**EDUCATIVE FEATURES OF UPPER ELEMENTARY
EUREKA MATH CURRICULUM**

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University of Missouri

In Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy

by

Amy Dawn Dwiggin

Dr. Samuel Otten, Dissertation Advisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the
dissertation entitled

EDUCATIVE FEATURES OF
UPPER ELEMENTARY EUREKA MATH CURRICULUM

Presented by Amy Dawn Dwiggin,

A candidate for the degree of Doctor of Philosophy,

And hereby certify that in their opinion it is worthy of acceptance.

Professor Samuel Otten

Professor Susan Empson

Professor Charles Munter

Professor David Bergin

For Caleb, Wilson, and Ian

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ABSTRACT

This two-phase study examined the educative features of upper elementary Eureka Math curriculum as well as examined take-up of those educative features in two classrooms at each grade level. Using an analytical framework based on Males (2011) and Quebec Fuentes and Ma (2018) in the first phase, I coded the educative features of a module at both the third- and fourth-grade levels for educative features for the category of content knowledge (Shulman, 1986) being addressed, the location of the feature in the lesson, and the type of guidance (Enactment or Rationale) being provided. An examination of the data in this phase revealed that most educative features addressed *Pedagogical Content Knowledge for Mathematics Topics*, was located in the Concept Development of the lesson, and provided guidance for enacting the feature. In the second phase, using two target lessons at each grade level, I identified key educative features to follow through the phases of curriculum use (Stein, Remillard, & Smith, 2007) in order to examine the take-up of those educative features by practicing teachers. An examination of the data in this phase revealed differences and similarities in take-up of educative features. In particular, differences in take-up of Enactment Guidance for Experiences seemed to heavily influence differences in take-up of Enactment Guidance for Facilitating Discourse and Enactment Guidance for Participation Structures. Additionally, similarities in take-up of Enactment Guidance for Representations and Enactment Guidance for Developing Mathematical Terminology were revealed across teachers at the same grade level. Implications for curriculum development and recommendations for further research are offered.

CHAPTER 1: INTRODUCTION

Since Ball and Cohen's call in 1996 to design curriculum materials that support both teacher and student learning and Davis and Krajcik's (2005) creation of design heuristics for the development of educative curriculum materials nine years later, mathematics curriculum designers have given considerable attention to the educative features in written curriculum that support teachers in reading, interpreting, and adapting the curriculum for their own students. In response to their efforts, greater understanding and knowledge about what educative features are actually taken up by teachers and incorporated into work with students is needed in mathematics education.

This need for understanding is especially important given the emergence of new open educational resources, especially those that are freely available. One such curricula is Engage^{NY}/Eureka Math, which was originally created through a partnership between the New York State Education Department and the nonprofit organization Great Minds (Great Minds, 2018). Although the curriculum was released nationally in 2013, a 2016 report by the RAND Corporation (Opfer, Kaufman, & Thompson, 2016) found that Eureka Math had already become the most commonly used published instructional resource for elementary mathematics teachers. Yet little is known about how teachers are using this curriculum and drawing upon the resources it contains.

Open educational resources have the potential for new educative features relative to traditionally-published texts, or the same educative features but embedded in the written curriculum in new ways. Although two frameworks (Males, 2011; Quebec Fuentes & Ma, 2018) for analyzing educative features in mathematics curriculum have been proposed, these frameworks have not been applied to open educational resources

such as the Eureka Math curriculum. Moreover, knowledge about what educative features are taken up by teachers in their lesson planning (intended curriculum) or drawn upon while teaching (enacted curriculum) is limited. This study seeks to understand how teachers are using this curriculum. In particular, this study will investigate educative features in the Eureka Math curriculum throughout the phases of curriculum use (Stein et al., 2007) by identifying educative features in the written curriculum materials as well as considering how those educative features are taken up by teachers in the intended and enacted curriculum. An in-depth analysis allows the mathematics education community to have a greater understanding of how educative features and curriculum use may be different with this widely used open educational resource.

Research Questions

This study is motivated by the emergence of open educational resources in general as well as the widespread use that the Eureka Math curriculum has seen in particular. Although research has investigated teacher use of content supports found in physical curriculum materials, teacher use of content supports in curriculum materials such as Eureka Math has yet to be investigated. Specifically, with removal of publishing restraints and further enhancement through digital opportunities, the ways in which opportunities for teacher learning are presented and, subsequently, how teachers make use of them may be changing. Investigation of the educative features contained in curriculum like Eureka Math and taken up by teachers has the potential to transform how curriculum developers incorporate opportunities for teacher learning. As such, this study seeks to answer the following questions:

- 1.) What educative features are present in Eureka Math materials?

- 2.) How do teachers take up the educative features in their planning and enactment of Eureka Math curricula?

The following sections provide an overview of how these research questions were studied. I first describe the theoretical perspectives that frame the study, followed by a brief description of the study and its significance.

Theoretical Perspectives

This study drew on two theoretical perspectives, one regarding teacher content knowledge (Shulman, 1986) and one regarding curriculum use (Stein et al., 2007). Shulman's framework of teacher content knowledge was used in this study because current analytical frameworks for identifying educative features in curriculum materials focus on promoting teacher learning by developing a teacher's content knowledge. Moreover, curriculum materials can serve as a site for teacher learning through the planning, altering, implementing, and assessing of lessons that takes place on an almost daily basis (Davis & Krajcik, 2005), which necessitates a theoretical perspective on curriculum use. Thus, I employed Stein and colleagues' perspective on curriculum use to examine the processes teachers engage in when drawing upon their content knowledge, among other things, to "participate with" the written curriculum materials (Remillard, 2000), plan for instruction, and enact lessons with students.

Teacher Content Knowledge

Previous analytical frameworks for identifying educative features (Beyer, Delgado, Davis, & Krajcik, 2009; Males, 2011; Quebec Fuentes & Ma, 2018) focused on how teacher learning of content knowledge was promoted through the written text. These frameworks were informed by Shulman's (1986) categories of content knowledge as well

as content knowledge specific to the teaching and learning of a particular subject area, which, in the case of mathematics, has been articulated as Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). In this study, a framework for teacher content knowledge was needed to examine the breadth of ways that teacher learning can be promoted through the educative features contained in written curriculum materials.

Shulman (1986) suggested three categories of teacher content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge “refers to the amount and organization of knowledge per se in the mind of the teacher” (p. 9), going beyond facts and concepts in mathematics, for example, to an understanding of the structures of mathematics. Moreover, content knowledge includes knowledge of “the variety of ways in which basic concepts and principles of the discipline are organized to incorporate its facts” as well as knowledge of the ways that “truth or falsehood...are established” (p. 9). Drawing on Shulman’s ideas, Ball, Thames, and Phelps (2008) also made the point that a teacher’s subject matter content knowledge is even more specialized than that of a mathematician, or specialized in different ways. In particular, teacher knowledge in this category goes beyond “knowing how knowledge is generated and structured in the discipline” to include “how such considerations matter in teaching” (p. 402). In this way, deep knowledge of the subject matter has tremendous influence on pedagogical judgments and curricular decisions.

Shulman’s (1986) pedagogical content knowledge pertains to “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (p. 9). In other words, pedagogical content knowledge is the knowledge that

teachers have to choose appropriate representations, analogies, illustrations, examples, explanations, and demonstrations that make the subject matter comprehensible for others. Moreover, teachers with strong pedagogical content knowledge have knowledge of how students come to understand the subject matter, their misconceptions about the subject matter, and strategies to overcome these misconceptions. This type of knowledge, as Ball, Thames, and Phelps (2008) put it, is “a kind of amalgam of knowledge of content and pedagogy that is central to the knowledge needed for teaching” (p. 392).

Curricular knowledge, according to Shulman, involves knowledge of the range of programs designed for teaching a particular subject or particular grade level. Shulman expects the “mature” teacher to draw on “those tools of teaching that present or exemplify particular content and remediate or evaluate the adequacy of student accomplishments” (p. 10). In addition, curricular knowledge includes knowledge of what students are simultaneously studying in other disciplines (lateral curriculum knowledge) as well as what students have studied in previous or upcoming courses in the same content area (vertical curriculum knowledge).

Within written curriculum materials, there may be some implicit opportunities for teachers to build content knowledge and curricular knowledge from the mathematical explanations for students and the organization of the curriculum within and across grade levels, but there is also the potential for explicit material specifically intended for the teacher’s benefit. For example, insights into common misconceptions or a vignette containing sample student-teacher discourse may build a teacher’s pedagogical content knowledge, while connections to other subject areas may build a teacher’s curricular knowledge.

Curriculum Use

Mere knowledge of the curriculum without attention to enactment of the curriculum in practice limits our understanding of curriculum. Stein, Remillard, and Smith's (2007) phases of curriculum use (see Figure 1) conceptualizes "curriculum as unfolding in a series of temporal phases from the printed page (*the written curriculum*), to the teachers' plans for instruction (*the intended curriculum*), to the actual implementation of curricular-based tasks in the classroom (*the enacted curriculum*)" (p. 321). In this conceptualization, teachers use interpretive and interactive processes to transform the curriculum in each phase into student learning in the end. Mediating these processes are the complex and multi-faceted interactions of teachers, professional communities, organizational and policy contexts, and classrooms of students. As a result, the curriculum materials themselves are not the only influence on student learning; rather, teachers, students, and their interactions and interpretations during phases of curriculum use all shape opportunities for student learning.

Figure 1. Temporal phases of curriculum use (Stein, Remillard, & Smith, 2007, p. 322)

Specifically, this dissertation study sought to examine those interactions and interpretations through the written, intended, and enacted curricula by examining what educative features are present in the written curriculum and how teachers drew on those educative features to design the intended curriculum and collectively create the enacted curriculum with students. Therefore, the relationship between the teacher and the curriculum in this study is “dynamic” and involves participation by both the teacher and the curriculum with the teacher positioned as an active designer of curriculum (Remillard, 2005). Remillard describes this perspective of curriculum use as “participation with the text” (p. 221). In this perspective, the focus of analysis is “on the activity of using or participating with the curriculum resource and on the dynamic relationship between the teacher and curriculum” (p. 221) rather than on “the nature of teachers’ interpretations, the factors that influence them, and the resulting classroom

practices” (p. 220). Analyzing curriculum use through the participatory relationship between teachers and the curriculum across the phases of curriculum use allows for greater understanding of the educative features contained in Eureka Math materials.

Overview of the Study

I conducted this study in two phases, with the first phase focused on the educative features present in the written curriculum and the second phase focused on how those educative features are taken up in the enacted curriculum. In the first phase, data was drawn from two modules, Grade 3 Module 7 *Geometry and Measurement Word Problems* and Grade 4 Module 6 *Decimal Fractions*, and was utilized to answer the first research question. Using a framework adapted from the work of Males (2011) to include updated language and mathematics practices from Quebec Fuentes and Ma (2018), I coded the lessons for educative features in four categories of content knowledge:

- (1) *Mathematics Subject Matter Content Knowledge*,
- (2) *Pedagogical Content Knowledge for Mathematics Topics*,
- (3) *Pedagogical Content Knowledge for Mathematics Practices*, and
- (4) *Mathematics Curricular Content Knowledge*.

I also considered whether the educative features included Enactment Guidance or Rationale Guidance in each of these categories as well as where the feature was located within the lesson (Fluency Practice, Application Problem, Concept Development, or Student Debrief). Data from this phase was analyzed to investigate trends within and across the categories of content knowledge and within and across grade levels as well as according to the location of the feature in the lesson or the type of guidance (Enactment or Rationale) provided.

In order to answer the second research question, key educative features from two target lessons in each module were identified and followed through the phases of curriculum use. Four teachers (two in Grade 3 and two in Grade 4) participated in the study and I collected video and audio data from their planning, enactment, and reflection upon these two lessons. Analysis in this phase focused on how key educative features within the same lesson were taken up by different teachers at the same grade level, how key educative features were taken up across the lessons at the same grade level, and how key educative features were taken up across both grade levels.

Significance of the Study

Research concerning educative features in mathematics curriculum has typically focused on singular aspects such as identification or evaluation of those features (e.g., Males, 2011; Quebec Fuentes & Ma, 2018) or in work with preservice teachers to develop teacher knowledge through curriculum materials (Drake, Land, & Tyminski, 2014). However, the present study, in focusing on in-service teachers using newly-developed curriculum materials, is able to connect the educative features in the written curriculum to the ways in which they are taken up (or not) by practicing teachers in the enacted curriculum. This study contributes to the mathematics education research community by investigating one of the most widely used elementary mathematics curricula.

Dissertation Outline

This dissertation is organized into six chapters. This first chapter was meant to provide an introduction to the study by sharing the motivation for the questions under study as well as the theoretical perspectives framing the study. In Chapter 2, I present a

review of the current literature in areas central to this study: educative curriculum materials, open educational resources, and teacher use of educative curriculum materials. Chapter 3 contains a description of the methodology for the two phases of this study, including details of data collection and analysis in each phase. In Chapters 4 and 5, the results of the two phases are presented. Chapter 4 focuses on the results and findings from the analysis of the written curriculum, whereas Chapter 5 focuses on the findings from the analysis of the enacted curriculum. Finally, in Chapter 6, I discuss how the findings of this study answer the research questions and build upon prior research. Additionally, I address limitations of the study, implications for curriculum development, and recommendations for further research.

CHAPTER 2: LITERATURE REVIEW

Given the vastness of the literature on curriculum and its use, this review of literature will focus on that which is most central to my study. In particular, I begin by describing what is meant by educative curriculum materials and open educational curriculum materials. Next, I discuss studies of teachers' use of educative curriculum materials related to mathematics.

Curriculum Materials

Although curriculum in general constitutes both student and teacher materials, the term curriculum materials refers to those materials designed for teachers' use to guide their instruction (Stein et al., 2007). Moreover, curriculum materials can include a variety of printed and digital materials available from both vetted and non-vetted sources, including full K–12 curricula available in print and through websites of textbook publishers and those available as single lessons or units through Teachers Pay Teachers, Educents, and Teachwise, among others. Prior curriculum analysis has focused on analyses of printed materials in terms of content and sequence (e.g., Flanders, 1987; Hook, Bishop, & Hook, 2007) and in terms of mathematical content areas such as fractions (e.g., Charalambous, Delaney, Hsu, & Mesa, 2010; Watanabe, 2003; Yang, Reys, & Wu, 2010). In addition, some research has focused on teacher materials in various ways: teaching opportunities attending to reason (Newton & Newton, 2006), cross-national comparison of mathematical ideas (Li, 2004; Watanabe, 2001), and teacher opportunities to learn (Castro, 2006; Stein & Kim, 2009; Stylianides, 2007).

The vein of research central to this dissertation study traces its roots to Ball and Cohen's (1996) call for curriculum materials to be conduits for teachers' learning; that is,

Ball and Cohen called for curriculum materials to support teacher learning in areas of understanding and anticipating student thinking, content knowledge, pedagogical designs, and mapping out instruction. To this end, Ball and Cohen coined the term “educative” to describe such curriculum materials.

Educative Curriculum Materials

Educative curriculum materials are distinct from typical teacher’s guides, where the focus is on supports for teaching strategies, and from general K–12 curriculum materials, where the focus is primarily on student learning. Even though teachers can and do learn through typical curriculum materials (Grossman & Thompson, 2008), the term educative curriculum materials is meant to identify an intentionality on the part of curriculum developers. More specifically, these materials are intended to help teachers acquire new content knowledge for teaching (Ball et al., 2008) while situating and integrating that new knowledge into their current practice (Putnam & Borko, 2000). In particular, these curriculum materials contain features that support teachers in knowing *what* concepts to teach and *how* to teach them (implementation or enactment guidance) as well as the mathematical or pedagogical justifications for *why* certain approaches are appropriate (rationale guidance) (Beyer et al., 2009). In this way, these materials are “designed to speak to teachers, not merely through them” (Remillard, 2000, p. 347) by providing rationales revealing the developer’s pedagogical judgments or providing detailed recommendations for how to use certain aspects of the curriculum, for example.

Drawing on Ball and Cohen (1996), Davis and Krajcik (2005) developed a set of “high-level guidelines” that address specific ways curriculum materials could promote teacher learning. First, educative curriculum materials could help teachers understand and

interpret student thinking and how students may respond to instructional activities.

Second, educative curriculum materials could support teachers' subject matter content knowledge. Additionally, educative curriculum materials could help teachers learn ways to relate units of student learning throughout and across the years of schooling as well as promote teacher learning by including rationales for pedagogical judgments that allow for integrating theory and practice. Lastly, Davis and Krajcik call for educative curriculum materials to promote a teacher's pedagogical design capacity (Brown, 2002), "defined as a teacher's capacity to perceive and mobilize existing resources in order to craft instructional episodes" (Brown, 2009, p. 29). Table 1 shows how Davis and Krajcik's "high-level guidelines" may coincide with Shulman's categories of content knowledge.

Table 1

Connections between Davis and Krajcik's (2005) "high-level guidelines" and Shulman's (1986) categories of content knowledge

"High-level guidelines"	Category of content knowledge
"curriculum materials could help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities" (Davis & Krajcik, 2005, p. 5)	Pedagogical content knowledge
"curriculum materials could support teachers' learning of subject matter"	Subject matter content knowledge
"curriculum materials could help teachers consider ways to relate units during the year"	Curricular knowledge
curriculum materials could "make visible the developers' pedagogical judgments"	Pedagogical content knowledge and Curricular knowledge
curriculum materials could "promote a teacher's pedagogical design capacity"	Pedagogical content knowledge

More recently, Davis, Palincsar, Smith, Arias, and Kademian (2017) identified themes across multiple studies and compared those themes to their own research to create a set of design principles for educative features in curriculum materials. However, these

principles focus more on the design of educative features, including adaptations teachers may make and ready-to-use teaching tools as well as particular characteristics of narratives, graphics, and definitions; whereas, the first phase of this study focuses on the nature (i.e., the intent for teacher learning) of the supports contained in the educative features of Eureka Math curricula.

Several studies have examined the nature of educative features in curricular materials. One of those studies investigated introductory biology curriculum (Beyer et al., 2009), while other studies investigated middle school mathematics textbooks (Males, 2011) and fourth-grade geometry units (Quebec Fuentes & Ma, 2018). As previously described, analytical frameworks in each of these studies were informed by Shulman's (1986) categories of content knowledge and considered whether the guidance was focused on how or why teachers may enact a particular instructional strategy.

In their study of the educative features in two units for eight introductory high school biology programs, Beyer, Delgado, Davis, and Krajcik (2009) found that most educative features were contained in the category of pedagogical content knowledge for science topics (55%). This category included support for teachers "in engaging students with scientific phenomena, using scientific instructional representations, and anticipating and dealing with students' ideas about science" (p. 982). Subject matter content knowledge accounted for 32% of educative features found in these textbooks, and pedagogical content knowledge for scientific inquiry accounted for 13% of the educative features. In this study, Beyer and colleagues considered whether the guidance in the pedagogical content knowledge categories focused on implementation or rationale guidance for the teachers and found more than twice as much guidance helping teachers

understand how to use a particular strategy (implementation guidance, 48%) than guidance helping teachers understand why they might want to use them (rationale guidance, 20%).

In mathematics education research, two studies have proposed frameworks to investigate educative features in mathematics textbooks. Males (2011) examined units on introduction to variables and transformations in her study of four middle school math textbooks. Building on Beyer and colleagues, Males adapted their framework to reflect mathematics education and also added a category for Shulman's curricular knowledge. For three of the four textbooks in her study, educative features in the category of *Pedagogical Content Knowledge for Mathematics Practices* were most prevalent, with *Pedagogical Content Knowledge for Mathematics Topics* and *Curricular Content Knowledge* the next most prevalent and almost evenly split between them. Additionally, *Subject Matter Content Knowledge* accounted for no more than seven percent of educative features in any curriculum except for a lone unit from the University of Chicago School Mathematics Project's *Transition Mathematics* book. Males also found that Rationale Guidance comprised no more than six percent of guidance for any curriculum, much less than the rationale guidance (20%) found in the study of high school biology textbooks by Beyer and colleagues (2009). More recently, Quebec Fuentes and Ma (2018) proposed a framework aligned with current standards document (e.g., *Principles to Actions* and CCSS) with the purpose to evaluate mathematics curriculum materials with respect to the included educative features. Rather than report on relative frequencies of educative features as Males and Beyer and colleagues did, Quebec Fuentes and Ma reported on how they used their framework to learn how the

educative features in a particular curriculum may need to be enhanced or supplemented. For example, in the dimension of teacher knowledge of technology use in mathematics, content in the *enVisionMATH* curriculum includes digital materials; however, it does not include any educative features to support the development of teacher knowledge of these digital materials.

Each of the studies described have investigated the educative features in curricula that were developed in two traditional manners—by publishing companies or through projects funded by groups such as the National Science Foundation. However, these traditional methods are no longer the only avenues for large-scale curriculum development.

Open Educational Curriculum Materials

Efforts to make curriculum free and widely accessible began in the mid-1990s with the release of MERLOT, a web-based, curated repository of curriculum materials or intellectual content for teaching and learning in higher education (Bliss & Smith, 2017). Since then, some universities across the United States, including the Massachusetts Institute of Technology, have freely shared the content of their courses with the world. This period marked the beginning of a new era of education innovation, and in 2002 at a meeting of the United Nations Educational, Scientific and Cultural Organization, the name Open Educational Resource (OER) was coined to describe this education innovation.

The Hewlett Foundation, an investor in OER since 2002, defines OER as “teaching, learning and research materials¹ in any medium—digital or otherwise—that reside in the public domain or have been released under an open license that permits no-cost access, use, adaptation and redistribution by others with no or limited restrictions” (The William and Flora Hewlett Foundation, 2016). Other foundations such as the Shuttleworth, Gates, and Open Society Foundations have also invested in OER over the years; notably, Khan Academy, a well-known producer of OER and widely utilized educational resource for all ages, has received support from the Gates Foundation (Bliss & Smith, 2017). Following a new strategic plan in 2010, the Hewlett Foundation turned its attention to supporting the development of market-ready, full end-to-end K–12 curricula in the United States. In particular, the Hewlett Foundation supported the implementation of the Common Core State Standards (CCSS) by providing support to Engage^{NY}, a provider of openly licensed, CCSS-aligned curricula in K–12 mathematics and English language arts (Bliss & Smith, 2017).

Originally created through a partnership with the New York State Education Department, the nonprofit organization Great Minds released the Eureka Math² curriculum nationally in 2013 (Great Minds, 2018). Authored by a team of teachers and mathematicians, the free, CCSS-aligned online curricula have since drawn considerable attention, particularly in grades K–8, for its usability and alignment with the CCSS as well as attention to aspects of rigor (conceptual understanding, fluency and procedures,

¹ According to the William and Flora Hewlett Foundation, OER materials can include “full courses, course materials, modules, textbooks, streaming videos, tests, software and any other tools, materials or techniques used to support access to knowledge” (as cited in Bliss & Smith, 2017, p. 1).

² Eureka Math refers to the mathematics portion of the broader Engage^{NY} curriculum, which also includes English language arts curriculum. Using the name Eureka Math in this paper refers to only the mathematics curriculum, whereas using the name Engage^{NY} refers to the broader curriculum.

and applications) (EdReports, 2014). Although EdReports' methodology, evaluation tool, and process have been criticized by leading mathematics education organizations (Briars & Mills, 2015), Eureka Math continues to be widely chosen for both English language arts and mathematics curriculum in grades K–8 and beyond. A 2016 report by the RAND Corporation found that Eureka Math was the most commonly used published instructional resource for elementary mathematics teachers (Opfer, Kaufman, & Thompson, 2016) even though rigor, pacing, and inadequate supports for learners (e.g., minimal scaffolding and differentiated materials) are presenting significant challenges for teachers (Kaufman et al., 2017).

In their case study on the use of both the Eureka Math and English language arts curriculum materials from Engage^{NY}, Kaufman and colleagues (2017) sought to understand teachers' use of Engage^{NY} by examining Google Analytics, survey findings from the October 2015 survey of the American Teacher Panel, and interviews with California and New York teachers. Findings from this case study indicated that Engage^{NY} materials were accessed in every state and use was particularly high in states that had adopted standards similar to the CCSS. In addition, Eureka Math curriculum materials were used at about three times the rate of English language arts materials, although survey data indicates that the English language arts teachers who used the materials may have done so more comprehensively than mathematics teachers. Interestingly, mathematics teachers did not cite availability of the open online resource as an influence on their use of Eureka Math materials as compared to other instructional resources though it may have been a factor nonetheless. In addition, compared to other instructional materials, teachers responded that Eureka Math materials gave students more

opportunities to engage in CCSS-aligned practices (e.g., explaining and justifying their work) and addressed conceptual understanding, procedural skills, and application to real-life contexts with equal time and intensity. Reflecting these comparisons, 88% of teachers interviewed stated that Eureka Math materials had improved their teaching even though nearly half of the teachers took issue with the vast amount of curriculum materials and content of individual lessons that, subsequently, required them to move too quickly through the materials.

Uptake or Use of Educative Features

Even if the written curriculum includes educative features, research indicates that teachers draw upon educative features or content supports in various ways as they prepare lessons and enact them with students (e.g., Collopy, 2003; Land, Tyminski, & Drake, 2013; Remillard, 2000). Davis and colleagues (2017) define uptake of educative features as “the ways teachers adopt, in language and/or action, ideas or practices recommended in the educative features” (p. 294). This definition seems to suggest that evidence of uptake must come from something a teacher says or does as part of the enacted curriculum, yet several of the Mathematical Tasks of Teaching (Ball et al., 2008) occur outside of the enacted curriculum and could also offer evidence of uptake of educative features. Detailed analysis of teachers reading the written curriculum and planning for the intended curriculum could also offer evidence of uptake of educative features.

Several studies in science education have attended to aspects of uptake of educative features in the intended and enacted curricula. In their quasi-experimental study of 53 preservice elementary teachers, Beyer and Davis (2009) investigated the use of

different forms of educative supports and how those forms help preservice teachers alter lesson plans. They found that lesson-specific narrative supports (e.g., a story of how Kendra used the principle of probing student thinking to modify her science lesson) helped preservice teachers identify specific ways to alter the intended curriculum; however, preservice teachers were not able to transfer these supports to other lessons. In contrast, more general expository supports (e.g., a general description of probing student thinking applicable to any academic discipline) helped preservice teachers identify principles of practice by which to analyze a variety of lessons, but those supports did not lead to more specific ways to alter the lessons. That study pointed to the need for further analysis of the forms and locations of educative supports in the written mathematics curriculum as these supports are taken up by teachers in the intended curriculum.

Differential use of the educative features in the enacted curriculum was also found in a study by McNeill (2009) of chemistry curriculum materials. In this study of the classroom enactments with six teachers and 568 students, McNeill found that teachers engaged in a range of instructional practices to support students in the construction of scientific arguments even though the written curriculum included educative features designed to specifically address scientific explanation (e.g., providing instructional practices and presenting exemplary and non-exemplary examples of students' explanations). These studies extend the findings about the written curriculum previously discussed into the intended and enacted curriculum phases of curriculum use. Moreover, these studies point to the need to understand the processes by which teachers participate with the written curriculum, create the intended curriculum, and enact curriculum with students.

Several studies have found that the ways in which teachers read or encounter the educative features in the written curriculum can greatly influence uptake of those features. Namely, teachers' expectations about the curriculum materials they are reading or the location of the educative features themselves may diminish the educative nature of the materials. Collopy (2003), for example, contrasted the expectations of two teachers. While one teacher, Ms. Ross, expected the materials to support her learning about teaching and resulted in the uptake of ideas to help students communicate their mathematical ideas, the other teacher, Ms. Clark, expected the materials to provide instructional activities rather than educative features, resulting in reduced information about alternative strategies in problem solving. In this way, Ms. Clark did not read the educative features in educative ways, as Land, Tyminski, and Drake (2013) also found in their study of prospective teachers. The location of content supports in educative curriculum materials have also been found to influence uptake in educative science curriculum. Schneider and colleagues (Schneider, 2006; Schneider & Krajcik, 2002; Schneider, Krajcik, & Blumenfeld, 2005) have consistently found that teachers utilized supports presented at the lesson level more than those at the unit level. This finding, together with Sherin and Drake's (2009) finding that teachers first attended to details of the curriculum rather than to overarching themes or big ideas, has significant implications for the design of educative curriculum materials. More research, specifically that which considers mathematics curriculum, is needed to investigate the take-up of educative features by practicing teachers.

Summary

In this chapter, I reviewed literature regarding what is meant by both educative curriculum materials and open educational resources as well as reviewed what is known about the presence of educative features and the take-up of educative features in the enacted curriculum. This review has shown that there is a lack of knowledge about educative features in this new era of open educational mathematics resources and that research should consider the take-up of educative features in the mathematics classroom. This study attempts to address each of these areas by investigating the educative features in a widely used open educational mathematics resource and following those educative features into a classroom to observe how teachers are taking them up. Moreover, because this study follows a single curriculum into enactment, this study will extend what is known in mathematics education about educative features in general and how practicing teachers draw upon these features with students.

CHAPTER 3: METHODOLOGY

In the sections that follow, I describe the setting of the study followed by details about the two phases of analysis in terms of how I addressed the research questions:

- 1.) What educative features are present in Eureka Math materials?
- 2.) How do teachers take up the educative features in their planning and enactment of Eureka Math curricula?

Setting

This study took place in a rural Missouri elementary school during the third year of full implementation of the Eureka Math curriculum. According to the Missouri Comprehensive Data System, enrollment consisted of 532 students in Kindergarten to fifth grade in the 2018-2019 school year. The large majority of these students were White (87.80%) with the sample sizes for other ethnicities being too small for reporting. Additionally, 43.9% of elementary students qualified to receive free and reduced lunches. Data at the district level indicated a special education incidence rate of 9.33% with Black students comprising 10.8% and White students comprising 78.3% of those that qualified for services.

Prior to the implementation of Eureka Math, teachers, typically working in grade-level teams, used their own discretion to tailor the sequence of units and content of instructional activities in the curriculum so long as the assessment standards held by the state were addressed throughout the school year. After several years of not seeing sustained success on state testing measures (see Table 2), instructional leaders and administrators in the district felt that adopting a new curriculum school-wide would

provide the coherence, focus, and attention to standards that many felt may have been lacking in the school.

Table 2

Percent of 3rd and 4th grade students scoring advanced or proficient on state mathematics assessment

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019
<i>3rd Grade</i>									
School	52.3	61	51.5	45.1	56.7	50.5	69.9	66.7	69.2
State	50.2	52.5	51.4	50.8	51.7	52.2	53.2	46.9	46.0
% away from State	+2.1	+8.5	+0.1	-5.7	+5	-1.7	+16.7	+19.8	+23.2
<i>4th Grade</i>									
School	46.6	40	36.9	33.6	35	44.4	46.5	62.3	57.9
State	51.2	51.2	50.7	42.9	49.4	52.5	53.9	45.9	46.8
% away from State	-4.6	-11.2	-13.8	-9.3	-14.4	-8.1	-7.4	+16.4	+11.1

Note. 2017 was the first year of Eureka Math implementation.

Eureka Math was chosen for several reasons. First, Eureka Math is fully aligned to the Common Core State Standards (CCSS) (National Governors Association Center for Best Practices & The Council of Chief State School Officers, 2010), which served as a basis for the latest Missouri Learning Standards that were implemented in state testing measures during the 2017–2018 school year. Second, at least two other comparable schools in the state, with similar test scores to this school district and also professional connections to teachers in the school, had adopted this curriculum and had improved and sustained their performance. Finally, one of the Kindergarten teachers at the time of adoption was a member of the Eureka Math writing team during the development of the curriculum and could serve as a mentor to fellow teachers during the implementation process.

Since the adoption of Eureka Math, measures of interest on the state mathematics assessment have seen positive growth; however, this trend does not necessarily establish

a causal relationship between Eureka Math and these achievement scores. Table 2 shows the percent of students scoring advanced or proficient on the state mathematics test for third and fourth grade as well as a comparison to corresponding state percentages.

Although adoption of Eureka Math may explain some of the growth starting in 2017, another notable contribution to this growth is that the state test had transitioned to assessing the newly adopted Missouri Learning Standards in 2018. These Standards were grounded in the CCSS, which served as the set of standards to which Eureka Math was aligned at its development.

Discussions with mathematics instructional leaders provided further insight into organizational and implementation factors important to this study such as the focus on third- and fourth-grade curriculum. In particular, state testing in Missouri begins in the third grade and continues through fourth and fifth grade. Additionally, implementation of Eureka Math at the third- and fourth-grade levels was reportedly going well, the teachers were comfortable with the curriculum, and teacher buy-in was high. The fourth-grade level also offered an opportunity to see the curriculum taught in a departmentalized setting. At this grade level, two of the five teachers taught mathematics and science only, while the two other teachers taught language arts and social science only; the remaining teacher's classroom was self-contained.

During this study, I was employed by the district as a high school mathematics teacher and had worked in the district on two separate occasions for a total of 14 years. Most recently, I was in my second year with the district after a two-year leave to complete my doctoral coursework at the University of Missouri. Although I had worked in the district and most colleagues were aware of my personal history prior to the study,

relationships with colleagues in the elementary school were grounded in school and community events as social acquaintances instead of in the teaching and learning of mathematics in the district as professional collaborators. In particular, vertical discussions across grade levels, and certainly those that would extend into high school, had not taken place for over five years and, when they did occur during my first tenure with the district, were limited to a half day meeting every few years focused on discussions with the grade level immediately preceding or immediately following a teacher's current grade level. That is, prior to this study, my professional interactions with the elementary teaching staff were very limited.

Nevertheless, conducting this study with my colleagues at the elementary school may have provided challenges for this study, such as atypical curriculum uses by the teachers during study observations. But these threats exist even with outside observers, and my view is that the relationships allowed opportunities for the participants to be more open and honest with me about issues involving the written and enacted curriculum. Teachers were unguarded in sharing their journeys with Eureka Math from implementation to the present and welcomed me and my recorders into their classrooms. In many respects, this study opened up opportunities for more meaningful professional relationships to take place and instigated further discussions about curriculum that are currently taking place throughout the district.

Research Design

In order to answer the research questions in this study, a two-phase design was necessary with phase one focusing on the written curriculum and phase two focusing on the intended and enacted curriculum. In particular, the first phase investigated the

educative features present in the written Eureka Math materials, while the second phase traced those educative features identified in phase one through the phases of curriculum use.

Phase One – Written Curriculum

During the first phase I investigated the educative features present in written Eureka Math curriculum materials, particularly Grade 3 Module 7 *Geometry and Measurement Word Problems* (6 topics, 30 lessons³) and Grade 4 Module 6 *Decimal Fractions* (5 topics, 16 lessons).

Curriculum sample. At the elementary level, the Eureka Math curriculum addresses the four domains of Number, Geometry, Number and Geometry/Measurement, and Fractions (see Figure 4). The Number and Geometry/Measurement domain focuses primarily on concepts of measurement, but incorporates concepts of number and geometry to solve problems of measurement. The curriculum at each grade level is further organized into six to eight modules within these domains. Each module begins with an overview, connecting the concepts in the module to those previously learned. In addition, the module overview contains notes on pacing, alignment to CCSS, terminology and symbols important to the module, and suggested tools and representations for use in the module. Each topic within the module begins with a summary detailing the CCSS of focus as well as links from previous modules in earlier grades, links to future modules in later grades, and how the current topic relates to prior topics within the same module. Each lesson in this curriculum is typically divided into four components (Figure 2): Fluency Practice, Application Problem, Concept Development, and Student Debrief.

³ The module includes 34 lessons in total. The final four lessons are intended to be used as review rather than introducing new content and were not included in this study.

Suggested Lesson Structure

Fluency Practice	(12 minutes)
Application Problem	(7 minutes)
Concept Development	(31 minutes)
Student Debrief	(10 minutes)
Total Time	(60 minutes)

Figure 2. Example of lesson component breakdown from Grade 3 Lesson 4.

Most Fluency Practice and Concept Development components include a sample vignette of a discussion between students and teachers (Figure 3). In addition, each lesson is followed by the Problem Set, Exit Ticket, Homework, and any templates described in the lesson.

Problem 3: Compare pairs of decimal numbers representing volume.

T: (Place all four graduated cylinders in front of the class.) Express the volume of the liquid in tenths or hundredths liter. (Use the document camera to project the side of Cylinder A so students can see the liter measurements. If this is not possible, select a student to read the volume aloud.)

S: $\frac{3}{10}$ liter. $\rightarrow \frac{30}{100}$ liter.

T: Record this volume in the table on the measurement record.

Repeat the process for the remaining water samples.

T: If we want to order these samples from least volume to greatest volume, what would the order be? Talk with your partner, and record your thinking on the measurement record. (Circulate to encourage use of the place value chart as students compare the measurements.)

S: (Complete the task.)

S: 0.09 liter, 0.15 liter, 0.29 liter, 0.3 liter.

T: How did you determine the order?

S: The place value chart made it easy to compare the decimals. $\rightarrow$ We compared the digits in the largest place first. That was the tenths. $\rightarrow$ In 0.3, there are 3 tenths. That is more than the others. 0.29 comes next, followed by 0.15 and 0.09.

T: (Select a student volunteer to order cylinders from least volume to greatest volume.) Let's look at the cylinders. Do they appear to match the order we determined?

S: Yes!

Volume of Liquid (liters)

Cylinder	ones	.	tenths	hundredths
A	0	.	3	
B	0	.	1	5
C	0	.	2	9
D	0	.	0	9

0.09 L, 0.15 L, 0.29 L, 0.3 L

Figure 3. Sample vignette in Concept Development for Grade 4 Lesson 9.

Curriculum Map for Grades PK-5					2015-16*							
					1st QUARTER		2nd QUARTER		3rd QUARTER		4th QUARTER	
					Grade 5		Grade 4		Grade 3		Grade 2	
					M1: Place Value and Decimal Fractions (20 days)		M1: Place Value, Rounding, Multiplication and Division and Algorithms for Addition and Subtraction (25 days)		M1: Properties of Multiplication and Division and Solving Problems with Units of 2-5 and 10 (25 days)		M1: Sums and Differences to 100 (10 days)	
					M2: Multi-Digit Whole Number and Decimal Fraction Operations (35 days)		**M2: Unit Conversions (7 days)		M2: Place Value and Problem Solving with Units of Measure (25 days)		M2: Addition and Subtraction of Length Units (12 days)	
					M3: Addition and Subtraction of Fractions (22 days)		M3: Multi-Digit Multiplication and Division (48 days)		M3: Multiplication and Division with Units of 0, 1, 6-9, and Multiples of 10 (25 days)		M3: Place Value, Counting, and Comparison of Numbers to 1,000 (25 days)	
					M4: Multiplication and Division of Fractions and Decimal Fractions (38 days)		M4: Angle Measure and Plane Figures (20 days)		M4: Multiplication and Area (20 days)		M4: Addition and Subtraction Within 200 with Word Problems to 100 (35 days)	
					M5: Addition and Multiplication with Volume and Area (25 days)		M5: Fraction Equivalence, Ordering, and Operations (45 days)		M5: Fractions as Numbers on the Number Line (35 days)		M5: Foundations of Multiplication and Division (24 days)	
					M6: Problem Solving with the Coordinate Plane (40 days)		M6: Collecting and Displaying Data (10 days)		M6: Collecting and Displaying Data (10 days)		M6: Problem Solving with Length, Money, and Data (30 days)	
					M7: Exploring Measurement with Multiplication (20 days)		M7: Geometry and Measurement Word Problems (40 days)		M7: Geometry and Measurement Word Problems (40 days)		M7: Time, Shapes, and Fractions as Equal Parts of Shapes (20 days)	
					M8: Addition and Subtraction Stories and Counting to 20 (35 days)		M8: Addition and Subtraction Stories and Counting to 20 (35 days)		M8: Addition and Subtraction Stories and Counting to 20 (35 days)		M8: Addition and Subtraction Stories and Counting to 20 (35 days)	
					M9: Addition and Subtraction Stories and Counting to 20 (35 days)		M9: Addition and Subtraction Stories and Counting to 20 (35 days)		M9: Addition and Subtraction Stories and Counting to 20 (35 days)		M9: Addition and Subtraction Stories and Counting to 20 (35 days)

Approx. test date for grades 3-5

Key:		
Number	Geometry	Fractions
Number and Geometry, Measurement		

Figure 4. Curriculum map for elementary Eureka Math curriculum

In looking at the Eureka Math Curriculum Map for Grades PK–5 (see Figure 4), I noticed that two domains—*Number and Geometry/Measurement* and *Fractions*—were heavily-targeted domains of learning in the upper elementary grades; that is, 15 of 20 modules in grades 3–5 reside in these two domains, with the remaining five modules targeting the Number domain. Choosing a module from Number and Geometry/Measurement and one from Fractions offered an opportunity to gain insight into the ways in which Eureka Math presented content across the scope of mathematics learning at the upper elementary grades. To gain clarity on which specific modules within the two domains to choose, I met with each grade-level team in early spring. Discussions from these meetings indicated that modules remaining in the current school year—that is, Grade 3 Module 7 *Geometry and Measurement Word Problems* (six topics, 34 lessons) and Grade 4 Module 6 *Decimal Fractions* (five topics, 16 lessons)—would serve as reasonable examples of the ways that Eureka Math presents content in the two targeted domains and offer a few unique opportunities (e.g., first time teaching lessons in the module) as well. Additionally, Grade 3 Module 7 allowed me a glimpse into some content from the Geometry domain found only at lower grade levels, while Grade 4 Module 6 offered opportunities to anticipate content at the fifth-grade level.

Analytic frameworks. With roots in Davis and Krajcik's (2005) Design Heuristics for Educative Curriculum Materials and Ball and Cohen's (1996) proposals for improved curriculum design, two frameworks have been proposed to analyze content supports for teachers in mathematics curriculum. The *Teacher Learning Opportunities in Mathematics Curriculum Materials (TLO-Math)* framework (Quebec Fuentes & Ma, 2018) and Males' (2011) framework for analyzing content supports for teachers in

mathematics curriculum both seek to determine what opportunities for teacher learning are available through the written curriculum. Both frameworks distinguish between the support of giving teachers a process to implement and how to implement that process (Enactment Guidance) or giving teachers the mathematical or pedagogical reasons for such implementation (Rationale Guidance) (Remillard, 2000). In the sections that follow, I discuss each framework, including the development and an instantiation of its use by the developers, and conclude with a comparison of the two frameworks leading to the coding scheme used in this study.

TLO-Math framework. Quebec Fuentes and Ma (2018) developed the *TLO-Math* “framework for designing and evaluating mathematics curriculum materials for their educative features” (p. 352). This framework was created through a synthesis of both theoretical and empirical literature on educative curriculum and refined for consistency and alignment with mathematics standards documents (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & The Council of Chief State School Officers, 2010; NCTM, 2014) in order to address the educative features unique to mathematics curriculum. The framework (see Appendix A for full framework) consists of seven dimensions with accompanying definitions and guiding questions to describe each dimension. The seven dimensions are as follows:

- (1) mathematics content knowledge for teaching,
- (2) teacher knowledge of student thinking in mathematics,
- (3) teacher knowledge of disciplinary discourse in mathematics,
- (4) teacher knowledge of assessment in mathematics,
- (5) teacher knowledge of differentiated instruction in mathematics,

(6) teacher knowledge of technology use in mathematics, and

(7) teacher knowledge of mathematical community. (p. 357)

To illustrate a component of the framework, Table 3 shows the definition and guiding questions for dimension (2). The guiding questions for the other dimensions have similar intent; that is, the first question seeks to find what supports exemplify the given dimension while the second and third questions seek to determine the level of presence, clarity, and appropriateness of rationales and implementation guidance, respectively.

Table 3

Teacher knowledge of student thinking in mathematics dimension from TLO-Math framework

<u>Dimension</u>	<u>Definition</u>	<u>Guiding questions</u>
Teacher Knowledge of Student Thinking in Mathematics	The curriculum materials support the development of teacher knowledge in anticipating and understanding student thinking.	<p>What supports are provided for anticipating and understanding student mathematical thinking (e.g., common perceptions and misconceptions)?</p> <p>To what extent are rationales for addressing the significance of students thinking in relation to understanding the mathematics content present, clear, and appropriate?</p> <p>To what extent is implementation guidance for anticipating and understanding student mathematical thinking present, clear, and appropriate?</p>

In their articulation of the framework, Quebec Fuentes and Ma (2018) also demonstrated its use with fourth-grade geometry units from the *enVisionMATH* (Charles, Caldwell, Cavanagh, & Pearson Education, Inc., 2012) and *Investigations in Number, Data, and Space* (Russell, Economopoulos, Pearson/Scott Foresman, & TERC (Firm), 2008) curricula. These programs were chosen for contrast, with the former considered “a

commercially published curriculum program” (Quebec Fuentes & Ma, 2018, p. 366) and the latter being a National Science Foundation-funded, NCTM *Standards*-based curriculum program. Their analysis revealed differences in teacher learning opportunities throughout the curriculum materials both in the range of educative features across the dimensions of the framework and in the nature of educative features in various locations (i.e., embedded in or supplemental to the teachers’ guides). In the dimensions of teacher knowledge of student thinking and teacher knowledge of disciplinary discourse, the framework revealed that both the embedded and supplemental support contained in the *Investigations* curriculum offered a variety of open-ended questions and sample student responses as well as rationales for teacher moves during discussions, whereas the exclusively embedded support in the *enVisionMATH* curriculum contained closed questions and rarely included sample student responses.

Males’ framework. As part of her dissertation study, Males (2011) developed a framework for analyzing content supports of teachers’ guides, specifically in mathematics curricula. Her framework drew from theoretical literature (Ball & Cohen, 1996; Davis & Krajcik, 2005; Shulman, 1986) and research on mathematics and science teacher learning and curriculum use. Also influencing the development of her framework was prior analysis of curriculum materials and her experiences as a teacher and researcher. The structure of Males’ framework was adapted from a framework by Beyer, Delgado, Davis, and Krajcik (2009) for analyzing biology textbooks. In addition, Males’ framework reflects “practices and processes that are integral to the teaching and learning of mathematics” (Males, 2011, p. 48).

Males' framework consists of four domains and nine categories⁴ within those domains (see Appendix B for the full framework). The four domains are:

- (1) *Mathematics Subject Matter Content Knowledge*,
- (2) *Pedagogical Content Knowledge for Mathematics Topics*,
- (3) *Pedagogical Content Knowledge for Mathematics Practices*, and
- (4) *Mathematics Curricular Content Knowledge*.

With the exception of one, each category distinguishes between *Rationale Guidance* and *Enactment Guidance* as the intent of the content support. The "Support Teachers in Anticipating and Using Students' Mathematical Ideas" category is the only category for which rationale guidance was not included. Table 4 shows an excerpt from the domain of *Pedagogical Content Knowledge for Mathematics Topics* to exemplify enactment guidance and rationale guidance.

Table 4

Excerpt of Males' Pedagogical Content Knowledge for Mathematics Topics domain

II. Pedagogical Content Knowledge for Mathematics Topics

Support Teachers in Engaging Students with Topic-Specific Mathematics

Enactment Guidance for:

Experiences: Engaging students in particular experiences. [E-Experiences]

Possible Student Pitfalls: Potential pitfalls with specific mathematical experiences (including describing where students may have difficulty or potential misconceptions) [E-Pitfalls]

Engaging in Problematic Experiences: Engaging students in experiences that may potentially be problematic [E-Problem]

Activity Sequences: Engaging in productive activity sequences [E-Sequence]

Rationale Guidance for:

Experiences: Why particular experiences are appropriate, including why a particular problem or activity is useful in a students' mathematical development. [R-Experiences]

Activity Sequences: Why particular activity sequences are appropriate or useful. [R-Sequence]

⁴ I assume that domain IV is considered both a domain and a category given that Males articulates that categories include codes for both Enactment and Rationale Guidance. Domain I only includes the code for the presence of mathematical content information.

**Support Teachers in Anticipating and Using Students' Mathematical Ideas
Enactment Guidance for:**

Anticipating Student Ideas: Anticipate student ideas within a topic [I-Anticipate]

Using Student Ideas: Using students' ideas during the enactment of the lesson [I-Use]

Males chose to analyze an algebra unit (introduction to variables) and a geometry unit (geometric transformations) from each of four middle school mathematics curricula: *Connected Mathematics Project 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), Glencoe's *Math connects: Concepts, skills, and problem solving (Michigan Edition)* (Day, Frey, Howard, Hutchens, Luchin, McClain, et al., 2009), *Mathematics in Context* (Wisconsin Center for Education & the Freudenthal Institute, 2010), and *Transition Mathematics* (Viktora, Cheung, Highstone, Capuzzi, Heeres, Metcalf, et al., 2008). Males' (2011) findings revealed differences in opportunities for teacher learning across the curricula and between the units analyzed for each curriculum as well as differences in the type of support present (Rationale Guidance vs. Enactment Guidance). For example, one curriculum attended more to *Curricular Knowledge* and *Pedagogical Content Knowledge for Topics*, whereas the others focused most prominently on *Pedagogical Content Knowledge for Practices*. Also, Enactment Guidance for Representations was frequently the focus of support in the introduction to variables units, whereas in the transformation units it was Enactment Guidance for Tools and also Justification, Reasoning and Proof that tended to be the focus of support. Throughout all four curricula, little *Subject Matter Content Knowledge* or Rationale Guidance as a whole was present.

Comparing and contrasting the frameworks. Both frameworks incorporate broad categories of content knowledge (Shulman, 1986); however, the depth, details, and organization of these categories differ fairly significantly. As discussed previously,

Shulman (1986) distinguished among three categories of content knowledge for teaching: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. In the sections that follow, distinctions in the frameworks are explored with respect to each of these categories to provide justification for the coding scheme used in this study.

Differences in depth, details, and organization for subject matter content knowledge and curricular knowledge. Although both frameworks account for subject matter content knowledge and curricular knowledge, the frameworks differ in their treatment of these domains. The *TLO-Math* framework goes beyond Males (2011) and Shulman (1986) in its depth and details for subject matter content knowledge; however, the depth, details, and attention to curricular knowledge are lacking. In their first dimension, Mathematics Content Knowledge for Teaching, subject matter content knowledge is viewed as being in line with Ball, Thames, and Phelps' (2008) specialized content knowledge, “mathematical knowledge not typically needed for purposes other than teaching” (p. 400) and knowledge that is even more specialized than a mathematician’s content knowledge. This same dimension also contains the only attention given to curricular knowledge in the framework; Quebec Fuentes and Ma (2018) view “the inclusion of, flow of, and connections between lessons and units” (p. 359) as a way to develop subject matter content knowledge. In this way, the *TLO-Math* framework only attends to curricular knowledge as a pathway to subject matter content knowledge and not a separate domain on its own merit.

In contrast, Males’ (2011) framework lacks the depth and details in subject matter content knowledge but compensates for it in the domain of curricular content knowledge.

An entire domain of Males' framework is devoted to curricular knowledge and includes sub-categories to consider supports specifically for curricular philosophy, overview, features, storyline, and goals. Capturing supports for curricular philosophy, overview, and storyline in Males' framework may also allow teachers to see the developer's pedagogical judgments, one of Davis and Krajcik's (2005) "high-level guidelines." These types of supports are essential to helping students and teachers see mathematics as a connected discipline (Fosnot & Jacob, 2010 and Ma, 2010; cited in NCTM, 2014), rather than as discrete subjects.

Modernizing Males' pedagogical content knowledge categories. While Males' details about curricular knowledge are informative, aspects of pedagogical content knowledge, differentiation, and technology have been further developed since Males' original work in 2011. As one might expect, both frameworks heavily emphasize developing a teacher's pedagogical content knowledge. Within their respective domains or dimensions, both frameworks specifically attend to supports for anticipating and using students' mathematical thinking, addressing student misconceptions, utilizing questioning to further student ideas, incorporating participation structures, choosing appropriate instructional representations, and drawing upon some of the desired disciplinary practices in mathematics (e.g., NCTM Process Standards and CCSS Standards of Mathematical Practice).

Although only seven years passed between the development of Males' framework (2011) and the *TLO-Math* framework (2018), influential works such as NCTM's *Principles to Actions* (2014) and the CCSS Standards of Mathematical Practice (SMPs) (National Governors Association Center for Best Practices & The Council of Chief State

School Officers, 2010) were brought to the fore in the mathematics education community during this time period. As such, some categories contained in Males' framework allude to elements of pedagogical content knowledge described in these works but is unable to capture them in the same spirit and using the same language as the *TLO-Math* framework. Consider, for example, the excerpt from Males' *Pedagogical Content Knowledge for Mathematics Topics* domain in Table 4. Enactment and Rationale Guidance for Activity Sequences and Experiences combined with the Enactment Guidance for Anticipating and Using Student Ideas anticipate the Mathematics Teaching Practices (NCTM, 2014) related to implementing tasks that promote reasoning and problem solving, facilitating meaningful mathematical discourse, and eliciting and using evidence of student thinking. However, Males' framework is in need of modernization to more fully encompass broader dimensions and language contained in *Principles to Actions* and the SMPs.

In the domain of pedagogical content knowledge, the *TLO-Math* framework attends to differentiated instruction and technology use. According to Quebec Fuentes and Ma (2018), content supports for developing teacher knowledge of differentiated instruction should focus on “*opportunities to learn meaningful mathematics and differentiated supports* [that incorporate] problem-solving, discussions, and high-level tasks” (p. 363, italics in original). Males' framework does not capture this type of teacher knowledge of differentiated instruction. In terms of technology use, Males' framework does capture supports for using mathematical tools (e.g., rulers, compasses, calculators), but stops short from capturing the strategic technology use that builds students'

mathematical understanding known as technological pedagogical content knowledge (TPCK) that is captured in the *TLO-Math* framework.

Coding scheme in this study. Although the *TLO-Math* framework and Males' framework are both rooted in what is known about educative curriculum materials (Ball & Cohen, 1996; Davis & Krajcik, 2005), the frameworks differ in the aspects of teacher knowledge that are illuminated. What is captured quite well by one framework may be minimally captured, captured in another way, or not captured at all by the other framework. In addition, these frameworks may adequately capture educative features in NCTM Standards-based texts or more "traditional" (Stein, Remillard, & Smith, 2007) texts; however, the frameworks may not adequately capture educative features in curricula like Eureka Math where physical publishing restraints are virtually eliminated and new web-based features are possible.

The coding scheme used in this study (see Table 5 for a full description of codes) seeks to update Males' framework by drawing upon the dimensions of teacher knowledge of disciplinary discourse and differentiated instruction from the *TLO-Math* framework. Males' framework was chosen as the basis for my framework due to the specificity of the categories in each domain as well as the attention to specific aspects of curricular knowledge. Categories in Males' framework related to anticipating and using students' mathematical ideas and engaging in students' questions have been updated and combined to reflect the meaningful mathematical discourse described in *Principles to Actions* and included in the *TLO-Math* framework. Using the phrase meaningful mathematical discourse encompasses a broader range of ways that the written curriculum may promote mathematical learning of the whole class; that is, as students have opportunities to share

their thinking, clarify understandings, construct arguments, utilize mathematical vocabulary, and consider others' viewpoints, the discourse created in the classroom promotes the mathematical learning of all. In addition, the dimension of teacher knowledge of differentiated instruction in the *TLO-Math* framework inspired the need to update a portion of Males' *Pedagogical Content Knowledge for Math Topics* domain.

The possible student pitfalls and problematic experiences addressed in Males' framework should be broadened to consider the many ways students engage in the mathematics specific to each topic. Using the term differentiation allows for wider attention to be given to the many ways that the written curriculum can promote teacher knowledge of students' diverse learning needs.

Table 5
Coding scheme for the study

Subject Matter Content Knowledge

Enactment Guidance for:

Content: Inclusion of subject matter beyond the level needed by students (E-Content)

Rationale Guidance for:

Content: Why subject matter beyond the level needed by students is necessary (R-Content)

Pedagogical Content Knowledge for Mathematics Topics

Topic-Specific Mathematics

Enactment Guidance for:

Experiences: Engaging students in instructional activities that reveal the mathematics rather than tell the mathematics or engaging students in experiences unique to a mathematics classroom (e.g., choral counting, using manipulatives to discover the mathematics). (E-Experiences)

Differentiation: Considering the ways students may have difficulty or need further challenge with specific mathematical experiences, accounting for possible misconceptions, or providing scaffolded learning. (E-Differentiation)

Activity Sequences: Engaging in an ordered sequence of instructional activities to build mathematical knowledge. Could be a particular sequence within an activity or problem or across multiple problems or activities to achieve a mathematical goal. If the activities could have taken place in any order, do not use this code. (E-Sequence)

Rationale Guidance for:

Experiences: Why particular experiences (as described above) are appropriate, including why a particular problem or activity is useful in a students' mathematical development. (R-Experiences)

Differentiation: Why students may have difficulty or need further challenge with specific mathematical experiences, misconceptions, or how tips for scaffolding lead to the mathematical purpose. (R-Differentiation)

Activity Sequences: Why particular activity sequences (as described above) are appropriate or useful. (R-Sequence)

Using Mathematical Instructional Representations

Enactment Guidance for:

Representations: Navigating within and among representations (contextual, physical, visual, symbolic, and verbal), as described in *Principles to Actions* (NCTM, 2014) (E-Representations)

Rationale Guidance for:

Representations: Why particular representations, as described above, are appropriate including the advantages and disadvantages for using particular representations. (R-Representations)

Using Mathematical Tools

Enactment Guidance for:

Tools: Strategically using particular mathematical tools (e.g., ruler, string) or technology in a way that builds students' mathematical understanding. (E-Tools)

Rationale Guidance for:

Tools: Why particular mathematical tools/technology are appropriate including the advantages and disadvantages for using particular tools (R-Tools)

Pedagogical Content Knowledge for Mathematics Practices

Facilitating Meaningful Mathematical Discourse

Enactment Guidance for:

Facilitating Discourse: Engaging students in “a purposeful exchange of ideas” (NCTM, 2014, p. 29) using questions, anticipating and using student ideas, or comparison of solution approaches or arguments in the learning activities or engaging students in mathematical talk (E-Discourse)

Rationale Guidance for:

Facilitating Discourse: Why mathematical talk, questions, students' ideas, and comparison of solution approaches or arguments lead to meaningful learning experiences (R-Discourse)

Engaging Students in Justification, Reasoning, and Proof

Enactment Guidance for:

Justification, Reasoning, & Proof: Using approaches for engaging students in justification, reasoning, and proof (e.g., reasoning about connections between concepts not previously discussed) (E-Reasoning)

Rationale Guidance for:

Justification, Reasoning, & Proof: Why approaches to engaging in justification, reasoning, and proof are appropriate or useful (R-Reasoning)

Developing Students' Mathematical Vocabulary/Terminology

Enactment Guidance for:

Developing Mathematical Terminology: Developing students' mathematical terminology (including vocabulary and notation), recognizing colloquial words that may help students understand context or promote/hinder students' conceptions, or explaining vocabulary terms in context of some concept under study (E-Terms)

Rationale Guidance for:

Developing Mathematical Terminology: Why developing mathematical vocabulary or terminology is important and why particular strategies are appropriate or useful (R-Terms)

Engaging Students in Appropriate Participation Structures

Enactment Guidance for:

Participation Structures: Using appropriate participation structures (e.g., individual, small-group, whole-class) (E-Structures)

Rationale Guidance for:

Participation Structures: Why particular participation structures are appropriate or useful (R-Structures)

Curricular Content Knowledge

Enactment Guidance for:

Curricular Overview: Using unit, section, or lesson content in the intended ways (by providing a description of what will unfold in the unit, section, or lesson) (E-Overview)

Curricular Features: Using features of the curriculum (e.g., guidance for how or when to use extraneous materials to achieve a lesson goal) (E-Features)

Curricular Storyline: Connecting previous and future mathematics content, including within and across disciplines, courses, units, sections, and lessons (E-Storyline)

Curricular Goals: Using the goals of the unit, section or lesson (including knowing what the goals are and how to achieve them) (E-Goals)

Rationale Guidance for:

Curricular Philosophy: Why particular features carry out the philosophy of the curriculum (R-Philosophy)

Curricular Features: Why curricular features are appropriate or useful (including explaining these features) (R-Features)

Curricular Storyline: Why particular pieces of content or processes are appropriate in the storyline (R-Storyline)

Curricular Goals: Why particular goals are appropriate or useful (R-Goals)

Coding process. Two modules—Grade 3 Module 7 *Geometry and Measurement*

Word Problems and Grade 4 Module 6 *Decimal Fractions*—were coded for this study.

For each module, I began by reading the module overview and topic summaries to get a sense of the objectives of the module and topics and how those objectives were achieved throughout the module; however, I did not code these overview sections until after the lessons for each topic had been coded in order fully understand how the lessons unfolded within each topic. In coding individual lessons, I read the entire lesson first to understand the objective of the lesson and develop a sense of the coherence across the lesson components. Next, I read the lesson sentence by sentence and applied appropriate code(s)

using the aforementioned framework. I coded the modules in MAXQDA, accounting for all applicable codes (see Figure 5 for an example of coded text, Table 6 for examples of each code, or Appendix C for a full coded lesson).

Problem 2: Compare pairs of decimal numbers representing mass.

T: (Place Rice Bag A on the scale.) What is the mass of this bag of rice?

S: Zero point one kilogram. $\rightarrow \frac{1}{10}$ kilogram. $\rightarrow \frac{10}{100}$ kilogram (see image below).

T: Record the mass in the table on the measurement record.

Repeat this process for the remaining bags.

T: (Leave Bag D, weighing 0.46 kg, on the scale.) Which bags are heavier than Bag D? How do you know?

S: Bags B and C were heavier than Bag D. $\rightarrow$ Bag B was 0.65 kg, and Bag C was 0.7 kg. Those numbers are both larger than 0.46 kg, so the bags are heavier.
 $\rightarrow$ I looked at my chart, from left to right. In the tenths column, I could see that Bag A was lighter. It had only 1 tenth. Bags B and C were heavier than D because they both had more tenths.

T: Let's look at Bags B and C. Make a statement comparing their mass.

S: 0.65 kilogram is lighter than 0.7 kilogram.
 $\rightarrow$ 0.7 kilogram is heavier than 0.65 kilogram.

T: How do you know?

S: I could just see that the bag was fuller and feel that the bag has more mass. $\rightarrow$ At first, I thought 65 hundredths was more because it looks like you are comparing 65 and 7, and 65 is greater than 7. But then we saw that it was 7 tenths, which is more than 6 tenths. $\rightarrow$ I realized that 7 tenths is 70 hundredths, and that is greater than 65 hundredths.

T: With your partner, make another statement to compare the bags. You can compare just two items, or you can compare more than two items.

S: (Responses will vary.)

T: Based on these comparisons, what is the mass of the bags in order from heaviest to lightest?

S: 0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg.

T: (Select a student volunteer.) Arrange the bags from heaviest to lightest. Looking at the bags, does it appear that we have properly ordered the bags from heaviest to lightest? Do they match the order we determined?

S: Yes.

NOTES ON TERMINOLOGY:

Mass is a fundamental measure of the amount of matter in an object. While weight is a measurement that depends upon the force of gravity (one would weigh less on the moon than one does on Earth), mass does not depend upon the force of gravity. Both words are used here, but it is not important for students to recognize the distinction in mathematics at this time.

Rice Bag	ones	.	tenths	hundredths
A	0	.	1	0
B	0	.	6	5
C	0	.	7	
D	0	.	4	6

0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg

Figure 5. Sample coded text from Grade 4 Lesson 9 Concept Development.

Table 6
Examples of codes given

Subject Matter Content Knowledge

E-Content

"Mass is a fundamental measure of the amount of matter in an object. While weight is a measurement that depends upon the force of gravity (one would weigh less on the moon than one does on the Earth), mass does not depend upon the force of gravity. Both words are used here, but it is not important for students to recognize the distinction in mathematics at this time." Grade 4 Lesson 9 p. 147

R-Content

No codes given.

Pedagogical Content Knowledge for Mathematics Topics

E-Experiences	"Students work in pairs at a station with 10 circular objects, applying what they learned in Part 1 to find the perimeters of those objects. (There may be more than one pair per station.) They use string and a ruler as tools...Remind students that they are working with a partner and that they need to be sure the work is done cooperatively." Grade 3 Lesson 16 p. 209
E-Differentiation	"A centimeter stencil that students can easily trace may be beneficial. In addition to having students interact with a to-scale centimeter (such as a cube), it may help to project teacher modeling with an overhead projector or document camera, if available." Grade 4 Lesson 1 p. 16
E-Sequences	"See Lesson 1 for the directions for administration of a Multiply-by Pattern Sheet. T: (Write $7 \times 7 = \underline{\quad}$.) Let's skip-count up by sevens. I'll raise a finger for each seven. (Raise a finger for each number to track the count.)...T: (Distribute the Multiply by 7 Pattern Sheet.) Let's practice multiplying by 7. Be sure to work left to right across the page. Grade 3 Lesson 26 p. 341-342
E-Representation	"On the measurement record, shade the tape diagram to represent the length of the yellow tape on the meter stick. Write the length of the tape in decimal form." Grade 4 Lesson 9 146
E-Tools	"T: (Distribute the white string.) Work with your partner to wrap the string around the outside edges of your square...S: (Switch roles and mark the string.)" Grade 3 Lesson 10 p. 141
R-Experiences	"While this strategy has more steps than the strategy in the vignette, it does not require students to know or figure out half of the perimeter. Finding half of the perimeter can become tricky when students start to work with larger perimeters." Grade 3 Lesson 20 p. 270
R-Differentiation	"Adjust the Order Decimal Numbers fluency activity so that English language learners gain more practice in oral comprehension and transcribing." Grade 4 Lesson 13 p. 196
R-Sequences	"This activity builds fluency with multiplication facts using units of 7. It works toward students knowing from memory all products of two one-digit numbers. Grade 3 Lesson 26 p. 341
R-Representation	"While converting decimals to fractions before adding may not seem as quick as just adding decimals (e.g., 0.3 and 0.57), doing so strengthens student understanding of the fraction and decimal relationship, increases their ability to think flexibly, and prepares them for greater success with fractions and decimals in Grade 5." Grade 4 Lesson 13 p. 197

R-Tools "The purpose of using personal white boards is for ease of sharing work. If necessary, slip out the Problem Set, and show the work on the clear plastic sleeve using an overhead projector." Grade 3 Lesson 1 p. 16

Pedagogical Content Knowledge for Mathematics Practices

E-Discourse "Facilitate a discussion in which students analyze this work. Choose any combination of the following questions to help guide the conversation:
Was the drawing helpful? What makes the drawing helpful or unhelpful?
Did Student A represent all the important information in his drawing? Why or why not?
... What suggestion would you make to Student A to improve his work?"

Grade 3 Lesson 3 p. 41-42

E-Reasoning "Project the sequence below, and ask students to compare using $<$, $>$, or $=$. With each pair of numbers, ask students to share their reasoning with a partner. They may use the area model, a number line, a place value chart, or other reasonable strategies.

6.24 _____ 5.24

...2.31 _____ 23 tenths 5 hundredths" Grade 4 Lesson 10 p. 161

E-Terms "T: Use your finger to trace around the edge of the piece you cut out. We call the boundary of the shape its **perimeter**. Say the word to yourself as you trace. S: Perimeter. (Trace with finger.)" Grade 3 Lesson 10 p. 142

E-Structures "T: (Invite a few students to the front of the room. Distribute two to three bags to each student." As we add each bag, count and see how the scale shows the weight in decimal form, and record it on your number line." Grade 4 Lesson 1 p. 15

R-Discourse "Select two or three groups to share their thinking with the rest of the class. Choose groups strategically to spark discussion and push learning in terms of both modeling and oral explanation. Selections could include a group with an exemplary choice, a group with an unusual choice, or a group with an excellent explanation." Grade 3 Lesson 2 p. 29

R-Reasoning No codes given.

R-Terms "Mass is a fundamental measure of the amount of matter in an object. While weight is a measurement that depends upon the force of gravity (one would weigh less on the moon than one does on the Earth), mass does not depend upon the force of gravity. Both words are used here, but it is not important for students to recognize the distinction in mathematics at this time." Grade 4 Lesson 9 p. 147

R-Structures	"T: Work with your group to draw at least two different ways to represent the problem...Each of you has a different color marker so that your participation shows on your poster. Make sure each member of your group contributes." Grade 3 Lesson 2 p. 28
<i>Curricular Content Knowledge</i>	
E-Overview	"In today's lesson, students apply their skill with adding decimals by first converting them to fraction form. The first two problems are single-step problems. Encourage students to use the RDW process because, in doing so, they again realize that part-whole relationships are the same whether the parts are whole numbers, fractions, or mixed numbers." Grade 4 Lesson 14 p. 208
E-Features	"Project Template 1 as shown, and give a copy of the shape to each pair of students." Grade 3 Lesson 6 p. 84
E-Storyline	"Topic C presents two possible optional lessons related to perimeter, including this lesson. The second option is a culminating lesson using the text below, along with the lesson plan found at the end of that text...With the Marilyn Burns option, the recommended sequence is after Lesson 17 and before the Mid-Module Assessment because the lesson explores the relationship between perimeter and area." Grade 3 Lesson 11 p. 150
E-Goals	"This problem is designed to bridge learning from prior lessons and lead up to the Concept Development for the current lesson." Grade 3 Lesson 9 p. 126
R-Philosophy	No codes given.
R-Features	"Some problems do not specify a method for solving. This is an intentional reduction of scaffolding that invokes Mp.5, Use Appropriate Tools Strategically." Grade 4 Lesson 1 p. 17
R-Storyline	"This Application Problem anticipates decimal fraction addition and reinforces the concept of how many more to make one." Grade 4 Lesson 3 p. 41
R-Goals	"Comparing decimals in the context of measurement supports their justifications of their conclusions and begins their work with comparison at a more concrete level." Grade 4 Topic C Overview p. 141

Using the features of MAXQDA, I was able to nest the codes by color within their category of content knowledge. As one can see, this coded excerpt contained codes from every category of content knowledge in the coding scheme, with a diversity of codes across *Pedagogical Content Knowledge for Mathematics Practices*. This excerpt also

contained the only code in *Subject Matter Content Knowledge* in the entire study and provides an example of the guidance found outside of the vignettes in boxes along the margin. In addition, this excerpt contained a sample measurement record that a student would be completing during this lesson. Although the picture of the record was not coded in this case, this feature of the curriculum was coded for at the beginning of the excerpt.

Nearly every lesson in these modules contained vignettes of sample teacher-student interactions of how to teach the concepts in the lesson; that is, not only did the lessons include what to teach, but they also included detailed recommendations for how to teach the concepts. For guidance to be coded as Enactment Guidance, both what to teach and how to teach it needed to be present. In Figure 5 above, the concept being taught was comparing decimal numbers representing mass. The vignette contained the detailed recommendations for how a teacher may utilize representations of mass, participation structures, discourse, reasoning, and features of the curriculum to teach comparison of decimal numbers that allowed for pieces of the vignette to be coded as Enactment Guidance. In contrast, some lessons, particularly those labelled as problem-solving lessons, contained minimal Enactment Guidance. Figure 6 below shows an excerpt from Grade 3 Lesson 23 that did not receive any codes from the coding scheme. Although the excerpt contains the problems to work with students, it does not contain any guidance for teachers on how to enact these problems with students.

Problem 1: Gale makes a miniature stop sign, a regular octagon, with a perimeter of 48 centimeters for the town he built with blocks. What is the length of each side of the stop sign?

Students might solve by first drawing an octagon, then labeling a side length with a letter, and dividing the perimeter (48 cm) by the number of sides on an octagon (8).

Problem 2: Travis bends wire to make rectangles. Each rectangle measures 34 inches by 12 inches. What is the total length of the wire needed for two rectangles?

This is a two-step problem. Students find the perimeter of one rectangle and then add to find the perimeter of two rectangles. They should recognize that the total perimeter of two rectangles is equal to the total length of wire needed. To solve the first step, students might draw a rectangle, label the side lengths, and find the perimeter. In the second step, students might use a variety of strategies to solve $92 + 92$, including using the standard algorithm or adding like units: 9 tens plus 9 tens equals 18 tens, and 2 ones plus 2 ones equals 4 ones, and 18 tens 4 ones equals 184.

Figure 6. Excerpt of Concept Development receiving no codes.

Additionally, I utilized the comments area of the code in MAXQDA to describe the nature of the feature. The presence of comments for a particular code is indicated by the filled-in circle next to the code name. In Figure 5 above, the comment on the E-Representation code at the beginning of the excerpt includes the type of representations (symbolic and physical) utilized in this code, while the comment on the E-Features code provides guidance to utilize the template included in this lesson. As I included these comments, I also attended to instances where the nature of the code seemed similar to other instances in the written curriculum. For example, I repeatedly used the phrase “choral counting” in the comments area to describe the nature of the Experience code in this instance.

Trustworthiness of codes. In phase one, I used two additional coders who were doctoral candidates in mathematics education to check the trustworthiness of my codes. Each of these additional individuals coded a random sample of four lessons in addition to at least one of the target lessons chosen in phase two. When their coding was complete, we met face-to-face to compare codes assigned and resolve any disagreements in coding or difficulties with the coding scheme. The nature of the disagreements seemed to differ by coder; that is, disagreements with one coder were focused on where a particular code

should begin and end, while disagreements with the second coder were focused on elements of the coding scheme that seemed difficult or unclear to them.

For example, in Figure 5 above, consider the first E-Reasoning code in the excerpt. While one of my additional coders just coded the sentence of “How do you know?” I coded both the original statement by the teacher and the response by the student in order to capture the reasoning and justification that was meant by the teacher-student interaction in the vignette. Similarly, for the E-Structures code at the bottom of the excerpt, the same additional coder just coded the sentence “Select a student volunteer” while I coded the remainder of this teacher-student interaction. These choices were intended to allow for follow-up analysis on the nature of these codes and others, which the additional coder agreed was important to answering the research question.

Disagreements with the second coder were focused on interpretation of the coding scheme. This coder experienced some of the same dilemmas I had experienced during my initial coding, which prompted much discussion with both coders about what was meant by some of the codes and how those may appear in this curriculum. For example, one area of discussion centered on codes for engaging students in Experiences. For this particular code, the second coder experienced difficulty differentiating Experiences from traditional instructional activities. Our discussion revealed that the original description, “Engaging students in particular experiences,” was not descriptive enough and, as written, could encompass any instructional activity. As a result, the description was altered to indicate that Experiences codes were reserved for those instructional activities intended to reveal, rather than tell, the mathematics or those that would be unique to a mathematics classroom.

Similar discussions for the codes for Activity Sequences, Representations, and Tools and brief comments about wording in other codes resulted in the additional coders and I revising the coding scheme as we met. After achieving consensus on the coding scheme, I reviewed the modules a second time, re-coding portions, if necessary, to reflect the revised coding scheme.

Cross-lesson data analysis. Once the final coding was complete, the codes in each lesson and topic or module overview were compiled for analysis in order to answer the first research question, “What educative features are present in upper elementary Eureka Math curriculum?” Because this question seeks investigation of the nature of the features within this curriculum rather than an accounting for and comparison of features to other curricula, analysis proceeded by collapsing codes to the lesson or overview in which they were located.

The first round of analysis focused on investigating trends in the educative features within and across categories of content knowledge as well as within and across grade levels. This round of analysis also influenced which lessons were chosen to be target lessons for phase two of the study. To aid in this round of analysis, an Excel spreadsheet was created categorizing the codes in each lesson or overview by their category of content knowledge. Table 7 contains an excerpt of this spreadsheet for the lessons in Topic B of third grade in the category of *Pedagogical Content Knowledge for Mathematics Topics*. An “x” indicates that a particular code was found in the lesson.

Table 7

Excerpt of analysis spreadsheet showing codes in lesson organized by category of content knowledge

Pedagogical Content Knowledge for Mathematics Topics						
Topic-Specific Mathematics						
<i>Lesson</i>	Enactment Guidance			Rationale Guidance		
	<i>Experiences</i>	<i>Differentiation</i>	<i>Sequences</i>	<i>Experiences</i>	<i>Differentiation</i>	<i>Sequences</i>
4	x	x	x	x	x	x
5	x	x	x	x		x
6	x	x		x		
7	x	x	x	x		x
8	x	x	x	x		x
9	x	x	x	x		x

Note. An “x” in the table indicates that a particular code was found in the lesson.

Investigation of trends in the full spreadsheet also led me to more closely analyze the individual codes in MAXQDA by creating queries for each individual code and sorting the retrieved segments according to the comments I had given. This sorting revealed instances of codes that contained similar, if not identical, wording across lessons and grade levels. The Student Debrief from Grade 3 Lesson 4 (see Figure 7) contains examples of some of these instances. Each Student Debrief across both grade levels begins with repeating the lesson objective and similar beginning paragraphs. Although this guidance necessitates a code to be given, the guidance provided is not unique to the lesson, which seemed important to the analysis of the nature of the educative feature. To capture the repetitive nature of these features, I replaced the “x” in the aforementioned Excel spreadsheet with “***,” as seen in the complete spreadsheet in Appendix D.

Figure 7. Beginning paragraphs of coded Student Debrief. Taken from Grade 3 Lesson 4.

Analysis in this second round investigated trends in the location of the educative feature in the lesson across the categories of content knowledge. As a result, the second round of analysis necessitated a second Excel spreadsheet (see Appendix E for complete spreadsheet) to aid in closer analysis of the trends both within and across categories of content knowledge as they pertained to the location of the educative features in the lesson. Table 8 shows an excerpt of this spreadsheet for lessons in Topic B of third grade in the category of *Pedagogical Content Knowledge for Mathematics Topics*. F indicates the feature was found in the Fluency Practice activities, A indicates the feature was found in the Application Problem, and C indicates the feature was found in the Concept Development activities. The full spreadsheet contains a mark of D for features found in the Student Debrief activities as well as a mark of LESSON for features that applied to an entire lesson that were not part of a particular lesson component.

Table 8

Excerpt of analysis spreadsheet showing location of codes in lessons organized by category of content knowledge

Pedagogical Content Knowledge for Mathematics Topics						
Topic-Specific Mathematics						
<i>Lesson</i>	Enactment Guidance			Rationale Guidance		
	<i>Experiences</i>	<i>Differentiation</i>	<i>Sequences</i>	<i>Experiences</i>	<i>Differentiation</i>	<i>Sequences</i>
4	F C	F A	F	F	C	F
5	F C	F C		F		F
6	F C	F C	F	F A		
7	F C	C	F C	F C		F
8	F C	F C	F	F		F C
9	F C	F C	F	F		F

Note. F indicates the code was found in the Fluency Practice of the lesson. C indicates the code was found in the Concept Development of the lesson. A indicates the code was found in the Application Problem of the lesson.

The third round of analysis focused on investigating trends in the educative features by the type of guidance (Enactment or Rationale) both within and across categories of content knowledge as well as within individual codes. This round of analysis was heavily influenced by previous rounds of analysis; that is, seeing the presence of some educative features in nearly every lesson (e.g., E-Experiences and E-Differentiation) in the first round as well as the location of those features within the lesson in the second round provided a clearer picture of the nature of specific educative features in this curriculum. In this round, I drew upon the spreadsheet from the second round of analysis as well as the written curriculum itself and a curriculum overview spreadsheet (see Appendix F) I had created at the beginning of the study detailing the objectives and lesson activities for the lessons in each module.

Phase Two – Curriculum Use

The second phase of this study sought to investigate teachers' curriculum use (Stein et al., 2007) as the educative features present in the written curriculum are incorporated (or not) into the intended and enacted curricula.

Participants. Participants were recruited by attending grade-level meetings to explain the purpose of the study and its expectations for participants. Two teachers at each grade level volunteered to participate. Their teaching backgrounds are presented in Table 9. All participants were veteran teachers and had used the Eureka Math curriculum since its adoption at their school two years earlier. With the exception of Emily, whose undergraduate education was in child development and initial teacher certification was in family and consumer sciences, all participants had completed a Bachelor's degree in elementary education, and Holly had also completed a Bachelor's degree in special education. Several participants had worked toward or completed advanced degrees during their years as a teacher. Rachel had completed several hours of graduate-level work in areas of curriculum and instruction, Kim was finishing her Specialist's degree in school administration, and Emily had completed a Specialist's degree in mental health practices in schools.

Table 9
Summary of teacher participants

Teacher	Grade Level	Years of Teaching Experience	Years at Current Grade Level	Years Teaching Eureka Math
Emily	3 rd	11	2	3 – 1 year at Kindergarten
Kim	3 rd	20	6	3 – all at 3 rd grade
Holly	4 th	9	4	3 – all at 4 th grade
Rachel	4 th	27	22	3 – all at 4 th grade

Across the sample, all participants expressed support for using Eureka Math, even though all participants described how difficult the first year of adoption had been in terms of preparing and enacting lessons as well as guiding students and their parents through the transition. In their pre-study interviews, teachers commented on various benefits and drawbacks they had experienced since adoption. Both Kim and Holly expressed gratitude for how Eureka Math includes complex, multi-step problem solving throughout the curriculum, describing how their current students are unafraid of these types of problems and feel “equipped,” “powerful,” and “empowered” to solve them. Both Emily and Rachel commented on the how they appreciate the sequencing and depth of concepts in the curriculum. Comparing Eureka Math to her own elementary mathematics experience, Emily stated:

I wasn't taught the why. I was taught the steps and never really had a number sense until I started teaching this because I had to teach myself how to teach this and now I understand why it all works and I wish I would have been taught that... It was just very rote and it was routine instead of actually understanding why it works, which is what I love about Eureka Math. (pre-study interview, March 19, 2019)

All participants expressed frustration with adapting their lessons to fit the time constraints they experienced; that is, the participants found it challenging to include all lesson components (Fluency Practice, Application Problem, Concept Development, and Student Debrief) and keep their mathematics lesson close to an hour. Over time, the participants have become more selective in their instructional activities and continually look for ways to improve their instruction. One aspect of Rachel's instruction that she

continues to work on is to find new ways to increase student engagement in the lessons as she feels that the lessons presented in the curriculum are “very much just sit and get kind of stuff” with “no group work, no physical activity getting up out of your seat to do something.”

A notable difference between the third- and fourth-grade levels is their organization of the grade levels by class. While each class at third grade is completely self-contained where one teacher teaches all subjects to the same set of students, the fourth grade is a mix of self-contained and departmentalized. The fourth grade consists of four departmentalized classes and a fifth class that is completely self-contained and whose teacher was not a participant in this study. In the departmentalized classes, Rachel and Holly teach mathematics and science while the other two fourth-grade teachers teach English language arts and social studies. Therefore, each teacher at the fourth-grade level was observed for two enactments of each target lesson while each teacher at the third-grade level was observed for a single enactment.

Choice of target lessons. After the initial coding of the written curriculum, I identified several lessons for enactment analysis according to the diversity of content supports in the lessons. I selected two lessons per module for enactment analysis with the primary consideration given to lessons with diverse content supports coded and secondary considerations given to how the particular lesson represented the span of topics in the module, the teachers’ recommendations for lessons in the module, and the timeline for carrying out the study. Once a lesson was chosen, the written materials were also examined for key educative features contained in the lesson that aided in identifying teachers’ uptake of content supports. Table 10 shows the key educative features chosen

for each target lesson as well as the location (Fluency Practice, Application Problem, Concept Development, or Debrief) of the educative feature in the written curriculum.

Table 10

Key educative features for target lessons observed in phase two of the study

Key educative feature	3 rd Grade		4 th Grade	
	Lesson 4	Lesson 23	Lesson 3	Lesson 9
<i>Pedagogical Content Knowledge for Mathematics Topics</i>				
E-Experiences	F C		F C	C
E-Differentiation	F A	C	C	
R-Differentiation	C			
E-Sequences	F			
R-Sequences	F			
E-Representation	A	C	F C D	F C D
R-Representation	A			
E-Tools	C		C D	C
R-Tools	C			
<i>Pedagogical Content Knowledge for Mathematics Practices</i>				
E-Discourse	C	C	C	C
E-Reasoning	C		D	C D
E-Terms	C D		C D	C
E-Structures	C	C D	C D	C
<i>Curricular Content Knowledge</i>				
E-Features	C		C D	F C D
R-Storyline	F A	F C	F A	F A

Note. F indicates the code was found in the Fluency Practice of the lesson. A indicates the code was found in the Application Problem of the lesson. C indicates the code was found in the Concept Development of the lesson. D indicates the code was found in the Student Debrief of the lesson.

Grade 3 lessons. At the 3rd grade level, Lesson 4 and Lesson 23 were chosen as target lessons. Lesson 4 was chosen for multiple reasons. First, this lesson contained 18 of 28 codes, the highest amount of codes in any grade three lesson in phase one of the study. Second, both third-grade teachers recommended this lesson due to the nature of its activities. They also thought this lesson was representative of the other lessons in the module focusing on attributes of polygons. From my perspective, this lesson offered opportunities to investigate teacher's uptake of diverse educative features, especially in

the category of *Pedagogical Content Knowledge for Mathematics Topics*. In addition, this lesson offered opportunities to see the enactment of choral counting activities and Pattern Sheets in the Fluency Practice components, a sorting activity and analysis of attributes of quadrilaterals in the Concept Development component, and a focus on developing mathematical terminology and mathematical talk with partners throughout the lesson.

Lesson 23 was uniquely positioned after students had explored the concept of perimeter as the boundary of a shape using a variety of polygons and circles and after they had used unit square tiles to construct rectangles of a given perimeter. Lesson 23 was chosen as a target lesson due, in part, to its connections to the span of topics in the module as well as its focus on problem-solving, which one third-grade teacher had described as a major strength of the written curriculum. From my perspective, the Concept Development of this lesson in the written curriculum was wholly different than other prior lessons in the module; that is, this lesson, described as a “problem-solving lesson,” suggested a three-step problem-solving process in which the students would work in pairs or independently on the Problem Set problems while the teacher facilitated conversation among students demonstrating problems at the board and the full class. Additionally, this lesson was one that teachers had never taught before due to coming up against the state testing window, which offered a unique opportunity to observe teachers plan, enact, and reflect on a new lesson.

Grade 4 lessons. At the 4th grade level, Lesson 3 and Lesson 9 were chosen as target lessons. Lesson 3 was chosen primarily because it contained codes for all but one code for Enactment Guidance in categories of pedagogical content knowledge. This

lesson was also recommended by the teachers as one that would offer opportunities to gain insight into concepts and representations of place value and fractions learned in prior modules. Additionally, the teachers thought this lesson encompassed many of the concepts, activities, and representations contained in the first half of the module. From my perspective, the enactment of this lesson could have been rote and procedural due to the instructional objective and the nature of the instructional activities even though the lesson activities included opportunities for students to build mathematical knowledge through the use of manipulatives and opportunities to talk and reason about representations of decimal fractions. In terms of this study, observing and analyzing the enactment of this lesson seemed important in investigating how teachers were taking up the educative features contained in the written curriculum.

Lesson 9 was chosen for multiple reasons. First, this lesson contained 14 of 28 codes, the highest amount of codes in any grade four lesson in phase one of the study. Second, Rachel described this lesson as one of the more engaging lessons in this module. For her, engaging lessons were ones that included group work or physical activity to learn the mathematics rather than “sitting in our seats and doing stuff.” Upon closer examination of the lesson’s activities, I agreed with Rachel. This lesson, whose objective was targeted at comparison of decimals, incorporated a variety of mathematical tools and physical representations to compare measurements expressed as decimals. Moreover, the lesson included multiple opportunities for student involvement and reasoning about comparison of decimals.

Data collection. Video and audio data from teacher participants and their classrooms were collected during teacher pre-study interviews, target lesson planning

meetings, target lesson enactments with students, target lesson reflection meetings, and a post-study questionnaire. In addition, my personal field notes from the lesson enactments as well as digital artifacts such as Google Slides presentations from teachers were collected. I describe these in more detail below.

The pre-study teacher interview (see Appendix G for interview protocol) focused on obtaining information about the teacher's background generally as well as her experiences teaching from the Eureka Math curriculum and others. Additionally, this pre-study interview was used to see how the teacher would typically prepare the intended curriculum and plan for the enacted curriculum. Using the first lesson in each module, I asked teachers to describe how they would plan for this lesson. My approach was the clinical interview method (Ginsburg, 1997). The flexible questioning techniques in the clinical interview method allowed for insight into each individual teacher's curriculum use. Pre-study interviews were transcribed and imported into MAXQDA for analysis. Transcripts of the pre-study interviews were open coded in order to capture key themes in teachers' curriculum use in general (Creswell, 2009).

Video and audio recordings of planning and reflection meetings with teachers were collected. These meetings were held largely because of the study; however, there is evidence that all teachers typically spend time, albeit less formally, planning and reflecting on their lessons. Planning meetings focused on observing teachers talk through the written curriculum in terms of what they planned to do with their students with my questioning focused on clarification rather than rationales so as not to influence the teachers' decision-making process about what to include or exclude from the written curriculum. Reflection meetings focused on sharing video of the enactment of some key

educative features with the teachers and asking them about some aspect of the enactment. In order to prepare for this meeting, I watched the enactment videos, took additional notes on my field notes, and prepared clips with accompanying questions to discuss with the teachers. These questions may have focused on affordances or constraints of particular key educative features, rationales for including or eliminating certain activities in the lesson, or aspects of student interaction observed in the video. Recordings of the planning and reflection meetings were transcribed, imported into MAXQDA, and coded according to the key educative features previously described. In addition, analytic memos were written and organized in an Excel spreadsheet for each teacher by the key educative feature being observed as well as the phase of curriculum use in which it was observed. Table 11 shows an excerpt from Kim's spreadsheet for Lesson 4.

Table 11

Excerpt of Kim's spreadsheet of analytic memos from Lesson 4

Planning	Enactment	Reflection
<i>E-Experiences</i>		
F: Not doing Equivalent Counting by Units of 5 because "we can all count by 5's just fine." CD:	F: Does one string of choral counting to count up to 28, but that is all. CD: The sorting experience here is governed by the term under study, starting with identifying all of the shapes with right angles and continuing with parallel lines. Students do get involved by going to the board with their index card to check on right angle or not.	F: Wanted to move on because the amount of time needed for the CD. Also standards-based grading dictates that kids are already fluent in 2's, 5's, and 10's, so choosing Multiply by 4 over Counting by 5's was main factor. CD: Would not cut out the shapes. Has considered getting out the quadrilaterals in the pattern blocks and using index cards or rulers to determine right angles and parallel lines. Talks about how Zearn has a really great representation of the sorting. Went to Zearn the next day because she wanted to beef up the sorting.
<i>E-Differentiation</i>		
F:	F: Does ask students to work across Pattern Sheet.	F: 18:30 talked about differential processing for students in the Pattern Sheets
App:	App: Not observed...students seemed to understand the less than language.	App:

Note. F indicates the memo is from the Fluency Practice of the lesson. App indicates the memo addresses the Application Problem of the lesson. CD indicates the memo addresses the Concept Development of the lesson.

Video and audio recordings during the target lesson enactment focused on both whole-group and small-group activities. Much of the video recordings focused on whole-group interaction; however, either myself or a trained videographer was present at each lesson enactment to capture small-group activities or board work. Two audio recorders were also utilized at each enactment to capture voices that were further away from the video camera. These recordings were viewed and timestamped according to the key educative feature observed. Analytic memos for the enactment were written for each key

educative feature observed and added to the Excel spreadsheet of analytic memos for the planning and reflection meetings (see Table 11) to create coherence across the phases of curriculum use for each lesson and key educative feature.

Field notes were captured on printed copies of the written curriculum and focused on copying work on the board or document camera, small group topics for further examination, and questions I had while observing the enactment. In addition, I took note of some affective factors observed during enactment (e.g., “kids seem ok with not finishing the Problem Set – no sad faces, grumbling, etc.”).

A post-study questionnaire was sent to teachers through email and included the following questions:

1. How does your knowledge of the curriculum as a whole influence your planning and enacting the curriculum with students?
2. What have you learned (could be about mathematics and/or pedagogy) through using this curriculum and from what aspect of your work (e.g., enacting lessons, examining student work, etc.) as a teacher did that learning come?
3. In what ways has your learning been limited by using this curriculum?
4. Oftentimes personal factors (e.g., experience or beliefs or values about mathematics education) and organizational factors (e.g., classroom context, school context) contribute to a teacher's curriculum use. How would you describe the biggest influencers of your curriculum use?

Although individual post-study interviews would have been most desirable for this study, timing and organizational factors inhibited the feasibility of individual interviews. By the

time that reflection meetings for the final target lessons were completed, all teachers in this study were in the midst of preparing their classrooms to be moved to the new upper elementary school building addition that was recently completed and were quite reluctant to devote undesignated time at school to completing the post-study interview originally intended. Weekly staff meetings regarding the upcoming move also limited opportunities to meet with these teachers. To mitigate the limitations of a written questionnaire instead of a follow-up interview, the teachers were quite willing to provide additional information and insight should the need arise throughout the remainder of the study.

Data analysis. Analysis during the second phase of the study was intended to answer the second research question, “How do teachers take up the educative features in their planning and enactment of Eureka Math curriculum?” This study drew upon the definition for take-up of educative features used by Davis and colleagues (2017), who defined uptake of educative features as “the ways teachers adopt, in language and/or action, ideas or practices recommended in the educative features” (p. 294). This definition allowed for both language and actions to serve as evidence that teachers take up educative features; therefore, any language or action used by teachers that align with educative features in the written curriculum will be considered as evidence of take-up. Being able to describe teachers’ curriculum use of educative features with the Eureka Math materials has the potential to inform what is known about educative features in general (Davis & Krajcik, 2005) and teacher uptake of educative features (Davis, Palincsar, Smith, Arias, & Kademian, 2017) as well as contribute to knowledge of educative features specifically in mathematics curriculum (Drake et al., 2014; Males, 2011; Quebec Fuentes & Ma, 2018).

Throughout all rounds of analysis, the key educative feature served as the unit of analysis. The descriptions of the rounds of analysis below illustrate how the constant comparative method (Glaser & Strauss, 1967) was used to arrive at themes of take up for the key educative features. The first round of analysis focused on investigating how educative features in the same lesson were taken up by different teachers at the same grade level. The aforementioned Excel spreadsheet of analytic memos was reorganized according to the key educative feature by lesson (e.g., E-Discourse codes for Grade 3 Lesson 4) to aid in looking across the teachers' planning, enactment, and reflection. Notable similarities and differences in the take up of the key educative features were recorded in a separate analytic memo. For example, in a memo from this round of analysis, I wrote, "In Lesson 4, students in Emily's class are able to talk to each other about how they are sorting their quadrilaterals. This stands in contrast to Kim's class where she seems to be leading the generation of ideas and topics of discussion." Additionally, I consulted enactment video and meeting transcripts to verify patterns seen in the analytic memos.

The second round of analysis focused on investigating how educative features were taken up across both lessons within the same grade level. Again, the Excel spreadsheet of analytic memos was reorganized to include all target lessons within the same grade level (e.g., E-Experiences for Grade 4) with notable similarities and differences being recorded in a separate analytic memo. Attention in this round of analysis was also given to examining patterns of take up by key educative feature for each teacher. To illustrate this analysis, Lessons 3 and 9 at the fourth-grade level, there were nine common key educative features previously identified. This round of analysis

provided an opportunity to identify patterns of take up for Rachel or Holly across lessons that may give insight into their take up of educative features in general. Analysis from these first two rounds greatly informed the final analysis looking at the take up of the key educative features across both grade levels.

The third round of analysis focused on investigating how educative features were taken up across both grade levels. In this analysis, classification by teacher and grade level were eliminated with primary attention given to the key educative feature as a whole. As a result, the list of key educative features was collapsed to the 10 features (see Table 12) found in at least three of the four target lessons.

Table 12

Key educative features found in at least three of the four target lessons observed in phase two of the study

Key educative feature	3 rd Grade		4 th Grade	
	Lesson 4	Lesson 23	Lesson 3	Lesson 9
<i>Pedagogical Content Knowledge for Mathematics Topics</i>				
E-Experiences	F C		F C	C
E-Differentiation	F A	C	C	
E-Representation	A	C	F C D	F C D
E-Tools	C		C D	C
<i>Pedagogical Content Knowledge for Mathematics Practices</i>				
E-Discourse	C	C	C	C
E-Reasoning	C		D	C D
E-Terms	C D		C D	C
E-Structures	C	C D	C D	C
<i>Curricular Content Knowledge</i>				
E-Features	C		C D	F C D
R-Storyline	F A	F C	F A	F A

Note. F indicates the code was found in the Fluency Practice of the lesson. A indicates the code was found in the Application Problem of the lesson. C indicates the code was found in the Concept Development of the lesson. D indicates the code was found in the Student Debrief of the lesson.

Although this round of analysis is worthy of consideration in its own right, themes of take up in the first and second rounds of analysis allowed for examination of the nuances of

take up as influenced by the instructional activities and the individual teachers enacting the curriculum.

Trustworthiness. To ensure trustworthiness of findings, several methods were employed. First, data was collected over a prolonged period of time and trust was built with the participants (Lincoln & Guba, 1985). Although only two lessons were observed, I met with each participant no less than six times throughout the study. Moreover, being present at all but one lesson enactment allowed the participants to become comfortable with me being in their classroom and teaching in ways that were typical for them. Second, peer debriefing and member checks were utilized throughout analysis (Creswell, 2009; Lincoln & Guba, 1985). I met with my advisor on several occasions during data collection and analysis to discuss ongoing analysis. These debriefings led to further refinement of my methodology and claims. The reflection meetings with study participants served as a type of member check during the study. At these reflection meetings, participants were able to verify or deny their intentionality, correct interpretation errors I had made, and offer additional information regarding the lesson enactments. All of these serve to establish trustworthiness of the findings in phase two.

CHAPTER 4: RESULTS AND FINDINGS OF THE WRITTEN CURRICULUM

The purpose of this study is to describe the educative features present in Eureka Math curriculum materials and subsequently investigate how teachers are taking up these educative features in the classroom. This chapter focuses on characterizing the educative features present in the written curriculum materials. I begin by sharing the findings as revealed through the coding scheme used, followed by highlights of the findings I consider the most important as well as descriptions of how these findings relate to the various components of the written curriculum.

Results of the Written Curriculum

As described in the previous chapter, each Eureka Math module contains not only the individual lessons but also an overview for the entire module and a summary for each topic. Thus, there were 36 opportunities (1 module, 5 topics, and 30 lessons) for each code to appear in sections of the third-grade module and 22 opportunities (1 module, 5 topics, and 16 lessons) for each code to appear in sections of the fourth-grade module. However, not all lessons contained unique opportunities for codes to appear. In particular, the Problem Set (a portion of the Concept Development), Student Debrief, and Exit Ticket components (see Appendix C for coded lesson) contained identical or similar wording from lesson to lesson, necessitating a code to be given in each instance but not necessarily providing a unique piece of guidance for teachers. For example, the Student Debrief typically begins with the sentence, “The Student Debrief is intended to invite reflection and active processing of the total lesson experience.” While this sentence provides a rationale for the debrief experience, its presence in nearly every lesson does not provide a unique piece of guidance for teachers. As a result, I made adjustments to

five codes (E-Experiences, E-Differentiation, R-Experiences, E-Structures, and E-Features). Table 13 shows the number of sections receiving each code, as well as adjustments to reflect unique opportunities, organized by grade level and type of content knowledge (see Appendix D for full data set).

Table 13

Number of sections containing features by grade level and type of content knowledge

<i>Codes by Content Knowledge</i>	<i>Third Grade</i>	<i>Third Adjusted</i>	<i>Fourth Grade</i>	<i>Fourth Adjusted</i>	<i>Total</i>	<i>Total Adjusted</i>
<i>Subject Matter Content Knowledge</i>					<i>1</i>	<i>1</i>
E-Content	0	0	1	1	1	1
R-Content	0	0	0	0	0	0
<i>Pedagogical Content Knowledge for Mathematics Topics</i>					<i>261</i>	<i>224</i>
E-Experiences	30	26	16	9	46	35
E-Differentiation	31	29	17	15	48	44
E-Sequences	18	18	1	1	19	19
E-Representation	25	25	16	16	41	41
E-Tools	16	16	5	5	21	21
R-Experiences	30	21	18	5	48	26
R-Differentiation	5	5	5	5	10	10
R-Sequences	18	18	1	1	19	19
R-Representation	3	3	3	3	6	6
R-Tools	3	3	0	0	3	3
<i>Pedagogical Content Knowledge for Mathematics Practices</i>					<i>150</i>	<i>149</i>
E-Discourse	30	30	16	16	46	46
E-Reasoning	23	23	10	10	33	33
E-Terms	14	14	7	7	21	21
E-Structures	30	29	16	16	46	45
R-Discourse	2	2	0	0	2	2
R-Reasoning	0	0	0	0	0	0
R-Terms	0	0	1	1	1	1
R-Structures	1	1	0	0	1	1
<i>Curricular Content Knowledge</i>					<i>148</i>	<i>138</i>
E-Overview	14	14	7	7	21	21
E-Features	29	20	16	15	45	35
E-Storyline	15	15	9	9	24	24
E-Goals	1	1	2	2	3	3
R-Philosophy	0	0	0	0	0	0

R-Features	1	1	1	1	2	2
R-Storyline	32	32	18	18	50	50
R-Goals	2	2	1	1	3	3

As an illustration of how to interpret this table, consider the first two rows in Curricular Content Knowledge: E-Overview and E-Features. All educative features receiving an E-Overview code in both third and fourth grade provided unique guidance for teachers; that is, the Total and Total Adjusted columns for E-Overview are equal. However, not all educative features receiving an E-Features code were unique. In particular, the Total and Total Adjusted columns differ by ten, with nine of those adjustments at the third-grade level and one adjustment at the fourth-grade level. The findings in the next section arise from analysis of data in the adjusted columns, reflecting the unique educative features in the written curriculum.

Findings of the Written Curriculum

This section describes the major findings of the curriculum study and provides illustrations of these findings from the written curriculum materials. This study revealed that educative features most often offered guidance in the category of *Pedagogical Content Knowledge for Mathematics Topics*, that most of these features were found in the Concept Development portions of the lesson, and that Enactment Guidance far surpassed Rationale Guidance in the types of guidance provided. I begin with the results related to the type of content knowledge, then discuss the results related to the location of the educative features in the lesson, and finish with results related to the type of guidance (i.e., Enactment or Rationale). Differences between grade levels and specific connections to the lesson components are noted throughout this section.

Educative Features by Type of Content Knowledge

Within the modules included in this study, educative features addressing *Pedagogical Content Knowledge for Mathematics Topics and Practices* as well as *Curricular Content Knowledge* were found across the Eureka Math curriculum at both the third- and fourth-grade levels; however, *Subject Matter Content Knowledge* was only found in a single lesson at the fourth-grade level. The most prevalent category of content knowledge, accounting for over 43% of all educative features across both grade levels, was *Pedagogical Content Knowledge for Mathematics Topics*. At the third-grade level, the prevalence of educative features in this category comprised nearly 50% of all educative features, while the fourth-grade percentages were considerably more balanced across the categories of content knowledge. Table 14 shows the percent of educative features for all categories of content knowledge across both grade levels.

Table 14

Percent of educative features by category of content knowledge and grade level

Category of Content Knowledge	3 rd Grade	4 th Grade	Total
Subject Matter Content Knowledge	0.00%	0.61%	0.20%
Pedagogical Content Knowledge for Mathematics Topics	47.13%	36.59%	43.75%
Pedagogical Content Knowledge for Mathematics Practices	28.45%	30.49%	29.10%
Curricular Content Knowledge	24.43%	32.32%	26.95%

Educative features in the category of *Pedagogical Content Knowledge for Mathematics Topics* included mostly those designed to guide teachers in enacting differentiation ideas for students of different ability levels and for English Language Learners (ELLs), followed closely by those guiding teachers in carrying out instructional activities considered to be mathematical experiences and using diverse mathematical representations. Figure 8 contains guidance in *Pedagogical Content Knowledge for*

Mathematics Topics (pink codes), specifically for differentiation in the call-out box at the bottom of the figure as well as Enactment Guidance for Experiences and Representations.

..E-Representati

..E-Tools

..E-Discours

..E-Experien

..E-Represer

..E-Structure

..E-Terms

..E-Different

Concept Development (35 minutes)

Materials: (T/S) Whole number place value disks (tens and ones), decimal place value disks (tenths), personal white board, tenths on a number line (Template)

Problem 1: Make groups of 10 tenths to rename as ones. Write the number in decimal form.

T: With a partner, use place value disks to show 21 units of 1 tenth in five-group formation.

S: (Lay out 21 disks, all tenths, in five-group formation, as shown.)

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Lesson 3: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the numberline, and in expanded form.

41

A STORY OF UNITS

Lesson 3

4•6

T: Talk with your partner. Is there any way we can use *fewer* disks to show this same value?

S: We can bundle 10 tenths to make one. → There are 2 groups of 10 tenths, so we can show 21 tenths as 2 ones 1 tenth. → In the five-groups, I can see 2 groups of 10 disks. 10 tenths is 1 whole. We have 1 (circling group with finger), 2 (circling group with finger) groups that make 2 ones, and then 1 tenth (touching final 0.1 disk.)

T: Let's group 10 tenths together and trade them for...?

S: 1 one.

T: How many times can we do this?

S: 1 more time. → 2 times.

T: What disks do we have now?

S: 2 ones and 1 tenth.

T: Express this number in decimal form.

S: (Write 2.1.)

T: How many more tenths would we have needed to have 3 ones?

S: 9 tenths more. → 0.9.

Repeat the process using disks to model 17 tenths. Then, continue the process having the students draw disks for 24 tenths. Have students circle the disks being bundled.

NOTES ON MULTIPLE MEANS FOR ACTION AND EXPRESSION:

Be sure to enunciate /th/ at the end of tenths to help English language learners distinguish tenths and tens. Try to speak more slowly, pause more frequently, or couple language with a tape diagram. Check for student understanding and correct pronunciation of fraction names.

Figure 8. Excerpt containing features in *Pedagogical Content Knowledge for Mathematics Topics*. Taken from Grade 4 Lesson 3.

The E-Differentiation at the bottom of Figure 8 is directed at how teachers may help ELLs distinguish between tenths and tens. The figure also contains an E-Experiences code and an E-Representations code across Problem 1 for its use of place value disks to engage students in bundling 10 tenths disks to make one whole.

Notably, at the fourth-grade level, educative features in *Curricular Content Knowledge* were more prevalent than those in *Pedagogical Content Knowledge for Mathematics Practices*. Educative features providing rationales for the storyline across lessons and throughout the module as well as those providing guidance for enacting various features of the curriculum made up the large majority of the features in this category of content knowledge. Figure 9 shows a Fluency Practice activity from Grade 4 containing Rationale Guidance for Curricular Storyline and Enactment Guidance for Curricular Features.

Decompose Larger Units (3 minutes)

Materials: (S) Personal white board, place value chart (Lesson 7 Template)

Note: This fluency activity reviews Lesson 8.

T: (Write 1.) Say the number in unit form.
 S: 1 one.
 T: Draw 1 one on your place value chart.
 S: (Draw 1 one disk.)
 T: (Write 1 one = ____ tenths.) Rename 1 one for tenths.
 S: (Cross out the one disk, and draw 10 tenth disks.)

Continue this process using the following possible sequence:

- Rename 1 one 2 tenths for tenths.
- Rename 1 tenth for hundredths.
- Rename 1 tenth 2 hundredths for hundredths.
- Rename 2 ones 3 tenths for tenths (leads into the next fluency activity).

ones	tenths	hundredths
	● ● ● ● ● ● ● ●	

1 one = 10 tenths

Figure 9. Sample guidance in the category of *Curricular Content Knowledge*. Taken from Grade 4 Lesson 9.

As with most Fluency Practice activities, a rationale is provided describing why the particular activity fits into the sequence of lessons (e.g., this activity reviews the prior lesson). This activity also contains guidance for features included in the curriculum. The vignettes also typically include specific guidance as to when and how to use the template rather than just giving teachers the information that the template is available.

Recall that the coding scheme (see Table 5) contains 10 codes for *Pedagogical Content Knowledge for Mathematics Topics*, 8 codes for *Pedagogical Content*

Knowledge for Mathematics Practices, and 8 codes for *Curricular Content Knowledge*. Although *Pedagogical Content Knowledge Mathematics Topics* contained more codes than *Pedagogical Content Knowledge for Mathematics Practices* or *Curricular Content Knowledge*, accounting for the differences in the amount of codes in each category of content knowledge yields similar percentages of prevalence; that is, the higher prevalence of guidance in *Pedagogical Content Knowledge for Mathematics Topics* across both grade levels is not due to the higher amount of codes in that category of content knowledge. To illustrate this, consider the comparison (see Table 15) of the codes given to the total codes possible if every lesson or overview were to receive every code in a given content knowledge category. Comparing the percentages in the Total column here with the percentages in the Total column of Table 14 shows that the relative prevalence across the categories of content knowledge are similar, which means that the higher amount of codes in the category of *Pedagogical Content Knowledge for Mathematics Topics* is not the sole reason for its prevalence across the categories of content knowledge.

Table 15

Comparison of codes given to codes possible for the categories of content knowledge

Category of Content Knowledge	Codes Given	Codes Possible	Percent	Total
<i>Pedagogical Content Knowledge for Mathematics Topics</i>				38.62%
3 rd Grade	164	360	45.56%	
4 th Grade	60	220	27.27%	
<i>Pedagogical Content Knowledge for Mathematics Practices</i>				31.47%
3 rd Grade	96	288	33.33%	
4 th Grade	50	176	28.41%	
<i>Curricular Content Knowledge</i>				29.74%
3 rd Grade	85	288	29.51%	
4 th Grade	53	176	30.11%	

Educative Features by Location in Lesson

Each lesson in this curriculum is typically divided into four components: Fluency Practice, Application Problem, Concept Development, and Student Debrief. Most Concept Development and Fluency Practice components are written as a sample discussion, or vignette, between students and teachers and provide an example of how teachers may enact those components with students. In addition, each lesson's resources are located immediately after the details of the lesson and include any templates (e.g., place-value chart) used during the lesson as well as the Problem Set, Exit Ticket, and Homework for the lesson.

Table 16

Placement of educative features in third- and fourth-grade lessons

Category of Content Knowledge	Fluency Practice	Application Problem	Concept Development	Student Debrief
Subject Matter Content Knowledge	0	0	1	0
Pedagogical Content Knowledge for Mathematics Topics	138	12	134	31
Pedagogical Content Knowledge for Mathematics Practices	8	4	121	60
Curricular Content Knowledge	51	30	59	27
<i>Total</i>	<i>197</i>	<i>46</i>	<i>315</i>	<i>118</i>

Analysis of the placement of educative features in the lesson reveals that, as one may expect, most educative features were found in the Concept Development component of the lesson (see Table 16) with over 46% of all educative features found here. However, the prevalence of educative features in the Fluency Practice (29%) and Student Debrief (17%) components should be noticed given that the amount of space in each lesson devoted to these components was typically much smaller than that devoted to the Concept Development (see Appendices C, H, I, and J). Additionally, these components often involve reviewing concepts either from previous lessons or modules, in the case of

the Fluency Practice activities, or from the current lesson, in the case of the Student Debrief component. One exception to the pattern of shorter Fluency Practice and Student Debrief sections occurred in third grade during Lesson 12 where, although the Concept Development (see Figure 10) was limited to one page, the three Fluency Practice activities took one and a half pages.

..E-Features

..E-Storyline

..E-Different

..E-Structure

..E-Terms

..E-Tools

..E-Discours

..E-Represen

..E-Experi

..E-Terms

..E-Differentia

..E-Structures

Concept Development (20 minutes)

Materials: (S) Personal white board, shapes (Template) (pictured below), ruler

T: (Pass out the Template.) Yesterday you learned that the boundary of a shape is the shape's perimeter. What forms the boundary of Shape A? Talk to a partner.

S: The outside edges of the shape. → The sides of the shape form the boundary.

T: The sides form the boundary of Shape A. Trace the perimeter of Shape A with your finger. (Allow students time to trace.) Your finger just traveled around the perimeter of Shape A. What tool can you use to figure out how many centimeters your finger traveled?

S: A ruler!

T: Measure and label the side lengths of Shape A in centimeters. (Allow students time to work, and then project Shape A with the side lengths labeled.) Check your side lengths against mine. Write and solve a number sentence to show how to find the total of Shape A's side lengths.

S: (Possible number sentences include the following:
 $10 + 10 + 4 + 4 = 28$. → $(2 \times 10) + (2 \times 4) = 20 + 8 = 28$.
→ $(10 + 4) \times 2 = 28$.)

T: What strategy did you use to find the total of the side lengths?

S: I doubled 10 to get 20 and doubled 4 to get 8. Then, I added 20 and 8 to get 28. → That's like what I did. I thought of it as 2 tens plus 2 fours. → I added 10 and 4 to get 14. I knew there were 2 fourteens, so I doubled 14 to get 28.

T: What is 28 centimeters a measurement of?

S: The perimeter!

T: What kind of polygon is Shape A?

S: A quadrilateral because it has four sides.
→ A parallelogram because it has two sets of parallel lines. → A rectangle because the opposite sides are equal and the corners look like right angles.

Repeat the process with Shapes B through E. Students measure the side lengths in centimeters, calculate the perimeter, discuss strategies for finding the total, and name each shape. When they are ready, release them to work independently or with a partner.

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

When asking, "What forms the boundary of Shape A?" use gestures to convey the meaning clearly to English language learners. For example, while saying the word *boundary*, project or hold the shape, and trace its sides.

Shapes Template

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Outlining the shapes on the Template with glue that dries and makes a hard boundary may be beneficial to some students as they measure, particularly students with low vision.

Figure 10. Concept Development from Grade 3 Module 7 Lesson 12.

In this case, although this Concept Development was shorter than most lessons, the educative features contained herein show the distribution across the categories of content knowledge; that is, there are four different features from *Mathematics Topics* (pink codes), three different features from *Mathematics Practices* (yellow codes), and two

different features from *Curricular Content Knowledge* (green codes). This excerpt also shows how the curriculum uses call-out boxes in addition to the vignettes to deliver some educative features. The Notes on Multiple Means of Representation box at the top has been coded with the first E-Differentiation code found along the left side.

The Fluency Practice activities in this lesson included Multiply by 7 and Equivalent Counting with Units of 3 (described in the next section) as well as Area and Perimeter (see Figure 11). These activities show how the Fluency Practice is sometimes used to review previous lessons (e.g., Area and Perimeter) and sometimes used to build fluency with multiplication facts, among others. In particular, the focus of educative features in the Fluency Practice activities is on providing guidance for the focused review of concepts rather than on teaching new concepts.

Area and Perimeter (3 minutes)

Materials: (S) Grid paper

Note: This activity reviews Lesson 10.

T: On your grid paper, shade a rectangle that is 2 units wide by 3 units long.
 S: (Shade a 2 unit by 3 unit rectangle.)
 T: What is the area of the rectangle?
 S: 6 square units!
 T: Draw a line around the perimeter of the rectangle.
 S: (Draw line around the perimeter.)
 T: At the signal, show your paper. (Signal.)
 S: (Show paper with the perimeter marked.)

Continue with the following possible sequence: 4 units by 2 units and 5 units by 3 units.

Figure 11. Fluency Practice activity from Grade 3 Module 7 Lesson 12.

In addition, placement of educative features in the lesson was found to differ by category of content knowledge. That is, Fluency Practice activities most often contained features addressing *Pedagogical Content Knowledge for Mathematics Topics* (70%) while Concept Development activities almost equally contained features addressing both *Pedagogical Content Knowledge for Mathematics Topics* (43%) and *Pedagogical Content Knowledge for Mathematics Practices* (38%). Given that Fluency Practice

activities were focused on review of concepts, these activities did not provide guidance for discourse, participation structures, or reasoning as described in the coding scheme (see Table 5). Conversely, Concept Development activities were focused on development of students' mathematical understanding of new concepts and included features from both *Pedagogical Content Knowledge for Mathematics Topics* and *Pedagogical Content Knowledge for Mathematics Practices*. In this way, the development of students' understanding of mathematics topics is complimented by students' engagement in the mathematics practices. Interestingly, Student Debrief activities most often contained guidance addressing *Pedagogical Content Knowledge for Mathematics Practices* (50%). These activities were focused on solidifying students' understanding and making connections across components of the curriculum; therefore, suggested questions in the Student Debrief often contained guidance for talking to a partner (i.e., participation structures) and making connections between concepts not previously discussed (i.e., reasoning). In the next section, I will provide more details about the placement of educative features in relation to the category of content knowledge.

Educative Features by Type of Guidance

Educative features were classified as Enactment Guidance (guidance directed at helping teachers carry out instructional activities with students) or Rationale Guidance (guidance directed toward the mathematical or pedagogical reasons for carrying out instructional activities). Across grade levels and categories of content knowledge, educative features more often provided Enactment Guidance (76% of all guidance) than Rationale Guidance (24% of all guidance), a finding consistent with research in both mathematics education (Males, 2011) and science education (Beyer et al., 2009).

However, differences across grade levels and categories of content knowledge exist for each type of guidance, as seen in Table 17. I describe these differences next.

Table 17

Percent of codes given in each category of content knowledge by type of guidance and grade level

	Pedagogical Content Knowledge for Mathematics Topics	Pedagogical Content Knowledge for Mathematics Practices	Curricular Content Knowledge
Enactment Guidance			
3rd Grade	43.85%	36.92%	19.23%
4th Grade	35.66%	37.98%	25.58%
Rationale Guidance			
3rd Grade	56.82%	3.41%	39.77%
4th Grade	40.00%	2.86%	57.14%
<i>Total</i>	43.75%	29.10%	26.95%

Enactment guidance. Prevalence of Enactment Guidance by category of content knowledge differed across grade levels. Although the enactment categories related to pedagogical content knowledge were most prevalent across both grade levels, *Pedagogical Content Knowledge for Mathematics Topics* was significantly favored in third grade while *Pedagogical Content Knowledge for Mathematics Practices* was slightly more prevalent than *Pedagogical Content Knowledge for Mathematics Topics* in fourth grade.

Figure 12 below shows an example of the Concept Development from Grade 3 Module 7 Lesson 22 containing more Enactment Guidance in *Pedagogical Content Knowledge for Mathematics Topics* (pink codes) than in *Pedagogical Content Knowledge for Mathematics Practices* (yellow codes). This lesson was the final lesson of Topic D,

which was focused on using line plots to draw conclusions about area and perimeter measurements.

Concept Development (37 minutes)

Materials: (T) Line plot (Template 1) (S) Problem Set, ruler, data chart from Lessons 20–21, line plot from Lesson 19, scissors, 11-inch piece of string (per pair), rectangles (Template 2) (per pair)

Problem 1: Draw a line plot representing measurement data.

Guide students through the process of recording the number of rectangles they made for each given perimeter on the line plot in Problem 1 of the Problem Set.

- Use a ruler to partition equal intervals.
- Label the number line to show the different perimeters.
- Record the data on the line plot using X's to represent one rectangle.

Problem 2: Observe and interpret data on a line plot.

T: Study the perimeter measurements on your line plot. Are they even, odd, or both?

S: They're all even!

T: Why do you think that is? Discuss with your table.

S: The teacher just made them up that way. → To get the perimeter of a rectangle, we add four sides. Maybe the totals have to be divisible by 4, so they have to be even because 4 is even? → But we don't know how to divide 10, 14, 18, or 22 by 4. → Wait. To get perimeter, we find the sum of the width and length. Then, we double it. If you double a number, I think it's always even because you have to multiply by 2. Let me try a few to check. Yep! Everything I multiply by 2 has an even product.

T: You're close! All the rectangles that we made had whole number side lengths. When we add whole number side lengths and double them, the perimeter will be even. (Pass out an 11-inch-long piece of string to each pair.) Use your ruler to measure the length of the string in inches.

S: (Measure the string.)

T: How long is the string?

S: 11 inches!

T: Work with your partner to shape your string into a rectangle.

S: (Make a rectangle with the string.)

T: What is the perimeter of your rectangle? How do you know?

S: 11 inches because it's the same as the length of the string.

T: Is 11 an odd or even number?

S: Odd!

T: So, do all rectangles have an even perimeter?

S: No!

NOTES ON MULTIPLE MEANS OF ACTION AND EXPRESSION:

Provide the option of wire, in addition to string, because it may be easier for students to measure and shape wire into a rectangle. However, students may need assistance cutting off an inch of wire. Gluing string or anchoring wire (so that it does not move) may decrease possible frustration as students measure side lengths. If gluing is chosen, provide additional pieces of string.

Figure 12. Excerpt of Grade 3 Lesson 22.

In Problems 1 and 2 of this particular lesson, students were creating and interpreting a line plot of data they had collected during the previous two lessons, necessitating the E-Representations code. In addition, guidance was provided to use an 11-inch piece of string to examine if perimeter has to be an even number, necessitating the E-Tools and E-Differentiation codes shown as well as the E-Experiences code shown by the pink code extending off the page.

In contrast to the lesson from third grade, Figure 13 below shows an example of the Concept Development from Lesson 9 containing more Enactment Guidance in *Pedagogical Content Knowledge for Mathematics Practices* (yellow codes) than in *Pedagogical Content Knowledge for Mathematics Topics* (pink codes).

Problem 2: Compare pairs of decimal numbers representing mass.

T: (Place Rice Bag A on the scale.) What is the mass of this bag of rice?

S: Zero point one kilogram. $\rightarrow \frac{1}{10}$ kilogram. $\rightarrow \frac{10}{100}$ kilogram (see image below).

T: Record the mass in the table on the measurement record.

Repeat this process for the remaining bags.

T: (Leave Bag D, weighing 0.46 kg, on the scale.) Which bags are heavier than Bag D? How do you know?

S: Bags B and C were heavier than Bag D. $\rightarrow$ Bag B was 0.65 kg, and Bag C was 0.7 kg. Those numbers are both larger than 0.46 kg, so the bags are heavier.
 $\rightarrow$ I looked at my chart, from left to right. In the tenths column, I could see that Bag A was lighter. It had only 1 tenth. Bags B and C were heavier than D because they both had more tenths.

T: Let's look at Bags B and C. Make a statement comparing their mass.

S: 0.65 kilogram is lighter than 0.7 kilogram.
 $\rightarrow$ 0.7 kilogram is heavier than 0.65 kilogram.

T: How do you know?

S: I could just see that the bag was fuller and feel that the bag has more mass. $\rightarrow$ At first, I thought 65 hundredths was more because it looks like you are comparing 65 and 7, and 65 is greater than 7. But then we saw that it was 7 tenths, which is more than 6 tenths. $\rightarrow$ I realized that 7 tenths is 70 hundredths, and that is greater than 65 hundredths.

T: With your partner, make another statement to compare the bags. You can compare just two items, or you can compare more than two items.

S: (Responses will vary.)

T: Based on these comparisons, what is the mass of the bags in order from heaviest to lightest?

S: 0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg.

T: (Select a student volunteer.) Arrange the bags from heaviest to lightest. Looking at the bags, does it *appear* that we have properly ordered the bags from heaviest to lightest? Do they match the order we determined?

S: Yes.

NOTES ON TERMINOLOGY:

Mass is a fundamental measure of the amount of matter in an object. While weight is a measurement that depends upon the force of gravity (one would weigh less on the moon than one does on Earth), mass does not depend upon the force of gravity. Both words are used here, but it is not important for students to recognize the distinction in mathematics at this time.

Rice Bag	ones	.	tenths	hundredths
A	0	.	1	0
B	0	.	6	5
C	0	.	7	
D	0	.	4	6

0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg

Figure 13. Excerpt from Grade 4 Lesson 9.

In this lesson, which was also a target lesson in phase two of this study, students were comparing and reasoning about decimal numbers using measurements from mathematical tools (meter sticks, scales, and graduated cylinders), necessitating the E-Reasoning codes. In addition, the guidance directed teachers to have students work with partners to write comparison statements and arrange the bags of rice, necessitating the E-Structures codes. The call-out box comparing the terms of mass and weight and their important distinction

also necessitated the E-Terms and R-Terms codes. In this lesson, the reasoning opportunities and development of terminology as mathematics practices complemented the development of students' understanding of decimal numbers as a mathematics topic.

It should be noted that *Pedagogical Content Knowledge for Mathematics Topics* contained one more Enactment code in the analytic framework than both *Pedagogical Content Knowledge for Mathematics Practices* and *Curricular Content Knowledge*; however, this difference in number of codes in the category was not the sole source for the difference across grade levels. Differences can be attributed to three areas within the category of *Pedagogical Content Knowledge for Mathematics Topics*: Experiences, Sequences, and Tools. For each of these areas, the percent of sections containing these pieces of Enactment guidance at third grade is at least 20% more than the sections containing this Enactment Guidance at fourth grade (see Table 18).

Table 18
Percent of codes given in each category of content knowledge by type of guidance and grade level

	Grade 3	Grade 4
E-Experiences	72.22%	40.91%
E-Sequences	50.00%	4.55%
E-Tools	44.44%	22.73%

Further analysis in these three areas revealed that differences relate to the Fluency Practice activities and the mathematical content of the module in general. Fluency Practice activities, namely Multiply by ____ and Equivalent Counting by Units of ____ at the third-grade level, account for 23% of the 31% difference between grade levels in Enactment Guidance for Experiences. Similarly, the Multiply by ____ Fluency Practice activity accounts for all of the difference between grade levels in Enactment Guidance for Sequences.

Analysis of codes at both grade levels for Enactment Guidance for Using Mathematics Tools revealed that the difference between grade levels is related to the mathematical content of the module. At the third-grade level, the purpose of the mathematical tools (e.g., index card as a right angle tool, ruler, string, tetrominoes, unit tiles, tangrams) was to investigate characteristics of special quadrilaterals or circles, explore area and perimeter, or compose and decompose polygons. In this way, the Concept Development portion of the lesson focused on using the tools to engage students in a mathematical experience as supported by twelve of fourteen lessons coded with E-Tools also receiving an E-Experience code. For example, consider an excerpt from the Concept Development of Lesson 16 (see Figure 14).

..E-Tools

..E-Features

.E-Structures

.E-Structures

.E-Experiences

..E-Discourse

..E-Tools

.E-Structures

Concept Development (34 minutes)

Materials: (T) Circles (Template) (copied on cardstock), white string, black marker, ruler
(S) White string (per pair), ruler, black marker, circle (Template) (one circle per pair, copied on cardstock), personal white board

Note: The white string in the materials list is used in both parts of this lesson. Be sure to cut the string so it is long enough to go around the item with the greatest perimeter in Part 2.

Part 1: Use string and a ruler to find the perimeter of a circle.

T: (Pass out a circle to each pair of students.) Talk to your partner.
Does this circle have a perimeter?

S: I don't think so because it doesn't have straight lines. → Remember when we made tessellations? Those shapes didn't have straight lines, but they still had perimeters. → The black line shows the boundary of the circle, so that's the circle's perimeter.

T: Can you find the perimeter of the circle in inches using just your ruler?

S: I don't think so.

T: (Pass out white string to each pair of students.) Work with your partner to wrap the string around the perimeter of the circle. (Model.) Partner A, hold the string in place. Partner B, use the black marker to mark the string where it meets the end after going all the way around once.

S: (Mark the string.)

EUREKA MATH™

Lesson 16: Use string to measure the perimeter of various circles to the nearest quarter inch.

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A STORY OF UNITS

Lesson 16 3•7

.E-Structures

T: What does the string around the circle represent?

S: The perimeter of the circle.

T: How can you use this string and your ruler to find the perimeter of the circle?

S: We can straighten the string out and measure it. → Since the string has about the same length as the perimeter, we can use our rulers to measure the length of the string to the black mark.

T: Work with your partner to measure the length of the string from the end to where you made the mark. Record your measurement on your personal white board to the nearest quarter inch.

S: (Measure the string.)

T: What is the length of the string to the nearest quarter inch?

S: $10\frac{1}{4}$ inches!

T: So, the perimeter of the circle is...?

S: $10\frac{1}{4}$ inches!

Figure 14. Using mathematical tools to engage in mathematical experiences in Grade 3 Lesson 16.

In this lesson, students are using a string and ruler to measure the circumferences of circles. Using the mathematical tools to physically measure circumference rather than simply telling the students how to find the circumference of a circle necessitates an E-Experiences code for this lesson as well.

At the fourth-grade level, the purpose of the mathematical tools (digital scales, meter sticks, rulers, bags of rice, etc.) was to provide a visual model as motivation for the

symbolic representation of decimal fractions. For the four lessons coded with E-Tools, activities in the Concept Development typically began by showing a measurement on one of the mathematical tools, determining a fractional representation for the measurement, and then relating that fractional representation to the decimal representation for the measurement (see Figure 15 and Appendix H). In Figure 15 below, students are given a physical representation of a meter stick in the form of a strip of paper that they then partition into 10 pieces, relating the portion they chose to shade to both its fraction and decimal symbolic representations. In this way, the mathematical tool (the meter-long strip of paper) was used to motivate the fraction and decimal measurement the students shaded.

Figure 15. Using mathematical tools to motivate representation in Grade 4 Lesson 1.

Contributing significantly to the high percentage of *Pedagogical Content Knowledge for Mathematics Topics* across both grade levels, Enactment Guidance for Differentiation was found in nearly every lesson, predominantly in call-out boxes (see Figure 16) of the Concept Development sections (23 of 30 third-grade lessons and 12 of 16 fourth-grade lessons).

Figure 16. Call-out boxes containing Enactment Guidance for Differentiation. Taken from Grade 3 Lessons 26 and 11 and Grade 4 Lessons 4 and 1, respectively.

Pieces of guidance addressed ways to scaffold learning or adjust specific tasks for students working below and above grade level as well as considerations for English Language Learners and students with special needs (e.g., low vision). Although the coding scheme did not contain codes specific to English Language Learners, further analysis of the notes included with each E-Differentiation code revealed that guidance directed at helping these students was often coupled with guidance for students working below grade (see Figure 17) when, in reality, the guidance could have benefited all students. It should also be noted that no guidance in these modules addressed aspects of racial or socioeconomic diversity, nor any other dimensions of equity other than those previously mentioned.

Figure 17. Call-out box containing Enactment Guidance for Differentiation directed at English Language and below grade level learners. Taken from Grade 4 Lesson 5.

Notably, Enactment Guidance for Facilitating Discourse and Engaging Students in Appropriate Participation Structures seem to work in concert with one another at both grade levels in the Concept Development and Student Debrief sections of the lessons and contributed significantly to the high percentage in the category of *Pedagogical Content Knowledge for Mathematics Practices*. The Discourse coded for in the Concept Development sections was highly contextualized given the instructional objective for the lesson (see Appendix I). Most often the Enactment Guidance for both Discourse and Structures in the Concept Development contained explicit guidance through the vignettes, as described previously. However, both grade levels also contained more implicit guidance in these categories outside of the vignettes in the Concept Development. Figure 18 below shows one example of such implicit guidance at third grade. Instead of the guidance containing a question and sample student response as in the vignettes, the guidance here describes a general process for students to share solutions to problems on the Problem Set in a small group. Additionally, it provides sentence starters for those hearing the student solutions so that they may respond in ways that promote mathematical discourse. In the Student Debrief sections, Enactment Guidance for

Structures was more explicit (e.g., “With your partner...” or “Explain to your partner...”) while Enactment Guidance for Facilitating Discourse was more implicit due to the intent of the section as a whole.

Part 2: Work independently to solve and present problems using multiple solution paths.

Assign each student two problems from the Problem Set. Challenge them to record more than one way to draw for each problem they solve. Ask students to share their work with the members of their groups from Part 1. When sharing, students should include answers to the following questions:

- How does your drawing represent the problem clearly?
- How did your drawing help you decide on a way to solve?
- Why does the equation that you used to model make sense with your drawing and with the problem?
- How do you know you answered the question?

Have students share their work in groups of three or four. Encourage group members to practice asking questions of the presenter. They might ask some of the questions listed below.

- I’m not sure what you mean. Can you say more about that?
- Why did you decide _____?
- What do you think about _____ instead?
- Which other way did you try to draw the problem?

One way to close this process is to have students write a compliment to another presenter. If time allows, students may solve problems on the Problem Set that they have not already completed on their own before the Student Debrief.

A NOTE ON STANDARDS ALIGNMENT:

Problems 2 and 5 on the Problem Set, the Exit Ticket, and Problems 1 and 5 on the Homework are two-step word problems involving milliliters and grams. The masses and volumes are given in the same units in each problem. Standard **3.MD.2** specifically states that students “solve one-step problems involving masses or volumes that are given in the same units.” However, these problems look ahead to **4.MD.2**. Students working above grade level might enjoy the challenge of solving these two-step word problems involving milliliters and grams. To make these problems accessible to students working below grade level, modify the problems so they can be solved with one step.

MP.3

Use the **MP.3** process to solve. Use a letter to represent the unknown in each problem.

1. Leanne needs 120 tiles for an art project. She has 54 tiles. If tiles are sold in boxes of 6, how many more

Figure 18. Excerpt from Grade 3 Lesson 2.

Rationale guidance. Rationale guidance accounted for almost 25% of all supports, a finding consistent with research in science education (Beyer et al., 2009) where a similar of supports in Biology textbooks were found to provide Rationale Guidance. On the contrary, this finding is inconsistent with research in mathematics education (Males, 2011) where considerably less supports in four middle school mathematics textbooks were found to provide Rationale Guidance.

Considering all of the sections of the written curriculum containing Rationale Guidance, features in the categories of *Pedagogical Content Knowledge for Mathematics Topics* and *Curricular Content Knowledge* (see Table 17) contained the strong majority of that guidance at both grade levels. Three areas within these categories are to credit for more than 77% of all sections containing Rationale Guidance: Experiences, Sequences, and Storyline. Differences in the area of Experiences were noticeable between grade levels with 21 of 36 sections at third grade containing Rational Guidance for Experiences but only five of 22 sections at fourth grade. Differences in the area of Sequences were also noticeable between grade levels with 18 of 36 sections at third grade and only one of 22 sections at fourth grade containing Rationale Guidance for Sequences. Further analysis of these types of guidance showed that two specific types of Fluency Practice, Equivalent Counting by Units of ____ (see Figure 19) and Multiply by ____ (see Figure 20), at the third-grade level contributed most significantly to these differences.

..R-Storyline

..E-Differentiation

..R-Experiences

..E-Experiences

Equivalent Counting with Units of 5 (4 minutes)

Note: This activity builds fluency with multiplication facts using units of 5. The progression builds in complexity. Work students up to the highest level where they can confidently participate.

T: Count to 10. (Write as students count. See the chart below.)

S: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

1	2	3	4	5	6	7	8	9	10
1 five	2 fives	3 fives	4 fives	5 fives	6 fives	7 fives	8 fives	9 fives	10 fives
5	10	15	20	25	30	35	40	45	50
1 five	10	3 fives	20	5 fives	30	7 fives	40	9 fives	50
5	2 fives	15	4 fives	25	6 fives	35	8 fives	45	10 fives

T: (Write 1 five beneath the 1.) Count to 10 fives. (Write as students count.)

S: 1 five, 2 fives, 3 fives, 4 fives, 5 fives, 6 fives, 7 fives, 8 fives, 9 fives, 10 fives.

T: Count by fives to 50. (Write as students count.)

S: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

T: (Write 1 five beneath the 5. Write 10 beneath the 10.) I'm going to give you a challenge. Let's alternate between saying the units of five and the number. (Write as students count.)

S: 1 five, 10, 3 fives, 20, 5 fives, 30, 7 fives, 40, 9 fives, 50.

T: (Write 5 beneath 1 five and 2 fives beneath the 10.) Let's alternate again. (Write as students count.)

S: 5, 2 fives, 15, 4 fives, 25, 6 fives, 35, 8 fives, 45, 10 fives.

Figure 19. Equivalent Counting with Units of 5 Fluency Practice activity from Grade 3 Lesson 4.

Multiply by 4 (8 minutes)

Materials: (S) Multiply by 4 (6–10) Pattern Sheet

Note: This activity builds fluency with multiplication facts using units of 4. It works toward students knowing from memory all products of two one-digit numbers. See Lesson 1 for the directions for administration of a Multiply-By Pattern Sheet.

T: (Write $7 \times 4 = \underline{\quad}$.) Let's skip-count up by fours. I'll raise a finger for each four. (Raise a finger for each number to track the count.)

S: 4, 8, 12, 16, 20, 24, 28.

T: Let's skip-count up by fours starting at 20. Why is 20 a good place to start?

S: It's a fact we already know, so we can use it to figure out a fact we don't know.

T: (Track with fingers as students say the numbers.)

S: 20 (5 fingers), 24 (6 fingers), 28 (7 fingers).

T: Let's see how we can skip-count down to find the answer, too. Start at 40 with 10 fingers, 1 for each four. (Count down with fingers as students say the numbers.)

S: 40 (10 fingers), 36 (9 fingers), 32 (8 fingers), 28 (7 fingers).

Continue with the following possible sequence: 9×4 , 6×4 , and 8×4 .

T: (Distribute the Multiply by 4 Pattern Sheet.) Let's practice multiplying by 4. Be sure to work left to right across the page.

Figure 20. Multiply by 4 Fluency Practice activity from Grade 3 Lesson 4.

The Note at the beginning of the Equivalent Counting by Units of 5 activity in Figure 19 was coded as Rationale Guidance for Experiences given that the choral counting in this activity is an experience unique to the mathematics classroom and the note provides a reason (building multiplication fact fluency) for using the activity. The Note at the beginning of the Multiply by 4 activity in Figure 20 was similarly coded for Rationale Guidance for Experiences in addition to being coded for Rationale Guidance for Sequences given that the activity contained a choral counting activity similar to the Equivalent Counting activity as well as a reason (building multiplication fact fluency) for the sequence of the choral counting followed by the Pattern Sheet. Notably, only four sections at the third-grade level contained Rationale Guidance for Experiences found outside these particular Fluency Practice activities.

Notably, Rationale Guidance for Storyline appeared in more sections than any other specific educative feature overall. While multiple overviews for both grade levels contained this type of guidance, most of these instances occurred in the Fluency Practice (see Figure 19 and Figure 20) and Application Problem (see Figure 21) sections of the

individual lessons. For example, this Application Problem explicitly points out that the problem connects back to Topic A from earlier in the module.

Application Problem (7 minutes)

The third graders raised \$437 in a fundraiser. The fourth graders raised \$68 less than the third graders. How much money did the two grade levels raise altogether?

Diagram showing \$437 and \$68 less, leading to a subtraction problem: $437 - 68 = 369$.

Tape diagram showing \$437 and \$369, leading to an addition problem: $437 + 369 = 806$.

Note: This problem reviews two-step word problems from Topic A.

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Modeling a tape diagram for the money fourth graders raise, as well as the total money raised, helps English language learners and students working below grade level better grasp the meaning of the phrase \$68 less than the third graders. Ask, "Who raised less money? Did the fourth graders raise \$68? Use the model to estimate about how much the fourth graders raised."

Figure 21. Application Problem from Grade 3 Lesson 4.

Including rationales for the Fluency Practice and Application Problem sends a message to teachers about the importance of these sections, especially when teachers may be apt to eliminate them from the lesson in order that more time may be spent on the Concept Development. In addition, seven lessons across both grade levels also contained this type of guidance in their Concept Development. Most of these lessons (see Appendix J) were “problem-solving lessons” designed for students to work in pairs or independently on the Problem Set with the teacher facilitating conversations about the problems. In these instances, the Rationale Guidance provided reasons why these problems were chosen and how they contributed to students’ mathematical development during the lesson.

Summary

The purpose of this chapter was to characterize the educative features contained in the written curriculum. The results of this phase of the study were that educative features most often provided guidance in the category of *Pedagogical Content Knowledge for Mathematics Topics*; however, guidance in the categories of *Pedagogical Content*

Knowledge for Mathematics Practices and *Curricular Content Knowledge* should not go unnoticed. Additionally, most educative features were found in the Concept Development portion of the lessons, and those features were almost evenly split between the categories of pedagogical content knowledge. That is, the features in the Concept Development provided guidance to help teachers develop students' mathematical understanding while simultaneously incorporating mathematics practices into those activities. Finally, educative features in the Eureka Math curriculum most often provided Enactment Guidance describing how to carry out instructional activities with students. These types of guidance were the focus of the second phase of the study, where take-up of the educative features in classrooms was studied.

CHAPTER 5: FINDINGS FROM THE ENACTED CURRICULUM ANALYSIS

The purpose of this study was to describe the educative features present in Eureka Math curriculum materials and subsequently investigate how teachers are taking up these educative features in the classroom. This chapter focuses on describing how teachers take up the educative features in the written curriculum as they plan, enact, and reflect on two lessons. Recall that take-up of educative features considers the ways that teachers' language or actions align with the ideas or practices present in the educative features in the written curriculum. Therefore, the findings presented here do not assume with certainty that a teachers' implementation of ideas or practices present in the written curriculum are a direct result of teacher learning from the curriculum; rather, the findings point to an alignment between the educative features found in the written curriculum and the teachers' language or actions in the enacted curriculum. I begin this chapter by stating my clearest and most significant findings. The remainder of the chapter illustrates these findings using data from the target lesson enactments with students and planning and reflection meetings with teachers.

First and foremost, analysis of target lesson enactment revealed differences in take-up of several key educative features across teachers. This was true for key educative features in both *Pedagogical Content Knowledge for Mathematics Topics and Practices* as well as *Curricular Content Knowledge*.

In the category of *Pedagogical Content Knowledge for Mathematics Topics*, differences in take-up of Enactment Guidance for Experiences occurred across teachers at the same grade level; that is, the ways that teachers led students to engage in the instructional activities offered different opportunities for students to experience learning

the mathematical content. Similarly, in the category of *Pedagogical Content Knowledge for Mathematics Practices*, differences in take-up of Enactment Guidance for Facilitating Discourse and Participation Structures also occurred across teachers at the same grade level. Take-up of these features were typically dependent on each other, meaning that take-up of Enactment Guidance for Facilitating Discourse and Participation Structures were taken up simultaneously or not at all. In the category of *Curricular Content Knowledge*, differences in take-up of Enactment Guidance for Curricular Features seemed to be associated with differences in take-up in the two preceding categories of content knowledge. In particular, how teachers utilized the templates and measurement records in these lessons was connected to how teachers engaged students in the instructional activities as well as the nature of the discourse and participation structures utilized during enactment.

Second, analysis of target lesson enactments showed similarities across teachers for some key educative features. In particular, Enactment Guidance for Representations in the category of *Pedagogical Content Knowledge for Mathematics Topics* seemed to be taken up across all teachers at fourth grade, with evidence of the features being drawn upon even when the lessons in the written curriculum did not explicitly include the educative features. Additionally, Enactment Guidance for Developing Mathematical Terminology seemed to be taken up across teachers at the same grade level.

Finally, analysis of target lesson enactments showed that teachers take-up key educative features from nearly all components of the curriculum. Educative features located in the Fluency Practice, Application Problem, and Concept Development

components of the curriculum were taken up by teachers; however, not all instructional activities provided in the written curriculum were utilized during enactment.

In the sections that follow, I illustrate these two major findings and provide narratives from the enactment of the lesson or the planning and reflection meetings to support these findings. Additionally, I describe other notable observations made during enactment of the target lessons.

Differences in Take-up of Key Educative Features Across Teachers

Analysis revealed differences in take-up of key educative features across teachers for both categories of pedagogical content knowledge. These differences in take-up for both categories of pedagogical content knowledge seemed to follow two patterns with take-up by Emily (3rd grade) and Rachel (4th grade) following a similar pattern and take-up of key educative features by Kim (3rd grade) and Holly (4th grade) following a similar pattern. In particular, take-up of Enactment Guidance for Experiences in the Concept Development portion of the lessons seemed to be associated with take-up of Enactment Guidance for Participation Structures and Facilitating Discourse. To illustrate this now, consider Table 19 which shows differences in take-up in Lesson 4 for third grade (Emily and Kim) and Lesson 9 for fourth grade (Rachel and Holly).

Table 19

Summary of differences in take-up of educative features.

Teacher	E-Experiences	E-Structures	E-Discourse
Emily	<u>Student-led</u> sorting of quadrilaterals at each desk	Student sorting with <u>small group</u> sharing and full group share-out	Reasoning about and noticing attributes; multiple solution strategies
Kim	<u>Teacher-led</u> sorting of quadrilaterals by math term at board	<u>Teacher-led</u> sorting with students identifying certain attributes at board	Identification of quadrilaterals with certain attributes
Rachel	Using measurement tools and number disks	<u>Student groups</u> ordering measurements; “walking out” decimal lengths	<u>Student group talk</u> including reasoning about ordering decimals and reading measurements
Holly	Using pictorial representations of tools and disks on Problem Set	<u>Teacher-student sharing</u> and telling tasks within table groups	<u>Teacher-student talk</u> with “telling” tasks such as “Somebody explain to me how 0.3 can be more than 0.27.”

I begin this section by describing the differences in take-up within each category of pedagogical content knowledge shown in Table 19. Discussion of differences in take-up in the category of Curricular Content Knowledge will be included throughout the remainder of this chapter as take-up of Enactment Guidance for Curricular Features and Rationale Guidance for Curricular Storyline seemed to be heavily influenced by the choice of instructional activities for the class rather than taken up based on their own merit.

Pedagogical Content Knowledge for Mathematics Topics

In this category of content knowledge, differences in take-up occurred for the key educative feature of Enactment Guidance for Experiences, which was coded in the Concept Development of Grade 3 Lesson 4 and both lessons at 4th grade as well as the

Fluency Practice of Grade 3 Lesson 4 and Grade 4 Lesson 3. I consider the grade levels separately to make connections to the lessons clearer, followed by a summary of this category as a whole.

Grade 3 lessons. Both Fluency Practice activities and the first part of the Concept Development for Lesson 4 were coded for Enactment Guidance for Experiences. In the Fluency Practice activities, the guidance in the written curriculum called for teachers to engage students in choral counting experiences, with the Multiply by 4 activity (see Figure 22) focused on skip-counting up to or down from a convenient value like 20 or 40 in preparation for working through the Pattern Sheet and the Equivalent Counting with Units of 5 focused on counting in amounts of fives up to 50. Both Emily and Kim engaged students in aspects of the Multiply by 4 activity; however, their enactments were quite different and resulted in different experiences for students.

Multiply by 4 (8 minutes)

Materials: (S) Multiply by 4 (6–10) Pattern Sheet

Note: This activity builds fluency with multiplication facts using units of 4. It works toward students knowing from memory all products of two one-digit numbers. See Lesson 1 for the directions for administration of a Multiply-By Pattern Sheet.

T: (Write $7 \times 4 = \underline{\quad}$.) Let's skip-count up by fours. I'll raise a finger for each four. (Raise a finger for each number to track the count.)

S: 4, 8, 12, 16, 20, 24, 28.

T: Let's skip-count up by fours starting at 20. Why is 20 a good place to start?

S: It's a fact we already know, so we can use it to figure out a fact we don't know.

T: (Track with fingers as students say the numbers.)

S: 20 (5 fingers), 24 (6 fingers), 28 (7 fingers).

T: Let's see how we can skip-count down to find the answer, too. Start at 40 with 10 fingers, 1 for each four. (Count down with fingers as students say the numbers.)

S: 40 (10 fingers), 36 (9 fingers), 32 (8 fingers), 28 (7 fingers).

Continue with the following possible sequence: 9×4 , 6×4 , and 8×4 .

T: (Distribute the Multiply by 4 Pattern Sheet.) Let's practice multiplying by 4. Be sure to work left to right across the page.

Figure 22. Fluency Practice activity from Grade 3 Lesson 4.

In the Multiply by 4 activity, both Emily and Kim began with the problem $7 \times 4 = \underline{\quad}$ and asked students to skip-count with them to find the answer. After skip-

counting to 28, both teachers presented the idea to the class about starting at a more convenient number like 20.

Table 20 includes the dialogue for the presentation of this idea.

Table 20

Emily and Kim's Multiply by 4 Fluency Practice Activity

Emily's Class	Kim's Class
<i>Emily:</i> (reading from slides) Let's skip-count by fours starting at 20. Why is 20 a good place to start? That's a good question. Who can use really good math words and explain that to us?	<i>Kim:</i> Instead of counting from 4×1 to get to 28, where could I have started in my counting by fours?
<i>Student 1:</i> Because 20 is halfway between 4 and 28.	<i>Student 1:</i> 4×2
<i>Emily:</i> Ok. Think about that property that we use where we ...(makes breaking motion with hands).	<i>Kim:</i> What's a bigger problem or a problem with a bigger answer that I could have started with that I know (snaps finger) just like that to get to 7×4 ?
<i>Students:</i> (several students in unison) break it apart and put it back together.	<i>Student 2:</i> You could of, um, done 4×4 and then do that answer...
<i>Emily:</i> Yes, break it apart and put it back together. That's the...	<i>Kim:</i> So we could have started at 4×4 . We maybe know that answer, but I'm thinking of an answer that maybe we know even better and even more quickly than 4×4 .
<i>Students:</i> (several students in unison) Distributive Property	<i>Student 3:</i> 4×6
<i>Emily:</i> Very good. So if we were doing that, what would we break 7 into? (makes 5 and 2 on her hands)	<i>Kim:</i> Maybe 4×6 . We're kind of hovering right around it.
<i>Student 2:</i> 5 and 2.	<i>Student 4:</i> 5×4
<i>Emily:</i> And what's 5×4 ?	<i>Kim:</i> 5×4 because when I multiply times 5, I know that my answer is going to end in one of two numbers. What are my numbers going to be that my answer's going to end in when I multiply times 5?
<i>Students:</i> (several students in unison) 20	<i>Student 5:</i> 5 or 0
<i>Emily:</i> And then we've got what leftover?	<i>Kim:</i> Very good. So I know that 4×5 ...even if I don't know it right off the top
<i>Student 3:</i> 2 and 4.	
<i>Emily:</i> Which is what?	

<p><i>Student 4:</i> 8.</p> <p><i>Emily:</i> So that would work too. You can break it apart and put it back together. That would work.</p>	<p>of my head, I could go 5, 10, 15, 20 (counting with fingers). Ok, 20, and then I can count by fours up to 28 to get to 7×4.</p>
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In this dialogue, Emily uses the Distributive Property and Kim uses students' knowledge of multiplication by 5 to help students find a convenient starting point. However, the work that followed these strategies provided different experiences for students. Kim's class went directly into completing the Pattern Sheet with no further exploration of starting at a convenient number like 20 or exploration that starting at any known multiplication fact of 4, ideas presented by two different students, would work. Emily's class did not complete the Pattern Sheet, but practiced skip-counting and starting at 20 for 9×4 , 6×4 , and 8×4 , with Emily repeatedly mentioning "break it apart and put it back together." Because of this, Emily's students had an opportunity to build fluency with their strategy of starting at a convenient number and with their multiplication facts, whereas Kim's students were introduced to the strategy of starting at a convenient number, but had limited opportunity to build fluency with this strategy.

Both Emily and Kim drew upon some aspects of the Enactment Guidance for Experiences in the written curriculum to build fluency with multiplication facts, but the enactments of this guidance resulted in different experiences for students. Kim focused on using the Pattern Sheet to build fluency, while Emily focused on the use of oral skip-counting and starting at a convenient number to build fluency. Although Emily's connection to the Distributive Property was not part of the written curriculum, her use of

this property with students offered an opportunity for students to think flexibly about its use in another context.

Part 1 of the written Concept Development in Lesson 4 focused on having students physically sort 12 quadrilaterals cut out from a template and developing the mathematical terminology (e.g., right angle, parallel, parallelogram) to sort the quadrilaterals as various groupings are constructed. Emily's enactment of this Concept Development began by reviewing the terms polygon and right angle as well as directing students to use the index card as a tool for finding right angles and the chart of terms, definitions, and accompanying pictures she had drawn as a reference throughout the lesson. Emily then instructed the students to "look at your shapes and see how you might categorize these or how you might sort your shapes. What would you do?" She encouraged students to talk to their neighbors and move the shapes around to look for similarities or differences in the shapes. As students were working, Emily circulated around the room, observing students' work, and asking questions to elicit student sorting strategies, redirecting the focus to mathematical ideas (e.g., "Look at the shapes. What do you notice about the shapes?") when students seemed to focus on non-mathematical ideas (e.g., the letters on the shapes). After about five minutes of exploration and students sharing out their sorting strategies, Emily then channeled the sorting of the quadrilaterals according to the attributes on the Problem Set, taking opportunities for students to test, for example, the presence of right angles or parallel lines in the shapes at their desks. Through this enactment, Emily had taken up the Enactment Guidance for Experiences in the written curriculum that allowed students to explore the quadrilaterals in such a way that the connections among them are revealed to the students rather than directed by her.

Kim's enactment provided a contrasting experience for her students. After the class reviewed the Application Problem on the carpet, Kim provided an index card for each student and described its use as a right-angle tool, reviewing the terms angle and right angle with the class. Using the full sheet template of quadrilaterals projected by the document camera, Kim asked students to find the shapes with a right angle or "a sharp, square corner." Students suggested such quadrilaterals, and using her set of cut-out shapes and the index card (see Figure 23), Kim determined whether the quadrilateral suggested by the student had a right angle. For some student suggestions, those students were asked to test the presence of a right angle in the quadrilateral on the template projected on the board (see Figure 24).

Figure 23. Kim's use of the index card to determine right angles.

Figure 24. Student demonstrating use of index card for class.

After all quadrilaterals with right angles had been found, Kim repeated the process with the vocabulary terms parallel, trapezoid, and parallelogram. In Kim's enactment of this Concept Development, students were led to examine the quadrilaterals according to the order described in the written curriculum; however, students were not given opportunities to explore sorting the quadrilaterals on their own, nor were they given opportunities to share how they see connections among the quadrilaterals. Kim's take-up of the Enactment Guidance for Experiences here is quite different than Emily's, and in the end, led to different opportunities for students to engage in the mathematics.

Grade 4 lessons. The Concept Development of both lessons (Lessons 3 and 9) as well as the Count by Tenths Fluency Practice activity in Lesson 3 at fourth grade contained codes for Enactment Guidance for Experiences. The Concept Development in Lesson 3 focused on students bundling tenths (0.1) number disks in groups of 10 and exchanging them for a 1 whole number disk to write, for example, 21 tenths as the decimal 2.1, while the Concept Development in Lesson 9 focused on three problems where students were engaged in reading measurements from a meter stick, scale, and a set of graduated cylinders to compare the decimals arising from those measurements.

For both lessons observed in this study, Holly utilized the Problem Set in the place of the Concept Development activities in the written curriculum. That is, rather than utilizing the number disks and measurement tools to learn about renaming tenths and comparing decimals, Holly used the representations of number disks and measurements on the Problem Set as the instructional activities to learn about comparing decimals. In contrast, Rachel used most of the mathematical tools in the instructional activities. These differences in take-up of the key educative features for Enactment Guidance for

Experiences resulted in similar learning goals but different opportunities for the students to experience or engage with the mathematics. For Holly's classroom, using the Problem Set as the Concept Development was typical, and she has transitioned to this over time because students were not getting very far into the Problem Set before their time for math was over, making Homework unnecessarily difficult for students, in her view, and creating potential differences between students in what was learned since problems on the Problem Sets are organized in increasing difficulty. In the Lesson 3 reflection meeting, Holly provided justification for using the Problem Set as the Concept Development,

When we first started implementing [Eureka Math], I just saw tons of frustration in my low-level kids because they couldn't get past number 3 ever on the page. Well that's annoying for Eureka Math to say, oh, it's fine. You don't have to get done with it. Well, they're 9, and they want [to finish] it all. It's really frustrating to them and I would see them shut down a lot, which is when I started doing the Problem Set together as the Concept Development. Then we can do the Exit Ticket as homework or then we can do the homework as guided practice...there have been times when I've been like, okay, so we've done the Problem Set together and now you have the option to work on your own or come sit on the floor with me and we'll do the homework page together. (April 11, 2019)

For Holly, doing the Problem Set in the place of the written Concept Development activities provided the affirmation she felt her students needed in order to be successful on other lesson components.

To illustrate the differences in take-up of Enactment Guidance for Experiences that created different ways that the mathematics was experienced by students, I compared

interactions from both teachers' enactment of Lesson 9, specifically from Holly's enactment of comparing decimal numbers representing length to Rachel's enactment of comparing decimal numbers representing mass in this lesson. For both enactments, students experienced comparing a smaller number of tenths (e.g., 3 tenths) with a larger number of hundredths (e.g., 27 hundredths), requiring them to think about how the place value affects the number.

In Holly's enactment of comparing decimal numbers representing length, she used students' fraction knowledge developed in the previous module to examine how the visual representation of a meter is partitioned into pieces and then compared the lengths of those pieces to one another on Problem 1 of the Problem Set (see Figure 25). After counting the pieces for the first meter and determining it is split into tenths, Holly asked the class to write the decimal value to the side of the meter (see Figure 26). Similarly, she worked with the students to determine how the second meter was partitioned and wrote a decimal value for that meter.

1. Express the lengths of the shaded parts in decimal form. Write a sentence that compares the two lengths. Use the expression *shorter than* or *longer than* in your sentence.

Figure 25. Problem 1 from Lesson 9 Problem Set.

Figure 26. Holly's comparison of decimal lengths.

The following interaction revealed how Holly utilized students' fraction knowledge and the visual representation to compare decimal numbers:

Holly: So how do I compare these? (Points to both meters.) First of all, which is more? Top or bottom?

Students: (several in unison) Bottom. (several in unison) Top.

Holly: Which one has more shaded? Top or bottom?

Students: (several in unison) The top.

Holly: The top has more shaded. Ooh, somebody explain to me how 3 tenths can be more than 27 hundredths. 3 is less than 27, so how can 3 tenths be more than 27 hundredths, [Student 1]?

Student 1: Because 3 is a bigger number than 2...zero and 27 hundredths and the smaller number has more shaded than the bigger number.

Holly: Ok. You are on the right track. Let's add a little more to that to compare them, [Student 2].

Student 2: Well, [inaudible] it's 2 and 27 hundredths.

Holly: It's 2 and 27 hundredths?

Student 2: Wait, no. It's...

Holly: 2 and 27 hundredths would mean I have 2 whole meters.

Student 2: Well, if you broke it down, it would be 2 and 7...because there's...so basically, the top one is 30 and the bottom one is 27.

Holly: Oh! The top one, 3 tenths, is equal to what, [Student 2]? (Holly adds a post-it note to the Problem Set paper under the document camera and writes $0.3 = 0.30$.)

Student 2: 30 hundredths.

(Holly uses the post-it note, see Figure 27, to review also that the fractions $3/10$ and $30/100$ are equivalent.)

Holly: Does everyone see how 3 tenths is equal to 30 hundredths?

Students: (several in unison) Yes.

Holly: Oh, I heard it. I heard somebody go, "Oh!" So 3 tenths (points to 0.3 on post-it note) is equal to 30 hundredths (points to $30/100$), and 30 hundredths is more than 27 hundredths.

(Holly works through part b on the Problem Set to determine the lengths represented in the other tape diagrams to be 0.35 m and 0.4 m in preparation for part c to order all four lengths from both parts from least to greatest.)

$$0.3 = 0.30$$

$$\frac{3}{10} = \frac{30}{100}$$

Figure 27. Holly's review of equivalent fractions.

In contrast, Rachel's enactment of comparing decimal numbers representing mass did not involve students measuring the mass of the bags because her scale showed the decimal places to the thousandths place for all measurements. Instead, she instructed students to fill in the measurement record on the template with 0.1 for Bag A, 0.37 for Bag B, 0.78 for Bag C, and 0.4 for Bag D. Rachel asked students to order the masses from least to greatest, discussing the order with their groups. This initial interaction established that students were split between two orders of mass for the bags: 0.1, 0.4, 0.37, 0.78 and 0.1, 0.37, 0.4, 0.78. The interaction that follows reveals how Rachel utilized a physical experience of walking out the place values and a visual experience of seeing the amount of beans in each bag to consider how the place value impacts the ordering of decimals:

Rachel: Which one of these (circling 0.37 and 0.4 on the board) is bigger? That's the debate. Some of you think D is bigger. Some of you think B is bigger.

(Rachel asks students to walk out the value of each place in the numbers, using normal steps for the number in the tenths place and "teeny, tiny" steps for the

number in the hundredths place. Rachel also used this strategy in Lesson 3.) For which number did you go farther?

Students: (several in unison) Four.

Rachel: Four normal size steps is going to be bigger than three normal size steps every time. It doesn't really matter how many teeny, tiny steps (pointing to the 7 in 0.37) you take here. Four normal size steps is going to be bigger than three normal size steps every time. (Pauses.) Alright, let's look at this another way. This scale actually goes out to thousandths. So Bag A...[Student 1], come up and read what it actually says.

Student 1: Zero dot one zero zero.

Rachel: (Writes two 0's at end of 0.1.) Zero dot one zero zero is what it actually says and here's how many beans are in that one. (Holds up Bag A. [...] Holds up Bag B.) This is B. [Student 2], come up and tell us what B says. [The scale reading] changed on me. Pretend the last number is a 0.

Student 2: Zero point three seven zero.

Rachel: (Writes 0 at end of 0.37.) Ok. It actually has a 1 there, not that it makes much difference, but...(Holds up Bag B and Bag D.) Ok, take a look up here. This is Bag B and this is Bag [D]. (Rachel circulates the room giving students a closer look at the amount of beans in the bags.) These two amounts are pretty close together, but can you see that there's a little bit more in one than the other? So if we put Bag [D] on the scale...[Student 3] come up and read what it says.

Student 3: Zero point four zero zero.

Rachel: Zero point four zero zero. (Writes two 0's at end of 0.4. Administrative talk.) So now take a look at these two. (Points to 0.3700 and 0.400 on board.) So now does everyone agree that D is bigger? D was bigger than B. When I take these zeros off (erases zeros at end of 0.370 and 0.400), D is still bigger than B. I didn't change how many beans were in the bags.

Student 4: Are you sure about that?

Rachel: Yep. [Student 4], you don't believe me?

Student 4: I do. I do.

Rachel: You might, but other people are still going, "What are you talking about?" Four tenths is how many hundredths?

Students: (several in unison) 40 hundredths.

Rachel: 40. 37 hundredths (pointing to the value). 40 hundredths (pointing to the value). [...] That's (pointing to 0.40) always going to be more.

In both Holly and Rachel's enactment of this lesson, students had the opportunity to learn to order decimals; however, the ways that the mathematics was revealed to students were quite different. While Holly drew upon students' fraction knowledge as well as seeing the relative lengths and the numbers themselves, Rachel engaged students in physically experiencing the place value as well as seeing the measurements on the scale and relative masses of the bags. Both enactments incorporated aspects of the written curriculum. The Problem Set used in Holly's class was taken directly from the curriculum materials and contained problems similar to the written Concept Development activities (e.g., measurements on meter sticks, scales, and graduated cylinders). The activities in Rachel's enactment of reading the measurements from the meter sticks and scales were

taken directly from the written Concept Development materials and adapted, in the case of the scales, for the mathematical tools in her classroom. Working from fraction knowledge to determine the decimal value for the shaded lengths in Holly's enactment and physically walking out the decimal values in Rachel's enactment were modifications these fourth-grade teachers made to the written curriculum materials.

The Count by Tenths Fluency Practice activity in Lesson 3 also contained a code for Enactment Guidance for Experiences at fourth grade. This activity focused on choral counting by tenths up to and down from 50/10, working up to identifying the whole numbers in the sequence. While neither Rachel nor Holly utilized this activity as written in Lesson 3, there was evidence of take-up of choral counting experiences in Holly's classroom during both lessons. In Lesson 3, Holly stopped the lesson in her section after lunch when students were displaying tiredness and inattention to do jumping jacks counting by five tenths up to 60 tenths. At 60 tenths, Holly stopped the class and asked them how many wholes is 60 tenths. In Lesson 9, Holly chose to use the Sprint from the omitted Lesson 8 and engaged students in a similar quick choral counting experience of five hundredths starting at 0 between the two sides of the Sprint. During the Lesson 3 reflection, Holly expressed the value that she has found in choral counting experiences:

I think that [choral counting] kind of helps keep it fresh because if they're having to think about 1, and today we [counted by four hundredths to 1 starting at 0], they just have to keep thinking about it. My hope would be that they're seeing it. If they're able to see it in their head as they say it, then they're able to kind of know what it looks like and then be able to write it down better and whatever. Also, it's something that feels easy to them to be able just to do. Maybe what

we're doing on paper is hard, but if I can stand up and have a brain break for 10 seconds where I can count by 5 hundredths, you know, that feels easy and maybe I won't checkout. That's always my fear is that they would feel like this is too hard for me and then they would just check out and not want to do anything. (April 11, 2019)

In contrast, Rachel expressed that she does not find the choral counting experiences beneficial given the amount of time one would spend on enacting them. She has used them in the past in Module 5 *Fraction Equivalence, Ordering, and Operations* to examine fraction equivalence and would use them currently if there was more time available with both of her sections. Currently, Holly and Rachel have only an hour for math with each of their sections.

Each of these enactments, because of the experiences chosen by the teachers, also resulted in differences in take-up of Enactment Guidance for Using Mathematical Tools and Enactment Guidance for Curricular Features as well as differences in take-up of Enactment Guidance for Participation Structures and Facilitating Discourse, which are discussed in the next section.

Pedagogical Content Knowledge for Mathematics Practices

In this category of content knowledge, differences in take-up occurred for the key educative features of Enactment Guidance for Participation Structures and Enactment Guidance for Facilitating Discourse. Enactment Guidance for Facilitating Discourse was coded in the Concept Development and Student Debrief for all target lessons, while Enactment Guidance for Participation Structures was coded in the Concept Development for all target lessons and in the Student Debrief of one target lesson at each grade level. I

first consider the two key educative features at each grade level separately to make clearer connections between the written and enacted curriculum, followed by a summary of this category of content knowledge as a whole.

Enactment Guidance for Participation Structures. This key educative feature provides guidance on appropriate ways that students can participate in lessons. These features in the Eureka Math curriculum may provide suggestions for how to group students and share out student solution strategies for the duration of the Concept Development as well as suggestions for teachers to have students compare solutions or explain an idea to a partner or small group.

Grade 3 lessons. While the participation structures code in Lesson 4 focused only on discussion with partners, the Suggested Problem-Solving Steps (see Figure 28) in Lesson 23 provided a full structure for student participation to solve a series of six-word problems of perimeter. The guidance suggested that teachers select pairs of students to work at the board while others work independently or in pairs at their desks. I use Lesson 23 to illustrate differences in take-up of Enactment Guidance for Participation Structures between Emily and Kim.

1. Read and model.

Review the following questions after reading the first problem:

- Can you draw something?
- What can you draw?
- What conclusions can you make from your drawing?

When most students have finished, have the pairs of students at the board share only their labeled diagrams. Have the demonstrating students receive and respond to feedback and questions from their peers.

**NOTE ON
MULTIPLE MEANS
OF REPRESENTATION:**

Students who have difficulty drawing an octagon to solve Problem 1 might choose instead to model with a tape diagram consisting of 8 equal parts and having a total value of 48 cm.

2. Write an equation, calculate to solve, and write a statement.

Allow time for students to finish work on the question. They then retrace the steps of their thinking as they share their work with a partner or another pair. Students write their equations and statements on their own Problem Sets. Demonstrating students can model this process for others.

3. Assess the solution for reasonableness.

Give students one to two minutes to validate and explain the reasonableness of their solutions. Two or three of the following sentence starters may be provided to guide them in this work:

- I reread the question to remind myself that it asks for ____.
- My answer matches what the question asks because ____.
- The units of my answer make sense because ____.
- I know my answer is neither too small nor too big because ____.
- My answer would not make any sense if it was ____ because ____.

Figure 28. Problem-Solving Steps from Grade 3 Lesson 23 (p. 311)

In Emily's class, the Concept Development began with Emily creating groups of two or three students to work together on the problems. For each problem, a group was asked to work together at the board; however, sharing the solution strategy of the group working at the board with the rest of the class only happened a few times as the class reviewed the entire Problem Set together toward the end of class time. As students were working around the room, Emily was able to observe solution strategies of those working at the board and at their desks. This aspect of her participation structure allowed for Emily to point out the solution strategies she saw students using around the room as the entire Problem Set was reviewed, which is described in the discussion of Enactment Guidance for Facilitating Discourse.

As a whole, this participation structure allowed Emily to be "focused instead of sitting at one group and then getting sidetracked because somebody has a question"

(Lesson 23 reflection meeting, April 30, 2019). Additionally, in the reflection meeting, Emily described using a multi-ability grouping structure, saying “If it's something like this where I want them to have an opportunity, those kids are good teachers too...They're actually getting some information from the other students and it's something that's going to actually help them.” This participation structure, although somewhat different than the written curriculum, afforded Emily closer observation of student solution strategies across the room from which to present to the entire class rather than a single strategy presented by a single group.

In Kim’s class, as each problem was projected on the board, she led the problem-solving process by first reading the problem and then decoding the words in the problem for the students. Students took notes for the solution to the problem in their math notebooks and answered Kim’s closed questions about each problem. I discuss this further in the upcoming section on Enactment Guidance for Facilitating Discourse, but here it suffices to note that the participation structure used by Kim offered no opportunities for students to work together on problem solving, limited opportunities for students to figure out a solution for themselves, and utilized little of the Enactment Guidance for Participation Structures in the written curriculum.

Grade 4 lessons. The participation structure codes in the written vignettes of Lesson 3 were restricted to working with a partner to discuss or complete a task given by the teacher such as “Discuss the total value of the number represented by the disks with your partner.” Participation structure codes in Lesson 9 included similar partner work; however, also included in this lesson were opportunities for the teachers to select a student volunteer to order the bags of rice or graduated cylinders for the class. I will

examine consistencies within each teacher's classroom across both lessons to illustrate differences in take-up of these types of enactment guidance between Holly and Rachel.

Both fourth-grade classrooms observed in this study are organized into groups of four desks oriented in sets of two desks facing one another and perpendicular to the projection area (see Figure 29).

Figure 29. Sample fourth-grade classroom layout.

In addition, teachers in both classrooms offered opportunities for students to participate both within these table groups and as a class, but no participation across table groups was observed in these lessons. In Lesson 9, the written curriculum included guidance for teachers to have students share with a partner about how the place value chart helps with comparison, to work with a partner to order the decimal measurements of rice bags' masses and graduated cylinders' volumes, and to select a student volunteer to physically order the rice bags and graduated cylinders. The participation structures that distinguish Holly's classroom from Rachel's classroom is the level to which students work with others at their table groups and the tasks on which the table groups work.

In Holly's class, most participation was directed back and forth between the teacher and students with a variety of question types (e.g., recall, explanation) rather than

providing students a chance to discuss concepts among themselves. For example, in Lesson 3, tasks given to table groups were “telling” tasks such as “Tell your teammates how to write 3 and 9 tenths as a decimal” and “Tell the people at your team how many tenths make one whole” rather than tasks that would utilize the thinking of various people at the table groups.

In Rachel’s class, I observed participation with the full class and among students at table groups as well as Rachel selecting student volunteers for tasks. Notably, Rachel’s choices for instructional activities may have necessitated participation structures that were utilized in her lesson. Consider Lesson 9 when Rachel chose to have students compare decimal measurements taken from mathematical tools. In the first activity, each group had a meter stick to read and the group was charged with determining a length in meters for the tape. Additionally, in the second activity, students were asked to work in their table groups to order the masses from least to greatest before the whole class discussed the ordering as well as student volunteers were chosen to use the scale to find the mass for the class. Students were also engaged in “walking out” the decimals 0.4 and 0.37. These participation structures, influenced by the instructional activities, offered different opportunities for students and also influenced take-up of guidance for facilitating classroom discourse.

Enactment Guidance for Facilitating Discourse. This key educative feature focused on guiding teachers to lead students in “a purposeful exchange of ideas” (NCTM, 2014, p. 29) using questions, student ideas, and student solution approaches to engage in mathematical talk. In the written Concept Development of these lessons, this exchange of ideas was typically presented as a vignette of this exchange between teachers and

students. In the written Student Debrief of these lessons, the guidance was presented as a list of questions that could be used to lead a discussion at the conclusion of the lesson.

Grade 3 lessons. While the discourse codes in the Student Debrief section of the written curriculum contained the typical list of questions, the discourse codes in the Concept Development at third grade were quite different when comparing Lessons 4 and 23. In Lesson 4, the discourse was presented as a vignette between the teacher and students with questions such as “What do you notice about the polygons you grouped?”, “Look at Polygon I...why isn’t it included in the last category?”, and “Why is a square a rhombus?”, simultaneously engaging students in reasoning about the sorting experience and mathematical terminology. In Lesson 23, the discourse in this problem-solving lesson was two-fold, with the first coded segment providing guidance to have students who are demonstrating at the board receive and respond to feedback and questions from their peers and the second coded segment guiding teachers to “Encourage diversity with solution strategies to make for interesting conversation about the problem.”

In Lesson 4, differences in take-up of the sorting experience by Emily and Kim resulted in differences in take-up of Enactment Guidance for Facilitating Discourse. As Emily’s students were engaged in sorting the quadrilaterals, they were encouraged to talk to one another about their work and share their ideas with the class. Moreover, while Emily elicited student suggestions for certain characteristics, she also included questions engaging students in reasoning about their sorting (e.g., “Why isn’t shape I a square?”) and noticing characteristics of the polygons (e.g., “What do you notice about all of the polygons?”). In contrast, the directed presentation of the sorting experience in Kim’s class limited the discourse to eliciting student suggestions of quadrilaterals meeting

certain characteristics and little to no opportunities for engaging in reasoning about the quadrilaterals' characteristics.

These differences in take-up of Enactment Guidance for Facilitating Discourse were also seen in Lesson 23. Although Emily and Kim offered minimal opportunities for feedback and questions about particular solution strategies, Emily's participation structure while going over the Problem Set encouraged the Enactment Guidance for Facilitating Discourse ("Encourage diversity with solution strategies to make for interesting conversation about the problem") on Problem 4 as well as others in the set. On the other hand, Kim's questions arising from her decoding of the problems directed the solution strategies with each problem. To illustrate this, I juxtapose Emily's class discourse while going over addition strategies for Problem 4 (see Figure 30) with Kim's class discourse while working on Problem 4 together in Table 21 Kim used a rectangle for Problem 4 that was 2 square tiles wide and 3 square tiles long, and while the side lengths and total perimeter are different, the comparison of discourse is being illustrated here.

4. Raj uses 6-inch square tiles to make a rectangle, as shown below. What is the perimeter of the rectangle in inches?

Figure 30. Problem 4 from the Problem Set of Grade 3 Module 7 Lesson 23.

Table 21

Excerpts of discourse from Lesson 23 in third grade

Emily's Class	Kim's Class
<p><i>Emily:</i> Ok, so what is the perimeter of the rectangle? I'm going to go ahead and underline and circle that so I remember I'm doing perimeter. What do we do when we do perimeter?</p> <p>(Several responses are given including length times width, length plus width, and add 30 + 18.)</p> <p><i>Student 1:</i> You add all the sides up.</p> <p><i>Emily:</i> Add all the sides up, not just length plus width. You've got to add up all the lengths and all the widths, right? So let's do it. What's 30 plus 30? (Writes $30 + 30$ and shows grouping the 30's together to make 60 in.)</p> <p><i>Students:</i> (several in unison) 60.</p> <p><i>Emily:</i> Plus 18 plus 18. (Completes $30 + 30 + 18 + 18$ and shows grouping the 18's together.)</p> <p><i>Student 3:</i> Equals 36.</p> <p><i>Emily:</i> How did you do that?</p> <p><i>Student 4:</i> Because I know that 9 times 2 is 18 (pauses) and 9 times 4 equals 36, so it's 36.</p> <p><i>Emily:</i> So say that one more time how you got to it.</p> <p><i>Student 4:</i> I know that 9 times 2 is 18 (Emily writes $9 \times 2 = 18$) and I know that if we double that, it's 9×4 and 9×4 is 36 (Emily writes $9 \times 4 = 36$).</p> <p><i>Emily:</i> Ok (happily). Very good. (Clapping by other students in background.) Who else could tell me how they can figure out what $18 + 18$ is? [Student 5].</p>	<p><i>Kim:</i> How do I find the perimeter of this rectangle in inches, [Student 1]?</p> <p><i>Student 1:</i> 18 plus 12 plus 18 plus 12.</p> <p><i>Kim:</i> Very good. 18 plus 12 plus 18 plus 12 (Writes this vertically on the board.) Do you see any tens?</p> <p><i>Student 2:</i> 18 and 12.</p> <p><i>Kim:</i> Ok. 8 and 2 is 10 (grouping the first 8 and 2 and showing 10) and 8 and 2 is 10 (grouping similarly). So down here I have 20 (writes 0 in the ones column) and carry my 2 (writes 2 at top of tens column). [Student 3], what's my answer for my perimeter? (Administrative talk)</p> <p><i>Student 3:</i> 60 (says quietly)</p>

Student 5: I would do 8 plus 8 and that's 16 and then I would do 3 ones.

Emily: Ok. Did anyone else get it a different way? (pauses) Did anyone else draw it out the standard way? 18 plus 18 like this? (Writes the addition sentence vertically.)

Students: (several in unison) No.

Emily: No. Ok. Now am I done?

Students: (several in unison) No.

Emily: Ok, so what do I have to do?

Student 6: You have to do 60 plus 36 equals 96.

Emily: Ok, so 60 plus 36 equals 96 inches (writes on board).

In examining the discourse of the addition strategies, Emily used the thinking of two different students to add 18 and 18, the first of which provided an interesting strategy decomposing each 18 into two 9's. Emily then provided a third strategy when she asked if any students had added the numbers vertically and demonstrated the strategy for students. On the other hand, Kim did not encourage a diversity of solution strategies, choosing instead to show the standard algorithm of stacking for addition. In the other problems on the Problem Set, Emily encouraged discourse surrounding multiple solution strategies, while Kim's solution methods were more traditional in nature.

Interestingly, both Emily and Kim described their classroom discourse surrounding multiple solution strategies in similar ways during the Lesson 23 reflection meetings, yet their enactments and origins for this idea differ. Crediting Eureka Math with promoting number sense and herself for promoting multiple solution strategies,

Emily stated, “every single day of math, you have to tell me how you got it. It's helped everyone else because they're listening to other people's ideas and, sometimes, they give me ideas.” Kim goes even further:

From the very beginning of the year, we look at multiple ways to solve a problem and then, about mid-year, a light bulb comes on and they go, I can solve it this way too. Oh wait, I can solve it this way. Oh wait, I can do... That's something that I think Eureka is really good with is allowing [students] to find different ways to come up with the same answer.

While Kim credits Eureka Math with promoting multiple solution strategies, her enactment of both Lesson 8 and 23 minimized the opportunities for students to display multiple solution strategies.

Grade 4 lessons. The discourse codes in the target lessons at fourth grade contained the typical vignettes in the Concept Development and the typical list of questions in the Student Debrief. However, the nature of the discourse presented in the written Concept Development was different. In Lesson 3, the exchange of ideas contained mostly closed questions on procedural tasks on grouping tenths together to rename decimals and navigating among various symbolic and visual representations of decimals, whereas in Lesson 9, the exchange of ideas focused on comparing decimals arising from measurements on meter sticks, scales, and graduated cylinders. In this lesson, the exchange of ideas engaged students in reasoning about the comparison using questions such as “How do you know?” and “How did you determine the order?”

Similar to enactments in the third grade, the take-up of Enactment Guidance for Experiences influenced the take-up of Enactment Guidance for Facilitating Discourse,

specifically in the enactments of Lesson 9. As alluded to in the preceding section, the differences in take-up of discourse in fourth grade were mainly due to differences in take-up of Enactment Guidance for Participation Structures. Recall that, in this lesson, Holly used the Problem Set as the Concept Development of the lesson and Rachel utilized measurements taken from tape on meter sticks and bags of beans on a scale to compare decimals. There was considerable discourse between students in Rachel's class as they worked together to determine the lengths of tape and order the masses of the bags of beans; however, the discourse in Holly's class was relegated to "telling" tasks directed between the students and the teacher.

In Rachel's class, students had multiple opportunities to talk with each other to reason about the mathematics they were learning. For the task involving the meter sticks, students quickly said 67 cm of the meter stick was covered with yellow tape; however, when Rachel pressed them to describe how many meters were covered, students encountered some difficulty. She encouraged them to talk to each other at their table groups to decide on a value. In one group, the exchange of ideas included statements such as "I think it's 67 hundredths." and "I think you are probably right because there is a hundred here." as well as gesturing toward the values all along the meter stick. Additionally, Rachel asked students to discuss in their table groups how to shade 0.67 m on the tape diagram found on the measurement record before actually shading it. For the task involving the bags of beans, Rachel provided the masses and asked students to talk in their groups to order the masses from least to greatest. In one group, all three students shared their ordering with each other, with one student asking another whether he ordered least to greatest or greatest to least. In another group, all three students shared their

orderings, but the group could not come to a consensus. One student was adamant that 0.4 was less than 0.37, even after a pre-service teacher observing in the class put her finger over the 7 and asked the student to read the decimals. Rachel's take-up of the Enactment Guidance for Facilitating Discourse in this lesson allowed students to reason with others about how much of a meter was covered and how the masses of the bags were ordered.

In Holly's class, discourse was relegated to "telling" tasks directed primarily between individual students and the teacher. Holly used questions such as "Somebody explain to me how 0.3 can be more than 0.27." Although the chosen student's response to this question as well as the additional responses given by others (as described in the Grade 4 section on Enactment Guidance for Experiences) offered students as a group to reason about the comparison of the decimals, the nature of the discourse was typically directed back to the teacher instead of among the students.

Although the discourse in both classes for Lesson 9 allowed opportunities for students to reason about the comparison of decimals, the nature of the discourse was different. Moreover, take-up of the Enactment Guidance for Facilitating Discourse was influenced by the Enactment Guidance for Participation Structures and Enactment Guidance for Experiences; that is, Rachel's use of the instructional activities and participation structures allowed the nature of discourse in her class to include student-to-student discourse.

Similarities in Take-up of Key Educational Features

Although there were clear differences in take-up of Enactment Guidance for Experiences, Participation Structures, and Facilitating Discourse between teachers at the

same grade level, analysis also revealed similar take-up of some educative features across teachers at the fourth-grade level for Enactment Guidance for Representations, even when the written curriculum in these observed lessons did not include features describing these representations. Additionally, analysis revealed similarities across teachers at the same grade level for Enactment Guidance for Developing Mathematical Terminology. I consider each of these key educative features separately and then summarize the similarities at the conclusion of this section.

Enactment Guidance for Representations

At the fourth-grade level, both target lessons included Enactment Guidance for Representations in the Fluency Practice and Concept Development components of the lessons. The majority of this guidance was directed at helping teachers navigate within and among visual and symbolic representations (NCTM, 2014). For each type of representation, I share enactment data of how Holly and Rachel took up these educative features.

Visual representations. The target lessons in the written curriculum provide guidance for using number lines and drawings of place value disks in Lesson 3 and place value charts and tape diagrams in Lesson 9. Both Rachel and Holly utilized these representations in the written curriculum in similar ways; however, the use of number bonds as visual representations will be the focus for the analysis of visual representations here. Number bonds are a representation used to show decomposing numbers into parts and can be used to decompose numbers by addition or multiplication. The guidance for using number bonds was not included in these written lessons; however, both Holly and Rachel alluded to their use in previous modules at fourth grade and Rachel knew of their

use as early as first grade, saying that number bonds are now the visual representation both she and her students gravitate toward. Holly had observed students latching on to the number bond representation and using it independently, especially those who struggled “because numbers can feel really abstract to kids if it’s not their strong point” (Lesson 3 reflection meeting).

In Lesson 3, the Fluency Practice activity asked students to “Write 13 tenths as a mixed number.” After seeing what students had written on their personal white boards to solve this problem, Holly drew a number bond (see Figure 31) on the board and then led a discussion with the class:

Figure 31. Number bond from Holly's Lesson 3 enactment.

Holly: Some of us kind of drifted away from the basics here. 13 tenths.

Remember when we were looking at how to do this back in our fractions unit, we did a number bond. (See Figure 31. Holly writes and circles 13/10 on the board.)

[Student 1], how many tenths to make a whole?

Student 1: 10.

Holly: 10 tenths to make a whole (writes and circles 10/10) and what's left over, [Student 2]?

Student 2: 3

Holly: 3 is left over (writes and circles 3/10). So how many whole?

Students: (several in unison) 1.

Students: 1 and 3 tenths. (Holly writes $1 \frac{3}{10}$.)

This whole-class interaction took place during the Fluency Practice portion of the lesson, which means that this should have been review for the students. However, many students initially had some difficulty renaming the 13 tenths to a mixed number. Providing the visual representation of a number bond that students had become familiar with during the previous module helped them navigate the renaming in a way that seemed more natural. In the Lesson 3 reflection meeting, Holly described the use of number bonds in the curriculum in general and her reasons for using them with students:

[N]umber bonds is all through the fractions things. When we're decomposing, there's a heavy theme of decomposing the fractions and so they do a lot of the number bonds and stuff and that was something that seemed to make sense to this group. ... to be able to draw a picture like I had 13 and I have 10 over here and 3 over here. I think that helps a lot, especially for my [specific student] and my [specific student] that I called on who are really, really struggling most of the time. I knew that they knew that part and so to call on them to kind of throw them confidence a little bit.

For Holly, the use of number bonds as visual representations of thinking in working with decimals helped students who typically struggle participate in the discourse of the

classroom. In addition, the use of a familiar representation such as number bonds could boost their confidence in connecting to a new concept such as decimal numbers.

In Lesson 9, Rachel used number bonds on multiple occasions with Fluency Practice problems to decompose 2 ones and 3 tenths into tenths (see Figure 32) and to write 3.1 as a fraction with denominator of 100.

$$2\frac{3}{10} = \frac{23}{10}$$

$$\begin{array}{c} \wedge \\ \frac{10}{10} \quad \frac{10}{10} \quad \frac{3}{10} = \end{array}$$

Figure 32. Decomposing a mixed number in Rachel's Lesson 9 enactment.

Throughout these Fluency Practice problems, one student asked Rachel to “give me a challenge,” so she asked students to write 64.3 as a fraction with denominator of 10. In the discourse that followed, Rachel used a number bond reference to help students see that the number of tenths in 64.3 should be quite large:

Rachel: (Pointing to $64\frac{3}{10}$, see Figure 33) I am not going to draw all of this out, (smiles at the class) but if we start, each one of these is how many tenths?

(Various responses are given, including 6, 1, and 10. Rachel shakes her head no at 6 and 1.)

Rachel: 10 tenths. One whole is 10 tenths, right? (Writes $10/10$.)

Student 1: There's 6 tenths.

Rachel: No, there's SIXTY-FOUR (pauses) 10 tenths.

Student 1: Oooohhhh.

Student 2: So you'd have to draw that 64 times?

Rachel: Yes, so you would have to do this 64 times. I'm not going to do that though. Can anybody do that math in their head. 64 times 10 is what we're trying to figure out, right? So I've got 64, 10 times.

Figure 33. Number bond in Rachel's Lesson 9 enactment.

In this interaction, Rachel used a number bond idea to help all students engage in this difficult problem, rather than focusing just on the symbolic representation.

Both Holly and Rachel's enactments demonstrated how the visual representation of number bonds has filtered into their instruction, even when the written curriculum in these lessons did not mention the use of number bonds. In this way, their enactments served as examples of how educative features, specifically in the area of representations, can be transferred to new lessons or modules, and in doing so, may have allowed students opportunities to see and experience stronger connections between concepts.

Symbolic representations. The target lessons in the written curriculum provided enactment guidance for symbolic representations in both Fluency Practice and Concept Development activities. In both Lesson 3 and Lesson 9, the Fluency Practice activities provided guidance focused on navigating among the various symbolic representations of a decimal value (e.g., 2 ones and 3 tenths, 2.3, and $\frac{23}{10}$). The guidance in the Concept Development activities differed, with those in Lesson 3 focused on the various symbolic representations of expanded forms (e.g., 4.1 as $(4 \times 1) + (1 \times 0.1)$ or $4 + 0.1$) and those in

Lesson 9 focused on measurements expressed as decimals or fractions. To demonstrate the similarities in enactment, I focused on the symbolic representations of expanded form in Lesson 3 and the teachers' rationales for using multiple representations.

Although Holly used the Problem Set as the Concept Development in this lesson and Rachel used the instructional activities in the written Concept Development, both teachers drew upon multiple symbolic representations of expanded form, as seen in Figure 34 and Figure 35, respectively.

17.5	$17\frac{5}{10}$	$10 + 7 + 0.5$
	$74\frac{7}{10}$	$(7 \times 10) + (4 \times 1) + (7 \times \frac{1}{10})$ $70 + 4 + \frac{7}{10}$
	$22\frac{2}{10}$	
		$(8 \times 10) + (8 \times 0.1)$ $80 + 0.8$

Figure 34. Expanded form in Holly's Lesson 3 enactment.

24 ones 6 tenths
2 tens 4 ones

$$(2 \times 10) + (4 \times 1) + (6 \times 0.1)$$

$$20 + 4 + 0.6$$

Figure 35. Expanded form in Rachel's Lesson 3 enactment.

In Holly's class, the use of multiple representations came about as she was helping students work through a table comparing various representations of a number. In the first line in Figure 34, a number line had been shown with the location 17.5 marked, and Holly worked with the students to create the addition expanded form expression in the

third column. The class previously worked with the addition expanded form expression in the first module of the year where the focus is on place value of whole numbers. In the second and fourth rows of the figure, Holly worked with students to understand both the fraction and decimal expanded forms, respectively. In Rachel's class, the use of multiple representations in Figure 35 came about when she explicitly asked students to write 24 ones 6 tenths in expanded form, specifically she asked them to write 24 ones 6 tenths as an addition expression and as a multiplication expression. After students had worked on the problem individually, Rachel went to the board and began working through the multiplication expression seen in the third line. Initially, students had difficulty separating the 2 tens from the 4 ones, and Rachel connected the representation to money, asking students if they could go to a bank with 24, one-dollar bills and receive two, ten-dollar bills and four, one-dollar bills in return. She asked students to tell her the addition expression, shown in the fourth line, and connected this representation to both the multiplication expression and to the original 24 ones 6 tenths.

These activities afforded more opportunities for students to navigate among the representations for expanded form specifically, in addition to connecting those to the fraction and decimal representations of numbers generally. For Holly, talking about the multiple representations allowed students to see the value as a mixed number, decimal, and in expanded form, "I think that's huge because I think that empowers them to do a lot of things. I think that empowers them to solve a lot of problems because they can read that." In the Lesson 3 reflection meeting, Rachel's thoughts about multiple representations focused on being successful on the state test:

You never quite know how the [state] math test is going to work because the standards just say do expanded form. So I used to only teach addition. But it seems like I have seen...it as multiplication. I'm like I need to make sure my kids are familiar with that way as well...so I try to emphasize it more.

Rachel also described how a multi-select question on the state mathematics test could require students to choose all the symbolic representations equivalent to a certain value and recalled related conversations with others as she participated in state training related to item specifications for fourth-grade content a few years ago.

Although their reasons for drawing upon multiple symbolic representations differed, their take-up of these representations were similar. In both Rachel and Holly's enactments, students had opportunities to represent decimals and fractions visually and symbolically, expanding their understanding through whole-class discussion, making connections among these forms, and in Holly's words, "empower[ing] them to solve a lot of problems."

Enactment Guidance for Developing Mathematical Terminology

Recall from the coding framework that Enactment Guidance for Developing Mathematical Terminology provides guidance for teachers to develop students' vocabulary and notation as well as consider colloquial or contextual situations that complement the vocabulary and notation. At fourth grade, this guidance in the written curriculum focused on helping students distinguish between tenths and tens in Lesson 3 and mass and weight in Lesson 9. Neither Holly nor Rachel drew upon the guidance in Lesson 9; however, both teachers worked with students on multiple occasions across the target lessons to distinguish tenths from tens. For example, in Lesson 3, Rachel made

connections to money as students worked to expand 24 ones 6 tenths (see Figure 35). She asked, “if I have 24, \$1 bills, can I take them to the bank and say I need 2 [\$10 bills] and 4 [\$1 bills]?” Juxtaposing this idea with 6 tenths of a dollar helped students see the relative size differences between tenths and tens.

Perhaps the most interesting similarity in take-up of Enactment Guidance for Developing Mathematical Terminology occurred at the third grade when both Kim and Emily chose to take-up, without knowledge of the other’s choice, the exclusive definition of trapezoid. In the planning meetings with Kim and Emily, I purposely did not ask which definition for trapezoid would be used so as not to influence either teacher’s decision nor greatly influence how they might engage in developing students’ mathematical terminology. In a call-out box in the written curriculum (see Figure 36), developers included both definitions of trapezoid and shared that the K-5 Eureka Math curriculum uses the inclusive definition. In addition, the narrative portion of the Concept Development near the call-out box contains the inclusive definition. Both Emily and Kim did not notice the exclusive definition in the call-out box or in the narrative as their understanding of trapezoid when preparing and enacting the lesson was that a trapezoid has exactly one pair of parallel sides; that is, they were unaware that this was even an issue and did not recall encountering this discrepancy in previous years.

Figure 36. Definitions of trapezoid in Eureka Math. Taken from Grade 3 Module 7 Lesson 4 (p. 57).

The realization about the possibility of there being a second definition came the day following the Lesson 4 enactment when Emily continued with the next lesson and Kim spent time reviewing the Lesson 4 vocabulary using Zearn.org. Emily was teaching Lesson 5 where, in a Fluency Practice activity, a parallelogram, rectangle, rhombus, and square are projected sequentially as part of the Google Slides presentation and determined to be trapezoids because they have at least one pair of parallel sides. In the Lesson 4 reflection meeting Emily describes how she navigated this situation with her students during the Lesson 5 enactment:

[The Module 7 book] said it and my PowerPoint said it had to have at least 1 pair of parallel sides. I said I think it's only 1 pair of parallel sides. Then I'm like, let me Google this. I'm like, this is a perfect opportunity to see that maybe I've made a mistake. Maybe I taught you wrong. So I did that and everything I was reading said only 1 pair of parallel sides. Eureka Math says at least 1 pair of parallel sides. So I'm like, hang on. I go over to Kim, I'm like, what did you do on this? She said,

‘I was confused too. I don't know.’ I said, well I'm going to stick with what I've always known to say only 1 pair of parallel sides is a trapezoid.

Emily used this situation as a decision point for both herself and her students, seeking out the knowledge of others (e.g., Kim and websites) and researching it to make a determination on the definition.

During Response-to-Intervention time on the day following the Lesson 4 enactment, Kim wanted to spend time reviewing the previous lesson's vocabulary and chose to work through the video and exercises on Zearn.org, a website with free resources similar to activities in the Eureka Math curriculum. In this Zearn lesson, the inclusive definition of trapezoid is used in a sorting activity (see Figure 37) similar to the Lesson 4 Concept Development. Kim's reactions were similar to Emily's, deciding that she would move forward with the exclusive definition as well. As a result, neither Emily nor Kim utilized the educative feature in the written curriculum addressing both definitions for trapezoid.

Figure 37. Zearn sorting activity and definition of trapezoids.

Other Notable Observations from Enactment

Educative features located in the Fluency Practice, Application Problem, and Concept Development components of the curriculum were taken up by teachers; however, not all instructional activities provided in the written curriculum were utilized during enactment. When Eureka Math was initially adopted, most of the teachers felt compelled to enact all components in the written curriculum from the Fluency Practice activities to the Exit Ticket and, as a result, were spending as much as a third of their day on the mathematics lesson. Over time, teachers in this study have become more selective about the instructional activities chosen for their classrooms or have begun using some components of the curriculum in different ways. For the target lessons I observed, the following modifications to the components of the curriculum were made:

- All teachers utilized versions of the Fluency Practice and Concept Development activities for each lesson.
- Holly utilized the Application Problem in both lessons as a filler activity after the students were finished with their homework of the day. Rachel, Kim, and Emily spoke in the reflection meetings about the Application Problem as a way to keep concepts in students' minds in preparation for the state mathematics test.
- Other than Emily, no teacher utilized any type of Student Debrief activity. In Lesson 4, Emily's students came to the class carpet and reviewed the vocabulary from the lesson as their debrief activity.
- The Problem Set and Homework included in the written curriculum were not utilized as written across both grade levels. Some teachers gave portions of the Problem Set as homework, some teachers gave portions of the Homework as

homework, some teachers gave the remaining Sprint problems as homework, and some gave no homework at all.

- I did not observe the Exit Ticket being utilized at the fourth-grade level. At the third-grade level, Emily sometimes uses it as homework, while Kim binds all Exit Tickets into a packet and uses them at the end of the module to review for the end of module test as well as an opportunity to show parents what students in her class are learning.
- Both Kim and Emily utilize resources outside of the written curriculum. Both teachers draw upon Zearn.org activities from time to time as well as utilize modified Google Slides presentations available from the EMBARC.online website.
- Rachel and Holly utilize portions of the curriculum for the Response-to-Intervention time.

Over time, the teachers in this study have progressed from being overwhelmed by the volume of activities to now feeling freer to modify the instructional activities.

Summary

Throughout the analysis of target lesson enactments, differences and similarities in take-up of educative features were observed across the four teacher participants. Differences in take-up of Enactment Guidance for Experiences created different opportunities for revealing mathematics. Although take-up of the key educative features (e.g., sorting the quadrilaterals in Grade 3 Lesson 4 or using the measurements from mathematical tools in Grade 4 Lesson 9) typically resulted in similar learning goals, the ways students experienced learning was different between teachers at each grade

level. Additionally, the differences in take-up of Enactment Guidance for Experiences created differences in take-up of Enactment Guidance for Participation Structures and Enactment Guidance for Facilitating Discourse. That is, teachers' choices for how the mathematics was revealed through the instructional activities influenced the ways students participated (e.g., small group work, full class interaction) and how the discourse was facilitated. Enactment Guidance for Participation Structures in these modules contained guidance for participation among small groups of students as well as that for the whole class.

Similarities in take-up of educative features occurred for the key educative features of Enactment Guidance for Representations and Enactment Guidance for Developing Mathematical Terminology. In the take-up of guidance at the fourth-grade level, the repeated use of number bonds as visual representations and multiple versions of expanded form as symbolic representations allowed students to experience connections between their prior understanding of fractions and place value and their developing understanding of decimals. Moreover, the attention to distinguishing between tenths and tens connected their understanding of place value and decimals. In the take-up of guidance at the third-grade level, the teachers' choice of the exclusive definition of trapezoid influenced how students came to relate, or not, trapezoids to parallelograms and allowed them an opportunity to see that not everyone agrees on this definition. Similarities in take-up of the educative features in the categories of representations and mathematical terminology seem to be important when thinking about the scope and sequence of the elementary curriculum as a whole.

CHAPTER 6: DISCUSSION

In this chapter, I return to the focus of my study and summarize what I have learned in relation to the research questions:

- 1.) What educative features are present in Eureka Math materials?
- 2.) How do teachers take up the educative features in their planning and enactment of Eureka Math curricula?

To answer these questions, I conducted a two-phase study of Eureka Math curriculum modules at the third- and fourth-grade levels. The first phase of this study investigated the educative features present in the written curriculum of one module at each grade level, and the second phase of the study investigated how two teachers at each grade level took up (or not) the educative features in the Eureka Math curriculum.

In the sections that follow, I review the findings from each phase and discuss how these findings connect to and build on prior research. I also include a discussion that looks across the findings from each phase. Additionally, I address limitations of the study, implications for curriculum development, and recommendations for future research.

Discussion of the Findings

Phase One – Written Curriculum

As reported in Chapter 4, the analysis of the written curriculum revealed three major findings:

- 1.) Educative features most often provided guidance in the content knowledge category of *Pedagogical Content Knowledge for Mathematics Topics*.

- 2.) Educative features were most often found in the Concept Development portion of the lesson.
- 3.) Educative features were most often classified as Enactment Guidance as opposed to Rationale Guidance.

Guidance in the category of *Pedagogical Content Knowledge for Mathematics Topics*, the most prevalent category, was typically focused on how to enact instructional activities as mathematical experiences, differentiate for learners, or utilize representations.

Guidance in the categories of *Pedagogical Content Knowledge for Mathematics Practices* (e.g., utilizing participation structures and facilitating discourse) and *Curricular Content Knowledge* (e.g., providing rationales for storyline and utilizing curricular features) almost evenly split the remainder of the educative features. Guidance was also found to differ by the location in the lesson. In particular, guidance in the Concept Development portion was almost evenly split between *Pedagogical Content Knowledge for Mathematics Topics* and *Pedagogical Content Knowledge for Mathematics Practices*, while most guidance found in the Fluency Practice portion was typically from the category of *Pedagogical Content Knowledge for Mathematics Topics*. Additionally, Rationale Guidance, though relatively rare compared to Enactment Guidance, typically involved *Pedagogical Content Knowledge for Mathematics Topics* (e.g., rationales for a particular mathematical experience or instructional sequence) and *Curricular Content Knowledge* (e.g., rationales for the inclusion of instructional activities).

In comparing the results of this study with prior research, Beyer and colleagues' (2009) analysis of high school biology textbooks found most educative features in the category of *Pedagogical Content Knowledge for Science Topics*, which is somewhat akin

to my finding of most educative features in the category of *Pedagogical Content Knowledge for Mathematics Topics*. Males (2011), on the other hand, found most educative features in the category *Pedagogical Content Knowledge for Mathematics Practices* with *Pedagogical Content Knowledge for Mathematics Topics* and *Mathematics Curricular Content Knowledge* following. Recall, however, that Males added the domain of *Mathematics Curricular Content Knowledge* to the analytical framework developed by Beyer and colleagues. In addition, she adapted the framework to incorporate practices specific to mathematics education.

Although this study's finding was more similar (with regard to prevalence) to the research from science education, consideration of its relationship to Males' finding from mathematics education research should be closely examined. Males' study and this study differed in three major ways: grade level, units of focus, and curricular alignment to standards. Males' study focused on middle school level units on transformations and introduction to variables, while this study focused on elementary level units on geometry and measurement and decimal fractions. In her study, Males found differences in prevalence across units, particularly for the educative features of Enactment Guidance for Using Mathematical Tools and Enactment Guidance for Engaging Students in Justification, Reasoning, and Proof (more prevalent in the transformations units) as well as for Enactment Guidance for Representations (more prevalent in the variables units). Although not a specific area of analysis in this study, differences across grade levels for these particular educative features were also observed here; therefore, the grade level and unit of focus do not fully explain this finding.

Perhaps the greatest difference in findings between this study and Males' (2011) study can be attributed to the alignment of the curricula in each study to respective standards. Males considered the curricula in her study to be designed to align with the NCTM Standards (NCTM, 2000), which she claimed heavily aligned with her *Pedagogical Content Knowledge for Mathematics Practices* category. The Eureka Math curriculum in this study is aligned to the CCSS (National Governors Association Center for Best Practices & The Council of Chief State School Officers, 2010), which even though these standards include the Standards for Mathematical Practice, the focus when developing this curriculum was toward "the new college- and career-ready standards, which emphasize deeper learning, critical thinking, and conceptual understanding of math" (Great Minds, 2020). Although research suggests that deeper learning and critical thinking are also accomplished through such mathematical practices as discourse and participation structures (NCTM, 2014), the prevalence of Eureka Math educative features in the category of *Pedagogical Content Knowledge for Mathematics Topics* may suggest that the Eureka Math curriculum gives greater attention to developing the mathematics content through topic-specific instructional activities, including the use of representations and mathematical tools, whereas curricula such as the *Connected Math Project*, UCSMP's *Transition Mathematics*, and Glencoe's *Math Connects* (studied in Males, 2011) consistently gave greater attention to mathematics practices across both units.

Although Beyer et al. (2009) and this study had a similar finding with regard to the prevalence of *Pedagogical Content Knowledge for Topics*, one major difference is the prevalence of educative features addressing *Subject Matter Content Knowledge*. Specifically, *Subject Matter Content Knowledge* accounted for 32% of all educative

features in the study by Beyer and colleagues, whereas in this study, there was only a single instance of this type of feature. This single instance in the Eureka Math curriculum described the difference between mass and weight after using both terms in the written curriculum and goes on to clarify that this distinction is not important for students at the fourth-grade level. One reason for this difference in prevalence of *Subject Matter Content Knowledge* may be that the mathematics content of third or fourth grade may be interpreted as common knowledge for elementary teachers, while the content of a high school biology may be interpreted as highly specialized knowledge that requires greater depth and more connections to be made beyond the high school level.

With respect to the type of guidance, prior literature, as in this study, has noted a disparity between Enactment Guidance and Rationale Guidance. Using the term implementation guidance, Beyer and colleagues (2009) found nearly two and a half times as much implementation guidance as rationale guidance. Even more pronounced, Males found that Rationale Guidance made up no more than 6% of all features for any textbook in her study. Although the present study shares similar findings with Beyer and colleagues, these findings also point to a lack of progress being made toward curriculum materials revealing developer's pedagogical judgments. Ball and Cohen's (1996) call and Davis and Krajcik's (2005) "high-level" guidelines for curriculum materials pointed out that curriculum should include discussions weighing the strengths and weaknesses of pedagogical choices rather than simply guiding the teachers' actions. That is, the curriculum should help teachers "integrate their knowledge base and make connections between theory and practice" (Davis & Krajcik, 2005, p. 5) rather than simply provide new ideas for teachers to implement. Although Rationale Guidance in this study

accounted for 24% of all educative features, the majority of these rationales did not include the types of discussions mentioned; rather the Rationale Guidance in this study focused almost exclusively on the mathematical intentions of the instructional activity.

Unlike prior research, this study offers an in-depth analysis of educative features in a single curriculum, which has offered insight into the placement of educative features in a lesson as well as how those educative features are connected to certain types of instructional activities. On the surface, this study's finding that the majority of educative features were found in the Concept Development portion of the lesson seems trivial and predictable; however, closer consideration of the analysis with regard to placement in the lesson and its connections to the categories of content knowledge provides insight into how the written Eureka Math curriculum supports teachers throughout the lesson and how these supports reveal the developers' pedagogical judgments. While the majority of educative features were found in the Concept Development portion of the lesson, those features predominantly addressed features in *Pedagogical Content Knowledge for Mathematics Topics* and *Pedagogical Content Knowledge for Mathematics Practices*. This coupling of support for teachers in developing mathematics topics as well as mathematics practices uncovers pedagogical judgments that mimic the ideals set forth in the Mathematics Teaching Practices (NCTM, 2014). In particular, the educative features in this curriculum provide support for enacting instructional activities that provide opportunities for students to not only learn the mathematical goals of the lesson, but engage them in solving and discussing tasks, using and connecting mathematical representations, and collectively build mathematical understanding through discourse.

Phase Two – Enacted Curriculum

Prior research in science education has noted that teachers take up educative features in various ways in the enactment of curriculum (e.g., McNeill, 2009), while prior research in mathematics education has attended to different aspects of the written curriculum in different ways in their planning (e.g., Collopy, 2003; Land, Tyminski, & Drake, 2013; Remillard, 2000). The findings in this study complement research in science education and build upon prior mathematics education research by investigating how educative features were taken up by the teachers in classrooms, rather than simply considering how educative features were taken up in planning for instruction. That is, this study analyzed the take-up of specific educative features found in the written curriculum as they were considered while planning instruction, enacted with students, and reflected upon following instruction.

A major finding of this study is that teachers take up educative features in different ways during enactment. The differences in take-up of educative features here seemed to be influenced primarily by how the details of the instructional activities, located at the lesson level, were carried out by the teachers. In other words, teachers in this study did attend to details contained in the educative features of the written curriculum; however, this attention did not translate into direct implementation of these details. This finding builds on work by Schneider and colleagues (Schneider & Krajcik, 2002) in science education who found that teachers give greater attention to features of the educative curriculum materials in their study during enactment at the lesson level than at the unit level. This finding also builds on work by Sherin and Drake (2009) in

mathematics education who found that teachers first attend to details (not necessarily educative features) of the curriculum rather than overarching themes.

More specifically, for this study, differences in take-up of details regarding Enactment Guidance for Experiences seemed to heavily influence the take-up of Enactment Guidance for Participation Structures and Enactment Guidance for Facilitating Discourse; that is, the ways that teachers engaged students in learning the mathematics in the instructional activities seemed to necessitate certain ways that students participated and communicated their knowledge in the classroom both with each other and with the teacher. For Rachel, in her use of measurements from mathematical tools to compare decimals, and Emily, in her use of the sorting quadrilaterals activity, the take-up of details contained in the written curriculum provided opportunities for the take-up of specific participation structures and mathematical discourse to occur.

In this study, similarities in take-up of educative features occurred for Enactment Guidance for Representations and Enactment Guidance for Developing Mathematical Terminology. Recall that at the fourth-grade level, both teachers used number bonds as a mathematical representation, even when not included in the written curriculum, while at the third-grade level, both teachers used the exclusive definition of trapezoid, even when not chosen in the written curriculum, without knowledge of the other's choice. The use of number bonds at fourth grade provides evidence that teachers take-up and transfer educative features across lessons and modules. This evidence stands in contrast to a study by Beyer and Davis (2009), who found that preservice teachers were not able to transfer educative features written as lesson-specific narratives to other lessons. The ability of the teachers in this study to transfer the use of number bonds from another lesson to this

lesson seems to suggest that similar take-up of certain educative features adds significantly to the coherence of the elementary curriculum. The same could be said for educative features addressing mathematical terminology. Similarities in the take-up of educative features of mathematical terminology both within and across grade levels adds to the coherence of the curriculum.

Implications for Curriculum Development

A unique aspect to this study was its focused consideration of the educative features present in a single written curriculum coupled with how those educative features are taken up by practicing teachers in the classroom. In a series of studies, prior research in science education (e.g., Arias, Bismack, Davis, & Palincsar, 2016) documented the variation in take-up of educative features as well as the variation in the interactions between the teacher and the curriculum; however, research in mathematics education has focused on singular aspects of educative features such as identification or evaluation of features in curricula (e.g., Males, 2011; Quebec Fuentes & Ma, 2018) or in work with preservice teachers to develop teacher knowledge through use of the materials (Drake et al., 2014). Looking across the phases of this study provides insight for developers of the Eureka Math curriculum.

The Eureka Math curriculum has been criticized for its rigor, pacing, and inadequate supports for learners (Kaufman et al., 2017), which points to a need for developers to include features in the written curriculum that support a teacher's capacity to read, understand, and adapt the curriculum materials for use with students. In particular, the connections between how Enactment Guidance for Experiences influenced the take-up of Enactment Guidance for Participation Structures and Enactment Guidance

for Facilitating Discourse point to a need for developers to make more explicit how these aspects of the instructional activities can mitigate the challenges that rigor poses for teachers. Moreover, features that recognize teachers' tendencies to adapt curriculum materials in order to address concerns about pacing and students' capabilities and experiences are needed. Narratives detailing the essential components of each lesson are needed so that teachers can make informed decisions about what activities to include and exclude. In this study, no teacher engaged students in the written Student Debrief activities even though teachers had been convinced that this element of the curriculum was an essential component when the curriculum was adopted three years prior.

The lack of Rationale Guidance and guidance addressing *Subject Matter Content Knowledge* found in this study and others (e.g., Beyer et al., 2009; Males, 2011) may also be contributing to the challenges that rigor and inadequate supports for learners create for teachers implementing the Eureka Math curriculum. Developers should include more detailed rationales addressing why certain mathematical representations are used, why differentiation may be necessary, or how certain grouping structures promote all students' learning. For example, rationales such as those that may advocate for a certain type of representation may offer teachers a different perspective on the use of that representation and allow teachers opportunities to integrate its use into their practice without dismissing it prior to understanding its purpose. In this way, the challenge that rigor presents to implementing the Eureka Math curriculum may be minimized through clearer understanding of the developer's pedagogical judgments.

In addition, although the mathematical content of these third- and fourth-grade modules may be considered common knowledge, research (e.g., Hill et al., 2008) has

documented the strong links between teachers' knowledge (including their subject matter content knowledge) and their instructional practice. The lack of guidance addressing *Subject Matter Content Knowledge* may also be contributing to the criticisms regarding rigor; that is, if a teacher is not versed in the structures and meanings in mathematics, their ability to read, understand, and adapt the curriculum materials in productive ways for use with students is minimized, leaving rigor and inadequate supports for learners a challenge.

Limitations of the Study

As a whole, this study has the potential to broaden our understanding of educative features in mathematics curriculum. In particular, this study provides an analysis of these features across the phases of curriculum use for the mathematics education community as well as offers insight into how teachers are engaging with one of the most commonly used curricula. Even so, there are limitations to this study.

First, this study only analyzed a single module at two grade levels of the Eureka Math curriculum, creating a considerably contextualized study. Although these modules came from different strands of mathematics (geometry and fractions), the prevalence of educative features in the written curriculum may not be generalizable across the corpus of elementary Eureka Math modules. Similarly, the take-up of educative features by teachers in this study may not be generalizable across all teachers using the Eureka Math curriculum. Nonetheless, we have gained insight into some patterns from the written curriculum of Eureka Math, and we have observed some things that are true in the enacted curriculum. In particular, we know that teachers can differ in their take-up of educative features down to the specific details of certain Enactment Guidance, and we

know that teachers can carry forward educative features even when they are not explicitly written into each lesson. Albeit contextualized, an advantage of this study is that it broadens our knowledge of this widely used curriculum.

Additionally, this study analyzed the educative features in the written curriculum in the spirit of prior research; that is, the analytical framework utilized in this study was adapted from studies that analyzed physical textbooks rather than open educational resources. Although the Eureka Math curriculum is an open resource, its structure and form are similar to traditional physical textbooks and resources. There are digital resources available both free of charge and for purchase through the Great Minds website; however, those resources were not chosen for analysis in this study due to prior knowledge that most of these resources had not been available to the teachers in the study since the first year of implementation.

Finally, I would be remiss to not admit that personal biases may have colored my analysis or interpretation of the written and enacted curriculum in this study. As I was coding the modules for educative features in this study, I was also reading and incorporating some of the instructional activities from the Eureka Math Algebra II curriculum into the Algebra II class I was teaching at the time. I felt my own algebra content knowledge was impacted through this process as I worked to integrate the written curriculum into my practice. Additionally, I found myself reflecting on how instructional practices (e.g., discourse moves) in the enactments that I was analyzing could inform my own teaching practice.

Recommendations for Future Research

This study utilized the Eureka Math curriculum to study how the educative features provided in the written curriculum are taken up by practicing teachers. Similar studies are needed at other grade levels and with other curricula to continue the work started by Males (2011) and Quebec Fuentes and Ma (2018) focused on mathematics curricula. These studies would serve two higher purposes: to examine the use of educative features specific to the learning and teaching of mathematics and to begin a new cycle of improvement for curriculum as a whole. Research conducted in science education found two science-specific design principles for educative curriculum materials. Similar research in the use of educative features in mathematics curriculum may reveal mathematics-specific design principles to inform the development of mathematics curriculum.

Additionally, the study did not examine the digital features available for subscription purchase through the Great Minds website such as the Eureka Navigator or the Teach Eureka Video Series. These digital resources offer personal professional development opportunities for teachers to learn about, for example, key instructional practices in the curriculum. Analysis of the educative features in these resources as well as how teachers are taking up these educative features in the classroom would provide even further insight into what is known about the Eureka Math curriculum specifically and educative features and educative curriculum materials in general.

Conclusion

Eureka Math curriculum has quickly become the most commonly used instructional resource for elementary teachers, yet the mathematics education research

community has largely not considered how teachers across the United States are using this resource. Perhaps the use of open educational resources such as Eureka Math is viewed as an educational trend that will eventually fade away and does not warrant study. However, this does not seem likely as school districts' purse strings tighten across the nation, leaving administrators and teachers to search for new ways to minimize expenses across the budget. Open educational resources such as Eureka Math can minimize costs associated with curriculum and other areas. Supplement this fact with the findings of this study and school districts may also be able to minimize professional development costs specific to mathematics by using the educative features of the written curriculum and how these features are being taken up by teachers as a source for professional growth through already organized grade-level teams.

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Appendix A

Teacher Learning Opportunities in Mathematics Curriculum Materials framework

Excerpted from Quebec Fuentes & Ma, 2018

Dimension	Definition	Guiding questions
Mathematics Content Knowledge for Teaching	The curriculum materials support the development of mathematics content knowledge for teaching and for the inclusion of, flow of, and connections between lessons and units.	<p>What supports are provided for the development of mathematics content knowledge for teaching and for the inclusion of, flow of, and connections between lessons and units?</p> <p>To what extent are rationales for the development of mathematics content knowledge for teaching and for the inclusion of, flow of, and connections between lessons and units present, clear, and appropriate?</p> <p>To what extent is implementation guidance for the development of mathematics content knowledge for teaching and for the inclusion of, flow of, and connections between lessons and units present, clear, and appropriate?</p>
Teacher Knowledge of Student Thinking in Mathematics	The curriculum materials support the development of teacher knowledge in anticipating and understanding student thinking.	<p>What supports are provided for anticipating and understanding student mathematical thinking (e.g., common perceptions and misconceptions)?</p> <p>To what extent are rationales for addressing the significance of student thinking in relation to understanding the mathematics content present, clear, and appropriate?</p> <p>To what extent is implementation guidance for anticipating and</p>

Teacher Knowledge of Disciplinary Discourse in Mathematics	The curriculum materials support the development of teacher knowledge in fostering disciplinary discourse in mathematics.	<p>understanding student mathematical thinking present, clear, and appropriate?</p> <p>What supports are provided for fostering disciplinary discourse in mathematics (e.g., focus and direction or step-by-step script)?</p> <p>To what extent are rationales for the inclusion of a discussion and its relationship to the development of student understanding of the mathematics content present, clear, and appropriate?</p> <p>To what extent is implementation guidance for fostering disciplinary discourse in mathematics present, clear, and appropriate?</p>
Teacher Knowledge of Assessment in Mathematics	The curriculum materials support the development of teacher knowledge in the use of assessments in mathematics to make instructional decisions.	<p>What supports are provided for using curricular assessments in mathematics and outcomes of these assessments to make instructional decisions?</p> <p>To what extent are rationales for the inclusion and grading of particular curricular assessments in mathematics and for the use of the outcomes of these assessments to make instructional decisions present, clear, and appropriate?</p> <p>To what extent is implementation guidance for using curricular assessments in mathematics and outcomes of these assessments to make instructional decisions present, clear, and appropriate?</p>
Teacher Knowledge of Differentiated Instruction in Mathematics	The curriculum materials support the development of teacher knowledge in the use of differentiated instruction to address the	<p>What supports are provided for how to differentiate instruction in mathematics?</p> <p>To what extent are rationales for these means for differentiating</p>

	mathematical needs of all students.	instruction made present, clear, and appropriate? To what extent is implementation guidance for how to differentiate instruction in mathematics present, clear, and appropriate?
Teacher Knowledge of Technology Use in Mathematics	The curriculum materials support the development of teacher knowledge in the strategic use of technology in mathematics.	What supports are provided for the strategic use of technology in mathematics? To what extent are rationales for the incorporation of a technological tool and its relationship to the development of student understanding of the mathematics content present, clear, and appropriate? To what extent is implementation guidance for strategic technology use in mathematics present, clear, and appropriate?
Teacher Knowledge of Mathematics Community	The curriculum materials support the development of teacher knowledge in building a mathematical community that establishes norms for a productive learning environment.	What supports are provided for building a mathematical community? To what extent are rationales justifying the type of mathematical community cultivated by the structure of the curriculum materials present, clear, and appropriate? To what extent is implementation guidance for building a mathematical community present, clear, and appropriate?

Appendix B

A framework for analyzing the content supports of teachers' guides

Excerpted from Males, 2011

I. Mathematics Subject Matter Content Knowledge

Support Teachers in the Development of Subject Matter Knowledge

Content: Present mathematical content information [Con]

II. Pedagogical Content Knowledge for Mathematics Topics

Support Teachers in Engaging Students with Topic-Specific Mathematics

Enactment Guidance for:

Experiences: Engaging students in particular experiences. [E-Experiences] Possible Student Pitfalls: Potential pitfalls with specific mathematical experiences (including describing where students may have difficulty or potential misconceptions) [E-Pitfalls] Engaging in Problematic Experiences: Engaging students in experiences that may potentially be problematic [E-Problem] Activity Sequences: Engaging in productive activity sequences [E-Sequence]

Rationale Guidance for:

Experiences: Why particular experiences are appropriate, including why a particular problem or activity is useful in a students' mathematical development. [R-Experiences] Activity Sequences: Why particular activity sequences are appropriate or useful. [R-Sequence]

Support Teachers in Using Mathematical Instructional Representations

Enactment Guidance for:

Representations: Using representations (verbal, equations, tables, graphs, pictures) with students. [E-Representations]

Rationale Guidance for:

Representations: Why particular representations (verbal, equations, tables, graphs, pictures) are appropriate including the advantages and disadvantages for using particular representations. [R-Representations]

Support Teachers in Using Mathematical Tools

Enactment Guidance for:

Tools: Using particular tools (e.g., rulers, compasses, calculator) with students. [E-Tools]

Rationale Guidance for:

Tools: Why particular tools are appropriate including the advantages and disadvantages for using particular tools [R-Tools]

Support Teachers in Anticipating and Using Students' Mathematical Ideas

Enactment Guidance for:

Anticipating Student Ideas⁵: Anticipate student ideas within a topic [I-Anticipate]⁶

Using Student Ideas: Using students' ideas during the enactment of the lesson [I- Use]

III. Pedagogical Content Knowledge for Mathematics Practices

Support Teachers in Engaging Students in Questions

Enactment Guidance for:

Using Questions: Using questions with students (Suggested answers provided) [I-QuestionsA]

Using Questions: Using questions with students (Support for answers not provided) [I-QuestionsNA]

Helping Students Ask/Answer Questions: Having students ask/answer their own questions [I-Ask]

Rationale Guidance for:

Questions: Why particular questions or series of questions are appropriate or useful [I-Ask]

Support Teachers in Engaging Students in Justification, Reasoning, and Proof Enactment

Enactment Guidance for:

Justification, Reasoning, & Proof: Using approaches for engaging students in justification, reasoning and proof [E-Proof]

Rationale Guidance for:

Justification, Reasoning, & Proof: Why approaches to engaging in justification, reasoning and proof are appropriate or useful [R-Proof]

Support Teachers in Developing Students' Mathematical Vocabulary/Terminology Enactment

Enactment Guidance for:

Developing Mathematical Terminology: Developing students' mathematical terminology (including vocabulary and notation) and recognizing colloquial words that may help students understand context or promote/hinder students' conceptions [E-Terminology]

Rationale Guidance for:

Developing Mathematical Terminology: Why developing mathematical vocabulary or terminology is important and why particular strategies are appropriate or useful [R-Terminology]

Support Teachers in Engaging Students in Appropriate Participation Structures

Enactment Guidance for:

Participation Structures: Using appropriate participation structures (e.g., individual, small-group, whole-class) [E-Participation]

Rationale Guidance for:

⁵ In the write-up later on in Males' dissertation, this code is written as "Anticipating Student Thinking" as well as the next one is "Using Student Thinking."

⁶ The codes used are not consistent with other coding schemes in the other categories. Enactment guidance seems to be coded as E-_____ and rationale guidance seems to be coded as R-_____ for other categories.

Participation Structures: Why particular participation structures are appropriate or useful [R-Participation]

IV. Mathematics Curricular Content Knowledge

Support Teachers in the development of Curricular Knowledge

Enactment Guidance for:

Curricular Overview: Using unit, section, or lesson content in the intended ways (by providing a description of what will unfold in the unit, section, or lesson) [E-Overview]

Curricular Features: Using features of the curriculum [E-Features]

Curricular Storyline: Connecting previous and future mathematics content, including within and across disciplines, courses, units, sections, and lessons [E-Storyline]

Curricular Goals: Using the goals of the unit, section or lesson (including knowing what the goals are and how to achieve them) [E-Goals]

Rationale Guidance for:

Curricular Philosophy: curricular philosophy. [R-Philosophy]

Curricular Features: Why curricular features are appropriate or useful (including explaining these features) [R-Features]

Curricular Storyline: Why particular pieces of content or processes are appropriate in the storyline [R-Storyline]

Curricular Goals: Why particular goals are appropriate or useful. [R-Goals]

Appendix C

Full Coded Lesson Grade 4 Lesson 3

Lesson 3

Objective: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

Suggested Lesson Structure

Fluency Practice	(10 minutes)
Application Problem	(5 minutes)
Concept Development	(35 minutes)
Student Debrief	(10 minutes)
Total Time	(60 minutes)

Fluency Practice (10 minutes)

- Write the Decimal or Fraction **4.NF.6** (5 minutes)
- Count by Tenths **4.NF.6** (5 minutes)

Write the Decimal or Fraction (5 minutes)

Materials: (S) Personal white board

Note: This fluency activity reviews Lessons 1–2.

T: (Write $\frac{1}{10}$.) Say the fraction.

S: 1 tenth.

T: (Write $\frac{1}{10} = \underline{\quad}$.) Write 1 tenth as a decimal to complete the number sentence.

S: (Write $\frac{1}{10} = 0.1$.)

Continue with the following possible sequence: $\frac{2}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$.

T: (Write $0.3 = \underline{\quad}$.) Write zero point three as a fraction to complete the number sentence.

S: (Write $0.3 = \frac{3}{10}$.)

Continue with the following possible sequence: 0.4, 0.8, and 0.6.

T: (Write $\frac{10}{10}$.) 10 tenths equals what whole number?

S: 1.

..R-Storyline

A STORY OF UNITS

Lesson 3 4•6

T: (Write $\frac{10}{10} = 1$. Beneath it, write $\frac{30}{10}$.) How many ones is 30 tenths?

S: 3 ones.

T: (Write $\frac{50}{10}$.) How many ones is 50 tenths?

S: 5 ones.

T: (Write $\frac{13}{10}$.) Write 13 tenths as a mixed number.

S: (Write $\frac{13}{10} = 1\frac{3}{10}$.)

T: (Write $\frac{13}{10} = 1\frac{3}{10} = \underline{\hspace{1cm}}$.) Write $1\frac{3}{10}$ in decimal form.

S: (Write $\frac{13}{10} = 1\frac{3}{10} = 1.3$.)

Continue with the following possible sequence: $\frac{17}{10}$, $\frac{37}{10}$, $\frac{34}{10}$, and $\frac{49}{10}$.

T: (Write 2.1.) Write two point one as a mixed number.

S: (Write $2.1 = 2\frac{1}{10}$.)

Continue with the following possible sequence: 3.1, 5.1, 5.9, and 1.7.

Count by Tenths (5 minutes)

Materials: (T) Personal white board

Note: This fluency activity reviews Lessons 1–2.

T: Count by fives to 50, starting at zero.

$\frac{0}{10}$	$\frac{5}{10}$	$\frac{10}{10}$	$\frac{15}{10}$	$\frac{20}{10}$	$\frac{25}{10}$	$\frac{30}{10}$	$\frac{35}{10}$	$\frac{40}{10}$	$\frac{45}{10}$	$\frac{50}{10}$
0		1		2		3		4		5

S: 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

T: Count by 5 tenths to 50 tenths, starting at 0 tenths. (Write as students count.)

S: $\frac{0}{10}$, $\frac{5}{10}$, $\frac{10}{10}$, $\frac{15}{10}$, $\frac{20}{10}$, $\frac{25}{10}$, $\frac{30}{10}$, $\frac{35}{10}$, $\frac{40}{10}$, $\frac{45}{10}$, $\frac{50}{10}$.

T: 1 is the same as how many tenths?

S: 10 tenths.

T: (Beneath $\frac{10}{10}$ write 1.)

Continue the process, identifying the number of tenths in 2, 3, 4, and 5.

T: Let's count by 5 tenths again. This time, when you come to a whole number, say the whole number. Try not to look at the board.

S: $\frac{0}{10}$, $\frac{5}{10}$, 1 , $\frac{15}{10}$, 2 , $\frac{25}{10}$, 3 , $\frac{35}{10}$, 4 , $\frac{45}{10}$, 5 .

T: Count backward by 5 tenths, starting at 5.

S: $5, \frac{45}{10}, 4, \frac{35}{10}, 3, \frac{25}{10}, 2, \frac{15}{10}, 1, \frac{5}{10}, 0$

T: Count by 5 tenths again. This time, stop when I raise my hand.

S: $0, \frac{5}{10}, 1, \frac{15}{10}$

T: (Raise hand.) Say 15 tenths using digits.

S: One point five.

Continue the process counting up to 5 and down from 5, asking students to say the improper fractions using digits.

Application Problem (5 minutes)

Ed bought 4 pieces of salmon weighing a total of 2 kilograms. One piece weighed $\frac{4}{10}$ kg, and two of the pieces weighed $\frac{5}{10}$ kg each. What was the weight of the fourth piece of salmon?

The 4th piece of salmon weighed $\frac{6}{10}$ kg.

Solution A

$$\frac{4}{10} + \frac{5}{10} + \frac{5}{10} = 1 \frac{4}{10}$$

$$2 - 1 \frac{4}{10} = \frac{6}{10}$$

$$1 \frac{10}{10} - 1 \frac{4}{10} = \frac{6}{10}$$

$w = \frac{6}{10}$ kg

Solution B

$$\frac{5}{10} \text{ kg} + \frac{5}{10} \text{ kg} = 1 \text{ kg}$$

$$1 \frac{4}{10} \text{ kg} + \frac{6}{10} \text{ kg} = 2 \text{ kg}$$

$w = \frac{6}{10}$ kg

Note: This Application Problem anticipates decimal fraction addition and reinforces the concept of how many more to make one.

Concept Development (35 minutes)

Materials: (T/S) Whole number place value disks (tens and ones), decimal place value disks (tenths), personal white board, tenths on a number line (Template)

Problem 1: Make groups of 10 tenths to rename as ones. Write the number in decimal form.

T: With a partner, use place value disks to show 21 units of 1 tenth in five-group formation.

S: (Lay out 21 disks, all tenths, in five-group formation, as shown.)

A STORY OF UNITS

Lesson 3 4•6

- T: Talk with your partner. Is there any way we can use *fewer* disks to show this same value?
- S: We can bundle 10 tenths to make one. → There are 2 groups of 10 tenths, so we can show 21 tenths as 2 ones 1 tenth. → In the five-groups, I can see 2 groups of 10 disks. 10 tenths is 1 whole. We have 1 (circling group with finger), 2 (circling group with finger) groups that make 2 ones, and then 1 tenth (touching final 0.1 disk.)
- T: Let's group 10 tenths together and trade them for...?
- S: 1 one.
- T: How many times can we do this?
- S: 1 more time. → 2 times.
- T: What disks do we have now?
- S: 2 ones and 1 tenth.
- T: Express this number in decimal form.
- S: (Write 2.1.)
- T: How many more tenths would we have needed to have 3 ones?
- S: 9 tenths more. → 0.9.

Repeat the process using disks to model 17 tenths. Then, continue the process having the students draw disks for 24 tenths. Have students circle the disks being bundled.

**NOTES ON
MULTIPLE MEANS
FOR ACTION AND
EXPRESSION:**

Be sure to enunciate /th/ at the end of tenths to help English language learners distinguish tenths and tens. Try to speak more slowly, pause more frequently, or couple language with a tape diagram. Check for student understanding and correct pronunciation of fraction names.

Problem 2: Represent mixed numbers with units of tens, ones, and tenths in expanded form.

- T: Hold up a place value disk with a value of 1 ten. We say the value of this disk is...?
- S: 1 ten. → Ten.
- T: (Draw or show 4 tens disks.) The total value of 4 of these is...?
- S: 4 tens. → Forty.
- T: 4 tens written as a multiplication expression is?
- S: 4×1 ten. → 4×10 .
- T: (Write the expression below the disks, as pictured to the right.) 4×10 is...?
- S: 40.
- T: (Complete the number sentence. Draw or show 2 ones disks.) The total value of these 2 disks is...?
- S: 2 ones. → Two.
- T: 2 ones written as a multiplication expression is...?
- S: 2×1 .
- T: (Write the expression below the disks, as pictured to the right.) $(4 \times 10) + (2 \times 1)$ is...?
- S: 42.

$$4 \times 10 = 40$$

$$(4 \times 10) + (2 \times 1) = 42$$

MP.4

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Lesson 3: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

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..E-Discourse
..E-Representation

..E-Representati

..E-Structures

..E-Structures

..E-Differentiation

..E-Features

A STORY OF UNITS

Lesson 3 4•6

T: (Complete the number sentence. Draw or show a tenth disk.) This place value disk says zero point one on it. We say the value of this disk is...?

S: 1 tenth.

T: (Draw or show 6 one-tenth disks in five-group formation.) The total value of 6 of these disks is ...?

S: 6 tenths.

T: 6 tenths written as a multiplication expression is...?

S: $6 \times \frac{1}{10}$

T: (Write the expression below the disks, as pictured above.) Discuss the total value of the number represented by the disks with your partner.

S: Do what is in the parentheses first, and then find the sum. $40 + 2 + \frac{6}{10}$ is $42\frac{6}{10}$. → 4 tens, 2 ones, 6 tenths. → It is like expanded form.

T: We have written $42\frac{6}{10}$ in expanded form, writing each term as a multiplication expression. Just like with whole numbers, the expanded form allows us to see the place value unit for each digit.

T: (Point to $(4 \times 10) + (2 \times 1) + (6 \times \frac{1}{10}) = 42\frac{6}{10}$.) Talk with your partner. How could you write this using **decimal expanded form** instead of **fraction expanded form**? Explain how you know.

S: (Work with partners, and write $(4 \times 10) + (2 \times 1) + (6 \times 0.1) = 42.6$.) I know that 1 tenth can be written as zero point one, and 42 and 6 tenths can be written as forty-two point six. → We looked at our disks. We had 4 tens, 2 ones, and 6 disks that had 0.1 on them. → We knew it was $42 + 0.6$, so that helped us rewrite $42\frac{6}{10}$ as 42.6.

Continue the process of showing a mixed number with place value disks, and then writing the expanded fraction form and expanded decimal form for the following numbers: 24 ones 6 tenths, 13 ones 8 tenths, and 68 ones 3 tenths. Challenge students to think how much each number needs to get to the next one.

Problem 3: Use the number line to model mixed numbers with units of ones and tenths.

T: (Distribute the Lesson 3 Template, tenths on a number line, and insert it into personal whiteboards.) Label the larger intervals from 0 to 5.

T: The segment between each whole number is divided up into how many equal parts?

S: 10 equal parts.

T: Plot a point on the number line to represent 4 and 1 tenth.

T: In the chart below your number line, let's plot the same number on a shorter number line partitioned into tenths. What will the endpoints of this shorter number line be?

S: 4 and 5.

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Lesson 3: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

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..E-Discourse

..E-Representation

..E-Structures

..E-Structures

..E-Features

A STORY OF UNITS

Lesson 3 4•6

T: (Fill out the chart to show 4.1 plotted on a number line between 4 and 5, in decimal form, as a mixed number, and in expanded form.)

Point	Number Line	Decimal Form	Mixed Number (ones and fraction form)	Expanded Form (fraction or decimal form)	How much more is needed to get to the next one?
a.		4.1	$4\frac{1}{10}$	$(4 \times 1) + (1 \times \frac{1}{10}) = 4\frac{1}{10}$ or $(4 \times 1) + (1 \times 0.1) = 4.1$	$\frac{9}{10}$ or 0.9
b.		32.5	$32\frac{5}{10}$	$(3 \times 10) + (2 \times 1) + (5 \times \frac{1}{10}) = 32\frac{5}{10}$ or $(3 \times 10) + (2 \times 1) + (5 \times 0.1) = 32.5$	$\frac{5}{10}$ or 0.5
c.		40.7	$40\frac{7}{10}$	$(4 \times 10) + (7 \times \frac{1}{10}) = 40\frac{7}{10}$ or $(4 \times 10) + (7 \times 0.1) = 40.7$	$\frac{3}{10}$ or 0.3
d.		90.9	$90\frac{9}{10}$	$(9 \times 10) + (9 \times \frac{1}{10}) = 90\frac{9}{10}$ or $(9 \times 10) + (9 \times 0.1) = 90.9$	$\frac{1}{10}$ or 0.1

tenths on a number line

S: (Write 4 ones and 1 tenth, $4\frac{1}{10}$, $4\frac{1}{10} = (4 \times 1) + (1 \times \frac{1}{10}) = 4.1$. $\rightarrow (4 \times 1) + (1 \times \frac{1}{10}) = 4\frac{1}{10}$)

T: How many more tenths are needed to get to 5? Explain to your partner how you know, and complete the final column of the chart.

S: 9 tenths. $\rightarrow \frac{9}{10}$. $\rightarrow 0.9$. $\rightarrow$ I know because it takes 10 tenths to make a one. If we have 1 tenth, we need 9 more tenths to make 1.

Repeat the process by naming the following points for students to plot. Then, have them complete and share their charts. The longer number line with 5 whole number intervals can either be relabeled to show a broader range of numbers than those included in the chart or omitted for parts (b)–(d) below.

- b. 3 tens 2 ones and 5 tenths
- c. 4 tens 7 tenths
- d. 9 tens 9 tenths

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Lesson 3: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

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G3-M3-TE-1.3.0-08.2018

..E-Features
..E-Experiences
..E-Differentiat
..E-Experiences

..R-Experiences

..E-Features
..E-Structures
..E-Differentiat

..E-Representati

..E-Repres
..E-Tools

..E-Structures
..E-Reasoning

..E-Features
..E-Experience

..E-Features
..E-Terms

..E-Structures
..E-Discourse

Problem Set (10 minutes)

Students should do their personal best to complete the Problem Set within the allotted 10 minutes. For some classes, it may be appropriate to modify the assignment by specifying which problems they work on first. Some problems do not specify a method for solving. Students should solve these problems using the RDW approach used for Application Problems.

Student Debrief (10 minutes)

Lesson Objective: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

Any combination of the questions below may be used to lead the discussion.

- Look at Problem 3(b). Today, we showed mixed numbers in **decimal expanded form** and **fraction expanded form**. How could you represent this number with place value disks? With an area model? Draw a line that is 17.5 cm in length.
- Look at Problem 3(a). How would you represent this number using only tenths? With your partner, use the number line or centimeter ruler to prove that 39 tenths is the same as 3 ones and 9 tenths.
- Look at Problems 2(d) and 3(c). How are these two problems alike?
- In Problems 2(c), 2(d), and 3(e) we have the same number of tens as tenths. Explain to your partner the difference in value between the tens place and the tenths place. Notice that the ones are sandwiched between the tens and tenths.

Name: Jack Date: _____

1. Draw place value disks to represent each number. Write the number in decimal form and fraction form. Then show the expanded form of the number in decimal form and fraction form.

a. 39 tenths (3.9)

Place value disks: Decimal form: 3.9 Fraction form: $\frac{39}{10}$

b. 39 tenths (3.9)

Place value disks: Decimal form: 3.9 Fraction form: $\frac{39}{10}$

2. Draw disks to represent each number using tens, ones, and tenths. Then show the expanded form of the number in decimal form and fraction form. The first one has been completed for you.

a. 4 ones (4.0)

Expanded form: $4 \times 10^0 = 4 \times 1 = 4$ Decimal form: 4.0 Fraction form: $\frac{40}{10}$

b. 3 tenths (0.3)

Expanded form: $3 \times 10^{-1} = 3 \times \frac{1}{10} = \frac{3}{10}$ Decimal form: 0.3 Fraction form: $\frac{3}{10}$

3. Complete the table.

	Number line	Place value disks	Expanded form (tens, ones, and tenths)	Expanded form (tens, ones, and tenths)	How many tenths are in the number?
a.	3.9	$3 \times 10 + 9 \times 1$	$(3 \times 10) + (9 \times 1)$	39	
b.	17.5	$1 \times 10 + 7 \times 1 + 5 \times \frac{1}{10}$	$(1 \times 10) + (7 \times 1) + (5 \times \frac{1}{10})$	175	
c.	79.7	$7 \times 10 + 9 \times 1 + 7 \times \frac{1}{10}$	$(7 \times 10) + (9 \times 1) + (7 \times \frac{1}{10})$	797	
d.	22.5	$2 \times 10 + 2 \times 1 + 5 \times \frac{1}{10}$	$(2 \times 10) + (2 \times 1) + (5 \times \frac{1}{10})$	225	
e.	80.8	$8 \times 10 + 0 \times 1 + 8 \times \frac{1}{10}$	$(8 \times 10) + (0 \times 1) + (8 \times \frac{1}{10})$	808	

..E-Representati

..R-Experiences

A STORY OF UNITS

Lesson 3 4•6

- How did you locate points on the number line?

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help with assessing students' understanding of the concepts that were presented in today's lesson and planning more effectively for future lessons. The questions may be read aloud to the students.

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Lesson 3: Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the number line, and in expanded form.

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Appendix D

Tallies of codes pages

Note: The first two pages contain information about 3rd grade and the second two pages contain information about 4th grade. An “x” indicates that the code was found in that particular lesson. The “***” indicates that the only instance of the code in the lesson was one that was not unique; that is, the instance contained similar wording to other lessons in the module and was adjusted for in the analysis.

3 rd Grade	Subject Matter Content Knowledge		Pedagogical Content Knowledge for Mathematics Topics									
			Topic-Specific Mathematics						Representations		Tools	
Lesson	E-Content	R-Content	E-Experiences	E-Differentiation	E-Sequences	R-Experiences	R-Differentiation	R-Sequences	E-Representation	R-Representation	E-Tools	R-Tools
Module Overview				x								
A Overview												
1			x	x	x	x	x	x	x		x	x
2			x	x	x	x	x	x	x			
3			x	x	x	x		x	x			
B Overview												
4			x	x	x	x	x	x	x	x	x	x
5			x	x	x	x		x			x	
6			x	x		x					x	
7			x	x	x	x		x	x		x	
8			x	x	x	x		x	x		x	
9			x	x	x	x		x			x	x
C Overview							x					
10			x	x	x	x		x	x		x	
11			x	x		***					x	
12			x	x	x	x		x	x		x	
13			x	x	x	x		x	x			
14			x	x	x	x	x	x	x			
15			x	x	x	x		x	x			
16			x	x	x	x		x	x		x	
17			x	***		x			x	x	x	
D Overview												
18			x	x		***			x		x	
19			x	x		***			x			
20			x	x		x			x	x	x	
21			***	x		***			x			
22			x	x		***			x		x	
E Overview												
23			***	x		***			x			
24			x	x	x	x		x	x			
25			***	***		***					x	
26			x	x	x	x		x	x			
27			x	x		***			x			
28			x	x	x	x		x	x			
29			***	x		***			x			
30			x	x	x	x		x	x			

3 rd Grade	Pedagogical Content Knowledge for Mathematics Practices								Curricular Content Knowledge							
	Discourse		Reasoning		Mathematical Vocabulary		Participation Structures									
Lesson	<i>E-Discourse</i>	<i>R-Discourse</i>	<i>E-Reasoning</i>	<i>R-Reasoning</i>	<i>E-Terms</i>	<i>R-Terms</i>	<i>E-Structures</i>	<i>R-Structures</i>	<i>E-Overview</i>	<i>E-Features</i>	<i>E-Storyline</i>	<i>E-Goals</i>	<i>R-Philoso.</i>	<i>R-Features</i>	<i>R-Storyline</i>	<i>R-Goals</i>
Module									x		x				x	
A Overview									x						x	
1	x	x			x		x			x				x	x	
2	x	x	x		x		x	x							x	
3	x		x		x		x			x					x	
B Overview									x						x	
4	x		x		x		x		x	x	x				x	
5	x				x		x			x					x	
6	x		x		x		x			x					x	
7	x		x		x		x			x					x	
8	x		x		x		x			***	x				x	
9	x		x				x			***		x			x	
C Overview									x							x
10	x		x		x		x			***	x				x	
11	x		x		x		x			***	x					
12	x		x		x		x		x	x	x				x	x
13	x				x		x			x	x				x	
14	x				x		x			x					x	
15	x		x				x			x					x	
16	x						x			x					x	
17	x		x		x		x			x	x				x	
D Overview									x							
18	x		x				x			x					x	
19	x		x				x			***	x				x	
20	x		x				x		x	x	x				x	
21	x		x				x			x	x				x	
22	x		x				x			x					x	
E Overview									x							
23	x						x		x	***					x	
24	x		x				x		x	***	x				x	
25	***		x				x		x	***	x				x	
26	x		x				x			x	x				x	
27	x		x				x			x					x	
28	***		x				x		x	x					x	
29	***		x				***		x	x	x				x	
30	x						x			***					x	

4 th Grade	Subject Matter Content Knowledge		Pedagogical Content Knowledge for Mathematics Topics									
			Topic-Specific Mathematics						Representations		Tools	
Lesson	E-Content	R-Content	E-Experiences	E-Differentiation	E-Sequences	R-Experiences	R-Differentiation	R-Sequences	E-Representation	R-Representation	E-Tools	R-Tools
Module				x								
A Overview						x				x		
1			x	x		x	x		x		x	
2			x	x		***			x		x	
3			x	x		***			x		x	
B Overview						x						
4			x	x		***			x		x	
5			x	x		x	x		x	x		
6			x	x		***			x			
7			x	x		***	x		x			
8			***	x		***			x			
C Overview												
9	x		x	***		***			x		x	
10			***	***	x	***		x	x			
11			x	x		x			x			
D Overview												
12			***	x		***	x		x			
13			***	x		***	x		x			
14			***	x		***			x	x		
E Overview												
15			***	x		***			x			
16			***	x		***			x			

4 th Grade	Pedagogical Content Knowledge for Mathematics Practices								Curricular Content Knowledge							
	Discourse		Reasoning		Mathematical Vocabulary		Participation Structures									
<i>Lesson</i>	<i>E-Discourse</i>	<i>R-Discourse</i>	<i>E-Reasoning</i>	<i>R-Reasoning</i>	<i>E-Terms</i>	<i>R-Terms</i>	<i>E-Structures</i>	<i>R-Structures</i>	<i>E-Overview</i>	<i>E-Features</i>	<i>E-Storyline</i>	<i>E-Goals</i>	<i>R-Philoso.</i>	<i>R-Features</i>	<i>R-Storyline</i>	<i>R-Goals</i>
Module									x		x				x	
A Overview									x							
1	x				x		x			x	x			x	x	
2	x						x			x					x	
3	x		x		x		x			x					x	
B Overview									x						x	
4	x				x		x			x	x				x	
5	x		x		x		x			x	x				x	
6	x		x		x		x			x					x	
7	x		x		x		x			x					x	
8	x		x				x			x					x	
C Overview									x							x
9	x		x		x	x	x			x	x				x	
10	x		x				x			x	x				x	
11	x		x				x			x					x	
D Overview									x							
12	x						x			x	x				x	
13	x						x			x	x				x	
14	x						x		x	x		x			x	
E Overview									x							
15	x		x				x			x					x	
16	x		x				x			***	x	x			x	

Appendix E

Excel spreadsheet F A C D page

Note: The first two pages contain lesson information for third grade, and the second two pages contain lesson information for fourth grade. F indicates the code was found in the Fluency Practice of the lesson. A indicates the code was found in the Application Problem of the lesson. C indicates the code was found in the Concept Development of the lesson. D indicates the code was found in the Student Debrief of the lesson. LESSON indicates the code was found at the beginning of the lesson and addressed the lesson as a whole rather than a particular lesson component.

3 rd Grade	Subject Matter Content Knowledge		Pedagogical Content Knowledge for Mathematics Topics									
			Topic-Specific Mathematics						Instructional Representations		Mathematical Tools	
Lesson	E-Content	R-Content	E-Experiences	E-Differentiation	E-Sequences	R-Experiences	R-Differentiation	R-Sequences	E-Representation	R-Representation	E-Tools	R-Tools
1			F	FC	F	FC	C	F	FCD		C	C
2			F	FC LESSON	F	F	C LESSON	F	FCD			
3			FC	F	F	F		F	FCD			
4			FC	FA	F	F	C	F	A	A	C	C
5			FC	FC	F	F		F			C	
6			FC	FC		FA					C	
7			FC	C	F	FC		F	D		C	
8			FC	FC	FC	F		FC	FC		C	
9			FC	FC	F	F		F			C	C
10			FC	A	F	F		F	A		C	
11			C	C							C	
12			FC	C	F	F		F	FCD		CD	
13			FC	AC	F	F		F	FCD			
14			FC	C	F	F	C	F	CD			
15			F	C	F	F		F	FCD			
16			FC	C	F	F		F	F		CD	
17			FC			F			FCD	FC	C	
18			C	AC					FAC		C	
19			C	AC					FAC			
20			C	C		C			C	C	C	
21				A					CD			
22			C	FC					FCD		CD	
23				C					C			
24			F	FC	F	F		F	F			
25											C	
26			F	FC	F	F		F	F			
27			C	F					F			
28			F	C	F	F		F	FCD			
29				C					FCD			
30			FC	C	F	F		F	C			

3 rd Grade	Pedagogical Content Knowledge for Mathematics Practices															
	Mathematical Discourse		Reasoning & Justification		Mathematical Vocabulary		Participation Structures		Curricular Content Knowledge							
Lesson	E-Discourse	R-Discourse	E-Reasoning	R-Reasoning	E-Terms	R-Terms	E-Structures	R-Structures	E-Overview	E-Features	E-Storyline	E-Goals	R-Philosophy	R-Features	R-Storyline	R-Goals
1	C	C			F		C D			F				C	F	
2	C	C	C		F		C	C							F	
3	C		C		F		C			C					F	
4	C		C		C D		C		C	C	C				F A	
5	C				F C D		C D			C D					F	
6	C		C		F D		F C			C D					F	
7	C		C		F C		C D			C					F	
8	C		D		D		D				C				F	
9	C		D				C D					A			F A	
10	C		C D		C D		C D				C				F A	
11	C		D		C		C D				C					
12	C		D		C		C D		LESSON	C D	C D				F A	LESSON
13	A C				D		A C D			C	D				F A	
14	C				C D		C D			D					F A	
15	C		C				C D			D					F A	
16	C						C D			C					F A	
17	C		C		D		C D			D	LESSON				F A LESSON	
18	C		C D				C			C D					F A	
19	C		C D				C				C				F A	
20	A C		C				C D		C	C	C				F A	
21	A C		D				C			C D	C				F A	
22	C		C D				C			C					F	
23	C						C D		C						F C	
24	C		D				C D		C		C D				F	
25			D				D		C		C				F	
26	C		C D				C			C	C				F A	
27	C		D				C			C					F	
28			D				D		C	C					F C	
29			D						C	D	C				F	
30	C						C								F	

4th Grade	Subject Matter Content Knowledge		Pedagogical Content Knowledge for Mathematics Topics									
			Topic-Specific Mathematics						Instructional Representations		Mathematical Tools	
Lesson			E-Experiences	E-Differentiation	E-Sequences	R-Experiences	R-Differentiation	R-Sequences	E-Representation	R-Representation	E-Tools	R-Tools
1			C	C		C	C		F C D		C	
2			F C	C					F C D		C	
3			F C	C					F C D		C D	
4			F	C					C D		C	
5			F C	F C		F	F		F C D	F		
6			F	F C					F C D			
7			F	C			C		F C D			
8				C					F C D			
9	x		C						F C D		C	
10					C			C	F C D			
11			C	F		F			F C D			
12				F C			C		F C			
13				F			F		F C D			
14				F					F	C		
15				C					F C D			
16				C					F D			

4th Grade	Pedagogical Content Knowledge for Mathematics Practices															
	Mathematical Discourse		Reasoning & Justification		Mathematical Vocabulary		Participation Structures									
Lesson	E-Discourse	R-Discourse	E-Reasoning	R-Reasoning	E-Terms	R-Terms	E-Structures	R-Structures	E-Overview	E-Features	E-Storyline	E-Goals	R-Philosophy	R-Features	R-Storyline	R-Goals
1	C				C D		C			D	D			C	F	
2	C						C			C D					F A	
3	C		D		C D		C D			C D					F A	
4	C				C D		C			C	A				F A	
5	C		C D		F		C D			C D	A C				F A	
6	C		D		C D		C D			F C					F A	
7	C		C		C D		C			F C D					F A	
8	C		C D				C			C D					F A	
9	C		C D		C	C	C			F C D	F				F A	
10	C		C D				C D			C D	A				F A	
11	C		C				C			C					F A	
12	C						C			C D	D				F A C	
13	C						C D			D	F D				F C	
14	C						C		C	D		C			F C	
15	C		D				C			D					F A C	
16	C		D				C D				C	C			F C	

Appendix F

Curriculum Overview

3rd Grade

		Activity					Application		Problems or					Problem			
Topic	Lesson Objective	Notes boxes	Activity 1	Activity 2	Activity 3	Notes boxes	Template	Problem	Notes boxes	Activities?	Activity 1	Activity 2	Activity 3	Activity 4	Notes boxes	Template	Set
A	Solve word problems in varied contexts using a letter to represent the unknown. Solve word problems in varied contexts using a letter to represent the unknown.		Name the Shape	Pattern Sheet: Multiply by 3	Equivalent Counting with Units of 2					Problem Set	Problem Set #1				Engagement; Representation		L1 (25 minutes)
A	Solve word problems in varied contexts using a letter to represent the unknown. Share and critique peer solution strategies to varied word problems	Standards Alignment	Name the Shape	Pattern Sheet: Multiply by 3	Equivalent Counting with Units of 4					Problem Set	Work independently to solve and present problems using multiple solution paths.	Work cooperatively to identify multiple solution paths.			Standards Alignment		
A	3 problems		Name the Shape	Pattern Sheet: Multiply by 4	Equivalent Counting with Units of 3					Problem/Problem Set	Assess sample student work for accuracy and efficiency.	Assess peer work for accuracy and efficiency. Through Problem Set			Action and Expression	student work samples	
B	Compare and classify quadrilaterals		Pattern Sheet: Multiply by 4	Equivalent Counting with Units of 5				Reviews Topic A	Representation	Activity/Problem Set	Group polygons by attributes	Analyze quadrilaterals into two triangles.	Decompose quadrilaterals into two triangles.		Vocabulary; Representation; Trapezoids	polygons (A-L)	
B	Compare and classify other polygons.		Pattern Sheet: Multiply by 5	Equivalent Counting with Units of 6	Classify the Polygon	Engagement				Activity/Problem Set	Group polygons by attributes	Compare polygons.				polygons (M-X)	
B	Draw polygons with specified attributes to solve problems.		Equivalent Counting with Units of 7	Classify the Polygon	Physiometry	Representation		MP.3 & revisiting classification of squares as rectangle		Activity	Name attributes of a given shape (Amy's words)	Can we draw a polygon with these 3 characteristic s game? (Amy's words)			Action and Expression; Engagement	polygon and game cards	excerpt of L1 (10 minutes)

B	Reason about composing and decomposing polygons using tetrominoes	Pattern Sheet: Multiply by 5	Physiometry	Classify the Shape			Problem Set	Use tetrominoes to compose polygons.	Use tetrominoes to compose polygons with given areas.	Representation; Engagement	tetrominoes	excerpt of L1 (10 minutes)
B	Create a tangram puzzle and observe relationships among the 8 shapes.	Pattern Sheet: Multiply by 6	Equivalent Counting with Units of 8	Shade Rectangles of Equal Area			Problem/Problem Set	Create a tangram puzzle.	Recompose shapes to form a square	Engagement; Representation		
B	Reason about composing and decomposing polygons using 9 tangrams	Pattern Sheet: Multiply by 6	Equivalent Counting with Units of 9		bridge learning from prior lessons & pre for CD		Problem/Problem Set	Use at least two tangram pieces to make and draw two of each of the following shapes. Draw lines to show where the tangram pieces meet.	Use your two smallest triangles to create a square, a parallelogram, and a triangle. Show how you created them below.	Create your own shape on a separate sheet of paper using all seven pieces. Reflect on your experience below. What was easy? What was challenging?	Action and Expression	
C	Decompose quadrilaterals to understand perimeter as the boundary of a shape. Tessellate to understand perimeter	Pattern Sheet: Multiply by 7	Equivalent Counting with Units of 2		L8 & L9 & Area (Module 4)	Representation	Problem/Problem Set	Decompose a square to define perimeter.				excerpt of L1 (5 minutes)
C	Representations						Problem/Problem Set	Tessellate to explore perimeter.	Use a string to measure and compare perimeters			excerpt of L1 (5 minutes)

	boundary of a shape. (Optional)								
	Measure side lengths in whole number units to determine the perimeter of 12 polygons.	Lesson Sequence	Pattern Sheet: Multiply by 7	Equivalent Counting with Units of 3	Area and Perimeter				
C	Explore perimeter as an attribute of plane figures and solve 13 problems. Determine the perimeter of regular polygons and rectangles when whole number measurements are unknown. Solve word problems to determine perimeter with given side lengths.								
			Pattern Sheet: Multiply by 8	Equivalent Counting with Units of 4	Find the Perimeter				
C						L11	Activity	Measure perimeter of polygons in centimeters and name the shape (Amy's words)	Representation; Representation shapes
								Practice calculating the perimeter of various shapes with given side lengths.	excerpt of L1 (10 minutes)
								Calculate perimeter with given side lengths.	Engagement quiz-quiz-trade cards
			Pattern Sheet: Multiply by 8	Equivalent Counting with Units of 7				Find the perimeter of rectangles with unknown side lengths.	
C						L13	Problem	Find the perimeter of regular polygons with one side length given.	Engagement; Representation
									excerpt of L1 (10 minutes)
			Pattern Sheet: Multiply by 9	Equivalent Counting with Units of 5	Find the Perimeter			Solve perimeter word problems with rectangles.	
C						adding side lengths to find perimeter	Problem Set	Solve perimeter word problems with regular polygons.	Engagement; Action and Expression; Representation
									excerpt of L1 (10 minutes)

C	Use string to measure the perimeter of various circles to the nearest quarter inch. Use all four operations to solve problems involving perimeter and unknown measurement	Pattern Sheet: Multiply by 9	Equivalent Counting with Units of 6	Find the Perimeter		Activity/Problem Set	Use string and a ruler to find the perimeter of a circle. Use string and a ruler to find the perimeter of circular objects.	Action and Expression	circles	excerpt of L1 (10 minutes)
C	17	Factors	Equivalent Counting with Units of 8	Find the Perimeter	L15	Problem	Find the perimeter of irregular figures with some unknown side lengths	Action and Expression		excerpt of L1 (10 minutes)
D	18. Construct rectangles from a given number of unit squares and determine the perimeters	Find the Unknown Factors	Draw Tape Diagrams	Find the Area and Perimeter	activates prior knowledge about determining factors of a specific product contributes to growing number of examples that help students conclude there is no relationship between area and perimeter	Representation	Use unit squares to construct a given rectangular area and calculate perimeter (Amy's words)	Action and Expression	grid paper	excerpt of L1 (10 minutes)
D	19. Use a line plot to record the number of rectangles constructed from a given number of unit squares.	Draw Tape Diagrams	Find the Perimeter		perimeter	Action and Expression	Problem Set	Create a line plot to display how many rectangles can be made with a given number of unit squares. Use unit square tiles to make rectangles with a given number of unit squares.	Action and Expression	

D	Construct rectangles with a given perimeter using unit squares and determine their areas. 20	Sprint: Multiply or Divide by 2			L18 & L19	Problem Set	Use unit square tiles to make rectangles with a given perimeter and determine area (Amy's words)	Action and Expression	Data Sheet for L20 & L21	excerpt of L1 (10 minutes)
D	Construct rectangles with a given perimeter using unit squares and determine their areas. 21	Sprint: Multiply or Divide by 3			L20	Action and Expression	Problem Set	Use unit square tiles to make rectangles with a given perimeter and determine area (Amy's words)	Data Sheet for L20 & L21; centimeter grid paper	excerpt of L1 (10 minutes)
D	Use a line plot to record the number of rectangles constructed in Lessons 2220 & 21	Sprint: Multiply or Divide by 4	Find the Perimeter and Area	Engagement			Problem Set	Draw a line plot representing measurement data. Compare and interpret data on a line plot.	Action and Expression	excerpt of L1 (10 minutes)
E	Solve a variety of word problems with perimeter. Use rectangles to draw a robot with specified perimeter measurements, and reason about the different areas that may be produced. 23	Sprint: Multiply or Divide by 5					Doing the Problem Set as the problems for the CD. Set in a problem-solving scheme of 1. Read and model. 2. Write an equation, calculate to solve, and write a statement. 3. Assess the solution for reasonableness.	Engagement; Representation		
E		Pattern Sheet: Multiply by 6	Find the Side Lengths	Representation			Problem Set	Carry out the problem set. Today is planning. They are given certain perimeters for body part sizes and get to choose the width and length to match that perimeter.	Action and Expression	excerpt of L1 (23 minutes)

E	Solve a variety of word problems involving area and perimeter using all four operations. Share and critique peer strategies for problem solving.	29	Sprint: Multiply or Divide by 8. Find the Perimeter.				Doing the Problem Set as the problems for the CD. Set in a problem-solving scheme of 1. Read and model. 2. Write an equation, calculate to solve, and write a statement. 3. Assess the solution for reasonableness.	Representation; Action and Expression	
E		30	Pattern Sheet: Multiply by 9. Multiply and Divide.				Analyze sample student work for accuracy and efficiency. (Problem Set)	Representation; Engagement	student work sample images

4th Grade

Grade 5						Grade 5								Grade 5			
Topic	Lesson Objective	Activity 1	Activity 2	Activity 3	Notes boxes	Template?	Application Problem	Problems or Activities?	Activity 1	Activity 2	Activity 3	Activity 4	Notes boxes	Template?	Problem Set	Student Exit Debrief	Exit Ticket
A	Use metric measurement to model the decomposition of one whole into tenths.	Divide by 10	Sprint: Divide by 10					Activity	Compose and decompose 1 kilogram, representing tenths in fraction form and decimal form.	Decompose 1 centimeter, representing tenths in fraction form and decimal form.			Engagement; Representation		L1	L1	L1
A	Use metric measurement and area models to represent tenths as fractions greater than 1 and decimal numbers.	Divide by 10	Write the Decimal or Fraction	Count by Tens			Mod 5 & L1	Problem	Draw line segments of given lengths, and express each segment as a mixed number and a decimal.	Use the area model to represent tenths as fractions greater than 1 and as decimal numbers.			tenths area model	excerpt of L1	L1	L1	
A	Represent mixed numbers with units of tens, ones, and tenths with place value disks, on the	Write the Decimal or Fraction	Count by Tens				Anticipates decimal fraction addition & how many more to make 1	Problem	Make groups of 10 tenths to rename as ones. Write the number in decimal form.	Represent mixed numbers with units of tens, ones, and tenths in expanded form.	Use the number line to model mixed numbers with units of ones and tenths.		Action and Expression	tenths on a number line	excerpt of L1	L1	L1

B	Use understanding of fraction equivalence to investigate decimal numbers on the place value chart expressed in 8 different units	Sprint: Write Fractions and Decimals	Expanded Form		Reviews expanded form and patterns of ten in place value chart (Mod 1); prepares for today's Concept Development	Problem	Represent numbers in unit form in terms of different units using the area model	Represent numbers in unit form in terms of different units using place value disks.	Decompose mixed numbers to express as smaller units.	Action and Expression	area model and place value chart	excerpt of L1	L1	L1	
C	Use the place value chart and metric measurement to compare decimals and answer comparison questions	Decompose Larger Units	Decimal Fraction Equivalence	Rename the Decimal	Reviews decomposition of a number with mixed units (maybe Mod 2)	Problem	Compare pairs of decimal numbers representing length	Compare pairs of decimal numbers representing mass.	Compare parts of decimal numbers representing volume.	Terminology	measurement record	excerpt of L1	L1	L1	
C	Use area models and the number line to compare decimal numbers, and record comparisons using <, >, 10 and =.	Decompose Larger Units	Decimal Fraction Equivalence	Rename the Decimal	Reviews L9	Problem	Compare pairs of decimal numbers using an area model. Record the comparison using <, >, and =.	Compare decimal numbers on a number line. Record the comparison using <, >, and =.	Compare decimal numbers using <, >, and =.		comparing with area models	excerpt of L1	L1	L1	
C	Compare and order mixed numbers in 11 various forms.	Expanded Form	Rename the Decimal	Compare Decimal Numbers	Representation	L10	Problem	Arrange mixed numbers, fractions, and decimals on a number line.	Arrange mixed numbers, fractions, and decimals in order from greatest to least.	Compare and order mixed numbers in the context of a word problem	decimal numbers flash cards	excerpt of L1	L1	L1	
D	Apply understanding of fraction equivalence to add tenths and 12 hundredths.	Add and Subtract	Compare Decimal Number	Order Decimal Numbers	Expression	Mod 5 & prep for Concept Development	Problem	Add tenths and hundredths written in unit form using pictorial models	Add tenths and hundredths by converting using multiplication. Express the sum as a decimal.	Add tenths and hundredths with sums greater than 1. Express the sum as a decimal	Representation	area model and place value chart	excerpt of L1	L1	L1

D	13	Add decimal numbers by converting to fraction form. Solve word problems involving the addition of measurements in decimal form.	Order Decimal Numbers	Write in Decimal and Fraction Notation	Action and Expression		Problem	Add two decimal numbers less than 1 by converting to fraction form.	Add two decimal numbers involving whole numbers and like fractional units by converting to fractional form.	Add two decimal numbers involving whole numbers, tenths, and hundredths with unlike units by converting to fractional form.	Converting Decimals to Fractions to Add	excerpt of L1	L1	L1	
D	14		State the Value of the Coins	Add Decimals	Write in Decimal and Fraction Notation	Engagement		Doing the Problem Set as the problems for the CD. Set in a problem-solving scheme of 1. Model the problem. 2. Calculate to solve and write a statement. 3. Assess the solution for reasonableness.			Representation		L1	L1	
E	15	Express money amounts given in various forms as decimal numbers. Solve word problems involving	Add Fractions	State the Value of the Coins			Grade 2 Mod 7	Problem	Express pennies, dimes, and quarters as fractional parts of a dollar.	Express the total value of combinations of pennies, dimes, and quarters in fraction and decimal form.	Find the sum of two sets of bills and cents using whole number calculations and unit form.	Action and Expression; Action and Expression; Reading Decimals and Fractions of a Unit	excerpt of L1	L1	L1
E	16	money.	Sprint: Add Decimal Fractions	State the Value of a Set of Coins				Doing the Problem Set as the problems for the CD. Set in a problem-solving scheme of 1. Model the problem. 2. Calculate to solve and write a statement. 3. Assess the solution for reasonableness.			Engagement; Action and Expression		L1	L1	

Appendix G

Pre-study Interview Protocol

1. Tell me about yourself.
2. Tell me about your teaching experience.
3. Tell me about your experiences in teaching math before using Eureka Math.
4. Tell me about when you first started teaching from Eureka Math.
 - a. What were your first thoughts about it?
 - b. When did this occur?
 - c. What were your greatest challenges?
 - d. What supports did you draw from?
5. Tell me about any feedback you got regarding Eureka Math from students?
 - a. From parents?
 - b. From administrators?
 - c. From peers?
6. How has your teaching of mathematics changed in the time you have used Eureka Math?
7. In retrospect, talk about whether or not you are satisfied with this choice of curriculum.
 - a. Why?
8. What advice do you have for other teachers or districts considering adopting Eureka Math?
9. Do you have anything else to add?

Appendix H
Grade 4 Lesson 9

Lesson 9

Objective: Use the place value chart and metric measurement to compare decimals and answer comparison questions.

Suggested Lesson Structure

Fluency Practice	(10 minutes)
Application Problem	(5 minutes)
Concept Development	(35 minutes)
Student Debrief	(10 minutes)
Total Time	(60 minutes)

Fluency Practice (10 minutes)

- Decompose Larger Units **4.NF.5** (3 minutes)
- Decimal Fraction Equivalence **4.NF.5** (5 minutes)
- Rename the Decimal **4.NF.5** (2 minutes)

Decompose Larger Units (3 minutes)

Materials: (S) Personal white board, place value chart (Lesson 7 Template)

Note: This fluency activity reviews Lesson 8.

- T: (Write 1.) Say the number in unit form.
 S: 1 one.
 T: Draw 1 one on your place value chart.
 S: (Draw 1 one disk.)
 T: (Write 1 one = ____ tenths.) Rename 1 one for tenths.
 S: (Cross out the one disk, and draw 10 tenth disks.)

ones	tenths	hundredths
●	●●●●●●●●	

1 one = 10 tenths

Continue this process using the following possible sequence:

- Rename 1 one 2 tenths for tenths.
- Rename 1 tenth for hundredths.
- Rename 1 tenth 2 hundredths for hundredths.
- Rename 2 ones 3 tenths for tenths (leads into the next fluency activity).

..R-Storyline

..E-Features

..E-Representation

..E-Representation

..E-Storyline

Decimal Fraction Equivalence (5 minutes)

Materials: (S) Personal white board, place value chart (Lesson 7 Template)

Note: This fluency activity reviews Lesson 8. For 4 ones 23 hundredths, 1 ten 7 tenths, and 3 tens 4 ones 12 hundredths, have the students express their answers in tenths and hundredths.

- T: (Write 2 ones and 3 tenths.) Write the number in digits on your place value chart.
- S: (Write the digit 2 in the ones place and the digit 3 in the tenths place.)
- T: (Write $2.3 = \underline{\hspace{1cm}}$.) Write the number as a mixed number.
- S: (Write $2.3 = 2\frac{3}{10}$.)
- T: (Write $2.3 = 2\frac{3}{10} = \frac{\hspace{1cm}}{\hspace{1cm}}$.) Write the number as a fraction greater than 1.
- S: (Write $2.3 = 2\frac{3}{10} = \frac{23}{10}$.)

ones	.	tenths	hundredths
2	.	3	

$$2.3 = 2\frac{3}{10} = \frac{23}{10}$$

Continue this process for the following possible sequence: 4 ones 23 hundredths, 1 ten 7 tenths, and 3 tens 4 ones 12 hundredths.

Rename the Decimal (2 minutes)

Materials: (S) Personal white board

Note: This fluency activity reviews Lesson 8.

- T: (Write 3.1.) Write the decimal as a mixed number.
- S: (Write $3\frac{1}{10}$.)
- T: (Write $3.1 = 3\frac{1}{10} = \frac{\hspace{1cm}}{\hspace{1cm}}$.) Complete the number sentence.
- S: (Write $3.1 = 3\frac{1}{10} = \frac{31}{10}$.)
- T: (Write $3.1 = 3\frac{1}{10} = \frac{31}{10} = \frac{\hspace{1cm}}{\hspace{1cm}}$.) Complete the number sentence.
- S: (Write $3.1 = 3\frac{1}{10} = \frac{31}{10} = \frac{310}{100}$.)

Continue this process for the following possible sequence: 9.8, 10.4, and 64.3.

Application Problem (5 minutes)

Kelly's dog weighs 14 kilograms 24 grams. Mary's dog weighs 14 kilograms 205 grams. Hae Jung's dog weighs 4,720 grams.

- Order the weight of the dogs in grams from least to greatest.
- How much more does the heaviest dog weigh than the lightest dog?

K 14 kg 24g

M 14 kg 205g

HJ 4,720g

$$a) 4,720g < 14,024g < 14,205g$$

$$b) \begin{array}{r} 14,205g \\ - 4,720g \\ \hline 9,485g \end{array}$$

Mary's dog weighs 9,485g more than Hae Jung's dog.

Note: This Application Problem reviews decomposition of a number with mixed units. Students need to convert the weight of Kelly's dog to 14,024 grams. The weight of Mary's dog may help them avoid the common error of 1,424 grams because of its inclusion of 205 grams.

Concept Development (35 minutes)

Materials: (T) 2 meter sticks, 2 rolls of different color masking tape (e.g., yellow and blue), metric scale, 4 graduated cylinders, bags of rice, water, food coloring, document camera (S) Personal white board, measurement record (Template)

Materials Note:

- Prepare 2 meter sticks by taping colored masking tape onto the edge of each meter stick to the following lengths: 0.67 m (yellow tape), 0.59 m (blue tape). Do not cover the hash marks or the numbers on the meter sticks.
- Prepare and label 4 bags of rice weighing 0.10 kg (Bag A), 0.65 kg (Bag B), 0.7 kg (Bag C), and 0.46 kg (Bag D).
- Prepare and label four graduated cylinders with water measuring 0.3 liter (Cylinder A), 0.15 liter (Cylinder B), 0.29 liter (Cylinder C), and 0.09 liter (Cylinder D). Use food coloring to help students read the measurements.

**NOTES ON
MULTIPLE MEANS
OF ACTION AND
EXPRESSION:**

If a document camera, overhead projector, interactive white board, or other means of magnifying the image of the meter stick is not available, consider having students use pre-marked meter sticks at their desks. Certain hardware and home furnishings stores and websites offer meter sticks or tape for free. A meter tape template is also available in Grade 2 Module 2 Lesson 6.

..R-Storyline

..E-Tools

..E-Features
..E-Tools

Problem 1: Compare pairs of decimal numbers representing length.

- T: (Hold up the meter stick with the yellow tape that measures 0.67 m, and then place it under the document camera.) Express the length of this yellow tape as a fraction of a meter.
- S: $\frac{67}{100}$ meter.
- T: On the measurement record, shade the tape diagram to represent the length of the yellow tape on the meter stick. Write the length of the tape in decimal form.
- T: (Hold up the meter stick with blue tape that measures 0.59 m, and then project the portion of the meter stick that shows the length of the blue tape under the document camera.) Express the length of this blue tape as a fraction of a meter.
- S: $\frac{59}{100}$ meter.
- T: On the measurement record, shade the tape diagram to represent the length of the blue tape on the meter stick. Write the length of the tape in decimal form. Record both lengths in a place value chart. (Allow students time to complete the task.)
- T: Use the words *longer than* or *shorter than* to compare these two lengths of tape.
- S: 0.67 meter is longer than 0.59 meter. $\rightarrow$ 0.59 meter is shorter than 0.67 meter.
 $\rightarrow$ 67 centimeters is longer than 59 centimeters, so I know 0.67 meter is longer than 0.59 meter.

- T: Share with a partner. How can the place value chart help you compare these numbers?
- S: We can compare the digits in the largest place first. Both measures have 0 in the ones place, so we move to the tenths place. The first tape has 6 tenths. That's greater than 5 tenths. $\rightarrow$ You don't even need to look at the hundredths place. Once you see that 6 tenths is greater than 5 tenths, you know that the first tape is longer.

ones	•	tenths	hundredths
0	.	6	7
0	.	5	9

Remove enough tape from each meter stick to create the following lengths: 0.4 m and 0.34 m. Repeat the above process.

Problem 2: Compare pairs of decimal numbers representing mass.

- T: (Place Rice Bag A on the scale.) What is the mass of this bag of rice?
 S: Zero point one kilogram. $\rightarrow \frac{1}{10}$ kilogram. $\rightarrow \frac{10}{100}$ kilogram (see image below).
 T: Record the mass in the table on the measurement record.

Repeat this process for the remaining bags.

- T: (Leave Bag D, weighing 0.46 kg, on the scale.) Which bags are heavier than Bag D? How do you know?
 S: Bags B and C were heavier than Bag D. $\rightarrow$ Bag B was 0.65 kg, and Bag C was 0.7 kg. Those numbers are both larger than 0.46 kg, so the bags are heavier.
 $\rightarrow$ I looked at my chart, from left to right. In the tenths column, I could see that Bag A was lighter. It had only 1 tenth. Bags B and C were heavier than D because they both had more tenths.
 T: Let's look at Bags B and C. Make a statement comparing their mass.
 S: 0.65 kilogram is lighter than 0.7 kilogram.
 $\rightarrow$ 0.7 kilogram is heavier than 0.65 kilogram.
 T: How do you know?

- S: I could just see that the bag was fuller and feel that the bag has more mass. $\rightarrow$ At first, I thought 65 hundredths was more because it looks like you are comparing 65 and 7, and 65 is greater than 7. But then we saw that it was 7 tenths, which is more than 6 tenths. $\rightarrow$ I realized that 7 tenths is 70 hundredths, and that is greater than 65 hundredths.

- T: With your partner, make another statement to compare the bags. You can compare just two items, or you can compare more than two items.

- S: (Responses will vary.)

- T: Based on these comparisons, what is the mass of the bags in order from heaviest to lightest?

- S: 0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg.

- T: (Select a student volunteer.) Arrange the bags from heaviest to lightest. Looking at the bags, does it *appear* that we have properly ordered the bags from heaviest to lightest? Do they match the order we determined?

- S: Yes.

NOTES ON TERMINOLOGY:

Mass is a fundamental measure of the amount of matter in an object. While weight is a measurement that depends upon the force of gravity (one would weigh less on the moon than one does on Earth), mass does not depend upon the force of gravity. Both words are used here, but it is not important for students to recognize the distinction in mathematics at this time.

Mass of Rice Bags (kilograms)

Rice Bag	ones	.	tenths	hundredths
A	0	.	1	0
B	0	.	6	5
C	0	.	7	
D	0	.	4	6

0.7 kg, 0.65 kg, 0.46 kg, 0.1 kg

Problem 3: Compare pairs of decimal numbers representing volume.

T: (Place all four graduated cylinders in front of the class.) Express the volume of the liquid in tenths or hundredths liter. (Use the document camera to project the side of Cylinder A so students can see the liter measurements. If this is not possible, select a student to read the volume aloud.)

S: $\frac{3}{10}$ liter. $\rightarrow \frac{30}{100}$ liter.

T: Record this volume in the table on the measurement record.

Repeat the process for the remaining water samples.

T: If we want to order these samples from least volume to greatest volume, what would the order be? Talk with your partner, and record your thinking on the measurement record. (Circulate to encourage use of the place value chart as students compare the measurements.)

S: (Complete the task.)

S: 0.09 liter, 0.15 liter, 0.29 liter, 0.3 liter.

T: How did you determine the order?

S: The place value chart made it easy to compare the decimals. $\rightarrow$ We compared the digits in the largest place first. That was the tenths. $\rightarrow$ In 0.3, there are 3 tenths. That is more than the others. 0.29 comes next, followed by 0.15 and 0.09.

T: (Select a student volunteer to order cylinders from least volume to greatest volume.) Let's look at the cylinders. Do they appear to match the order we determined?

S: Yes!

Volume of Liquid (liters)			
Cylinder	ones	tenths	hundredths
A	0	3	
B	0	1	5
C	0	2	9
D	0	0	9

0.09 L, 0.15 L, 0.29 L, 0.3 L

Problem Set (10 minutes)

Students should do their personal best to complete the Problem Set within the allotted 10 minutes. For some classes, it may be appropriate to modify the assignment by specifying which problems they work on first. Some problems do not specify a method for solving. Students should solve these problems using the RDW approach used for Application Problems.

Student Debrief (10 minutes)

Lesson Objective: Use the place value chart and metric measurement to compare decimals and answer comparison questions.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

name: Jack

1. Measure the length of the shaded parts in decimal form. Write the answer in the comparison chart. Explain. Use the expanded number line or number line to assist.

a.

0.5m	0.3 meters is larger than 0.2 meters.
0.57m	0.57m

b.

0.95m	0.95 meters is shorter than 1 meter.
0.4m	0.4m

c. Use all four lengths from part (a) to compare.

0.27m, 0.5m, 0.95m, 0.4m

d. Compare the mass of each item in grams to the 100-gram scale. Put each item in the box that is heavier than the others.

100g	100g	100g	100g
------	------	------	------

..E-Features
..E-Structures
..E-Differentiation

..E-Representational
..E-Experiences
..E-Reasoning
..E-Discourse
..E-Representational
..E-Reasoning
..E-Features

..R-Experiences

A STORY OF UNITS

Lesson 9 4•6

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

Any combination of the questions below may be used to lead the discussion.

- How do the tape diagrams in Problem 1 support your statements? Make a statement comparing a length from part (a) to a length from part (b).
- Share one of your statements for Problem 2(c). Explain your reasoning.
- How did the place value chart help to compare and order the different measurements in Problem 3?
- How is comparing decimal measurements of length, mass, and volume similar? How is it different?
- How did the Application Problem connect to today's lesson?

1. Answer the questions about each measurement value chart.

Unit	Label	Value	Measurement
Length	10	1	10
Mass	10	1	10
Volume	10	1	10
Length	10	1	10

2. Complete the statements below using the words **greater than** or **less than** in your statements.

The mass of the pencil is greater than the mass of the paper.

The length of the pencil is greater than the length of the paper.

3. Record the volume of water in each graduated cylinder on the place value chart below.

Label	Unit	Value	Measurement
A	10	1	10
B	10	1	10
C	10	1	10
D	10	1	10
E	10	1	10
F	10	1	10

4. Compare the values using **>**, **<**, or **=**.

A > B, B < C, C < D, D < E, E < F

5. Write the volume of water in each graduated cylinder in order from least to greatest.

0.1 L, 0.2 L, 0.3 L, 0.4 L, 0.5 L, 0.6 L

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help with assessing students' understanding of the concepts that were presented in today's lesson and planning more effectively for future lessons. The questions may be read aloud to the students.

Name _____ Date _____

1. Express the lengths of the shaded parts in decimal form. Write a sentence that compares the two lengths. Use the expression *shorter than* or *longer than* in your sentence.

- c. List all four lengths from least to greatest.

2. a. Examine the mass of each item as shown below on the 1-kilogram scales. Put an X over the items that are heavier than the avocado.

- b. Express the mass of each item on the place value chart.

Mass of Fruit (kilograms)

Fruit	ones	.	tenths	hundredths
avocado				
apple				
bananas				
grapes				

- c. Complete the statements below using the words *heavier than* or *lighter than* in your statements.

The avocado is _____ the apple.

The bunch of bananas is _____ the bunch of grapes.

3. Record the volume of water in each graduated cylinder on the place value chart below.

Volume of Water (liters)

Cylinder	ones	.	tenths	hundredths
A				
B				
C				
D				
E				
F				

Compare the values using $>$, $<$, or $=$.

- $0.9 \text{ L} \underline{\hspace{1cm}} 0.6 \text{ L}$
- $0.48 \text{ L} \underline{\hspace{1cm}} 0.6 \text{ L}$
- $0.3 \text{ L} \underline{\hspace{1cm}} 0.19 \text{ L}$
- Write the volume of water in each graduated cylinder in order from least to greatest.

Name _____

Date _____

1. a. Doug measures the lengths of three strings and shades tape diagrams to represent the length of each string as show below. Express, in decimal form, the length of each string.

- b. List the lengths of the strings in order from greatest to least.
2. Compare the values below using $>$, $<$, or $=$.
- a. 0.8 kg _____ 0.6 kg
- b. 0.36 kg _____ 0.5 kg
- c. 0.4 kg _____ 0.47 kg

Name _____ Date _____

1. Express the lengths of the shaded parts in decimal form. Write a sentence that compares the two lengths. Use the expression *shorter than* or *longer than* in your sentence.

a.

b.

- c. List all four lengths from least to greatest.

2. a. Examine the mass of each item as shown below on the 1-kilogram scales. Put an X over the items that are heavier than the volleyball

- b. Express the mass of each item on the place value chart.

Mass of Sport Balls (kilograms)

Sport Balls	ones	.	tenths	hundredths
baseball				
volleyball				
basketball				
soccer ball				

- c. Complete the statements below using the words *heavier than* or *lighter than* in your statements.

The soccer ball is _____ the baseball.

The volleyball is _____ the basketball.

3. Record the volume of water in each graduated cylinder on the place value chart below.

Volume of Water (liters)

Cylinder	ones	.	tenths	hundredths
A				
B				
C				
D				
E				
F				

Compare the values using $>$, $<$, or $=$.

a. $0.4 \text{ L} \underline{\hspace{1cm}} 0.2 \text{ L}$

b. $0.62 \text{ L} \underline{\hspace{1cm}} 0.7 \text{ L}$

c. $0.2 \text{ L} \underline{\hspace{1cm}} 0.28 \text{ L}$

- d. Write the volume of water in each graduated cylinder in order from least to greatest.

--	--	--	--	--	--	--	--	--	--

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--	--	--	--	--	--	--	--	--	--

--	--	--	--	--	--	--	--	--	--

Mass of Rice Bags (kilograms)

Rice Bag	ones	.	tenths	hundredths
A				
B				
C				
D				

Volume of Liquid (liters)

Cylinder	ones	.	tenths	hundredths
A				
B				
C				
D				

measurement record

Appendix I**Grade 3 Lesson 4**

Lesson 4

Objective: Compare and classify quadrilaterals.

Suggested Lesson Structure

Fluency Practice	(12 minutes)
Application Problem	(7 minutes)
Concept Development	(31 minutes)
Student Debrief	(10 minutes)
Total Time	(60 minutes)

Fluency Practice (12 minutes)

- Multiply by 4 **3.OA.7** (8 minutes)
- Equivalent Counting with Units of 5 **3.OA.7** (4 minutes)

Multiply by 4 (8 minutes)

Materials: (S) Multiply by 4 (6–10) Pattern Sheet

Note: This activity builds fluency with multiplication facts using units of 4. It works toward students knowing from memory all products of two one-digit numbers. See Lesson 1 for the directions for administration of a Multiply-By Pattern Sheet.

- T: (Write $7 \times 4 = \underline{\quad}$.) Let's skip-count up by fours. I'll raise a finger for each four. (Raise a finger for each number to track the count.)
- S: 4, 8, 12, 16, 20, 24, 28.
- T: Let's skip-count up by fours starting at 20. Why is 20 a good place to start?
- S: It's a fact we already know, so we can use it to figure out a fact we don't know.
- T: (Track with fingers as students say the numbers.)
- S: 20 (5 fingers), 24 (6 fingers), 28 (7 fingers).
- T: Let's see how we can skip-count down to find the answer, too. Start at 40 with 10 fingers, 1 for each four. (Count down with fingers as students say the numbers.)
- S: 40 (10 fingers), 36 (9 fingers), 32 (8 fingers), 28 (7 fingers).

Continue with the following possible sequence: 9×4 , 6×4 , and 8×4 .

- T: (Distribute the Multiply by 4 Pattern Sheet.) Let's practice multiplying by 4. Be sure to work left to right across the page.

..R-Experi
..R-Storyline
..R-Sequenc

..E-Experi
..E-Sequenc

..R-Storyline
..E-Differentiatk
..R-Experiences

..E-Experiences

..R-Representat
..E-Representati
..E-Differentiatk

..R-Storyline

A STORY OF UNITS

Lesson 4 3•7

Equivalent Counting with Units of 5 (4 minutes)

Note: This activity builds fluency with multiplication facts using units of 5. The progression builds in complexity. Work students up to the highest level where they can confidently participate.

T: Count to 10. (Write as students count. See the chart below.)

S: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

1	2	3	4	5	6	7	8	9	10
1 five	2 fives	3 fives	4 fives	5 fives	6 fives	7 fives	8 fives	9 fives	10 fives
5	10	15	20	25	30	35	40	45	50
1 five	10	3 fives	20	5 fives	30	7 fives	40	9 fives	50
5	2 fives	15	4 fives	25	6 fives	35	8 fives	45	10 fives

T: (Write 1 five beneath the 1.) Count to 10 fives. (Write as students count.)

S: 1 five, 2 fives, 3 fives, 4 fives, 5 fives, 6 fives, 7 fives, 8 fives, 9 fives, 10 fives.

T: Count by fives to 50. (Write as students count.)

S: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

T: (Write 1 five beneath the 5. Write 10 beneath the 10.) I'm going to give you a challenge. Let's alternate between saying the units of five and the number. (Write as students count.)

S: 1 five, 10, 3 fives, 20, 5 fives, 30, 7 fives, 40, 9 fives, 50.

T: (Write 5 beneath 1 five and 2 fives beneath the 10.) Let's alternate again. (Write as students count.)

S: 5, 2 fives, 15, 4 fives, 25, 6 fives, 35, 8 fives, 45, 10 fives.

Application Problem (7 minutes)

The third graders raised \$437 in a fundraiser. The fourth graders raised \$68 less than the third graders. How much money did the two grade levels raise altogether?

$$\begin{array}{r}
 \$437 \\
 - \$68 \\
 \hline
 \$369
 \end{array}$$

The 2 grade levels raise \$806 altogether.

Note: This problem reviews two-step word problems from Topic A.

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Modeling a tape diagram for the money fourth graders raise, as well as the total money raised, helps English language learners and students working below grade level better grasp the meaning of the phrase \$68 less than the third graders. Ask, "Who raised less money? Did the fourth graders raise \$68? Use the model to estimate about how much the fourth graders raised."

Concept Development (31 minutes)

Materials: (T) 2 rulers (S) Index card for use as right angle tool, polygons (A–L) (Template), ruler, Problem Set, scissors

Part 1: Group polygons by attributes.

Pass out the index cards and Template.

- T: We'll use these cards as tools. Put a finger on each corner.
S: (Touch each corner.)
- T: Remember from second grade that we call the point where sides meet to make a corner an **angle**. These are **right angles** because they have square corners. We'll use our cards as right angle tools to help us find other shapes that have right angles. (Save the right angle tools for the entire module.)
- T: Now, cut out shapes A–L on your template.
S: (Cut.)
- T: Look at your shapes. Discuss with a partner: What are some different ways we can group these shapes together?
- S: We can group them by name, like all the squares together. → We can group them by the number of sides. → We can also group them by the number of angles.
- T: Remember from second grade that closed shapes like these that have no gaps or overlaps between the straight sides are called **polygons**. Polygons with four straight sides are called **quadrilaterals**. Tell your partner what a quadrilateral is, and then find and group the quadrilaterals.
- S: A quadrilateral is a polygon with four sides. (Group the quadrilaterals.)
- T: What do you notice about the polygons you grouped?
- S: They don't look the same. → Some are slanted, and some are boxy. → Some are squares and rectangles, but others are strange looking. → One polygon even looks like a boomerang. → They have four angles.
- T: The polygons look different, but they share the attributes of having four sides and four angles. Complete the first row of the chart on the Problem Set. Make sure to sketch one polygon from the group.

NOTES ON VOCABULARY:

All of the bold-type terms in this lesson were introduced in Grade 2 Module 8. However, given the specificity of the words and the time that has passed since students' work in Grade 2 Module 8, it may be best to approach teaching the vocabulary as if it were new.

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Clarify for English language learners and others the term **group**. In past modules, they have grouped like units, such as 10 tens to make 100 or 3 fives to make 15. Here, grouping does not mean bundling units that are exactly the same but rather sorting polygons by one or more shared attribute.

4-sided:

- T: Next, we'll find and group **trapezoids**. These are quadrilaterals that have at least one set of **parallel** sides. Think of parallel sides like the two side lines of a capital *H*, or a slanted *H*, since not all parallel sides stand vertical. (Demonstrate using two rulers.) Imagine these two lines go on forever. Do you think they will ever cross? Why or why not?
- S: I don't think they will cross. → No, they won't cross because they're straight and going in the same direction all the time.
- T: (Slant the rulers so they are not parallel anymore but are still not touching.) These lines are not touching. Are they parallel? Why or why not?
- S: No. The sides don't look like an *H* anymore. → If we imagine the lines keep going, they will eventually cross!
- T: If trapezoids must have at least one set of parallel sides, can they have more than one set?
- S: Yeah. At least means one or more.
- T: Group the trapezoids. Complete the second row of the chart on the Problem Set. Make sure to sketch one polygon from the group.
- S: (Group all shapes, except A, D, and K, and sketch one shape.)
- T: What do you notice about the polygons you grouped?
- S: I found a bunch! → No. There's only one shape that has only one set of parallel sides. Polygon E! → Remember, though, a trapezoid has at least one set! That's almost all of them!
- T: Now we'll find and group **parallelograms**. These are four-sided polygons that have two sets of parallel sides.
- T: Group the parallelograms. Then, complete the next row of the chart on your Problem Set.
- S: (Group the polygons, and complete the chart.)
- T: Now, use your right angle tool to measure and group all the polygons that have four right angles. Then, complete the chart.
- S: (Measure, group, and complete the chart.)
- T: Next, find and group all the squares. Which attributes make squares special?
- S: They have four equal sides and four right angles.
- T: Use your ruler and right angle tool to confirm that with these polygons. Then, complete the chart.
- S: (Measure, group, and complete the chart.)

NOTES ON TRAPEZOIDS:

According to the K–6 Geometry Progressions, the term trapezoid can have two different meanings:

- **Exclusive Definition:** A trapezoid is a quadrilateral with exactly one pair of parallel sides.
- **Inclusive Definition:** A trapezoid is a quadrilateral with at least one pair of parallel sides.

A Story of Units uses the inclusive definition. Therefore, a parallelogram is also considered a trapezoid.

..E-Reasoning
..E-Discourse

..E-Terms

..E-Tools
..E-Terms

..E-Discourse

Part 2: Analyze quadrilaterals.

- T: In our set of polygons A–L, did the number of polygons get smaller or larger as we added attributes?
- S: It got smaller.
- T: Discuss with your partner why you think the number of polygons in each group got smaller as we added attributes.
- S: I think it's because the attributes in our chart become more special. The last category only includes the most special polygon, a square, because it has to have four right angles *and* four equal sides.
→ Each time we added a new attribute, fewer polygons belonged to the group.
- T: As the attributes become more specific, fewer shapes in our set share all of the attributes. Look at Polygons C and F. They are included in every group. Why do you think that is?
- S: They have four sides, two sets of parallel lines, and four right angles.
- MP.3 T: Why aren't Polygons B and H included in the last category? These specific rectangles have four sides, two sets of parallel lines, and four right angles.
- S: Polygons B and H don't have all equal sides.
- T: Look at Polygon I. It has four equal sides and two sets of parallel lines. Why isn't it included in the last category?
- S: It doesn't have four right angles. → It needs to have them all, not just one attribute.
- T: Let's make a new category, one that has shapes with 4 equal sides. Work with your partner.
- T: (Move Polygons C, F, and I to form a new group.) A shape with 4 equal sides is called a **rhombus**.
- T: Why is a square a rhombus?
- S: Because it has 4 equal sides!
- T: Why isn't shape I a square?
- S: Because it doesn't have right angles!

Part 3: Decompose quadrilaterals into two triangles.

- T: Problem 4 asks you to use a straightedge to draw a line between opposite corners in each quadrilateral you drew in the chart. This kind of line is called a **diagonal** line. Do that now.
- S: (Draw diagonals in each polygon.)
- T: Which new polygons did you make by drawing the diagonal line?
- S: Triangles.
- T: Complete Problem 4 on your Problem Set.
- T: Pick other polygons we used that you did not draw on your chart. Draw diagonal lines inside the polygons. Do you still get two triangles? (Allow time for students to draw.)
- S: Yes!
- T: All quadrilaterals are made up of two triangles.

Students should now go back and finish Problems 2 and 3 on the Problem Set.

..R-Experiences

..E-Features

..E-Structures

..E-Differentiation

..E-Experiences

..E-Discourse

..E-Terms

..R-Experiences

Student Debrief (10 minutes)**Lesson Objective:** Compare and classify quadrilaterals.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

Any combination of the questions below may be used to lead the discussion.

- How does grouping quadrilaterals by **attributes**, like you did in Problem 1, help us see the similarities and differences between the **polygons**?
- Share sketches of parallelograms from Problem 3. Have students describe **parallel** lines through their color-coded tracing.
- For Problem 4, share drawings of different quadrilaterals to reinforce how every quadrilateral can be decomposed into two triangles.
- What math vocabulary did we use today to name polygons with four sides? (**Quadrilateral**.) At least one set of parallel sides? (**Trapezoid**.) Two sets of parallel sides? (**Parallelogram**.) A shape with 4 equal sides? (**Rhombus**.) An angle that makes square corners? (**Right angle**.) The line between opposite corners in each quadrilateral? (**Diagonal**.)

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help with assessing students' understanding of the concepts that were presented in today's lesson and planning more effectively for future lessons. The questions may be read aloud to the students.

Name: Sina Date: _____

1. Sort each of the polygons (A-L) in the box below. Then use the polygons to complete the following chart.

Attributes	Sketch the names of the polygons in the box below	Sketch a polygon from the group
Quadrilaterals	Polygons: A, B, C, D, E, F, G, H, I, J, K, L All are 4-sided!	
Parallelograms	Polygons: B, C, E, F, G, H, I, J, K, L	
2 sets of parallel sides	Polygons: B, C, E, F, G, H, I, J, K, L	
1 right angle	Polygons: B, C, E, F, H	
4 right angles and 4 equal sides	Polygons: C, F	

2. Write the names of the polygons that are quadrilaterals. Explain how you know these polygons are quadrilaterals.

All of the polygons A-L are quadrilaterals because they all have 4 sides.

3. Sketch a polygon from the group that has 2 sets of parallel sides. Trace 1 pair of parallel sides and trace the other pair of parallel sides. What makes parallel sides different from sides that are not parallel?

The parallel sides look like the 2 sides of a big H. They're never touch or cross, even if we make them longer and longer.

4. Draw a diagonal line from one corner to the opposite corner of each polygon you drew in the chart using a straightedge. What new polygons did you make by drawing the diagonal?

I made 2 triangles in each shape!

Multiply.

$4 \times 1 =$ _____	$4 \times 2 =$ _____	$4 \times 3 =$ _____	$4 \times 4 =$ _____
$4 \times 5 =$ _____	$4 \times 6 =$ _____	$4 \times 7 =$ _____	$4 \times 8 =$ _____
$4 \times 9 =$ _____	$4 \times 10 =$ _____	$4 \times 5 =$ _____	$4 \times 6 =$ _____
$4 \times 5 =$ _____	$4 \times 7 =$ _____	$4 \times 5 =$ _____	$4 \times 8 =$ _____
$4 \times 5 =$ _____	$4 \times 9 =$ _____	$4 \times 5 =$ _____	$4 \times 10 =$ _____
$4 \times 6 =$ _____	$4 \times 5 =$ _____	$4 \times 6 =$ _____	$4 \times 7 =$ _____
$4 \times 6 =$ _____	$4 \times 8 =$ _____	$4 \times 6 =$ _____	$4 \times 9 =$ _____
$4 \times 6 =$ _____	$4 \times 7 =$ _____	$4 \times 6 =$ _____	$4 \times 7 =$ _____
$4 \times 8 =$ _____	$4 \times 7 =$ _____	$4 \times 9 =$ _____	$4 \times 7 =$ _____
$4 \times 8 =$ _____	$4 \times 6 =$ _____	$4 \times 8 =$ _____	$4 \times 7 =$ _____
$4 \times 8 =$ _____	$4 \times 9 =$ _____	$4 \times 9 =$ _____	$4 \times 6 =$ _____
$4 \times 9 =$ _____	$4 \times 7 =$ _____	$4 \times 9 =$ _____	$4 \times 8 =$ _____
$4 \times 9 =$ _____	$4 \times 8 =$ _____	$4 \times 6 =$ _____	$4 \times 9 =$ _____
$4 \times 7 =$ _____	$4 \times 9 =$ _____	$4 \times 6 =$ _____	$4 \times 8 =$ _____
$4 \times 9 =$ _____	$4 \times 7 =$ _____	$4 \times 6 =$ _____	$4 \times 8 =$ _____

multiply by 4 (6–10)

Name _____ Date _____

1. Cut out all the polygons (A–L) in the Template. Then, use the polygons to complete the following chart.

Attribute	Write the letters of the polygons in this group.	Sketch 1 polygon from the group.
<i>Example:</i> 3 Sides	Polygons: Y, Z
4 Sides	Polygons:	
At Least 1 Set of Parallel Sides	Polygons:	
2 Sets of Parallel Sides	Polygons:	
4 Right Angles	Polygons:	
4 Right Angles and 4 Equal Sides	Polygons:	

2. Write the letters of the polygons that are quadrilaterals. Explain how you know these polygons are quadrilaterals.
3. Sketch a polygon below from the group that has 2 sets of parallel sides. Trace 1 pair of parallel sides red. Trace the other pair of parallel sides blue. What makes parallel sides different from sides that are not parallel?
4. Draw a diagonal line from one corner to the opposite corner of each polygon you drew in the chart using a straightedge. What new polygon(s) did you make by drawing the diagonal lines?

Name _____

Date _____

List as many attributes as you can to describe each polygon below.

1.

2.

Name _____

Date _____

1. Complete the chart by answering true or false.

Attribute	Polygon	True or False
Example: 3 Sides		True
4 Sides
2 Sets of Parallel Sides
4 Right Angles
Quadrilateral

2. a. Each quadrilateral below has at least 1 set of parallel sides. Trace each set of parallel sides with a colored pencil.

- b. Using a straightedge, sketch a different quadrilateral with at least 1 set of parallel sides.

polygons (A–L)

Appendix J**Grade 3 Lesson 23**

Lesson 23

Objective: Solve a variety of word problems with perimeter.

Suggested Lesson Structure

■ Fluency Practice	(10 minutes)
■ Concept Development	(40 minutes)
■ Student Debrief	(10 minutes)
Total Time	(60 minutes)

Fluency Practice (10 minutes)

- Sprint: Multiply or Divide by 5 **3.OA.7** (10 minutes)

Sprint: Multiply or Divide by 5 (10 minutes)

Materials: (S) Multiply or Divide by 5 Sprint

Note: This Sprint builds fluency with multiplication and division facts using units of 5.

Concept Development (40 minutes)

Materials: (S) Problem Set

In this problem-solving lesson, students work in pairs or independently to solve the six problems on the Problem Set. The teacher facilitates conversation and may provide structure for problem solving using Steps 1–3 (on the next page). Specific information about each problem follows and can be used to facilitate conversation.

MP.1

Suggested Problem-Solving Steps

For each problem, select two pairs of students to work at the class board or central space. Other students work independently or in pairs at their tables.

NOTES ON MULTIPLE MEANS OF ENGAGEMENT:

Give everyone a fair chance to be successful by providing appropriate scaffolds. Demonstrating students may use peer translators or sentence frames to present and respond to feedback. Models shared may include concrete manipulatives or adaptive materials.

If the pace of the lesson is a consideration, prepare presenters beforehand. Problem 4 may be most approachable for students working below grade level.

..R-Storyline

..E-Overview

..E-Differentiation

..E-Structures

..E-Differentia

..E-Represent

..E-Discourse

A STORY OF UNITS

Lesson 23 3•7

1. Read and model.

Review the following questions after reading the first problem:

- Can you draw something?
- What can you draw?
- What conclusions can you make from your drawing?

When most students have finished, have the pairs of students at the board share only their labeled diagrams. Have the demonstrating students receive and respond to feedback and questions from their peers.

NOTE ON
MULTIPLE MEANS
OF REPRESENTATION:

Students who have difficulty drawing an octagon to solve Problem 1 might choose instead to model with a tape diagram consisting of 8 equal parts and having a total value of 48 cm.

2. Write an equation, calculate to solve, and write a statement.

Allow time for students to finish work on the question. They then retrace the steps of their thinking as they share their work with a partner or another pair. Students write their equations and statements on their own Problem Sets. Demonstrating students can model this process for others.

3. Assess the solution for reasonableness.

Give students one to two minutes to validate and explain the reasonableness of their solutions. Two or three of the following sentence starters may be provided to guide them in this work:

- I reread the question to remind myself that it asks for _____.
- My answer matches what the question asks because _____.
- The units of my answer make sense because _____.
- I know my answer is neither too small nor too big because _____.
- My answer would not make any sense if it was _____ because _____.

Problem 1: Gale makes a miniature stop sign, a regular octagon, with a perimeter of 48 centimeters for the town he built with blocks. What is the length of each side of the stop sign?

Students might solve by first drawing an octagon, then labeling a side length with a letter, and dividing the perimeter (48 cm) by the number of sides on an octagon (8).

Problem 2: Travis bends wire to make rectangles. Each rectangle measures 34 inches by 12 inches. What is the total length of the wire needed for two rectangles?

This is a two-step problem. Students find the perimeter of one rectangle and then add to find the perimeter of two rectangles. They should recognize that the total perimeter of two rectangles is equal to the total length of wire needed. To solve the first step, students might draw a rectangle, label the side lengths, and find the perimeter. In the second step, students might use a variety of strategies to solve $92 + 92$, including using the standard algorithm or adding like units: 9 tens plus 9 tens equals 18 tens, and 2 ones plus 2 ones equals 4 ones, and 18 tens 4 ones equals 184.

..E-Representations

..R-Experiences

..E-Features

..E-Structures

..E-Differentiation

..E-Experiences

..E-Structures

..E-Discourse

..Student metac

..R-Experiences

Problem 5: Mischa makes a 4-foot by 6-foot rectangular banner. She puts ribbon around the outside edges. The ribbon costs \$2 per foot. What is the total cost of the ribbon?

Students recognize that the length of the ribbon is equal to the perimeter of the banner, so they find the perimeter of the banner (20 feet). They might calculate the cost of the ribbon by multiplying its length (20 feet) by the cost (\$2 per foot). Students can use a variety of strategies to solve, including turning it into a doubles addition fact or thinking of it as 2 tens times 2. Students might also calculate the cost of the ribbon for each side and then add to find the total cost. Encourage drawing the rectangular banner with the side lengths labeled. In the second step, encourage using a letter to represent the unknown cost of the ribbon.

Problem 6: Colton buys a roll of wire fencing that is 120 yards long. He uses it to fence in his 18-yard by 24-yard rectangular garden. Will Colton have enough wire fencing left over to fence in a 6-yard by 8-yard rectangular play space for his pet rabbit?

To solve, students need to find the perimeter of the garden, the difference between the length of the wire fencing and the perimeter of the garden, and the perimeter of the rabbit's play space. Students then compare the amount of leftover fencing to the perimeter of the rabbit's play space to determine whether or not Colton has enough left over.

Student Debrief (10 minutes)

Lesson Objective: Solve a variety of word problems with perimeter.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

Any combination of the questions below may be used to lead the discussion.

- How was setting up the problem to solve Problem 1 different from setting up the other problems? What did you need to know about the stop sign before you could solve?
- Explain to a partner how knowing the perimeter and the width helped you find the length of the rectangle in Problem 3.
- Explain to a partner how you were able to find the perimeter of the rectangle in Problem 4 without knowing either side length.
- How does knowing the perimeter of the banner in Problem 5 help you find the cost of the ribbon?
- You found that Colton has enough fencing to complete both projects in Problem 6. How much fencing will be left over after he fences in his garden and a play space for his rabbit?
- Which problem did you find most difficult? Why?

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help with assessing students' understanding of the concepts that were presented in today's lesson and planning more effectively for future lessons. The questions may be read aloud to the students.

A

Number Correct: _____

Multiply or Divide by 5

1.	$2 \times 5 =$	
2.	$3 \times 5 =$	
3.	$4 \times 5 =$	
4.	$5 \times 5 =$	
5.	$1 \times 5 =$	
6.	$10 \div 5 =$	
7.	$15 \div 5 =$	
8.	$25 \div 5 =$	
9.	$5 \div 5 =$	
10.	$20 \div 5 =$	
11.	$6 \times 5 =$	
12.	$7 \times 5 =$	
13.	$8 \times 5 =$	
14.	$9 \times 5 =$	
15.	$10 \times 5 =$	
16.	$40 \div 5 =$	
17.	$35 \div 5 =$	
18.	$45 \div 5 =$	
19.	$30 \div 5 =$	
20.	$50 \div 5 =$	
21.	$\underline{\quad} \times 5 = 25$	
22.	$\underline{\quad} \times 5 = 5$	

23.	$\underline{\quad} \times 5 = 50$	
24.	$\underline{\quad} \times 5 = 10$	
25.	$\underline{\quad} \times 5 = 15$	
26.	$50 \div 5 =$	
27.	$25 \div 5 =$	
28.	$5 \div 5 =$	
29.	$10 \div 5 =$	
30.	$15 \div 5 =$	
31.	$\underline{\quad} \times 5 = 30$	
32.	$\underline{\quad} \times 5 = 35$	
33.	$\underline{\quad} \times 5 = 45$	
34.	$\underline{\quad} \times 5 = 40$	
35.	$35 \div 5 =$	
36.	$45 \div 5 =$	
37.	$30 \div 5 =$	
38.	$40 \div 5 =$	
39.	$11 \times 5 =$	
40.	$55 \div 5 =$	
41.	$15 \div 5 =$	
42.	$60 \div 5 =$	
43.	$12 \times 5 =$	
44.	$70 \div 5 =$	

B

Number Correct: _____

Improvement: _____

Multiply or Divide by 5

1.	$1 \times 5 =$	
2.	$2 \times 5 =$	
3.	$3 \times 5 =$	
4.	$4 \times 5 =$	
5.	$5 \times 5 =$	
6.	$15 \div 5 =$	
7.	$10 \div 5 =$	
8.	$20 \div 5 =$	
9.	$5 \div 5 =$	
10.	$25 \div 5 =$	
11.	$10 \times 5 =$	
12.	$6 \times 5 =$	
13.	$7 \times 5 =$	
14.	$8 \times 5 =$	
15.	$9 \times 5 =$	
16.	$35 \div 5 =$	
17.	$30 \div 5 =$	
18.	$40 \div 5 =$	
19.	$50 \div 5 =$	
20.	$45 \div 5 =$	
21.	$___ \times 5 = 5$	
22.	$___ \times 5 = 25$	

23.	$___ \times 5 = 10$	
24.	$___ \times 5 = 50$	
25.	$___ \times 5 = 15$	
26.	$10 \div 5 =$	
27.	$5 \div 5 =$	
28.	$50 \div 5 =$	
29.	$25 \div 5 =$	
30.	$15 \div 5 =$	
31.	$___ \times 5 = 15$	
32.	$___ \times 5 = 20$	
33.	$___ \times 5 = 45$	
34.	$___ \times 5 = 35$	
35.	$40 \div 5 =$	
36.	$45 \div 5 =$	
37.	$30 \div 5 =$	
38.	$35 \div 5 =$	
39.	$11 \times 5 =$	
40.	$55 \div 5 =$	
41.	$12 \times 5 =$	
42.	$60 \div 5 =$	
43.	$13 \times 5 =$	
44.	$65 \div 5 =$	

Name _____ Date _____

1. Gale makes a miniature stop sign, a regular octagon, with a perimeter of 48 centimeters for the town he built with blocks. What is the length of each side of the stop sign?
2. Travis bends wire to make rectangles. Each rectangle measures 34 inches by 12 inches. What is the total length of the wire needed for two rectangles?
3. The perimeter of a rectangular bathroom is 32 feet. The width of the room is 8 feet. What is the length of the room?

4. Raj uses 6-inch square tiles to make a rectangle, as shown below. What is the perimeter of the rectangle in inches?

5. Mischa makes a 4-foot by 6-foot rectangular banner. She puts ribbon around the outside edges. The ribbon costs \$2 per foot. What is the total cost of the ribbon?
6. Colton buys a roll of wire fencing that is 120 yards long. He uses it to fence in his 18-yard by 24-yard rectangular garden. Will Colton have enough wire fencing left over to fence in a 6-yard by 8-yard rectangular play space for his pet rabbit?

Name _____

Date _____

Adriana traces a regular triangle to create the shape below. The perimeter of her shape is 72 centimeters. What are the side lengths of the triangle?

Name _____ Date _____

1. Rosie draws a square with a perimeter of 36 inches. What are the side lengths of the square?
2. Judith uses craft sticks to make two 24-inch by 12-inch rectangles. What is the total perimeter of the 2 rectangles?
3. An architect draws a square and a rectangle, as shown below, to represent a house that has a garage. What is the total perimeter of the house with its attached garage?

4. Manny draws 3 regular pentagons to create the shape shown below. The perimeter of 1 of the pentagons is 45 inches. What is the perimeter of Manny's new shape?

5. Johnny uses 2-inch square tiles to make a square, as shown below. What is the perimeter of Johnny's square?

6. Lisa tapes three 7-inch by 9-inch pieces of construction paper together to make a happy birthday sign for her mom. She uses a piece of ribbon that is 144 inches long to make a border around the outside edges of the sign. How much ribbon is leftover?

VITA

Amy Dwiggins grew up in Shelbyville, Missouri, the youngest daughter of Jim and Glenna Wilson. She completed her elementary and secondary education in Shelbyville, graduating from North Shelby High School in 1998. She continued her education at Truman State University, graduating in 2002 with a Bachelor of Arts degree in Mathematics and in 2004 with a Master of Arts in Education – Secondary Mathematics. She taught various courses at Macon High School, Macon, Missouri, for 12 years prior to entering the Ph.D. program at the University of Missouri. Amy earned her Ph.D. in Learning, Teaching, and Curriculum with a focus on Mathematics Education in May 2020. During her Ph.D. program, she worked on the development and delivery of online Master's level classes on equity and technology in mathematics education and taught both mathematics methods courses for preservice elementary teachers in the College of Education at the University of Missouri. In addition, she worked on research projects examining online mathematics modules for Professional Learning Communities and examining ELL resources in algebra curricula as well as worked with Dr. Sam Otten and Dr. Zandra de Araujo in the beginning stages of the Two-Minute Teacher's Guide project. Amy is married to Caleb Dwiggins, and they have two sons, Wilson and Ian Dwiggins.