PERSISTENT HOMOLOGY: CATEGORICAL STRUCTURAL THEOREM AND STABILITY THROUGH REPRESENTATIONS OF QUIVERS

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by

KILLIAN MEEHAN

Dr. Calin Chindris, Dissertation Supervisor

Dr. Jan Segert, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Killian Meehan, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Associate Professor Calin Chindris

Associate Professor Jan Segert

Assistant Professor David C. Meyer

Associate Professor Mihail Popescu

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My parents, for teaching me that the best ideas are never found in an intellectual vacuum.

My grandparents, for asking me the questions that led me to where I am.

Contents

ABSTRACT

The purpose of this thesis is to advance the study and application of the field of persistent homology through both categorical and quiver theoretic viewpoints. While persistent homology has its roots in these topics, there is a wealth of material that can still be offered up by using these familiar lenses at new angles.

There are three chapters of results.

Chapter 3 discusses a categorical framework for persistent homology that circumvents quiver theoretic structure, both in practice and in theory, by means of viewing the process as factored through a quotient category. In this chapter, the widely used persistent homology algorithm collectively known as *reduction* is presented in terms of a matrix factorization result.

The remaining results rest on a quiver theoretic approach.

Chapter 4 focuses on an algebraic stability theorem for generalized persistence modules for a certain class of finite posets. Both the class of posets and their discretized nature are what make the results unique, while the structure is taken with inspiration from the work of Ulrich Bauer and Michael Lesnick.

Chapter 5 deals with taking directed limits of posets and the subsequent expansion of categories to show that the discretized work in the second section recovers classical results over the continuum.

Chapter 1

Introduction

This research is motivated by the crossover between pure mathematics and real world problem solving. I have focused largely on topological methods of analyzing data, and in particular the device of *persistent homology* [ZC05a], [EH10]. At its inception, this field gathered algebraic topology, category theory, and quiver theory to provide a new theory of robust analysis of data that is independent of scale. Even in the short time since its inception, persistent homology has grown to give and take from a staggering number of disciplines—mathematical and otherwise.

All the results featured in this manuscript are the result of collaborative efforts. The nature of the material and the coauthors with which it was derived is split into two groups.

The first result (Chapter 3) is from the paper [MPS17] and was written with my dissertation advisor Jan Segert and fellow graduate student Andrei Pavlichenko. We show that the category of filtered chain complexes is in fact Krull-Schmidt, and that this allows for decomposition *before* applying the homology functor. Furthermore, this abstract result—which we name the Categorical Structural Theorem—constitutes a non-constructive proof of the Matrix Structural Theorem, which is the matrix factorization result that is key to *reduction*, a common persistent homology algorithm in the literature. The Categorical Structural Theorem is the foundation for an alternate workflow for the persistent homology process that circumvents quiver theoretic decomposition results entirely, using instead a quotient category to the category of filtered chain complexes.

The last three chapters of results were all obtained in collaboration with postdoctoral researcher David Meyer. They stem from the papers [MM17a] (see Chapter 4) and [MM17b] (see Chapter 5), and Chapter 6 is the result of our ongoing research. All these results are related to algebraic stability between the interleaving metric on persistence modules and bottleneck metrics on barcodes. One overarching goal of all three chapters is to bypass the wall raised by quiver theory on the road to multi-dimensional persistence: that the immaculate decomposition result over the $\mathbb{A}_n$ quiver fails for all but a very small list of additional quivers. In particular, the posets that would be used for multi-dimensional persistence have, as quivers, representation theory that is known to be unsolvable. Through discretization of the usual $\mathbb{R}$-indexed persistent homology and a restriction on the permitted category of indecomposables, we obtain algebraic stability results for a large class of finite posets that are not totally ordered.

Chapter 2

Preliminaries

2.1 Category Theory

Definition 2.1.1. A category $\mathcal{C}$ is:

- a class of objects, $\mathrm{Obj}(\mathcal{C})$, which we will simply denote as $\mathcal{C}$ itself,
- a set of morphisms $\mathrm{Hom}(X, Y)$ for all pairs of objects $X, Y \in \mathcal{C}$,
- an identity morphism $1_x \in \mathrm{Hom}(X, X)$ for all $X \in \mathcal{C}$, and
- a composition map

  $$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

  for all triplets $X, Y, Z \in \mathcal{C}$, such that

  1. for all $\phi \in \mathrm{Hom}(X, Y)$, $1_y \circ \phi = \phi = \phi \circ 1_x$, and
  2. for all

     $$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

     $$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition 2.1.2. We define the following morphism types:

- A *category with zero morphisms* is one in which for every pair of objects X, Y there exists a morphism we will call $0_{X,Y}$ such that, for any triple X, Y, Z and any morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$0_{Y,Z} \circ f = 0_{X,Z} = g \circ 0_{X,Y}.$$

- A *monomorphism* is a homomorphism that cancels on the left.

$$\text{If } f \circ g = f \circ h, \text{ then } f = h.$$

  In a concrete category (one with additional underlying structure), injective maps are monomorphisms.

- An *epimorphism* is a homomorphism that cancels on the right.

$$\text{If } f \circ h = g \circ h, \text{ then } f = g.$$

  In a concrete category, surjective maps are epimorphisms.

Definition 2.1.3. A *functor* $\mathcal{F} : C \to D$ is a map $\mathcal{F} : obj(C) \to obj(D)$ and a collection of maps of morphisms $\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(\mathcal{F}(X), \mathcal{F}(Y))$ for every pair of objects X, Y in C, such that $\mathcal{F}$ satisfies:

- $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ for all X in C, and
- $\mathcal{F}$ commutes with composition. I.e., for any composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ in C,

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(Z).$$

A functor as defined above is also called a *covariant* functor. A *contravariant* functor is defined similarly save that the map of homomorphisms is backwards,

$\mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(\mathcal{F}(Y), \mathcal{F}(X))$, and composition is subsequently reversed as well, $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Definition 2.1.4. A *natural transformation* is a morphism of (covariant) functors. A natural transformation Γ between functors $F, G : C \to D$ is a collection of morphisms $\Gamma(X) : F(X) \to G(X)$ for every object X in C, such that for every X, Y in C and every morphism $f \in \mathrm{Hom}_C(X,Y)$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \Gamma(X)}\downarrow & & \downarrow{\scriptstyle \Gamma(Y)} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

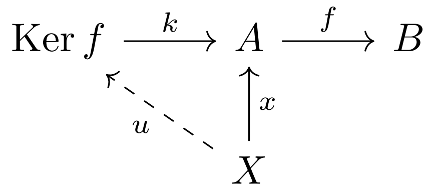
That is, $G(f) \circ \Gamma(X) = \Gamma(Y) \circ F(f)$.

2.1.1 Universal Properties: Kernels and Cokernels

Many basic categorical definitions are phrased as *universal properies:* for us, a pair consisting of an object and morphism(s) that satisfy a given statement. The following definitions are constructions that solve certain universal properties.

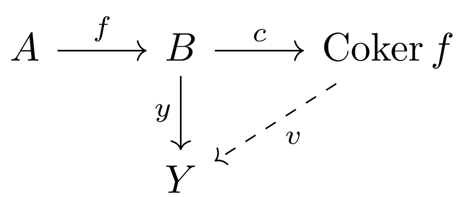
Definition 2.1.5. Let C be a category with zero morphisms. Let $f \in \mathrm{Hom}_C(A, B)$.

- A *kernel* of f is an object $\mathrm{Ker}(f)$ and a morphism $k \in \mathrm{Hom}(\mathrm{Ker}(f), A)$ that satisfy the universal property:
  - $f \circ k$ is a zero morphism, and
  - for any other pair X and $x : X \to A$ such that $f \circ x$ is a zero morphism, x factors through $\mathrm{Ker}(f)$. I.e., there exists a unique morphism $u : X \to \mathrm{Ker}(f)$ such that $k \circ u = x$.



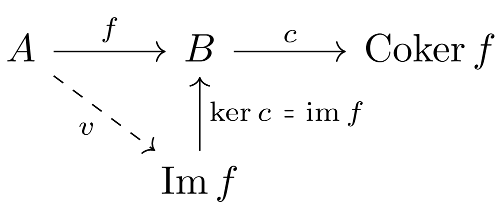
Any kernel morphism is a monomorphism.

- A *cokernel* of f is an object $\mathrm{Coker}(f)$ and a morphism $c \in \mathrm{Hom}(B, \mathrm{Coker}(f))$ that satisfy the universal property:
  - $c \circ f$ is a zero morphism, and
  - for any other pair Y and $y : B \to Y$ such that $y \circ f$ is a zero morphism, y factors through $\mathrm{Coker}(f)$. I.e., there exists a unique morphism $v : \mathrm{Coker}(f) \to Y$ such that $c \circ v = y$.

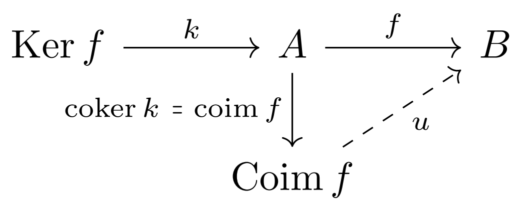


Any cokernel morphism is an epimorphism.

- An *image* of f is a kernel of a cokernel of f. By definition, the pair $(\mathrm{Im}(f), \mathrm{im}(f))$ pre-composes with c to be zero. As this is also true of f, there is a unique morphism $v : A \to \mathrm{Ker}(c)$ such that $\ker(c) \circ v = f$.



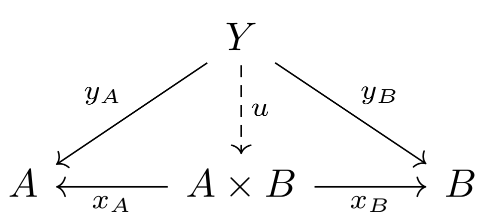
- A *coimage* of f is a cokernel of a kernel of f. By definition, the pair $(\mathrm{Coim}(f), \mathrm{coim}(f))$ post-composes with k to be zero. As this is also true of f, there is a unique morphism $u : \mathrm{Coker}(k) \to B$ such that $u \circ \mathrm{coker}(k) = f$.



Note that these definitions are not inherently unique, nor do they necessarily exist.

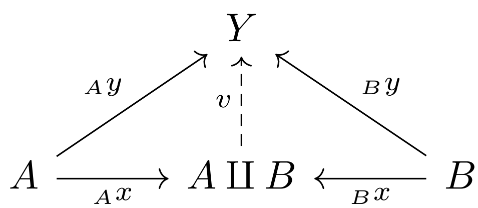
Definition 2.1.6. Let $\mathcal{C}$ be a category and A, B be objects.

- A *product* of A and B is (X, x_A, x_B) with $x_A \in \mathrm{Hom}(X, A)$, $x_B \in \mathrm{Hom}(X, B)$ satisfying the universal property:
  - for every (Y, y_A, y_B) with $y_A \in \mathrm{Hom}(Y, A)$, $y_B \in \mathrm{Hom}(Y, B)$, there exists a unique $u : Y \to X$ such that the following commutes:



  As in the above, such an object X is frequently denoted $A \times B$.

- A *coproduct* of A and B is $(X, {}_Ax, {}_Bx)$ with ${}_Ax \in \mathrm{Hom}(A, X)$, ${}_Bx \in \mathrm{Hom}(B, X)$ satisfying the universal property:
  - for every $(Y, {}_Ay, {}_By)$ with ${}_Ay \in \mathrm{Hom}(A, Y)$, ${}_By \in \mathrm{Hom}(B, Y)$, there exists a unique $v : X \to Y$ such that the following commutes:



As in the above, such an object X is frequently denoted $A \amalg B$.

- If product and coproduct agree in a category, we refer to both (either) of them as the *direct sum*.

Definition 2.1.7. A category C is *pre-abelian* if

- Every Hom set can be endowed with the structure of an abelian group such that composition

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

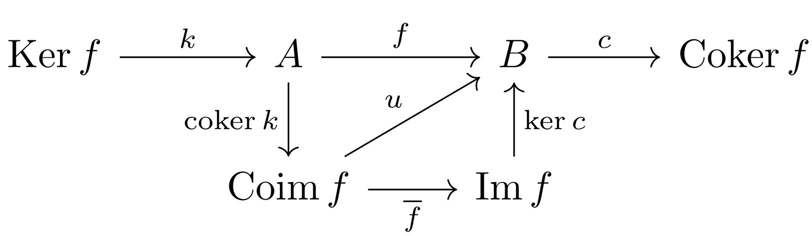
is a bilinear map,

- it has all finite products and coproducts,
- every morphism has kernels and cokernels.

Definition 2.1.8. A category C is *abelian* if it is pre-abelian, and for any morphism $f \in \mathrm{Hom}(A, B)$ the *canonical morphism*

$$\mathrm{Coim}(f) \to \mathrm{Im}(f)$$

is an isomorphism.



The canonical morphism is $\bar{f}$ in the diagram above induced in the following way from universal properties:

- The morphism u is induced by $\mathrm{Coim}(f)$.
- As $c \circ f$ is zero and $u \circ \mathrm{coker}(k) = f$, the composition $c \circ u \circ \mathrm{coker}(k)$ is zero. But, as $\mathrm{coker}(k)$ is an epimorphism, it can be canceled on the right, and so $c \circ u$ is zero.
- Then as $\mathrm{Im}(f)$ is a kernel of c, u factors uniquely through $\mathrm{Im}(f)$ via $\bar{f}$.

2.1.2 Additive and Krull-Schmidt Categories

Definition 2.1.9. A category $\mathcal{C}$ is *additive* if the following hold.

- Every Hom set can be endowed with the structure of an abelian group and composition is bilinear (as in 2.1.7).
- There is a zero object, 0, with the properties that

  $$\mathrm{Hom}(0, X) \text{ and } \mathrm{Hom}(X, 0) \text{ are singletons}$$

  for all objects X.
- Any finite collection of objects $X_1, \ldots, X_n$ has a direct sum $X_1 \oplus \ldots \oplus X_n$.

In an additive category, product and coproduct always have a canonical isomorphism: so, direct product.

Definition 2.1.10. In an additive category, a non-zero object X is *indecomposable* if $X = Y \oplus Z$ guarantees that one of Y or Z is zero.

An object is *decomposable* if it is not indecomposable.

Lemma 2.1.11. *An object is indecomposable if it has a local endomorphism ring.*

A ring R is *local* if it is unitary ($1 \neq 0$) and for every element $f \in R$, either f or $1 - f$ is a unit. The *endomorphism ring* of an object X in an additive category is simply $\mathrm{Hom}(X, X)$.

Definition 2.1.12. A category $\mathcal{C}$ is *Krull-Schmidt* if every object is isomorphic to a direct sum of indecomposables, and every indecomposable has local endomorphism ring.

Remark 1. A Krull-Schmidt category has the property that every object is isomorphic to a direct sum of indecomposables, such that if

$$X \cong V_1 \oplus \ldots \oplus V_n \cong W_1 \oplus \ldots \oplus W_m,$$

then $n = m$ and there is a permutation σ such that $V_i \cong W_{\sigma(i)}$ for all $1 \leq i \leq n$.

Note that the definition of a Krull-Schmidt category does not assume that the additive category is abelian, or even pre-abelian. Later on we will take interest in the linear category of filtered chain complexes, which is not abelian. But we will use Abelian categories and their subcategories to show that this category is nonetheless Krull-Schmidt. Atiyah's Criterion [Ati56, Kra15] provides a very general sufficient condition for an Abelian category to be Krull-Schmidt. Since many of the categories of interest are linear, we will be able to use the following special result:

Theorem 2.1.13. *(Atiyah's Criterion) A linear Abelian category with finite-dimensional Hom spaces is Krull-Schmidt.*

Though Atiyah's Criterion is powerful, its proof is nonconstructive. It neither provides an algorithm for decomposing a given object as a direct sum of indecomposables, nor a classification of those indecomposables, both of which we will be interested in later on.

2.2 Quiver Theory

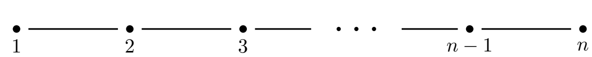
The recommended general reference for this section is the book *An Introduction to Quiver Representations* by Derksen and Weyman [HD17].

Definition 2.2.1. A *quiver* Q is a quadruple $Q = (Q_0, Q_1, h, t)$ where

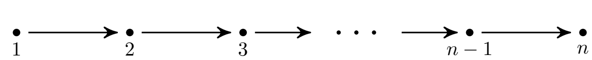
- Q_0 is a finite set of vertices,
- Q_1 is a finite set of arrows,
- h, t are functions from $Q_o \to Q_1$ and for an arrow $a \in Q_1$, ha, ta are called the *head* and *tail* of a, respectively.

A quiver, then, is a directed graph permitting multiple arrows and loops.

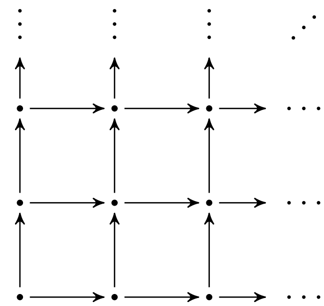
Example 2.2.2. The $\mathbb{A}_n$ quiver is the quiver with the following shape, and with any orientation on the arrows:

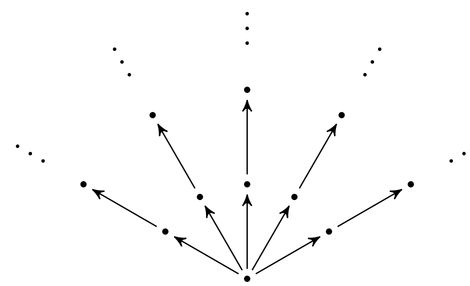


When all the edges are oriented $i \to i + 1$, we call it the equi-oriented $\mathbb{A}_n$ quiver, and denote it by $\overrightarrow{\mathbb{A}_n}$.



Example 2.2.3. Consider the two quivers drawn below: a grid of vertices m vertices wide and n vertices tall with arrows draw outward, and the concatenation of multiple $\overrightarrow{\mathbb{A}_n}$ quivers at a common minimal.





While there is a wealth of further examples in the literature, the results in this document focus around the $\mathbb{A}_n$ quiver with arbitrary orientations. The two quivers above are examples of interest that will be referred to often for the sake of contrast, in that the theory of representations over these quivers is immensely more complex than that over the $\mathbb{A}_n$ quiver.

Definition 2.2.4. A *respresentation* V of Q is an assignment

- of a K-vector space $V(i)$ to each vertex $i \in Q_0$,
- and a K-linear map $V(ta) \to V(ha)$ for each $a \in Q_1$.

Example 2.2.5. Consider $Q = \mathbb{A}_3$.

$$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xrightarrow{b} \underset{3}{\bullet}$$

A representation V of Q is any triplet of finite-dimensional vector spaces $V(1), V(2), V(3)$ and linear maps $V(a), V(b)$

$$V(1) \xrightarrow{V(a)} V(2) \xrightarrow{V(b)} V(3)$$

where $V(a) \in M_{\dim(V(2))\times\dim(V(1))}(K)$ and $V(b) \in M_{\dim(V(3))\times\dim(V(2))}(K)$.

Definition 2.2.6. For a quiver Q and two representations V, W, a *morphism* $V \to W$ is a family $\phi = \{\phi_i\}_{i\in Q_0}$ where each $\phi_i \in \mathrm{Hom}(V(i), W(i))$, such that the following diagram commutes for all $a \in Q_1$:

$$\begin{array}{ccc} V(ta) & \xrightarrow{V(a)} & V(ha) \\ {\scriptstyle \phi(ta)}\downarrow & & \downarrow{\scriptstyle \phi(ha)} \\ W(ta) & \xrightarrow[W(a)]{} & W(ha) \end{array}$$

Composition of quiver morphisms are coordinate-wise. It is immediate that for any quiver Q, the collection of representations and morphisms form a category, denoted $\mathrm{Rep}(Q)$. If we wish to specify the category of *finite-dimensional* representations (a representation in which each $V(i)$ is a finite-dimensional K-vector space), we will write lowercase $\mathrm{rep}(Q)$.

2.2.1 The Path Algebra

A *path* in Q is a sequence of arrows $p = a_1a_2\ldots a_n$ such that $ta_i = ha_{i+1}$. That is, arrows are composed similarly to functions.

$$a_1a_2\ldots a_n \;=\; \xleftarrow{a_1}\xleftarrow{a_2}\xleftarrow{a_3}\;\ldots\;\xleftarrow{a_n}$$

For a representation V of Q and a path $p = a_1 \ldots a_n$, $V(p) = V(a_1) \circ \ldots \circ V(p_n)$. The length of a path p is the number of arrows in p, and $hp = ha_1, tp = ta_n$. Let e_i denote the trivial path of length zero that begins and ends at the vertex i. These trivial paths are important objects for defining the following.

Definition 2.2.7. The *path algebra* KQ of a quiver Q is the K-linear span of all distinct paths in Q. That is, KQ has a basis consisting of $\{e_1, \ldots, e_n, p_1, p_2, \ldots\}$ where the $\{p_i\}$ are all distinct paths in Q. Multiplication is given by concatenation $p{\cdot}q = pq$ if $tp = hq$, and zero otherwise.

Remark 2. The path algebra KQ is finite-dimensional if and only if Q is *acyclic*: that is, having no oriented cycles.

2.2.2 Quivers with Relations

In the case of quivers arising from posets, we are concerned with accepting only representations that agree over parallel paths. That is, if $x, y \in Q_0$ and p, q are two paths with $tp = tq = x$ and $hp = hq = y$, then we would like to consider only representations V with the property that $V(p) = V(q)$.

Definition 2.2.8.

1. A *relation* in KQ is an element of the form $\sum_{i=1}^{n} \lambda_i p_i$ with each $\lambda_i \in K$ and such that *all* the p_i share the same tail and same head: $tp_1 = \ldots = tp_n$ and $hp_1 = \ldots = hp_n$.

2. A *quiver with relations* is a pair Q, R where Q is a quiver and R is a finite set of relations in the path algebra.

3. A *bound quiver* is a quiver with relations (Q, R) such that

   - for every $r = \sum \lambda_i p_i \in R$, the length of each p_i is at least 2, and
   - there is some natural number N such that any path of length N or longer belongs to $\langle R \rangle$, the two-sided ideal generated by R. The path algebra of a bound quiver (Q, R) is $A = KQ/\langle R \rangle$.

Definition 2.2.9. For a quiver with relations (Q, R), a (Q, R)-representation is a Q-representation V such that $V(r) = 0$ for all $r \in R$.

Definition 2.2.10 (Hasse Quiver)**.** Let P be a finite poset and Q_P be the quiver given by the following:

- There is a vertex for every point in the poset, i.e., $(Q_p)_0 = P$.

- There is an arrow $a \in (Q_P)_1$ with $ta = x, ha = y$ whenever $x < y$ and there is no $t \in P$ with $x < t < y$.

For any poset P, Q_P has no oriented cycles. Let R_P be the set of relations consisting of all $p - q$ where p, q are parallel paths in Q_P. The bound quiver (Q_P, R_P) is called the *Hasse quiver* of the finite poset P.

2.2.3 Representation Type

The *representation type* of a quiver is a classification of its indecomposable representations. There are three representation types commonly used in the literature. We list them here with brief, colloquial definitions.

The following three representation types partition the collection of all possible quivers.

- **Finite representation type:** $\mathrm{Rep}(Q)$ contains finitely many isomorphism classes of indecomposables (resp., finitely many isomorphism classes of A-modules, where A is the path algebra). There is a short, finite list of quiver shapes that are of finite representation type.

- **Tame:** though $\mathrm{Rep}(Q)$ may possess infinitely many isomphism classes of any given dimension, it is possible to parametrize them in a productive manner.

- **Wild:** it is demonstrably impossible to organize or classify the isomorphism classes of indecomposables in $\mathrm{Rep}(Q)$ in any meaningful way.

The classification of indecomposables is of great interest due to the following:

Proposition 2.2.11. *For any quiver Q, the space of representations $\mathrm{Rep}(Q)$ has the Krull-Schmidt property.*

Definition 2.2.12. Let Q be the Hasse quiver of a poset P (in particular, all parallel paths are modded out in the path algebra). Then by an *interval* module, we mean a convex collection of vertices I (i.e., if $v_i, v_j \in I$ and $v_i \leq x \leq v_j$ for some $x \in P$, then $x \in I$), that determine precisely the following representation:

- $V(I)(i) = K$ for all $i \in I$,
- $V(I)(i \leq j) = 1_K$ for all $i, j \in I$.

In the case that $Q = \mathbb{A}_n$, the set I is necessarily an interval $[x, y]$.

$$\ldots \leftrightarrow 0 \leftrightarrow K \leftrightarrow \ldots \leftrightarrow K \leftrightarrow 0 \leftrightarrow \ldots$$

From this point forward, we frequently identify indecomposables of $\mathbb{A}_n$ (the interval modules) with their interval of support, with endpoints given by the vertex labels on the poset.

The quiver $Q = \mathbb{A}_n$ is representation-finite. In fact, according to the full result of [Gab72], the complete collection of indecomposables for $Q = \mathbb{A}_n$ is precisely the interval representations over $\mathbb{A}_n$. Combining these statements, we have that any representation V of $\mathbb{A}_n$ is of the form

$$V = \bigoplus_{i=1}^{n} [x_i, y_i].$$

In the situation opposite the $\mathbb{A}_n$ quiver, the Hasse quivers of the posets in example 2.2.3 are wild. This says definitively that there is no parallel decomposition result to be had for these quivers, as there is not even a classification to be had for the indecomposables themselves.

2.2.4 Equivalence of Categories

Theorem 2.2.13. *Let (Q, R) be a quiver with relations. There is an equivalence of categories between $\mathrm{Rep}(Q)$ and the category of left modules over the path algebra KQ. This equivalence restricts to finite-dimensional representations and finite-dimensional left modules.*

For this reason we will frequently refer to representations of a quiver Q as (left) modules over the path algebra, and vice versa.

2.2.5 Auslander-Reiten Theory

The recommended reference for this section is the book *Quiver Representations* by Ralf Schiffler [Sch14].

Definition 2.2.14. For a quiver Q with finite representation type, we define a new quiver called the *Auslander-Reiten* (A-R) quiver that relates the indecomposable representations of the original quiver in the following way.

- The set of vertices is Σ_Q, which is defined to be the set of isomorphism classes of indecomposable representations of Q.
- For any two vertices (so, Q-indecomposables) σ and τ, there is an arrow $\sigma \to \tau$ precisely when there exists an *irreducible morphism* from σ to τ.

We require that Q have finite representation type as otherwise the set Σ_Q is infinite, whereas a quiver requires a finite vertex set.

We leave the specifics of the existence of irreducible morphisms to the text mentioned above.

2.3 Persistent Homology

The reference for the following two sections is the book *Elements of Algebraic Topology* by James R. Munkres [Mun96].

Persistent homology is an approach to large-scale data analysis that has found application in studying atomic configurations [HNH+16], analyzing neural activity [GPCI15], and filtering noise in sensor networks [BG10]. Persistent homology identifies clustering, holes, and higher-dimensional structures in data. This information is obtained independent of scale. The flexibility and routes for expansion within this topological approach to understanding data are what continue to draw interest.

2.3.1 Simplicial Complexes

Suppose a fixed collection of 'vertices' (points in some $\mathbb{R}^m$) denoted by $V = \{v_1, \ldots, v_n\} = \{v_i\}_{i \in I}$, where I is some finite indexing set.

- A *k-simplex* is the convex hull in $\mathbb{R}^m$ of a collection of $k+1$ vertices of V. We denote a k-simplex by $v_{i_0 \ldots i_k}$, or by v_J where $J = \{v_{i_0}, \ldots, v_{i_k}\}$.

- A *face* of some k-simplex v_J is any $v_{J'}$ where $J' \subset J$.

- A *complex* is a collection K of vertices and simplices such that if $v_J \in K$, then every face of v_J is also in K.

- A *k-chain* in K is

$$\sum a_{i_0\ldots i_k} v_{i_0\ldots i_k}$$

where the $a_{i_0\ldots i_k}$ are in $\mathbb{F}$ with at least one non-zero. Let C_k denote the vector space of k-chains (which has a K-basis consisting of all k-simplices). If V is a collection of n vertices, then for $0 \leq k \leq n-1$,

$$\dim(C_k) = \binom{n}{k+1}.$$

For all values of $k < 0$ or $k \geq n$, C_k is the zero vector space.

- The *boundary map* ∂ is a linear operator

$$\bigoplus_{k=0}^{n-1} C_k \to \bigoplus_{k=0}^{n-1} C_k$$

that is homogenous of degree -1. That is, we can choose a basis such that ∂ is block-superdiagonal and nilpotent of degree 2, and we write $\partial_k : C_k \to C_{k-1}$ for all k.

The boundary operator $\partial_k : C_k \to C_{k-1}$ for any k is defined on elementary k-chains (the basis elements for C_k) by

$$\partial_k v_{i_1\ldots i_k} = \sum_{j=1}^{n} (-1)^j v_{i_1\ldots \hat{i_j}\ldots i_k}$$

For example, $\partial_1 v_{ab} = v_b - v_a$ and $\partial_2 v_{abc} = v_{bc} - v_{av} + v_{ab}$.

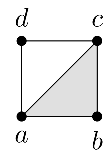
2.3.2 Simplicial Homology

The n^{th}-Homology of a simplicial complex is the vector space

$$H_n = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}},$$

where we mean the linear algebraic definitions of Kernel and Image (rather than the more general categorical ones, though this construction exists over arbitrary pre-Abelian categories where quotient is taken as coknerel of the subobject morphism). The dimension of $H_0(K)$ corresponds to the number of distinct connected components of K. For $n > 1$, $H_n(K)$ counts the number of $n+1$-dimensional holes.

Consider the following simplicial complex. The shading of triangle denotes that the 2-simplex abc is contained in the complex, but acd is not.



- $C_0 = \langle a, b, c, d \rangle$
- $C_1 = \langle ab, ac, ad, bc, cd \rangle$
- $C_2 = \langle abc \rangle$
- $C_k = 0$ for all $k \neq 0, 1, 2$

Combining all of the above, and ordering our basis by dimension (with remaining ties broken lexicographically), we have the following matrix representation of the boundary operator ∂ :

	a_0	b_0	c_0	d_0	ab_1	ac_1	ad_1	bc_1	cd_1	abc_2
a_0	0	0	0	0	−1	−1	−1	0	0	0
b_0	0	0	0	0	1	0	0	−1	0	0
c_0	0	0	0	0	0	1	0	1	−1	0
d_0	0	0	0	0	0	0	1	0	1	0
ab_1	0	0	0	0	0	0	0	0	0	1
ac_1	0	0	0	0	0	0	0	0	0	−1
ad_1	0	0	0	0	0	0	0	0	0	0
bc_1	0	0	0	0	0	0	0	0	0	1
cd_1	0	0	0	0	0	0	0	0	0	0
abc_2	0	0	0	0	0	0	0	0	0	0

Degree of each basis simplex is denoted by its subscript. Breaking the matrix up into its respective partials, we have $\partial_1 : C_1 \to C_0$ and $\partial_2 : C_2 \to C_1$:

$$\partial_1 = \begin{array}{c} \\ a_0 \\ b_0 \\ c_0 \\ d_0 \end{array} \begin{array}{c} \begin{array}{ccccc} ab_1 & ac_1 & ad_1 & bc_1 & cd_1 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

$$\partial_2 = \begin{array}{c} \\ ab_1 \\ ac_1 \\ ad_1 \\ bc_1 \\ cd_1 \end{array} \begin{array}{c} abc_2 \\ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array}.$$

- H_0: As $C_{-1} = 0$, $\text{Ker } \partial_0 = C_0$ which has dimension 4. From the matrix representation for ∂_1 above it can be readily computed that the column-space $\text{Im } \partial_1$ has dimension 3. So
$$H_0 \cong \frac{K^4}{K^3} \cong K.$$
The intuition for this is that for any 1-simplex v_ij, $\partial_1(v_ij) = v_j - v_i$, and so all connected 0-simplices are in the same equivalence class. For this example, this means that in $\text{Ker } \partial_0/\text{Im } \partial_1$, $[a] \cong [b] \cong [c] \cong [d]$, and that we have equivalence class of connected components.

- H_1: For $i \geq 1$, $\partial_i(v_{i_1} \ldots v_{i_j}) = 0$ if and only if $v_{i_1} \ldots v_{i_j}$ is a closed cycle. So $\text{Ker } \partial_1 = \langle bc - ac + ab, cd - ad + ac \rangle$. As $\text{Im } \partial_2 = \langle bc - ac + ab \rangle$, we see that
$$H_1 \cong \frac{K^2}{K} \cong K.$$

- H_2: As $C_3 = 0$, $\text{Im } \partial_3 = 0$, as does $\text{Ker } \partial_2 = 0$, and so $H_2 \cong 0$.

2.3.3 GPMs

Definition 2.3.1. A *generalized persistence module* (GPM) M over a poset P with values in a category $\mathcal{D}$ is an assignment

- $x \to M(x) \in \mathcal{D}$ for all $x \in P$,

- $(x \leq y) \to M(x \leq y) \in \mathrm{Hom}(M(x), M(y))$ for all $x \leq y$ in P,

such that $M(x \leq z) = M(y \leq z) \circ M(x \leq y)$ whenever $x \leq y \leq z \in P$.

For two GPMs M, N over P with values in $\mathcal{D}$, a *morphism* of $M \to N$ is a collection $\{\phi(x)\}_{x \in P}$ where each $\phi(x) \in \mathrm{Hom}(M(x), N(x))$, such that the following diagram commutes for all $x \leq y \in P$:

$$\begin{array}{ccc} M(x) & \xrightarrow{M(x \leq y)} & M(y) \\ \downarrow{\scriptstyle \phi(x)} & & \downarrow{\scriptstyle \phi(y)} \\ N(x) & \xrightarrow[N(x \leq y)]{} & N(y) \end{array}$$

With this structure, we denote by $\mathcal{D}^P$ the category of GPMs over P with values in $\mathcal{D}$.

Remark 3. The similarity of this commutative diagram to the one for natural transformations (and quiver morphisms) is not an accident. A GPM is a functor from the poset category of P to the category $\mathcal{D}$, and a morphism between GPMs is a natural transformation. Quivers (without relations, at least) are simply GPMs with values in $\mathcal{D} = \mathrm{vect}$, the category of finite vector spaces.

2.3.4 Process

Classical persistent homology transforms a data set (finite point cloud) into a set of topological invariants (the barcode). The classical workflow for this process and the objects obtained along the way are summarized below. Some nuances, evolutions, and algorithmic implementations of this process are discussed throughout Chapter 3, while the following is cursory overview of the fundamentals.

- Obtain an n-dimensional finite point cloud I.
- For each $\epsilon \in [0, \infty) \subset \mathbb{R}$, construct the following simplicial complex, $K(\epsilon)$:
  - $K_0(\epsilon)$ is the collection of all 0-dim simplices (points/vertices) of the data set.
  - $K_1(\epsilon)$ is the collection of all 1-dim simplices (edges) $v_{i_1}v_{i_2}$ such that $d(v_{i_1}, v_{i_2}) \leq \epsilon$.
  - $K_n(\epsilon)$ is the collection of the n-dim simplicies $v_{i_1} \dots v_{i_{n+1}}$ such that $v_{i_1} \dots \hat{v}_i \dots v_{i_{n+1}} \in K_{n-1}(\epsilon)$ for all $1 \leq i \leq n+1$. That is, include every n-dim simplex such that each of its $n-1$-dim faces are contained in $K_{n-1}(\epsilon)$. (This inductive definition terminates, as $K_{|K_0|}(\epsilon)$ is the last possible non-zero collection of simplicies.)
- Taking this collection for all $\epsilon \in [0, \infty)$, we have a GPM over $[0, \infty)$ with values in the category of simplicial complexes.
- It is here that one may *discretize* in the following sense. Let E be the collection

of ϵ such that

$$\lim_{\epsilon' \to \epsilon^-} K(\epsilon') \neq K(\epsilon).$$

As we started with a finite data set, there are only finitely many such values where the simplicial complex changes. So, the set E is one-to-one with some subset of the natural numbers, $\{1, \ldots, n\} \subset \mathbb{N}$. Viewing only the simplicial complexes over values in E, we have a GPM over the finite totally ordered set E.

- For any $\epsilon_1 \leq \epsilon_2$, it is clear that $K(\epsilon_1)$ is a subobject of $K(\epsilon_2)$. Let $f_{\epsilon_1,\epsilon_2}$ be the subobject morphism of simplicial complexes $K(\epsilon_1) \hookrightarrow K(\epsilon_2)$.
- Having attached these morphisms, apply the desired *homology functor*.
- If one has discretized by now, we at last have a persistence vector space: a GPM over a totally ordered poset with values in the category of vector spaces. That is to say, we have a representation of the quiver $Q = \mathbb{A}_n$.
- By Gabriel's decomposition result, this object decomposes into interval modules with multiplicities. The multiset of these indecomposable interval modules is the *barcode* of the data set.

The barcode is usually the object of greatest interest, and is a topological invariant of the data set. It highlights topological features that persist across a range of scale values. This process is robust under considerations of noise in the data collection, in the following sense:

- The elements of the barcode that are long—persist across a wide range of scale values relative to others—are true features of the data set.

- Noise and other inaccuracies result in intervals that are short, and thus not indicative of topological trends in the data.

2.4 Stability

Stability is an answer to the question: if a certain object in the persistent homology workflow is perturbed slightly, how similar are the resulting objects? For example, in the literature the two primary types of stability are:

- Soft stability: distance between persistence modules (quiver representations) based on perturbations of the data set.

- Hard stability: distance between barcodes based on distance between persistence modules.

Chapters 4 and onward are all at least partially concerned with hard stability results. Before any statements of hard stability can be discussed, however, we must have notions of distances between persistence modules and between barcodes.

2.5 Interleaving Metric

We begin the construction of the *interleaving metric* of Bubenik, de Silva, and Scott: a metric on generalized persistence modules.

We measure interleaving distance by *translations*. A translation Λ on a poset P is a monotone morphism. I.e.,

- $\Lambda(x) \geq x$,

- $\Lambda(x) \leq \Lambda(y)$ for all $x \leq y$ in P.

For example, in the totally ordered set $\{1 \leq 2 \leq 3 \leq 4\}$, any map of vertices that sends $1 \to 4$ and $2 \to 3$ is immediately not a translation. Any translation that sends $1 \to 4$ must be the constant translation $\Lambda(1) = \Lambda(2) = \Lambda(3) = \Lambda(4) = 4$.

The set of translations $\mathcal{T}(P)$ is itself a poset under the relation: $\Lambda \leq \Gamma$ when $\Lambda(x) \leq \Gamma(x)$ for all $x \in P$.

The *height* of a translation Λ is

$$h(\Lambda) = \max\{\delta(x, \Lambda(x)) : x \in P\},$$

where δ is some metric on the poset.

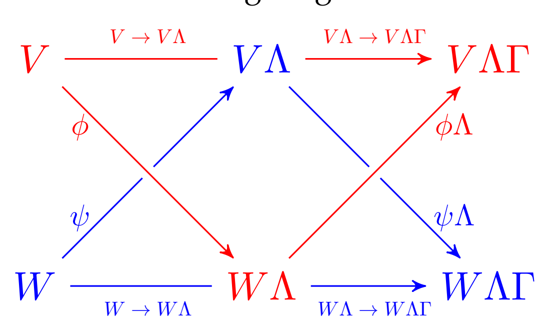
Let $\mathcal{D}$ be a category. Then $\mathcal{T}(P)$ acts on $\mathcal{D}^P$ on the right by

- $(V\Lambda)(p) = V(\Lambda p)$, and
- $V\Lambda(x \leq y) = V(\Lambda x \leq \Lambda y)$.

Likewise, for a morphism of GPMs $F : V \to W$ in $\mathcal{D}^P$, $(F\Lambda) : V \to W$ is given by $(F\Lambda)(i) = F(\Lambda i) : V(\Lambda i) \to W(\Lambda i)$.

Remark 4. The action of translations on GPMs is that of a natural transformation acting between functors: $\Gamma : V \to \Gamma V$. (See Remark 3.)

Definition 2.5.1. For two GPMs V, W in $\mathcal{D}^P$ and two translations Λ, Γ on P, a Λ, Γ-*interleaving* between V and W is a pair of morphisms $\phi : V \to W\Lambda$, $\psi : W \to V\Lambda$ such that both triangles of the following diagram commute.



The *interleaving distance* between V and W is

$$D(V,W) = \min\{\epsilon : \max\{h(\Lambda), h(\Gamma)\} = \epsilon \text{ and } V, W \text{ are } (\Lambda, \Gamma) - \text{interleaved.}\}$$

If there is no interleaving between V and W for any translations, then we say that the interleaving distance is *infinite*.

2.5.1 Fixed Points

For a translation Λ, the *fixed points* of Λ are precisely those points $p \in P$ for which $\Lambda(p) = p$. For example, any maximal element of P is fixed under all translations.

Lemma 2.5.2. *Let V, W be GPMs over some poset P, Λ, Γ be translations and $p \in P$ be a fixed point of both translations. Then if $\dim_K V(p) \neq \dim_K W(p)$, V, W cannot be (Λ, Γ)-interleaved.*

Proof. Suppose $\dim_K V(p) < \dim_K W(p)$. Then the commutativity condition of the triangle beginning at $W(p)$,

$$\begin{array}{ccc} W(p) & \xrightarrow{\quad 1_{W(p)} \quad} & W(\Lambda\Gamma p) = W(p) \\ & {\scriptstyle \phi(p)} \searrow \quad \nearrow {\scriptstyle \psi(\Lambda p) = \psi(p)} & \\ & V(\Lambda p) = V(p) & \end{array}$$

cannot possibly hold, as there is no linear map of full rank from $V(p) \to W(p)$. ∎

To put the preceding lemma in more direct form:

Proposition 2.5.3. *Let P be a finite poset. Let V, W be two GPMs over P with values in a category $\mathcal{D}$. Then the interleaving distance between V and W is finite if and only if $V(p) \cong W(p)$ for all points $p \in P$ that are fixed by the set of translations.*

So if a point $p \in P$ (such as a maximal element) is fixed for all translations, then any two modules with different dimensions at p have *no* interleaving.

Specifically, for $Q = \mathbb{A}_n$ and two interval modules with similar supports that are away from n, interleaving distance is small (see 2.5.6). But the moment one includes n in the support of one and not the other, distance is infinite. In $\mathbb{R}$ this is not an issue, as there are no maximal elements. In representations of finite posets, which is the topic of interest in this document, the issue of infinite interleaving distances must be addressed.

2.5.2 Submodules and Quotient Modules

Before going any further, we address what morphisms between persistence modules look like in specific settings, and in particular which pairs of persistence modules even possess non-zero Hom-sets.

Let P be a poset and I, J be interval modules—that is, modules fully described by their supports. Consider briefly the following two non-commuting triangles.

$$\begin{array}{ccccc} \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ V(p) = & K & \xrightarrow{\phi(p)} & K & = W(p) \\ \uparrow & {\scriptstyle 1_K}\uparrow & & \uparrow & \uparrow \\ V(q) = & K & \longrightarrow & 0 & = W(q) \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \end{array} \qquad \begin{array}{ccccc} \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ V(r) = & 0 & \longrightarrow & K & = W(r) \\ \uparrow & \uparrow & & \uparrow{\scriptstyle 1_K} & \uparrow \\ V(p) = & K & \underset{\phi(p)}{\longrightarrow} & K & = W(p) \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \end{array}$$

Lemma 2.5.4. *Let V, W be interval modules in $\mathcal{D}^P$. Suppose $p \in \operatorname{supp}(V) \cap \operatorname{supp}(W)$.*

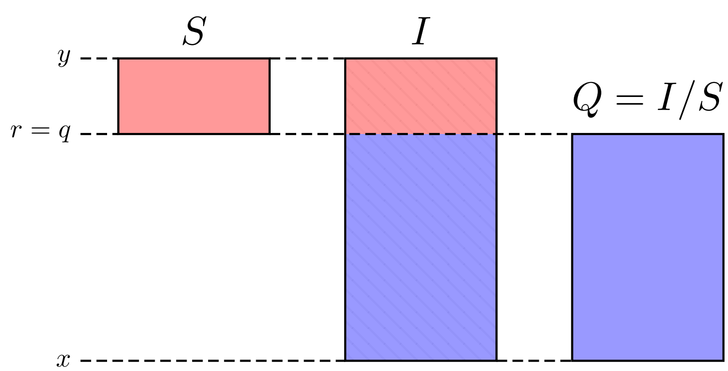
- *If $q \in \operatorname{supp}(V)$, $q \leq p$, and $q \notin \operatorname{supp}(W)$, it must be that $\mathrm{Hom}(V, W) = 0$.*
- *If $r \in \operatorname{supp}(W)$, $p \leq r$, and $r \notin \operatorname{supp}(V)$, it must be that $\mathrm{Hom}(V, W) = 0$.*

Extending on this idea, a *submodule* of a module V is a module S that is an additive subgroup of V such that there exists a morphism $V \to S$. A *quotient* of a module V is a module Q that is an additive subgroup of V such that there exists a morphism $Q \to V$.

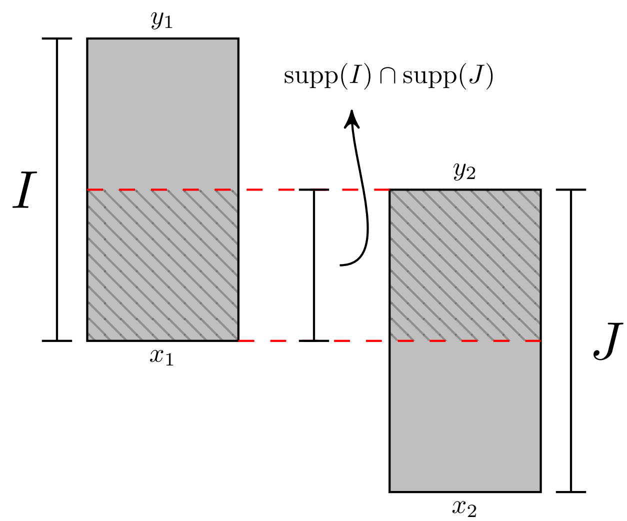
In the case of a totally ordered poset $\{1, \ldots, n\}$ and some interval module $I \cong [x, y]$,

- any submodule of I is of the form $S = [x, r]$ where $x \leq r \leq y$,

- and any quotient of I is of the form $Q = [q, y]$ where $x \leq q \leq y$.



When we have two interval modules $I = [x_1, y_1], J = [x_2, y_2]$ over a totally ordered set, there exists a non-zero morphism $I \to J$ if and only if $y_2 \leq y_1 \leq x_2 \leq x_1$.



In this case, define $\Phi_{I,J}$ to be the morphism

- $\Phi_{I,J}(i) = 1_K(i)$ if $i \in \operatorname{supp}(I) \cap \operatorname{supp}(J)$,
- $\Phi_{I,J}(i) = 0$ otherwise.

Clearly, $\operatorname{Hom}(I, J) = {}_K\langle \Phi_{I,J} \rangle$.

Example 2.5.5. Let $P = \{1, \ldots, 10\}$. Let $V = [6, 8], W = [3, 7]$. We need to choose translations Λ, Γ such that

$$V \to W\Lambda \to V\Lambda^2 = V \to V\Lambda^2$$

and

$$W \to V\Lambda \to W\Lambda^2 = W \to W\Lambda^2.$$

First, $\operatorname{Hom}(V, W\Gamma) \neq 0$ for Γ equal to the identity translation.

$\operatorname{Hom}(W, V\Lambda) \neq 0$ only for Λ larger than or equal to the translation that sends $8 \to 9$ and $3 \to 6$. This is due to the fact that, if $\Lambda(8) = 8$, then there exists no morphism $W \to V\Lambda$ by the properties of quotient modules (i.e., $V\Lambda$ it too high at the top). Similarly, if $\Lambda(3) < 6$, $V\Lambda$ is not low enough at the bottom.

Define Λ to be precisely the minimal translation given by the above constraints.

- $\Lambda(3) = 6$,
- $\Lambda(4) = 6$,
- $\Lambda(5) = 6$,
- $\Lambda(8) = 9$,
- and all other vertices are fixed.

With this choice of translation,

$$V\Lambda = V\Lambda^2 = W\Lambda = W\Lambda^2 = [3, 7].$$

Commutativity beginning at W is trivial. For V, the internal morphism $V \to V\Lambda^2 = \Phi_{V,V\Lambda^2} = 1_{[6,7]}$. The morphisms on the other sides of the triangle are $V \to W\Lambda = 1_{[6,7]}$, $W\Lambda \to V\Lambda = 1_{[3,7]}$ which compose to $1_{[6,7]}$ above, satisfying commutativity.

As a comment that will be argued formally in a later chapter, one might guess that in a totally ordered set, interleaving two indecomposable modules is strictly a matter of minimally aligning upper and lower endpoints such that there exist non-zero morphisms in both directions.

2.5.3 Weights and Suspension at Infinity

As previously mentioned, in the poset $P = \mathbb{R}$, the interleaving distance between any two finite persistence modules (i.e., excluding the indecomposables of type $[a, \infty)$ or $(-\infty, b]$) is finite. However, when P is a finite poset, we begin to encounter infinite interleaving distances due to fixed points.

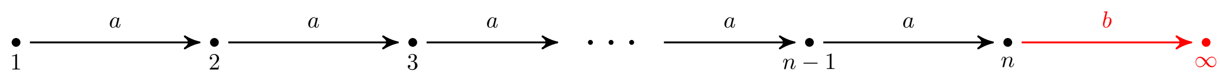
Example 2.5.6. Consider the totally ordered poset $P = \{1 \leq \ldots \leq n\}$ and the category vect of finite-dimensional vector spaces over a base field K. Consider the interval representations $V = [1, n]$ and $W = [1, n-1]$. Despite agreeing on all vertices save for n, they have infinite interleaving distance by Prop 2.5.3.

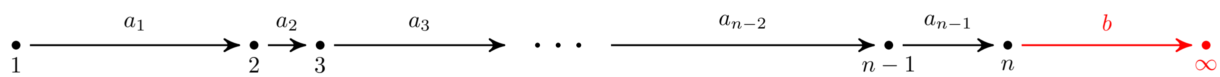
To correct this, we shift to a setting in which maximals—which are always fixed by the usual set of translations—are permitted to move, but we associate a new (and possibly large) finite cost to the operation.

For a poset P, define the new poset P^+ to be the collection of vertices $P \cup \{\infty\}$ with the original relations of P plus the new relations $p \leq \infty$ for all $p \in P$. Now there exist interleavings in P^+ between any GPMs over P.

The Hasse quiver for P^+ is the Hasse quiver for P with extra edges $p \to \infty$ for all maximal elements p of P. We assign *weights* to the edges of this new quiver/poset to adjust the relative costs of translations acting within the original poset against those translations that are capable of moving points outside the original poset.

There are two ways to add weights the poset P^+ that will both be used in later chapters. The first is with so-called *democratic* weights, in which all original edges of the poset have weight a, and all new edges to ∞ have weight b. To capture even more information, we call *general* weights those which assign possibly unique weights to every individual edge of P^+. See the examples below, in which both weight-types are illustrated on $\overrightarrow{\mathbb{A}_n}$ (democratic weights on the top, general weights below).





Example 2.5.7. Consider the same setup as in Example 2.5.6 but over the poset P^+ with *democratic* weights. The indecomposables $V = [1, n]$ and $W = [1, n-1]$ are interleaved in P^+ by the translations Λ, Γ that send n to ∞, and that fix all other vertices. Since

$$
\begin{aligned}
V &\sim [1, n] & W &\sim [1, n-1] \\
V\Lambda &\sim [1, n-1] & W\Lambda &\sim [1, n-1] \\
V\Lambda^2 &\sim [1, n-1] & W\Lambda^2 &\sim [1, n-1]
\end{aligned}
$$

it is easy to see that the interleaving morphisms commute. As any interleaving between them must send $b \to \infty$, V and W have an interleaving distance of b.

2.6 Bottleneck Metric

A *bottleneck metric* provides a metric structure on the set of isomorphsim classes of GPMs, which automatically yields a metric on barcodes. In general, a bottleneck metric acts on multisubsets of a set Σ. It requires a metric d on Σ and a function $W : \Sigma \to (0, \infty)$ compatible in the following way: for any $\sigma_1, \sigma_2 \in \Sigma$,

$$|W(\sigma_1) - W(\sigma_2)| \leq d(\sigma_1, \sigma_2).$$

Following [BL13] and [BL16], define a *matching* between two multisets S, T of Σ to be a bijection $f : S' \to T'$ between multisubsets $S' \subseteq S$ and $T' \subseteq T$. For $\epsilon \in (0, \infty)$, we say a matching f is an ϵ-matching if the following conditions hold;

(i) if $W(s) > \epsilon$, then $s \in S'$

(ii) if $W(t) > \epsilon$, then $t \in T'$, and

(iii) $d(s, f(s)) \leq \epsilon$, for all $s \in S$.

Intuitively, W measures the *width* of an element of Σ. In an ϵ-matching, elements of S and T which are actually identified are within ϵ of each other (according to d), while all those not identified have width at most ϵ (according to W).

For us, Σ will be a sub-collection of indecomposable modules of a quiver Q and $\mathcal{C}$ the subcategory generated by direct sums over Σ.

Suppose Σ is the *full* collection of indecomposables of Q. Then multisubsets of Σ are one to one with sums of indecomposables, and so, the collection of finite-dimensional representations of Q. In particular, given a finite-dimensional representation V of Q, the barcode of V, denoted $B(V)$, is the multisubset of the collection of isomorphism classes that consists of precisely the indecomposable summands of V according to their corresponding multiplicities.

Definition 2.6.1. Let S, T be two finite multisubsets of any set Σ. Suppose d and W are compatible. Then the bottleneck distance between S and T is defined by

$$D_B(S,T) = \inf\{\epsilon \in \mathbb{R} : \text{there exists and } \epsilon\text{-matching between } S, T\}$$

Let Σ be any fixed subset of isomorphism classes of indecomposable representations over Q. If V, W are finite-dimensional representations of Q with the property that every indecomposable summand of V or W is isomorphic to an element of Σ, then we may identify V, W with their barcodes $B(V), B(W)$: two multisubsets of

Σ. Then define

$$D_B(V, W) := D_B(B(V), B(W)).$$

While there are many examples of bottleneck metrics in the literature, in this document we will examine the bottleneck metric based that is the pairwise (or, diagonal) interleaving metric. In this situation, we will restrict ourselves to the subcategory generated by Σ, the set of *convex modules* for the poset.

2.7 The Category Generated by Convex Modules

Since most posets have the property that their Hasse quiver is of wild representation type, a characterization of all of the isomorphism classes of its indecomposable modules will subsequently be impossible. Moreover, in all but the simplest of posets, an indecomposable module will not be determined by its support (see Definition 2.2.12). Fix a poset P and let Ω denote the set of isomorphism classes of indecomposable representations. The function

$$\Omega \xrightarrow{Supp} \mathcal{P}(P) \text{ , which sends } M \xrightarrow{Supp} \text{Supp}(M)$$

may have infinite (and unknowable) domain, but always has finite range. Motivated by one-dimensional persistent homology, we take the perspective that the width of an indecomposable should be determined only by its support. We will therefore restrict our attention to the category $\mathcal{C}$ generated by an appropriate set Σ of indecomposable *thin* modules—modules whose dimension vector consists of only zeros and ones.

The following is a generalization of the notion of interval modules to non-totally-ordered posets.

Definition 2.7.1. An indecomposable module M is *convex* if it is thin ($\dim(V(i)) \leq 1$ for all i) and isomorphic to a module M' where M' satisfies:

- for all $x, y \in \mathrm{Supp}(M')$ with $x \leq y$, the linear map $M'(x \leq y)$ is given by Id_K.

In this document the terms *interval module* and *convex module* are used interchangeably.

Clearly, when we restrict our attention to the set of isomorphism classes of convex modules, the function $M \to \mathrm{Supp}(M)$ is one-to-one. Of course, the function is not onto, as not every subset of P is the support of a convex module. One easily checks that if S is some subset of the poset P, then there exists a convex module M (unique up to isomorphism) with Supp$(M) = S$ if and only if

(i) For all $s_1, s_2 \in S$ there exists an unoriented path in the Hasse quiver of P that connects s_1 and s_2 staying entirely within S , and

(ii) For all $s_1, s_2 \in S$ the set $\{p \in P : s_1 \leq p \leq s_2\} = [s_1, s_2] \subseteq S$.

If S satisfies (i), we say S is *connected*, and if S satisfies (ii), we say S is *interval convex*. Regardless of the representation type of the poset P, $\Sigma = \{[\sigma] : \sigma \text{ is convex } \}$ is finite.

While the class of posets we will restrict to in Chapter 4 contain many posets of wild representation type, they all have the property that every indecomposable thin is convex.

Chapter 3

Categorical Framework

3.1 Motivation

This chapter is based on a collaboration with Andrei Pavlichenko and Jan Segert [MPS17].

3.2 Manifestations of the Structural Theorem

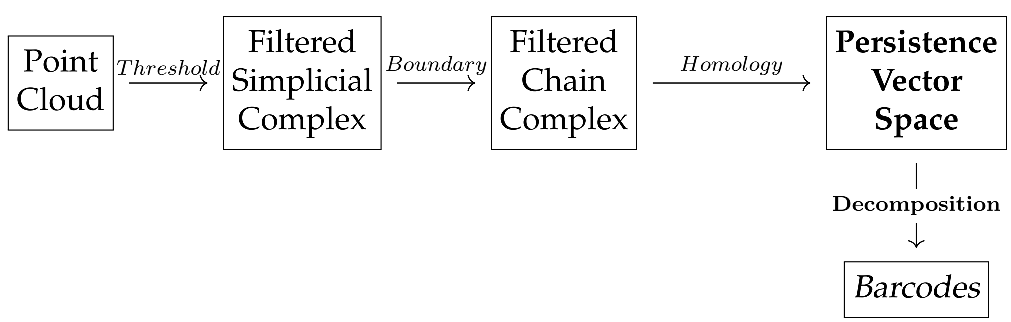
3.2.1 Introduction

Decomposition plays a central role both in the theory and in the applications of persistent homology. The ubiquitous "barcode diagrams" encode a decomposition in terms of the types and multiplicities of indecomposable summands. This data is an invariant, independent of the choice of decomposition. The summands represented by long barcodes contain important characteristic information, while the summands represented by short barcodes only contain random "noise" and may be disregarded. A number of "stability theorems" [DCS07, Oud15] provide a firm foundation for this intuitively appealing interpretation of the long and short barcode invariants. In this paper we consider the interplay between the algorithmic and the categorical underpinnings for decomposition in persistent homology.

It is helpful to first review analogous decomposition issues for the much more familiar context of finite-dimensional vector spaces (over a fixed field $\mathbb{F}$). The ordinary Gaussian elimination algorithm can construct a basis for a vector space. Any choice of basis then constitutes a decomposition of the vector space, wherein the linear span of each basis element is a one-dimensional vector space. The direct sum of these one-dimensional summands is canonically identified (naturally isomorphic) to the original vector space. A one-dimensional vector space cannot be further decomposed as a sum of nonzero (dimensional) summands. This means that one-dimensional vector spaces are the *indecomposable objects*, in the the category of vector spaces. Since all one-dimensional vector spaces are mutually isomorphic, there is just one type of indecomposable (object) in the category of vector spaces. The familiar dimension of a vector space is just the multiplicity of the indecomposable (one-dimensional) summands in a decomposition, and this multiplicity is an invariant independent of the choice of decomposition.

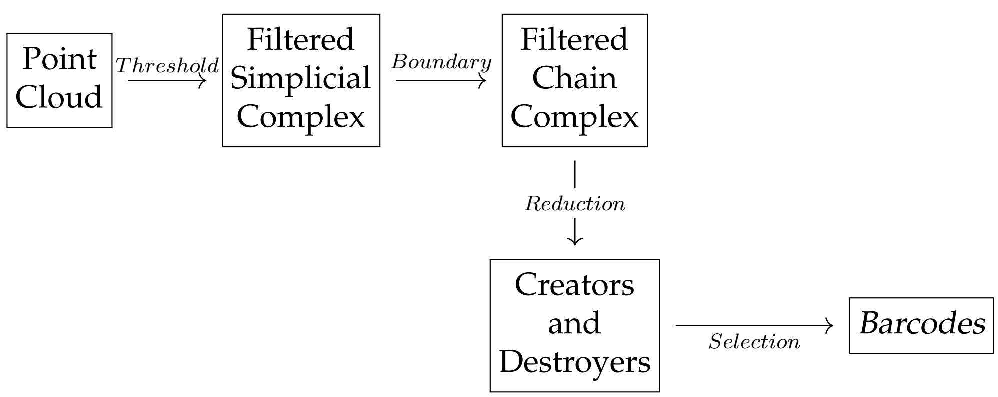
Now setting aside what we know about Gaussian elimination, we ask more abstractly *why* is it that any vector space is actually decomposable? We first observe that decomposability is a categorical property, since it involves both objects (vector spaces) and morphisms (linear maps). The theory of *Krull-Schmidt categories* [Kra15] provides an appropriate, albeit abstract, categorical setting for questions of decomposability. The axioms of a Krull-Schmidt category guarantee that every object admits an essentially unique decomposition as a finite sum of indecomposable objects. For the category of vector spaces, this essential uniqueness encodes the familiar fact that the dimension is an invariant independent of the choice of decomposition. The goal then becomes to verify (and understand) the Krull-Schmidt property for the category of vector spaces. A concrete constructive verification of the Krull-Schmidt axioms for the category of vector spaces follows easily from basic properties of Gaussian elimination and linearity, but this is more in line with describing *how* to perform a decomposition rather than *why* vector spaces are decomposable. Fortunately there is a complementary abstract tool available. A theorem of Atiyah [Ati56] dating back to the early years of category theory provides a very useful criterion for verifying the Krull-Schmidt property of a category. For the category of vector spaces, Atiyah's criterion reduces to checking certain elementary properties of linear maps. So Atiyah's theorem nonconstructively answers the abstract question of *why* any vector space admits a decomposition, complementing our understanding of *how* to constructively decompose a given vector space via Gaussian elimination.

In this paper we consider analogous questions of *how* and *why* decomposition works in persistent homology. The following picture summarizes one common description of the transformation from point cloud data to barcodes invariants:



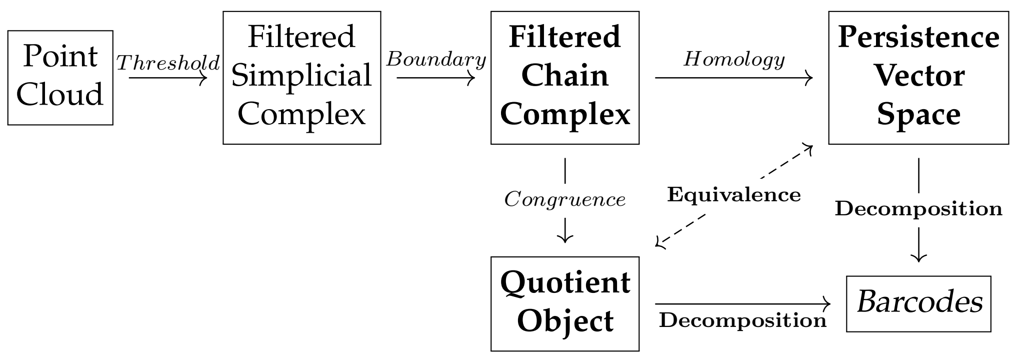
The initial stages, going from a point cloud to a filtered chain complex, will be briefly reviewed at the end of this section. Our primary focus will be the final stages, going from filtered chain complexes to barcodes. At the homology step, the homology functor H_n of the chosen dimension/degree n takes a *filtered chain* *complex* (which is a diagram of chain complexes) to a *persistence vector space* (which is a diagram of vector spaces). The key final step is to compute barcode invariants by decomposition of a persistence vector space. An important insight [Car09] is that persistence vector spaces are quiver representations. A concrete consequence is the applicability of decomposition algorithms from quiver representation theory, showing how to decompose a persistence vector space and compute the barcodes. An abstract consequence is that the appropriate category of quiver representations is Krull-Schmidt by Atiyah's theorem, showing *why* all of this works. So the Krull-Schmidt property of persistence vector spaces nicely ties together the theoretical and computational aspects. But there is one problem with this picture.

The standard computational algorithms for persistent homology [HE02, ZC05b, ZC08] do *not* work by decomposing a persistence vector space. The following picture summarizes *how* barcodes are normally computed:



The initial stages of the picture, going from a point cloud to a filtered chain complex, are unchanged. The key *reduction* step [HE02, ZC05b, ZC08] is the construction of a special type of basis. Each basis element is interpreted as either a *creator* or as a *destroyer* of a homology class. The *selection* step consists of keeping those creators and destroyers that correspond to nonzero barcodes of the desired homology dimension/degree n, and discarding the remaining basis elements. The question remains of *why* there should exist such algorithms operating on filtered complexes, rather than on persistence vector spaces.

In this paper we provide a categorical framework for the standard persistent homology algorithms, using an equivalence of categories to unify the two pictures above:

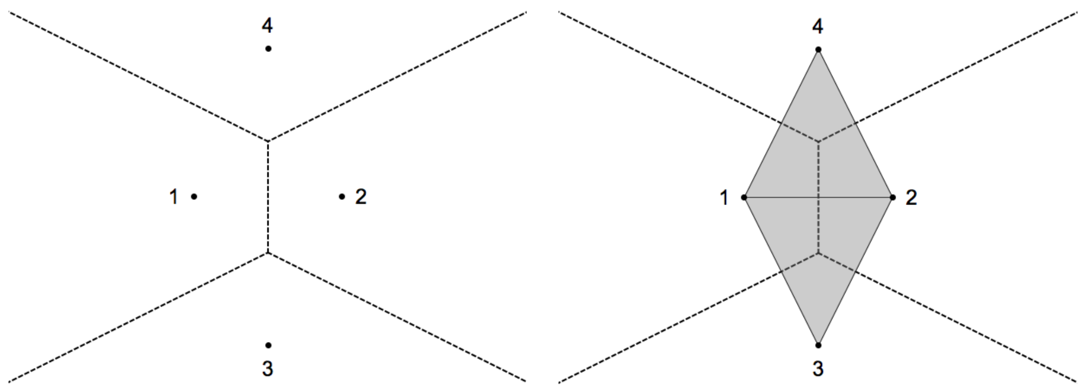


The foundation is the *Categorical Structural Theorem* (Theorem 1.6), which asserts that the category of filtered chain complexes is Krull-Schmidt. This gives an alternate framework for persistent homology, where the barcodes describe the Krull-Schmidt decomposition of an object in a quotient of the category of filtered chain complexes. These are still the same barcodes as in the standard framework, because the quotient category is equivalent to the category of persistence vector spaces. This gives a unified answer for *why* and *how* decomposition actually works in persistent homology. We no longer need to rely on the Krull-Schmidt property of the category of persistence vector spaces as an indirect theoretical foundation for decomposition and barcodes, since we can directly appeal to the Krull-Schmidt property already in the category of filtered chain complexes.

The Categorical Structural Theorem is the abstract version of what we call the *Structural Theorem of Persistent Homology*. We give a nonconstructive categorical proof of the Categorical Structural Theorem, indirectly using Atiyah's criterion. Combining the Categorical Structural Theorem with a classification of indecomposable filtered chain complexes then yields a novel *nonconstructive* proof of what we call the *Matrix Structural Theorem* (Theorem 1.4). The Matrix Structural Theorem characterizes the output of any of the various standard persistent homology algorithms in terms of a matrix factorization rather than the more common description in terms of creators and destroyers for homology. In this sense, any of the standard algorithms can be thought of as constituting a *constructive* proof of the Matrix Structural Theorem. In Section 3.7, we give a self-contained description of one such algorithm in terms of elementary matrix operations.

This paper focuses on the final stages of topological data analysis, going from a filtered chain complex to barcode invariants. We give below a simple example to illustrate the first stage, going from a point cloud to a filtered simplicial complex. In our example, we use the α-complex construction [EH09, Ede], which is suitable for low dimensions. We note that for large point clouds in high dimensions, the Vietoris-Rips construction [Car14, Oud15] is often preferable.

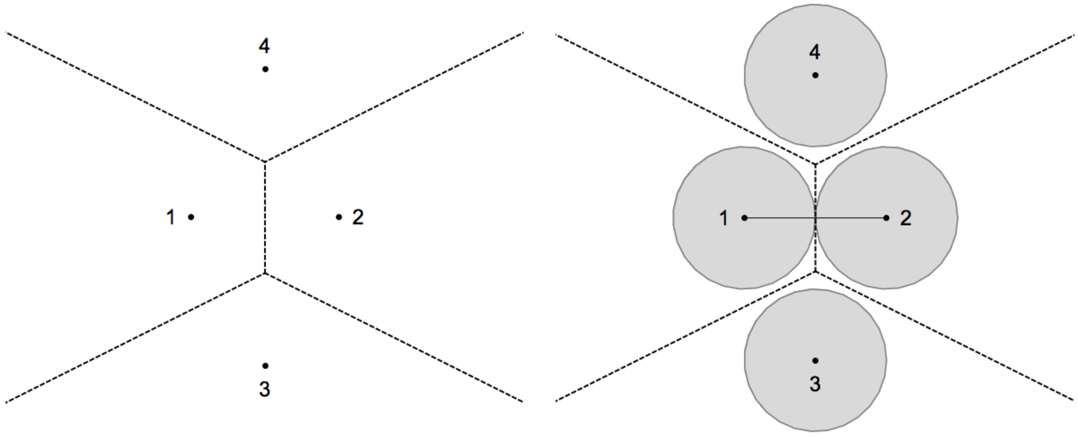
Example 3.2.1. The first step is to construct a Delaunay complex, the second step is to construct a filtration of the Delaunay complex. Figure 1 illustrates the construction of the Delaunay simplicial complex associated to a point cloud. The left image shows a point cloud consisting four planar points labeled by $n \in \{1, 2, 3, 4\}$, together with the Voronoi cell $V(n)$ of each labeled point. We recall [EH09] that a Voronoi cell $V(n)$ contains all the points $x \in \mathbb{R}^2$ such that n is the closest labeled point to x (or one of the closest if several are equidistant). The right image shows the Delaunay simplicial complex encoding the intersections of the Voronoi cells. We recall that the simplex $[n_0, \ldots, n_k]$, where $n_i \in \{1, 2, 3, 4\}$ and $n_0 < \cdots < n_k$, is included in the Delaunay complex iff $V(n_0) \cap \cdots \cap V(n_k) \neq \emptyset$. For example, the simplex $[1, 2]$ is included because $V(1) \cap V(2) \neq \emptyset$, but the simplex $[3, 4]$ is not included because $V(3) \cap V(4) = \emptyset$.



Voronoi cells Delaunay Simplicial Complex

Figure 3.1: Delaunay Complex

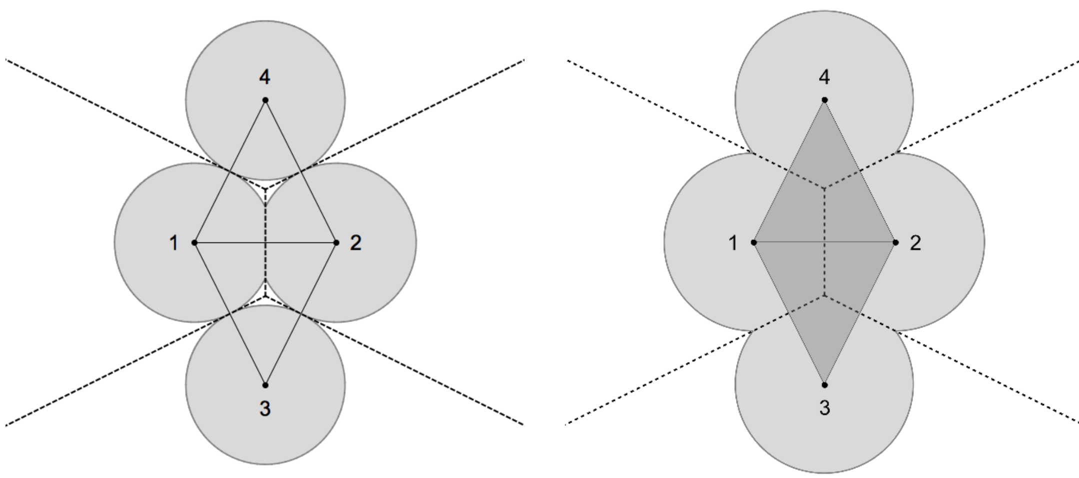
The α construction assigns to each Delaunay simplex $[n_0, \ldots, n_k]$ a real nonnegative "birth parameter" $b([n_0, \ldots, n_k])$. Let $B_r(n)$ denote the closed ball of radius r centered at the labeled point n, and consider the subset $A_r(n) = B_r(n) \cap V(n)$ of the Voronoi cell $V(n)$. The birth parameter of the Delaunay simplex $[n_0, \ldots, n_k]$ is defined to be the smallest value of r such that $A_r(n_0) \cap \cdots \cap A_r(n_k) \neq \emptyset$. A value of r is called a"threshold' if it is the birth parameter for some Delaunay simplex. The integer "level" p indexes the thresholds in increasing order, as illustrated in Figures 2 and 3:



Level $p = 1$ is the threshold $r = 0 = b([1]) = b([2]) = b([3]) = b([4])$.

Level $p = 2$ is the threshold $r = 1 = b([1, 2])$.

Figure 3.2: α-complex at level $p = 1$ and level $p = 2$.



Level $p = 3$ is the threshold $r = 1.12 = b([1, 3]) = b([1, 4]) = b([2, 3]) = b([2, 4])$.

Level $p = 4$ is the threshold $r = 1.25 = b([1, 2, 3]) = b([1, 2, 4])$

Figure 3.3: α-complex at level $p = 3$ and level $p = 4$.

The α construction produces a filtration of the Delaunay complex, and the simplicial homology [Hat01] of this filtered complex is described in terms of the barcode invariants [Oud15, EH09, Zom05]. Conventionally the filtration and the corresponding barcodes are indexed by the real-valued threshold parameter r, which for our example yields the H_1 barcode diagram of Figure 4:

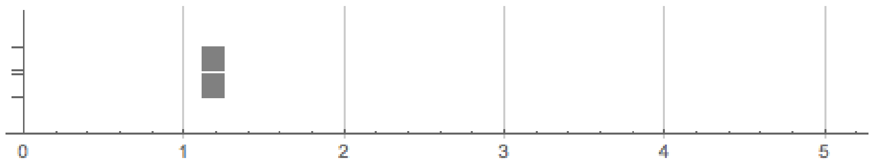


Figure 3.4: H_1 barcodes indexed by the real threshold parameter r.

The diagram indicates that the first homology H_1 detects two one-dimensional "holes" that appear at $r = 1.12$ and are filled in at $r = 1.25$. In this paper we will index filtrations and the corresponding barcodes by the integer-valued level p, which for our example yields the H_1 barcode diagram of Figure 5:

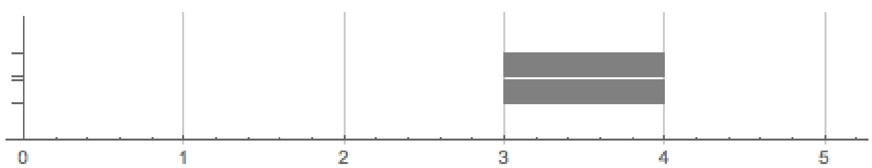


Figure 3.5: H_1 barcodes indexed by the integer level p.

This diagram indicates the same information, namely that the first homology H_1 detects two one-dimensional "holes" that appear at $p = 3$ (which corresponds to $r = 1.12$) and are filled in at $p = 4$ (which corresponds to $r = 1.25$).

3.2.2 Matrix Structural Theorem

For simplicity, we start with the ungraded version of the structural theorem. A *differential matrix* is a square matrix D satisfying $D^2 = 0$. We'll say a differential matrix is *Jordan* if it is in Jordan normal form, meaning it decomposes as a block-diagonal matrix built from copies of the two differential Jordan block matrices

$$\mathrm{J} = \begin{bmatrix} 0 \end{bmatrix}, \qquad \mathrm{K} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We'll say a differential matrix $\underline{D}$ is *almost-Jordan* if there exists a permutation matrix P such that the differential matrix $P^{-1}\underline{D}P$ is Jordan. Given an almost-Jordan differential matrix $\underline{D}$, it is trivial to construct such a permutation matrix P. We will say a square matrix B is *triangular* if it is upper-triangular and invertible.

The standard algorithm for computing persistent homology is based on the papers [HE02, ZC05b, ZC08]. The result of a persistent homology computation, not depending on a choice of algorithm, is conveniently summarized [VdS11, Oud15] as a matrix factorization:

Theorem 3.2.2. *(Ungraded Matrix Structural Theorem) Any differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is an almost-Jordan differential matrix and B is a triangular matrix.*

It is the triangular condition that makes this interesting: without the triangular condition, this would follow immediately from the ordinary Jordan normal form. Furthermore, the matrix $\underline{D}$ is unique by a standard result from Bruhat factorization, as reviewed in Section 3.6. We'll call $\underline{D}$ the *persistence canonical form* of the differential matrix D. A column of the triangular matrix B is in $\ker D$ iff the corresponding column of $\underline{D}$ is zero. We will say that B is *normalized* if each such column has diagonal entry equal to 1. It is always possible to normalize B by scalar multiplication of columns, but even with normalization B is not unique in general. A constructive proof of Theorem 3.2.2 follows from any of the algorithms for computing persistent homology. In Section 3.7 we discuss a simple version of the standard algorithm, with complete proofs using ordinary linear algebra of matrices.

Example 3.2.3. Consider the filtered simplicial complex:

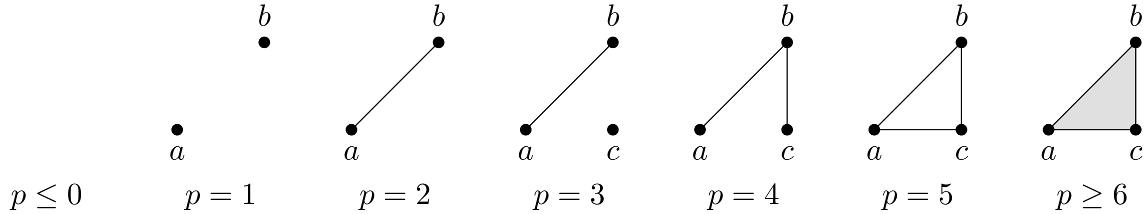


Figure 3.6: Filtered Simplicial Complex Number 1

With the usual convention for an adapted basis, the ordering of basis elements prioritizes the level of the filtration over the degree/dimension of the simplex. The initial basis of simplices is then ordered so the level (denoted by prescript) is nondecreasing, and within each level the degree (denoted by postscript) is nondecreasing. Using lexicographic order to break any remaining ties, the initial adapted basis is ${}_1a_0, {}_1b_0, {}_2ab_1, {}_3c_0, {}_4bc_1, {}_5ac_1, {}_6abc_2$, and the boundary operator over the field $\mathbb{F} = \mathbb{Q}$ of rationals is represented by the differential matrix

$$\mathrm{D} = \begin{array}{r} \\ {}_1a_0 \\ {}_1b_0 \\ {}_2ab_1 \\ {}_3c_0 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1a_0 & {}_1b_0 & {}_2ab_1 & {}_3c_0 & {}_4bc_1 & {}_5ac_1 & {}_6abc_2 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The persistence canonical form is

$$\underline{\mathrm{D}} = \begin{array}{r} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_2\underline{ab}_1 \\ {}_3\underline{c}_0 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array},$$

as verified by checking that $\underline{D} = B^{-1}DB$ for the triangular (and normalized) matrix

$$
B = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_2ab_1 \\ {}_3c_0 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \end{array} \\ \left[\begin{array}{ccccccc} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \end{array} .
$$

The persistence canonical form $\underline{D}$ is almost-Jordan in general, and in this example it happens to be actually Jordan. The matrix B represents the basis change to the new adapted basis ${}_1\underline{a}_0, {}_1\underline{b}_0, {}_2\underline{ab}_1, {}_3\underline{c}_0, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. The level remains nondecreasing because B is triangular. Each basis element retains pure degree, although Theorem 3.2.2 does not explicitly address issues of degree. The matrix $\underline{D}$ represents the boundary operator relative to the new adapted basis.

We prefer to prioritize degree over level in ordering the elements of an adapted basis. This has the advantage of encoding the degree in the matrix block structure. The following version of the structural theorem is then manifestly compatible with the grading by degree:

Theorem 3.2.4. *(Matrix Structural Theorem) Any block-superdiagonal differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is a block-superdiagonal almost-Jordan differential matrix and B is a block-diagonal triangular matrix.*

The block-diagonal structure of B ensures that the transformed basis elements retain pure degree. The persistence canonical form $\underline{D}$ inherits the block-superdiagonal structure of the differential D. It is always possible to normalize B by scalar multiplication of columns as in the ungraded case. Any of the algorithmic proofs of Theorem 3.2.2 [HE02, ZC05b, ZC08] can be used to prove Theorem 3.2.4 by keeping track of degrees. We discuss this point for the standard algorithm in Subsection 3.7.2.

Example 3.2.5. We again consider the filtered chain complex of Example 3.2.3, but with basis order prioritizing degree over level. Now the degree of basis elements (denoted by postscript) is nondecreasing, and within a degree the level (denoted by prescript) of basis elements is nondecreasing. Using lexicographic order to break any remaining ties, the initial adapted basis is now ${}_1a_0, {}_1b_0, {}_3c_0, {}_2ab_1, {}_4bc_1, {}_5ac_1, {}_6abc_2$, and the boundary operator over the field $\mathbb{F} = \mathbb{Q}$ of rationals is now represented by the block-superdiagonal differential matrix

$$D = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_3c_0 \\ {}_2ab_1 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1a_0 & {}_1b_0 & {}_3c_0 & {}_2ab_1 & {}_4bc_1 & {}_5ac_1 & {}_6abc_2 \end{array} \\ \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The persistence canonical form inherits the block-superdiagonal structure

$$\underline{D} = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_3\underline{c}_0 \\ {}_2\underline{ab}_1 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_3\underline{c}_0 & {}_2\underline{ab}_1 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \end{array} \\ \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

as verified by checking that $\underline{D} = B^{-1}DB$ for the block-diagonal triangular (and normalized) matrix

$$B = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_3c_0 \\ {}_2ab_1 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array} \begin{array}{c} \begin{array}{ccccccc} {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_3\underline{c}_0 & {}_2\underline{ab}_1 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \end{array} \\ \left[\begin{array}{ccc|ccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \end{array}$$

The persistence canonical form $\underline{D}$ is almost-Jordan, but not actually Jordan in this example. The matrix B represents the basis change to the new adapted basis ${}_1\underline{a}_0, {}_1\underline{b}_0, {}_3\underline{c}_0, {}_2\underline{ab}_1, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. Since B is block-diagonal, each basis element retains pure degree, and the degree remains nondecreasing. Since B is triangular, the level remains nondecreasing within each degree. The computation of this matrix B via the standard algorithm is worked out in Subsection 3.7.2.

3.2.3 Categorical Structural Theorem and Structural Equivalence

A Krull-Schmidt category is an additive category where objects decompose nicely as direct sums of indecomposable objects. A filtered complex will be called *basic* if its boundary operator can be represented by differential matrix consisting of a single Jordan block. We will use nonconstructive categorical methods to prove the following structural theorem for the category of filtered complexes:

Theorem 3.2.6. *(Categorical Structural Theorem) The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.*

In section 3.4 we will prove the equivalence of the matrix and the categorical versions of the structural theorem. One direction is proved in Section 3.1:

Proposition 3.2.7. *(Forward Structural Equivalence) The Matrix Structural Theorem implies the Categorical Structural Theorem.*

This is followed by a detailed example of a Krull-Schmidt decomposition. The other direction is proved in Subection 3.4.2:

Proposition 3.2.8. *(Reverse Structural Equivalence) The Categorical Structural Theorem implies the Matrix Structural Theorem.*

Combining the Categorical Structural Theorem 3.2.6 and the Reverse Structural Equivalence Proposition 3.2.8 yields a *nonconstructive* categorical proof of the Matrix Structural Theorem 3.2.4. This contrasts with the various *constructive* algorithmic proofs of Theorem 3.2.4, one of which is reviewed in Subsection 3.7.2. The constructive algorithmic proofs explain *how* persistent homology works, the nonconstructive proof explains *why* persistent homology works.

3.3 Proving the Categorical Structural Theorem

3.3.1 Persistence Objects and Filtered Objects

Persistence objects [Car09] and filtered objects [Aut] are described by categorical diagrams. Suppose that $\mathcal{X}$ is a linear Abelian category (and therefore Krull-Schmidt by Theorem 2.1.13). We will study persistence indexed by an integer $p \in \mathbb{Z}$, with $\leq$ denoting the standard partial order. A *persistence object* in $\mathcal{X}$ is a diagram ${}_{\bullet}X$ in the category $\mathcal{X}$ of type

$$\cdots \longrightarrow {}_{p-1}X \longrightarrow {}_{p}X \longrightarrow {}_{p+1}X \longrightarrow \cdots .$$

A morphism of persistence objects ${}_{\bullet}f : {}_{\bullet}X \to {}_{\bullet}X'$ is a commutative diagram of "ladder" type

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & {}_{p-1}X & \longrightarrow & {}_{p}X & \longrightarrow & {}_{p+1}X & \longrightarrow & \cdots \\
& & \Big\downarrow {\scriptstyle {}_{p-1}f} & & \Big\downarrow {\scriptstyle {}_{p}f} & & \Big\downarrow {\scriptstyle {}_{p+1}f} & & \\
\cdots & \longrightarrow & {}_{p-1}X' & \longrightarrow & {}_{p}X' & \longrightarrow & {}_{p+1}X' & \longrightarrow & \cdots
\end{array}$$

The category of persistence objects in $\mathcal{X}$ is Abelian, with pointwise kernels, cokernels, and direct sums. The set of morphisms ${}_\bullet X \to {}_\bullet X'$ between two persistence objects is a vector space, but not finite-dimensional in general. We will say a categorical diagram is *tempered* if all but finitely many of its arrows are iso(morphisms). We remark that the term "tame" is often used for conditions of this type, but we prefer the specificity afforded by the uses of the nonstandard term "tempered". Since $\mathcal{X}$ is linear, the set of morphisms ${}_\bullet X \to {}_\bullet X'$ between two tempered persistence objects is a finite-dimensional vector space. The tempered persistence objects comprise a strictly full Abelian subcategory of the persistence objects. Theorem 2.1.13 now yields:

Proposition 3.3.1. *Let $\mathcal{X}$ be a linear Abelian category. The category of tempered persistence objects in $\mathcal{X}$ is Krull-Schmidt.*

We next discuss subobjects in a linear Abelian category $\mathcal{X}$. We make the additional assumption that the category $\mathcal{X}$ is concrete, meaning that an object in $\mathcal{X}$ is a set with some additional structure, and a morphism in $\mathcal{X}$ is a map of sets compatible with the additional structure. For example, the linear Abelian category $\mathcal{V}$ of (finite-dimensional) vector spaces is a concrete linear Abelian category. An *inclusion* $X \hookrightarrow X'$ in $\mathcal{X}$ is an arrow that is an inclusion of the underlying sets. We say X is a *subobject* of X' iff such an inclusion arrow exists. An inclusion arrow is monic [Mac71, Awo10], and the composition of inclusion arrows is an inclusion arrow. Any object X' in $\mathcal{X}$ has a zero subobject $0 \hookrightarrow X'$, and is its own subobject $X' \hookrightarrow X'$. A subobject $X \hookrightarrow X'$ is *proper* if $X \neq X'$. A nonzero object is said to be *simple* if it does not have a proper nonzero subobject. A simple object is obviously indecomposable, but an indecomposable object need not be simple.

A filtered object in a concrete linear Abelian category $\mathcal{X}$ is a special type of tempered persistence object in $\mathcal{X}$. We say a tempered persistence object ${}_\bullet X$ in $\mathcal{X}$ is *bounded below* if there exists an integer j such that ${}_j X = 0$ whenever $p \leq j$. We say ${}_\bullet X$ is a *filtered object* if it is bounded below and if every arrow is an inclusion arrow:

$$\cdots \hookrightarrow {}_{p-1}X \hookrightarrow {}_pX \hookrightarrow {}_{p+1}X \hookrightarrow \cdots .$$

The filtered objects in $\mathcal{X}$ comprise a strictly full subcategory of the tempered persistence objects. The properties of monics have several consequences. A filtered object diagram has a categorical limit and a colimit [Mac71, Awo10]. The limit is 0 since the diagram is bounded below. The colimit X is ${}_kX$ for k sufficiently large (satisfying ${}_pX = X$ whenever $k \leq p$). Finally, any summand of a filtered object is isomorphic to a filtered object. Combining these facts with Lemma 3.3.1 yields:

Lemma 3.3.2. *Let $\mathcal{X}$ be a linear Abelian category. The category of filtered objects in $\mathcal{X}$ is Krull-Schmidt. A filtered object ${}_\bullet X$ in $\mathcal{X}$ is indecomposable iff its colimit X is an indecomposable object in $\mathcal{X}$.*

We note that the filtered objects comprise a subcategory of the tempered persistence objects, but this subcategory is not Abelian because a morphism of filtered objects may have a kernel and/or cokernel that is not a filtered object. So

Lemma 3.3.2 is not merely a corollary of Theorem 2.1.13. Finally we observe that the category of persistence objects in a (concrete) linear Abelian category is itself a (concrete) linear Abelian category, to which Lemma 3.3.2 applies.

3.3.2 Chain Complexes and Filtered Chain Complexes

A *persistence vector space* is a persistence object in the (concrete) linear Abelian category $\mathcal{X} = \mathcal{V}$ of (finite-dimensional) $\mathbb{F}$-vector spaces. Tempered persistence vector spaces are well-understood via the theory of quiver representations. A nonempty subset $I \subseteq \mathbb{Z}$ will be called an *interval* if $c \in I$ whenever $a \leq c \leq b$ with $a \in I$ and $b \in I$. We associate to an interval $I \subseteq \mathbb{Z}$ the *interval persistence vector space* ${}_\bullet I$ constructed as follows: ${}_pI = \mathbb{F}$ whenever $p \in I$, ${}_pI = 0$ whenever $p \notin I$, and every arrow $\mathbb{F} \to \mathbb{F}$ is the identity morphism 1. We will often omit the bullet prescript when context allows. For example, the interval persistence vector space $[1,4) = {}_\bullet[1,4)$ is the diagram of vector spaces

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{F} & \xrightarrow{1} & \mathbb{F} & \xrightarrow{1} & \mathbb{F} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & \end{array}$$

associated to the interval $[1,4) = \{1,2,3\} \subseteq \mathbb{Z}$. Proposition 3.3.1 applies to the linear Abelian category of tempered persistence vector spaces. Furthermore, the well-studied representation theory of A_n quivers (see e.g. [Sch14]) carries over by a limiting argument to prove the following structural theorem for the category of tempered persistence vector spaces:

Theorem 3.3.3. *The category of tempered persistence vector spaces is Krull-Schmidt. A tempered persistence vector space is indecomposable iff it is isomorphic to an interval.*

Theorem 3.3.3 can be applied to cochain complexes. A *cochain complex*, or *cocomplex* for short, is a tempered persistence vector space ${}_{\bullet}V$

$$\cdots \longrightarrow {}_{p-1}V \xrightarrow{\partial^p} {}_{p}V \xrightarrow{\partial^{p+1}} {}_{p+1}V \longrightarrow \cdots$$

with the property that the composition of successive arrow $\partial^{p+1} \circ \partial^p$ is zero. The kernel of a morphism between cocomplexes is a cocomplex, as is the cokernel, so the cocomplexes comprise a strictly full Abelian subcategory of the tempered persistence vector spaces. Theorem 2.1.13 and Theorem 3.3.3 now yield the structural result:

Proposition 3.3.4. *The category $\mathcal{C}^{op}$ of cocomplexes is linear and Abelian, and therefore Krull-Schmidt. A cocomplex is indecomposable iff it is isomorphic to an interval cocomplex.*

Chain complexes are dual to cochain complexes. A *complex* (short for chain complex) is a tempered diagram $V_{\bullet}$ in $\mathcal{V}$ of type

$$\cdots \longleftarrow V_{n-2} \xleftarrow[\partial_{n-1}]{} V_{n-1} \xleftarrow[\partial_n]{} V_n \longleftarrow \cdots$$

with the property that the composition of successive arrows $\partial_{n-1} \circ \partial_n$ is zero. A morphism of complexes $f_{\bullet} : V_{\bullet} \to V'_{\bullet}$ is a commutative ladder diagram. The category $\mathcal{V}$ of vector spaces is isomorphic to its opposite category $\mathcal{V}^{op}$ via the duality functor that takes a vector space to its dual and a linear map to its transpose/adjoint [Mac71, Awo10]. Duality takes the category $\mathcal{C}^{op}$ of cocomplexes to the category of complexes $\mathcal{C}$. A complex is called an *interval complex* if its dual is an interval cocomplex, and Proposition 3.3.4 becomes:

Proposition 3.3.5. *The category $\mathcal{C}$ of complexes is linear and Abelian, and therefore Krull-Schmidt. A complex is indecomposable iff it is isomorphic to an interval complex.*

The interval complexes are easily classified. An interval complex $I_\bullet$ is associated to an interval $I \subseteq \mathbb{Z}$ as follows: $n \in I$ whenever $I_n = \mathbb{F}$, and $n \notin I$ whenever $I_n = 0$. Since adjacent nonzero arrows in a complex cannot be iso(morphisms), the interval complexes are in bijective correspondence with the intervals $I \subseteq \mathbb{Z}$ of cardinality at most two. We will often omit the bullet postscript when the context allows. We denote by $J[n] = \{n\} \subseteq \mathbb{Z}$ the intervals of cardinality one. For example, the complex $J[1] = J[1]_\bullet$ is the diagram of vector spaces

$$\begin{array}{ccccccccc} \cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \cdots \\ & n=0 & & n=1 & & n=2 & & n=3 & \end{array}$$

The indecomposable complex $J[n]$ is simple. We denote by $K[n] = [n, n+1] \subseteq \mathbb{Z}$ the intervals of cardinality two. For example, the complex $K[1] = K[1]_\bullet$ is the diagram of vector spaces

$$\begin{array}{ccccccccc} \cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \underset{1}{\longleftarrow} & \mathbb{F} & \longleftarrow & 0 & \longleftarrow \cdots \\ & n=0 & & n=1 & & n=2 & & n=3 & \end{array}$$

The indecomposable complex $K[n]$ has exactly one nonzero proper subobject $J[n] \hookrightarrow K[n]$. For example, the inclusion of complexes $J[1] \hookrightarrow K[1]$ is the commutative ladder diagram

$$\begin{array}{ccccccccc} \cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \cdots \\ & \big\downarrow & & \big\downarrow{\scriptstyle 1} & & \big\downarrow & & \big\downarrow & \\ \cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \underset{1}{\longleftarrow} & \mathbb{F} & \longleftarrow & 0 & \longleftarrow \cdots \\ & & & & & & & & \\ & n=0 & & n=1 & & n=3 & & n=4 & \end{array}$$

We now return to the the Categorical Structural Theorem 3.2.6. A filtered complex is a diagram in the category $\mathcal{C}$ of complexes

$$\cdots \hookrightarrow {}_{p-1}V_\bullet \hookrightarrow {}_pV_\bullet \hookrightarrow {}_{p+1}V_\bullet \hookrightarrow \cdots .$$

We will say a filtered complex is *basic* if its colimit $V_\bullet$ is isomorphic to an interval complex. The first statement of Proposition 3.3.5 tells us that the category $\mathcal{C}$ of complexes is linear and Abelian. Then Lemma 3.3.2 tells us that the category of filtered complexes is Krull-Schmidt. The second statement of Proposition 3.3.5 classifies the indecomposable filtered complexes, completing the proof of:

Theorem 1.6. (Categorical Structural Theorem) The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.

The basic filtered complexes are easily classified since we know all proper subobjects of interval complexes, namely $0 \hookrightarrow J[n]$, $0 \hookrightarrow K[n]$, and $J[n] \hookrightarrow K[n]$. Details and examples of basic filtered complexes appear in Section 3.5.

3.4 Categorical Frameworks for Persistent Homology

3.4.1 Standard Framework using Persistence Vector Spaces

The structural theorem for the category of tempered persistence vector spaces, Theorem 3.3.3, is the foundation for the standard framework for persistent homology.

For each integer n, the *homology* of degree n is a functor $H_n : \mathcal{C} \to \mathcal{V}$ from the category $\mathcal{C}$ of complexes to the category $\mathcal{V}$ of vector spaces. An object C in $\mathcal{C}$ is a diagram of vector spaces

$$\cdots \longleftarrow V_{n-1} \xleftarrow[\partial_n]{} V_n \xleftarrow[\partial_{n+1}]{} V_{n+1} \longleftarrow \cdots$$

where $\partial_n \circ \partial_{n+1} = 0$. Then $\mathrm{im}\, \partial_{n+1} \hookrightarrow \ker \partial_n$ is a subobject inclusion of vector spaces, and the corresponding quotient vector space is the homology

$$H_n(C) = \ker \partial_n / \mathrm{im}\, \partial_{n+1}.$$

More generally, the homology functor H_n takes a diagram in $\mathcal{C}$ to a diagram in $\mathcal{V}$.

Denote by $\mathcal{F}$ the category of filtered complexes. An object F in $\mathcal{F}$ is a diagram of complexes

$$\cdots \hookrightarrow {}_{p-1}V_\bullet \hookrightarrow {}_{p}V_\bullet \hookrightarrow {}_{p+1}V_\bullet \hookrightarrow \cdots,$$

which is tempered and bounded below, and which has monic arrows. Denote by $\mathcal{P}$ the category of tempered persistence vector spaces. The homology functor H_n takes the the diagram F to the diagram of vector spaces

$$\cdots \longrightarrow H_n({}_{p-1}V_\bullet) \longrightarrow H_n({}_{p}V_\bullet) \longrightarrow H_n({}_{p+1}V_\bullet) \longrightarrow \cdots,$$

which is tempered and bounded below, but which need not have monic arrows in general. An object F in $\mathcal{F}$ then goes to an object $P_n(F)$ in $\mathcal{P}$. Similarly a morphism in $\mathcal{F}$, which is a commutative ladder diagram of complexes, goes to a morphism in $\mathcal{P}$, which is a commutative ladder diagram of vector spaces. The resulting functor $P_n : \mathcal{F} \to \mathcal{P}$ is the *persistent homology* of degree n.

The standard framework for studying the persistent homology functors $P_n : \mathcal{F} \to \mathcal{P}$ is based on the structural theorem for the category $\mathcal{P}$, Theorem 3.3.3. It suffices to work with an appropriate Krull-Schmidt subcategory of the Krull-Schmidt category $\mathcal{P}$. A filtered complex F is studied by decomposing the persistence vector space $P_n(F)$ as a sum of indecomposables. Since the diagram $P_n(F)$ is bounded below, all of its indecomposables are bounded below. The persistence vector spaces that are bounded below comprise the a full Abelian subcategory of $\mathcal{P}$, which we will denote by $\operatorname{im} P_n$. Despite the notation, the category $\operatorname{im} P_n$ does not depend on n; it is always the same subcategory of $\mathcal{P}$. The isomorphism class of an indecomposables in the Krull-Schmidt category $\operatorname{im} P_n$ is described by the familar barcode. An interval $I \subseteq \mathbb{Z}$ will be called a *barcode* if it is bounded below. A *barcode persistence vector space* is a persistence vector space ${}_\bullet I$ corresponding to a barcode $I \subseteq \mathbb{Z}$.

Theorem 3.4.1. *The persistent homology functor $P_n : \mathcal{F} \to \mathcal{P}$ factors as*

$$\mathcal{F} \to \operatorname{im} P_n \to \mathcal{P}.$$

The category $\operatorname{im} P_n$ is Krull-Schmidt. An object in $\operatorname{im} P_n$ is indecomposable iff it is isomorphic to a barcode persistence vector space.

We can now express the standard framework for persistent homology in terms of the functor $\mathcal{F} \to \operatorname{im} P_n$ which takes a filtered complex to a persistence vector space in $\operatorname{im} P_n$. The Krull-Schmidt property of $\operatorname{im} P_n$ then allows decomposition as a sum of indecomposables. Each indecomposable in $\operatorname{im} P_n$ is a barcode persistence vector space, which is specified up to isomorphism by its barcode $I \subseteq \mathbb{Z}$. An object in $\operatorname{im} P_n$ is determined up to isomorphism by its set of barcodes.

3.4.2 Alternate Framework using Quotient Categories

The structural theorem for the category of filtered complexes, Theorem 3.2.6, is the foundation for an alternate framework for persistent homology.

We will work with an appropriate Krull-Schmidt quotient category of the Krull-Schmidt category $\mathcal{F}$. Recall in general [Mac71] that an object of a quotient category of $\mathcal{F}$ is an object of $\mathcal{F}$, and a morphism is an equivalence class of morphisms of $\mathcal{F}$.

Our quotient category $\operatorname{coim} P_n$ is defined via the following equivalence relation (congruence) on morphisms: two morphisms f and f' in $\mathcal{F}$ are equivalent iff the morphisms $P_n(f)$ and $P_n(f')$ in $\mathcal{P}$ are equal. Note that the category $\operatorname{coim} P_n$ now does depend on the integer n; each $\operatorname{coim} P_n$ is a different quotient category of $\mathcal{F}$.

Example 3.4.2. We recall the filtered simplicial complex of Example 3.2.3:

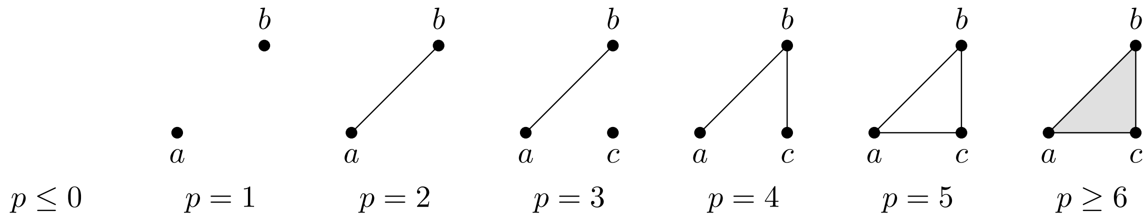


Figure 3.7: Repeat of Filtered Simplicial Complex Number 1

We first consider the subobject:

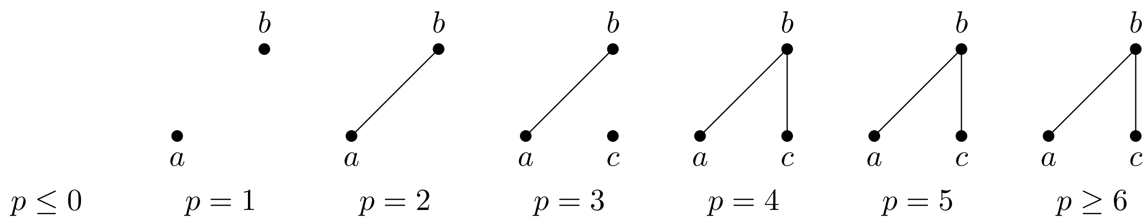


Figure 3.8: Filtered Simplicial Complex Number 2

In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\operatorname{coim} P_0$, the subobject inclusion becomes an isomorphism between nonzero objects. In the quotient category $\operatorname{coim} P_1$, this becomes an a proper zero subobject.

Now consider another subobject:

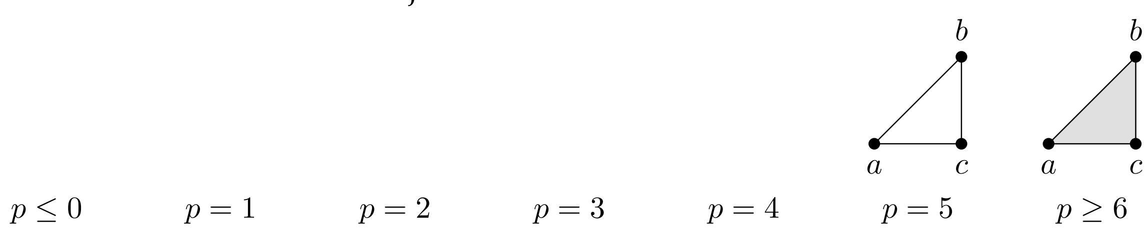


Figure 3.9: Filtered Simplicial Complex Number 3

In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\operatorname{coim} P_0$, this remains a proper nonzero subobject. In the quotient category $\operatorname{coim} P_1$, the inclusion morphism becomes an isomorphism between nonzero objects.

We recall that a quotient of a Krull-Schmidt category is Krull-Schmidt in general. This is because an indecomposable in $\mathcal{F}$ becomes either a zero object or an indecomposable with a local endomorphism ring in the quotient category (see e.g. [Liu10] p. 431). The classification of indecomposables in the quotient category $\operatorname{coim} P_n$ is now easily obtained from Theorem 3.2.6. This is independent of the well-known classification of indecomposables in the category of persistence vector spaces (Theorem 3.3.3). Using the classification of indecomposables in each of the Krull-Schmidt categories $\operatorname{coim} P_n$ and $\operatorname{im} P_n$, it is now easy to verify that the functor $\operatorname{coim} P_n \to \operatorname{im} P_n$ is full, faithful, and essentially surjective. Recalling [Mac71, Awo10] that a functor satisfying these conditions is an equivalence of categories, we have:

Theorem 3.4.3. *The persistent homology functor $P_n : \mathcal{F} \to \mathcal{P}$ factors as*

$$\mathcal{F} \to \operatorname{coim} P_n \to \operatorname{im} P_n \to \mathcal{P},$$

where the functor $\operatorname{coim} P_n \to \operatorname{im} P_n$ is an equivalence of categories.

The isomorphism class of an indecomposable in the Krull-Schmidt category $\operatorname{coim} P_n$ can be specified by I_n, where the integer n is the degree of the homology, and $I \subseteq \mathbb{Z}$ is a barcode specifying the isomorphism class of the corresponding indecomposable in $\operatorname{im} P_n$. The example in Section 3.5 will show how the barcode I can be understood directly in terms of the indecomposable filtered complex, without reference to homology.

We can now express the alternate framework for persistent homology in terms of the functor $\mathcal{F} \to \operatorname{coim} P_n$ which takes a filtered complex in $\mathcal{F}$ to the same filtered complex viewed as an object in $\operatorname{coim} P_n$. The Krull-Schmidt property of $\operatorname{coim} P_n$ then allows decomposition as a sum of indecomposables in $\operatorname{coim} P_n$. Each indecomposable is specified up to isomorphism by I_n. An object in $\operatorname{coim} P_n$ is determined up to isomorphism by the collection of of intervals $I_n \subseteq \mathbb{Z}$ indexing its decomposition. This framework obviates the need for auxiliary objects such as persistence vector spaces, while providing exactly the same information about filtered complexes as the standard framework.

3.5 Proving Structural Equivalence

3.5.1 Forward Structural Equivalence

We now consider in more detail matrix representations of a filtered complex and its automorphisms. The first step is to associate to a filtered complex a finite-dimensional vector space with an appropriately adapted basis. A filtered complex is a diagram of complexes indexed by the integer level p, displayed below together with its colimit:

$$\cdots \hookrightarrow {}_{-1}V_\bullet \hookrightarrow {}_{0}V_\bullet \hookrightarrow {}_{1}V_\bullet \hookrightarrow {}_{2}V_\bullet \hookrightarrow {}_{3}V_\bullet \hookrightarrow \cdots \qquad \underset{\text{colim}}{V_\bullet}$$

A filtered complex becomes a "lattice" diagram of finite-dimensional vector spaces:

$$
\begin{array}{ccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\cdots \hookrightarrow & {}_{-1}V_2 & \hookrightarrow & {}_{0}V_2 & \hookrightarrow & {}_{1}V_2 & \hookrightarrow & {}_{2}V_2 & \hookrightarrow & {}_{3}V_2 & \hookrightarrow \cdots & & V_2 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow{\scriptstyle \partial_2} \\
\cdots \hookrightarrow & {}_{-1}V_1 & \hookrightarrow & {}_{0}V_1 & \hookrightarrow & {}_{1}V_1 & \hookrightarrow & {}_{2}V_1 & \hookrightarrow & {}_{3}V_1 & \hookrightarrow \cdots & & V_1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow{\scriptstyle \partial_1} \\
\cdots \hookrightarrow & {}_{-1}V_0 & \hookrightarrow & {}_{0}V_0 & \hookrightarrow & {}_{1}V_0 & \hookrightarrow & {}_{2}V_0 & \hookrightarrow & {}_{3}V_0 & \hookrightarrow \cdots & & V_0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow{\scriptstyle \partial_0} \\
\cdots \hookrightarrow & {}_{-1}V_{-1} & \hookrightarrow & {}_{0}V_{-1} & \hookrightarrow & {}_{1}V_{-1} & \hookrightarrow & {}_{2}V_{-1} & \hookrightarrow & {}_{3}V_{-1} & \hookrightarrow \cdots & & V_{-1} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \text{colim}
\end{array}
$$

In the colimit complex, the composition $\partial_{n-1} \circ \partial_n : V_n \to V_{n-2}$ is zero for all n. Since the diagram is tempered, $\partial_{n-1} \circ \partial_n$ is an isomorphism for all but finitely many n. It follows that the complex is bounded, meaning that the vector space V_n is zero-dimensional for all but finitely many n. The direct sum $V = \oplus_{n \in \mathbb{Z}} V_n$ is then a finite-dimensional vector space associated to the filtered complex. A vector $v \in V$ is said to have *pure degree* iff $v \in V_n \subseteq V$ for some integer n. The integer n is then called the *degree* of the pure degree vector v, and is encoded by a postscript v_n. The *(filtration) level* of a degree n vector $v_n \in V$ is the smallest integer p such that $v_n \in {}_pV_n \subseteq V_n$. The level of the degree n vector v_n is encoded by a prescript ${}_pv_n$. An *adapted basis* of a filtered complex is a basis of the vector space V satisfying the three conditions:

- Every basis element has pure degree.

- For each n and p, the vector space ${}_pV_n$ is spanned by the basis vectors with degree equal to n and level less than or equal to p.

- The basis elements are ordered so that degree is nondecreasing, and within each degree the level is nondecreasing.

A block-diagonal triangular matrix B transforms an adapted basis to a new adapted basis, representing an automorphism of the filtered complex. Here we assume that the block structure of the matrix is compatible with degrees of the basis elements.

The *colimit boundary* $\partial = \oplus_{n \in \mathbb{Z}} \partial_n$ of a filtered complex is a linear endomorphism $\partial : V \to V$. The colimit boundary ∂ is represented by a matrix D relative to an adapted basis. The matrix representative D is block-superdiagonal because ∂ is homogeneous of degree -1, and $D^2 = 0$ because $\partial^2 = 0$. If additionally the matrix representative is almost-Jordan, we will say the adapted basis is *special*. The Matrix Structural Theorem 3.2.4 yields:

Proposition 3.5.1. *A filtered complex admits a special adapted basis.*

Proof. Choose an adapted basis. Let D be the block-superdiagonal differential matrix representing ∂ relative to the adapted basis. Theorem 3.2.4 provides a block-diagonal triangular matrix B such that $\underline{D} = B^{-1}DB$ is almost-Jordan. So the matrix B transforms the original adapted basis to a special adapted basis. ∎

Corollary 3.5.2. *A filtered complex admits a finite decomposition as sum of basic filtered complexes.*

Proof. Choose a special adapated basis, and denote by $\underline{D}$ the corresponding almost-Jordan block-superdiagonal differential matrix representative. Let P be a permutation matrix such that the matrix $P^{-1}\underline{D}P$ is Jordan. Each Jordan block of this matrix represents a basic subobject of the filtered complex. The decomposition into Jordan blocks represents the decomposition of the filtered complex as a direct sum of basic filtered complexes. ∎

To verify the Krull-Schmidt property, we will also need:

Lemma 3.5.3. *A basic filtered complex has local endomorphism ring.*

Proof. We first show that the colimit complex of a basic filtered complex has local endomorphism ring. The colimit complex is isomorphic to an interval complex. An interval complex is an indecomposable in the linear Abelian category of complexes, so it has local endomorphism ring by Atiyah's Criterion 2.1.13. (Or less abstractly, it is easy to check that the endomorphism ring of an interval complex is isomorphic to the field $\mathbb{F}$.)

The proof is completed by checking that that the endomorphism ring of a basic filtered complex maps isomorphically to the endomorphism ring of its colimit interval complex. In general, the endomorphism ring of a filtered object maps *injectively* to the endomorphism ring of its colimit. We need to show that the endomorphism ring of a basic filtered complex maps *surjectively* to the endomorphism ring of its colimit. It suffices to show that an endomorphism of an interval complex restricts to an endomorphism of any subobject. There are two types of interval complexes to consider. If the interval complex is isomorphic to $J[n]$, then the subobjects are 0 and $J[n]$, and any endomorphism restricts. If the interval complex is isomorphic to $K[n]$, then the subobjects are 0, $J[n]$, and $K[n]$, and any endmorphism restricts. ∎

Assembling the pieces proves the main result of this section:

Proposition 1.7. (Forward Structural Equivalence) The Matrix Structural Theorem implies the Categorical Structural Theorem.

Proof. We first prove that a filtered complex is indecomposable iff it is basic. A basic filtered complex has a local endomorphism ring by Lemma 3.5.3, so it is indecomposable by Lemma 2.1.11. An indecomposable filtered complex is a finite direct sum of basic filtered complexes by Corollary 3.5.2. The direct sum cannot have more than one summand, because that would contradict the indecomposability. So an indecomposable filtered complex is basic.

Now it remains to check the two conditions of Definition 2.1.12. Since a basic filtered complex is indecomposable, Corollary 3.5.2 asserts that every filtered complex admits a finite decomposition as a sum of indecomposables. Since an indecomposable filtered complex is basic, Lemma 3.5.3 asserts that every indecomposable has a local endomorphism ring. ∎

Example 3.5.4. Let F be the filtered complex of Example 3.2.5. The initial adapted basis consists of appropriately ordered simplices: ${}_1a_0, {}_1b_0, {}_3c_0, {}_2ab_1, {}_4bc_1, {}_5ac_1, {}_6abc_2$. The block-superdiagonal differential matrix D represents the colimit boundary operator relative to the initial adapted basis.

The triangular block-diagonal matrix B represents an automorphism of the filtered complex. This automorphism takes the initial adapted basis to the transformed adapted basis ${}_1\underline{a}_0, {}_1\underline{b}_0, {}_3\underline{c}_0, {}_2\underline{ab}_1, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. This transformed adapted basis is special, because the block-superdiagonal differential matrix representative $\underline{D} = B^{-1}DB$ is almost-Jordan:

$$
\underline{D} = \begin{array}{c|ccccccc|}
 & {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_3\underline{c}_0 & {}_2\underline{ab}_1 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\
\hline
{}_1\underline{a}_0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_1\underline{b}_0 & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 & 0 \\
{}_3\underline{c}_0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 \\
{}_2\underline{ab}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_4\underline{bc}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_5\underline{ac}_1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
{}_6\underline{abc}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

We have retained the shading denoting the super-diagonal blocks, and we have also boldfaced the nonzero entries and the diagonal entries of zero columns. An almost-Jordan differential matrix $P^{-1}\underline{D}P$ is Jordan iff the matrix $\underline{D}P$, which is related to $\underline{D}$ by a permutation of columns, has each boldfaced **1** immediately following the boldfaced **0** in the same row. Permuting columns 3 and 4 suffices for this example:

$$
P = \begin{array}{c|ccccccc|}
 & {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\
\hline
{}_1\underline{a}_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_1\underline{b}_0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
{}_3\underline{c}_0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
{}_2\underline{ab}_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
{}_4\underline{bc}_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
{}_5\underline{ac}_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
{}_6\underline{abc}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

produces the Jordan matrix

$$
P^{-1}\underline{D}P = \begin{array}{c|ccccccc|}
 & {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\
\hline
{}_1\underline{a}_0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_1\underline{b}_0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 \\
{}_2\underline{ab}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_3\underline{c}_0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 \\
{}_4\underline{bc}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}_5\underline{ac}_1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
{}_6\underline{abc}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} .
$$

The decomposition of the Jordan matrix into its Jordan blocks represents the decomposition of the filtered complex into indecomposable/basic summands. We now list the indecomposable summands, denoting by $\langle v \rangle$ the linear span of a vector $v \in V$:

- The Jordan block matrix $\begin{array}{cc} & {}_1\underline{a}_0 \\ {}_1\underline{a}_0 & \left[\begin{array}{c} \mathbf{0} \end{array}\right] \end{array}$ represents the filtered complex

$$
\begin{array}{ccccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\cdots \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\cdots \hookrightarrow & 0 & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow \cdots & & \langle {}_1\underline{a}_0 \rangle. \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\cdots \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\
 & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & & \text{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccccc}
\cdots \hookrightarrow & 0 & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow \cdots & & J[0], \\
 & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & & \text{colim}
\end{array}
$$

which has the interval complex $J[0]$ as colimit. The quotient functor $\mathcal{F} \to \operatorname{coim} P_0$ takes the filtered complex to an indecomposable in the isomorphism class $[1, \infty)_0$. This corresponds under the equivalence $\operatorname{coim} P_0 \to \operatorname{im} P_0$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[1, \infty)$,

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow \cdots . \\
 & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 &
\end{array}
$$

For any $n \neq 0$, the quotient functor $\mathcal{F} \to \operatorname{coim} P_n$ takes the filtered complex to a zero object.

- The Jordan block matrix $\begin{array}{cc} & \begin{array}{cc} {}_1\underline{b}_0 & {}_2\underline{ab}_1 \end{array} \\ \begin{array}{c} {}_1\underline{b}_0 \\ {}_2\underline{ab}_1 \end{array} & \left[\begin{array}{cc} \mathbf{0} & \mathbf{1} \\ 0 & 0 \end{array}\right] \end{array}$ represents the filtered complex

$$
\begin{array}{ccccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & \langle {}_2\underline{ab}_1 \rangle & \hookrightarrow & \langle {}_2\underline{ab}_1 \rangle & \hookrightarrow & \langle {}_2\underline{ab}_1 \rangle & \hookrightarrow & \langle {}_2\underline{ab}_1 \rangle & \hookrightarrow \cdots & \langle {}_2\underline{ab}_1 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & \langle {}_1\underline{b}_0 \rangle & \hookrightarrow & \langle {}_1\underline{b}_0 \rangle & \hookrightarrow & \langle {}_1\underline{b}_0 \rangle & \hookrightarrow & \langle {}_1\underline{b}_0 \rangle & \hookrightarrow & \langle {}_1\underline{b}_0 \rangle & \hookrightarrow \cdots & \langle {}_1\underline{b}_0 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \mathrm{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccccc}
\cdots & \hookrightarrow & 0 & \hookrightarrow & J[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow \cdots & K[0], \\
 & & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \mathrm{colim}
\end{array}
$$

which has the interval complex $K[0]$ as colimit. The quotient functor $\mathcal{F} \to \operatorname{coim} P_0$ takes the filtered complex to an indecomposable in the isomorphism class $[1,2)_0$. This corresponds under the equivalence $\operatorname{coim} P_0 \to \operatorname{im} P_0$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[1,2)$,

$$
\begin{array}{cccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots . \\
 & & p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 &
\end{array}
$$

For any $n \neq 0$, the quotient functor $\mathcal{F} \to \operatorname{coim} P_n$ takes the filtered complex to a zero object.

- The Jordan block matrix $\begin{array}{c|cc} & {}_3\underline{c}_0 & {}_4\underline{bc}_1 \\ \hline {}_3\underline{c}_0 & \mathbf{0} & \mathbf{1} \\ {}_4\underline{bc}_1 & 0 & 0 \end{array}$ represents the filtered complex

$$
\begin{array}{ccccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & \langle {}_4\underline{bc}_1 \rangle & \hookrightarrow & \langle {}_4\underline{bc}_1 \rangle & \hookrightarrow & \langle {}_4\underline{bc}_1 \rangle & \hookrightarrow \cdots & \langle {}_4\underline{bc}_1 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & \langle {}_3\underline{c}_0 \rangle & \hookrightarrow & \langle {}_3\underline{c}_0 \rangle & \hookrightarrow & \langle {}_3\underline{c}_0 \rangle & \hookrightarrow & \langle {}_3\underline{c}_0 \rangle & \hookrightarrow \cdots & \langle {}_3\underline{c}_0 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6 & & \text{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccccc}
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & J[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow \cdots & K[0] \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6 & & \text{colim}
\end{array}
$$

which has the interval complex $K[0]$ as colimit. The quotient functor $\mathcal{F} \to \operatorname{coim} P_0$ takes the filtered complex to an indecomposable in the isomorphism class $[3,4)_0$. This corresponds under the equivalence $\operatorname{coim} P_0 \to \operatorname{im} P_0$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[3,4)$,

$$
\begin{array}{cccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots . \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6 &
\end{array}
$$

For any $n \neq 0$, the quotient functor $\mathcal{F} \to \operatorname{coim} P_n$ takes the filtered complex to a zero object.

- The Jordan block matrix $\begin{array}{c|cc} & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\ \hline {}_5\underline{ac}_1 & \mathbf{0} & \mathbf{1} \\ {}_6\underline{abc}_2 & 0 & 0 \end{array}$ represents the filtered complex

$$
\begin{array}{ccccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & \langle {}_6\underline{abc}_2 \rangle & \hookrightarrow \cdots & \langle {}_6\underline{abc}_2 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & \langle {}_5\underline{ac}_1 \rangle & \hookrightarrow & \langle {}_5\underline{ac}_1 \rangle & \hookrightarrow \cdots & \langle {}_5\underline{ac}_1 \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6 & & \mathrm{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccccc}
\cdots & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow & J[1] & \hookrightarrow & K[1] & \hookrightarrow \cdots & K[1] \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6 & & \mathrm{colim}
\end{array}
$$

which has the interval complex $K[1*]$ as colimit. The quotient functor $\mathcal{F} \to \operatorname{coim} P_1$ takes the filtered complex to an indecomposable in the isomorphism class $[5,6)_1$. This corresponds under the equivalence $\operatorname{coim} P_1 \to \operatorname{im} P_1$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[5,6)$,

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \longrightarrow \cdots \\
 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & p=6
\end{array}
$$

For any $n \neq 1$, the quotient functor $\mathcal{F} \to \operatorname{coim} P_n$ takes the filtered complex to a zero object.

This completes the decomposition of the filtered complex F in the category $\mathcal{F}$. As an object in the quotient category $\operatorname{coim} P_0$, the filtered complex F is isomorphic to

$[0,\infty)_0 \oplus [1,2)_0 \oplus [3,4)_0$. As an object in the quotient category $\operatorname{coim} P_1$, the filtered complex F is isomorphic to $[5,6)_1$. For any other value of n, the filtered complex F is a zero object in the quotient category $\operatorname{coim} P_n$.

3.5.2 Reverse Structural Equivalence

Special adapted bases help to intermediate between the Matrix Structural Theorem and Categorical Structural Theorem. In Proposition 3.5.1, we established the existence of a special adapted basis using the Matrix Structural Theorem 3.2.4. Now in the reverse direction, we establish the existence of a special adapted basis using the Categorical Structural Theorem 3.2.6:

Proposition 3.5.5. *A filtered complex admits a special adapted basis.*

Proof. The Categorical Structural Theorem decomposes the filtered complex as a finite direct sum of indecomposables. Each indecomposable summand is a basic filtered complex, so it admits a special adapted basis. With appropriate ordering, the union over the summands of these basis elements is a special adapted basis for the direct sum filtered complex. ∎

An automorphism of a filtered complex transforms an adapted basis to another adapted basis. The change of basis is represented by a matrix B, which is block-diagonal because an automorphism preserves the degree of basis elements. But the matrix B need not be triangular in general. We call a filtered complex *nondegenerate* if $\dim_{p+1} V_n \leq 1 + \dim_p V_n$ for any p and any n.

Lemma 3.5.6. *If a filtered complex is nondegenerate, then any change of adapated basis is represented by a triangular matrix B.*

Proof. An automorphism takes a basis element of degree n and level p to a linear combination of basis elements of degree n and level at most p. A filtered complex is nondegenerate iff an adapted basis contains no pair of elements with the same degree and same level. In this case the linear combination does not contain any basis elements that appear later in the ordering of the basis. The matrix B is then triangular, since it has no nonzero entries below the diagonal. ∎

We will construct nondegenerate filtered complexes by using the upper-left submatrices of a differential matrix. We illustrate submatrices with an example:

Example 3.5.7. The upper-left submatrices are indicated below for a block-superdiagonal differential matrix $D : \mathbb{Q}^7 \to \mathbb{Q}^7$.

$$\mathrm{D} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that for each integer $0 < p < 7$, the upper-left submatrix ${}_pD : \mathbb{Q}^p \to \mathbb{Q}^p$ is itself a block-superdiagonal differential matrix. We remark that the matrix D had appeared previously in Example 3.2.5, representing the degenerate (not nondegenerate) filtered complex of Example 3.2.3.

Lemma 3.5.8. *Any block-superdiagonal differential matrix D represents the colimit boundary of some nondegenerate filtered complex.*

Proof. Let $D : \mathbb{F}^m \to \mathbb{F}^m$ be a block-superdiagonal differential matrix. We construct a filtered complex

$$\cdots \hookrightarrow {}_{-1}V_\bullet \hookrightarrow {}_0V_\bullet \hookrightarrow {}_1V_\bullet \hookrightarrow {}_2V_\bullet \hookrightarrow {}_3V_\bullet \hookrightarrow \cdots \qquad \underset{\text{colim}}{V_\bullet}$$

by specifying for each integer p the complex ${}_pV_\bullet$ at level p:

- For $p \leq 0$, the complex is the zero complex.
- For $1 < p < m$, the complex is specified by the block-superdiagonal differential submatrix ${}_pD : \mathbb{F}^p \to \mathbb{F}^p$.
- For $m \leq p$, the complex is specified by the initial block-superdiagonal differential matrix $D : \mathbb{F}^m \to \mathbb{F}^m$.

The arrows are the subobject inclusions ${}_pV_\bullet \hookrightarrow {}_{p+1}V_\bullet$. Then the diagram is a filtered complex since the zero complex is a limit and the complex $D : \mathbb{F}^m \to \mathbb{F}^m$ is a colimit. It only remains to observe that the filtered complex is nondegenerate, and that the matrix D represents its colimit boundary. ∎

Note that the block structure of the differential matrix D is important in the preceding proof. If a differential matrix does not have block-superdiagonal structure, then an upper-left submatrix need not be a differential matrix in general.

Now we have assembled the ingredients to prove:

Proposition 1.8. (Reverse Structural Equivalence) The Categorical Structural Theorem implies the Matrix Structural Theorem.

Proof. Let D be a block-superdiagonal differential matrix. Lemma 3.5.8 lets us choose a nondegenerate filtered complex that is represented by D. Proposition 3.5.5 lets us make a change of basis to a special adapted basis. The block-diagonal matrix B representing the basis change is triangular by Lemma 3.5.6. Finally, the block-superdiagonal differential $\underline{D} = B^{-1}DB$ is almost-Jordan because the adapted basis is special. ∎

3.6 Bruhat Uniqueness Lemma

Here we establish the uniqueness of the persistence canonical form $\underline{D}$ appearing in the Matrix Structural Theorem 3.2.4, as well as in the ungraded version Theorem 3.2.2. Our result generalizes the uniqueness statement for the usual Bruhat factorization of an invertible matrix [JA95, Gec03].

It is convenient to make the following definitions. We call an (upper) triangular matrix U *unitriangular* if it is unipotent, meaning that each diagonal entry is 1. We call a matrix M *quasi-monomial* if each row has at most one nonzero entry and each column has at most one nonzero entry. We remark that a unitriangular matrix is always square, but a quasi-monomial matrix need not be square. The key to proving uniqueness is:

Lemma 3.6.1. *Suppose $M_1\,U = V\,M_2$, where M_1 and M_2 are quasi-monomial and U and V are unitriangular. Then $M_2 = M_1$.*

In the following proof, the term *row-pivot* denotes a matrix entry that is the leftmost nonzero entry in its row, and *column-pivot* denotes a matrix entry that is the bottommost nonzero entry in its column.

Proof. The first half of the proof consists of showing that every nonzero entry of M_2 is also an entry of M_1. A nonzero entry of the quasi-monomial matrix M_2 is a column-pivot. Similarly a nonzero entry of the quasi-monomial matrix M_1 is a row-pivot. It now suffices to show that a column-pivot of M_2 is a row-pivot of M_1. Since V is unitriangular, VM_2 has the same column-pivots as M_2. Similarly since U is unitriangular, M_1U has the same row-pivots as M_1. It now suffices to prove that a column-pivot of $S = VM_2$ is a row-pivot of $S = M_1U$. Suppose to the contrary that some column-pivot of S is not a row-pivot of S. Let x be the leftmost such column-pivot. Since x is not a row-pivot, there exists a row-pivot y to the left of x in the same row. If y were a column-pivot of $S = VM_2$, then it would be a column-pivot of M_2. But the quasi-monomial matrix M_2 cannot have two nonzero entries y and x in the same row. So y is not a column-pivot of S, and there exists a column-pivot z below y in the same column. If z were a row-pivot of $S = UM_1$, then it would be a row-pivot of M_1. But the quasi-monomial matrix M_1 cannot have two nonzero entries z and y in the same column. So z is a column-pivot of S that is not a row-pivot of S, and z is to the left of (and below) x. This is a contradiction, because x is the leftmost such column-pivot.

The second half of the proof consists of showing that every nonzero entry of M_1 is also an entry of M_2. This is analogous to the first half, and we omit the details. The two matrices then have the same nonzero entries, so they must also have the same zero entries. Since all the entries of the two matrices are the same, we have proved $M_2 = M_1$. ∎

Recall that a matrix M is *Boolean* if every non-zero entry is 1. An almost-Jordan differential matrix $\underline{D}$ is Boolean and quasi-monomial.

Proposition 3.6.2. *Suppose $P_1A = BP_2$ where P_1 and P_2 are Boolean quasi-monomial and A and B are invertible triangular. Then $P_2 = P_1$.*

Proof. Factor $A = T_1U$ as the product of an invertible diagonal matrix T_1 and a unitriangular matrix U. Factor $B = VT_2$ as the product of a unitriangular matrix V and an invertible diagonal matrix T_2. Then $(P_1T_1)U = V(T_2P_2)$, with (P_1T_1) and (T_2P_2) quasi-monomial. Lemma 3.6.1 then gives the $P_1T_1 = T_2P_2$. Since the quasi-monomial matrices P_1 and P_2 are Boolean, the conclusion follows. ∎

We remark that a permutation matrix P is also Boolean and quasi-monomial, so Proposition 3.6.2 generalizes the standard uniqueness result for Bruhat factorization of an invertible matrix [JA95, Gec03].

The uniqueness of the persistence canonical form $\underline{D}$ appearing in Theorem 3.2.2 and in the Matrix Structural Theorem 3.2.4 now follows easily:

Corollary 3.6.3. *Suppose D is a differential matrix and B_1 and B_2 are invertible triangular matrices. If both differential matrices $\underline{D}_1 = B_1^{-1}DB_1$ and $\underline{D}_2 = B_2^{-1}DB_2$ are almost-Jordan, then $\underline{D}_2 = \underline{D}_1$.*

Proof. $\underline{D}_1(B_1^{-1}B_2) = (B_1^{-1}B_2)\underline{D}_2$, and the result follows from Proposition 3.6.2. ∎

3.7 Constructively Proving the Matrix Structural Theorem

3.7.1 Linear Algebra of Reduction

In this section we discuss column-reduction of a matrix $M : \mathbb{F}^m \to \mathbb{F}^n$, including its application to describing the kernel and image of the matrix. Column-reduction of a differential matrix D is a standard tool in the computation of persistent homology, where it is usually just called *reduction* [HE02, ZC05b, ZC08, DCS06]. We prefer the more precise terminology in order to maintain the distinction with row-reduction, since both are used for Bruhat factorization [JA95, Gec03, Lus].

As in Section 3.6, the term *column-pivot* denotes a matrix entry that is the bottommost nonzero entry in its column. A matrix R is said to be *column-reduced* if each row has at most one column-pivot.

Definition 3.7.1. A column-reduction of a matrix M is an invertible triangular matrix V such that $R = MV$ is column-reduced.

A column-reduction V exists for any matrix M, but is not unique in general. Column-reduction algorithms used for persistent homology [HE02, ZC05b, ZC08, VdS11] usually prioritize computational efficiency. For our computational examples, we will use a column-reduction algorithm that is popular for Bruhat factorization [JA95, Gec03]. This algorithm is easy to implement, but is not very efficient computationally. The algorithm starts at the leftmost column of M and proceeds rightward by successive columns as follows:

- If the current column is zero, do nothing.
- If the current column is nonzero, add an appropriate multiple of the current column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

Stop if the current column is the rightmost column, otherwise proceed to the column immediately to the right and repeat. By design, the resulting matrix R has the property that any column-pivot has only zeros to the right of it (in the same row). So a row of R cannot contain more than one column-pivot, implying that R is column-reduced. The invertible triangular column-reduction matrix V is constructed by performing the same column operations on the identity matrix I, where I has same number of columns as M.

We digress briefly to discuss some linear-algebraic properties of column-reduction. A column-reduction easily yields a basis for the kernel of a matrix as well as a basis for the image. By contrast, Gaussian elimination easily yields a basis for the image a matrix, but requires additional back-substitution to produce a basis for the kernel. Column-reduction algorithms are therefore a convenient alternative to Gaussian elimination for matrix computations in general, and this fact seems to be underappreciated. We use a variant of the usual adapted basis for a filtered vector space, disregarding the ordering of basis elements. We'll say that a basis of a finite-dimensional vector space X is *almost-adapted* to a subspace $Y \subseteq X$ if Y is spanned by the set of basis elements that are contained in Y. Proposition 3.7.1 yields:

Corollary 3.7.2. *Let* $V : \mathbb{F}^m \to \mathbb{F}^m$ *be a column-reduction of a matrix* $M : \mathbb{F}^m \to \mathbb{F}^n$. *Then:*

1. *The nonzero columns of the column-reduced matrix* $R = MV$ *are a basis of* $\operatorname{im} M$.

2. *The columns of the invertible triangular matrix* V *are a basis of* $\mathbb{F}^m$*, and this basis is almost-adapted to* $\ker M$.

Proof.

1. The nonzero columns of R span $\operatorname{im} M$. The nonzero columns of R are linearly independent because R is column-reduced.

2. The columns of V are a basis of $\mathbb{F}^m$ because V is invertible. This basis is

almost-adapted to $\ker M$ because the nonzero columns of $R = MV$ are linearly independent.

∎

Example 3.7.3. We compute in detail a column-reduction of the matrix $M : \mathbb{Q}^4 \to \mathbb{Q}^3$, which is presented below with a column augmentation by the identity matrix I.

$$\frac{M}{I} = \left[\begin{array}{cccc} 1 & -2 & 0 & -8 \\ 2 & -4 & 6 & 2 \\ 1 & -2 & 2 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc} 1 & 0 & -2 & -6 \\ 2 & 0 & 2 & 6 \\ \mathbf{1} & 0 & 0 & 0 \\ \hline 1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc} 1 & 0 & -2 & 0 \\ 2 & 0 & \mathbf{2} & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ \hline 1 & 2 & -2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] = \frac{R}{V}$$

The result of the computation is a factorization $R = MV$, where R is column-reduced and V is invertible triangular (and unipotent). We describe each step of the computation:

1. The first column of M is nonzero, so it has a column-pivot. At the next processing step, boldface the column pivot for clarity, and add an appropriate multiple of the first column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

2. At this point the second column is zero, so requires no processing step.

3. At this point the third column is nonzero, so it has a column-pivot. At the next processing step, boldface the column pivot, and add an appropriate multiple of the third column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

4. At this point the fourth column is zero, so requires no processing step.

The columns of V are a basis of $\mathbb{Q}^4$ that is amost-adapted to $\ker M$. Columns 2 and 4 of V are the columns corresponding to zero columns of R, so they are a basis of $\ker M$. Columns 1 and 3 of R are the nonzero columns, so they are a basis of $\operatorname{im} M$.

3.7.2 Matrix Structural Theorem via Reduction

The standard algorithm of persistent homology [HE02, ZC05b, ZC08] starts with a differential matrix D and constructs a matrix B satisfying the conditions of:

Theorem 1.2. (Ungraded Matrix Structural Theorem) Any differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is an almost-Jordan differential matrix and B is a triangular matrix.

The matrix formulation of the *standard algorithm* constructs a matrix $B = \hat{V}$ from a column-reduction V of a differential matrix D, as discussed in [VdS11, JDB17] for $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Since $R = DV$ is column-reduced, there exists at most one nonzero column of R that has its column-pivot in row k. Here $1 \leq k \leq m$ where m is the number of rows of the square matrix D. $\hat{V}$ is constucted one column at a time using the following rule:

- If there exists a nonzero column of R that has its column-pivot in row k, then column k of $\hat{V}$ is equal to this column of R.
- If there does not exist a nonzero column of R that has its column-pivot in row k, then column k of $\hat{V}$ is equal to column k of V.

The matrix $\hat{V}$ is invertible triangular, because each column is nonzero and has its column-pivot on the diagonal.

We will now prove that $B = \hat{V}$ satisfies the conditions of Theorem 3.2.2. The key is to recognize when a differential matrix $\underline{D} : \mathbb{F}^m \to \mathbb{F}^m$ is almost-Jordan. The columns of any invertible matrix $G : \mathbb{F}^m \to \mathbb{F}^m$ comprise a basis of $\mathbb{F}^m$. When $G = I$ is the identity matrix, we have:

Lemma 3.7.4. *Let $\underline{D}$ be a differential matrix, and let I be the identity matrix of the same size. The differential matrix $I^{-1}\underline{D}I = \underline{D}$ is almost-Jordan iff the following two conditions hold:*

1. *Every nonzero column of $\underline{D}I = \underline{D}$ is equal to some column of I.*
2. *The nonzero columns of $\underline{D}I = \underline{D}$ are distinct (meaning no two are equal).*

Proof. Suppose the differential matrix $\underline{D}$ is almost-Jordan. Then for some permutation P the differential matrix $P^{-1}\underline{D}P$ is Jordan. Each of the two conditions holds for a Jordan differential matrix. Each of the two conditions is preserved by conjugation with a permutation, so each of the two conditions holds for the differential matrix $\underline{D}$.

Suppose the two conditions hold. Any permutation of the columns of $\underline{D}$ is expressed as the matrix product $\underline{D}P$ where P is the associated permutation matrix. It is possible to choose P so that any column of $\underline{D}P$ that is not in $\ker \underline{D}$ immediately follows the column that is its image under $\underline{D}$. Then the differential matrix $P^{-1}\underline{D}P$ is Jordan, so $\underline{D}$ is almost-Jordan. ∎

Now we can recognize basis changes that make a differential matrix almost-Jordan:

Corollary 3.7.5. *Let D be a differential matrix, and let G be an invertible matrix of the same size. The differential matrix $G^{-1}DG$ is almost-Jordan iff the following two conditions* *hold:*

1. *Every nonzero column of DG is equal to some column of G.*

2. *The nonzero columns of DG are distinct.*

We can now give a constructive proof of Theorem 3.2.2:

Proof. Let $B = \hat{V}$ be an invertible triangular matrix constructed from the differential matrix D by the standard algorithm. A column of $B = \hat{V}$ is either equal to the corresponding column of V or to some nonzero column of $R = DV$. Then a column of $D\hat{V}$ is either equal to the corresponding column of $DV = R$ or to some column of $DR = D^2V = 0$. So a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. We can use this fact to verify two conditions of Corollary 3.7.5:

1. We know that a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. Since R is column-reduced, the standard algorithm ensures that every nonzero column of R is equal to some column of $\hat{V}$. So any nonzero column of $D\hat{V}$ is equal to some column of $\hat{V}$.

2. We know that a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. The nonzero columns of R are distinct since R is column-reduced. So the nonzero columns of $D\hat{V}$ are distinct.

∎

Note that an invertible triangular matrix $\hat{V}$ produced by the standard algorithm is not normalized in general. But it is easy to construct a diagonal matrix T such that the invertible diagonal matrix $\hat{V}T$ is normalized. This will be illustrated in the example at the end of the section.

We now consider the graded case:

Theorem 1.4. (Matrix Structural Theorem) Any block-superdiagonal differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is a block-superdiagonal almost-Jordan differential matrix and B is a block-diagonal triangular matrix.

Proof. Let D be a block-superdiagonal differential matrix D. Then the invertible triangular column-reduction marix V produced by a reduction algorithm, such as [HE02, ZC05b, ZC08] or our Subsection 3.7.1, is block-diagonal. If V is block-diagonal, then so is the invertible triangular matrix $B = \hat{V}$ constructed by the standard algorithm from V and $R = DV$. ∎

The following example of a standard algorithm computation illustrates both block-structure and normalization.

Example 3.7.6. We work with block-superdiagonal differential $D : \mathbb{Q}^7 \to \mathbb{Q}^7$ of Example 3.2.5, which is presented below with a column augmentation by the identity matrix I. The identity matrix is block-diagonal with respect to the grading structure inherited from D. We first compute a column-reduction of D.

$$\frac{D}{I} = \left[\begin{array}{ccccccc} 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \cdots \mapsto \left[\begin{array}{ccccccc} 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \frac{R}{V}$$

The result of the computation is a factorization $R = DV$, where R is column-reduced and V is invertible triangular (and unipotent). The intervening steps are omitted for brevity.

Next we use the standard algorithm to construct $\hat{V}$ as a modification of V. Each nonzero column of R replaces the column of V that has its column-pivot in the same row. $\hat{V}$ inherits the block-diagonal structure of V:

$$V = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \cdots \mapsto \left[\begin{array}{ccccccc} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \hat{V}$$

We list the columns of $\hat{V}$ that are equal to columns of R:

- Column 2 of $\hat{V}$ is equal to column 4 of R.
- Column 3 of $\hat{V}$ is equal to column 5 of R.
- Column 6 of $\hat{V}$ is equal to colum 7 of R.

Each of the remaining columns of $\hat{V}$ is equal to the corresponding column of V. Now by Corollary 3.7.5, the block-superdiagonal differential matrix $\underline{D} = \hat{V}^{-1} D \hat{V}$ is almost-Jordan:

$$\underline{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The invertible triangular matrix $\hat{V}$ is not normalized: column 6 of $\hat{V}$ corresponds to a zero column of $\underline{D}$, but its diagonal entry is not equal to 1. We can normalize by scalar multiplication of the appropriate columns. Let T be the diagonal matrix with 1 in the first five diagonal entries and -1 in the last two. Then the invertible triangular matrix $B = \hat{V} T$ is normalized, and this is the matrix that appears in Example 3.2.5. Note that $B^{-1} D B = \underline{D} = \hat{V}^{-1} D \hat{V}$ by Corollary 3.6.3.

Chapter 4

An Isometry Theorem for Generalized Persistence Modules

This chapter is based on a collaboration David C. Meyer [MM17a].

4.1 Motivation

4.1.1 Algebraic Stability

The main result of this chapter is an *algebraic stability theorem*. This is a particular kind of hard stability theorem (section 2.4) in which one endows a collection of generalized persistence modules with two metric structures—for us, interleaving and bottleneck—and an automorphism on the collection that is then shown to be a contraction or isometry between the space with the two different metrics equipped. Algebraic stability theorems such as this are common ([Les11], [BL16], [BL13], [CZ09]). It is worth noting that in the literature the term 'interleaving' is broader, while in this document it will consistently refer to the interleaving metric proposed by Bubenik, de Silva, and Scott [BdS13] as this metric is the most general and categorical in nature. The interleaving metric is well defined on any poset P, and when $P = (0, \infty)$, this definition reduces to the interleaving metric that is often used elsewhere [BL13].

4.1.2 Connection to Finite-dimensional Algebras

This chapter concerns algebraic stability studied using techniques from the representation theory of algebras. As seen in section 2.3.3, any persistence module arising from data is a representation of a finite totally ordered set after discretizing the $\mathbb{R}$-indexed persistence module.

There are two issues to point out when discretizing. First, a finite data set D gives rise to not only a generalized persistence module, but also to its algebra. Two persistence modules may not be able to be compared simply because they are not modules over the same algebra. Second, information about the width of the interval $[\epsilon_i, \epsilon_{i+1})$ in relevant to the analysis, but seems to be lost by discretizing and indexing by $\{1, \ldots, n\}$. Both of these issues are addressed in the following chapter.

While one-dimensional persistence modules will always discretize to a generalized persistence module for a finite totally ordered set, representations of many other infinite families of finite posets also have a physical interpretation in the literature (see [BL16], [CZ09], [EH14]). For example, multi-dimensional persistence modules (see [CZ09]) will discretize in an analagous fashion to representations of a different family of finite posets. This is relevant because of the categorical equivalence between the generalized persistence modules for a finite poset P with values in K-mod, and the module category of the finite-dimensional K-algebra $A(P)$, the poset (or incidence) algebra of P. The module theory (representation theory) of such algebras has been widely studied (see, for example [ACMT05], [Bac72], [Cib89], [Fei76], [Kle75], [BdlPS11], [Lou75], [Naz81], [Yuz81], [IK17], and many others). Thus, by passing to the jump discontinuities of a filtration of simplical complexes one may apply techniques from the representation theory of finite-dimensional algebras.

This perspective, however, suggests the need for caution. While it is well-known that the set of isomorphism classes of indecomposable modules for the algebra $A(P_n)$ (i.e., $\mathrm{Rep}(\mathbb{A}_n)$) is *finite*, we have seen that this situation is far from typical.

Because of this, studying *arbitrary* generalized persistence modules in complete generality is hopeless. Indeed, if a possibly infinite poset discretizes to a finite poset P, and the module category for $A(P)$ is undecidable, the same holds for generalized persistence modules for the original poset. Moreover, our intuition from persistent homology tells us that indecomposable modules should come with a notion of widths which can be measured, in order to decide whether they should be kept or interpreted as noise. In order to reconcile these two issues, we pass from the full category of all $A(P)$-modules, to a more manageable full subcategory where we can make sense of what it means for indecomposable modules to be "wide." This suggests the following template for a representation-theoretic algebraic stability theorem:

Let P be a finite poset of some prescribed type, and let K be a field. Choose a full subcategory $\mathcal{C} \subseteq A(P)$-mod, and let D and D_B be two metrics on $\mathcal{C}$ where;

(i.) D is the interleaving distance of [BdS13] restricted to $\mathcal{C}$, and

(ii.) D_B is a bottleneck metric on $\mathcal{C}$ which incorporates some algebraic information.

Prove that $(\mathcal{C}, D) \xrightarrow{Id} (\mathcal{C}, D_B)$ is an isometry.

In addition, the class of posets covered should contain all the posets $P_n, n \in \mathbb{N}$. In addition, the category $\mathcal{C}$ should reduce to the full module category when $P = P_n$. When this is the case, the theorem should be a discrete version of the classical isometry theorem [BL13]. If possible, elements of $\mathcal{C}$ should have a nice physical description.

4.2 A Particular Class of Posets

In this section we confine our discussion to a certain class of finite posets. Though easy to describe, most such posets are of wild representation type (see the discussion in Subsection 2.7.1). We will restrict to $\mathcal{C}$, the full subcategory of $A(P)$-modules which are isomorphic to a direct sum of convex modules.

Let P be a finite poset such that:

1. P has a unique minimal element m,
2. for every maximal element $M_i \in P$, the interval $[m, M_i]$ is totally ordered, and
3. $[m, M_i] \cap [m, M_j] = \{m\}$ for all $i \neq j$.

As a technical convenience, we sometimes also assume

4. there exists an i_0 with $\big|[m, M_{i_0}]\big| > \big|[m, M_i]\big|$, for all $i \neq i_0$.

That is, P is a tree which branches only at the its unique minimal element and has one totally ordered segment longer than the others.

Definition 4.2.1. If P satisfies conditions (1), (2), (3), we say P is an n-Vee, where n denotes the number of maximal elements in P. If, in addition, P satisfies (4) we say that P is an asymmetric n-Vee.

Clearly, a 1-Vee is exactly a finite totally ordered set. It is easy to see that every 1-Vee is an asymmetric. We will prove our isometry theorem for n-Vees.

Remark 5. The convex modules for n-Vees have some nice properties. Note that if P is any finite poset, then, the following two statements are equivalent:

(i) P has a unique minimal element m and every maximal interval in P, $[m, M_i]$ is totally ordered.

(ii) the support of every convex module has a unique minimal element.

That is to say, finite posets satisfying only properties (1) and (2) in the definition for n-Vees are precisely those posets for which the support of a convex module always has a unique minimal element. The proof is easy, but we include it.

Proof. First, if P is as above, from the characterization of convex modules in Subsection 2.7 it is clear that the support of each convex module has a unique minimal element. On the other hand, for a contradiction suppose P satisfies (ii), but not (i). Let $S \subseteq P$ denote the support of a potential convex module. If P has at least two minimals, then set $S = P$. Thus it must be the case that P has a unique minimal m. If there is a maximal interval $[m, M_j]$ contained in P with $[m, M_j]$ not totally ordered. Then, there exist $x, y \in [m, M_j]$ with x, y not comparable. But then $S = [x, M_j] \cup [y, M_j]$ is the support of a convex module contradicting (ii). ∎

We will now establish some properties of the collection of translations of an asymmetric n-Vee. Much (but not all) carries over to (general) n-Vees (see the end of the proof of Theorem 4.5.6).

Lemma 4.2.2. *Let P be an asymmetric n-Vee, and let (a, b) be any weights. Let $d = d_{a,b}$ denote the weighted graph metric on the Hasse quiver of P^+ corresponding to (a, b). Then,*

(i) For each $\epsilon \in \{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}$, the set $\{\Gamma \in \mathcal{T}(P) : h(\Gamma) = \epsilon\}$ has a unique maximal element Λ_ϵ.

(ii) The set $\{\Lambda_\epsilon\}$ is totally ordered, and $\Lambda_\epsilon \leq \Lambda_\delta$ if and only if $\epsilon \leq \delta$.

(iii) If $\Lambda, \Gamma \in \mathcal{T}(P)$ with $h(\Lambda), h(\Gamma) \leq \epsilon$ then there exists a Λ_δ with $\Lambda, \Gamma \leq \Lambda_\delta$, and $h(\Lambda_\delta) = \delta = \max\{h(\Lambda), h(\Gamma)\}$.

Proof. Let P be as above. First, say $n > 1$, then $P = \bigcup [m, M_i]$, with $[m, M_{i_0}]$ of maximal cardinality. Let $T_i = |[m, M_i]| - 1$, so by hypothesis, $T_{i_0} > T_i$ for all $i \neq i_0$. Let T = max$\{T_i : i \neq i_0\}$ (note that if P was not asymmetric $T = T_{i_0}$). Let $\epsilon \in \{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}$ and suppose $h(\Lambda) = \epsilon$. If $\Lambda m > m$, then $\epsilon \geq aT + b$, since;

$$\text{if } \Lambda m = \infty, \text{ then } h(\Lambda) = aT_{i_0} + b,$$

$$\text{if } \Lambda m \in (m, M_{i_0}], \text{ then } h(\Lambda) \geq aT + b, \text{ and}$$

$$\text{if } \Lambda m \in (m, M_i], i \neq i_0, \text{ then } h(\Lambda) = aT_{i_0} + b.$$

Therefore, if $\epsilon < aT + b$, $\Lambda m = m$. Then, $\Lambda \leq \Lambda_\epsilon$, where

$$\Lambda_\epsilon(x) = \begin{cases} m, \text{ if } x = m \\ \max\{y \in (m, M_i] \cup \{\infty\} : d(x, y) \leq \epsilon\}, \text{ if } x \in (m, M_i] \end{cases}$$

On the other hand, if $aT_{i_0} + b > \epsilon \geq aT + b$, then $\Lambda m \in [m, M_{i_0}]$, and $\Lambda((m, M_i]) = \infty$ for $i \neq i_0$. In this case, $\Lambda \leq \Lambda_\epsilon$, where

$$\Lambda_\epsilon(x) = \begin{cases} \infty, x \in (m, M_i], i \neq i_0 \\ \max\{y \in (m, M_{i_0}] \cup \{\infty\} : d(x,y) \leq \epsilon\}, \text{ if } x \in [m, M_{i_0}] \end{cases}$$

Lastly, if $\epsilon = aT_{i_0} + b$, then $\Lambda \leq \Lambda_\epsilon$, where $\Lambda_\epsilon(x) = \infty$, for all x.

Note that the formulae above are well defined, since $[m, M_i] \cap [m, M_j] = \{m\}$ for all $i \neq j$. Now, suppose that $n = 1$. Then $\Lambda \leq \Lambda_\epsilon$, where $\Lambda_\epsilon(x) = \max\{y \geq x : d(x,y) \leq \epsilon\}$ for any ϵ. This proves (i). The expressions for Λ_ϵ show that (ii) holds. Now let $\Lambda, \Gamma \in \mathcal{T}(P)$ with $h(\Lambda), h(\Gamma) \leq \epsilon$, and suppose $\max\{h(\Lambda), h(\Gamma)\} = \delta$. Without loss of generality, say $h(\Lambda) = \delta, h(\Gamma) \leq \delta$. Then, $\Lambda \leq \Lambda_\delta$ and $\Gamma \leq \Lambda_{h(\Gamma)} \leq \Lambda_\delta$, by (i), (ii) as required. ∎

The important observation is that although $\mathcal{T}(P)$ is not totally ordered, (for $n > 1$) it is directed in such a way that one may pass to a larger translation without increasing the height. In contrast, for an arbitrary finite poset P, $\mathcal{T}(P)$ will still be a directed set (because we suspended at infinity). It may be the case, however, that for all Λ_0 with $\Lambda, \Gamma \leq \Lambda_0$, $h(\Lambda_0) > \kappa > \max\{h(\Lambda), h(\Gamma)\}$. That is to say, one may have to pay a price when passing to any larger common translation. Lemma 4.2.2 shows that this does not happen for asymmetric n-Vees. We are now ready to define the width of a convex module.

Lemma 4.2.3. *Let P be an asymmetric n-Vee, and let (a,b) be a weight. Then for all I convex, the following are equal;*

(i) $W(I) = W_1(I) = \ min\{\epsilon : \exists \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon, and\ Hom(I, I\Lambda\Gamma) = 0\}$

(ii) $W_2(I) = min\{\epsilon : \exists \Lambda \in \mathcal{T}(P), h(\Lambda) \leq \epsilon,\ and\ Hom(I, I\Lambda^2) = 0\}$.

(iii) $W_3(I) = min\{\epsilon : \exists \Lambda_\epsilon \in \mathcal{T}(P) with\ Hom(I, I{\Lambda_\epsilon}^2) = 0\}$.

Before proving Lemma 4.2.3, we note that for any I convex and for any $\theta \in \mathcal{T}(P)$,

$$\mathrm{Hom}(I, I\theta) \neq 0 \iff \exists x \in \mathrm{Supp}(I), \theta x \in \mathrm{Supp}(I) \iff \theta x' \in \mathrm{Supp}(I), \text{ for } x' \text{ minimal in } \mathrm{Supp}(I).$$

This follows from general properties of module homomorphisms, and the observation in Remark 5 that convex modules for n-Vees have unique minimal elements. (See the Section 4.3 for a detailed analysis of homomorphisms and translations)

Using this fact, we see that if $\Lambda \leq \Gamma$ and $\mathrm{Hom}(I, I\Lambda) = 0$, then $\mathrm{Hom}(I, I\Gamma) = 0$. Thus this condition defining W produces an interval in $\{h(\Lambda) : \Lambda \in \mathcal{T}(P)\}$. We will now prove Lemma 4.2.3.

Proof. Let $\Lambda, \Gamma \in \mathcal{T}(P)$ with $h(\Lambda), h(\Gamma) \leq \epsilon$, and suppose $\mathrm{Hom}(I, I\Lambda\Gamma) = 0$, and $\delta = \max\{h(\Lambda), h(\Gamma)\}$. Then, by Lemma 4.2.2, there exists Λ_δ, with $h(\Lambda_\delta) = \delta$ and $\Lambda, \Gamma \leq \Lambda_\delta$. Then $\Lambda\Gamma \leq {\Lambda_\delta}^2$ so $\mathrm{Hom}(I, I{\Lambda_\delta}^2) = 0$, so $W_3(I) \leq W_2(I) \leq W(I)$. But $S \subseteq T \implies \inf(S) \geq \inf(T)$, thus $W_3(I) \geq W_2(I) \geq W(I)$, so all are equal. With this equivalence established, we define the width of a convex module. ∎

Definition 4.2.4. Let P be an asymmetric n-Vee and let (a, b) be a weight. Let I be convex. Then,

$$W(I) = W_1(I) = \min\{\epsilon : \exists \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon, \text{and } \mathrm{Hom}(I, I\Lambda\Gamma) = 0\}.$$

While this definition of the width of a module is formulated algebraically, and is natural considering the structure of $\mathcal{T}(P)$, it is not without complication. Intuitively $\frac{1}{2}|\mathrm{Supp}(I)|$ (or perhaps $\lceil\frac{1}{2}|\mathrm{Supp}(I)|\rceil$) is a first approximation of $W(I)$. Indeed, this is the discrete analogue of the width used in the classical isometry theorem [BL13], as their work corresponds to translations that are exactly constant shifts. This discrete analogue of this is the choice of weights $(a, b) = (1, 1)$ on a 1-Vee. For an n-Vee, however, modules with smaller support may happen to have large widths or the opposite. For example, if P is a 2-Vee and I is the simple convex module supported at m, then $W(I) = aT + b$. In contrast, if $x \in (m, M_i)$ and J is the convex module supported at $x, W(J) = a$. Moreover, any convex module supported at M_i for some i necessarily has width greater or equal to b. This is relevant, as no relation between a and b is specified.

The following Proposition will prove useful in Section 4.4 when we produce an explicit matching for 1-Vees. This result is an analogue of the corresponding statement in [BL13].

Proposition 4.2.5. *Let P be an asymmetric n-Vee, let $A = \bigoplus_i A_i, C = \bigoplus_j C_j$ be in $\mathcal{C}$. For any module M, let $B(M)$ denote the barcode of M viewed as a multiset, and let $\Lambda \in \mathcal{T}(P)$. Then,*

(i) If $A \overset{f}{\hookrightarrow} C$ is an injection, then for all $d \in P$, the set

$$|\{i : [-, d] \text{ is a maximal totally ordered subset of Supp}(A_i)\}| \leq$$
$$|\{j : [-, d] \text{ is a maximal totally ordered subset of Supp}(C_j)\}|, \text{ and}$$

(ii) If $A \overset{g}{\to} C$ is a surjection, then for all $b \in P$,

$$|\{j : [b, -] \text{ is a maximal totally ordered subset of Supp}(C_j)\}| \leq$$
$$|\{i : [b, -] \text{ is a maximal totally ordered subset of Supp}(A_i)\}|.$$

(iii) If A and C are (Λ,Λ)-interleaved, and $A \xrightarrow{\phi} C\Lambda$ is one of the homomorphisms, then for all I in $B(ker(\phi))$, $W(I) \leq h(\Lambda)$.

(iv) If A and C are (Λ,Λ)-interleaved, and $A \xrightarrow{\phi} C\Lambda$ is one of the homomorphisms, then for all J in $B(cok(\phi))$, $W(J) \leq h(\Lambda)$.

Before proving the Proposition 4.2.5, we state a Lemma.

Lemma 4.2.6. *Let P be an asymmetric n-Vee, say $P = \bigcup[m, M_i]$, with $[m, M_i]$ totally ordered. Let $m_i = \ min(m, M_i]$, and let $\mathcal{I}_j$ be the left ideal in $A(P)$ generated by $\{m_i : i \neq j\}$. Then,*

(i) For any M convex,
$$M/\mathcal{I}_jM \text{ is } \begin{cases} 0, \ \textit{if Supp}(M) \cap [m, M_j] = \phi \\ \textit{the convex module with support given by Supp}(M) \cap [m, M_j] \ \textit{otherwise.} \end{cases}$$

(ii) For $A, B \in \mathcal{C}$, If f is a homomorphism $A \xrightarrow{f} B/\mathcal{I}_jB$, then f factors through $A/\mathcal{I}_jA$.

Proof. (i) obvious. Statement (ii) is clear, since for $f : A \to B/\mathcal{I}_jB$, $w \in \mathcal{I}_j$, $f(w \cdot a) = w \cdot f(a) = 0$. ∎

Note that if $n = 1$, the left ideal $\mathcal{I}_i$ is identically zero, but the above is still true. We now prove Proposition 4.2.5.

Proof. Let A, C be as above. For all i, let $A_i = A(P)x_i$, $x_i \in \text{Supp}(A_i)$, and let $[x_i, X_i]$ be a maximal connected totally ordered subset of Supp(A_i) (We do not suppose $x_i = m$). Similarly, let y_j be such that $C_j = A(P)y_j$. For i, j let $f^i_j : A_i \to C_j$. Now suppose $A \xrightarrow{f} C$ is an injection. Fix i_0 and $[x_{i_0}, X_{i_0}]$ be maximal contained in Supp(A_{i_0}). Since $f^{i_0} = (f^{i_0}_j) : A_{i_0} \to \bigoplus_j C_j$ is an inclusion, for any $t \in [x_{i_0}, X_{i_0}]$, there exists $j(t)$ with $f^{i_0}_{j(t)} \neq 0$. Since $f^{i_0}_{j(t)}$ is a homomorphism, $f^{i_0}_{j(t)} \neq 0 \implies$

$f^{i_0}_{j(t)}(x_{i_0}) \neq 0$, and it is not that case that there exists $\ell > X_{i_0}$, with $\ell \in$Supp(C_{j_0}). Set $j_0 = j(X_{i_0})$. Therefore,

$$\{j : [x_{i_0}, X_{i_0}] \subseteq \text{Supp}(C_j) \text{ and for all } \ell, \ell > X_{i_0} \implies \ell \notin \text{ Supp}(C_j)\} \neq \phi.$$

Now, for $d \in P$, let

$$\begin{aligned} j(d) &= \{j : [-, d] \subseteq \text{Supp}(C_j), \ell \notin \text{Supp}(C_j) \text{ for } \ell > d\}, and \\ i(d) &= \{i : [-, d] \subseteq \text{Supp}(A_i), \ell \notin \text{Supp}(A_I) \text{ for } \ell > d\}. \end{aligned}$$

Clearly, $i(d) \neq \phi \implies j(d) \neq \phi$. Now, let $d \in P$, $d \neq m$ with $i(d) \neq \phi$. Say $d \in (m, M_k]$. Then,

$$\begin{aligned} \bigoplus_{i \in i(d)} A_i/\mathcal{I}_k A_i &\hookrightarrow \bigoplus_{\substack{j \in j(d') \\ d' \leq d}} C_j/\mathcal{I}_k C_j \hookrightarrow C/\mathcal{I}_k C \implies \\ \bigoplus_{i \in i(d)} (A_i/\mathcal{I}_k A)(d) &\hookrightarrow \bigoplus_{\substack{j \in j(d') \\ d' \leq d}} (C_j/\mathcal{I}_k C_j)(d) = \bigoplus_{j \in j(d)} (C_j/\mathcal{I}_k C_j)(d) \end{aligned}$$

where the above inclusions are induced from f and the inclusion of a submodule into a larger module respectively. Thus, $|i(d)| \leq |j(d)|$. If $d = m$, then,

$$\bigoplus_{i \in i(m)} A_i \hookrightarrow \bigoplus_{j \in j(m)} C_j \hookrightarrow C \implies \bigoplus_{i \in i(m)} A_i(m) \hookrightarrow \bigoplus_{j \in j(m)} C_j(m) \hookrightarrow C(m),$$

so $|i(m)| \leq |j(m)|$. This proves (i). The proof of (ii) is similar, though one inducts on the the cardinality of $\mathcal{S} = \{b : [b, -] \text{ is a maximal totally ordered subset of } A_i \text{ for some } A_i\}$.

Now we prove (iii). For a contradiction, suppose there exists an $I \in B(ker(\phi))$ with Hom$(I, I\Lambda^2) \neq 0$. But then the diagram below commutes.

$$\begin{array}{ccc} I & \longrightarrow & I\Lambda^2 \\ & {\scriptstyle \phi_{|I}} \searrow \quad \nearrow {\scriptstyle \psi\Lambda} & \\ & \bigoplus_j C_j & \end{array}$$

Thus, $\psi\Lambda\phi(I) \neq 0$, a contradiction. This proves (iii).

Now let $J \in B(cok(\phi))$. For a contradiction, suppose $W(J) > h(\Lambda)$. But then there exists $[x, X]$ a maximal subinterval in Supp(J) with $\Lambda^2 x \leq X$. Let $\{b_x + im(\phi), ...b_X + im(\phi)\}$ be the corresponding basis elements for J. But then there exists j such that

1. $[x, X]$ $\subseteq$Supp$(C_j\Lambda)$, and

2. $C_j\Lambda(y) \notin im(\phi)$, for $x \leq y \leq X$.

Then, $\Lambda x, \Lambda X$ $\in$Supp$(C_j) \implies \Lambda^2\Lambda x = \Lambda\Lambda^2 x \leq \Lambda X$ which is in the support of C_j. Therefore $W(C_j) \geq h(\Lambda)$. But then, the following diagram commutes.

$$\begin{array}{ccc} C_j(\Lambda x) & \longrightarrow & (C_j\Lambda^2)(\Lambda x) = (C_j\Lambda)(\Lambda^2 x) \\ & \searrow \quad \nearrow & \\ & J\Lambda(\Lambda x) & \end{array}$$

But then $(C_j\Lambda^2)(\Lambda x) \in im(\phi\Lambda)(\Lambda x)$, a contradiction. This proves (iv) and finishes the proof. ∎

The Example below shows that (i), (ii) in the Proposition 4.2.5 cannot be extended from maximal totally ordered intervals to convex subsets.

Example 4.2.7. Consider the 2-Vee $[m, M_1] \cup [m, M_2]$, where $m < x < M_1$ and $m < y < z < M_2$. Let C_1 be the convex module supported on $\{m, x, M_1\}$, and C_2 be the convex module supported on $\{m, y, z, M_2\}$. Say C_1 has basis $\{e_m, e_x, e_{M_1}\}$ and C_2 has basis $\{f_m, f_y, f_z, f_{M_2}\}$. Then the submodule of $C_1 \oplus C_2$ with basis $\{e_m + f_m, e_x, e_{M_1}, f_y, f_z, f_{M_2}\}$ is isomorphic to the convex module with full support. Thus, let A_1 be the convex module with full support. Then, $A_1 \hookrightarrow C_1 \oplus C_2$, and while one can make the claim in the Proposition for each maximal totally ordered subset of the support of A_1 separately, one cannot do so simultaneously.

In the next section we study homomorphisms and translations and their properties in $\mathcal{C}$.

4.3 Homomorphisms and Translations

In this section we investigate the relationship between homomorphisms and translations in the category $\mathcal{C}$. In the interest of generality, we will relax our hypotheses on the poset P. In this section, unless otherwise specified, P is any finite poset. The functions defined in Definitions 4.3.1, 4.3.2 are analogues of functions used by Bauer and Lesnick [BL13]. In this context, however, they fail to preserve W, and may annihilate a convex module.

Note that if $S \subseteq P$ is non-empty and interval convex, then it canonically determines the isomorphism class of an element of $\mathcal{C}$ under the identification;

$$S \to \bigoplus M_i, \text{ where } \mathrm{Supp}(M_i) \text{ is the } i\text{th connected component of } S.$$

We use this in the definition below.

Definition 4.3.1. Let P be any finite poset and M be convex. Say $\mathrm{Supp}(M) = \bigcup_i [a_i, b_i]$, where $[a_i, b_i]$ are maximal intervals in $\mathrm{Supp}(M)$, and let $\Gamma \in \mathcal{T}(P)$. Then, $M^{+\Gamma}$ is the element of $\mathcal{C}$ given by Supp$(M^{+\Gamma}) = S = \bigcup_i [\Gamma a_i, b_i]$.

That is, $M^{+\Gamma}$ is the direct sum determined by $S = \bigcup_i [\Gamma a_i, b_i]$. (Note that if $\Gamma a_i \not\leq bi$, then $[\Gamma a_i, b_i]$ is empty.) One easily checks that,

(i) If P is an n-Vee, $M^{+\Gamma}$ is convex, or 0, and

(ii) For a general poset P, $M^{+\Gamma}$ is a submodule of M.

Moreover, for $i \in P$,

$$M^{+\Gamma}(i) = \sum_x im(M(x \leq \Gamma x \leq i)) = im(M(x_0 \leq \Gamma x_0 \leq i)) \text{ for any } x_0 \leq \Gamma x_0 \leq i.$$

That is, $\theta = \theta_i \in M^{+\Gamma}(i) \implies \theta \in im(M(x_0 \leq \Gamma x_0 \leq i))$ for any $x_0 \leq \Gamma x_0 \leq i$. Now, for $M \in \mathcal{C}$ arbitrary, set

$$M^{+\Gamma} = \bigoplus_t M_t^{+\Gamma}, \text{ where } M = \bigoplus_t M_t.$$

If P has the property that for all $i \in P$, $(-\infty, i]$ is totally ordered, one can still find $x = x(i)$ such that $M^{+\Gamma}(i) = im(M(x \leq \Gamma x \leq i))$ is still valid. Thus, in particular, the result holds for as n-Vee. Note that if $(-\infty, i]$ is not totally ordered, then $M^{+\Gamma}(i) = \sum_x im(M(x \leq \Gamma x \leq i))$. When P is an n-Vee, we now make a dual definition.

Definition 4.3.2. Let P be an n-Vee and let M be a convex module. Say $\text{Supp}(M) = \bigcup_i [x, X_i]$ where each $x \leq X_i \leq M_i$. (Recall that since P is an n-Vee, the support of each convex module has a minimal element. So $X_i \neq x$ for more than one i implies $x = m$.) Let $\Gamma \in \mathcal{T}(P)$. Then, $M^{-\Gamma}$ is the convex modules with

$$\text{Supp}(M^{-\Gamma}) \text{ equal to } \bigcup_i [x, X_i^{N(i)}], \text{ where } X_i^{N(i)} = \max \{y : \Gamma y \leq X_i, y \geq x\}.$$

Note that if no such $X_i^{N(i)}$ exists, $M^{-\Gamma} = 0$

One easily checks that,

(i) If M is convex, $M^{-\Gamma}$ is either identically zero or convex, and

(ii) $M^{-\Gamma}$ is a quotient of M.

Because of (i) we may extend our definition from Σ to $\mathcal{C}$. For $M \in \mathcal{C}$, set

$$M^{-\Gamma} = \bigoplus_t {M_t}^{-\Gamma}, \text{ where } M = \bigoplus_t M_t.$$

Notice that the assignment $M \to M^{+\Gamma}$ moves the left endpoints of the support of a module to the right, while $M \to M^{-\Gamma}$ moves the right endpoints of the support to the left. A physical characterization of $M^{-\Gamma}$ is possible, but will prove unecessary for our purposes.

We now prove a useful proposition.

Proposition 4.3.3. *Let P be an n-Vee, and let $I, M \in \mathcal{C}$. Let (ϕ, ψ) be a (Λ, Λ)-interleaving betweem I and M. Say $\phi : I \to M\Lambda$. Then,*

(i) $I^{-\Lambda^2}$ is a quotient of both I and $im(\phi)$, and

(ii) $M^{+\Lambda^2}\Lambda$ is a submodule of both $M\Lambda$ and $im(\phi)$.

Proof. First, by the comments above, $I^{-\Lambda^2}$ is a quotient of I. Now, since I and M are (Λ, Λ)-interleaved, $\psi\Lambda \circ \phi = (I \to I\Lambda^2)$. Therefore,

$$(\psi\Lambda)(im(\phi)) = I^{-\Lambda^2},$$

and hence $I^{-\Lambda^2}$ is a homomorphic image, and hence a quotient of $im(\phi)$. This proves (i).

We now prove (ii). First, already $M^{+\Gamma}$ is a submodule of M . Moreover, $C \leq D$ implies τ, $C\tau \leq D\tau$, for any $\tau \in \mathcal{T}(P)$. Hence, $M^{+\Lambda^2}\Lambda$ is a submodule of $M\Lambda$. It remains to show that $M^{+\Lambda^2}\Lambda$ is a submodule of $im(\phi)$. Let $i \in \mathcal{P}$, $i \in \text{Supp}((M^{+\Lambda^2})\Lambda)$.

Then, as vector spaces,

$$((M^{+\Lambda^2})\Lambda)(i) \subset (im(\phi))(i) \oplus_K (coker(\phi))(i).$$

Let $\theta_i \in ((M^{+\Lambda^2})\Lambda)(i)$. Note that $\theta_i = \theta'_{\Lambda i}$. At the i level, $\theta_i = a_i + b_i$ with $a_i \in (im(\phi))(i)$ and $b_i \in (coker(\phi))(i)$. Then,

$$\theta_i \in im((M(x \leq \Lambda^2 x \leq \Lambda i))) \implies$$

$$\theta'_{\Lambda i} = M(x \leq \Lambda^2 x \leq \Lambda i)(a_x + b_x), a_x \in im(\phi), b_x \in coker(\phi) \implies$$

$$M(x \leq \Lambda^2 x \leq \Lambda i)(a_x) = a_i + \alpha_i, \text{ with } \alpha_i \in im(\phi), \text{ and}$$

$$M(x \leq \Lambda^2 x \leq \Lambda i)(b_x) = -\alpha_i + b_i.$$

But, by Proposition 4.2.5, $W(coker(\phi)) < h(\Lambda)$, thus $-\alpha_i + b_i = 0$, and therefore $\alpha_i = b_i = 0$. Hence, θ_i was fully contained in $(im(\phi))(i)$.

Thus,

$$im((M(x \leq \Lambda^2 x \leq \Lambda i))) \subset (im(\phi))(i) \text{ for all } i, \text{ hence}$$

$$((M^{+\Lambda^2})\Lambda(i) \subset (im(\phi)))(i) \text{ for all } i.$$

Therefore,

$$(M^{+\Lambda^2})\Lambda \leq im(\phi).$$

This proves (ii). ∎

We will now consider the action of $\mathcal{T}(P)$ on $\mathcal{C} \cup \{0\}$. We first point out that, in general, the monoid $\mathcal{T}(P)$ need not act on $\Sigma \cup \{0\}$.

Example 4.3.4. Let P be the diamond poset: $1 \leq 2, 3 \leq 4$, and let Λ be the translation $\Lambda 1 = 1, \Lambda 2 = 4, \Lambda 3 = 4, \Lambda 4 = \infty, \Lambda\infty = \infty$. Let J be the convex module with support equal to $\{2, 3, 4\}$. Then, $J\Lambda \cong S \oplus T$, where S is the simple supported on $\{2\}$, and T is the simple supported on $\{3\}$. Alternatively, let P be the poset $1, 2 \leq 3$, with $1, 2$ not comparable (i.e., just the upper part of the diamond). Let J be the convex module with full support, and Λ be given by $\Lambda 1 = 1, \Lambda 2 = 2, \Lambda 3 = \infty, \Lambda\infty = \infty$. Then, $J\Lambda$ is again a direct sum of two convex modules.

Example 4.3.4 shows that the action of $\mathcal{T}(P)$ on $\mathcal{C} \cup \{0\}$ need not restrict to $\Sigma \cup \{0\}$. In Lemma 4.3.6 we will see that when P is an n-Vee, however, the action does restrict. First, a quick observation.

Lemma 4.3.5. *Let P be any poset with a unique minimal element m, and suppose $\Lambda \in \mathcal{T}(P)$ with $\Lambda m = m$. Then for all convex J with $m \in$ Supp(J), $J\Lambda$ is convex.*

Proof. Let P be as above, M be convex, with $m \in$ Supp(M). Let Λ be a translation with $\Lambda m = m$. Clearly $M\Lambda$ is thin. Let t_1, t_2 be in the support of $M\Lambda$, and suppose $t_1 \leq t \leq t_2$. Then, $\Lambda t_1, \Lambda t_2 \in$ Supp$(M) \implies [\Lambda t_1, \Lambda t_2] \subseteq$Supp$(M)$, since M is convex. Since $\Lambda t \in [\Lambda t_1, \Lambda t_2]$, t is in the support of M, so $[t_1, t_2] \subseteq$ Supp$(M\Lambda)$. Now, since $\Lambda m = m$, $m \leq x$ for all x, Supp$(M\Lambda)$ is connected. ∎

Lemma 4.3.6. *Let P be an n-Vee, I a convex module and $\Lambda \in \mathcal{T}(P)$. Then, $I\Lambda$ is either the zero module or convex.*

Proof. First, from the proof of Lemma 4.3.5, if non-zero $I\Lambda$ is in $\mathcal{C}$. We now proceed in cases. First, suppose $m \in$ Supp(I). If $\Lambda m = m$, then m is in the support of $I\Lambda$, so $I\Lambda$ is convex. On the other hand, if $\Lambda m \in (m, M_i]$, then for all $j \neq i, (m, M_j] \cap$ Supp$(I\Lambda) = \phi$. But then Supp$(I\Lambda) \subseteq [m, M_i]$, hence it is convex or zero, since it is interval convex. If $m \notin$ Supp(I), then I is supported in $(m, M_j]$ for some j and the result follows. ∎

We will now work towards the characterization of homomorphism between convex modules when P is an n-Vee. In the interest of generality, we begin with an arbitrary finite poset P.

Definition 4.3.7. Let I, M be convex. Let $\{e_x : x \in \text{Supp}(I)\}$, $\{f_x : x \in \text{Supp}(M)\}$ be K-bases for I, M respectively. Consider the linear function $\Phi_{I,M}$, defined by

$$\Phi_{I,M}(e_y) = \begin{cases} f_y, & \text{if } y \in \text{Supp}(I) \cap \text{Supp}(M) \\ 0 \text{ otherwise.} \end{cases}$$

By inspection, $\Phi_{I,M}$ is a non-zero module homomorphism if and only if Supp$(I) \cap$ $\text{Supp}(M)$ satisfies,

(i) Supp$(I) \cap$ $\text{Supp}(M) \neq \phi$

(ii) $x \in$ Supp$(I) \cap$ Supp(M), $y \geq x, y \in$ $\text{Supp}(M) \implies y \in$ $\text{Supp}(I)$, and

(iii) $x \in$ Supp$(I) \cap$ Supp(M), $y \leq x, y \in$ $\text{Supp}(I) \implies y \in$ $\text{Supp}(M)$.

Note that even when it is not a module homomorphsim, $\Phi_{I,M}$ can be viewed as the linear extension of $\chi(\text{Supp}(I) \cap \text{Supp}(M))$, the characteristic function on the intersection of the supports of I and M.

The following two lemmas will allow us to conclude that when P is an n-Vee, up to a K-scalar, this is the only possible module homomorphisms from I to M.

Lemma 4.3.8. *Let P be any finite poset, and let I, M be convex. Let $S \subseteq$ Supp$(I) \cap$ Supp(M), with S nonempty. Suppose that there exists an $N \in \mathcal{C}$ with Supp$(N) = S$. Then, N is isomorphic to the image of a non-zero module homomorphsim from I to M if and only if*

(a) for all $x \in S$, if $y \in$ Supp(I) with $y \leq x$, then $y \in S$, and

(b) for all $x \in S$, if $y \in$ Supp(M) with $y \geq x$, then $y \in S$.

Proof. S corresponds to the support of a non-zero quotient module of I if and only if S satisfies (a). Similarly, S corresponds to a non-zero submodule of M if and only if S satisfies (b). Since any homomorphism can be factored into an injection after a surjection, the result follows. ∎

Lemma 4.3.9. *Let P be an n-Vee. Let I, M be convex modules. Then, Hom(I,M) $\cong K$ or 0 (as a vector space)*

Proof. First, let P be any finite poset, and I, M be convex. Suppose that g is any non-zero homomorpism from I to M. Then, by Lemma 4.3.8, $im(g) = I/ker(g)$ has support equal to $S \subseteq \mathrm{Supp}(I) \cap \mathrm{Supp}(M)$ satisfying (a), (b) from Lemma 4.3.8. We will show that any such S is a union of connected components of $\mathrm{Supp}(I) \cap \mathrm{Supp}(M)$. Since g is non-zero, S is non-empty. Now, let $s \in S$ and suppose that $y \in \mathrm{Supp}(I) \cap \mathrm{Supp}(M)$ with $y \geq s$. Then, by (b), $y \in S$. Similarly, if $y \in \mathrm{Supp}(I) \cap \mathrm{Supp}(M)$ with $y \leq s$, then by (a), $y \in S$. Therefore S contains the connected component of s in $\mathrm{Supp}(I) \cap \mathrm{Supp}(M)$. The result follows.

Now, if P is an n-Vee, $\mathrm{Supp}(I) \cap \mathrm{Supp}(M)$ is connected, so S must be the full intersection. As above, let $\{e_x : x \in \mathrm{Supp}(I)\}$, $\{f_x : x \in \mathrm{Supp}(M)\}$ be K bases for I, M respectively.

Then,

$$g(e_z) = \begin{cases} c_z f_z, \text{ if } z \in \mathrm{Supp}(I) \cap \mathrm{Supp}(M), c_z \in K \\ 0 \text{ otherwise,} \end{cases}$$

where Supp(I), Supp(M) satisfy conditions (i), (ii) and (iii) from below Definition 4.3.7. Clearly, every non-zero quotient of I must have support containing the minimal element t of Supp(I). Then, since I, M are convex,

$$g(I(t \leq y))e_t = ge_y = M(t \leq y)ge_t = M(t \leq y)c_t f_t = c_t f_y = c_y f_y.$$

Therefore, $g = c_t \Phi_{I,M}$. Of course, if g is identically zero, g is still in the span of $\Phi_{I,M}$. ∎

We now investigate the action of $\mathcal{T}(P)$ on Hom(I, M), when I, M are convex. From the observation in the proof of Lemma 4.3.9, we see that for P arbitrary, Hom(I, M) will have dimension equal to the number of connected components of Supp$(I) \cap \mathrm{Supp}(M)$. Still, for a fixed translation $\Lambda \in \mathcal{T}(P)$, Hom$(I\Lambda, M\Lambda)$ may be trivial (even when $I\Lambda, M\Lambda$ are non-zero). We now state a condition which ensures that Hom$(I, M) \cdot \Lambda \neq 0$. This can be done more generally, but we state the result only for P an n-Vee.

Lemma 4.3.10. *Let P be an n-Vee, and let I, M be convex. Let $\Lambda \in \mathcal{T}(P)$. Say Hom(I,M) $\neq 0$ and there exists t with $\Lambda t \in Supp(I) \cap Supp(M)$. Then Hom$(I\Lambda, M\Lambda) \neq 0$.*

Proof. Since $\Lambda t \in Supp(I) \cap Supp(M)$, $I\Lambda$ and $M\Lambda$ are not zero. Also, by Lemma 4.3.9, Supp(I) and Supp(M) satisfy the conditions (i), (ii) and (iii) from below Definiton 4.3.7. Since $I\Lambda, M\Lambda$ are convex, it is enough to show that the above still holds for $\mathrm{Supp}(I\Lambda)$ and $\mathrm{Supp}(M\Lambda)$. Again, $t \in \mathrm{Supp}(I\Lambda) \cap \mathrm{Supp}(M\Lambda)$, hence the intersection is nonempty. Now let $z \in \mathrm{Supp}(I\Lambda) \cap \mathrm{Supp}(M\Lambda)$, with $w \in$

$\mathrm{Supp}(M\Lambda), w \geq z$. Then, $\Lambda z \in \mathrm{Supp}(I) \cap \mathrm{Supp}(M), \Lambda w \in \mathrm{Supp}(M)$, and $\Lambda w \geq \Lambda z$. Therefore, $\Lambda z \in \mathrm{Supp}(I)$, so $z \in$ Supp$(I\Lambda)$. The last requirement is proved similarly. ∎

Note that conditions (ii) and (iii) are clearly inherited from I, M. The authors point out that the hypothesis above that the intersection of supports coincides with the image of the translation is required even on a totally ordered set (see Example 4.3.11 below).

Example 4.3.11. Let P be the totally ordered set $\{1, 2, 3, 4, 5, 6\}$ with its standard ordering, and let Λ be defined by $\Lambda 1 = 2, \Lambda 2 = 3, \Lambda 3 = 3, \Lambda 4 = 5, \Lambda 5 = 6, \Lambda 6 = 6$. Let I and M be the convex modules supported on $\{4, 5, 6\}$ and $\{3, 4\}$ respectively. Note that Hom$(I, M) \neq$, $I\Lambda$ and $M\Lambda$ are supported on $\{4, 5\}$ and $\{2, 3\}$ respectively. Clearly, the supports of $I\Lambda$ and $M\Lambda$ are disjoint, so Hom$(I\Lambda, M\Lambda) = 0$.

Remark 6. Let P be an n-Vee, $\Lambda \in \mathcal{T}(P)$ and say I is convex. We write Φ_I^Λ for $\Phi_{I,I\Lambda}$, as $I\Lambda$ is either zero or convex. For $I \in \mathcal{C}$, we write Φ_I^Λ for the canonical homomorphism as well, since it is necessarily diagonal. (Of course, as mentioned above, even if $I\Lambda$ is not trivial, it may be the case that $\Phi_I^\Lambda$is identically zero.)

We can now show that when P is an n-Vee the collection of interleavings between two elements of $\mathcal{C}$ will have the structure of an affine variety (not necessarily irreducible). Though the result still holds for more general posets, our proof is an application of the results of this section. Some examples are provided in Section 4.6

Proposition 4.3.12. *Let P be an n-Vee and let $I = \bigoplus I_s, M = \bigoplus M_t$ be two elements of*

$\mathcal{C}$. *Let* $\Lambda, \Gamma \in \mathcal{T}(P)$. *Then the collection of* (Λ, Γ)*-interleavings between* I *and* M *has the structure of an affine variety.*

Indeed, as stated above the result holds for any finite poset, though when P is an n-Vee the variety has a simpler description. We sketch the proof. Let P, I, M, Λ be as above, and let ϕ, ψ be any interleaving between I and M. Thus, we obtain the commutative triangles below.

$$\begin{array}{ccc} I & \xrightarrow{\Phi_I^{\Gamma\Lambda}} & I\Gamma\Lambda \\ & {\scriptstyle\phi}\searrow \quad \nearrow{\scriptstyle\psi\Lambda} & \\ & M\Lambda & \end{array} \qquad \begin{array}{ccc} & I\Gamma & \\ & {\scriptstyle\psi}\nearrow \quad \searrow{\scriptstyle\phi\Gamma} & \\ M & \xrightarrow[\Phi_M^{\Lambda\Gamma}]{} & M\Lambda\Gamma \end{array}$$

Therefore, as matrices of module homomorphisms;

$$[\psi_s^t\Lambda] \cdot [\phi_t^s] = [\Phi_{I_s}^{\Gamma\Lambda}], \text{ and } [\phi_t^s\Gamma] \cdot [\psi_s^t] = [\Phi_{M_t}^{\Lambda\Gamma}]$$

where ϕ, ψ decompose into their component homomorphisms $\phi_t^s : I_s \to M_t\Lambda$ and $\psi_s^t : M_t \to I_s\Gamma$ respectively. By Lemma 4.3.9, ϕ_t^s, ψ_s^t are in the span of $\Phi_{I_s, M_t\Lambda}$ and $\Phi_{M_t, I_s\Gamma}$ respectively. Hence, if $\mathrm{Hom}(I_s, M_t\Lambda)$ is not identically zero, $\phi_t^s = \lambda_t^s\Phi_{I_s, M_t\Lambda}$, where $\lambda_t^s \in K$, with a similar result holding for ψ_s^t. In addition, $(\lambda\Phi_{A,B})\Lambda_0 = \lambda(\Phi_{A\Lambda_0, B\Lambda_0})$ for all scalars λ, translations Λ_0, and all A, B convex.

Therefore, the interleavings between I and M correspond to the algebraic set given by values of λ_t^s, μ_s^t satisfying all quadratic relations obtained by evaluating the matrix equations above at all elements of P.

More precisely, first suppose $\mathrm{Hom}(I_s, M_t\Lambda), \mathrm{Hom}(M_t, I_s\Lambda) = 0$ for all s, t. In this case, the variety of interleavings $V^{\Lambda,\Gamma}(I, M)$ is given by

$$V^{\Lambda,\Gamma}(I, M) = \begin{cases} \text{the zero variety, if } W(I_s), W(M_t) \leq \max\{h(\Lambda), h(\Gamma)\} \text{ for all } s, t \\ \text{the empty variety, otherwise.} \end{cases}$$

The above cases correspond to whether or not setting all morphisms identically equal to zero corresponds to an admissible interleaving between I and M. On the other hand, suppose some of the relevant spaces of homomorphisms above are non-zero. Then, let r_t^s, q_s^t be given by

$$r_t^s = \lambda_t^s \cdot dim_K(\mathrm{Hom}(I_s, M_t\Lambda)), \text{ and } q_s^t = \mu_s^t \cdot dim_K(\mathrm{Hom}(M_t, I_s\Lambda)).$$

Also, let

$$\bar{r}_t^s = r_t^s \cdot dim_K(\mathrm{Hom}(I_s\Lambda, M_t\Lambda^2)), \text{ and } \bar{q}_s^t = q_s^t \cdot dim_K(\mathrm{Hom}(M_t\Lambda, I_s\Lambda^2)).$$

Let R denote the $|T| \times |S|$ matrix $R = [r_t^s \Phi_{I_s, M_t\Lambda}]$. Similarly, let Q denote the $|S| \times |T|$ matrix $Q = [q_s^t \Phi_{M_t, I_s\Lambda}]$. Also, set $\bar{R} = [\bar{r}_t^s \Phi_{I_s\Lambda, M_t\Lambda^2}], \bar{Q} = [\bar{q}_s^t \Phi_{M_t\Lambda, I_s\Lambda^2}]$.

Then, since $\phi_t^s = \lambda_t^s \Phi_{I_s, M_t\Lambda}$, and $\psi_s^t = \mu_s^t \Phi_{M_t, I_s\Lambda}$, the homomorphisms ϕ, and ψ correspond to an interleaving if and only if the equations below are satisfied, when evaluated at all elements of the poset P.

$$\bar{Q} \cdot R = [\bar{q}_s^t \Phi_{M_t\Lambda, I_s\Lambda^2}] \cdot [r_t^s \Phi_{I_s, M_t\Lambda}] = [\Phi_{I_s}^{\Gamma\Lambda}], \bar{R} \cdot Q = [\bar{r}_t^s \Phi_{I_s\Lambda, M_t\Lambda^2}] \cdot [q_s^t \Phi_{M_t\Lambda, I_s\Lambda^2}] = [\Phi_{M_t}^{\Lambda\Gamma}] \quad (4.1)$$

Therefore, in this situation $V^{\Lambda,\Gamma}(I, M)$ is the affine algebraic set with coordinate ring given by $K[\{\lambda_t^s : \mathrm{Hom}(I_s, M_t\Lambda) \neq 0\}, \{\mu_s^t : \mathrm{Hom}(M_t, I_s\Gamma) \neq 0\}]$ modulo the ideal given by all identities from (1). For some computations, see Examples 4.6.1, 4.6.2.

When P is not an n-Vee (or at least a tree branching only at a unique minimal element), the collection of interleavings still admits the structure of a variety, though the description is more cumbersome.

Remark 7. Using Proposition 4.3.12, we may visualize the interleaving distance between two elements of $\mathcal{C}$ as follows. Let (a, b) be any weight, and let $I, M \in \mathcal{C}$. For each $\epsilon \in \{h(\Lambda)\}$, let $V_\epsilon(I, M)$ denote the variety of $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleavings between I and M. Then,

$$D(I, M) = \min\{\epsilon : \text{ the variety } V^{\Lambda_\epsilon,\Lambda_\epsilon}(I, M) \text{ is non-empty}\}.$$

For some computations see Example 4.6.3.

We now observe that our width gives rise to a bottleneck metric when P is an n-Vee. For this W must be compatible with the interleaving distance in the sense of Section 2.6.

Proposition 4.3.13. *Let P be an n-Vee, and let (a, b) be weights. Let $D = D(d_{a,b})$ be the interleaving distance, and W be the width function. Then, for I, M convex,*

$$|W(I) - W(J)| \leq D(I, J).$$

The proof, which proceeds in cases, is omitted. Since W and D are compatible on Σ, we obtain a bottleneck metric on the category $\mathcal{C}$ (see Section 2.6). Let D_B denote this bottleneck metric. In the next section we will prove an isometry theorem for 1-Vees.

4.4 Isometry Theorem for Finite Totally Ordered Sets

We now prove the isometry theorem for finite totally ordered sets. We will fix notation in this section for our poset. Let $P = \{m < m_1 < m_2 < ... < n = M_1\} = [m, n]$ be totally ordered (a 1-Vee), and fix any weight (a, b). Note that, in this section only, n does not correspond to the number of maximal elements in P. We begin with some preliminary observations.

Lemma 4.4.1. *Let $P = \{m < m_1 < m_2 < ... < n = M_1\} = [m, n]$, and suppose Λ be a power of a maximal translation with given height. Then,*

(i) $im(\Lambda) \cap P = [\Lambda(m), n]$.

(ii) If $i \in [\Lambda(m), n)$, then $\Lambda^{-1}(i)$ is a singleton.

(iii) $\Lambda i = \Lambda j \in P \implies i = j$ *or* $\Lambda i = \Lambda j = n$.

The result follows from the form of the maximal translation Λ (see the proof of Lemma 4.2.2). Note that the power of a maximal translation need not be maximal. Moreover, $h(\Lambda^2)$ need not be $2h(\Lambda)$. The following Lemma follows from our characterization of the homomorphisms between convex modules in the last section (see Lemma 4.3.9).

Lemma 4.4.2. *If I, J are convex modules for $P = \{m < m_1 < m_2 < ... < n = M_1\} = [m, n]$, then $\mathrm{Hom}(I, J) \neq 0$ if and only if the endpoints of Supp$(I) = [x, X]$ and Supp$(J) = [y, Y]$ satisfy*

$$y \leq x \leq Y \leq X.$$

As previously mentioned, any homomorphism is a scalar in K times $\Phi_{I,J}$ (see Definition 4.3.7).

Lemma 4.4.3. *Let P be as above, and suppose $\Lambda = \Lambda_\epsilon$ is a maximal translation. Let A and B be convex, and suppose $A\Lambda, B\Lambda \neq 0$ and $\mathrm{Hom}(A, B) \neq 0$. Then $\mathrm{Hom}(A\Lambda, B\Lambda) \neq 0$.*

Proof. Let $s \in$Supp$(A) \cap$ Supp(B). If $s \in im(\Lambda)$ we are done by Lemma 4.3.10. Otherwise, $[x, Y] \cap im(\Lambda)$ is empty, where Supp$(A) = [x, X]$, Supp$(B) = [y, Y]$ with $y \leq x \leq Y \leq X$ as in Lemma 4.4.2. But then, by Lemma 4.4.1, $[y, Y]$ is disjoint from the image of Λ, therefore, $B\Lambda = 0$, a contradiction. ∎

Lastly, the following is an easy consequences of the results of the previous section (see Definitions 4.3.1, 4.3.2).

Lemma 4.4.4. *Let P be totally ordered. Then the following are equivalent:*

(i) $\mathrm{Hom}(J, J\Lambda^2) \neq 0$

(ii) *there is an* $x \in \mathrm{Supp}(J)$ *with* $\Lambda^2 x \in \mathrm{Supp}(J)$

(iii) $J^{+\Lambda^2} \neq 0$

(iv) $(J^{+\Lambda^2})\Lambda \neq 0$

(v) $J^{-\Lambda^2} \neq 0.$

We are now ready to prove that every interleaving induces a matching of barcodes when P is a 1-Vee. This is very much an algebraic reformulation of the results of Bauer and Lesnick in [BL13] applied to our framework. We will make use of their canonical matchings of barcodes induced by injective or surjective module homomorphisms.

Definition 4.4.5. (see Section 4 in [BL13]) Let P be totally ordered, and let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $\mathcal{C}$. Let f be a module homomorphism from $I \xrightarrow{f} M$. Then,

(i) if f is surjective, let $\Theta(f)$ from $B(M)$ to $B(I)$ be the canonical matching of barcodes.

(ii) if f is injective, let $\Theta(f)$ from $B(I)$ to $B(M)$ be the canonical matching of barcodes.

Recall from [BL13], Θ is categorical on injections or surjections. That is, $f = g \circ h$, f, g, h surjections implies $\Theta(f) = \Theta(h) \circ \Theta(g)$. And dually, $f = g \circ h$, f, g, h injections implies $\Theta(f) = \Theta(g) \circ \Theta(h)$.

The authors wish to emphasize that the above statements holds for any permissable enumeration on each barcode. That is to say, for each module M, all isomorphic elements of the barcode $B(M)$ may be enumerated arbitrarily. This enumeration is then fixed. In one instance, it will be convenient (though not necessary) to choose explicitly an enumeration for a particular barcode.

We now establish some additional properties of convex modules for 1-Vees.

Lemma 4.4.6. *Let $P = [m, n]$ be a 1-Vee, Λ a maximal translation on P. Let Σ be the set of isomophism classes of convex modues. Let F, G be the functions*

$$F, G : \Sigma \to \Sigma \cup \{0\}$$

where $F(\sigma) = \sigma^{-\Lambda^2}$ and $G(\sigma) = \sigma^{+\Lambda^2}\Lambda$.

Let $\Sigma_0 = \{\sigma : W(\sigma) > h(\Lambda)\}$, and $\bar{\Sigma}$ be $\Sigma_0 \cap \{\sigma \in \Sigma : Supp(\sigma) = [x, X], \text{ with } \Lambda^2 x = n\}$.

(i) $F(\Sigma_0) \subset \Sigma$, and F is one-to-one on Σ_0.

(ii) $G(\Sigma_0) \subset \Sigma$, and G is one-to-one on $\Sigma_0 - \bar{\Sigma}$. Also, $G(\bar{\Sigma}) = \{\sigma_n \Lambda\}$, where σ_n is the convex module with support $[n]$.

Proof. We will show that if $\sigma_1, \sigma_2 \in \Sigma_0$ with$F(\sigma_1) \cong F(\sigma_2)$, then $\sigma_1 \cong \sigma_2$. Since convex modules are characterized by their supports, say $F(\sigma_1), F(\sigma_2)$ have shared support $[x, X']$. Then X' is maximal such that $\Lambda^2 X' \leq X_1$ and also such that $\Lambda^2 X' \leq X_2$where σ_1, σ_2 have support given by $[x, X_1]$, and $[x, X_2]$ respectively. But then by Lemma 4.4.1, $X_1 = X_2$, so $\sigma_1 \cong \sigma_2$. This proves (i).

For (ii), we'll prove the contrapositive. Suppose $\sigma_1, \sigma_2 \in \Sigma_0 - \bar{\Sigma}$ have supports given by $[x_1, X_1], [x_2, X_2]$ respectively. Suppose $\Lambda^2 x_1 < \Lambda^2 x_2 \leq n$, then by Lemma 4.4.1, $\Lambda x_1 < \Lambda x_2$. Then, again by Lemma 4.4.1, $G(\sigma_1) = [\Lambda x_1, \cdot]$, $G(\sigma_2) = [\Lambda x_2, \cdot]$, which are distinct. On the other hand, if $x_1 = x_2$, $X_1 < X_2 \leq n$, then $\sigma_1^{\Lambda^2}, \sigma_2^{\Lambda^2}$ have supports given by $[\Lambda^2 x_1, X_1], [\Lambda^2 x_1, X_2]$ respectively. But then, since only X_2 is possibly equal to n, the right endpoint of the support of $G(\sigma_2)$ is strictly larger than the right endpoint of the support of $G(\sigma_1)$.

Clearly, if $\sigma \in \bar{\Sigma}$, the support of σ is $[x, n]$, with $\Lambda^2 x = n$. Then, by inspection, $G(\sigma) = \sigma_n \Lambda$. Moreover, it is clear from the proof that $G^{-1}(\sigma_n \Lambda) \subseteq \bar{\Sigma}$. ∎

Proposition 4.4.7. *Let P be totally ordered and let $\mathcal{C}$ be the full subcategory of $A(P)$-modules consisting of direct sums of convex modules. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be a weight and let D denote interleaving distance (corresponding to the weight (a, b)) restricted to $\mathcal{C}$. Let $W(M) = min\{\epsilon : Hom(M, M\Gamma\Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(\mathcal{P}), h(\Gamma), h(\Lambda) \leq \epsilon\}$, and let D_B be the bottleneck distance on $\mathcal{C}$ corresponding to the interleaving distance and W. Then, the identity is an isometry from*

$$(\mathcal{C}, D) \xrightarrow{Id} (\mathcal{C}, D_B).$$

This corresponds to the case that P is a 1-Vee in Theorem 4.5.6. The result follows from Theorem 4.4.8. We will proceed in the same fashion as [BL13]. Before continuing, we point out that Theorem 4.4.8 (and later Theorem 4.5.6) do not say that every interleaving is diagonal (see Examples 4.6.1, 4.6.2). Instead, they simply constrain the isomorphism classes of modules which admit an interleaving.

Theorem 4.4.8. *Let P be totally ordered (P is a 1-Vee) and let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $\mathcal{C}$. Let $\Lambda = \Lambda_\epsilon \in \mathcal{T}(P)$ be maximal with $h(\Lambda) = \epsilon$. Suppose there exists a (Λ, Λ)-interleaving between I and M. Then there exists a $h(\Lambda)$ matching from $B(I)$ to $B(M)$.*

The proof of the Theorem will consist of three parts.

1. If $W(I_s) > h(\Lambda)$, then I_s is matched.

2. If $W(M_t) > h(\Lambda)$, then M_t is matched.

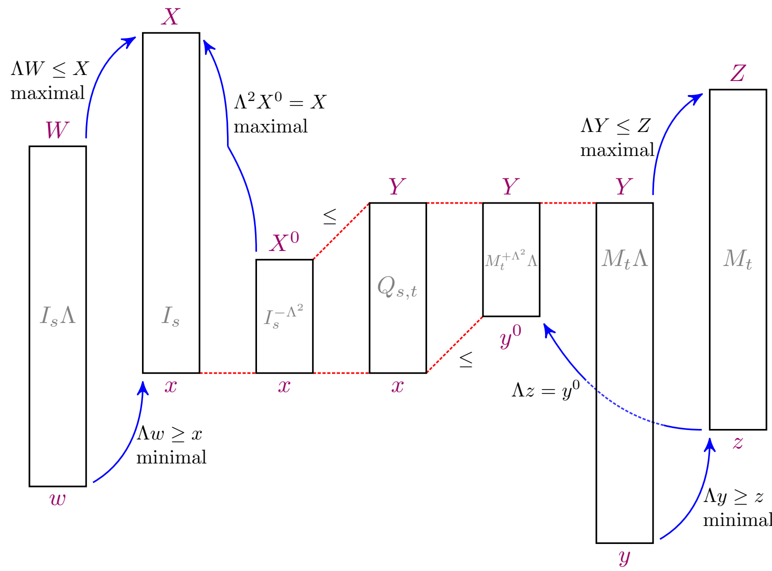
3. If I_s and M_t are matched (independent of W), then there is a (Λ, Λ)-interleaving between I_s and M_t.

Our matching is a slight modification of the matching in [BL13]. It is given by the following composition (see Definition 4.4.5)

$$B(I) \xrightarrow{\Theta(\rho)^{-1}} B(im(\phi)) \xrightarrow{\Theta(\iota)} B(M\Lambda) \xrightarrow{B(M, M\Lambda)} B(M),$$

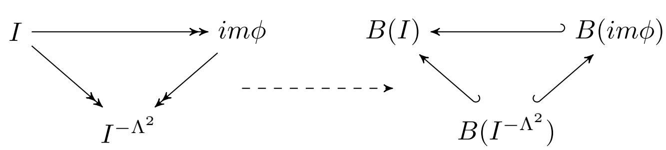
where ι is the inclusion from $im(\phi)$ into $M\Lambda$, ρ is the surjection from I to $im(\phi)$, and $B(M, M\Lambda)$ is the natural inclusion of barcodes induced from $M\Lambda$ to M given by $B(M, M\Lambda)(M_t\Lambda) = M_t$.

Note that in [BL13], it was only necessary to take the matching as far as $B(M\Lambda)$. There, that was justified since the assignment which we call $B(M, M\Lambda)$ was a bijection between barcodes which preserved W. In the present context neither of these properties hold. Specifically, $|B(M\Lambda)|$ may be strictly smaller than $|B(M)|$. Moreover, either of $W(M_t), W(M_t\Lambda)$ may be strictly larger than the other. The detailed schematic below displays all relevant convex modules. This will be useful in the proof.



We now prove Theorem 4.4.8.

Proof. First, say $I_s \in B(I)$ with $W(I_s) > h(\Lambda)$. Then, by Lemma 4.4.4, $I_s{}^{-\Lambda^2} \neq 0$. Additionally, by Proposition 4.3.3, and since induced matchings are categorical for surjections, we obtain the commutative triangle of barcodes below.

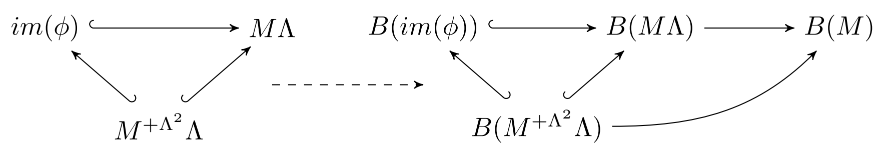


But the induced matching $B(I^{-\Lambda^2}) \to B(I)$ sends $I_s^{-\Lambda^2} \xrightarrow{\Theta} I_s$ up to isomorphism, therefore I_s is matched with an element of $B(im(\phi))$. That is, $I_s \in im(\Theta(\rho))$. But then, since $\Theta(\iota)$ and $B(\Phi_M^{\Lambda})$ are injections of barcodes, I_s is matched with some $M_t \in B(M)$. This establishes (1).

Next, suppose $M_t \in B(M)$ with $W(M_t) > h(\Lambda)$. Then, by Lemma 4.4.4, $M_t^{+\Lambda^2}\Lambda \neq 0$. Moreover, by Proposition 4.3.3, and since induced matchings are categorical for injections, we obtain a commutative diagram of barcodes for any choice of admissible emumeration. It is convenient to specify a particular enumeration for $B(M^{+\Lambda^2}\Lambda)$. This is done as follows;

- For $\sigma \in B(M^{+\Lambda^2}\Lambda), \sigma \ncong \sigma_n\Lambda$ (see Lemma 4.4.6), there is no restriction on the enumeration restricted to $\{\sigma\}$.
- For $\sigma \in B(M^{+\Lambda^2}\Lambda), \sigma \cong \sigma_n\Lambda$, enumerate $\{\sigma\}$, by $\sigma_1 = G(\tau_1) \leq G(\tau_2) = \sigma_2$ if and only if $\tau_1 \leq \tau_2$.

With this choice of enumeration, we obtain the commutative diagram below.



Since $B(\rho)$ is an injection of barcodes, this proves (2).

We now prove (3). First, note that if I_s and M_t are matched with $h(I_s), h(M_t) \leq \epsilon$, then setting ϕ, ψ both equal to zero, we obtain a (Λ, Λ)-interleaving between I_s and M_t. Therefore, let

$$S' = \{s : h(I_s) > \epsilon\}, \text{ and } T' = \{t : h(M_t) > \epsilon\}.$$

We will write $I_s \updownarrow M_t$ when I_s and M_t are matched. It remains to show that if $I_s \updownarrow M_t$, then there is a (Λ, Λ)-interleaving between I_s and M_t when,

(a) $s \in S', t \notin T'$,

(b) $s \notin S', t \in T'$, or

(c) $s \in S', t \in T'$

Note that because of the asymmetry associated with the matching, the cases (b.) and (c.) are not identical. Let the supports of $I_s, I_s\Lambda, M_t$ and $M_t\Lambda$ be given by $[w, W], [x, X], [y, Y]$, and $[z, Z]$ respectively. When $s \in S'$, let X^0 be maximal such that $\Lambda^2 X^0 \leq X$. That is, $I_s^{-\Lambda^2}$ has support given by $[x, X^0]$. Similarly, when $t \in T'$, let $y^0 = \Lambda z$, so then $M_t^{+\Lambda^2}\Lambda$ has support given by $[y^0, Y]$. Note that if $z \notin im(\Lambda)$, we have that $\Lambda z < \Lambda^2 y$ and $y = m$.

Proceeding as in [BL13], by Proposition 4.2.5, if $I_s \updownarrow M_t$, then we have the relations

$$y \leq x \leq Y \leq X.$$

Hence, there is a non-zero homomorphism from $I_s \to M_t\Lambda$. Therefore, set $\Phi_{I_s, M_t\Lambda} = \chi([x, Y]) = \phi'$. This will be one of our interleaving morphisms. We next define our second interleaving morphism. We must show that if one of (a), (b), or (c) is satisfied, we have the relations,

$$w \leq z \leq W \leq Z.$$

By inspection, it suffices to show the following statements:

(i) If $t \in T'$, then $w \leq z$.

(ii) If $s \in S'$, then $z \leq W$ and $W \leq Z$.

(iii) If $s \in S'$ and $t \notin T'$, then $w \leq z$.

(iv) If $s \notin S'$ and $t \in T'$, then $z \leq W$ and $W \leq Z$.

We now prove (i) through (iv). First, if $t \in T'$, then $\Lambda z = y_0$. Also, w is minimal such that $\Lambda w \geq x$. As $x \leq y_0$, $x \leq \Lambda z$, and so $w \leq z$ by minimality. This proves (i).

Next, say $s \in S'$. Then, $z \leq \Lambda y$ by definition. Also, since $x \leq y$ we have that $\Lambda y \leq \Lambda x$. As W is maximal such that $\Lambda W \leq X$, and $s \in S'$, we have $\Lambda^2 x \leq X$. Therefore, $\Lambda x \leq W$. Since $\Lambda(\Lambda x) \leq X$, $\Lambda x \leq W$. Therefore $z \leq \Lambda y \leq \Lambda x \leq W \text{ as required.}$ Continuing, since $s \in S'$, X_0 is maximal such that $\Lambda^2 X_0 = X$. By the maximality of W, $\Lambda X_0 = W$. But then we have $W = \Lambda X_0 \leq \Lambda Y \leq Z,$ since $im(\phi)$ includes into $I^{-\Lambda^2}$. This proves (ii).

Now, suppose $s \in S'$, $t \notin T'$. If $x \geq \Lambda m$, then $\Lambda w = x$, and so $\Lambda^3 w = \Lambda^2 x \leq X$, since $s \in S'$. But then, $W \geq \Lambda^2 w$. Hence, since $t \notin T'$, we have $\Lambda^2 w \leq W \leq Z < \Lambda^2 z$. The result follows from monoticity. On the other hand, if $x < \Lambda m$, then $w = m$, so $w \leq z$. Thus we have shown (iii).

Lastly, say $s \notin S', t \in T'$. We must establish $z \leq W$ and $W \leq Z$. First, since $t \in T'$, we have that $\Lambda z = y_0 \leq Y \leq X$. Since W is maximal with $\Lambda W \leq X$, it follows that $z \leq W$. Next, note that if $t \in T'$, then $\Lambda W = X$, since $\Lambda W \neq X \implies X \notin im(\Lambda)$. Then $\Lambda^2 z = \Lambda(\Lambda z) \leq Z$, so $Y \geq \Lambda z$. Therefore X is in $im(\Lambda)$ so it must be the case that $\Lambda W = X$. But then since $s \notin S', t \in T'$, we have $\Lambda W = X < \Lambda^2 x \leq \Lambda^2 y_0 = \Lambda^3 z \leq \Lambda Z$. Therefore, $\Lambda W \neq n$, so by monoticity $W < Z$ as required. This proves (iv).

Thus, we have shown that if $s \in S'$ or $t \in T'$,

$$w \leq z \leq W \leq Z.$$

Therefore, set $\Phi_{M_t, I_s\Lambda} = \chi([z, W]) = \psi'$. This will be our second interleaving morphism. It now remains only to show that

$$\psi'\Lambda \circ \phi' = \Phi_{I_s}^{\Lambda^2}, \text{ and } \phi'\Lambda \circ \psi' = \Phi_{M_t}^{\Lambda^2}.$$

Thus, we have;

$$\phi'\Lambda = \chi([x, Y])\Lambda = \chi([w, Y^*]), \text{ and } \psi'\Lambda = \chi([z, W])\Lambda = \chi([y, W^*]), \text{ where}$$

Y^* is maximal such that $\Lambda Y^* \leq Y$, and W^* is the maximal with $\Lambda W^* \leq W$.

We now proceed to establish the required commutativity conditions. First, say $s \in S'$. We will show that $\psi'\Lambda \circ \phi' = \Phi_{I_s, I_s\Lambda^2} = \chi([x, W^*])$. Note that, by definition, $\psi'\Lambda \circ \phi'$ is a composition of module homomorphisms, and, hence, a module homomorphism. Therefore, by Lemma 4.3.9, we need only show that the linear map $\chi([x, W^*])$ is non-zero at any vertex. To do this, we will establish that $x \leq W^*$ (that is, $\chi([x, W^*])$ is non-zero at x). But, $s \in S' \implies \Lambda^2 x \leq X$. As W^* is maximal with $\Lambda^2 W^* = X$, the inequality follows. Now say $s \notin S'$. Then, $\psi'\Lambda \circ \phi' = 0$ as required.

We now show the commutativity of the other triangle. First, suppose that $t \in T'$. As above, we will show that $\phi'\Lambda \circ \psi' = \Phi_{M_t, M_t\Lambda^2} = \chi([z, Y^*])$. Again, we need only demonstrate that $z \leq Y^*$. But $t \in T' \implies \Lambda^2 z \leq Z$. SInce Y^* is maximal with $\Lambda^2 Y^* = Z$, the result follows. Again, if $t \notin T'$, the result is trivial.

Therefore, if $I_s \updownarrow M_t$, then there is a (Λ, Λ)-interleaving between I_s and M_t as required. This proves (3) and finishes the proof of the theorem. ∎

In the next section we will use Theorem 4.4.8 to prove our main result.

4.5 Proof of Main Results

Before proving the main results, we establish some useful facts. This first result will allow us to make a "half matching."

Lemma 4.5.1. *Let S, T be sets with S finite, let $x : S \to \mathcal{P}(T)$ be a function such that for* *all* $\phi \neq S_0 \subseteq S$*,*

$$\Big|\bigcup_{s\in S_0} x(s)\Big| \geq |S_0|.$$

Then, there exists a function $F : S \to T$ *such that* F *is an injection, and for all* $s, F(s) \in x(s)$*.*

Proof. We prove the result by induction on $|S|$. If $|S| = 1$, the result is trivial. Now say $|S| > 1$ and the result holds for all sets with smaller cardinality. First, suppose there exists a non-empty subset $S_0 \subseteq S$ such that

$$\Big|\bigcup_{s\in S_0} x(s)\Big| = |S_0|.$$

Let S_0 be a minimal non-empty subset of S where equality holds. We will show that we can define an injection f from S_0 to T with $f(s) \in x(s)$. Pick $s_0 \in S_0$, $t_0 \in x(s_0)$ and set $f(s_0) = t_0$. If $S_0 = \{s_0\}$ we are done, so assume $S_0 \neq \{s_0\}$. Then, let $\bar{x} : S_0 - \{s_0\} \to \mathcal{P}(T)$, be defined by $\bar{x}(s) = x(s) - \{t_0\}$. Now let S' be a non-empty subset of $S_0 - \{s_0\}$. Then,

$$\Big|\bigcup_{s\in S'} \bar{x}(s)\Big| \;=\; \Big|\bigcup_{s\in S'} x(s) - \{t_0\}\Big| = \Big|\big(\bigcup_{s\in S'} x(s)\big) - \{t_0\}\Big| \geq |S'| + 1 - 1 = |S'|,$$

by the minimality of S_0. Thus, by induction, there exists a one-to-one function $f : S_0 - \{s_0\} \to T$ such that $f(s) \in \bar{x}(s)$. Clearly, f can be extended to an injection on all of S_0 . If $S_0 = S$, set $f = F$ and we are done. Otherwise, define

$$\bar{x} : S - S_0 \to \mathcal{P}(T) \text{ be defined by } \bar{x}(s) = x(s) - \{f(\sigma) : \sigma \in S_0\}.$$

Now, let $\bar{s_1}, \bar{s_2}, ...\bar{s_x} \in S - S_0$. Clearly, for all i, $x(\bar{s_i}) = \bar{x}(\bar{s_i}) \cup T_i$ for some set $T_i \subseteq \{f(\sigma) : \sigma \in S_0\}$. Note that

$$\Big|\bigcup_{i\leq k} \bar{x}(\bar{s_i})\Big| < k \implies \Big|\bigcup_{s\in S_0} x(s) \cup x(\bar{s_1}) \cup x(\bar{s_2}) \cup ... \cup x(\bar{s_k})\Big| < |S_0| + k,$$

a contradiction. Thus, by induction, there is an injection $\bar{f} : S - S_0 \to T$ with $\bar{f}(s) \in \bar{x}(s)$. By construction $F = f \cup \bar{f}$ is the desired function from all of S to T.

On the other hand, if S has the property that for all $S_0 \subseteq S$, $S_0 \neq \phi$,

$$\Big| \bigcup_{s \in S_0} x(s) \Big| > |S_0|,$$

pick $s_1 \in S, t_1 \in x(s_1)$ and set $f(s_1) = t_1$. Again, let

$$\bar{x} : S - \{s_1\} \to \mathcal{P}(T) \text{ be defined by } \bar{x}(s) = x(s) - \{t_1\}.$$

Then, for $S_0 \subseteq S - \{s_1\}$,

$$\Big| \bigcup_{s \in S_0} \bar{x}(s) \Big| = \Big| \bigcup_{s \in S_0} x(s) - \{t_1\} \Big| \geq |S_0| + 1 - 1 = |S_0|.$$

Since $|S - \{s_1\}| < |S|$, the result holds by induction. ∎

Example 4.5.2. $S = \{1, 2, 3, 4, 5\}, T = \{a, b, c, d, e\}$ the function x given by

$$1 \to \{a, b, d\}, 2 \to \{b, c, e\}, 3 \to \{a, c, d\}, 4 \to \{d\}, 5 \to \{e\}.$$

A matching is constructed by setting $f(4) = d$, and $f(5) = e$. Then, one can choose any bijection from $\bar{f} : \{1, 2, 3\} \to \{a, b, c\}$. We glue f and $\bar{f}$ to obtain an injection F from S to T.

Next we make a simple observation about interleavings.

Lemma 4.5.3. *Let P be any poset, $\Lambda, \Gamma \in \mathcal{T}(P)$. Let A, B, C, D be any $A(P)$-modules with ϕ, ψ a (Λ, Γ)-interleaving between $A \oplus B$ and $C \oplus D$. Then, if $Hom(A, D\Lambda) = 0 = Hom(C, B\Gamma)$, then A, C are (Λ, Γ)-interleaved and B, D are (Λ, Γ)-interleaved.*

Proof. Note that we do not assume any modules are in the category $\mathcal{C}$. For brevity, let f_A, f_B denote the canonical homomorphism from $A \to A\Gamma\Lambda$ and $B \to B\Gamma\Lambda$ respectively. Similarly, let g_C, g_D denote $C \to C\Lambda\Gamma$ and $D \to D\Lambda\Gamma$ respectively. By decomposing ϕ, ψ into their component homomorphisms, we have;

$$\begin{bmatrix} f_A & 0 \\ 0 & f_B \end{bmatrix} = \begin{bmatrix} \psi_A^C\Lambda & \psi_A^D\Lambda \\ 0 & \psi_B^D\Lambda \end{bmatrix} \begin{bmatrix} \phi_C^A & \phi_C^B \\ 0 & \phi_D^B \end{bmatrix} = \begin{bmatrix} \psi_A^C\Lambda\phi_C^A & \psi_A^C\Lambda\phi_C^B + \psi_A^D\Lambda\phi_D^B \\ 0 & \psi_B^D\Lambda\phi_D^B \end{bmatrix},$$

$$\begin{bmatrix} g_C & 0 \\ 0 & g_D \end{bmatrix} = \begin{bmatrix} \phi_C^A\Gamma & \phi_C^B\Gamma \\ 0 & \phi_D^B\Gamma \end{bmatrix} \begin{bmatrix} \psi_A^C & \psi_A^D \\ 0 & \psi_B^D \end{bmatrix} = \begin{bmatrix} \phi_C^A\Gamma\psi_A^C & \phi_C^A\Gamma\psi_A^D + \phi_C^B\Gamma\psi_B^D \\ 0 & \phi_D^B\Gamma\psi_B^D \end{bmatrix}.$$

Thus, by inspection, if we set $\phi_C^B, \psi_A^D = 0$, the required condition will still be satisfied. ∎

We point out that this does not say that the interleaving was initially diagonal (see Example 4.6.2).

Corollary 4.5.4. *Let P be a finite poset with a unique minimal element m. Let $X, Y \in \mathcal{C}$, and Λ be a translation. Suppose $X = \bigoplus_s X_s$, and $Y = \bigoplus_t Y_t$ are (Λ, Λ)-interleaved, and $\Lambda m = m$. Let $S_m = \{s \in S : X_s(m) \neq 0\}, T_m = \{t \in T : Y_t(m) \neq 0\}$. Then,*

$$\bigoplus_{s \in S_m} X_s, \bigoplus_{t \in T_m} Y_t \text{ are } (\Lambda, \Lambda)\text{-interleaved, and } \bigoplus_{s \notin S_m} X_s, \bigoplus_{t \notin T_m} Y_t \text{ are } (\Lambda, \Lambda)\text{-interleaved.}$$

Proof. This follows easily from Lemma 4.5.3. ∎

Proposition 4.5.5. *Let P be an n-Vee. Let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $\mathcal{C}$. Suppose for all s, t, I_s and M_t are supported at m. Let $\Lambda \in \mathcal{T}(P)$ with $\Lambda m = m$. Suppose there exists a (Λ, Λ)-interleaving between I and M. Then there exists a $h(\Lambda)$ matching (in the sense of Theorem 4.4.8) from $B(I)$ to $B(M)$.*

Proof. First, we show that $|B(I)| = |B(M)|$. Since m is fixed by Λ, the commutativity of the diagram below shows that $|B(I)| = rank(f) \leq dim(M(m)) = |B(M)|$.

$$\begin{array}{ccc} (\bigoplus_{s\in S} I_s)(m) & \longrightarrow & (\bigoplus_{s\in S} I_s\Lambda^2)(m) = (\bigoplus_{s\in S} I_s)(m) \\ & \searrow^{\phi} \qquad \nearrow_{\psi\Lambda=\psi} & \\ & (\bigoplus_{t\in T} M_t\Lambda)(m) = (\bigoplus_{t\in T} M_t)(m) & \end{array}$$

Thus, by symmetry $|B(I)| = |B(M)|$. Now, let $s \in S$. Since $I_s \to I_s\Lambda^2$ is nonzero, its image is in the image of $\psi\Lambda\phi_{|I}$. Thus in particular, there exists a $t \in T$ with $\psi_s^t\Lambda\phi_t^s \neq 0$. That is, $\mathrm{Hom}(M_t\Lambda, I_s\Lambda^2) \circ \mathrm{Hom}(I_s, M_t\Lambda) \neq 0$. But by Lemma 4.3.10 then $\mathrm{Hom}(I_s\Lambda, M_t\Lambda^2)$ is also not equal to zero. So, $\mathrm{Hom}(M_t, I_s\Lambda^2)$, $\mathrm{Hom}(I_s\Lambda, M_t\Lambda^2)$ are both nonzero, hence their composition is nonzero since it is defined at m. But then, up to a scalar, it is the composition $M_t \to M_t\Lambda^2$, since $\mathrm{Hom}(M_t, M_t\Lambda^2) = K$ by Lemma 4.3.9. Thus, there is a (Λ, Λ)-interleaving between I_s and M_t. We have shown that whenever $\psi_s^t\Lambda\phi_t^s$ is nonzero, there is a (Λ, Λ)-interleaving between I_s and M_t.

Now for $s \in S$, let $x(s) = \{t \in T : \psi_s^t\Lambda\phi_t^s \neq 0\}$. Let $S_0 \subseteq S$. Then, the diagram below commutes

$$\begin{array}{ccc} \bigoplus_{s\in S_0} I_s & \longrightarrow & \bigoplus_{s\in S_0} I_s\Lambda^2 \\ & \searrow \qquad \nearrow & \\ & \bigoplus_{\substack{t\in x(s) \\ \text{some } s\in S_0}} M_t & \end{array}$$

Hence, by evaluation at m, $|S_0| = rank\{(\bigoplus_{s\in S_0} I_s)(m) \to (\bigoplus_{s\in S_0} I_s\Lambda^2)(m)\} \leq |\bigcup_{s\in S_0} x(s)|$. Then, by Lemma 4.5.1, there is an injection f from S to T with $f(s) \in x(s)$ for all s. The result follows since $|S| = |T|$ and $t \in x(s)$ implies there is a (Λ, Λ)-interleaving between I_s and M_t. ∎

We are now ready to prove our main results.

Theorem 4.5.6. *Let P be an n-Vee and let $\mathcal{C}$ be the full subcategory of $A(P)$-modules consisting of direct sums of convex modules. Let $(a,b) \in \mathbb{N} \times \mathbb{N}$ be a weight and let D denote interleaving distance (corresponding to the weight (a,b)) restricted to $\mathcal{C}$.*
Let $W(M) = min\{\epsilon : Hom(M, M\Gamma\Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(\mathcal{P}), h(\Gamma), h(\Lambda) \leq \epsilon\}$, and let D_B be the bottleneck distance on $\mathcal{C}$ corresponding to the interleaving distance and W. Then, the identity is an isometry from

$$(\mathcal{C}, D) \xrightarrow{Id} (\mathcal{C}, D_B).$$

Proof. First, let P be an asymmetric n-Vee, $P = \bigcup [n, M_i]$ with $|[m, M_{i_0}]| > |[m, M_i]|$ for $i \neq i_0$, and fix the weight (a,b). We will prove that any (Λ_1, Γ_1)-interleaving between $I, M \in \mathcal{C}$ produces an ϵ-matching for $\epsilon = \max\{h(\Lambda_1), h(\Gamma_1)\}$. Once this is established, $D_B \leq D$. For the other inequality, note that an ϵ-matching yields (after inserting appropriate zero homomorphisms) a diagonal interleaving, thus $D \leq D_B$, and hence equality.

Let $I = \bigoplus_{s \in S} I_s, M = \bigoplus_{t \in T} M_t$ be in $\mathcal{C}$. If $\mathcal{V}$ is a partition of P, for $v \in \mathcal{V}$, let

$$S_v = \{s \in S : \text{ the minimal element of } \mathrm{Supp}(I_s) \text{ is in } v\}.$$

Similarly, define T_v. Now, suppose there is a (Λ_1, Γ_1)-interleaving between I and M. Then, by Lemma 4.2.2, there exists $\Lambda = \Lambda_\epsilon$ maximal, where $\epsilon = \max\{h(\Lambda_1), h(\Gamma_1)\}$. By [BdS13] since $\Lambda_1, \Gamma_1 \leq \Lambda$, there exists a (Λ, Λ)-interleaving between I and M.

First, if $\epsilon < aT + b$, then $\Lambda m = m$. In this case, consider the partition $\mathcal{V}$ of P given by

$$\mathcal{V} = \{(m, M_i]\} \cup \{\{m\}\}, \text{ and set } S_m = S_{\{m\}}, S_i = S_{(m, M_i]}.$$

Similarly, set $T_m = T_{\{m\}}$ and $T_i = T_{(m,M_i]}$. Since $\Lambda m = m$, for all M convex,

- $\mathrm{Supp}(M) \subseteq (m, M_i], M\Lambda \neq 0 \implies \mathrm{Supp}\, M\Lambda \subseteq (m, M_i]$, and
- $m \in \mathrm{Supp}(M) \implies m \in \mathrm{Supp}(M\Lambda)$.

Therefore, if $s \in S_m, t \in T_i$, then $\mathrm{Hom}(I_s, M_t\Lambda) = 0$. Similarly, if $t \in T_m, s \in S_i$, then $\mathrm{Hom}(M_t, I_s\Lambda) = 0$. Then, by Lemma 4.5.3, we may diagonalize, obtaining (Λ, Λ)-interleavings between

$$\bigoplus_{s\in S_m} I_s \text{ and } \bigoplus_{t\in T_m} M_t, \text{ and also between } \bigoplus_{s\notin S_m} I_s \text{ and } \bigoplus_{t\notin T_m} M_t.$$

We now diagonalize further. Again, since $\Lambda m = m$, for each $i \neq j$, $s \in S_i, t \in T_j \implies \mathrm{Hom}(I_s, M_t\Lambda) = 0$ as well as the symmetric condition. Thereofore, by applying Lemma 4.5.3 repeatedly, we obtain interleavings between

$$\bigoplus_{s\in S_v} I_s \text{ and } \bigoplus_{t\in T_v} M_t \text{ for all } v \in \mathcal{V}.$$

Hence, by Proposition 4.5.5, we get a matching between the elements of the barcodes supported at m. Also, for each i, $\Lambda_{|(m,M_i]}$ is a maximal translation on a totally oriented set. Therefore, for each i we acquire a matching between those elements of the barcode in S_i and T_i by Theorem 4.4.8. Thus, an ϵ-matching is produced piecewise.

Now, suppose $\epsilon = h(\Lambda) \geq aT + b$. Then, for all convex modules M, $M\Lambda$ is identically 0, or is a convex module supported in $[m, M_{i_0}]$. Then, $M\Lambda = M\Lambda/\mathcal{I}_{i_0}M\Lambda$, thus any homomorphism from $N \to M\Lambda$ factors through $N/\mathcal{I}_{i_0}N$.

Consider the partition

$$\mathcal{V} = \{[m, M_{i_0}]\} \cup \{(m, M_i] : i \neq i_0\}, \text{ and set } S_i = S_{(m,M_i]}, S_m = S_{[m,M_{i_0}]}.$$

Similarly, define T_m, T_i. Then, for $s \in S_m, t \in T_i$, $\text{Hom}(I_s, M_t\Lambda) = 0$, since $M_t\Lambda = 0$. Since the symmetric condition holds as well, again by Lemma 4.5.3 we obtain an interleaving between

$$\bigoplus_{s \in S_m} I_s \text{ and } \bigoplus_{t \in T_m} M_t, \text{ and between } \bigoplus_{s \notin S_m} I_s \text{ and } \bigoplus_{t \notin T_m} M_t \text{ respectively.}$$

Since the latter interleaving corresponds to convex modules N with $W(N) \leq \epsilon$, it suffices to match only convex modules with indices in S_m and T_m.

However, the morphisms

$$\bigoplus_{s \in S_m} I_s \xrightarrow{\phi} \bigoplus_{t \in T_m} M_t\Lambda, \text{ and } \bigoplus_{t \in T_m} M_t \xrightarrow{\psi} \bigoplus_{s \in S_m} I_s\Lambda$$

factor through $\bigoplus_{s \in S_m} (I_s/\mathcal{I}_{i_0} I_s)$ and $\bigoplus_{t \in T_m} (M_t/\mathcal{I}_{i_0} M_t)$ respectively. Thus since $\Lambda_{|[m, M_{i_0}]}$ is maximal, again the result follows from Theorem 4.4.8.

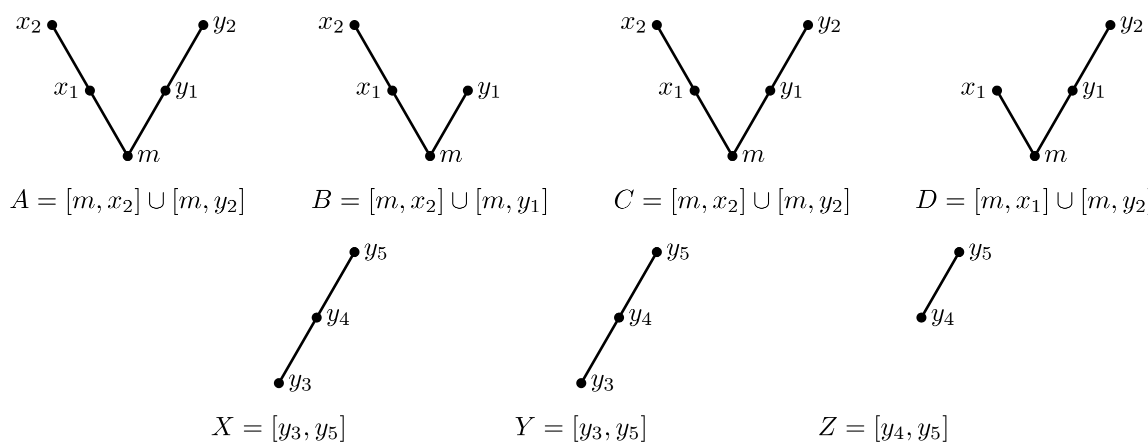
If P is an n-Vee but is not asymmetric, then we may not use Lemmas 4.2.2, and 4.2.3 explicitly. It is still the case, however, that for $\epsilon \in \{h(\Gamma) : \Gamma \in \mathcal{T}(P)\}$, with $\epsilon < aT + b$, the set $\{\Lambda : h(\Lambda) = \epsilon\}$ has a unique maximal element. Moreover, the set $\{\Lambda_\epsilon : \epsilon < aT + b\}$ is still totally ordered. Thus, if I, M are (Λ, Γ)-interleaved with $\max\{h(\Lambda), h(\Gamma)\} < aT + b$ the proof above still goes through. On the other hand, when P is not asymmetric, for all convex modules σ, $W(\sigma) \leq aT + b = aT_{i_0} + b$. Therefore, though there is not a unique translation with height corresponding to this value, interleavings of this height always produce empty matchings. ∎

4.6 Examples

We conclude with some examples. First, in Example 4.6.1, we decompose an interleaving as in the proof of Theorem 4.5.6. We also compute the varieties (see

Proposition 4.3.12) corresponding to two interleavings. Along the way, we construct some non-diagonal interleavings. In this section, if M is convex with support given by S, we write $M \sim S$.

Example 4.6.1. Let P be the 2-Vee, $P = [m, x_3] \cup [m, y_6]$ and let (a, b) be a weight. Let $\Lambda = \Lambda_a$. Consider the following convex modules.



$A = [m, x_2] \cup [m, y_2]$ $B = [m, x_2] \cup [m, y_1]$ $C = [m, x_2] \cup [m, y_2]$ $D = [m, x_1] \cup [m, y_2]$

$X = [y_3, y_5]$ $Y = [y_3, y_5]$ $Z = [y_4, y_5]$

Let $I = A \oplus B \oplus X$ and $M = C \oplus D \oplus Y \oplus Z$. We will decompose an arbitrary (Λ, Λ)-interleaving between I and M as in the proof of Theorem 4.5.6. Then, we will calculate the varieties (see Remark 4.3.12) corresponding to the "factored" interleavings the decomposition produces on the appropriate partition of the barcodes $B(I)$ and $B(M)$.

First, let ϕ', ψ' be any (Λ, Λ)-interleaving between I and M. By Lemma 4.5.3, since $\Lambda m = m$, there exist

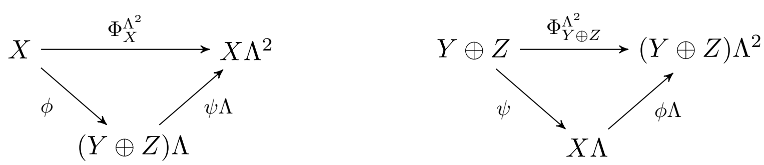
- a (Λ, Λ)-interleaving between $A \oplus B$ and $C \oplus D$, and
- a (Λ, Λ)-interleaving between X and $Y \oplus Z$.

We treat these separately, referring to each in turn as ϕ, ψ. First, we factor each ϕ, ψ

into their corresponding summands, adopting the previous notation. For example $\phi_Y^X : X \to Y\Lambda$. Since $I, M \in \mathcal{C}$, we know that

$$\phi_Y^X = \lambda\Phi_{X,Y\Lambda}, \text{ and } \phi_Y^X\Lambda = \lambda\Phi_{X\Lambda,Y\Lambda^2}.$$

Of course, all other similar identities hold as well. We first concentrate on the modules supported in $(m, M_y]$. Thus, we have the diagrams below.



Therefore, we have the matrices of module homorphisms

$$\phi = \begin{bmatrix} \phi_Y^X = \alpha\Phi_{X,Y\Lambda} \\ \phi_Z^X = \beta\Phi_{X,Z\Lambda} \end{bmatrix} \text{ and } \psi = \begin{bmatrix} \psi_X^Y = \lambda\Phi_{Y,X\Lambda} & \psi_X^Z = \mu\Phi_{Z,X\Lambda} \end{bmatrix}$$

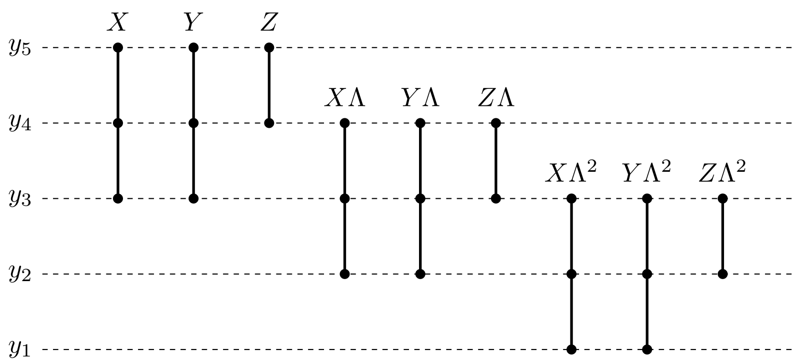
Since ϕ, ψ is a (Λ_a, Λ_a)-interleaving between X and $Y \oplus Z$ equations (2) and (3) below must hold. First,

$$\Phi_X^{\Lambda^2} = \begin{bmatrix} \Phi_X^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} (\psi_X^Y\Lambda)\phi_Y^X + (\psi_X^Z\Lambda)\phi_Z^X \end{bmatrix} = \begin{bmatrix} (\lambda\alpha + \mu\beta)\Phi_X^{\Lambda^2} \end{bmatrix}. \tag{4.2}$$

And then,

$$\Phi_{Y\oplus Z}^{\Lambda^2} = \begin{bmatrix} \Phi_Y^{\Lambda^2} & 0 \\ 0 & \Phi_Z^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} \alpha\lambda\Phi_{Y,Y\Lambda^2} & \alpha\mu\Phi_{Z,Y\Lambda^2} \\ \beta\lambda\Phi_{Y,Z\Lambda^2} & \beta\mu\Phi_{Z,Z\Lambda^2} \end{bmatrix}. \tag{4.3}$$

Note that the above equations define the variety $V^{\Lambda,\Lambda}(X, Y \oplus Z)$, because in the notation of Proposition 4.3.12, no variables are deleted between R and $\bar{R}$, and Q and $\bar{Q}$ respectively. For computational purposes, it is convenient to arrange the relevant convex modules side by side:



Notice that evaluating (2) at all elements of P we have

$$\left(\Phi_X^{\Lambda^2}\right)(y_3) = \begin{bmatrix}1\end{bmatrix} \text{ and } \left(\Phi_X^{\Lambda^2}\right)(i) = \begin{bmatrix}0\end{bmatrix} \text{ for any } i \neq y_3.$$

Similarly, evaluating (3) we have

$$\left(\Phi_{Y\oplus Z}^{\Lambda^2}\right)(y_3) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \text{ and } \left(\Phi_{Y\oplus Z}^{\Lambda^2}\right)(i) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \text{ for } i \neq y_3.$$

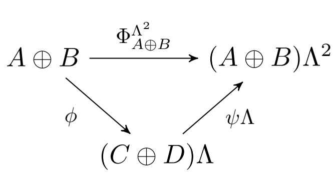
Therefore, we get the following system of equations:

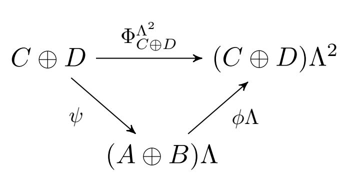
$$\lambda\alpha + \mu\beta = 1, \alpha\lambda = 1, \beta\lambda = 0.$$

Thus, $V^{\Lambda,\Lambda}(X, Y \oplus Z)$ is the affine variety with coordinate ring $K[\lambda, \mu, \alpha, \beta]$ modulo the ideal $\langle \lambda\alpha + \mu\beta - 1, \alpha\lambda - 1, \lambda\beta \rangle$ (see Proposition 4.3.12).

This corresponds to one choice of parameter in K^* and one in K.

Next consider the modules supported at m. We now compute the variety $V^{\Lambda,\Lambda}(A \oplus B, C \oplus D)$. Again, we must have the commutative triangles below.





Decomposing ϕ, ψ we have,

$$\phi = \begin{bmatrix} \phi_C^A = e\Phi_{A,C\Lambda} & \phi_C^B = f\Phi_{B,C\Lambda} \\ \phi_D^A = g\Phi_{A,D\Lambda} & \phi_D^B = h\Phi_{B,D\Lambda} \end{bmatrix} \text{ and } \psi = \begin{bmatrix} \psi_A^C = i\Phi_{C,A\Lambda} & \psi_A^D = j\Phi_{D,A\Lambda} \\ \psi_B^C = k\Phi_{C,B\Lambda} & \psi_B^D = l\Phi_{D,B\Lambda} \end{bmatrix}.$$

Since ϕ, ψ is a (Λ, Λ)-interleaving between $A \oplus B$ and $C \oplus D$ (and since no variables are eliminated from Q to $\bar{Q}$) we have,

$$\Phi_{A\oplus B}^{\Lambda^2} = \begin{bmatrix} \Phi_A^{\Lambda^2} & 0 \\ 0 & \Phi_B^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} (ei+gj)\Phi_{A,A\Lambda^2} & (fi+hj)\Phi_{B,A\Lambda^2} \\ (ek+gl)\Phi_{A,B\Lambda^2} & (fk+hl)\Phi_{B,B\Lambda^2} \end{bmatrix}$$

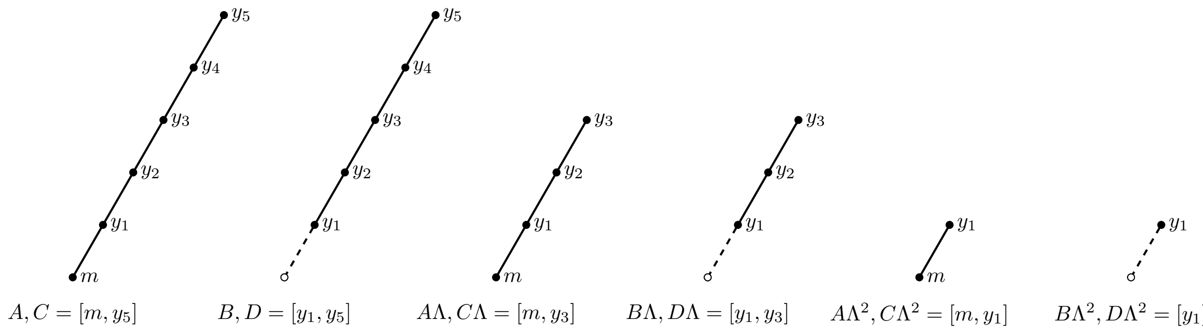
Evaluating everything at m, we obtain

$$\begin{bmatrix} ei+gj & fi+hj \\ ek+gl & fk+hl \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since evaluation at any other element of the poset makes all equations trivial, this identity (along with the redundant one obtained from $\Phi_{C\oplus D}^{\Lambda^2}$) is the only necessary condition. Therefore, the space of (Λ, Λ)-interleavings between $A \oplus B$ and $C \oplus D$, $V^{\Lambda,\Lambda}(A \oplus B, C \oplus D)$ is $Gl_2(K)$, the variety of invertible 2×2 matrices. In particular, many interleavings between $A \oplus B$ and $C \oplus D$ are as far from diagonal as possible.

By choosing a point in each variety separately, we obtain an interleaving between I and M. Note that although we produced many interleavings, we did not classify the (Λ, Λ)-interleavings between I and M. This is clear, as we passed from our original ϕ', ψ' to a pair of separate interleavings on a partition of each barcode. In the next example we compute the full variety of interleavings between two elements of $\mathcal{C}$.

Example 4.6.2. Let P be again be the 2-Vee, $P = [m, x_3] \cup [m, y_6]$. Let (a, b) be a weight, and set $\Lambda = \Lambda_{2a}$.



Let $I = A \oplus B$ and $M = C \oplus D$, and let $\Lambda = \Lambda_{2a}$. We will calculate$V^{\Lambda,\Lambda}(I, M)$, the variety corresponding to all (Λ, Λ)-interleavings between I and M (see Proposition 4.3.12). First, consider an arbitrary such interleaving, ϕ, ψ. Of course, as always this yields the standard commutative triangles. Since $\Lambda m = m$, there exist no non-zero morphisms from A into any module not containing the minimal m. Moreover, $m \in \mathrm{Supp}(\sigma\Lambda)$ if and only if $m \in \mathrm{Supp}(\sigma)$. Hence, using our standard notation, it must be the case that ϕ_D^A and ψ_B^C are identically zeros. Then, as matrices of module homomorphisms, we have

$$\phi = \begin{bmatrix} \phi_C^A & \phi_C^B \\ 0 & \phi_D^B \end{bmatrix} = \begin{bmatrix} \lambda\Phi_{C\Lambda}^A & \rho\Phi_{C\Lambda}^B \\ 0 & \mu\Phi_{D\Lambda}^B \end{bmatrix} \text{ and } \psi = \begin{bmatrix} \psi_A^C & \psi_A^D \\ 0 & \psi_B^D \end{bmatrix} = \begin{bmatrix} \alpha\Phi_{A\Lambda}^C & \gamma\Phi_{A\Lambda}^D \\ 0 & \beta\Phi_{B\Lambda}^D \end{bmatrix}$$

Since ϕ and ψ constitute a (Λ, Λ)-interleaving, we obtain equations (4) and (5) below.

$$\Phi_I^{\Lambda^2} = \Phi_{A\oplus B}^{\Lambda^2} = \begin{bmatrix} \Phi_A^{\Lambda^2} & 0 \\ 0 & \Phi_B^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} \alpha\lambda\Phi_{A,A\Lambda^2} & (\alpha\rho + \gamma\mu)\Phi_{B,A\Lambda^2} \\ 0 & \beta\mu\Phi_{B,B\Lambda^2} \end{bmatrix} \tag{4.4}$$

$$\Phi_M^{\Lambda^2} = \Phi_{C\oplus D}^{\Lambda^2} = \begin{bmatrix} \Phi_C^{\Lambda^2} & 0 \\ 0 & \Phi_D^{\Lambda^2} \end{bmatrix} = \begin{bmatrix} \lambda\alpha\Phi_{C,C\Lambda^2} & (\lambda\gamma + \rho\beta)\Phi_{D,C\Lambda^2} \\ 0 & \mu\beta\Phi_{D,D\Lambda^2} \end{bmatrix} \tag{4.5}$$

Evaluating (4), we see that

$$\left(\Phi_I^{\Lambda^2}\right)(m) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \left(\Phi_I^{\Lambda^2}\right)(y_1) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \text{ and } \left(\Phi_I^{\Lambda^2}\right)(x) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \text{ for } x \neq m, y_1$$

From the evaluation at y_1, we get the restrictions:

$$\alpha\lambda = 1, \beta\mu = 1, \alpha\rho + \gamma\mu = 0,$$

the last of which only appears because $\Phi_{B,A\Lambda^2} \neq 0$.

When evaluating at m, we obtain the redundant constraint $\alpha\lambda = 1$. Since every homomorphism in (4) has support contained in $[m, y_1]$, so there are no further restrictions from (4).

By inspection, evaluating (5) obtains no new conditions, since

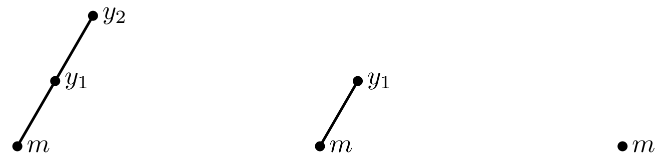
$$\alpha\rho + \gamma\mu = 0 \iff \rho = -\lambda\gamma\mu \iff \lambda\gamma + \rho\beta = 0.$$

Therefore, $V^{\Lambda,\Lambda}(I, M)$ is the affine variety with coordinate ring $K[\lambda, \mu, \rho, \alpha, \beta, \gamma]$ modulo the ideal $\langle \lambda\alpha - 1, \mu\beta - 1, \alpha\rho + \gamma\mu \rangle$.

Thus, the interleavings are parametrized by two elements of K^* and one element of K.

We point out that in both Examples 4.6.1 and 4.6.2 there were additional degrees of freedom for interleavings not seen when one passes to a matching. In Example 4.6.3 we realize the interleaving distance as the minimum height of a translation with non-empty variety.

Example 4.6.3. Let P be the 1-Vee $P = [m, x_3]$, and let (a, b) be a weight, with $a < b$. Consider the convex modules



$A \sim [m, y_2]$ $\quad$ $B \sim [m, y_1]$ $\quad$ $[m]$

We will calculate the variety $V^{\Lambda,\Lambda}(I, M)$ for various choices of translations. We will use these calculations to point out the interleaving distance as in Remark 7. Whatever translations we consider, we always set

$$\phi = \left[\phi_B^A = \alpha\Phi_{A,B\Lambda}\right] \text{ and } \psi = \left[\psi_A^B = \beta\Phi_{B,A\Lambda}.\right]$$

First, let $\Lambda = \Lambda_0$, the identity translation. We can see that there is no (Λ_0, Λ_0)-interleaving between A and B, since $\Phi_A^{\Lambda^2} \neq 0$ but $\mathrm{Hom}(B, A\Lambda) = 0$. The variety of (Λ_0, Λ_0)-interleavings therefore, must be the empty variety. We recover this via the equation,

$$[\Phi_A^{\Lambda^2}] = \bar{Q} \cdot R = [0] \cdot [\alpha\Phi_{A,B\Lambda}]$$

(using the notation of Proposition 4.3.12) which has no solution, say when evaluated at m. Thus, for $\Lambda = \Gamma = \Lambda_0$, the variety $V^{\Lambda,\Gamma}(I, M)$ is empty, as required

.

Now, say $\Lambda = \Lambda_a$. Since $a < b$, Λ_a is the next largest maximal translations. Then, $A\Lambda \cong B$ and $B\Lambda \cong C$. Moreover, $\mathrm{Hom}(A, B\Lambda), \mathrm{Hom}(B, A\Lambda) \cong K$. The space of interleavings are defined by the equations

$$\left[\Phi_A^{\Lambda^2}\right] = \bar{Q} \cdot R = \left[\alpha\beta(\Phi_{B\Lambda,A\Lambda^2} \cdot \Phi_{A,B\Lambda})\right], \left[\Phi_B^{\Lambda^2}\right] = \bar{R} \cdot Q = \left[0 \cdot \beta(\Phi_{A\Lambda,B\Lambda^2} \cdot \Phi_{B,A\Lambda})\right]$$

Note that the variable α is absent from $\bar{R}$, since $\mathrm{Hom}(A\Lambda, B\Lambda^2) = 0$. The first equation yields $\alpha\beta = 1$ by evaluating at m, and the second equation is consistent and trivially satisfied since $\Phi_B^{\Lambda^2} = 0$.

Therefore, the space of (Λ_a, Λ_a)-interleavings corresponds to the affine variety with coordinate ring $K[\alpha, \beta]$ modulo the ideal $\langle \alpha\beta - 1 \rangle$. This corresponds to a choice of one parameter in K^*. Also, since $\epsilon = a$ corresponds to the first non-zero variety, we can see that $D(A, B) = a$.

Now suppose that $\Lambda = \Gamma = \Lambda_{2a}$. Then, $A\Lambda \cong C$ and $B\Lambda \cong 0$. Also, $\mathrm{Hom}(A, B\Lambda) = 0$ and $\mathrm{Hom}(B, A\Lambda) \cong K$.

The space of interleavings are defined by the equations

$$\left[\Phi_A^{\Lambda^2}\right] = \bar{Q} \cdot R = \left[0 \cdot 0(\Phi_{B\Lambda, A\Lambda^2} \cdot \Phi_{A, B\Lambda})\right], \left[\Phi_B^{\Lambda^2}\right] = \bar{R} \cdot Q = \left[0 \cdot \beta(\Phi_{A\Lambda, B\Lambda^2} \cdot \Phi_{B, A\Lambda})\right].$$

Since $\Phi_A^{\Lambda^2}, \Phi_B^{\Lambda^2}$ are both identically zero, any value of β satisfies the above equations. Therefore, in this case $V^{\Lambda,\Gamma}(A, B)$ corresponds to the affine variety with coordinate ring $K[\beta]$. Of course, this corresponds to a choice of one parameter in K.

On the other hand, when $\Lambda = \Gamma = \Lambda_l$ for $l \geq 3a$, $A\Lambda = B\Lambda = 0$. In this case, the only interleaving between A and B is $\phi = \psi = 0$. That is to say, tfor such translations, $V^{\Lambda,\Gamma}(A, B)$ is the 0 variety.

Putting it all together, we obtain the curve below with values in the category of affine varieties. Only the jump discontinuities are labeled. In a slight abuse of notation, we write the coordinate rings instead of their corresponding varieties.

$$\emptyset \xrightarrow[\epsilon \in [0,1)]{} K^* \xrightarrow[\epsilon \in [1,2)]{} K \xrightarrow[\epsilon \in [2,3)]{} 0 \xrightarrow[\epsilon \in [3,\infty)]{} \cdots$$

Remark 8. In [MM17b] we investigate the non-democratic choice of weights on a finite totally ordered set. Here we will show that from the perspective of topological data analysis, both potential problems associated with discretizing one-dimensional persistence modules can be overcome. Moreover, we recover the (classical) interleaving distance as a limit of discrete distances. In future work, we will study the geometric formulation of interleaving distances (see Remark 7 and Example 4.6.3).

Chapter 5

The Interleaving Distance as a Limit

This chapter is based on a collaboration with David C. Meyer [MM17b].

5.1 Motivation

This chapter is rooted in the framework of the previous chapter, but we focus our attention on the two issues that arose with that analysis.

1. How do we go about comparing the persistence modules generated by two different data sets if they are modules over entirely different algebras after discretization (equivalently, if they are representations over different $\mathbb{A}_n$ quivers)?

2. How do we preserve information about the distances between discretized scale values, rather than discretizing to a set $\{1, \ldots, n\}$ in which all changes in the original simplicial complex are treated as being separated by scale distances of 1?

5.2 Restriction and Inflation

In this section we discuss restricting and inflating the module category for $A(P)$. We first make a preliminary observation. It is easy to see that for every $m \in \mathbb{R}$ and for every n, there is a one-to-one correspondence between the sets

$$\{(P_n, \{a_i\}, b) : (\{a_i\}, b) \text{ are weights for } P_n\} \overset{T_m}{\longleftrightarrow} \{(X, b) : X \subseteq \mathbb{R}, |X| = n, \min(X) = m\}$$

Specifically, T_m sends the tuple

$$\big(P_n, \{a_i\}, b\big) \xrightarrow{T_m} \big(\{m, m + a_1, m + a_1 + a_2, ...m + \sum_{i=1}^{n-1} a_i\}, b\big)$$

Clearly, this assignment is invertible. That is to say, once a left endpoint is fixed, the triple $(P_n, \{a_i\}, b)$ conveys the same information as the set $\{m + \sum_{i=1}^{k} a_i\}$ plus the choice of b. This is useful, as we may assume that our poset is given by the order type of the finite subset $X = \{x_1 < x_2 < ... < x_n\}$ with weights given by $a_i = x_{i+1} - x_i$ for $i \leq n - 1$ and b. Of course, the latter has a physical interpretation in context of the real line. In what follows, the points in X will include the jump discontinuities of the Vietoris-Rips complex of a data set. When a generalized persistence module comes from the Vietoris-Rips complex of a data set and X is a superset of the jump discontinuities of the complex, we will say that ${T_m}^{-1}(X)$ is the *natural choice of weights on the corresponding* P_n, for $n = |X|$. When this is the case, we will write P_X for P_n (with this choice of weights).

Definition 5.2.1. Let P be any poset, and let $X \subseteq P$. Suppose I is a generalized persistence module for P with values in the category $\mathcal{D}$. Let I^X be defined by the formulae;

- $I^X(x) = I(x) \in \mathcal{D}$, for all $x \in X$, and
- $I^X(x_1 \leq x_2) = I(x_1 \leq x_2) \in \text{Hom}_{\mathcal{D}}(I(x_1), I(x_2))$ for all $x_1 \leq x_2, x_1, x_2 \in X$.

Then I^X is a generalized persistence module for X (with restricted ordering) with values in the category $\mathcal{D}$. Moreover, it is clear that by restricting morphsims between generalized persistence modules in the obvious way we obtain a functor from $\mathcal{D}^P \to \mathcal{D}^X$. Of particular interest will be the case when $P = \mathbb{R}$, and $X \subseteq P$ is a finite subset. When this is the case, we write δ^X for the functor

$$\delta^X : \mathcal{D}^{\mathbb{R}} \to \mathcal{D}^X.$$

We now discuss from *inflating* from $A(P_X)$-mod to $A(P_Y)$-mod, when X, Y are finite subsets of $\mathbb{R}$ with $X \subseteq Y$. First, we work with translations.

Definition 5.2.2. Let X, Y be finite subsets of $\mathbb{R}$, with $X \subseteq Y$. Let $\Lambda \in \mathcal{T}(P_X)$. Let $\bar{\Lambda}$ be given by

$$\bar{\Lambda}(y) = \begin{cases} \max\big\{y,\ \max\{\Lambda x : x \in X, x \leq y\}\big\}, \text{ if there exists } x \in X, x \leq y \\ y \text{ , otherwise.} \end{cases}$$

Of course there is an assignment $\Lambda \to \bar{\Lambda}$ on $\mathcal{T}(P)$ for every X, Y with $X \subseteq Y$. When the context is clear, we supress the arguments X, Y. The following lemma shows that $\bar{\Lambda}$ is a translation on P_Y of the same height as Λ.

Lemma 5.2.3. *Let X, Y be finite subsets of $\mathbb{R}$, with $X \subseteq Y$. If $\Lambda \in \mathcal{T}(P_x)$, then $\bar{\Lambda} \in \mathcal{T}(P_Y)$ and $h(\bar{\Lambda}) = h(\Lambda)$. Moreover, $\bar{\Lambda}_{|X} = \Lambda$.*

Proof. First, by inspection, for all $y \in Y$, $\bar{\Lambda} y \geq y$. It is also easy to see that $\bar{\Lambda}$ restricts to Λ on X as a function. Since its height is attained on X, $h(\bar{\Lambda}) = h(\Lambda)$. Now, let $t_1, t_2 \in Y$, with $t_1 \leq t_2$. We must show that $\bar{\Lambda} t_1 \leq \bar{\Lambda} t_2$. Note that

$$\max\{\Lambda x : x \in X, x \leq y\} = \Lambda x_y, \text{ where } x_y = \max\{x \in X, x \leq y\}.$$

When they exist, let x_1, x_2 be maximal elements of X less than or equal to t_1, t_2 respectively.

First, suppose $t_1 \in X$, $t_2 \notin X$. Then,

$$t_1 \leq x_2, \text{ so } \bar{\Lambda} t_1 = \Lambda t_1 \leq \Lambda x_2 = \bar{\Lambda} t_2.$$

On the other hand, say $t_1 \notin X, t_2 \in X$. If $\bar{\Lambda} t_1 = t_1$, then $\bar{\Lambda} t_1 = t_1 \leq t_2 \leq \Lambda t_2 = \bar{\Lambda} t_2$. Otherwise, $\bar{\Lambda} t_1 = \Lambda x_1$. Then, $x_1 \leq t_1 \leq t_2$, so $\Lambda x_1 \leq \Lambda t_2$ and $\bar{\Lambda} t_1 = \Lambda x_1 \leq \Lambda t_2 = \bar{\Lambda} t_2$. The remaining cases are handled similarly. ∎

We now include the category $A(P_X)$-mod inside $A(P_Y)$-mod when $X \subseteq Y$ and X, Y are finite subsets of $\mathbb{R}$.

Definition 5.2.4. Let X, Y be finite subsets of $\mathbb{R}$, with $X \subseteq Y$. Let $I \in A(P_X)$-mod. Define $j(X,Y)I$ by the formulae;

$$j(X,Y)I(y) = \begin{cases} I(x_y), \ x_y \text{ maximal in } X, x_y \leq y, \text{ or} \\ 0, \text{ if no such } x_y \text{ exists.} \end{cases} \tag{5.1}$$

$$j(X,Y)I(y_1 \leq y_2) = \begin{cases} I(x_1 \leq x_2), \text{ where } x_i \text{ is maximal in } X, x_i \leq y_i, \text{ or} \\ 0, \text{ if either of the above do not exist.} \end{cases} \tag{5.2}$$

It is clear that Equations (5.1), (5.2) define a module for the algebra $A(P_Y)$. Note that if I is a convex module for P_X, $I \sim [a,b]$, then $j(X,Y)I \sim [a, b_y]$, where b_y is maximal in $Y \cap [b, b^{+1})$, where b^{+1} is the successor of b in X. That is to say, the right endpoint of the support of I may move to the right. We now discuss how morphisms in $A(P_X)$-mod can be extended to the image of $j(X,Y)$.

Definition 5.2.5. Let X, Y be finite subsets of $\mathbb{R}$, with $X \subseteq Y$. Let I, M be modules for $A(P_X)$, and let $\alpha \in \mathrm{Hom}(I, M)$. Let $j(X,Y)\alpha$ be defined by the formula

$$j(X,Y)\alpha(y) = \begin{cases} \alpha(x_y), \text{ where } x_y \text{ is maximal with } x_y \in X, x_y \leq y, \text{ or} \\ \text{the zero homomorphism, if no such element exists.} \end{cases}$$

Then, $j(X,Y)\alpha$ defines an $A(P_Y)$-module homomorphism from $j(X,Y)I$ to $j(X,Y)M$. The proof follows from the commutativity of the diagrams below.

$$\begin{array}{ccccccc} I(x_1) & \xrightarrow{\alpha(x_1)} & M(x_1) & & j(I(y_1)) & \xrightarrow{j(\alpha(y_1))} & j(M(y_1)) \\ {\scriptstyle I(x_1 \leq x_2)}\downarrow & & \downarrow{\scriptstyle M(x_1 \leq x_2)} & \overset{j}{\dashrightarrow} & {\scriptstyle j(I(y_1 \leq y_2))}\downarrow & & \downarrow{\scriptstyle j(M(y_1 \leq y_2))} \\ I(x_2) & \xrightarrow[\alpha(x_2)]{} & M(x_2) & & j(I(y_2)) & \xrightarrow{j(\alpha(y_2))} & j(M(y_2)) \end{array}$$

We are now ready to compare different one-dimensional persistence modules. Let $D_1, D_2, ..., D_n$ be finite data sets in some metric space. Let L_i be the set of jump discontinuities of the Vietoris-Rips complex of D_i, and let $L = L_1 \cup L_2 \cup ... \cup L_n$. Let $\Delta(D_1, D_2, ..., D_n)$ be the collection of all finite supersets of L. Clearly L is a directed set under the partial ordering given by containment. Note that for all i the one-dimensional persistence modules coming from the data set D_i admits the structure of an $A(P_X)$-module for any X in $\Delta(D_1, D_2, ..., D_n)$. Thus, we may compare discretized persistence modules which are a priori modules for *different* poset algebras. Since clearly any finite set of one-dimensional persistence modules can be compared in this way, from this point forward we write $\Delta(D)$ for $\Delta(D_1, D_2, ..., D_n)$.

Proposition 5.2.6. *Let I be a one-dimensional persistence module that comes from data. Say $D_1, \ldots, D_n$ are such that the jump discontinuities of the Vietoris-Rips complex of I*

are contained in the corresponding set L. Let $X, Y \in \Delta(D)$, with $X \subseteq Y$. Then, $j(X,Y)$ is a fully-faithful functor from $A(P_X)$-mod to its image in $A(P_Y)$-mod. Moreover, $j(X,Y)$ commutes with δ^X, δ^Y in the sense that $(j(X,Y) \circ \delta^X)I = \delta^Y I$.

Proof. One easily checks that $j(X,Y)(\beta \circ \alpha) = j(X,Y)\beta \circ j(X,Y)\alpha$. Now say $X, Y \in \Delta(D)$, where I is as above. We will show the commutativity of the triangle below.

$$\begin{array}{ccc} & I & \\ {\scriptstyle \delta^X} \swarrow & & \searrow {\scriptstyle \delta^Y} \\ \delta^X I & \xrightarrow{\;j(X,Y)\;} & \delta^Y I \end{array}$$

First, say I is convex, with $I \sim [t,T)$. Let $y \in Y \cap [t,T)$. Then, since $t \in X \cap Y$, there exists $x \in X$ with $x \leq y$. Thus, let x_y be maximal in X, with $x_y \leq y$. Since I is convex, it is enough to show that $j(X,Y)\delta^X I(y) = \delta^Y I(y) = K$. As $y \in [t,T)$, $t \leq x_y \leq y < T$, so $j(X,Y)\delta^X I(y) = \delta^X I(x_y) = I(x_y) = K$. Similarly, $\delta^Y I(y) = I(y) = K$ as required, so the result holds for I convex. The general case follows since all the above functors distribute through direct sums. $J(X,Y)$ is fully faithful by the characterization of homomorphisms between convex modules and since $j(X,Y)$ is one-to-one on isomorphism classes of objects. ∎

Moreover, the following lemma shows that $j(X,Y)$ is compatible with the assignment $\Lambda \to \bar{\Lambda}$ (for X, Y).

Lemma 5.2.7. *Let X, Y be finite subsets of $\mathbb{R}$, with $X \subseteq Y$. Let $\Lambda \in \mathcal{T}(P_X)$, and let I, M be $A(P_X)$-modules. Suppose $\alpha \in Hom(I,M)$. Then,*

(i.) $j(X,Y)(I\Lambda) = (j(X,Y)I)\bar{\Lambda}$, *and*

(ii.) $j(X,Y)(\alpha\Lambda) = (j(X,Y)\alpha)\bar{\Lambda}$.

Proof. Let $y \in Y$, z_y the maximal element of X such that $z_y \leq y$.

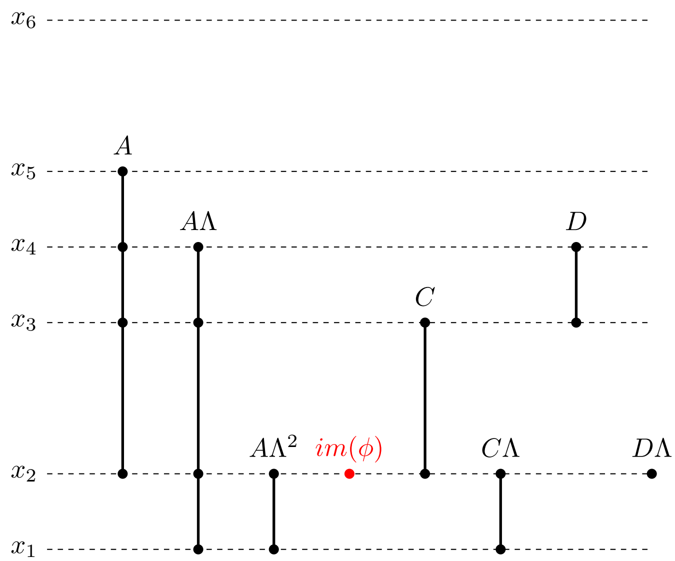
Then $(j(X,Y)I)\bar{\Lambda}(y) = (j(X,Y)I)\Lambda(z_y) = I(\Lambda(z_y)) = (I\Lambda)(z_y) = j(X,Y)(I\Lambda)(y)$. Similarly, $j(X,Y)\alpha\bar{\Lambda}(y) = j(X,Y)\alpha\Lambda(z_y) = \alpha\Lambda(z_y) = (\alpha\Lambda)(z_y) = j(X,Y)(\alpha\Lambda)(y)$. ∎

The results in this section are used in the next section where we prove our algebraic stability theorems.

5.3 The Shift Isometry Theorem

In this section we show that an interleaving need not produce on induced matching of the same height. Since the existence of such a matching is the key step in the proof of an isometry theorem, this provides an obstruction to proving an isometry theorem for $A(P_n)$-mod. We then prove a (shifted) isometry theorem by enlarging the category $A(P_n)$-mod. We begin with an example illustrating the failure of the "matching theorem." For a lengthier discussion, see Subsection 5.5.

Example 5.3.1. Let $P = P_6 = \{x_1 < ... < x_6\}$ with weights $a_1, a_3, a_4 = 1$ and $a_2, a_5 = 2$. Let $\Lambda = \Lambda_2$ and consider the convex modules with supports depicted below.



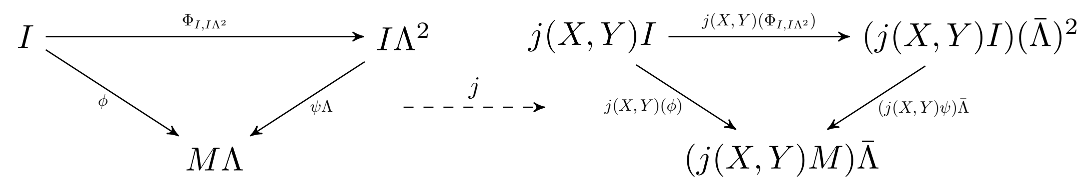
Let $\phi = \Phi_{A,D\Lambda}$ and $\psi = \Phi_{D,A\Lambda}$. It is immediate that ϕ, ψ correspond to a (Λ_2, Λ_2)-interleaving. However, the induced matching pairs $A \updownarrow C$ which are not (Λ_2, Λ_2)-interleaved as $\Phi_{C,A\Lambda} = 0$. Thus, the induced matching corresponding to ψ does not generate a matching of the correct height. We can cause both induced matchings to fail by taking $I = A \oplus (C \oplus D)$ and $M = (C \oplus D) \oplus A$ to be (Λ_2, Λ_2)-interleaved by morphisms $\phi' = \phi \oplus \psi$ and $\psi' = \psi \oplus \phi$.

It is important to note that this does not say that the interleaving distance between $A, C \oplus D$ is not the bottleneck distance. In fact, they are the same. This simply says that the only known algorithm for producing a matching with the same height as the interleaving fails in this situation.

We now work towards the proof of the shift isometry theorem. First we show that we enlarge the category, the functor j is a contraction.

Proposition 5.3.2. *For X, Y finite subsets of $\mathbb{R}$, $X \subseteq Y$, the functor $j(X, Y)$ is a contraction from $A(P_X)$-mod equipped with D^X to its image in $A(P_Y)$-mod equipped with D^Y.*

Proof. Let $I, M \in A(P_X)$-mod. Suppose I, M are $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaved in P_X. It suffices to show that $j(X,Y)I$ and $j(X,Y)M$ are (Γ, Γ)-interleaved in P_Y with $h(\Gamma) \leq \epsilon$. By Lemma 5.2.7, $j = j(X,Y)$ gives the following progression of diagrams.



The result is now obtained by letting $\Gamma = \bar{\Lambda}_\epsilon$ and noting that $h(\bar{\Lambda}) = \epsilon$ by Lemma 5.2.3. ∎

We will now construct a particular refinement of X. This will give rise to the appropriate enlarged module category. Let $X \subseteq \mathbb{R}$ be finite. Let $Y = X \cup \{x - \epsilon : x \in X, \epsilon \in N_1\}$ where N_1 denotes the set of all distances between points in X. Order the finite set $Y = \{y_1 > y_2 > \ldots > y_n\}$ by greatest to least on the real number line.

Let

$$Z_1 = Y \cup \{z_{y_1} - \epsilon : \text{ for all } \epsilon \in N_1, \text{ with } z_{y_1} \text{ maximal in } Y \text{ such that } z_{y_1} < y_1\}.$$

Next, let

$$Z_2 = Z_1 \cup \{z_{y_2} - \epsilon : \text{ for all } \epsilon \in N_1, \text{ with } z_{y_2} \text{ maximal in } Z_1 \text{ such that } z_{y_2} < y_2\}.$$

For the i-th step in the process, let

$$Z_{i+1} = Z_i \cup \{z_{y_{i+1}} - \epsilon : \text{ for all } \epsilon \in N_1, \text{ with } z_{y_{1+1}} \text{ maximal in } Z_i \text{ such that } z_{y_{i+1}} < y_{i+1}\}.$$

Since Y is finite, the process terminates after n steps. Let $Sh(X)$ be the set Z_n. We will call $Sh(X)$ the *shift refinement* of the set X.

Lemma 5.3.3. *Let X be a finite subset of $\mathbb{R}$. For $q \in Sh(X)$, let q^{+1}, q^{-1} denote subsequent and previous elements in $Sh(X)$ respectively, where applicable. Then, $Sh(X)$ has the property that for every $x \in X$ and $\epsilon \in N_1$, $(x-\epsilon)^{-1} - \epsilon \in Sh(X)$. Equivalently, for every $q \in Y$ and $\epsilon \in N_1$, $q^{-1} - \epsilon$ is in $Sh(X)$.*

Proof. Let $q \in Y$, $\epsilon \in N_1$. For any $1 \leq i \leq n$ and $k \geq i - 1$, by construction the maximal element of Z_k strictly less than y_i is in fact precisely $y_i^{-1} \in Sh(X)$. Hence, $q = y_i$ for some $1 \leq i \leq n$, and so $q^{-1} - \epsilon \in Sh(X)$. ∎

We are now ready to prove our main result.

Theorem 5.3.4 (Shift Isometry Theorem)**.** *Let $X_1 \subseteq \mathbb{R}$ be finite and $Sh(X_1)$ be its shift refinement. Let P_{X_1} and $P_{Sh(X_1)}$ be the corresponding totally ordered sets. Let C be the full subcategory of $A(P_{Sh(X_1)})$-mod given by $\mathcal{C} = im(j(X_1, Sh(X_1)))$ and let $D^{Sh(X_1)}$, $D_B^{Sh(X_1)}$ be the interleaving metric and bottleneck metric respectively. Let $j = j(X_1, Sh(X_1))$, and Id denote the identity. Then, the horizontal arrow is an isometry and the diagonal arrows are contractions.*

$$\begin{array}{ccc} & (A(P_X)\text{-mod}, D^X) & \\ {}^{j}\swarrow & & \searrow^{j} \\ (\mathcal{C}, D^{Sh(X)}) & \xrightarrow{\quad Id \quad} & (\mathcal{C}, D_B^{Sh(X)}) \end{array}$$

Proof. Let $X \subseteq \mathbb{R}$ be finite and $Sh(X)$ be its shift refinement. Let $\mathcal{C}$ be the subcategory given by $im(j(X, Sh(X)))$. First, by Proposition 5.3.2 the functor $j = j(X, Sh(X))$ is a contraction from

$$(A(P_X)\text{-mod}, D^X) \to (\mathcal{C}, D^{Sh(X)}).$$

Thus, it suffices to show that the identity is an isometry from

$$(\mathcal{C}, D^{Sh(X)}) \to (\mathcal{C}, D_B^{Sh(X)}).$$

Let $I, M \in im(j(X, Sh(X)))$. Since an ϵ-matching immediately produces a diagonal interleaving of the same height, $D \leq D_B$. To show the other inequality, we will prove that for any (Λ, Λ)-interleaving ϕ, ψ, the induced matching of the triangle beginning at I is a $h(\Lambda)$-matching. The proof of this inequality will consist of the following three parts.

1. If $W(I_s) > h(\Lambda)$, then I_s is matched.

2. If $W(M_t) > h(\Lambda)$, then M_t is matched.

3. If I_s and M_t are matched with each other (independent of their W values), then there is a (Λ, Λ)-interleaving between I_s and M_t.

The proof of (1), (2) proceed as in Theorem 2 [MM17a] with an additional consideration. Specifically, in the present situation it is possible for non-isomorphic convex modules in $B(I)$ to be matched with isomorphic convex modules in $B(I^{-\Lambda^2})$. Thus, we will choose a particular enumeration of each multisubset $I_{[x,-]}^{-\Lambda^2}$ of $\Sigma_{[x,-]}$ (see Section 4.4). If $\sigma_1^{-\Lambda^2} \cong \sigma_2^{-\Lambda^2}$ in $I_{[x,-]}^{-\Lambda^2}$, then we set $\sigma_1^{-\Lambda^2} < \sigma_2^{-\Lambda^2}$ if $\sigma_1 < \sigma_2$ in $I_{[x,-]}$. This was not a concern in [MM17a], since for the democratic choice of weights $\sigma_1 \cong \sigma_2$ if and only if $\sigma_1^{-\Lambda^2} \cong \sigma_2^{-\Lambda^2}$. In our present situation, the above choice of enumeration ensures commutativity of the appropriate triangle for (1). The proof of (2) is similar. Note that the arguments for (1), (2) do not require any special properties of the poset $P_{Sh(X)}$.

We will now prove (3). We show that if I_s and M_t are matched by the induce matching, then they are $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaved. Let $I_s \sim [u, U]$, $I_s\Lambda_\epsilon \sim [w, W]$, $M_t \sim [z, Z]$, and $M_t\Lambda_\epsilon \sim [v, V]$.

If $W(I), W(M) \leq \epsilon$, then I and M are immediately $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaved by $\phi, \psi = 0$. Assume that $W(I) > \epsilon$ or $W(M) > \epsilon$. Then, $s \in S'$ or $t \in T'$ using the notation in Theorem 2 [MM17a]. We will show that the following morphisms constitute a $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaving of I_s and M_t. Let $\phi' = \Phi_{I_s, M_t\Lambda}$ by the linearization of $\chi([u, V])$, and similarly $\psi' = \Phi_{M_t, I_s\Lambda}$. Proceeding as in [MM17a] we will show that

$$w \leq z \leq W \leq Z$$

whenever $s \in S'$ or $t \in T'$. It is enough to show that the following four statements hold.

(i.) If $t \in T'$, then $w \leq z$.

(ii.) If $s \in S'$, then $z \leq W$ and $W \leq Z$.

(iii.) If $s \in S'$ and $t \notin T'$, then $w \leq z$.

(iv.) If $s \notin S'$ and $t \in T'$, then $z \leq W$ and $W \leq Z$.

We now prove (i.) through (iv.). First, if $t \in T'$, we may define v_0 to be the lower endpoint of $M_t^{+\Lambda^2}\Lambda$, equivalently the minimal vertex such that $\Lambda v_0 \geq \Lambda^2 z$. As $\Lambda(\Lambda z) \geq \Lambda^2 z$, by minimality $v_0 \leq \Lambda z$. Furthermore, w is minimal such that $\Lambda w \geq u$. Now, as $u \leq v_0$ (by properties of induced matchings) and $v_0 \leq \Lambda z$ (by above), $u \leq \Lambda z$ and so minimality of w guarantees that $w \leq z$. This proves (i.).

For (ii.), note that by hypothesis v is minimal such that $\Lambda v \geq z$. By properties of induced matchings $v \leq u$, and so $\Lambda v \leq \Lambda u$. Combining these inequalities, $z \leq \Lambda u$. Then, $s \in S'$ guarantees that $\Lambda^2 u \leq U$. Since W is maximal such that $\Lambda W \leq U$, we have that $\Lambda u \leq W$. Combining this with the above inequality, it follows that $z \leq W$. To prove the second inequality in (ii.), we first claim that $W = \Lambda U_0$, where U_0 is the maximal element such that $\Lambda^2(U_0) \leq U$. Once this is established, since $\Lambda V \leq Z$ and $U_0 \leq V$ by the properties of induced matchings, we will have that

$$W = \Lambda U_0 \leq \Lambda V \leq Z.$$

Thus, we need only verify that $W = \Lambda U_0$. By definition, $\Lambda U_0 \leq W$. To show the opposite inequality we will first show that $W \in \operatorname{Im} \Lambda$. Since $[u, U] \in im(j(X, Sh(X)))$, it is immediate that $U^{+1} \in X$, and so $U^{+1} - \epsilon \in Y$. Note that since the distance from $U^{+1} - \epsilon$ to U^{+1} is precisely ϵ, we must have $W \leq (U^{+1} - \epsilon)^{-1}$. Furthermore, maximality of W ensures that $\Lambda W^{+1} > U$, so $W^{+1} \geq U^{+1} - \epsilon$, so $W \geq (U^{+1} - \epsilon)^{-1}$. This verifies that $W = (U^{+1} - \epsilon)^{-1}$. By Lemma 5.3.3, $W - \epsilon \in Sh(X)$. Since $\Lambda(W - \epsilon) = W$, we have that W is in the image of Λ, as desired. Hence, as $\Lambda^2(W - \epsilon) = \Lambda W \leq U$, it must be that $U_0 \geq W - \epsilon$, and so $\Lambda U_0 \geq W$. Thus, $W = \Lambda U_0$ as required. This completes the proof of (ii.).

We will now prove (iii.). First, $I\Lambda_\epsilon = [w, W],$ where $w = u - \epsilon$, and so $\Lambda w = u$. Since $s \in S'$ we know $\Lambda^2 u \leq U$, and so $\Lambda(\Lambda^2 w) \leq u$. By the maximality of W under the condition $\Lambda W \leq U$, we have that $\Lambda^2 w \leq W$. Finally, using (ii.) and the fact that $t \notin T'$ guarantees that $\Lambda^2 z > Z$, we have that $\Lambda^2 w \leq W \leq Z < \Lambda^2 z,$ so by monotonicity $w < z$. This proves (iii.).

For (iv.), one can check that $\Lambda W \leq U < \Lambda^2 u \leq \Lambda^2 v_0 = \Lambda^3 z \leq \Lambda Z,$ so by monoticity $W < Z$.

Thus, we have shown that $\psi' = \Phi_{M_t, I_s\Lambda} \neq 0$. To finish, by Corollary 5.4.2 in the next section, ϕ', ψ' comprise a $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaving between I_s and M_t, completing the proof of (3). This finishes the proof of Theorem 5.3.4. Note that the requirement that we work in $Sh(X)$ only appears in the latter half of (ii.) and in (iii.). ∎

5.4 Interleaving Distance as a Limit

We will now use the results from the last section to recover the classical interleaving distance as a limit.

Lemma 5.4.1. *Let I, M be convex modules for $A(P_X)$ (with its natural metric d) and say $I \sim [u, U]$ and $M \sim [z, Z]$. Then,*

$$D(I, M) \leq \min\{\max\{W(I), W(M)\}, \max\{d(u, z), d(U^{+1}, Z^{+1})\}\}.$$

Proof. Let γ denote the quantity on the right hand side above. If $\gamma = \max\{W(I), W(M)\}$, the result is obvious, since $\phi, \psi = 0$ constitute a $(\Lambda_\gamma, \Lambda_\gamma)$-interleaving between I and M.

On the other hand, suppose $\gamma < \max\{W(I), W(M)\}$. Let $\Lambda = \Lambda_\delta$ and let $\phi = \Phi_{I,M\Lambda}$ and $\psi = \Phi_{M,I\Lambda}$. Let $I\Lambda \sim [w, W]$ and $M\Lambda \sim [v, V]$. By choice of γ, it is immediate that $v \leq u$ and $V \leq U$. We will show that $u \leq V$. Similarly, we know that $w \leq z$ and $W \leq Z$ both hold, and will show that $z \leq W$. We will then establish the commutativity of both interleaving triangles.

By assumption, at least one of I, M has width strictly larger than γ. Suppose that $W(I) > \delta$. First, we'll show $u \leq V$, which means that $\Phi_{I,M\Lambda}$ is non-zero. Since $d(Z^{+1}, U^{+1}) \leq \gamma$ and $W(I) > \gamma$, it must be that $\Lambda^2 u < U^{+1}$, and so $d(\Lambda u, U^{+1}) > \gamma$.

This says that $\Lambda u < Z^{+1}$, i.e., $\Lambda u \leq Z$. By maximality of V, $u \leq V$. Hence, $\phi = \Phi_{I,M\Lambda}$, the linearization of $\chi([u,V])$ is not identically zero.

We next show that $\Phi_{M,I\Lambda}$ is non-zero. If $W(M) > \gamma$, we are done by symmetry. Thus, assume $W(M) \leq \gamma$, and $W(I) > \gamma$. Since $d(u,z) \leq \delta$ we have that $z \leq \Lambda u$. As $W(I) > \delta$, it must be that $\Lambda^2 u \leq U$, so by maximality of W, $\Lambda u \leq W$. Hence, combining inequalities we have $z \leq W$, and so $\psi = \Phi_{M,I\Lambda} \neq 0$.

Thus we have shown that when $\gamma < \max\{W(I), W(M)\}$, ϕ, ψ are both non-zero. It remains to show that ϕ, ψ give commutative interleaving triangles.

Suppose that $W(I) > \gamma$. To show that the triangle beginning with I commutes we need only show that $\psi\Lambda \circ \phi \neq 0$. By inspection, $(\psi\Lambda \circ \phi)(u) \neq 0$ as required.

By symmetry, if $W(M) > \gamma$, we are done. Thus, we need only show the commutativity of the other triangle when $W(M) \leq \gamma$. However, since $\mathrm{Hom}(M, M\Lambda^2) = 0$, $\phi\Lambda \circ \psi = 0$ as required.

Hence, ϕ, ψ are a $(\Lambda_\gamma, \Lambda_\gamma)$-interleaving between I and M, so $D(I,M) \leq \gamma$. ∎

Corollary 5.4.2. *Let Λ be a maximal translation of height $h(\Lambda)$. Let I, M be convex modules for $A(P_X)$. Say $I \sim [u,U]$ and $M \sim [z,Z]$. If $\Phi_{I,M\Lambda}$ and $\Phi_{M,I\Lambda}$ are both non-zero, then I, M are (Λ,Λ)-interleaved.*

Proof. Let $\Phi_{I,M\Lambda}$ and $\Phi_{M,I\Lambda}$ are both non-zero only if $h(\Lambda) \geq \max\{d(u,z), d(U^{+1}, Z^{+1})\} \geq \gamma$. By Lemma 5.4.1, $\gamma \geq D(I,M)$, hence I, M are $(\Lambda_\gamma, \Lambda_\gamma)$-interleaved, and so they are also (Λ,Λ)-interleaved as $h(\Lambda) \geq \gamma$. ∎

The next Proposition also follows from Lemma 5.4.1.

Proposition 5.4.3. *Let $X \subseteq \mathbb{R}$ be finite, I, M be indecomposables in $A(P_X)$-mod with*

$I \sim [u, U]$, $M \sim [z, Z]$. Then,

$$D(I, M) = \min\{\max\{W(I), W(M)\}, \max\{d(u, z), d(U^{+1}, Z^{+1})\}\}.$$

Proof. By Lemma 5.4.1, we need only show that $D(I, M) \geq \gamma$. Let $\epsilon = D(I, M)$ and let $\Lambda = \Lambda_\epsilon$. If $\phi, \psi = 0$ is a (Λ, Λ)-interleaving between I and M it must be that $h(\Lambda) \geq \max\{W(I), W(M)\}$, and so $D(I, M) \geq \gamma$.

Otherwise, it must be that, without loss of generality, $\Phi_{I, I\Lambda^2} \neq 0$. Hence, to have a commutative triangle beginning at I, it must be that ϕ, ψ are both non-zero. But, $\mathrm{Hom}(I, M\Lambda)$ and $\mathrm{Hom}(M, I\Lambda)$ are both non-zero only if $h(\Lambda) \geq \max\{d(u, z), d(U^{+1}, Z^{+1})\}$, and so $D(I, M) \geq \gamma$. ∎

We now connect our work to persistent homology. Again, let $D_1, D_2, ...D_n$ be finite data sets in some metric space, and let $L \subseteq \mathbb{R}$ be the corresponding union of the jump discontinuities of their Vietoris-Rips complexes. Let $X \in \Delta(D)$, and let D^X denote the corresponding interleaving metric and on the category $A(P_X)$-mod. Similarly, let W_X denote the width of a convex $A(P_X)$-module. We now work towards the proof of the following theorem:

Theorem 5.4.4. *Let I, M be persistence modules (for $\mathbb{R}$) whose endpoints lie in L. Then,*

$$\lim_{X \in \Delta(D)} \left(D^X(\delta^X I, \delta^X M)\right) = D(I, M).$$

This will show that the classical interleaving distance can be recovered as the limit over the directed set $\Delta(X)$. If I is a one-dimensional persistence module coming from data, we say that I has endpoints in L if the jump discontinuities of the Vietoris-Rips complex of I are contained in L.

Lemma 5.4.5. *Let σ be any convex one-dimensional persistence module (for $\mathbb{R}$) whose endpoints are contained in L, say $\sigma \sim [r, R)$. Then,*

$$\lim_{X \in \Delta(D_1, D_2)} \big(W^X(\delta^X \sigma)\big) = W(\sigma) = \frac{R-r}{2}.$$

Proof. Let $\epsilon > 0 \in \mathbb{R}$. Let $Y = Y(\epsilon)$ be any element of $\Delta(D)$ such that

(i.) the $\max(Y) > \max(L)$, and

(ii.) the difference between consecutive elements of $Y \cap [m, M+\epsilon]$ is less than $\frac{1}{2}\epsilon$.

Note that any superset of Y necessarily satisfies (i.), (ii.) as well. Let $X' \in \Delta(D)$ with $X' > Y$. Then,

$$W^{X'}(\delta^{X'}\sigma) = \min\big\{\big\{\max\big\{x - r, R - x\big\} : x \in X' \cap [r, R)\big\}.$$

Since $\frac{R-r}{2}$ must be within $\frac{1}{2}\epsilon$ of some x, clearly

$$\Big|W^{X'}(\delta^{X'}\sigma)) - W(\sigma)\Big| < \epsilon \text{ as required.}$$

∎

We point out that condition (i.) above removes the consideration of the weight "b" from the discussion. Next we will show that if σ, τ are convex one-dimensional persistence modules (for $\mathbb{R}$) then their interleaving distance can be recovered as a discrete limit as well. This establishes Theorem 5.4.4 for convex modules.

Lemma 5.4.6. *Let σ, τ be any convex one-dimensional persistence modules whose endpoints are contained in L. Say $\sigma \sim [r, R), \tau \sim [s, S)$. Then,*

$$\lim_{X \in \Delta(D)} \big(D^X(\delta^X \sigma, \delta^X \tau)\big) = D(\sigma, \tau).$$

Proof. Proceeding as in the proof of Lemma 5.4.6, let ϵ be positive and set $Y = Y(\epsilon) \in \Delta(D)$. Let $X' \in \Delta(D)$ with $X' > Y$. Then, by Proposition 5.4.3,

$$D^{X'}(\delta^{X'}\sigma, \delta^{X'}\tau) = \min\Big\{\ \max\{W^{X'}(\delta^{X'}\sigma), W^{X'}(\delta^{X'}\tau)\},\ \ \max\{|r-s|, |R-S|\}\Big\}.$$

Clearly, this is within ϵ of

$$D(\sigma, \tau) = \min\Big\{\ \max\{W(\sigma), W(\tau)\},\ \ \max\{|r-s|, |R-S|\}\Big\},$$

by Lemma 5.4.5. The result follows. ∎

Now that we have established Lemmas 5.4.5, 5.4.6, we are ready to prove Theorem 5.4.4

Proof. Let I, M be one-dimensional persistence modules whose endpoints lie in L. Let $\gamma > 0$, and let ϵ be such that

$$\epsilon - \frac{1}{2}\gamma < D(I, M) \leq \epsilon$$

Let $Y \in \Delta(D)$ be such that

(i.) max$(Y) > \max(L)$, and

(ii.) $y_{i+1} - y_i < \frac{1}{8}\gamma$, for $y_i \in Y \cap [m, M + \frac{1}{4}\delta]$

Let $X' \in \Delta(D)$, with $X' \supseteq Y$. We will show that

$$D(I, M) - \delta < D^{X'}(\delta^{X'}I, \delta^{X'}M) \leq D(I, M) \leq D(I, M) + \gamma$$

First, let $\delta^{X'}\sigma \in B(\delta^{X'}I) \cup B(\delta^{X'}M)$, and say $W^{X'}(\delta^{X'}\sigma) > \epsilon + \frac{1}{2}\gamma$. Since $\epsilon \geq D(I, M)$, there exists a $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaving between I and M. But then, by the isometry theorem of Bauer and Lesnick ([BL13]), there is an ϵ-matching between $B(I)$ and $B(M)$. But then, we have that

$$W(\sigma) + \frac{1}{2}\gamma > W^{X'}(\delta^{X'}\sigma) > \epsilon + \frac{1}{2}\gamma, \text{ so } W(\sigma) > \epsilon.$$

Therefore σ is matched with some τ in the opposite barcode. So by hypothesis,

$$D(\sigma, \tau) = \min\Big\{ \ \max\{W(\sigma), W(\tau)\}, \ \ \max\{|r-s|, |R-S|\}\Big\}$$

where $\sigma \sim [r, R)$ and $\tau \sim [s, S)$.

But,

$$\max\big\{W^{X'}(\delta^{X'}\sigma), W^{X'}(\delta^{X'}\tau)\big\} \leq \max\big\{W(\sigma) + \frac{1}{2}\gamma, W(\tau) + \frac{1}{2}\gamma\big\} =$$
$$\max\big\{W(\sigma), W(\tau)\big\} + \frac{1}{2}\gamma.$$

Therefore,

$$D^{X'}(\delta^{X'}\sigma, \delta^{X'}\tau) \leq D(\sigma, \tau) + \frac{1}{4}\gamma \leq \epsilon + \frac{1}{2}\gamma < D(I, M) + \gamma.$$

So, the assignment $\delta^{X'}\sigma \updownarrow \delta^{X'}\tau \iff \sigma \updownarrow \tau$ defines a diagonal interleaving (a matching) between $\delta^{X'}I$ and $\delta^{X'}M$ of height $\epsilon + \frac{1}{2}\gamma$. Thus, $D^{X}(\delta^{X'}I, \delta^{X'}M) \leq D(I, M) + \gamma$ as required.

Now, if $D^{X'}(\delta^{X'}I, \delta^{X'}M) \geq D(I, M) - \gamma$ we are done. Thus for a contradiction suppose that $D^{X'}(\delta^{X'}I, \delta^{X'}M) < D(I, M) - \gamma$. Then, there exists a weight ϵ for X' with $\epsilon < D(I, M) - \gamma$ and there exists a $(\Lambda_\epsilon^{X'}, \Lambda_\epsilon^{X'})$-interleaving between $\delta^{X'}I$ and $\delta^{X'}M$, where $\Lambda_\epsilon^{X'}$ is the maximal translation of height ϵ for $P_{X'}$. Then, by Theorem 5.3.4 there is an ϵ-matching between $B(j(X', Z)\delta^{X'}I)$ and $B(j(X', Z)\delta^{X'}M)$ for $Z = Sh(X')$. By Proposition 5.2.6,

$$j(X', Z)\delta^{X}I = \delta^{Z}I \text{ and } j(X', Z)\delta^{X}M = \delta^{Z}M.$$

Thus, there is an ϵ-matching between $B(\delta^Z I)$ and $B(\delta^Z M)$. Now, say $W(\sigma) > \epsilon + \frac{1}{2}\gamma$. Then,

$$W^Z(\delta^Z \sigma) \geq W(\sigma) > \epsilon + \frac{1}{2}\gamma > \epsilon.$$

Therefore, $\sigma^Z \updownarrow \tau^Z$ for some element $\delta^Z \tau$ of the opposite barcode, with

$$D^Z(\sigma^Z, \tau^Z) \leq \epsilon < D(I, M) - \gamma.$$

Therefore, define the matching $\sigma \updownarrow \tau \iff \sigma^Z \updownarrow \tau^Z$. But then,

$$D(\sigma, \tau) \leq D^Z(\sigma^Z, \tau^Z) + \frac{1}{2}\gamma \leq \epsilon + \frac{1}{2}\gamma < D(I, M) - \frac{1}{2}\gamma.$$

This matching shows that

$$D(I, M) \leq \epsilon + \frac{1}{2}\gamma < D(I, M) - \frac{1}{2}\gamma.$$

As this is clearly a contradiction, so it must be the case that $D^{X'}(\delta^{X'} I, \delta^{X'} M) \geq D(I, M) - \gamma$ as required. The result is proven. ∎

5.5 Regularity

As we have seen, the poset P_n with arbitrary choice of weights has the property that an interleaving between two modules need not produce an induced matching of barcodes of the same height (see Example 5.3.1). While this does not necessarily mean that the isometry theorem is false in this context, it is clearly an obstruction to its proof. In this section, we show that when we identify $(P_n, \{a_i\}, b)$ with P_X, $X \subseteq \mathbb{R}$, unless X satisfies certain regularity conditions there will always exist interleavings whose induced matchings do not have the same height. One would not expect such regularity for a poset P_X which comes from real world data.

In what follows, it is convenient to work with maximal translations. We will define what it means for a poset to be *regular* after examining some conditions on maximal translations.

Let $x_i < x_l$ in X, and let $\Lambda = \Lambda_{(x_l - x_i)}$. Suppose that

(a) $x_{l+1} < \Lambda(x_{i+1})$,

(b) $\Lambda(x_{l+1}) < \Lambda^2(x_{i+1})$, and

(c) $\Lambda(x_{i-1}) > x_{i-1}$.

Then, one can produce an interleaving whose induced matching has strictly larger height. If X is identically the set of jump discontinuities of a data set, one would expect the the existence of some $x_i < x_l$ satisfying the above.

On the other hand, if X avoids conditions (a) or (b) for all $x_i < x_l$ we say that X will be regular. Property (c) is a purely technical condition that will not be commented on further. Roughly speaking, a regular set has a periodicity associated both with its elements and the spaces between its elements. We now define regularity. After the definition, we connect regularity to the absence of a maximal translation of the form above.

Definition 5.5.1. Let X be a finite subset of $\mathbb{R}$, and let $x_i \in X = \{x_1 < x_2 < ... < x_n\}$. Then, X is regular at x_i if for all $x_l > x_i$, either

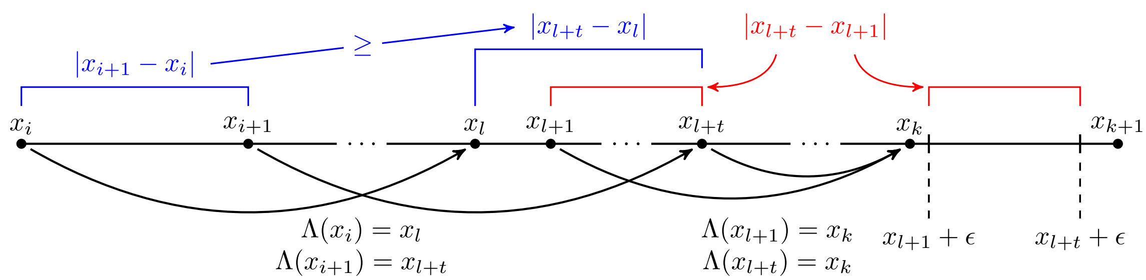
(i.) $x_{l+2} - x_l > x_{i+1} - x_i$, or

(ii.) $x_{l+2} - x_l \leq x_{i+1} - x_i$, and $x_{k+1} > x_{l+t} + x_l - x_i$, where t be maximal such that $x_{l+t} - x_l \leq x_{i+1} - x_i$, and $\Lambda_{(x_l - x_i)}(x_{l+1}) = x_k$.

We say that the set X is regular if X is regular at every $x_i \in X$.

We now explain regularity at x_i. Let $\epsilon = x_l - x_i$ and $\Lambda = \Lambda_\epsilon$, and note that $\Lambda(x_i) = x_l$ and $\Lambda(x_{l+1}) = x_k$. In addition, by the choice of t it is always the case that $\Lambda(x_{i+1}) = x_{l+t}$. Clearly, if x_i is regular and $x_i < x_l$, exactly one of (i.), (ii.) hold.

First, if (i.) holds at x_l the spacing of points in the poset X is uniform in the sense that the length of the edge from x_i to x_{i+1} is surpassed by the sum of the two consecutive edges x_l to x_{l+2}. In terms of the translation Λ, property (i.) says that $0 \leq t < 2$ or $\Lambda(x_{i+1}) \leq x_{l+1}$. This is the negation of (a) above. On the other hand, if (ii.) holds at x_l, the "hole" in X given by the edge from x_i to x_{i+1} is periodic. Specifically, there are no vertices in X contained in the real interval $(x_{l+1}+\epsilon, x_{l+t}+\epsilon]$ (see the figure below). In terms of the translation Λ, property (ii.) corresponds to the statement that $\Lambda(x_{l+1}) = \Lambda(x_{l+t})$. Of course, this is the negation of (b) above.

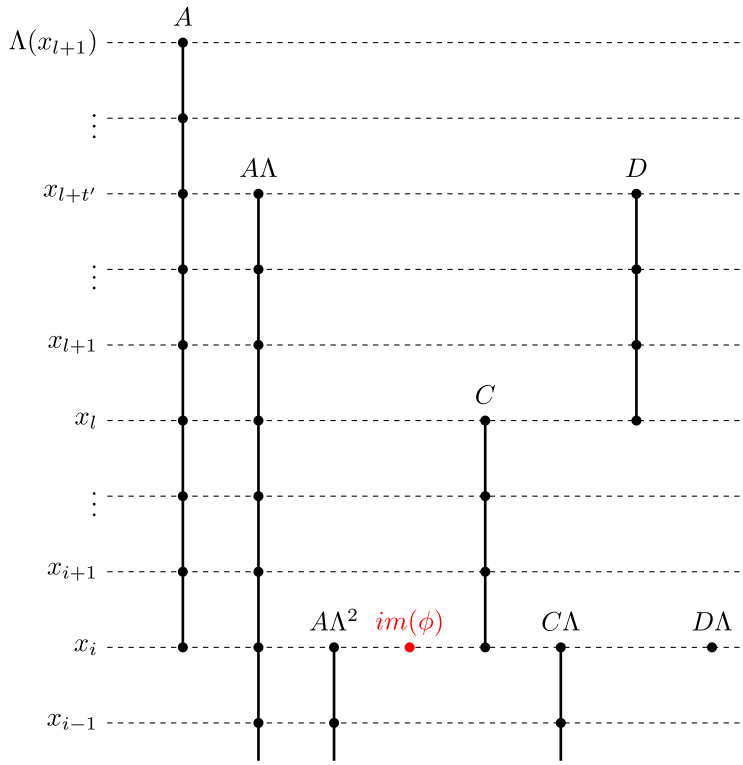


Of course, if X is regular, it is regular at every x_i. Of particular interest is the case in which $x_{i+1} - x_i$ is large. Then, for all x_l with $x_i < x_l$ where (i.) is satisfied, the sum of the two edges after x_l must be long. Thus, since $x_{i+1} - x_i$ is large, the distances $x_{l+2} - x_{l+1}, x_{l+1} - x_l$ taken together must also be large. Alternatively, if (ii.) is satisfied then a hole close to the size of $x_{i+1} - x_i$ must be repeated at a distance of exactly ϵ away from x_{l+1}. Since the distance $x_{i+1} - x_i$ is large, this says that large holes must be repeated regularly. We emphasize that the above statements must hold for all $x_i \in X$ if X is regular. Alternatively, if X fails to be regular (with an addition technical condition), then there always exist interleavings whose corresponding induced matchings have strictly larger heights.

Proposition 5.5.2. *If X is not regular at some $x_i < x_l$ where x_{i-1} is not fixed by $\Lambda_{(x_l - x_i)}$, then there exists an interleaving whose induced matching has strictly larger height (see Example* 5.3.1*).*

Proof. By the above remarks, let $x_i < x_l$ be such that the translation $\Lambda = \Lambda_{(x_l - x_i)}$ has the properties $x_{l+1} < \Lambda(x_{i+1})$, $\Lambda(x_{l+1}) < \Lambda(x_{l+t})$ and x_{i-1} is not fixed by Λ. Let $\epsilon = x_l - x_i$, and let $\Lambda = \Lambda_\epsilon$. Consider the following convex modules, $A \sim [x_i, \Lambda(x_{l+1})]$, $C \sim [x_i, x_l]$, and $D \sim [x_l, x_{l+t'}]$, where $1 \leq t' < t$ is maximal such that $\Lambda(x_{l+t'}) = \Lambda(x_{l+1})$. Note that the vertex $x_{l+t'}$ is also the upper endpoint of $A\Lambda$.

We then define the $(\Lambda_\epsilon, \Lambda_\epsilon)$-interleaving between A and $C \oplus D$ by the diagonal morphisms $\phi = \Phi_{A,D\Lambda}$, $\psi = \Phi_{D,A\Lambda}$. One easily checks that this is indeed an interleaving. However, the induced matching corresponding to the triangle beginning at ϕ matches A with C. Clearly A and C are not (Λ, Λ)-interleaved, as $W(A) > \epsilon$ but $\Phi_{C,A\Lambda} = 0$. Proceeding as in Example 5.3.1 by setting $I = A \oplus (C \oplus D), M = (C \oplus D) \oplus A$ with $\phi' = \phi \oplus \psi$ and $\psi' = \psi \oplus \phi$ we produce an interleaving where both induced matchings have strictly larger height. ∎



This analysis shows that proof of the ”matching theorem” is likely to fail for the poset P_X. Therefore, it is necessity (at this point) to enlarge the category to obtain an isometry on $A(P_X)$-mod in the sense of Theorem 5.3.4.

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VITA

Killian Meehan was born in 1988 in Rochester, New York. He grew up in the region and attended the Community College of the Finger Lakes for two years before attending the State University of New York at Potsdam, from which he received both his bachelor's and master's degrees in May 2011. Later that fall he moved to Columbia, Missouri where he pursued his Ph.D. in mathematics at the University of Missouri. In August 2018 he leaves the country for a position at the Kyoto University Institute for Advanced Study.