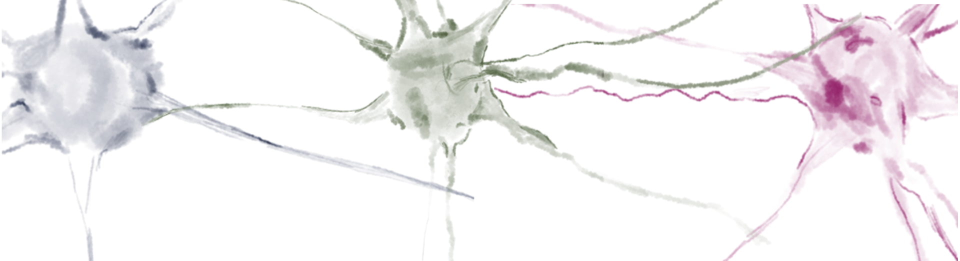


OSTROGRADSKY'S DIVERGENCE THEOREM

BY: WHITNEY WHITE

The divergence theorem, though not named as such, was first shown by Michael Ostrogradsky in Paris in 1826 (Stolze 440). While the current vector notation of the divergence theorem applied to 3-dimensional space looks as such,

(I)

$$\iiint_V (\nabla \cdot \vec{A})\, dV = \iint_S \vec{A} \cdot \vec{da}$$

Ostrogradsky's original statement took the following form (Katz 147):

(II)

$$\int \left(a\frac{\partial p}{\partial x} + b\frac{\partial q}{\partial y} + c\frac{\partial r}{\partial z}\right)\, \omega$$
$$= \int (ap\ cos\alpha + bq\ cos\beta + cr\ cos\gamma)\ \varepsilon$$

It is the goal of this paper to not only connect the original notation to the new, but to also connect Ostrogradsky's proof to our modern proof, as they run parallel. The vector notation we are currently familiar with is not present in Ostrogradsky's proofs, as vector calculus was not developed until later. The divergence theorem was first stated in its vector form by Oliver Heaviside in 1901. This is also when the theorem took on its modern name (Stolze 441). We will, without further reference to Heaviside's vector proof, connect the two equations above by using our current understanding of vector notation.

Note that in equality II above, *a*, *b*, and *c* are scalar constants. We will take for granted that scalar constants inside of the integral are multiples of the integral (so pulled out in front), and we will not address them in our proof. Thus II would become

(III)

$$\int \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}\right) \omega = \int (p\ cos\alpha + q\ cos\beta + r\ cos\gamma)\ \varepsilon$$

Ostrogradsky presented his theorem multiple times. The notation used in II is his original statement from 1826, in a presentation to the Academy of Sciences in Paris (Stolze 441). He then presented another formulation of his theorem to the Academy in 1827, though without proof (Stolze 440). Ostrogradsky returned to Russia after five years in France (1822-1827), where he continued to research and publish. Another of his publications contained the divergence theorem as only part of the presentation, used as a tool to study heat theory. This paper was presented to the Imperial Academy of Sciences of St. Petersburg in 1828, then published as part of the memoirs of the Academy in 1831. The notation of this last publication (IV below) is different than what appeared six years earlier in equation II and looks as follows (shown side by side for comparison. Note the absence of scalar constants in IV as discussed above):

(II)

$$\int \left(a\frac{\partial p}{\partial x} + b\frac{\partial q}{\partial y} + c\frac{\partial r}{\partial z}\right) \omega = \int (ap\ cos\alpha + bq\ cos\beta + cr\ cos\gamma)\ \varepsilon$$

(IV)

$$\int \left(\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz}\right) \omega = \int (P\cos\lambda + Q\cos\mu + R\cos v)\ s$$

Let's now connect the current vector notation in equality I to the notation of equality IV. To do this, recall that the scalar product of two vectors (a, b, c) and (d, e, f) is defined as $(a, b, c) \cdot (d, e, f) = ad + be + cf$.

(I)

$$\iiint\limits_V (\nabla \cdot \vec{A})\ dV = \iint\limits_S \vec{A} \cdot \overrightarrow{da}$$

Here, the vector function $\vec{A} = \big(A_x(x, y, z), A_y(x, y, z), A_z(x, y, z)\big)$, the operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $dV = dxdydz$, and $\overrightarrow{da} = \big((\vec{\imath} \cdot \vec{n}) + (\vec{\jmath} \cdot \vec{n}) + (\vec{k} \cdot \vec{n})\big)\ dxdy$, where $\vec{n} = \vec{n}(x, y, z)$ is the outward unit normal to the surface S and $\vec{\imath} = (1,0,0)$, $\vec{\jmath} = (0,1,0)$, $\vec{k} = (0,0,1)$.

(V)

$$\iiint\limits_V \left(\frac{\partial A_x(x,y,z)}{\partial x} + \frac{\partial A_y(x,y,z)}{\partial y} + \frac{\partial A_z(x,y,z)}{\partial z}\right) dxdydz$$
$$= \int\limits_S \left(A_x(x,y,z)(\vec{\imath}\cdot\vec{n}) + A_y(x,y,z)(\vec{\jmath}\cdot\vec{n})\right.$$
$$\left. + A_z(x,y,z)(\vec{k}\cdot\vec{n})\right) dxdy$$

$Note\ that\ \vec{\imath}\cdot\vec{n} = |\vec{\imath}|\cdot|\vec{n}|\cos\lambda = 1\cdot 1\cdot\cos\lambda,\ where$ $\lambda = \lambda(x,y,z)$ is the angle between vectors $\vec{\imath}$ and $\vec{n}$. The same is true for $\vec{\jmath}\cdot\vec{n}$ and $\vec{k}\cdot\vec{n}$.

(VI)

$$\iiint\limits_V \left(\frac{\partial A_x(x,y,z)}{\partial x} + \frac{\partial A_y(x,y,z)}{\partial y} + \frac{\partial A_z(x,y,z)}{\partial z}\right) dxdydz$$
$$= \iint\limits_S \left(A_x(x,y,z)\cos\lambda + A_y(x,y,z)\cos\mu\right.$$
$$\left. + A_z(x,y,z)\cos v\right) dxdy$$

For direct comparison, I rewrite Ostrogradsky's IV from above:

(IV)

$$\int \left(\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz}\right)\omega = \int (P\cos\lambda + Q\cos\mu + R\cos v)\, s$$

First, connect the left-hand side of IV (Ostrogradsky) and VI (modern). Ostrogradsky writes vector $\vec{A}$ in its three Cartesian components: $p(x,y,z) = A_x(x,y,z)$, $q(x,y,z) = A_y(x,y,z)$, and $r(x,y,z) = A_z(x,y,z)$, where $(x,y,z) \in V$. The operator ∇ is used consistently in both IV and VI, though Ostrogradsky does not use our partial derivative notation. The differential volume *dxdydz* in VI is denoted by Ostrogradsky as ω in IV, and we write three integral symbols where his notation has one.

Second, on the right-hand side of IV and V, Ostrogradsky writes $P = A_x(x,y,s(x,y))$, $Q = A_y(x,y,s(x,y))$, and $R = A_z(x,y,s(x,y))$, where $z = s(x,y)$ is the function describing the surface S. His differential surface area s is *dxdy*. Lastly, our specific surface double integral is intended by his single integral.

Thus we have connected the original notation to the new. Moving toward further clarity, I will use Ostrogradsky's *p, q, r, P, Q, R* notation, maintaining the same definitions as described above. Next, I wish to connect Ostrogradsky's proof to our modern proof, and I believe the most effective way to show this is by first stating the full modern proof. Within this modern proof, I will add superscripts, such as (1), (2), and so on, to be easily referenced while reading Ostrogradsky's original work.

I will prove:

$$\iiint_V \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}\right) dxdydz\ ^{(1)} = \iint_S (P\cos\lambda + Q\cos\mu + R\cos\nu)\, dxdy$$

It is only necessary to show:

$$\iiint_V \left(\frac{\partial r}{\partial z}\right) dxdydz\ ^{(2)} = \iint_S (R\cos\nu)\, dxdy\ ^{(3)}$$

From equation V above, this is:

$$\iiint\limits_{V} \left(\frac{\partial r}{\partial z}\right) dxdydz = \iint\limits_{S} \left(R(x,y,s(x,y))(\vec{k}\cdot\vec{n})\right) dxdy$$

<u>Modern Proof</u>:

Preliminary assumptions: we have a 3D solid region V and an oriented surface S enclosing it. Let $\vec{A} = p\,\vec{\imath} + q\,\vec{\jmath} + r\,\vec{k}$, each *p, q, r* a function of *x, y,* and *z,* where $(x,y,z) \in V$. Let $\vec{\imath} = (1,0,0)$, $\vec{\jmath} = (0,1,0)$, $\vec{k} = (0,0,1)$ and $\vec{n} = \vec{n}(x,y,s(x,y))$ be the unit normal surface vector, oriented outward at each point of the surface. We prove the theorem for the special case of a 3D region $V' = \{(x,y,z): (x,y) \in D, f_1(x,y) \le z \le f_2(x,y)\}$, where D is the projection of the solid region *V'* onto the *xy*-plane, and each *z* of the region *V'* is bounded by two surfaces f_1 (lower) and f_2 (upper).

The left-hand side will then be (using Ostrogradsky's notation *R* in the last term):

$$\iiint\limits_{V'} \left(\frac{\partial r}{\partial z}\right) dxdydz = \iint\limits_{D} \left[\int_{f_1}^{f_2} \frac{\partial r}{\partial z} dz\right] dxdy$$

$$= \iint\limits_{D} [(R(x,y,f_2) - R(x,y,f_1)] dxdy \quad (4)$$

The right-hand side will become the sum of three integrals, as we integrate over the lower surface f_1, the upper surface f_2, and the vertical side surface (if any). But, since $\vec{k} = (0,0,1)$, which is normal to the $\vec{n}$ of the vertical surface, the integral from this vertical surface is zero. We are only left with the integrals over the top and bottom surfaces. For the lower surface f_1 where $\vec{n}$

points down, $\vec{n} = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, -1\right)$. For the upper surface f_s where $\vec{n}$ points up, $\vec{n} = \left(-\frac{\partial f_2}{\partial x}, -\frac{\partial f_2}{\partial y}, 1\right)$.

So the right-hand side becomes:

$$\iint_S \left(R(x, y, s(x, y))(\vec{k} \cdot \vec{n})\right) dxdy$$

$$= \iint_D \left(R(x, y, f_1) \left((0,0,1) \cdot \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, -1\right)\right)\right) dxdy + \iint_D \left(R(x, y, f_2) \left((0,0,1) \cdot \left(-\frac{\partial f_2}{\partial x}, -\frac{\partial f_2}{\partial y}, 1\right)\right)\right) dxdy$$

$$= \iint_D (R(x, y, f_1)(-1))\, dxdy + \iint_D (R(x, y, f_2)(1))\, dxdy$$

$$= \iint_D [(R(x, y, f_2) - R(x, y, f_1)]dxdy \quad \blacksquare$$

Here are the first few pages of Ostrogradsky's 1831 publication containing one of his proofs for the divergence theorem (129-131). The translation is by me and comments I have appear in [square brackets].

Note on the Theory of Calculus

By M. Ostrogradsky

(Read November 5th, 1828)

[Introduction] The questions of mathematical physics lead most often 1) to integrate an equation with partial differentials that takes place for all points of space where the phenomenon occurs; 2) to satisfy a differential equation that only exists at the boundary of this space, that is to say, the area of the expanse where the phenomenon occurs; 3) we know all the peculiarities of the phenomenon at a given moment.

We first look for a particular solution which satisfies the equation relating to the interior as well as to that relating to the surface [this solution is the divergence theorem]; it happens that we find an infinity [of solutions]; each of them contains, as a factor, an arbitrary constant; all the particular solutions are added together, and an attempt is made to determine the arbitrary factors in order to satisfy the known state of the phenomenon corresponding to the given moment. It is to satisfy the last part of the problem that this note is intended.

[He now sets up his proof].

Imagine within the interior of a space [a volume] terminated [bounded] by any surface [its boundary], a differential [volume] element ω [$dxdydz$], designated by x, y, z, the rectangular coordinates [position] of this element and by p, q, r functions of x, y, z [$p(x,y,z)$, $q(x,y,z)$, and $r(x,y,z)$ are the components of the vector field, each a function of position] which remain in all the space that we have just imagined [the common domain of p, q, r is the volume described above].

Consider the triple integral [his single $\int$ meaning $\iiint$ across the volume]

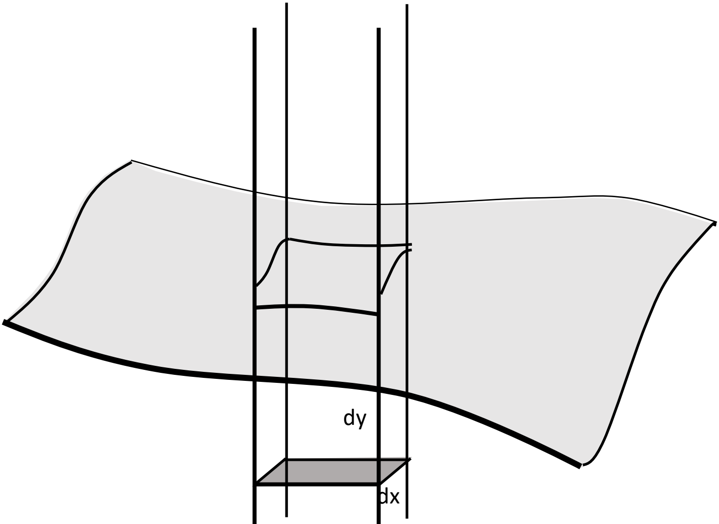
$$\int \left(\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz}\right) \omega \quad (1)$$

we can suppose ω = *dzdydx* [differential volume introduced above]. [This is the integral $\iiint \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}\right) dxdydz$, which can be broken into a sum of three integrals. Since the following process will yield analogous results for all three integrals, Ostrogradsky chooses to proceed with just $\iiint \frac{\partial r}{\partial z} dxdydz$.]

Let's take the integral first

$$\int dxdy \int \frac{dr}{dz} dz \quad (2).$$

[We would write $\iint \left[\int \frac{\partial r}{\partial z} dz\right] dxdy$.] To evaluate it, let us imagine a quadrangular prism perpendicular to the plane of *xy*, having as base, on this plane, the parallelogram *dydx*; this prism will completely cross the volume, and penetrate its surface in several points [imagine a prism punching vertically through the volume, drawn here through a boundary surface];



the number of these points will necessarily be even [entries and exits of the same surface appear in pairs], since the volume is assumed [to be] limited [by the boundary] on all sides. Let us denote by $z_1, z_2, z_3, \ldots, z_{2n}$ the values of z corresponding to the points where the prism penetrates the surface, these quantities are supposed to be arranged in order of their size, z [z_1] being the smallest. Let $\boldsymbol{R_1}$, $\boldsymbol{R_2}$, $\boldsymbol{R_3}$, … , $\boldsymbol{R_{2n}}$ be what r becomes when we successively make $z = z_1$, $z = z_2$, ….. $z = z_{2n}$ [parallel to the ***xy***-plane. So $\boldsymbol{R_1}$ is the graph of $r(x,y,z_1)$, a 2D surface, $\boldsymbol{R_2}$ the graph of $r(x,y,z_2)$,… etc. This corresponds to our two chosen surfaces f_1 (lower, odd subscripts) and f_2 (upper, even subscripts)] We will have,

$$\int dydx \int \frac{dr}{dz} dz = \int (R_2 + R_4 + \cdots + R_{2n}) dydx - \int (R_1 + R_3 + \cdots [+] R_{2n-1}) dydx \quad (4)$$

[The right-hand side of the equality is the difference between the volume under the upper surfaces $R_2 + R_4 + \cdots + R_{2n}$ and the volume under the lower surfaces $R_1 + R_3 + \cdots + R_{2n-1}$.]

let us denote by ν the angle normal to the surface, extended outside the volume [Here we would use our current convention of the normal surface vector $\vec{n}$ to obtain the same result by taking the scalar product of R and $\vec{n}$. Where ν is the angle between the z component of R and $\vec{n}$, we have $R(\vec{k} \cdot \vec{n}\,) = R|\vec{k}||\vec{n}\,|\cos\nu = R\cos\nu$], made with the positive half-axis z, and by s [ds], a differential element of the same surface: we shall have,

$$\int \frac{dv}{dz}\omega = \int R\cos\nu\ s; \ ^{(3)}$$

the integral of the second member of the last equation [the right-hand side] refers only to the points of the surface.

[He now claims the analogous results for the x and y components.] If we designate by P and Q what p and q become, the area of the volume, by μ and λ the angles normal to the surface, prolonged from inside to outside the spheroid [by which he means the volume], made with the half axes x and y positive, we will also have:

$$\int \frac{dp}{dx}\omega = \int P\cos\lambda\ s$$

$$\int \frac{d[q]}{dy}\omega = \int Q\cos\mu\ s$$

and consequently

$$\int \left(\frac{dp}{dx} + \frac{dy}{dy} + \frac{dr}{dz}\right)\omega = \int (P\cos\lambda + Q\cos\mu + R\cos v)s$$

Thus, we have shown and connected the modern proof of the divergence theorem to its historic 1828 proof by Ostrogradsky. As stated above, the divergence theorem was not given its current name until 1901, but this does not mean it was not used in its different forms. Its use spread quite rapidly after Ostrogradsky's presentations and publications. Its various applications by mathematicians and scientists in the same time-period made it difficult for historians to track down the source. I was exposed to the divergence theorem alongside names like George Green, Carl Friedrich Gauss, and George Stokes. Michael Ostrogradsky was a name unknown to me prior to my specific search for the original proof of the divergence theorem. It was a name I had to practice spelling.

Recall that Ostrogradsky's first presentation of the divergence theorem was in February of 1826 at The Academy of Sciences in Paris. Thirteen years prior, Carl Friedrich Gauss published a paper relating volume and surface integrals while studying forces of attraction and repulsion. Gauss' presentation contains three specific cases of the divergence theorem, yet today the divergence theorem is sometimes called Gauss' theorem even though he never stated or proved the general case as did Ostrogradsky (Stolze 439). In Gauss' first result, $p(x,y,z)=1$, $q(x,y,z)=0$ and $r(x,y,z)=0$. In the second, $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{\partial r}{\partial z} = 0$, and in the third, $p(x,y,z)$ is an unknown while $q(x,y,z)=0$ and $r(x,y,z)=0$. In 1833 and 1839, he published more special cases of the theorem (Katz 147).

In 1928, George Green published *An Essay on the Application of Mathematical Analysis to the Theories of Elasticity and Magnetism* in England. This essay contains what are now known as Green's identities, which also relate volume integrals and surface integrals. The first identity of his paper is quite close to the divergence theorem, and with a single substitution, it would be the exact same equation. The needed substitution would be $P = u\frac{\partial v}{\partial x}$, $Q = u\frac{\partial v}{\partial y}$ and $R = u\frac{\partial v}{\partial z}$ (Stolze 438), where the *P*, *Q*, and *R* are Ostrogradsky's values discussed above and the terms on the right side of the equalities are what appear in Green's identity.

The same year in Paris, 1828, Simeon Denis Poisson stated and proved a result identical to Ostrogradsky's, but without citation. Mathematical historian Victor Katz states that Poisson had "refereed" Ostrogradsky's 1827 paper, thus learning it from him. Katz reminds us that "references were not made then with the frequency that they are today," so there's no need to accuse anyone of theft (148).

Other evidence that attributes the divergence theorem to Ostrogradsky appears later in the work of James Clark Maxwell. In Maxwell's 1873 *Treatise on Electricity and Magnetism*, he uses the divergence theorem with reference to Ostrogradsky's work (Stolze 439).

Each person used their version of the divergence theorem as a means to an end. I have not found any information stating if the divergence theorem was named by anyone prior to Heaviside in 1901. Since the theorem was viewed as a secondary tool, this may be a reason why. Even Ostrogradsky did not present it as its own result, but as a benchmark toward a further goal. In the 1831 publication, discussed above, he states and proves the theorem in less than a page and a half. He proceeds without pause through the next eight pages, dedicating most of them to the theory of heat. "Let's apply these general results to the theory of heat" (134). The divergence theorem to Ostrogradsky was a simple and general result.

Katz states that each scientist used the theorem “in the middle of long papers” to achieve a further goal. Its applications were widespread, as “Gauss was interested in the theory of magnetic attraction, Ostrogradsky in the theory of heat, Green in electricity and magnetism, Poisson in elastic bodies,” and later Maxwell also in electricity and magnetism (149). Applications of the divergence theorem have continued to spread, and it is just as useful today, with the proof hardly changed, as it was when originally stated by Ostrogradsky nearly 200 years ago. Ostrogradsky’s work certainly deserves more recognition. As widespread as the divergence theorem is, Ostrogradsky’s name should be spread equally as far, even if it takes a few tries to spell it correctly.

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