_{n+1}} \boxtimes \overline{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \overline{V}_{\pi_n}) \rightarrow \mathbb{C},$$

defined by

$$\mathcal{P}(\phi_1, \phi_2; f_1, f_2) = \left(\int_{A_n \backslash G_n} \phi_1(g) f_1(g) \omega^{-1}(\nu(g)) dg \right) \left(\int_{A_n \backslash G_n} \overline{\phi_2(g) f_2(g) \omega^{-1}(\nu(g))} dg \right). \quad (1.1)$$

If $\phi_1 = \phi_2$, we call it ϕ and similarly for f . Then $\mathcal{P}(\phi_1, \phi_2 : f_1, f_2) = \mathcal{P}(\phi, f)$, which we call the global period integral. Similarly, we set

$$\mathcal{P}_v^\natural(\phi_v, f_v) := \int_{A_{n,v} \backslash G_{n,v}} \mathcal{B}_{\pi_{n+1},v}(\pi_{n+1,v}(g_{n,v})\phi_v) \mathcal{B}_{\pi_n,v}(\pi_{n,v}(g_{n,v})f_v) \omega^{-1}(\nu(g_{n,v})) dg_{n,v}.$$

(1.2)

The conjecture for general spin groups is:

Conjecture 1.3 (The Gan-Gross-Prasad conjecture for general spin groups). *Let $\pi_n \cong \otimes_v \pi_{n,v}$ and $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$ be irreducible tempered cuspidal automorphic representations of $G_n(\mathbb{A}_F)$ and $G_{n+1}(\mathbb{A}_F)$, respectively. Assume $\omega^2 = \omega_{\pi_{n+1}} \omega_{\pi_n}$ and let $\mathcal{P}$ be as in (1.1). Then,*

$$\mathcal{P} = \frac{\Delta_{G_{n+1}}}{2^\beta} \frac{L(1/2, \pi_{n+1} \otimes \omega^{-1} \times \pi_n)}{L(1, \pi_{n+1}, \text{Ad}) L(1, \pi_n, \text{Ad})} \prod_v \mathcal{P}_v(\phi_v, f_v)$$

where

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4) \cdots \zeta(2m) & \text{if } \dim V_i = 2m + 1, \\ \zeta(2)\zeta(4) \cdots \zeta(2m-2) L(m, \mu) & \text{if } \dim V_i = 2m. \end{cases}$$

Here $\mu = \mu_{V_i}$ is the Hecke character associated with the discriminant of the quadratic form associated to V_i , and 2^β is associated with the product of the cardinalities of the component groups attached to the L -parameters of π_{n+1} and π_n . The $L(1, \pi_i, \text{Ad})$ are the adjoint L -functions defined by (2.3) and (2.4).

To discuss the term 2^β , we first set up some notation. Let $\mathcal{L}_F$ be the hypothetical global Langlands group of F and let $\phi_i : \mathcal{L}_F \rightarrow {}^L G = \widehat{G} \rtimes W_F$. Set $S_{\phi_i} = \text{Cent}(\text{Im}(\phi_i), \widehat{G})$. Define

$$\mathcal{S}_{\phi_i} = S_{\phi_i} / S_{\phi_i}^0 Z(\widehat{G})^\Gamma,$$

where, $S_{\phi_i}^0$ is the identity component of the complex reductive group S_{ϕ_i} , $Z(\widehat{G})$ is the center of $\widehat{G}$ and $Z(\widehat{G})^\Gamma$ is the subgroup of invariants in $Z(\widehat{G})$ under the natural action of the Galois group $\Gamma = \text{Gal}(\overline{F}/F)$.

We conjecture that

Conjecture 1.4.

$$2^\beta = 4|\mathcal{S}_{\phi_n}||\mathcal{S}_{\phi_{n+1}}|$$

Remark 1.5. We work through the case where V_n is isomorphic to a hyperbolic plane in Section 4.3. In this case, we have $\mathrm{GSpin}(V_2) \subseteq \mathrm{GSpin}(V_3)$ and GSpin_2 is split. This case does not confirm Conjecture 1.3 or Conjecture 1.4.

1.2 Outline of Dissertation

In Chapter 2 we define general spin groups and prove two lemmas which are used throughout the dissertation. Next we review the structure theory for general spin groups and describe the root datum in the split case. We then discuss the general theory of L -groups and L -functions and describe each for GSpin groups. We conclude the chapter with a discussion on the component groups for the local Langlands parameters of GSpin in relation to Conjecture 1.4.

In Chapter 3, we begin by proving the convergence of the integral (1.2) for tempered representations at every place v using results of Waldspurger [Wal85] and Silberger [Sil79]. We then compute the integral for unramified data in the split case first, then the quasi-split but non-split case. In the course of the proof, we use the doubling method of Piatetski-Shapiro and Rallis [LR05] and results of Kato-Murase-Sugano [KMS03].

In Chapter 4, we prove the global Gan-Gross-Prasad conjecture proposed in Chapter 3 for $\mathrm{GSpin}_2 \times \mathrm{GSpin}_3$. We discuss the Waldspurger formula ([Wal85]) before utilizing the formula in the proof of the case where GSpin_2 is not split. We then work through the case when GSpin_2 is split. This is an exceptional case in that it does not

confirm Conjecture 1.3.

Chapter 2

Structure of GSpin Groups.

In this chapter, we first define general spin groups and then prove two lemmas needed in the thesis. Next, we review the structure theory for general spin groups and describe the root datum in the split case. We then discuss the general theory of L -groups and L -functions and describe each for GSpin groups. We conclude the chapter with a discussion on the component groups for the local Langlands parameters of GSpin in relation to Conjecture 1.4.

2.1 $\mathrm{GSpin}(V)$

We begin by defining GSpin groups and then discuss some general structure theory of GSpin groups. Good resources for the general theory of GSpin groups are [Del72].

Let V be a free module over a ring F , equipped with a quadratic form q . Denote by $C(V)$ the Clifford algebra of V . The module V is identified with a sub-module of $C(V)$. In $C(V)$, one has $v \cdot v = q(v)$. Denote by C^+ the even part of $C(V)$.

Suppose that F is a field of characteristic zero with a non-degenerate quadratic form q . The group $\mathrm{GSpin}(V)$, also referred to as the Clifford group or as $CSpin(V)$, is defined as

$$\mathrm{GSpin}(V) = \{g \in C^+(V) : gVg^{-1} = V\}.$$

There is a natural homomorphism from $\mathrm{GSpin}(V)$ to $\mathrm{SO}(V)$ sending $g \in \mathrm{GSpin}(V)$ to the map $v \mapsto gvg^{-1}$. The kernel consists of scalars, giving the short exact sequence of algebraic groups

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1. \quad (2.1)$$

The spinor group $\mathrm{Spin}(V)$ is the algebraic subgroup of $\mathrm{GSpin}(V)$ which is the kernel of the similitude map

$$\nu : \mathrm{GSpin} \rightarrow \mathrm{GL}_1.$$

Altogether this gives the commutative diagram similar to the one used by Deligne [Del72]

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & & \downarrow & & & \\
 & & & \mathrm{Spin}(V) & & & \\
 & & & \downarrow & \searrow \phi & & \\
 1 & \longrightarrow & \mathrm{GL}_1 & \longrightarrow & \mathrm{GSpin}(V) & \longrightarrow & \mathrm{SO}(V) \longrightarrow 1 \\
 & & \searrow x \mapsto x^2 & & \downarrow \nu & & \\
 & & & & \mathrm{GL}_1 & & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
 \end{array}$$

We also have the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

There is an identification

$$\mathrm{GSpin}(V) \cong \frac{\mathrm{GL}_1 \times \mathrm{Spin}(V)}{\mu_2}$$

where $\mu_2 = \ker \phi$ which is explicitly defined in the split case in Definition 2.4.

Lemma 2.1. *Let $\dim V \geq 3$, let A be the identity component of $\mathrm{GSpin}(V)$, and let $P : \mathrm{GSpin}(V) \rightarrow \mathrm{SO}(V)$. Then the kernel of P is A , so $A = \mathrm{GL}_1$.*

Proof. Denote Z to be the center of $\mathrm{GSpin}(V)$ and denote Z_{SO} to be the center of $\mathrm{SO}(V)$. Let $a \in Z$. From (2.1) we have

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GSpin}(V) \xrightarrow{P} \mathrm{SO}(V) \longrightarrow 1.$$

There is a homomorphism sending a to $\bar{a}$ in Z_{SO} . Recall that $Z_{\mathrm{SO}} = \{\pm 1\}$. The center of $\mathrm{GSpin}(V)$ must be in $P^{-1}(\{\pm 1\})$. Note that $P^{-1}(\{1\}) = \ker P$ and $\ker P = \mathrm{GL}_1$. Let A be the identity component of Z . Since $\{\pm 1\}$ is disconnected, and GL_1 is connected,

$$A \subseteq P^{-1}(\{1\}) = \mathrm{GL}_1 \subseteq Z$$

implies $A = \mathrm{GL}_1$. ■

Lemma 2.2. *Let V_n, V_{n+1} be quadratic spaces over F such that $V_n \subseteq V_{n+1}$. One has the commutative diagram*

$$\begin{array}{ccc} \mathrm{GSpin}(V_n) & \hookrightarrow & \mathrm{GSpin}(V_{n+1}) \\ \downarrow & & \downarrow \\ \mathrm{SO}(V_n) & \hookrightarrow & \mathrm{SO}(V_{n+1}) \end{array}$$

Proof. Let $V_n \subseteq V_{n+1}$. Then there exists a natural injection from $\mathrm{SO}(V_n)$ to $\mathrm{SO}(V_{n+1})$.

From (2.1) we have the exact sequence

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GSpin}(V_{n+1}) \longrightarrow \mathrm{SO}(V_{n+1}) \xrightarrow{P} 1$$

which gives the exact sequence

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow P^{-1}(\mathrm{SO}(V_n)) \longrightarrow \mathrm{SO}(V_n) \longrightarrow 1.$$

Because $P^{-1}(\mathrm{SO}(V_n))$ is a non-trivial extension of $\mathrm{SO}(V_n)$ by GL_1 , $P^{-1}(\mathrm{SO}(V_n)) = \mathrm{GSpin}(V_n)$. Thus, $\mathrm{GSpin}(V_n) \subseteq \mathrm{GSpin}(V_{n+1})$.

■

Next, we discuss some general structure theory of $\mathrm{GSpin}(V)$ which will be utilized in the proof of Proposition 3.3.

Let F be a local field with residual characteristic not 2. Let V be a non-degenerate quadratic space over F . Denote the anisotropic kernel of V by V^{an} and denote the dimension of V^{an} by d . Then there is a decomposition

$$V = X \oplus V^{an} \oplus Y$$

where X and Y are totally isotropic spaces. By definition, the Witt rank r equals the dimension of X or Y . By fixing a basis for X , we obtain a minimal parabolic subgroup $P_{\min} = M_{\min}N_{\min}$ of $\mathrm{GSpin}(V)$. The Levi factor $M_{\min}$ is isomorphic to the direct product of $(F^\times)^r \times \mathrm{GSpin}_{V^{an}}$. The split component $A_{\min}$ of $M_{\min}$ is isomorphic to $(F^\times)^r$ and the Weyl group $W(\mathrm{GSpin}(V), A_{\min})$ is of type B if d is odd and is of type D if d is even. Denote an element $A_{\min}$ by $x = (x_1, x_2, x_3, \dots, x_r)$. The simple roots of $(P_{\min}, A_{\min})$ are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \dots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1}$$

and

$$\alpha_r = \begin{cases} x_r & \text{if } d \text{ is odd} \\ x_{r-1} x_r & \text{if } d \text{ is even.} \end{cases}$$

Let T be a maximal torus such that $T \subset M_{\min}$. We view the α_i as elements of $\mathrm{Hom}(T, \mathrm{GL}_1)$ via the natural projection $M_{\min} \rightarrow T$. This allows us to view these

roots as a character of $M_{\min}$. Let $\delta_{P_{\min}}(x)$ be the modulus character of $P_{\min}$. Then

$$\delta_{P_{\min}}(x) = \prod_{i=1}^r |x_i|^{d+2r-2i}.$$

2.2 Root Datum for split GSpin

In this section, we describe split GSpin_n in terms of its root datum.

Proposition 2.3. *The root datum of GSpin_n , $n = 2m + 1$ or $2m$, can be described as the following. Let*

$$X = \mathbb{Z}e_o \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m,$$

$$X^\vee = \mathbb{Z}e_o^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_m^*,$$

and let

$$\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$$

be the standard pairing. The root datum for GSpin_n is $(X, R, X^\vee, R^\vee)$ with R and $R^\vee$ generated by the simple roots and coroots given by, respectively,

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{m-1} = e_{m-1} - e_m, \alpha_m = e_m\},$$

$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \dots, \alpha_{m-1}^\vee = e_{m-1}^* - e_m^*, \alpha_m^\vee = 2e_m^* - e_0^*\},$$

if $n = 2m + 1$ and by

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{m-1} = e_{m-1} - e_m, \alpha_m = e_{m-1} + e_m\},$$

$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \dots, \alpha_{m-1}^\vee = e_{m-1}^* - e_m^*, \alpha_m^\vee = e_{m-1}^* + e_m^* - e_0^*\},$$

if $n = 2m$.

Proof. See [Asg02, Section 2]. ■

The root datum listed in Proposition 2.3 corresponds with the root datum listed in Section 2.1. For example, $\alpha_1(x) = (e_1 - e_2)(x) = x_1 x_2^{-1}$. The root datum of Proposition 2.3 determines the split group GSpin_n uniquely, equipped with a Borel subgroup containing a maximal torus T . Note that $\alpha_i : T \rightarrow \mathrm{GL}_1$ and $\alpha_i^\vee : \mathrm{GL}_1 \rightarrow T$ is such that $\alpha_i \circ \alpha_j^\vee(x) = x^{\langle \alpha_i, \alpha_j^\vee \rangle}$.

When $n = 2m + 1$, GSpin_n has a Dynkin diagram of type B_m :

When $n = 2m$, GSpin_n has a Dynkin diagram of type D_m :

Now that the root structure for GSpin has been described explicitly, we can define split GSpin as follows:

Proposition 2.4. *For $n \geq 3$,*

$$\mathrm{GSpin}_n \cong \frac{\mathrm{GL}_1 \times \mathrm{Spin}_n}{\{(1, 1), (-1, c)\}}$$

where $c = \alpha_m^\vee(-1)$ if $n = 2m + 1$ and $c = \alpha_{m-1}^\vee(-1)\alpha_m^\vee(-1)$ if $n = 2m$.

Proof. See [Asg02, Section 2] ■

We define $\mathrm{GSpin}_0 = \mathrm{GSpin}_1 = \mathrm{GL}_1$. See the $n = 2$ case for a definition of GSpin_2 .

Proposition 2.5. *The center of GSpin_n is given by*

$$Z(\mathrm{G}_n) = \begin{cases} A_0 & \text{if } n = 2m + 1, \\ A_0 \cup (\zeta_0 A_0) & \text{if } n = 2m \end{cases}$$

where $A_0 = \{e_0^*(\lambda) : \lambda \in \mathrm{GL}_1\}$ and $\zeta_0 = e_1^*(-1)e_2^*(-1) \cdots e_m^*(-1)$.

Proof. See [AS06, Prop. 2.3]. ■

2.3 L -groups

In this section, we first discuss the general theory of L -groups, and then describe the L -groups of GSpin . For the concepts reviewed in this section, one may consult [Roh94] or [CKM04, Ch. 3].

Let k be a finite field of characteristic p and cardinality q . We denote $\bar{k}$ as the algebraic closure of k . The Weil group of k , denoted W_k , is the infinite cyclic subgroup of $\mathrm{Gal}(\bar{k}/k)$ generated by the inverse of the Frobenius automorphism.

Now consider a nonarchimedean local field F with residue field k . Denote the separable algebraic closure of F as $\bar{F}$ and F_{unr} for the maximal unramified extension of F . The inertia subgroup of $\mathrm{Gal}(\bar{F}/F)$ is defined to be $I_F = \mathrm{Gal}(\bar{F}/F_{unr})$. Identifying $\bar{k}$ with the residue field of $\bar{F}$ (or of F_{unr}), we have an exact sequence

$$1 \longrightarrow I_F \longrightarrow \mathrm{Gal}(\bar{F}/F) \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1,$$

where π is the decomposition map. Define W_F to be the inverse image of W_k under π :

$$W_F = \pi^{-1}(W_k).$$

Then W_F fits into the exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow W_k \longrightarrow 1.$$

Denote the inverse of the Frobenius automorphism as $Frob$, and one has the exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \langle Frob \rangle \longrightarrow 1. \tag{2.2}$$

Since $\langle Frob \rangle \subseteq \mathrm{Gal}(\bar{k}/k)$, W_F is a subgroup of $\mathrm{Gal}(\bar{F}/F)$. By a representation of W_F we mean a continuous homomorphism $W_F \rightarrow \mathrm{GL}(V)$, where V is a finite-dimensional

complex vector space. A representation σ of W_F , is unramified if σ restricted to I_F is trivial, and ramified otherwise.

Let G be a connected, reductive group over F . Let $\widehat{G}$ denote the complex dual of G . The L -group of G is denoted ${}^L G$ and ${}^L G = \widehat{G} \rtimes W_F$ or $\widehat{G} \rtimes \text{Gal}(\overline{F}/F)$. The following is a list of groups and the respective dual groups.

G	$\widehat{G}$	$\parallel$	G	$\widehat{G}$
GL_m	$\text{GL}_m(\mathbb{C})$	$\parallel$	SL_m	$\text{PGL}_m(\mathbb{C})$
SO_{2m+1}	$\text{Sp}_{2m}(\mathbb{C})$	$\parallel$	GSp_{2m}	$\text{GSpin}_{2m+1}(\mathbb{C})$
SO_{2m}	$\text{SO}_{2m}(\mathbb{C})$	$\parallel$	GSO_{2m}	$\text{GSpin}_{2m}(\mathbb{C})$

Hence, the L -groups of quasi-split GSpin groups are

$${}^L \text{GSpin}_n = \text{GSp}_{2m}(\mathbb{C}) \rtimes W_F$$

if $n = 2m + 1$, and

$${}^L \text{GSpin}_n = \text{GSO}_{2m}(\mathbb{C}) \rtimes W_F$$

if $n = 2m$.

When G is split, the semi-direct product is a direct product. If GSpin_n is not quasi-split, then there exists a quasi-split GSpin_n^* so that GSpin is an inner form of GSpin_n^* . Then

$${}^L \text{GSpin}_n = {}^L \text{GSpin}_n^*.$$

When F is a global field, G is quasi-split and unramified, i.e. split over an unramified extension of F_v for almost all places v of F ([Sha10, pg. 7]).

2.4 L -functions for GSpin

First we develop the local L -functions for split GSpin , and then quasi-split but non-split. We conclude the chapter by defining the global (or automorphic) L -functions for GSpin .

2.4.1 L -functions

In the general setting, let G be a connected reductive group over a number field F . Let π be an automorphic representation. For almost all v , the local representation π_v is an unramified representation of $G(F_v)$. Here v is a non-archimedean place. Let $G_v = G(F_v)$ in this section. To develop L -functions, we consider representations of the Weil-Deligne group WD_{F_v} which is defined to be

$$WD_{F_v} = W_{F_v} \times SL_2(\mathbb{C}).$$

The Local Langlands Conjecture asserts that there is a surjective, finite-to-one map from the set $\mathrm{Irr}(G_v)$ of isomorphism classes of irreducible smooth complex representations of $G(F_v)$ to the set of admissible homomorphisms $\phi : WD_{F_v} \rightarrow {}^L G_v$.

Now let $G_v = \mathrm{GSpin}_{2m+1}(F_v)$. Assume that G_v is split. Then $\widehat{G}_v = \mathrm{GSp}_{2m}(\mathbb{C})$. The Local Langlands Conjecture gives

$$\mathrm{Irr}(G_v) \twoheadrightarrow \{\phi : WD_{F_v} \rightarrow \mathrm{GSp}_{2m}(\mathbb{C})\}$$

sending $\pi_{n+1,v}$ to $\phi_{\pi_{n+1,v}}$.

Let $\pi_{n+1,v}$ be unramified. Conjecturally, $\phi_{\pi_{n+1,v}}$ is trivial on $SL_2(\mathbb{C})$ and on I_{F_v} . Then we have that

$$\phi_{\pi_{n+1,v}} : I_{F_v} \backslash W_{F_v} \rightarrow \mathrm{GSp}_{2m}(\mathbb{C}).$$

From the exact sequence (2.2), $I_{F_v} \backslash W_{F_v} \cong \langle Frob \rangle \cong \mathbb{Z}$. Denote ϖ_F as the uniformizer of F_v . Since $\pi_{n+1,v}$ is unramified, it is given by an unramified character Ξ of the maximal torus $T(F_v)$. This means that there are unramified characters $\Xi_0, \Xi_1, \dots, \Xi_m$ of $F_v^\times$ such that we have the following ([AS06, proof of Thm 1.1]):

$$\phi_{\pi_{n+1,v}}(Frob) = \begin{pmatrix} \Xi_1(\varpi_F) & & & & \\ & \ddots & & & \\ & & \Xi_m(\varpi_F) & & \\ & & & \Xi_1^{-1}(\varpi_F)\Xi_0(\varpi_F) & \\ & & & & \ddots \\ & & & & & \Xi_m^{-1}(\varpi_F)\Xi_0(\varpi_F) \end{pmatrix}$$

which is a subset of $\mathrm{GSp}_{2m}(\mathbb{C})$. Let q be the cardinality of the residue field. The standard local L -function for $\pi_{n+1,v}$ is

$$\begin{aligned} L(s, \pi_{n+1,v}) &= L(s, \phi_{\pi_{n+1,v}}) = \det(I - q^{-s} \phi_{\pi_v}(Frob))^{-1} \\ &= \prod_{i=1}^m \frac{1}{1 - q^{-s} \Xi_i(\varpi_F)} \frac{1}{1 - q^{-s} \Xi_i^{-1} \Xi_0(\varpi_F)} \\ &= \prod_{i=1}^m L(s, \Xi_i) L(s, \Xi_i^{-1} \Xi_0). \end{aligned}$$

Let $n = 2m$ be even and assume GSpin_n is split. Then similarly, if we assume $\pi_{n,v}$ is unramified, it is given by an unramified character ξ of the maximal torus $T(F_v)$. This means that there are unramified characters $\xi_0, \xi_1, \dots, \xi_m$ of $F_v^\times$ such that we have the following

$$\phi_{\pi_{n,v}}(Frob) = \begin{pmatrix} \xi_1(\varpi_F) & & & & \\ & \ddots & & & \\ & & \xi_m(\varpi_F) & & \\ & & & \xi_1^{-1}(\varpi_F)\xi_0(\varpi_F) & \\ & & & & \ddots \\ & & & & & \xi_m^{-1}(\varpi_F)\xi_0(\varpi_F) \end{pmatrix}$$

which is a subset of $\mathrm{GSO}_{2m}(\mathbb{C})$. The standard local L -function for $\pi_{n,v}$ is

$$L(s, \pi_{n,v}) = L(s, \phi_{\pi_{n,v}}) = \det(I - q^{-s} \phi_{\pi_v}(Frob))^{-1}$$

$$\begin{aligned}
&= \prod_{i=1}^m \frac{1}{1 - q^{-s}\xi_i(\varpi_F)} \frac{1}{1 - q^{-s}\xi_i^{-1}\xi_0(\varpi_F)} \\
&= \prod_{i=1}^m L(s, \xi_i) L(s, \xi_i^{-1}\xi_0).
\end{aligned}$$

Let $n = 2m$, and suppose π_v is unramified and assume GSpin_n is quasi-split, but not split. There is a one-to-one correspondence between π_v and the Satake parameters for GSpin_n given by [AS14, (3.32)]

$$\begin{pmatrix}
\chi_1(\varpi_F) & & & & \\
& \ddots & & & \\
& & \chi_m(\varpi_F)\mu & & \\
& & & \chi_1^{-1}(\varpi_F)\chi_0(\varpi_F) & \\
& & & & \ddots \\
& & & & & \chi_m^{-1}(\varpi_F)\chi_0(\varpi_F)
\end{pmatrix}$$

where $\mu = \mu_{V_n}$ is the Hecke character associated with the discriminant of the quadratic form associated to V_n . Then,

$$\begin{aligned}
L(s, \pi_v) &= L(s, \phi_{\pi_v}) = \det(I - q^{-s}\phi_{\pi_v}(\mathrm{Frob}))^{-1} \\
&= \frac{1}{1 - q^{-s}\chi_m(\varpi_F)\mu} \frac{1}{1 - q^{-s}\chi_m(\varpi_F)} \prod_{i=1}^{m-1} \frac{1}{1 - q^{-s}\chi_i(\varpi_F)} \frac{1}{1 - q^{-s}\chi_i^{-1}\chi_0(\varpi_F)} \\
&= L(s, \chi_m\mu) L(s, \chi_m^{-1}\chi_0) \prod_{i=1}^{m-1} L(s, \chi_i) L(s, \chi_i^{-1}\chi_0).
\end{aligned}$$

2.4.2 Tensor product L-functions

Assume that GSpin_n and GSpin_{n+1} are split and assume that π_{n+1} and π_n are unramified. Using the standard L -functions, the local tensor product L -functions, are

$$\begin{aligned}
L(s, \pi_{n+1} \times \pi_n) &= \prod_{1 \leq i, j \leq m} L(s, \Xi_i \xi_j) L(s, \Xi_i^{-1} \xi_j \Xi_0) \\
&\quad \times L(s, \Xi_i \xi_j^{-1} \xi_0) L(s, \Xi_i^{-1} \xi_j^{-1} \Xi_0 \xi_0).
\end{aligned}$$

2.4.3 Adjoint L -functions

Here we define what we mean by $L(s, \pi, \text{Ad})$ in Conjecture 1.3. Let $G = \text{GSpin}_{2n+1}$ then $\widehat{G} = \text{GSp}_{2n}(\mathbb{C})$ and $\widehat{\text{SO}}_{2n+1} = \text{Sp}_{2n}(\mathbb{C})$. Let π_v be an irreducible admissible representation of G_v . Then one has the following

$$WD_{F_v} \xrightarrow{\phi_{\pi_v}} \text{GSp}_{2n}(\mathbb{C}) \xrightarrow{\text{Ad}_{\mathfrak{gsp}}} \text{GL}(\text{Lie}(\widehat{G})).$$

Note that $\text{Lie}(\widehat{G}) = \mathfrak{gsp}_{2n}(\mathbb{C}) = \text{Lie}(\text{Sp}_{2n})(\mathbb{C}) \oplus \mathbb{C}$. Hence, $\text{Ad}_{\mathfrak{gsp}} = \text{Ad}_{\mathfrak{sp}} \oplus \mathbb{1}$. Then,

$$L(s, \pi_v, \text{Ad}_{\mathfrak{gsp}}) = L(s, \pi_v, \text{Ad}_{\mathfrak{sp}}) \zeta_v(s)$$

and we have

$$\begin{aligned} L(s, \pi, \text{Ad}_{\mathfrak{gsp}}) &:= \prod_v L(s, \pi_v, \text{Ad}_{\mathfrak{gsp}}) \\ &= \prod_v L(s, \pi_v, \text{Ad}_{\mathfrak{sp}}) \zeta_v(s) \\ &= L(s, \pi, \text{Ad}_{\mathfrak{sp}}) \zeta(s). \end{aligned}$$

We define

$$L(s, \pi, \text{Ad}) := L(s, \pi, \text{Ad}_{\mathfrak{sp}}) \tag{2.3}$$

Similarly, if $G = \text{GSpin}_{2n}$, in the split case we define

$$L(s, \pi, \text{Ad}) = L(s, \pi, \text{Ad}_{\mathfrak{so}}) \tag{2.4}$$

2.5 Component Groups for the local Langlands parameters

This goal of this section is to discuss the component groups for the local Langlands parameters in relation to Conjecture 1.4. We restate the notation from Chapter 1 for

the sake of the reader. Let $\mathcal{L}_F$ be the hypothetical global Langlands group of F and let $\phi_i : \mathcal{L}_F \rightarrow {}^L G = \widehat{G} \rtimes W_F$. Set $S_{\phi_i} = \text{Cent}(\text{Im}(\phi_i), \widehat{G})$. Define

$$\mathcal{S}_{\phi_i} = S_{\phi_i} / S_{\phi_i}^0 Z(\widehat{G})^\Gamma,$$

where, $S_{\phi_i}^0$ is the identity component of the complex reductive group S_{ϕ_i} , $Z(\widehat{G})$ is the center of $\widehat{G}$ and $Z(\widehat{G})^\Gamma$ is the subgroup of invariants in $Z(\widehat{G})$ under the natural action of the Galois group $\Gamma = \text{Gal}(\overline{F}/F)$.

Recall, we conjecture (Conjecture 1.4)

$$2^\beta = 4|\mathcal{S}_{\phi_n}||\mathcal{S}_{\phi_{n+1}}|.$$

We justify this conjecture with the following. Let π be a cuspidal representation of split GSpin_{2n+1} . The global Langlands conjecture states that

$$\phi_\pi : \mathcal{L}_F \rightarrow {}^L \text{GSpin}_{2n+1} = \text{GSp}_{2n}(\mathbb{C}) \times W_F$$

and $\text{Im}(\phi_\pi)$ is not in a proper Levi of $\text{GSp}_{2n}(\mathbb{C})$. Then

$$\phi_\pi : \mathcal{L}_F \rightarrow \text{GSp}_{2n}(\mathbb{C}) = \text{GSp}(W)$$

where $(W, \langle \cdot, \cdot \rangle)$ is a symplectic space over $\mathbb{C}$ of $\dim_{\mathbb{C}} W = 2n$.

For $l \in \mathcal{L}_F$,

$$\langle \phi_\pi(l)w, \phi_\pi(l)w' \rangle = \nu(l) \langle w, w' \rangle$$

where $\nu(l) \in \mathbb{C}^\times$ is the similitude character discussed in Chapter 2.

This implies that ϕ_π is a representation of $\mathcal{L}_F$ acting on W . Let $W_0 \subseteq W$ be an irreducible subspace of ϕ_π . Define

$$W_0^\perp = \{w \in W : \langle w, w_0 \rangle = 0 \text{ for all } w_0 \in W_0\}.$$

Note that $W_0^\perp$ is a subrepresentation of ϕ_π . To see this, let $l \in \mathcal{L}_F$ and $w \in W_0^\perp$. We want to show that $\phi_\pi(l)w \in W_0^\perp$. Consider

$$\langle \phi_\pi(l)w, w_0 \rangle = \langle \phi_\pi(l)w, \phi_\pi(l)\phi_\pi(l)^{-1}w_0 \rangle = \nu(l)\langle w, \phi_\pi(l)^{-1}w_0 \rangle = 0$$

Assume that $W_0 \cap W_0^\perp = 0$ or $W_0 \cap W_0^\perp = W_0$. If $W_0 \cap W_0^\perp = 0$ then this implies that $W = W_0 \oplus W_0^\perp$. If $W_0 \cap W_0^\perp = W_0$ then $W_0 \subseteq W_0^\perp$. But if $w_0, w'_0 \in W_0$ and $\langle w_0, w'_0 \rangle = 0$, then W_0 is totally isotropic. Hence W_0 is not a subset of $W_0^\perp$.

Note that $\text{Im}(\phi_\pi)$ fixes W_0 setwise. $\text{Im}(\phi_\pi)$ stabilizes the flag $0 \subseteq W_0 \subseteq W$ which implies $\text{Im}(\phi_\pi)$ is in the parabolic subgroup $P(W_0) \subseteq \text{GSp}(W)$. Then $\text{Im}(\phi_\pi)$ is in the Levi of $P(W_0)$, which is a contradiction.

Hence, $W = W_0 \oplus W_0^\perp$ which means

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

where each $(W_i, \langle \cdot, \cdot \rangle)$ is a smaller symplectic space. Then

$$\text{Im}(\phi_\pi) \subseteq (\text{GSp}(W_1) \times \cdots \times \text{GSp}(W_k)) \cap \text{GSp}(W).$$

and there are similitude characters ν_i corresponding to each of the $\text{GSp}(W_i)$. This forces

$$\text{Im}(\phi_\pi) \subseteq \{(g_1, g_2, \dots, g_k) \in \text{GSp}(W_1) \times \cdots \times \text{GSp}(W_k) : \nu(g_1) = \nu(g_2) = \cdots = \nu(g_k)\}.$$

Then,

$$\text{Cent}(\text{Im}(\phi_\pi)) = \{(a_1, a_2, \dots, a_k) : a_1^2 = a_2^2 = \cdots = a_k^2\} = \{(\pm a, \pm a, \pm a, \dots, \pm a)\}$$

and so $\text{Cent}(\text{Im}(\phi_\pi|_{W_1})) = \mathbb{C}^\times = Z(\text{GSp}(W_1))$. Then

$$S_\phi / Z(\text{GSp}(W)) = \{(\pm 1, \pm 1, \dots, \pm 1)\} = (\mathbb{Z}/2\mathbb{Z})^k. \text{ Finally,}$$

$$\mathcal{S}_\phi = (\mathbb{Z}/2\mathbb{Z})^{k-1}.$$

Chapter 3

Local Integrals of Matrix Coefficients

In this chapter, we begin by proving the integral 1.2 converges at every place v assuming the representations are tempered. The proof uses results of Waldspurger [Wal03] and Silberger [Sil79]. We then compute the integral for unramified data in the split case followed by the quasi-split but non-split case. In the course of the proof, we use the doubling method of Piatetski-Shapiro and Rallis [LR05] and results of Kato-Murase-Sugano [KMS03].

We utilize the notation from chapter one. Let $V_{n+1} \subseteq V_{n+2}$ be quadratic spaces of dimension $n+1$ and $n+2$, respectively. Let F be a number field with adele ring $\mathbb{A}_F$ and let $G_i = \mathrm{GSpin}(V_i)$. From Lemma 2.2, we consider the algebraic groups $\mathrm{GSpin}(V_{n+1}) \subseteq \mathrm{GSpin}(V_{n+2})$ defined over F . Let $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$ and $\pi_{n+2} \cong \otimes_v \pi_{n+2,v}$ be irreducible tempered cuspidal automorphic representations of G_{n+1} and G_{n+2} , respectively. Let S denote a sufficiently large finite set of bad places, including all even and archimedean places. Assume the following holds for all $v \notin S$:

1. $G_{i,v}$ is unramified over F_v
2. $K_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$

3. $K_{i,v} \subset K_{i+1,v}$
4. $\pi_{i,v}$ is an unramified representation of $G_{i,v}$
5. The vectors $f_{i,v} \in \pi_{i,v}$ are fixed by $K_{i,v}$ and $\|f_{i,v}\| = 1$
6. $\int_{K_{i,v}} dg_{i,v} = 1$.

For v not in S , we still fix a maximal compact subgroup K_i of G_i . For the remainder of this chapter, we omit the v from the notation, though everything is local.

Put

$$L_{\pi_{i+1}, \pi_i}(s) := \frac{L(s, \pi_{i+1} \otimes \omega^{-1} \times \pi_i)}{L(s + 1/2, \pi_{i+1}, \text{Ad}) L(s + 1/2, \pi_i, \text{Ad})}.$$

Consider the matrix coefficients

$$\Phi_{\phi_i, \phi'_i}(g_i) := \mathcal{B}_{\pi_i}(\pi_i(g)\phi_i, \phi'_i), \quad g_i \in G_i$$

for K_i -finite vectors $\phi_i, \phi'_i \in \pi_i$. When $\phi_i = \phi'_i$ we refer only to Φ_{ϕ_i} . Denote A_i as the identity component of the center of G_i and let ω_{π_i} be the central characters of π_i restricted to A_i and let $\omega^2 = \omega_{\pi_{n+2}}\omega_{\pi_{n+1}}$. Since ω_{π_i} is unramified, the square root exists, although not uniquely. However, there exists a unique unramified character χ such that $\chi_i^2 = \omega_{\pi_i}$, which is the ω_{π_i} we choose. Denote ν as the similitude character discussed in Section 2.1. Recall, $\nu : G_i \rightarrow \text{GL}_1$. Put

$$\mathcal{P}(\phi_{n+2}, \phi_{n+1}) := \int_{A_{n+1} \backslash G_{n+1}} \Phi_{\phi_{n+2}}(g) \Phi_{\phi_{n+1}}(g) \omega^{-1}(\nu(g)) dg.$$

Next we establish the convergence of the integral assuming each π_i is tempered.

3.1 Convergence of the Integral

In this section, we first introduce some definitions and prove a couple of technical lemmas before proceeding to the proof of Proposition 3.3.

Let G be a reductive group and fix a special maximal compact subgroup K of G .

Then we have a Cartan decomposition $G = KM_{\min}^+K$ where

$$M_{\min}^+ := \{m \in M_{\min} : |\alpha_i(m)| \leq 1 \text{ for } i = 1, \dots, r\}.$$

Let $\text{Hom}(G, \mathbb{C}^\times)$ be the group of continuous homomorphisms from G into $\mathbb{C}^\times$, and let $\chi \in \text{Rat}(G)$ be such that $|\chi|_F \in \text{Hom}(G, \mathbb{C}^\times)$ is the character defined by

$$|\chi|_F(g) = |\chi(g)|_F.$$

Define

$$G^1 := \bigcap_{\chi \in \text{Rat}(G)} \text{Ker} |\chi|_F.$$

Set $M_{\min}^1 := M_{\min} \cap G^1$. From equation (4) of [Wal03],

$$G = \coprod_{m \in M_{\min}^+ / M_{\min}^1} KmK. \quad (3.1)$$

This leads to the following lemma:

Lemma 3.1. *For $f \in L^1(G)$,*

$$\int_G f(g) dg = \int_{M_{\min}^+} \mu(m) \int_{K \times K} f(k_1 m k_2) dk_1 dk_2 dm$$

where $\mu(m) = A \cdot \text{Vol}(KmK)$ for some constant A .

Proof. Let $f \in C_c(G)$, we have ([Sil79, pg. 149])

$$\begin{aligned} \int_G f(x) dx &= \sum_{m \in M_{\min}^+ / M_{\min}^1} \int_{KmK} f(x) dx \\ &= \sum_{m \in M_{\min}^+ / M_{\min}^1} \int_{K \times K} \text{Vol}(KmK) f(k_1 m k_2) dk_1 dk_2 \\ &= \sum_{m \in M_{\min}^+ / M_{\min}^1} \text{Vol}(KmK) \int_{K \times K} f(k_1 m k_2) dk_1 dk_2 \end{aligned}$$

$$= \int_{M_{\min}^+/M_{\min}^1} \text{Vol}(KmK) \int_{K \times K} f(k_1mk_2) dk_1 dk_2.$$

Let $\tilde{F} : M_{\min}^+ \rightarrow \mathbb{C}$ be such that

$$\int_{M_{\min}^+} \tilde{F}(m) dm = \int_{M_{\min}^+/M_{\min}^1} \text{Vol}(KmK) \int_{K \times K} f(k_1mk_2) dk_1 dk_2.$$

We also have that

$$\int_{M_{\min}^+} \tilde{F}(m) dm = \int_{M_{\min}^+/M_{\min}^1} \int_{M_{\min}^1} \tilde{F}(mm_1) dm dm_1.$$

Then $\tilde{F}$ is such that

$$\int_{M_{\min}^1} \tilde{F}(mm_1) dm_1 = \text{Vol}(KmK) \int_{K \times K} f(k_1mk_2) dk_1 dk_2. \quad (3.2)$$

Next, we verify

$$\tilde{F}(m) = \frac{\text{Vol}(KmK)}{\text{Vol}(M_{\min}^1)} \int_{K \times K} f(k_1mk_2) dk_1 dk_2$$

by noting that $M_{\min}^1$ is a compact subset of K and considering

$$\begin{aligned} \int_{M_{\min}^1} \tilde{F}(mm_1) dm_1 &= \int_{M_{\min}^1} \frac{\text{Vol}(Kmm_1K)}{\text{Vol}(M_{\min}^1)} \int_{K \times K} f(k_1mm_1k_2) dk_1 dk_2 \\ &= \int_{M_{\min}^1} \frac{\text{Vol}(KmK)}{\text{Vol}(M_{\min}^1)} \int_{K \times K} f(k_1mk_2) dk_1 dk_2 \\ &= \int_{M_{\min}^1} dm_1 \cdot \left(\frac{\text{Vol}(KmK)}{\text{Vol}(M_{\min}^1)} \int_{K \times K} f(k_1mk_2) dk_1 dk_2 \right) \\ &= \text{Vol}(KmK) \int_{K \times K} f(k_1mk_2) dk_1 dk_2. \end{aligned}$$

which agrees with (3.2) when F is non-archimedean. Since $M_{\min}^1$ is compact, $\text{Vol}(M_{\min}^1)$ is finite. Letting $\mu(m) = A \cdot \text{Vol}(KmK)$, when F is non-archimedean the proof is complete.

In the archimedean case, the same integral formula holds for a non-negative function $\mu(m)$ on $M_{\min}^+$ ([Hel00, Th. 5.8]). ■

We introduce the final pieces of general theory needed for the proof of Proposition 3.8. Fix a suitable embedding $\eta : G \rightarrow GL_m$. Then the height function $\sigma(g)$ with respect to η is given by

$$\sigma(g) = \max_{1 \leq i, j \leq m} (\log |\eta(g)_{ij}|, \log |\eta(g^{-1})_{ij}|).$$

Let $\Xi(g)$ be Harish-Chandra's spherical function on G given by

$$\Xi(g) := \int_K h_0(kg) dk$$

where $h_0 \in \text{Ind}_{P_{\min}}^G \mathbf{1}$ is the function that is identically 1 on K .

Now we focus our attention on $G\text{Spin}$. Recall that

$$G^1 := \bigcap_{\chi \in \text{Rat}(G)} \text{Ker} |\chi|_F.$$

In our case,

$$G^1 = \{g \in G\text{Spin} : |\nu(g)| = 1\}.$$

Lemma 3.2. *Let $G = G\text{Spin}(V)$ and let A be the identity component of G . Let $f : G \rightarrow \mathbb{C}$ be such that $f(zg) = f(g)$. Then $\int_{A \setminus G} f(g) dg$ converges if and only if $\int_{G^1} f(g) dg$ converges.*

Proof. Let A be the identity component of the center of G . Let pr be the natural projection homomorphism from $A \setminus G$ to $G^1 A \setminus G$ sending $g \mapsto g'g$. Note that

$$\begin{aligned} \text{Ker } pr &= \{g \in A \setminus G : pr(g) = 1G^1 A\} \\ &= \{g \in A \setminus G : g'gG^1 A = 1G^1 A\} \\ &= \{g \in A \setminus G : g'g \in G^1 A\} \\ &= (G^1 \cap A) \setminus G^1. \end{aligned}$$

Hence,

$$G^1 A \backslash G \cong ((G^1 \cap A) \backslash G^1) \backslash (A \backslash G).$$

Since $A \cong F^\times$, we see that $A \cap G^1 \cong \mathcal{O}^\times$, which is compact. Moreover, since $G^1 A \backslash G$ is finite, $\int_{G^1} f(g) dg$ converges if and only if $\int_{A \backslash G} f(g) dg$ converges. ■

We utilize the notation from Section 2.1. Denote by r the the Witt rank. Let $P_{\min} = M_{\min} N_{\min}$ be the minimum parabolic subgroup. The Levi factor $M_{\min}$ is isomorphic to the direct product of $(F^\times)^r \times \mathrm{GSpin}_{Van}$. The split component $A_{\min}$ of $M_{\min}$ is isomorphic to $(F^\times)^r$ and the Weyl group $W(\mathrm{GSpin}(V), A_{\min})$ is of type B if d is odd and is of type D if d is even. Denote an element $A_{\min}$ by $x = (x_1, x_2, x_3, \dots, x_r)$. The simple roots of $(P_{\min}, A_{\min})$ are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \dots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1}$$

and

$$\alpha_r = \begin{cases} x_r & \text{if } d \text{ is odd} \\ x_{r-1} x_r & \text{if } d \text{ is even.} \end{cases}$$

Let $\delta_{P_{\min}}(x)$ be the modulus character of $P_{\min}$. Then

$$\delta_{P_{\min}}(x) = \prod_{i=1}^r |x_i|^{d+2r-2i}.$$

We define,

$$A_{\min}^{+, G^1} := \{a \in A_{\min} : |\alpha_i(a)| \leq 1 \text{ for } i = 1, \dots, r \text{ and } a \in G^1\}.$$

We are now ready to prove the following:

Proposition 3.3. *If π_{n+1} and π_{n+2} are tempered, then the local integral*

$$\mathcal{P}(\phi_{n+2}, \phi_{n+1}) = \int_{\overline{G_{n+1}}} \Phi_{\phi_{n+2}}(g) \Phi_{\phi_{n+1}}(g) \omega^{-1}(\nu(g)) dg$$

is absolutely convergent and $\mathcal{P}(\phi_{n+2}, \phi_{n+1}) \geq 0$ for any K_i -finite vector $\phi_{n+1} \in \pi_{n+1}$ and $\phi_{n+2} \in \pi_{n+2}$.

Proof. Let π_{n+1} and π_{n+2} be irreducible tempered representations of G_{n+1} and G_{n+2} , respectively. Let Φ_{ϕ_i} be a matrix coefficient of a tempered representation on G_i defined above. By Lemma 3.2,

$$\mathcal{P}(\phi_{n+2}, \phi_{n+1}) = \int_{G_{n+1}} |\Phi_{\phi_{n+2}}(g)| |\Phi_{\phi_{n+1}}(g)| |\omega^{-1}(\nu(g))| dg$$

converges if and only if

$$\int_{G^1} |\Phi_{\phi_{n+2}}(g)| |\Phi_{\phi_{n+1}}(g)| |\omega^{-1}(\nu(g))| dg$$

converges. Note that $|\omega^{-1}(\nu(g))| = 1$. Using (3.1) and that G^1 is a subset of G_{n+1} and K is a compact subset of G^1 one has the following:

$$G^1 = \left(\coprod_{m \in M_{\min}^+ / M_{\min}^1} KmK \right) \cap G^1 = \coprod_{m \in (M_{\min}^+ / M_{\min}^1) \cap G^1} KmK.$$

We define

$$M_{\min}^{+, G^1} := \{m \in M_{\min} : |\alpha_i(m)| \leq 1 \text{ for } i = 1, \dots, r \text{ and } m \in G^1\}.$$

By Lemma 3.1, the convergence of the integral is reduced to the convergence of

$$\int_{M_{\min}^{+, G^1}} \mu(m) \int_{K \times K} |\Phi_{\phi_{n+2}}(k_1 m k_2)| |\Phi_{\phi_{n+1}}(k_1 m k_2)| dk_1 dk_2 dm$$

where $\mu(m) = A \cdot \text{Vol}(KmK)$. Since $\Phi_{\phi_{n+1}}$ and $\Phi_{\phi_{n+2}}$ are matrix coefficients of tempered representations, they satisfy for any $g \in G$

$$|\Phi_{\phi_i}(g)| \leq A \Xi(g) (1 + \sigma(g))^B$$

for some positive constants A and B [Wal03, pg. 274]. Hence, the convergence of the integral reduces to the convergence of

$$\int_{M_{\min}^{+,G^1}} \mu(m) \int_{K \times K} \Xi_{n+2}(m) \Xi_{n+1}(m) (1 + \sigma(m))^{2B} dk_1 dk_2 dm$$

for $m \in M_{\min}^{+,G^1}$.

Per Theorem 4.2.1 in [Sil79, p.154] and [Wal03, Lemma II.1.1], there exists positive constants A and B such that

$$A^{-1} \delta_{P_{\min}}^{1/2}(m) \leq \Xi(m) \leq A \delta_{P_{\min}}^{1/2}(m) (1 + \sigma(m))^B$$

for any m in $M_{\min}^{+,G^1}$. So the convergence of the integral is reduced to the convergence of

$$\int_{M_{\min}^{+,G^1}} \mu(m) \int_{K \times K} (\delta_{n+2})^{1/2}(m) (\delta_{n+1})^{1/2}(m) (1 + \sigma(m))^{2B} dk_1 dk_2 dm.$$

But $K \times K$ is compact. Moreover, there exists a positive constant A such that $\mu(m) \leq A \delta_n^{-1}$ for any $m \in M_{\min}^{+,1}$ [Wal03, page 241]. So it is enough to show that

$$\int_{M_{\min}^{+,G^1}} (\delta_{n+1})^{-1/2}(m) (\delta_{n+2})^{1/2}(m) (1 + \sigma(m))^{2B} dm$$

converges.

When $W(G_{n+1}, A_{n+1,\min})$ is of type B , then $A_{n+1,\min}^+ \cap G^1$ sits inside of $A_{n+2,\min}^+ \cap G^1$. Thus the integral is reduced to the convergence of

$$\int_{A_{n+1,\min}^{+,G^1}} (\delta_{n+1})^{-1/2}(m) (\delta_{n+2})^{1/2}(m) (1 + \sigma(m))^{2B} dm.$$

GSpin has an extra root dealing with the similitude factor. But $|\nu(g)| = 1$, for all $g \in G^1$, so it has little effect in the following calculation. Hence, the convergence of the integral is reduced to

$$\int_{|x_1| \leq \dots \leq |x_{r_{n+1}}| \leq |x_0| \leq 1} |x_1 \cdots x_{r_{n+1}}|^{1/2} \left(1 - \sum_{j=1}^{r_{n+1}} \log |x_j| \right)^{2B} d^\times x_1 \cdots d^\times x_{r_{n+1}} d^\times x_0.$$

One can show that this integral converges following a result from calculus.

For the second case, when we are in type D , $A_{n+1,\min}^+ \cap G^1$ is not a subset of $A_{n+2,\min}^+ \cap G^1$. Hence, the convergence of the integral in this case is reduced to the convergence of

$$\begin{aligned} & \int_{|x_1| \leq \dots \leq |x_{r_{n+1}}| \leq |x_0| \leq 1} |x_1 \cdots x_{r_{n+1}}|^{1/2} \left(1 - \sum_{j=1}^{r_{n+1}} \log |x_j| \right)^{2B} d^\times x_1 \cdots d^\times x_{r_{n+1}} d^\times x_0 \\ & + \int_{|x_1| \leq \dots \leq |x_{r_{n+1}-1}| \leq |x_{r_{n+1}}|^{-1} \leq |x_0| \leq 1} |x_1 \cdots x_{r_{n+1}-1} x_{r_{n+1}}^{-1}|^{1/2} \\ & \left(1 - \sum_{j=1}^{r_{n+1}-1} \log |x_j| + \log |x_{r_{n+1}}| \right)^{2B} d^\times x_1 \cdots d^\times x_{r_{n+1}} d^\times x_0. \end{aligned}$$

This integral can be shown to be convergent similarly to the one above.

Just as in [II10], the second part of the proposition, that $\mathcal{P}(\phi_{n+2}, \phi_{n+1}) \geq 0$ for any K_i -finite vector $\phi_i \in \pi_i$, follows from a general theorem from He [He03, Theorem 2.1]. Recall that we have the isomorphism $A_{n+1} \setminus G_{n+1} \cong \mathrm{SO}_{n+1}$. Let Ξ_i be Harish-Chandra spherical functions for G_i . By the first part of the proposition,

$$g_{n+1} \rightarrow \Xi_{n+2}(g_n) \Xi_{n+1}(g_{n+1}) \omega^{-1}(\nu(g_{n+1}))$$

belongs to $L^1(A_{n+1} \setminus G_{n+1})$. Note that Harish-Chandra's spherical function is a matrix coefficient of a tempered representation. Then using He's theorem $\mathcal{P}(\phi_{n+2}, \phi_{n+1}) \geq 0$ for any K_i -finite vector $\phi_i \in \pi_i$. ■

3.2 Calculation of integrals in the unramified section

For the remainder of this chapter, we assume $v \notin S$ and we explicitly calculate two integrals, $\zeta(\Xi, \xi)$ and $S_{\Xi^{-1}, \xi^{-1}}(1)$, for such v . The product of $\zeta(\Xi, \xi)$ and $S_{\Xi^{-1}, \xi^{-1}}(1)$ is

the integral that we desire. Away from places in S , we are in either the split case, or the quasi-split but non-split case. We compute $\zeta(\Xi, \xi)$ and $S_{\Xi^{-1}, \xi^{-1}}(1)$ in each case.

Let E be the discriminant field. By class field theory, E is either equal to F or the unramified quadratic extension of F . Let q be the number of elements of the residue field of F . The local zeta function is

$$\zeta(s) := \frac{1}{1 - q^{-s}}.$$

Set $l_i := \lfloor i/2 \rfloor$. Let $\xi_1, \dots, \xi_{l_{n+1}}$ and $\Xi_1, \dots, \Xi_{l_{n+2}}$ be unramified characters of $F^\times$.

Let $B_i = T_i N_i$ be the Borel subgroups where the N_i are unipotent radicals and the T_i is a maximal torus of G_i . Then we can view ξ and Ξ as characters of B_{n+1} and B_{n+2} by extending them by 1 to N_i . Let $\mathcal{A}_i \subset T_i$ be the maximal split torus. Without loss of generality, assume $N_i \subset N_{i+1}$ and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. Away from S , let

$$\pi_{n+1} = \text{Ind}_{B_{n+1}}^{G_{n+1}}(\xi),$$

$$\pi_{n+2} = \text{Ind}_{B_{n+2}}^{G_{n+2}}(\Xi)$$

be unramified principal series of G_{n+1} and G_{n+2} respectively.

Choose the $\mathcal{B}_{\pi_i}$ so that $\Phi_{f_{\pi_i}}$ are the spherical matrix coefficients normalized so that $\Phi_{f_{\pi_i}}(k_i) = 1$ for all $k_i \in K_i$. Then we have

$$\Phi_{f_{\pi_i}}(g_i) = \int_{K_i} f_{\pi_i}(k_i g_i) dk_i$$

for all $g_i \in G_i$. Denote $\overline{G}_i = A_i \setminus G_i$. Consider the local integral

$$F(g_{n+2}) = \int_{\overline{G}_{n+1}} \Phi_{f_{\pi_{n+2}}}(g_{n+2}^{-1} g_{n+1}) \Phi_{f_{\pi_{n+1}}}(g_{n+1}) \omega^{-1}(\nu(g_{n+2}^{-1} g_{n+1})) dg_{n+1}. \quad (3.3)$$

Next, we review the theory of Shintani functions. Kato-Murase-Sugano [KMS03] showed that if both G_{n+1} and G_{n+2} are split, then there exists a Shintani function.

We will show in a moment that $F(g_{n+2})$ is indeed a Shintani function. Although the result is for the orthogonal groups, the argument follows for $A_i \setminus G_i$. Let $\mathcal{H}_i$ denote the Hecke algebra $\mathcal{H}(K_i \setminus G_i / K_i)$. Denote ω_i the algebra homomorphism given by the Satake isomorphism from $\mathcal{H}_i$ to the complexes.

A Shintani function S is a function that satisfies

1. $\mathcal{L}(k_{n+1})\mathcal{R}(k_{n+2})S = S$ for any $k_{n+2} \in K_{n+2}$ and $k_{n+1} \in K_{n+1}$,
2. $\mathcal{L}(\phi_{n+1})\mathcal{R}(\phi_{n+2})S = \omega_{\pi_{n+1}}(\phi_{n+1})\omega_{\pi_{n+2}}(\phi_{n+2})S$ for any $\phi_{n+1} \in \mathcal{H}_{n+1}$ and $\phi_{n+2} \in \mathcal{H}_{n+2}$.

Lemma 3.4. *Let F be as (3.3). Then F is a Shintani function for $\tilde{\pi}_{n+2}$ and $\tilde{\pi}_{n+1}$.*

Proof. We begin by verifying the first condition.

$$\begin{aligned}
\mathcal{L}(k_{n+1})\mathcal{R}(k_{n+2})F(g_{n+2}) &= \int_{\bar{G}_{n+1}} \Phi_{f_{\pi_{n+2}}}(k_{n+2}^{-1}g_{n+2}^{-1}k_{n+1}g_{n+1})\Phi_{f_{\pi_{n+1}}}(g_{n+1}) \\
&\quad \cdot \omega^{-1}(\nu(k_{n+2}^{-1}g_{n+2}^{-1}k_{n+1}g_{n+1}))dg_{n+1} \\
&= \int_{\bar{G}_{n+1}} \Phi_{f_{\pi_{n+2}}}(k_{n+2}^{-1}g_{n+2}^{-1}g_{n+1})\Phi_{f_{\pi_{n+1}}}(k_{n+1}^{-1}g_{n+1}) \\
&\quad \cdot \omega^{-1}(\nu(k_{n+2}^{-1}g_{n+2}^{-1}k_{n+1}g_{n+1}))dg_{n+1} \\
&= \int_{\bar{G}_{n+1}} \int_{K_{n+2}} f_{\pi_{n+2}}(k'_{n+2}k_{n+2}^{-1}g_{n+2}^{-1}g_{n+1}) \\
&\quad \cdot \int_{K_{n+1}} f_{\pi_{n+1}}(k'_{n+1}k_{n+1}^{-1}g_{n+1})\omega^{-1}(\nu(k_{n+2}^{-1}g_{n+2}^{-1}g_{n+1}))dg_{n+1} \\
&= \int_{\bar{G}_{n+1}} \Phi_{f_{\pi_{n+2}}}(g_{n+2}^{-1}g_{n+1})\Phi_{f_{\pi_{n+1}}}(g_{n+2})\omega^{-1}(\nu(g_{n+2}^{-1}g_{n+1}))dg_{n+1} \\
&= F(g_{n+2}).
\end{aligned}$$

Next, we verify the second condition.

$$\mathcal{L}(\phi_{n+1})\mathcal{R}(\phi_{n+2})F(g'_{n+2}) = \int_{G_{n+1}} \mathcal{R}(\phi_{n+1})F(g_{n+1}^{-1}g'_{n+2})\phi_{n+1}(g_{n+1})dg_{n+1}$$

$$\begin{aligned}
&= \int_{\overline{G}_{n+2}} \int_{\overline{G}_{n+1}} F(g_{n+1}^{-1} g'_{n+2} g_{n+2}) \phi_{n+1}(g_{n+1}) \phi_{n+2}(g_{n+2}) dg_{n+1} dg_{n+2} \\
&= \int_{\overline{G}_{n+2}} \int_{\overline{G}_{n+1}} \int_{\overline{G}_{n+1}} \Phi_{f_{\pi_{n+2}}}(g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1}) \Phi_{f_{\pi_{n+1}}}(g'_{n+1}) \\
&\quad \cdot \phi_{n+1}(g_{n+1}) \phi_{n+2}(g_{n+2}) \omega^{-1}(\nu(g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1})) dg_{n+1} dg_{n+2} \\
&= \int_{\overline{G}_{n+2}} \int_{\overline{G}_{n+1}} \int_{\overline{G}_{n+1}} \left(\int_{K_{n+2}} f_{\pi_{n+2}}(k_{n+2} g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1}) dk_{n+2} \right. \\
&\quad \cdot \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1} g'_{n+1}) dk_{n+1} \phi_{n+1}(g_{n+1}) \phi_{n+2}(g_{n+2}) \\
&\quad \cdot \omega^{-1}(\nu(g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1})) dk_{n+1} dg_{n+1} dg_{n+2} \\
&= \int_{\overline{G}_{n+1}} \int_{K_{n+2}} \int_{\overline{G}_{n+2}} f_{\pi_{n+2}}(k_{n+2} g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1}) \phi_{n+2}(g_{n+2}) dg_{n+2} dk_{n+2} \\
&\quad \cdot \int_{K_{n+1}} \int_{\overline{G}_{n+1}} f_{\pi_{n+1}}(k_{n+1} g'_{n+1}) \phi_{n+1}(g_{n+1}) dg_{n+1} dk_{n+1} dg_{n+1} \\
&\quad \cdot \omega^{-1}(\nu(g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1})) dg_{n+1} dk_{n+1} \\
&= \int_{\overline{G}_{n+1}} \int_{K_{n+2}} \int_{\overline{G}_{n+2}} f_{\pi_{n+2}}(k_{n+2} g'_{n+2} g_{n+1}) \phi_{n+2}(g_{n+2}) dg_{n+2} dk_{n+2} \\
&\quad \cdot \int_{K_{n+1}} \int_{\overline{G}_{n+1}} f_{\pi_{n+1}}(k_{n+1} g'_{n+1}) \phi_{n+1}(g_{n+1}) dg_{n+1} dk_{n+1} dg_{n+1} \\
&\quad \cdot \omega^{-1}(\nu(g_{n+2}^{-1} g'_{n+2} g_{n+1} g'_{n+1})) dg_{n+1} dk_{n+1} \\
&= \int_{\overline{G}_{n+1}} \int_{K_{n+2}} \int_{\overline{G}_{n+2}} f_{\pi_{n+2}}(k_{n+2} g'_{n+2} g_{n+1}) \phi_{n+2}(g_{n+2}) dg_{n+2} dk_{n+2} \\
&\quad \int_{K_{n+1}} \int_{\overline{G}_{n+1}} f_{\pi_{n+1}}(k_{n+1} g_{n+1} g'_{n+1}) \phi_{n+1}(g_{n+1}) dg_{n+1} dk_{n+1} dg_{n+1} \\
&\quad \cdot \omega^{-1}(\nu(g'_{n+2} g_{n+1})) dg_{n+1} dk_{n+1} \\
&= \omega_{\pi_{n+1}}(\phi_{n+1}) \omega_{\pi_{n+2}}(\phi_{n+2}) \int_{\overline{G}_{n+2}} \int_{\overline{G}_{n+1}} \int_{K_{n+2}} f_{\pi_{n+2}}(k'_{n+2} g_{n+2}) \\
&\quad \cdot \int_{K_{n+1}} \int_{\overline{G}_{n+1}} f_{\pi_{n+1}}(k_{n+1} g_{n+1}) \omega^{-1}(\nu(g'_{n+2} g_{n+1})) dg_{n+1} dk_{n+1} \\
&= \omega_{\pi_{n+1}}(\phi_{n+1}) \omega_{\pi_{n+2}}(\phi_{n+2}) F(g_{n+2}).
\end{aligned}$$

Hence, F is a Shintani function. ■

We need the following two lemmas before proceeding.

Lemma 3.5. *There exists a unique open orbit $B_{n+2} \backslash G_{n+2} / B_{n+2}$ which has a representative $\eta_{n+2} \in K_{n+2}$.*

Proof. Note that $B_{n+2} \backslash G_{n+2} / B_{n+1} = \{B_{n+2}gB_{n+1} : g \in G_{n+2}\}$. Clearly, $B_{n+2}gB_{n+1}$ is a subset of G_{n+2} . Let W denote the Weyl group of G_{n+2} and let N be the unipotent radical of G . Then we have the Bruhat decomposition as follows:

$$\begin{aligned} G_{n+2} &= \bigsqcup_{w \in W} B_{n+2}wB_{n+2} \\ &= \bigcup_{w \in W} B_{n+2}wT_{n+2}N_{n+2} \quad (B_{n+2} = T_{n+2}N_{n+2}) \\ &= \bigcup_{w \in W} B_{n+2}wT_{n+2}w^{-1}wN_{n+2} \quad (wT_{n+2}w^{-1} = T_{n+2}) \\ &= \bigcup_{w \in W} B_{n+2}wN_{n+2}. \end{aligned}$$

If w is the longest Weyl element, then $B_{n+2}wN_{n+2}$ is open and dense. Note that $B_{n+2}wN_{n+2} = B_{n+2}wB_{n+2} \supsetneq B_{n+2}wB_{n+1}$. The unipotent radical for GSpin is the same as for SO so the argument in Cor 7.1 in [GPSR97] holds for GSpin which concludes the proof. ■

Lemma 3.6. *Let A_{n+1} be the identity component of G_{n+1} and let $\eta_{n+2} \in K_{n+2}$. Then*

$$\eta_{n+2}^{-1}B_{n+2}\eta_{n+2} \cap B_{n+1} = A_{n+1}.$$

Proof. First, to see that

$$A_{n+1} \subseteq \eta_{n+2}^{-1}B_{n+2}\eta_{n+2} \cap B_{n+1},$$

let $Z(G_i)$ denote the center of G_i . Using Proposition 2.3 in [AS06], $A_{n+2} = A_{n+1} \subseteq Z(G_{n+1}) = Z(G_{n+1}) \cap Z(G_{n+2}) = Z(B_{n+1}) \cap Z(B_{n+2}) \subseteq B_{n+1} \cap \eta_{n+2}^{-1}B_{n+2}\eta_{n+2}$. Here we also use Corollary 6.4.6 in [Spr98] which shows the center of a group equals the center of the Borel subgroup. The last inclusion comes from $Z(B_{n+1}) \subseteq B_{n+1}$ and $Z(B_{n+2}) \subseteq \eta^{-1}B_{n+2}\eta$. Hence, $A_{n+1} \subseteq \eta^{-1}B_{n+2}\eta \cap B_{n+1}$.

It remains to show that $\eta_{n+2}^{-1}B_{n+2}\eta \cap B_{n+1} \subseteq A_{n+1}$. Since there exists the surjective map pr taking Borel subgroups in GSpin to Borel subgroups in SO , if $\eta^{-1}B_{n+2}\eta_{n+2} \cap B_{n+1}$ was not the identity component of the center, then this would contradict the claim of Ichino-Ikeda that $\eta^{-1}B_{n+2}\eta_{n+2} \cap B_{n+1} = \{1\}$ [II10, pg. 1389]. Hence,

$$A_{n+1} = \eta_{n+2}^{-1}B_{n+2}\eta \cap B_{n+1}.$$

■

Let $Y_{\Xi,\xi}$ be the function on G_{n+2} determined by the following conditions:

1. $Y_{\Xi,\xi}(b_{n+2}g_{n+2}b_{n+1}) = (\Xi^{-1}\delta_{n+2}^{1/2})(b_{n+2})(\xi\delta_{n+1}^{-1/2})(b_{n+1})Y_{\Xi,\xi}(g_{n+2})\omega(\nu(b_{n+2}))$ for any $b_i \in B_i$,
2. $Y_{\Xi,\xi}(\eta_{n+2}) = 1$,
3. $Y_{\Xi,\xi}(g_{n+2}) = 0$ if $g_{n+2} \notin B_{n+2}\eta_{n+2}B_{n+1}$.

Here δ_i is the modulus character for B_i . Note that $Y_{\Xi,\xi}$ is well-defined. To see this, let $b_{n+2}, c_{n+2} \in B_{n+2}$, and let $z \in A_{n+1}$. Note that if $b_{n+2}\eta_{n+2}b_{n+1} = c_{n+2}\eta_{n+2}c_{n+1}$, then $\eta_{n+2}^{-1}c_{n+2}^{-1}b_{n+2}\eta_{n+2} = c_{n+1}b_{n+1}^{-1} = z$. So we have,

$$c_{n+1} = zb_{n+1} \text{ and } c_{n+2} = z^{-1}b_{n+2}.$$

Thus,

$$\begin{aligned} Y_{\Xi,\xi}(c_{n+2}\eta c_{n+1}) &= \Xi^{-1}\delta_{n+2}^{1/2}(c_{n+2})(\xi\delta_{n+1}^{-1/2})(c_{n+1})\omega(v(c_{n+2}))Y_{\Xi,\xi}(\eta_{n+2}) \\ &= \Xi^{-1}\delta_{n+2}^{1/2}(z^{-1}b_{n+2})(\xi\delta_{n+1}^{-1/2})(zb_{n+1})\omega(v(z^{-1}b_{n+2}))Y_{\Xi,\xi}(\eta_{n+2}) \\ &= \Xi^{-1}\delta_{n+2}^{1/2}(z^{-1})\Xi^{-1}\delta_{n+2}^{1/2}(b_{n+2})(\xi\delta_{n+1}^{-1/2})(z)\xi\delta_{n+1}^{-1/2}(b_{n+1}) \\ &\quad \omega(z^{-2})\omega(v(b_{n+2}))Y_{\Xi,\xi}(\eta_{n+2}) \end{aligned}$$

$$\begin{aligned}
&= \Xi \delta_{n+2}^{-1/2}(z) \Xi^{-1} \delta_{n+2}^{1/2}(b_{n+2}) (\xi \delta_{n+1}^{-1/2})(z) \xi \delta_{n+1}^{-1/2}(b_{n+1}) \\
&\quad \omega(z)^{-2} \omega(v(b_{n+2})) Y_{\Xi, \xi}(\eta_{n+2}) \\
&= \omega_{\pi_{n+2}}(z) \Xi^{-1} \delta_{n+2}^{1/2}(b_{n+2}) \omega_{\pi_{n+1}}(z) \xi \delta_{n+1}^{-1/2}(b_{n+1}) \omega(z)^{-2} \\
&\quad \omega(v(b_{n+2})) Y_{\Xi, \xi}(\eta_{n+2}) \\
&= \omega_{\pi_{n+2}}(z) \Xi^{-1} \delta_{n+2}^{1/2}(b_{n+2}) \omega_{\pi_{n+1}}(z) \xi \delta_{n+1}^{-1/2}(b_{n+1}) \omega_{\pi_{n+2}}(z)^{-1} \\
&\quad \omega_{\pi_{n+1}}(z)^{-1} \omega(v(b_{n+2})) Y_{\Xi, \xi}(\eta_{n+2}) \\
&= \Xi^{-1} \delta_{n+2}^{1/2}(b_{n+2}) \xi \delta_{n+1}^{1/2}(b_{n+1}) \omega(v(b_{n+2})) Y_{\Xi, \xi}(\eta_{n+2}) \\
&= Y_{\Xi, \xi}(b_{n+2} \eta_{n+2} b_{n+1}).
\end{aligned}$$

Hence, $Y_{\Xi, \xi}$ is well defined.

Next, we define $l_{\Xi, \xi} \in \text{Hom}_{G_{n+1}}(\pi_{n+2}, \tilde{\pi}_{n+1})$ by

$$l_{\Xi, \xi}(\text{pr}_1(f))(g_{n+1}) = \int_{G_{n+2}} f(g_{n+2} g_{n+1}) Y_{\Xi, \xi}(g_{n+2}) dg_{n+2}, \quad g_{n+1} \in G_{n+1}.$$

Since f is compactly supported, the integral converges.

Here, $\text{pr}_1 : C_c^\infty(G_{n+1}) \rightarrow \pi_{n+2}$ is given by

$$\text{pr}_1(f)(g_{n+2}) = \int_{B_{n+1}} (\Xi_{n+2}^{-1} \delta_{n+2}^{1/2})(b_{n+2}) f(b_{n+2} g_{n+2}) db_{n+2}.$$

Let $\langle, \rangle$ be the natural pairing on $\pi_{n+1} \times \tilde{\pi}_{n+1}$ defined by

$$\langle \phi_{n+1}, \phi'_{n+1} \rangle = \int_{K_{n+1}} \phi_{n+1}(k_{n+1}) \phi'_{n+1}(k_{n+1}) dk_{n+1}$$

for $\phi_{n+1} \in \pi_{n+1}$ and $\phi'_{n+1} \in \tilde{\pi}_{n+1}$.

Choose $f = \mathbf{1}_{K_{n+2}}$, the characteristic function on K_{n+2} . Put

$$S_{\Xi, \xi}(g_{n+2}) = \langle f_{\pi_{n+1}}, l_{\Xi, \xi}(\pi_{n+2}(g_{n+2}) f_{\pi_{n+2}}) \rangle.$$

Then we have,

$$\begin{aligned}
S_{\Xi,\xi}(g_{n+2}) &= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) l_{\Xi,\xi}(\pi_{n+2}(g_{n+2}) f_{\pi_{n+2}})(k_{n+1}) dk_{n+1} \\
&= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) \int_{G_{n+2}} f(g'_{n+2} k_{n+1} g_{n+2}) Y_{\Xi,\xi}(g'_{n+2}) dg'_{n+2} dk_{n+1} \\
&= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) \int_{G_{n+2}} \mathbb{1}_{K_{n+2}}(g'_{n+2} k_{n+1} g_{n+2}) Y_{\Xi,\xi}(g'_{n+2}) dg'_{n+2} dk_{n+1} \\
&= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) \int_{G_{n+2}} \mathbb{1}_{K_{n+2}}(g'_{n+2} g_{n+2}^{-1} k_{n+1}^{-1} k_{n+1} g_{n+2}) \\
&\quad Y_{\Xi,\xi}(g'_{n+2} g_{n+2}^{-1} k_{n+1}^{-1}) dg'_{n+2} dk_{n+1} \\
&= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) \int_{G_{n+2}} \mathbb{1}_{K_{n+2}}(g'_{n+2}) Y_{\Xi,\xi}(g'_{n+2} g_{n+2}^{-1} k_{n+1}^{-1}) dg'_{n+2} dk_{n+1} \\
&= \int_{K_{n+1}} f_{\pi_{n+1}}(k_{n+1}) \int_{K_{n+2}} Y_{\Xi,\xi}(k_{n+2} g_{n+2}^{-1} k_{n+1}) dk_{n+2} dk_{n+1} \\
&= \int_{K_{n+2} \times K_{n+1}} Y_{\Xi,\xi}(k_{n+2} g_{n+2}^{-1} k_{n+1}) dk_{n+2} dk_{n+1}.
\end{aligned}$$

Note that $S_{\Xi,\xi}$ is also a Shintani Function. Define

$$T_{\Xi,\xi}(g_{n+2}) = \int_{\overline{G_{n+1}}} f_{\pi_{n+2}}(g_{n+2} g_{n+1}) f_{\pi_{n+1}}(g_{n+1}) \omega^{-1}(\nu(g_{n+2} g_{n+1})) dg_{n+1}$$

if $g_{n+2} \in B_{n+2} \eta_{n+2} B_{n+1}$ and 0 otherwise.

The relationship between $T_{\Xi,\xi}$ and $Y_{\Xi^{-1},\xi^{-1}}$ is

$$T_{\Xi,\xi}(g_{n+2}) = T_{\Xi,\xi}(\eta_{n+2}) Y_{\Xi^{-1},\xi^{-1}}(g_{n+2}).$$

To see this, let $g_{n+2} \in B_{n+2} \eta_{n+2} B_{n+1}$ and let $g_{n+2} = b_{n+2} \eta_{n+2} b_{n+1}$. Then we have,

$$\begin{aligned}
T_{\Xi,\xi}(g_{n+2}) &= T_{\Xi,\xi}(b_{n+2} \eta_{n+2} b_{n+1}) = \int_{\overline{G_{n+1}}} f_{\pi_{n+2}}(b_{n+2} \eta_{n+2} b_{n+1} g_{n+1}) f_{\pi_{n+1}}(g_{n+1}) \\
&\quad \omega^{-1}(\nu(b_{n+2} \eta_{n+2} b_{n+1} g_{n+1})) dg_{n+1} \\
&= \Xi(b_{n+2}) \delta^{1/2}(b_{n+2}) \omega^{-1}(\nu(b_{n+2})) \\
&\quad \int_{\overline{G_{n+1}}} f_{\pi_{n+2}}(\eta_{n+2} b_{n+1} g_{n+1}) f_{\pi_{n+1}}(g_{n+1}) \omega^{-1}(\nu(\eta_{n+2} b_{n+1} g_{n+1})) dg_{n+1}
\end{aligned}$$

$$\begin{aligned}
&= \Xi(b_{n+2})\delta_{n+2}^{1/2}(b_{n+2})\omega^{-1}(\nu(b_{n+2})) \\
&\quad \int_{\overline{G_{n+1}}} f_{\pi_{n+2}}(\eta_{n+2}g_{n+1})f_{\pi_{n+1}}(b_{n+1}^{-1}g_{n+1})\omega^{-1}(\nu(\eta_{n+2}g_{n+1}))dg_{n+1} \\
&= \Xi(b_{n+2})\delta_{n+2}^{1/2}(b_{n+2})\omega^{-1}(\nu(b_{n+2}))\xi^{-1}(b_{n+1})\delta_{n+1}^{-1/2}(b_{n+1}) \\
&\quad \int_{\overline{G_{n+1}}} f_{\pi_{n+2}}(\eta_{n+2}g_{n+1})f_{\pi_{n+1}}(g_{n+1})\omega^{-1}(\nu(\eta_{n+2}g_1))dg_1 \\
&= Y_{\Xi^{-1},\xi^{-1}}(g_{n+2})T_{\Xi,\xi}(\eta_{n+2}).
\end{aligned}$$

Note that $T_{\Xi,\xi}$ does not depend on the choice of representative of η_{n+2} so we define $\zeta(\Xi, \xi) := T_{\Xi,\xi}(\eta_{n+2})$. Hence, we have

$$\begin{aligned}
F(g_{n+2}) &= \int_{\overline{G_{n+1}}} \int_{K_{n+2}} \int_{K_{n+1}} f_{\pi_{n+2}}(k_{n+2}g_{n+2}^{-1}g_{n+1})f_{\pi_{n+1}}(k_{n+1}g_{n+1}) \\
&\quad \omega^{-1}(\nu(g_{n+2}^{-1}g_{n+1}))dk_{n+1}dk_{n+2}dg_{n+1} \\
&= \int_{\overline{G_{n+1}}} \int_{K_{n+2}} \int_{K_{n+1}} f_{\pi_{n+2}}(k_{n+2}g_{n+2}^{-1}k_{n+1}g_{n+1})f_{\pi_{n+1}}(g_{n+1}) \\
&\quad \omega^{-1}(\nu(g_{n+2}^{-1}k_{n+1}g_{n+1}))dk_{n+1}dk_{n+2}dg_{n+1} \\
&= \int_{\overline{G_{n+1}}} \int_{K_{n+2}} \int_{K_{n+1}} f_{\pi_{n+2}}(k_{n+2}g_{n+2}^{-1}k_{n+1}g_{n+1})f_{\pi_{n+1}}(g_{n+1}) \\
&\quad \omega^{-1}(\nu(k_{n+2}g_{n+2}^{-1}k_{n+1}g_{n+1}))dk_{n+1}dk_{n+2}dg_{n+1} \\
&= \int_{K_{n+2} \times K_{n+1}} T_{\Xi,\xi}(k_{n+2}g_{n+2}^{-1}k_{n+1})dk_{n+2}dk_{n+1} \\
&= T_{\Xi,\xi}(\eta_{n+2}) \int_{K_{n+2} \times K_{n+1}} Y_{\Xi^{-1},\xi^{-1}}(k_{n+2}g_{n+2}^{-1}k_{n+1})dk_{n+2}dk_{n+1} \\
&= \zeta(\Xi, \xi)S_{\Xi^{-1},\xi^{-1}}(g_{n+2}).
\end{aligned}$$

As in [II10, pg. 1390], $F(g_{n+2})$ is convergent for Ξ, ξ sufficiently close to the unitary axis. Indeed, Proposition 3.3 holds for such Ξ and ξ . Hence, $T_{\Xi,\xi}(k_{n+2}g_{n+2}^{-1}k_{n+1})$ converges for almost all k_{n+2}, k_{n+1} such that $k_{n+2}g_{n+2}^{-1}k_{n+1} \in B_{n+2}\eta_{n+2}B_{n+1}$. Moreover, $T_{\Xi,\xi}$ converges for some $g \in B_{n+2}\eta_{n+2}B_{n+1}$ precisely when it converges for all $g \in B_{n+2}\eta_{n+2}\eta_{n+1}$. Thus, $T_{\Xi,\xi}(\eta_{n+2})$ converges. Hence, $S_{\Xi^{-1},\xi^{-1}}(g_{n+2})$ converges as

well.

3.2.1 Calculation of $\zeta(\Xi, \xi)$ when $E = F$

We consider the case when E is equal to F , and the residual characteristic of F is not 2. As before, T_i is a maximal torus of G_i . Let $\mathcal{A}_i$ be the split component of the Levi factor M_i . Since we are in the case that each G_i is split, we have that

$$T_{n+2} = \mathcal{A}_{n+2} \simeq (F^\times)^{\lfloor (n+2)/2 \rfloor} \text{ and } T_{n+1} = \mathcal{A}_{n+1} \simeq (F^\times)^{\lfloor (n+1)/2 \rfloor}.$$

Let $\tilde{\Xi}, \xi, \Xi$ be characters of T_n, T_{n+1}, T_{n+2} , respectively, where

$$\tilde{\Xi} := (\Xi_1, \dots, \Xi_{\lfloor n/2 \rfloor}, \chi_0),$$

$$\xi := (\xi_1, \dots, \xi_{\lfloor (n+1)/2 \rfloor}, \chi_1),$$

$$\Xi := (\Xi_1, \dots, \Xi_{\lfloor (n+2)/2 \rfloor}, \chi_0).$$

Moreover, denote $l := \lfloor (n+2)/2 \rfloor$ and let χ_{n+1} be unramified and such that $\chi_{n+1}^2 = \omega_{\pi_{n+1}}$. Recall that $\omega^2 = \omega_{\pi_{n+2}} \omega_{\pi_{n+1}}$. Hence, $\omega^2 = (\chi_{n+1} \chi_n)^2$. Per Asgari [AS06, pg. 178], $\omega^{-1} \chi_{n+1} = \chi_0^{-1/2}$. We can view $\pi_n \otimes \chi_0^{-1/2}$ as a representation of $\overline{G_n} \cong \text{SO}_n$. Define $\overline{\pi_n} := \pi_n \otimes \chi_0^{-1/2}$. Take V_n to be a n -dimensional symmetric space, with associated symmetric form $\langle \cdot, \cdot \rangle_{V_n}$.

Let $e, f \in V$ such that $\langle e, e \rangle = 0, \langle f, f \rangle = 0$, and $\langle e, f \rangle = 1$. Then,

$V_{n+2} = V_n \oplus \text{Span}\{e, f\}$ and $V_{n+1} = V_n \oplus \text{Span}\{e + f\}$. Define

$$W := V_n \oplus V_n^- \oplus \text{Span}\{e, f\}.$$

Note that $\dim W = 2n + 2$ and so $\text{SO}(W) \cong \text{SO}_{2n+2}$. Let $Q := \text{Span}\{e, f\}$. Then,

$$\overline{G_n}(V_n) \times \overline{G_{n+1}}(V_{n+1}) \subseteq \text{SO}(W).$$

Choose a basis of W to be

$$\{v_1 \oplus -v_1, v_2 \oplus -v_2, \dots, v_n \oplus -v_n, e, v_1 \oplus v_1, v_2 \oplus v_2, \dots, v_n \oplus v_n, f\}.$$

Hence, $V^\dagger = \text{Span}\{v_i \oplus (-v_i), e\}$ where $v_i \in V_n$ and $-v_i \in V_n^-$. Note that $V^\dagger$ is a maximal isotropic subspace.

Denote $P := \text{Stab}_{\text{SO}(W)} V^\dagger$, the corresponding maximal parabolic subgroup. Note that P is the Siegel parabolic in $\text{SO}(W)$. Then $P = MN$, where M is the Levi subgroup and N is the unipotent radical. We have that M is isomorphic to $\text{GL}(V^\dagger)$. Consider the following induced representation of $\text{SO}(W)$:

$$I(\Xi_l) := \text{Ind}_P^{\text{SO}(W)}((\Xi_l \otimes \chi_0^{-1/2}) \circ \det_{V^\dagger})$$

where the induction is normalized. Let $f_0 \in I(\Xi_l)$ be the spherical vector, normalized so that $f_0(1) = 1$. Define $\tilde{f}_0$ to be f_0 restricted to $\overline{G_n} \times \overline{G_{n+2}}$. Let $\overline{f_{\pi_n}} \in \overline{\pi_n}$, and define

$$\Lambda(f_0, \overline{f_{\pi_n}})(g_{n+2}) := \int_{\overline{G_n}} \tilde{f}_0(g_n, g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n.$$

Hence,

$$\Lambda(f_0, \overline{f_{\pi_n}})(g_{n+2})(\overline{g_n}) := \int_{\overline{G_n}} \tilde{f}_0(g_n, g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}}(\overline{g_n}) dg_n.$$

The integral $\Lambda(f_0, \overline{f_{\pi_n}})$ converges for $\text{Re}(s) \gg 0$, and we use analytic continuation to define $\Lambda(f_0, \overline{f_{\pi_n}})$ elsewhere. Note that,

$$\Lambda(f_0, \overline{f_{\pi_n}}) \in \text{Ind}_{\text{GL}(V^\dagger \cap Q) \times \overline{G_n}}^{\overline{G_{n+2}}} \Xi_l \otimes \chi_0^{-1/2} \otimes \overline{\pi_n}.$$

To see this, let $(a, \overline{g_n}) \in \text{GL}(V^\dagger \cap Q) \times \overline{G_n}$. Then,

$$\Lambda(f_0, \overline{f_{\pi_n}})((a, \overline{g_n})g_{n+2}) = \int_{\overline{G_n}} \tilde{f}_0(g_n, (a, \overline{g_n})g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n$$

$$\begin{aligned}
&= \int_{\overline{G_n}} \tilde{f}_0(g_n, (a, 1)(1, \overline{g_n})g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n \\
&(a, 1) \in \mathrm{GL}(V^\dagger \cap Q) \subseteq \mathrm{GL}(V^\dagger) \text{ and } \det_{V^\dagger}(a, 1) = a \\
&= \Xi_l \chi_0^{-1/2}(a) \int_{\overline{G_n}} \tilde{f}_0(g_n, (1, \overline{g_n})g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n \\
&= \Xi_l \chi_0^{-1/2}(a) \int_{\overline{G_n}} \tilde{f}_0(g_n, \overline{g_n}g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n \\
&= \Xi_l \chi_0^{-1/2}(a) \int_{\overline{G_n}} \tilde{f}_0((\overline{g_n})^{-1}g_n, g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n \\
&= \Xi_l \chi_0^{-1/2}(a) \int_{\overline{G_n}} \tilde{f}_0(g_n, g_{n+2}) \overline{\pi_n}(\overline{g_n}g_n) \overline{f_{\pi_n}} dg_n \\
&= \Xi_l \chi_0^{-1/2}(a) \overline{\pi_n}(\overline{g_n}) \int_{\overline{G_n}} \tilde{f}_0(g_n, g_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n \\
&= \Xi_l \chi_0^{-1/2}(a) \overline{\pi_n}(\overline{g_n}) \Lambda(f_0, \overline{f_{\pi_n}})(g_{n+2}).
\end{aligned}$$

By the transitive property of induction, $\Lambda(f_0, \overline{f_{\pi_n}})$ may be viewed as an element of $\overline{\pi_{n+2}}$.

Proposition 3.7. *For any k in K_{n+2} , n a positive integer, and $\Lambda(f_0, \overline{f_{\pi_n}})$ defined as above, we have*

$$\Lambda(f_0, \overline{f_{\pi_n}})(k) = \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1})} \overline{f_{\pi_n}} \quad \text{for odd } n$$

and

$$\Lambda(f_0, \overline{f_{\pi_n}})(k) = \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{n/2} L(2j, \Xi_l^2 \otimes \chi_0^{-1})} \overline{f_{\pi_n}} \quad \text{for even } n.$$

Proof. Recall,

$$\Lambda(f_0, \overline{f_{\pi_n}})(k_{n+2}) := \int_{\overline{G_n}} \hat{f}_0(g_n, k_{n+2}) \overline{\pi_n}(g_n) \overline{f_{\pi_n}} dg_n.$$

Since f_0 is a spherical vector, Λ is constant on K_{n+2} . Denote

$$G^\square = \mathrm{SO}(V \oplus V^-)$$

which is isomorphic to SO_{2n} . Define $\hat{f}_0$ to be f_0 restricted to $G^\square$. In viewing $f_0 \in I(\Xi_l)$, then f_0 is a function on $\mathrm{SO}(W)$ to the complex numbers such that

$$f_0(pg) = \Xi_l \chi_0^{-1/2} \circ \det_{V^\dagger}(m) \delta_P^{1/2} f_0(g).$$

Let, $p \in P_{G^\square} \cap G^\square$ and let $g \in G^\square$. Then,

$$\begin{aligned} \hat{f}_0(pg) &= \Xi_l \chi_0^{-1/2} \circ \det_{V^\dagger}(p) \delta_P^{1/2}(p) \hat{f}_0(g) \\ &= \Xi_l \chi_0^{-1/2} \circ \det_{V^\dagger}(p) \left(\frac{\delta_P}{\delta_P \cap G^\square} \right)^{1/2} (p) \hat{f}_0(g) (\delta_P \cap G^\square)^{1/2}. \end{aligned}$$

The induction is normalized, so

$$\hat{f}_0 \in \mathrm{Ind}_{P_{G^\square} \cap G^\square}^{G^\square} \left(\frac{\delta_P}{\delta_P \cap G^\square} \right)^{1/2} \Xi_l \otimes \chi_0^{-1/2} \circ \det_{V^\dagger}.$$

We know that

$$\delta_P(p) = |\det_{V^\dagger}(p)|^n.$$

To calculate $\delta_{P \cap G^\square}(p)$, note that $P = \mathrm{Stab}(V^\dagger) = \mathrm{Span}\{v_i \oplus -v_i, e\}$. Hence, $P \cap G^\square = \mathrm{Stab}(\mathrm{Span}\{v_i \oplus -v_i\}) = P_{G^\square}$. Hence,

$$\delta_{P \cap G^\square}(p) = |\det_{V^\dagger}(p)|^{n-1}.$$

Using Proposition 3 from [LR05], and noting that

$$\Lambda(f_0, \overline{f_{\pi_n}})(1) = \Lambda(\hat{f}_0, \overline{f_{\pi_n}})(1),$$

the proposition follows. ■

The previous proposition gives the following inductive relationship between $\zeta(\Xi, \xi)$ and $\zeta(\xi, \tilde{\Xi})$.

Proposition 3.8. *For $n \geq 1$, we have*

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})} \times \begin{cases} L(1, \Xi_l^2 \otimes \chi_0^{-1})^{-1} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We consider the integral

$$\int_{\overline{G_{n+1}}} \hat{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) f_{\pi_n}(g_n) \omega^{-1}(\nu(g_n)) dg_n dg_{n+1}.$$

We will utilize Proposition 3 from [LR05] again, but they consider an integral over $G^\diamond \backslash (G \times G)$ where $G^\diamond = \{(g, g) \in G \times G\}$. The map between the sets $G^\diamond \backslash (G \times G) \rightarrow G$ defined by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is both injective and surjective. We use this isomorphism to consider the above integral instead. To see this, we follow the proof of R.N. Harris in [Har14], modifying to our case, and consider the pairing $\mathcal{T} : \pi_{n+1} \otimes \pi_n \rightarrow \mathbb{C}$ given by

$$\mathcal{T}(f_1, f_2) := \int_{\overline{G_n}} f_1(\eta_{n+1} g_n) f_2(g_n) \omega^{-1}(\nu(g_{n+1} g_n)) dg_n.$$

Recall that $T_{\Xi, \xi} = \zeta(\Xi, \xi)$, so $\mathcal{T}(f_{\pi_{n+1}}, f_{\pi_n}) = \zeta(\xi, \tilde{\Xi})$. Also, note that $g_{n+1} \mapsto \tilde{f}_0(1, g_{n+1})$ is bi-invariant under K_{n+1} . Choosing the Haar measure so that the volume on K_{n+1} is 1, we have

$$\begin{aligned} & \int_{\overline{G_{n+1}}} \tilde{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \mathcal{T}(g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dg_{n+1} \\ &= \int_{\overline{G_{n+1}}} \int_{K_{n+1}} \tilde{f}_0(1, k g_{n+1}) \chi_{n+1}^{-1}(\nu(k g_n^{-1} g_{n+1})) \mathcal{T}(g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dk dg_{n+1} \\ &= \int_{\overline{G_{n+1}}} \int_{K_{n+1}} \tilde{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \mathcal{T}(k^{-1} g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dk dg_{n+1} \\ &= \int_{\overline{G_{n+1}}} \tilde{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \int_{K_{n+1}} \mathcal{T}(k^{-1} g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dk dg_{n+1}. \end{aligned}$$

The inner integral gives a smooth linear form on π_{n+1} . More specifically,

$$f \mapsto \int_{K_{n+1}} \mathcal{T}(k^{-1} f, f_{\pi_n})$$

gives a map $L \in \pi_{n+1}^\vee$. Indeed, $L(g_{n+1}f_{\pi_{n+1}})$ gives a matrix coefficient on π_{n+1} which is bi-invariant under K_{n+1} . Thus, there is a constant A such that

$$L(g_{n+1}f_{\pi_{n+1}}) = A \cdot \mathcal{B}_{\pi_{n+1}}(g_{n+1}f_{\pi_{n+1}}, f_{\pi_{n+1}}).$$

To compute A , take $g_{n+1} = 1$ and we obtain $L(f_{\pi_{n+1}}) = A$ so $L(f_{\pi_{n+1}}) = \mathcal{T}(f_{\pi_{n+1}}, f_{\pi_n})$

and hence $\zeta(\xi, \tilde{\Xi}) = A$. Now using the result of [LR05], we have

$$\begin{aligned} & \int_{\overline{G_{n+1}}} \hat{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) f_{\pi_n}(g_n) \omega^{-1}(\nu(g_n)) dg_n dg_{n+1} \\ &= \frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{(n+1)/2} \zeta(\xi, \tilde{\Xi}) \\ & \quad \prod_{j=1} L(2j-1, \Xi_l^2 \otimes \chi_0^{-1}) \end{aligned}$$

for n odd, and

$$\begin{aligned} & \int_{\overline{G_{n+1}}} \hat{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) f_{\pi_n}(g_n) \omega^{-1}(\nu(g_n)) dg_n dg_{n+1} \\ &= \frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{n/2} \zeta(\xi, \tilde{\Xi}) \\ & \quad \prod_{j=1} L(2j, \Xi_l^2 \otimes \chi_0^{-1}) \end{aligned}$$

for n even. Using Proposition 3.8, we have

$$\begin{aligned} & \int_{\overline{G_{n+1}}} \hat{f}_0(1, g_{n+1}) \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) f_{\pi_n}(g_n) \\ & \quad \cdot \omega^{-1}(\nu(g_n)) dg_n dg_{n+1} \\ &= \int_{\overline{G_{n+1}}} \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) \hat{f}_0(g_n, g_n g_{n+1}) f_{\pi_n}(g_n) \\ & \quad \cdot \omega^{-1}(\nu(g_n)) \chi_{n+1}^{-1}(\nu(g_{n+1})) dg_n dg_{n+1} \\ &= \int_{\overline{G_n}} \int_{\overline{G_n}} f_{\pi_{n+1}}(\eta_{n+1} g_{n+1}) \hat{f}_0(g_n, g_{n+1}) f_{\pi_n}(g_n) \omega^{-1}(\nu(g_n)) \\ & \quad \cdot \chi_{n+1}^{-1}(\nu(g_n^{-1} g_{n+1})) dg_n dg_{n+1} \end{aligned}$$

$$\begin{aligned}
&= \int_{G_{n+1}} \int_{G_n} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \hat{f}_0(g_n, g_{n+1}) f_{\pi_n}(g_n) \omega^{-1}(\nu(g_n)) \chi_{n+1}(\nu(g_n)) \\
&\quad \cdot \chi_{n+1}^{-1}(\nu(g_{n+1})) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \chi_{n+1}^{-1}(\nu(g_{n+1})) \int_{G_n} \hat{f}_0(g_n, g_{n+1}) \pi_n(1) f_{\pi_n}(g_n) \\
&\quad \omega^{-1}(\nu(g_n)) \chi_{n+1}(\nu(g_n)) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \chi_{n+1}^{-1}(\nu(g_{n+1})) \int_{G_n} \hat{f}_0(g_n, g_{n+1}) \pi_n(1) \overline{f_{\pi_n}}(g_n) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \chi_{n+1}^{-1}(\nu(g_{n+1})) \Lambda(\hat{f}_0, \overline{f_{\pi_n}})(g_{n+1})(1) dg_{n+1} \\
&= \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1})} \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \chi_{n+1}^{-1}(\nu(g_{n+1})) \\
&= \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, (\Xi_l^2 \otimes \chi_0^{-1})} \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) f_{\pi_{n+2}}(g_{n+1}) \chi_{n+1}(\nu(g_{n+1})) \\
&\quad \cdot \omega^{-1}(\nu(g_{n+1})) \chi_{n+1}^{-1}(\nu(g_{n+1})) dg_{n+1} \\
&= \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1/2})} \int_{G_{n+1}} f_{\pi_{n+1}}(g_{n+1}) f_{\pi_{n+2}}(\eta_{n+1}^{-1}g_{n+1}) \\
&\quad \cdot \omega^{-1}(\nu(\eta_{n+1}^{-1}g_{n+1})) dg_{n+1} \\
&= \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1})} T_{\Xi, \xi}(\eta_{n+1}^{-1}) = \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1})} \zeta(\Xi, \xi)
\end{aligned}$$

for n odd. Since $T_{\Xi, \xi}$ is independent of the choice of η , $T_{\Xi, \xi}(\eta_{n+1}^{-1}) = \zeta(\Xi, \xi)$. Putting these results together, we have

$$\frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n+1)/2} L(2j-1, \Xi_l^2 \otimes \chi_0^{-1})} \zeta(\xi, \tilde{\Xi}) = \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{(n-1)/2} L(1+2j, \Xi_l^2 \otimes \chi_0^{-1})} \zeta(\Xi, \xi).$$

Solving for $\zeta(\Xi, \xi)$ completes the proof for n odd.

Similarly for n even, we have

$$\frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{n/2} L(2j, \Xi_l^2 \otimes \chi_0^{-1})} \zeta(\xi, \tilde{\Xi}) = \frac{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})}{\prod_{j=1}^{n/2} L(2j, \Xi_l^2 \otimes \chi_0^{-1})} \zeta(\Xi, \xi).$$

Solving for $\zeta(\Xi, \xi)$, the proposition follows. ■

By induction, we have the following Corollary.

Corollary 3.9. *Suppose $E = F$. If n is odd, then*

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^{(n+1)/2} L(1, \Xi_i^2 \chi_0^{-1})^{-1} \prod_{1 \leq i < j \leq (n+1)/2} L(1, \Xi_i \Xi_j \chi_0^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\ &\times \prod_{1 \leq i < j \leq (n+1)/2} L(1, \xi_i \xi_j \chi_1^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\ &\times \prod_{1 \leq i < j \leq (n+1)/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) \\ &\times \prod_{1 \leq j \leq i \leq (n+1)/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}). \end{aligned}$$

If n is even, then

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{1 \leq i < j \leq n/2+1} L(1, \Xi_i \Xi_j \chi_0^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\ &\times \prod_{i=1}^{n/2} L(1, \xi_i^2 \chi_1^{-1})^{-1} \prod_{1 \leq i < j \leq n/2} L(1, \xi_i \xi_j \chi_1^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\ &\times \prod_{1 \leq i < j \leq n/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) \\ &\times \prod_{1 \leq j \leq i \leq n/2+1} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}). \end{aligned}$$

Proof. The base case deals with G_1 and $G_1 = \text{GSpin}_1 = \text{GL}(1)$. By Lemma 2.5

$A_1 = \text{GL}(1)$, so $\overline{G_1}$ is trivial. The induction step follows from the previous proposition.

■

3.2.2 Calculation of $S_{\Xi^{-1}, \xi^{-1}}$ when $E = F$

On the other hand,

$$S_{\Xi^{-1}, \xi^{-1}}(1) = \int_{K_{n+2} \times K_{n+1}} Y_{\Xi, \xi}(k_{n+2} k_{n+1}) dk_{n+2} dk_{n+1}.$$

The similitude character is trivial on compact sets. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} Y_{\Xi, \xi} : K_{n+2} \times K_{n+1} & \xrightarrow{\quad} & \mathbb{C} \\ \downarrow & \nearrow & \\ (A_{n+2} \cap K_{n+2}) \backslash K_{n+2} \times (A_{n+1} \cap K_{n+1}) \backslash K_{n+1} & & . \end{array}$$

Hence, $Y_{\Xi, \xi}^{\text{GSpin}_m}$ restricted to $K_{n+2}^{\text{GSpin}} \times K_{n+1}^{\text{GSpin}}$ is isomorphic to $Y_{\Xi', \xi'}^{\text{SO}}$ restricted to $K_{n+2}^{\text{SO}} \times K_{n+1}^{\text{SO}}$. Here if $\xi = (\xi_1, \dots, \xi_{\lfloor (n+1)/2 \rfloor}, \chi_1)$ and $\Xi = (\Xi_1, \dots, \Xi_{\lfloor (n+2)/2 \rfloor}, \chi_0)$ are representations on GSpin , then $\xi' := (\xi_1, \dots, \xi_{\lfloor (n+1)/2 \rfloor})$ and $\Xi' := (\Xi_1, \dots, \Xi_{\lfloor (n+2)/2 \rfloor})$ are representations on SO .

Since both G_{n+2} and G_{n+1} are split and there exists the above relationship with the special orthogonal groups, we use [KMS03, Thm 10.8] to give the following:

$$\begin{aligned} S_{\Xi^{-1}, \xi^{-1}}(1) &= \Delta_{G_{n+2}} \zeta(1)^{-(n+1)} \prod_{i=1}^{(n+1)/2} L(1, \Xi_i^{-2} \chi_0)^{-1} \\ &\times \prod_{1 \leq i < j \leq (n+1)/2} L(1, \Xi_i^{-1} \Xi_j^{-1} \chi_0) L(1, \Xi_i \Xi_j^{-1})^{-1} \\ &\times \prod_{1 \leq i < j \leq (n+1)/2} L(1, \xi_i^{-1} \xi_j^{-1} \chi_1)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ &\times \prod_{1 \leq i < j \leq (n+1)/2} L(1/2, \Xi_i^{-1} \xi_j^{-1} \chi_0^{1/2} \chi_1^{1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}) \\ &\times \prod_{1 \leq j \leq i \leq (n+1)/2} L(1/2, \Xi_i^{-1} \xi_j^{-1} \chi_0^{1/2} \chi_1^{1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) \end{aligned}$$

for n odd, and

$$S_{\Xi^{-1}, \xi^{-1}}(1) = \Delta_{G_{n+2}} \zeta(1)^{-(n+1)} \prod_{i=1}^{n/2} L(1, \xi_i^{-2} \chi_1)^{-1}$$

$$\begin{aligned}
& \times \prod_{1 \leq i < j \leq n/2+1} L(1, \Xi_i^{-1} \Xi_j^{-1} \chi_0) L(1, \Xi_i \Xi_j^{-1})^{-1} \\
& \times \prod_{1 \leq i < j \leq n/2} L(1, \xi_i^{-1} \xi_j^{-1} \chi_1)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\
& \times \prod_{1 \leq i < j \leq n/2} L(1/2, \Xi_i^{-1} \xi_j^{-1} \chi_0^{1/2} \chi_1^{1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}) \\
& \times \prod_{1 \leq j \leq i \leq n/2+1} L(1/2, \Xi_i^{-1} \xi_j^{-1} \chi_0^{1/2} \chi_1^{1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2})
\end{aligned}$$

for n even and

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4) \cdots \zeta(2m) & \text{if } \dim V_i = 2m + 1, \\ \zeta(2)\zeta(4) \cdots \zeta(2m - 2) & \text{if } \dim V_i = 2m. \end{cases}$$

If n is odd, let $n + 2 = 2m + 1$. From (2.3) we have

$$\begin{aligned}
L(s, \pi_{n+2}, \text{Ad}) &= \left(\frac{1}{1 - q^{-s}} \right)^m \prod_{i \neq j} \frac{1}{1 - \Xi_i^{\pm 1} \Xi_j^{\mp 1} q^{-s}} \cdot \frac{1}{1 - \Xi_i^{\pm 1} \Xi_j^{\pm 1} \chi_0^{\mp 1} q^{-s}} \\
&\times \prod_{1 \leq i \leq (n+1)/2} \frac{1}{1 - \Xi_i^2 \chi_0^{-1} q^{-s}} \cdot \frac{1}{1 - \Xi_i^{-2} \chi_0 q^{-s}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
L(s, \pi_{n+2}, \text{Ad}) &= \zeta(s)^{(m)} \prod_{1 \leq i, j \leq (n+1)/2} L(s, \Xi_i \Xi_j^{-1}) L(s, \Xi_i \Xi_j \chi_0^{-1}) \\
&\times \prod_{1 \leq i, j \leq (n+1)/2} L(s, \Xi_i^{-1} \Xi_j) L(s, \Xi_i^{-1} \Xi_j^{-1} \chi_0) \\
&\times \prod_{1 \leq i \leq (n+1)/2} L(s, \Xi_i^2 \chi_0^{-1}) L(s, \Xi_i^{-2} \chi_0).
\end{aligned}$$

For $n + 1 = 2m$, we use (2.4) and we have

$$L(s, \pi_{n+1}, \text{Ad}) = \left(\frac{1}{1 - q^{-s}} \right)^m \prod_{i \neq j} \frac{1}{1 - q^{-s} \xi_i^{\pm 1} \xi_j^{\mp 1}} \cdot \frac{1}{1 - q^{-s} \xi_i^{\pm 1} \xi_j^{\pm 1} \chi_0^{\mp 1}}.$$

Hence,

$$\begin{aligned}
L(s, \pi_{n+1}, \text{Ad}) &= \zeta(s)^{(m)} \prod_{1 \leq i, j \leq (n+1)/2} L(s, \xi_i \xi_j^{-1}) L(s, \xi_i \xi_j \chi_1^{-1}) \\
&\times \prod_{1 \leq i, j \leq (n+1)/2} L(s, \xi_i^{-1} \xi_j) L(s, \xi_i^{-1} \xi_j^{-1} \chi_1).
\end{aligned}$$

Similarly for $n = 2m$, we have

$$\begin{aligned} L(s, \pi_{n+2}, \text{Ad}) &= \zeta(s)^{(m)} \prod_{i \leq i < j \leq n/2} L(s, \Xi_i \Xi_j^{-1}) L(s, \Xi_i \Xi_j \chi_0^{-1}) \\ &\times \prod_{i \leq i < j \leq n/2} L(s, \Xi_i^{-1} \Xi_j) L(s, \Xi_i^{-1} \Xi_j \chi_0). \end{aligned}$$

and

$$\begin{aligned} L(s, \pi_{n+1}, \text{Ad}) &= \zeta(s)^{(m)} \prod_{1 \leq i, j \leq (n+1)/2} L(s, \xi_i \xi_j^{-1}) L(s, \xi_i \xi_j \chi_1^{-1}) \\ &\times \prod_{1 \leq i, j \leq (n+1)/2} L(s, \xi_i^{-1} \xi_j) L(s, \xi_i^{-1} \xi_j \chi_1) \\ &\times \prod_{1 \leq i \leq (n+1)/2} L(s, \xi_i^2 \chi_1^{-1}) L(s, \xi_i^{-2} \chi_1). \end{aligned}$$

We also have,

$$\begin{aligned} L(s, \pi_{n+2} \times \pi_{n+1}) &= \prod_{1 \leq i, j \leq \lfloor (n+1)/2 \rfloor} L(s, \Xi_i \xi_j) L(s, \Xi_i^{-1} \xi_j \chi_0) \\ &\times L(s, \Xi_i \xi_j^{-1} \chi_1) L(s, \Xi_i^{-1} \xi_j^{-1} \chi_0 \chi_1). \end{aligned}$$

Tensoring by ω^{-1} gives

$$\begin{aligned} L(s, \pi_{n+2} \otimes \omega^{-1} \times \pi_{n+1}) &= \\ &\prod_{1 \leq i, j \leq \lfloor (n+1)/2 \rfloor} L(s, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(s, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}) \\ &\times \prod_{1 \leq i, j \leq \lfloor (n+1)/2 \rfloor} L(s, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) L(s, \Xi_i^{-1} \xi_j^{-1} \chi_0^{1/2} \chi_1^{1/2}). \end{aligned}$$

Thus, for each $v \notin S$,

$$\begin{aligned} \mathcal{P}^\natural(f_{\pi_{n+2}}, f_{\pi_{n+1}}) &= F(1) = \zeta(\Xi, \xi) S_{\Xi^{-1}, \xi^{-1}}(1) \\ &= \zeta(\Xi, \xi) \zeta(\Xi^{-1}, \xi^{-1}) \Delta_{G_{n+2}} \zeta(1)^{-(n+1)}. \\ &= \Delta_{G_{n+2}} \frac{L(1/2, \pi_{n+2} \otimes \omega^{-1} \times \pi_{n+1})}{L(1, \pi_{n+2}, \text{Ad}) (L(1, \pi_{n+1}, \text{Ad}))}. \end{aligned}$$

where

$$\Delta_{G_i} = \begin{cases} \zeta(2) \zeta(4) \cdots \zeta(2m) & \text{if } \dim V_i = 2m + 1, \\ \zeta(2) \zeta(4) \cdots \zeta(2m - 2) L(m, \mu) & \text{if } \dim V_i = 2m. \end{cases}$$

3.2.3 Calculations when E/F is an unramified quadratic extension

There are quasi-split non-split forms in the even case. Note that these do not exist in the odd case. As in [AS14], fix a splitting $(B, T, \{x_\alpha\}_{\alpha \in \Delta})$ where $\{x_\alpha\}$ is a collection of root vectors, one for each simple root of T in B . The quasi-split forms of $\mathrm{GSpin}(2m)$ over F are in one-to-one correspondence with homomorphisms from $\mathrm{Gal}(\overline{F}/F)$ to the group of automorphisms of the character lattice preserving Δ . The only two elements in this group are the trivial element and the one switching the roots $e_{n-1} - e_n$ and $e_{n-1} + e_n$ while keeping all other simple roots fixed. As we had stated before, class field theory tells us such homomorphisms correspond to quadratic characters

$$\mu : F^\times \rightarrow \{\pm 1\}.$$

Note that

$$\mathcal{A}_{n+2} \simeq (F^\times)^{[(n+1)/2]}, \quad \mathcal{A}_{n+1} \simeq (F^\times)^{[n/2]}.$$

The unramified characters Ξ and ξ are determined by their restriction to their associated maximal split subtorus. Let

$$\Xi = (\Xi_1, \Xi_2, \dots, \Xi_{[(n+1)/2]}, \chi_0)$$

$$\xi = (\xi_1, \xi_2, \dots, \xi_{[n/2]}, \chi_1).$$

Put $\tilde{\Xi} = (\Xi_1, \Xi_2, \dots, \Xi_{[(n-1)/2]}, \chi_0)$. Set $\zeta(\Xi, \xi) = T_{\Xi, \xi}(\eta)$.

Hence we have the same result as before. Namely, for a positive integer n ,

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, \overline{\pi_{n+1}} \otimes \Xi_l \otimes \chi_0^{-1/2})}{L(1, \overline{\pi_n} \otimes \Xi_l \otimes \chi_0^{-1/2})} \times \begin{cases} L(1, \Xi_l^2 \otimes \chi_0^{-1})^{-1} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Let $n = 2m$. Since π_{n+2} is unramified and G_{n+2} is quasi-split, but not split, there is a one-to-one correspondence between π_{n+2} and the Satake parameters for G_{n+2} given

by [AS14, Equation 3.32]

$$\begin{pmatrix} \Xi_1(\pi) & & & & & \\ & \ddots & & & & \\ & & \Xi_{n/2}(\pi) & & & \\ & & & \Xi_{n/2+1}(\pi)\mu & & \\ & & & & \Xi_{n/2+1}^{-1}(\pi)\chi_0(\pi) & \\ & & & & & \ddots \\ & & & & & & \Xi_1^{-1}(\pi)\chi_0(\pi) \end{pmatrix}$$

where μ is the quadratic character associated to E/F . By induction, when n is odd we have,

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^{(n+1)/2} L(1, \Xi_i^2 \chi_0^{-1})^{-1} \prod_{1 \leq i < j \leq (n+1)/2} L(1, \Xi_i \Xi_j \chi_0^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\ &\times \prod_{i=1}^{(n-1)/2} L(1, \xi_i \chi_1^{-1/2}) L(1, \mu \xi_i \chi_1^{-1/2}) \times \prod_{1 \leq i < j \leq (n-1)/2} L(1, \xi_i \xi_j \chi_1^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\ &\times \prod_{1 \leq i < j \leq (n-1)/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) \\ &\times \prod_{i=1}^{(n+1)/2} L(1/2, \Xi_i \chi_0^{1/2}) L(1/2, \mu \Xi_i \chi_0^{1/2}) \\ &\times \prod_{1 \leq j \leq i \leq (n+1)/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}). \end{aligned}$$

When n is even,

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^{n/2} L(1, \Xi_i \chi_0^{1/2})^{-1} L(1, \mu \Xi_i \chi_0^{1/2})^{-1} \\ &\prod_{1 \leq i < j \leq n/2} L(1, \Xi_i \Xi_j \chi_0^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\ &\times \prod_{i=1}^{n/2} L(1, \xi_i^2 \chi_1^{-1})^{-1} \prod_{1 \leq i < j \leq n/2} L(1, \xi_i \xi_j \chi_1^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\ &\prod_{1 \leq j \leq i \leq n/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i \xi_j^{-1} \chi_0^{-1/2} \chi_1^{1/2}) \\ &\times \prod_{1 \leq i < j \leq n/2} L(1/2, \Xi_i \xi_j \chi_0^{-1/2} \chi_1^{-1/2}) L(1/2, \Xi_i^{-1} \xi_j \chi_0^{1/2} \chi_1^{-1/2}) \\ &\times \prod_{i=1}^{n/2} L(1/2, \xi_i) L(1/2, \mu \xi_i). \end{aligned}$$

Using Lemma 5.1 in [II10], we have that

$$S_{\Xi, \xi}(1) = \Delta_{G_{n+2}} \zeta(1)^{-\dim \mathcal{A}_{n+2} - \dim \mathcal{A}_{n+1}} L(1, \mu)^{-1} \zeta(\Xi, \xi).$$

Note that,

$$L_{\pi_{n+2}, \pi_{n+1}}(1/2) = \zeta(1)^{-\dim(\mathcal{A}_{n+2}+1) - \dim(\mathcal{A}_{n+1}+1)} L(1, \mu)^{-1} \zeta(\Xi, \xi) \zeta(\Xi^{-1}, \xi^{-1}).$$

Using the results from the previous sections, we have the following theorem.

Theorem 3.10. *For each $v \notin S$*

$$\begin{aligned} \mathcal{P}^{\natural}(f_{\pi_{n+2}}, f_{\pi_{n+1}}) &= F(1) = \zeta(\Xi, \xi) S_{\Xi^{-1}, \xi^{-1}}(1) \\ &= \Delta_{G_{n+2}} \frac{L(1/2, \pi_{n+2} \otimes \omega^{-1} \times \pi_{n+1})}{L(1, \pi_{n+2}, \text{Ad}) L(1, \pi_{n+1}, \text{Ad})}. \end{aligned}$$

Chapter 4

The Global Gan-Gross-Prasad Conjecture for $\mathrm{GSpin}_2 \times \mathrm{GSpin}_3$

Let F be an algebraic number field with adele ring $\mathbb{A}_F$. Denote by D the quaternion algebra over F and define E as a quadratic field extension of F with a fixed embedding $E \hookrightarrow D$ over F . We consider the following exceptional isomorphisms of GSpin [AS14, section 4.1]:

$$\mathrm{GSpin}_2 = \begin{cases} \mathrm{Res}_{E/F} \mathrm{GL}_1 & \text{if quasi-split} \\ \mathrm{GL}_1 \times \mathrm{GL}_1 & \text{if split} \end{cases}$$

and

$$\mathrm{GSpin}_3 = \begin{cases} \mathrm{GL}_2 & \text{if } D = M_2(F) \\ D^\times & \text{if } D \text{ is non-split.} \end{cases}$$

We use the Waldspurger formula [Wal85] to prove the cases where GSpin_2 is

$\mathrm{Res}_{E/F} \mathrm{GL}_1$ and D is either split or non-split. The case where $\mathrm{GL}_1 \times \mathrm{GL}_1 \subseteq D^\times$ does not occur. When GSpin_2 is split, there is an embedding $\mathrm{GL}_1 \times \mathrm{GL}_1 \subseteq \mathrm{GL}_2$. We work through this case, but it does not confirm Conjecture 1.3 or 1.4.

4.1 Waldspurger Formula

The following example is due to Waldspurger [Wal85]. First, we give a brief discussion of the Waldspurger formula ([Wal85]). Another good source for the Waldspurger formula is [YZZ13, Ch. 1].

4.1.1 Linear forms over local fields

Let F be a local field and D a quaternion algebra over F . Then D is either isomorphic to $M_2(F)$ or the unique division quaternion algebra over F . Let E be either $F \oplus F$ or a quadratic field extension over F , with a fixed embedding $E \hookrightarrow D$ of algebras over F . Let μ be the quadratic character associated to E/F by class field theory. Let π be an irreducible cuspidal representation of $D^\times$ with central character ω_π , and let $\chi : E^\times \rightarrow \mathbb{C}$ be a character on $E^\times$ such that $\omega_\pi \cdot \chi$ restricted to $F^\times$ is trivial.

Fix a Haar measure dt on $E^\times/F^\times$. Define a bilinear form

$$\alpha : \pi \otimes \tilde{\pi} \rightarrow \mathbb{C}$$

by

$$\alpha(f_1, f_2) = \frac{L(1, \pi, \text{Ad})L(1, \mu)}{\zeta(2)L(1/2, \pi \otimes \chi)} \int_{F^\times \backslash E^\times} \mathcal{B}_\pi(\pi(t)f_1, f_2) dt, \quad f_1 \in \pi, f_2 \in \tilde{\pi}.$$

where $\mathcal{B}_\pi(\pi(t)f_1, f_2)$ is the fixed inner product discussed in chapter one.

4.1.2 Waldspurger formula

Now let F be a number field with adele ring $\mathbb{A}_F$. Denote by D the quaternion algebra over F and define E as a quadratic field extension of F with fixed embedding $E \hookrightarrow D$ over F . Let $\mu : \mathbb{A}_F^\times/F^\times \rightarrow \mathbb{C}$ be the quadratic character associated to the extension E/F . Let π be an irreducible cuspidal automorphic representation of $D^\times$ with central character ω_π and χ a Hecke character on $\mathbb{A}_E^\times/E^\times$. Define the period integral

$$\widetilde{P}_\chi(f) = \left| \int_{E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}^\times} f(t) \chi(t) dt \right|^2$$

where dt is the Tamagawa measure, which has total volume 2 on $E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}^\times$. Let

$\mathcal{B}_{\pi_v}$ and $\mathcal{B}_{\chi_v}$ be local pairings. We choose local measures dt_v so that $dt = \prod_v dt_v$. Let

$\mathcal{B}(\cdot, \cdot)$ be the Petersson inner product on π . That is,

$$\mathcal{B}_\pi(\pi(t)f_1, f_2) = \int_{D^\times \backslash D(\mathbb{A}^\times)/\mathbb{A}^\times} f_1(b) \overline{f_2(b)} db, \quad f_1 \in \pi, f_2 \in \tilde{\pi}$$

where db is the Tamagawa measure. Let $BC(\pi)$ denote the base change of π to E .

Theorem 4.1. (*Waldspurger [Wal85]*) *For any $f_1 \in \pi$ and $\overline{f_1} \in \tilde{\pi}$, we have*

$$\widetilde{P}_\chi(f_1) = \frac{\zeta(2)L(1/2, BC(\pi) \otimes \chi)}{4L(1, \pi, \text{Ad})L(1, \mu)^2} \alpha(f_1 \otimes \overline{f_1})$$

The global bilinear form

$$\alpha : \pi \times \tilde{\pi} \rightarrow \mathbb{C}$$

is the tensor product of the local bilinear forms $\alpha : \pi_v \otimes \tilde{\pi}_v \rightarrow \mathbb{C}$ introduced in the last section.

Remark 4.2. The Waldspurger formula above looks slightly different from the original in [Wal85] or from the Waldspurger formula listed in [YZZ13]. Waldspurger chose the global Haar measure used in the period integral such that the $\text{Vol}(E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}^\times) = 2L(1, \mu)$. The authors of [YZZ13] chose the global Haar measure such that the volume is 1. Then each chose local measures to be compatible with their choice of global measure. With our choice of measures, our formulation is equivalent.

Remark 4.3. Waldspurger assumed the central character of π is trivial, but the authors of [YZZ13] removed this condition.

4.2 Proof of Conjecture 1.3 for $\text{GSpin}_2 \times \text{GSpin}_3$: Waldspurger Formula Case

As in the last section, let F be a number field with adele ring $\mathbb{A}_F$. Denote D to be the quaternion algebra over F and define E as a quadratic field extension of F with

fixed embedding $E \hookrightarrow D$ over F . Let $\mu : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}$ be the quadratic character associated to the extension E/F . Denote the completed zeta function of F as $\xi_F(s)$.

As stated in the introduction of this chapter, here we consider the case where GSpin_2 is quasi-split, but not split, so $\mathrm{GSpin}_2 \cong \mathrm{Res}_{E/F} \mathrm{GL}_1$ and

$$\mathrm{GSpin}_3 = \begin{cases} \mathrm{GL}_2 & \text{if } D = M_2(F) \\ D^\times & \text{if } D \text{ is non-split.} \end{cases}$$

The quotient $D^\times / \mathrm{GL}_1$ can be considered as a special orthogonal group associated to a 3-dimensional quadratic space over F . Recall that $\Delta_{\mathrm{G}_3} = \xi_F(2)$. Let $T = \mathrm{SO}_2 \cong A_2 \backslash \mathrm{GSpin}_2$ be an anisotropic torus of $A_3 \backslash \mathrm{GSpin}_3$. Let E be a splitting field of T over F . One has the following exact sequences:

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{Res}_{E/F} \mathrm{GL}_1 \longrightarrow T \longrightarrow 1$$

$$\left(1 \longrightarrow A_2 \longrightarrow \mathrm{GSpin}_2 \longrightarrow \mathrm{SO}_2 \longrightarrow 1 \right)$$

and

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow D^\times \longrightarrow D^\times / \mathrm{GL}_1 \longrightarrow 1$$

$$\left(1 \longrightarrow A_3 \longrightarrow \mathrm{GSpin}_3 \longrightarrow \mathrm{SO}_3 \longrightarrow 1 \right)$$

Let π be an irreducible cuspidal automorphic representation of $D^\times$ with central character ω_π and χ a Hecke character on $\mathbb{A}_E^\times / E^\times$. Note that

$$\omega_\pi : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times, \quad \chi|_{\mathbb{A}_F^\times} : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times.$$

Assume there exists $\omega : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times$ such that $\omega^2 = \omega_\pi \chi|_{\mathbb{A}_F^\times}$. Then

$$(\omega^{-2} \omega_\pi) \cdot \chi|_{\mathbb{A}^\times} = 1.$$

Define $\pi \otimes \omega^{-1}$ by $(\pi \otimes \omega^{-1})(g) = \pi(g)\omega^{-1}(N(g))$ where N is the norm map

$$N : D^\times \rightarrow \mathbb{A}^\times.$$

Let z be in the center of D . Then

$$\begin{aligned} (\pi \otimes \omega^{-1})(z) &= \pi(z)\omega^{-1}(N(z)) \\ &= \omega_\pi(z)\omega^{-1}(z^2) \\ &= \omega_\pi(z)\omega^{-2}(z) \\ &= \omega_\pi\omega^{-2}(z). \end{aligned}$$

Hence, $\omega_{\pi \otimes \omega^{-1}} = \omega^{-2}\omega_\pi$. Using $\pi \otimes \omega^{-1}$ in Waldspurger's formula we obtain

$$\begin{aligned} \mathcal{P}(f_\pi, f_\chi) &= \frac{\xi_F(2)L(1/2, BC(\pi \otimes \omega^{-1}) \otimes \chi)}{4L(1, \mu)^2 L(1, \pi \otimes \omega^{-1}, \text{Ad})} \prod_v \mathcal{P}(f_{\pi,v}, f_{\chi,v}) \\ &= \frac{\xi_F(2)L(1/2, BC(\pi \otimes \omega^{-1}) \otimes \chi)}{4L(1, \chi, \text{Ad})L(1, \pi, \text{Ad})} \prod_v \mathcal{P}(f_{\pi,v}, f_{\chi,v}) \end{aligned}$$

Note that ([Bum97, pg. 102])

$$L(s, BC(\pi \otimes \omega^{-1}) \otimes \chi) = L(s, \pi \otimes \omega^{-1} \times \theta_\chi)$$

where θ_χ is the automorphic induction of χ . Hence, one has

$$\mathcal{P}(f_\pi, f_\chi) = \frac{\xi_F(2)L(1/2, \pi \otimes \omega^{-1} \times \chi)}{4L(1, \chi, \text{Ad})L(1, \pi, \text{Ad})} \prod_v \mathcal{P}(f_{\pi,v}, f_{\chi,v}),$$

confirming Conjecture 1.3. Moreover, $|\mathcal{S}_{\phi_2}| = |\mathcal{S}_{\phi_3}| = 1$, so $2^\beta = 4|\mathcal{S}_{\phi_2}||\mathcal{S}_{\phi_3}|$, confirming Conjecture 1.4.

4.3 Jacquet-Langlands Case

For the case when GSpin_2 is split, then we have $\text{GL}_1 \times \text{GL}_1 \subseteq \text{GL}_2$. Let π_2 be an irreducible cuspidal automorphic representation of $\text{GL}_1 \times \text{GL}_1$ and set $\pi_2 = \chi_1 \boxtimes \chi_2$.

Let π_3 be an irreducible cuspidal automorphic representation of GL_2 with central character ω_{π_3} . Assume $\omega^2 = \chi_1 \chi_2 \omega_{\pi_3}$ and let ϕ be in π_3 .

First consider the integral,

$$\begin{aligned}
& \int_{[\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)]} \phi \begin{pmatrix} x & \\ & y \end{pmatrix} \chi_1(x) \chi_2(y) \omega(xy)^{-1} dx dy \\
&= \int_{\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)} \phi \left(\begin{pmatrix} y^{-1} & \\ & y^{-1} \end{pmatrix} \begin{pmatrix} x & \\ & y \end{pmatrix} \right) \\
&\quad \cdot \omega_{\pi_3}(y) \chi_1(x) \chi_2(y) \omega(xy)^{-1} dx dy \\
&= \int_{\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)} \phi \begin{pmatrix} xy^{-1} & \\ & 1 \end{pmatrix} \\
&\quad \cdot \omega_{\pi_3}(y) \chi_1(x) \chi_2(y) \omega(xy)^{-1} dx dy
\end{aligned}$$

Since $\omega_{\pi_3}(y) \chi_2(y) = \omega^2(y) \chi_1^{-1}(y) = \omega(y^2) \chi_1^{-1}(y)$, we replace in the integral and use

[Bum97, pg. 338] to obtain

$$\begin{aligned}
& \int_{\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)} \phi \begin{pmatrix} xy^{-1} & \\ & 1 \end{pmatrix} \chi_1(xy^{-1}) \omega(xy^{-1})^{-1} dx dy \\
&= \int_{F^\times \setminus \mathbb{A}^\times} \phi \begin{pmatrix} z & \\ & 1 \end{pmatrix} \chi_1(z) \omega(z)^{-1} dz \\
&= Z(1/2, \pi_3, \chi_1 \omega^{-1}).
\end{aligned}$$

Now consider the following integral, recalling $\tilde{\pi}_3 \cong \pi_3 \otimes \omega_{\pi_3}^{-1}$,

$$\begin{aligned}
& \int_{[\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)]} \overline{\phi \begin{pmatrix} x & \\ & y \end{pmatrix} \chi_1(x) \chi_2(y) \omega(xy)^{-1} dx dy} \\
&= \int_{\Delta \mathrm{GL}_1 \setminus (\mathrm{GL}_1 \times \mathrm{GL}_1)} \bar{\phi} \begin{pmatrix} xy^{-1} & \\ & 1 \end{pmatrix} \chi_1(xy^{-1})^{-1} \omega(xy^{-1}) dx dy \\
&= \int_{F^\times \setminus \mathbb{A}^\times} \bar{\phi} \begin{pmatrix} z & \\ & 1 \end{pmatrix} \chi_1(z)^{-1} \omega(z) dz \\
&= Z(1/2, \tilde{\pi}_3 \times \chi_1^{-1} \omega) = Z(1/2, \pi_3 \times \omega_{\pi_3}^{-1} \chi_1^{-1} \omega) \\
&= Z(1/2, \pi_3 \times \chi_2 \omega^{-1}).
\end{aligned}$$

Per [Wal85, Prop.6], $\mathcal{B}_{\pi_3}(\phi, \phi) = 2\zeta(2)^{-1} L(1, \mathrm{Ad}, \pi_3) \prod \mathcal{W}_{\pi_{3,v}}(\phi_v, \phi_v)$. Note that

$\mathcal{B}_{\pi_2}(\chi_1 \boxtimes \chi_2) = 1$ and $L(s, \pi_2, \mathrm{Ad}) = \zeta(s)^2$. Putting this together, one has

$$\frac{\mathcal{P}(\phi, \chi_1 \boxtimes \chi_2)}{\mathcal{B}_{\pi_3}(\phi, \phi) \mathcal{B}_{\pi_2}(\chi_1 \boxtimes \chi_2)} = \frac{\zeta(2) L(1/2, \pi_3 \otimes \omega^{-1} \times \pi_2)}{2L(1, \text{Ad}, \pi_3)} \prod_v \frac{\mathcal{P}_v(\phi_v, \chi_{1,v} \boxtimes \chi_{2,v})}{\mathcal{W}_{\pi_3}(\phi_v, \phi_v) \mathcal{B}_{\pi_2}(\chi_{1,v} \boxtimes \chi_{2,v})}$$

Moreover, $|\mathcal{S}_{\phi_2}| = |\mathcal{S}_{\phi_3}| = 1$. Note that we do not have $L(s, \pi_2, \text{Ad})$ in the denominator, so this case does not confirm Conjecture 1.3. Moreover, $2^\beta \neq 4|\mathcal{S}_{\phi_2}||\mathcal{S}_{\phi_3}|$ and so this case does not appear to confirm Conjecture 1.4 either.

Bibliography

- [AS06] Mahdi Asgari and Freydoon Shahidi, *Generic transfer for general spin groups*, Duke Math. J. **132** (2006), no. 1, 137–190.
- [AS14] ———, *Image of functoriality for general spin groups*, Manuscripta Math. **144** (2014), no. 3-4, 609–638.
- [Asg02] Mahdi Asgari, *Local L -functions for split spinor groups*, Canad. J. Math. **54** (2002), no. 4, 673–693.
- [Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508
- [CKM04] James W. Cogdell, Henry H. Kim, and M. Ram Murty, *Lectures on automorphic L -functions*, Fields Institute Monographs, vol. 20, American Mathematical Society, Providence, RI, 2004.
- [Del72] Pierre Deligne, *La conjecture de Weil pour les surfaces $K3$* , Invent. Math. **15** (1972), 206–226. MR 0296076
- [GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the*

- representation theory of classical groups*, Astérisque (2012), no. 346, 1–109, Sur les conjectures de Gross et Prasad. I.
- [GI11] Wee Teck Gan and Atsushi Ichino, *On endoscopy and the refined Gross-Prasad conjecture for $(\mathrm{SO}_5, \mathrm{SO}_4)$* , J. Inst. Math. Jussieu **10** (2011), no. 2, 235–324.
- [GJR05] David Ginzburg, Dihua Jiang, and Stephen Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L -functions. II*, Automorphic representations, L -functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 157–191. MR 2192823
- [GP92] Benedict H. Gross and Dipendra Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* , Canad. J. Math. **44** (1992), no. 5, 974–1002.
- [GP94] ———, *On irreducible representations of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$* , Canad. J. Math. **46** (1994), no. 5, 930–950.
- [GPSR97] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis, *L functions for the orthogonal group*, Mem. Amer. Math. Soc. **128** (1997), no. 611, viii+218.
- [Gro97] Benedict H. Gross, *On the motive of a reductive group*, Invent. Math. **130** (1997), no. 2, 287–313.
- [Har14] R. Neal Harris, *The refined Gross-Prasad conjecture for unitary groups*, Int. Math. Res. Not. IMRN (2014), no. 2, 303–389.

- [He03] Hongyu He, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121.
- [Hel00] Sigurdur Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000, Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original. MR 1790156
- [Ich08] Atsushi Ichino, *Trilinear forms and the central values of triple product L -functions*, Duke Math. J. **145** (2008), no. 2, 281–307.
- [II10] Atsushi Ichino and Tamutsu Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425.
- [KMS03] Shin-ichi Kato, Atsushi Murase, and Takashi Sugano, *Whittaker-Shintani functions for orthogonal groups*, Tohoku Math. J. (2) **55** (2003), no. 1, 1–64.
- [LR05] Erez M. Lapid and Stephen Rallis, *On the local factors of representations of classical groups*, Automorphic representations, L -functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 309–359.
- [Roh94] David E. Rohrlich, *Elliptic curves and the Weil-Deligne group*, Elliptic curves and related topics, CRM Proc. Lecture Notes, vol. 4, Amer. Math. Soc., Providence, RI, 1994, pp. 125–157. MR 1260960

- [Sha10] Freydoon Shahidi, *Eisenstein series and automorphic L -functions*, American Mathematical Society Colloquium Publications, vol. 58, American Mathematical Society, Providence, RI, 2010. MR 2683009
- [Sil79] Allan J. Silberger, *Introduction to harmonic analysis on reductive p -adic groups*, Mathematical Notes, vol. 23, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979, Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971–1973.
- [Spr98] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998.
- [Wal85] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242.
- [Wal03] ———, *La formule de Plancherel pour les groupes p -adiques (d’après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333. MR 1989693
- [YZZ13] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang, *The Gross-Zagier formula on Shimura curves*, Annals of Mathematics Studies, vol. 184, Princeton University Press, Princeton, NJ, 2013.

VITA

Melissa Emory was born in San Jose, California to Duanne Leroy Bosma and Sandra Elaine (Finnie) Bosma. She grew up in Las Vegas, Nevada and has also lived in Germany, Ohio, and Nebraska. Melissa received her B.S. and M.A. in mathematics at the University of Nebraska at Omaha in 2011 and 2013, respectively. Currently, Melissa is a Ph.D. candidate in mathematics at the University of Missouri in Columbia, Missouri where she studies number theory and representation theory under the supervision of Professor Shuichiro Takeda. In July 2018, Melissa begins a position as a Postdoctoral Fellow at the University of Toronto under the supervision of Professor James Arthur.