Generalization of Kerr spacetime

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A class of exact solutions of the Einstein-Maxwell equations is presented. It represents the exterior gravitational field of a charged rotating mass. The class contains a generalization of the charged Kerr metric involving an infinite set of parameters that pertain to an axisymmetric deformation of the source.

The Newtonian theory of gravitation provides an adequate description of the gravitational field of conventional astrophysical objects. However, the discovery of exotic systems such as quasars and pulsars together with the theoretical possibility of continued gravitational collapse to a black hole points to the importance of relativistic gravitation in astrophysics. Moreover, advances in space exploration and the development of modern measuring techniques have made it necessary to take relativistic effects into account even in the Solar System. It is therefore important to describe the relativistic gravitational fields of astrophysical bodies in terms of their multipole moments in close analogy with the Newtonian theory. In this connection, we presented a class of stationary axisymmetric vacuum solutions of the gravitational field equations which could be used to describe the exterior field of a rotating deformed mass.[1,2] The Kerr solution is included in this class for a particular choice of parameters. However, it appears from the observational data that strong electromagnetic fields are associated with gravitationally collapsed systems. This is consistent with the fact that the general final state of gravitational collapse of matter is described by the *charged* Kerr spacetime. The electrodynamics of a gravitationally collapsed configuration is rather complex;[3] however, any significant charge separation that may develop is expected to be short lived since the charging mechanism would be self-limiting. Therefore, on the average, the net charge on an astrophysical object is expected to be small.[4] A general description of the exterior electromagnetic state of a collapsed system interacting with its surrounding matter is not available; hence, it seems useful as a preliminary step to find a generalization of the charged Kerr spacetime containing all mass multipole moments. In this paper previous work[5] is generalized in order to incorporate the Kerr-Newman black hole in the general solution. Therefore, we present a class of exterior solutions of the Einstein-Maxwell equations containing an infinite set of gravitational multipole moments as well as an additional parameter which represents the electric charge per unit mass of the source.

The formation of a black hole is a complex process: The catastrophic collapse of matter is generally accompanied by the emission of gravitational radiation. In the last stages of collapse, the system might undergo oscillations as it divests itself of the vestiges of its original state and settles down to a black-hole state.[6] To study the transition from the original system to a black hole as reflected in the multipole structure of the exterior field, it is advantageous to have exact solutions corresponding to both configurations within a given class. In a collapsing configuration, the effective-mass moments would depend upon time; thus the stationary axisymmetric solutions presented in this paper may only suggest certain approximate forms for the description of the near field of a collapsing system. The main physical result of this paper is a nonlinear superposition of the stationary Kerr-Newman metric with the static generalized Erez-Rosen metric, which represents the exterior gravitational field of a static deformed mass with arbitrary multipole moments. It should be emphasized that this extension of the charged Kerr black hole is simply a special case of the general class of solutions presented in this paper.

The general stationary axisymmetric line element in prolate spheroidal coordinates (t,x,y,ϕ) is given by $(x\geq 1,\ -1\leq y\leq 1)$

$$ds^2=f(dt-\omega\, d\phi)^2 -\sigma^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right] +(x^2-1)(1-y^2)d\phi^2\right], \qquad (1)$$

where σ is a constant length scale and the metric functions f, ω, and γ depend only on x and y. The charged solution is obtained through the application of a Harrison transformation to a neutral seed solution. In fact, the Harrison transformation generates an Ernst potential that is proportional to the Ernst potential of the neutral solution with a proportionality factor of $\eta^{-1}=(1-e^2)^{1/2}$, where e is a constant that turns out to be the charge per unit mass of the source.[7] The requirement that $e^2<1$ is satisfied by all realistic astrophysical configurations. Once the Ernst potential of the new solution is known, the corresponding metric coefficients can be calculated *algebraically* from those of the neutral solution.[5,8] The methods developed by Kinnersley, Hoenselaers, and Xanthopoulos, usually referred to as HKX transformations,[9] have already been used to obtain the neutral solution. It is a stationary axisymmetric vacuum solution possessing an infinite set of parameters which may be described as follows: (i) q_n $(n=0,1,2,\ldots)$ are proportional to the Newtonian multipole moments of an axisymmetric mass distribution; (ii) α_1 and α_2 are two independent parameters which can be fixed in order to obtain the Kerr-Taub-NUT (Newman-Unti-Tamburino) metric as a special case, (iii) δ is the Zipoy-Voorhees parameter and generalizes each specific solution to a one-parameter class of solutions by taking different real values, and (iv) τ corresponds to an Ehlers transformation. This general solution contains, as a special case, a nonlinear superposition of the Kerr spacetime with the generalized Erez-Rosen static spacetime,[10,11] which involves an infinite set of parameters $\{q_n\}$ that takes the static deformation of the source into account. Thus q_n determine in part the gravitoelectric multipole moments of the source. The deformation of the source due to rotation is also reflected in the infinite set of gravitomagnetic multipole moments which involve $\{q_n\}$ and the Kerr parameter a. In the general case, the parameters q_n, α_1, α_2, δ, and τ may be used to obtain different sets of gravitoelectric and gravitomagnetic multipole moments corresponding to different configurations of the source. For instance, a source distribution with equatorial symmetry would have only even gravitoelectric and odd gravitomagnetic moments, etc.

The metric functions of the general charged solution can be expressed as

$$f=R\,(c_+^2L_+e^{2\delta\hat{\psi}}+c_-^2L_-e^{-2\delta\hat{\psi}}+2c_+c_-R)^{-1}\ , \quad (2)$$

$$\omega=\eta K_1\left[1+\frac{\kappa}{2}\right]-\eta\frac{\sigma}{R}\left[\left[2+\frac{\kappa}{2}\right]M_+e^{2\delta\hat{\psi}}+\frac{\kappa}{2}M_-e^{-2\delta\hat{\psi}}\right]\ , \quad (3)$$

and

$$e^{2\gamma}=K_2(x^2-1)^{-\delta}R\times\exp\left[2\delta^2\sum_{m,n=0}^{\infty}(-1)^{m+n}q_mq_n\Gamma^{(mn)}\right]\ , \quad (4)$$

where K_1 and K_2 are constants which can be fixed by demanding asymptotic flatness of the metric, the constants $c_\pm$ and κ are defined as

$$2c_\pm=1\pm\eta,\quad \kappa=(2-e^2)\eta-2\ , \quad (5)$$

and the functions $\hat{\psi}$, R, $L_\pm$, and $M_\pm$ are given by

$$\hat{\psi}=\sum_{n=1}^{\infty}(-1)^nq_nQ_n(x)P_n(y)\ , \quad (6)$$

$$R=a_+a_-+b_+b_-,\quad L_\pm=a_\pm^2+b_\pm^2\ , \quad (7)$$

and

$$M_\pm=(x\pm1)^{\delta-1}[x(1-y^2)(\lambda+\mu)a_\pm+y(x^2-1)(1-\lambda\mu)b_\pm]\ . \quad (8)$$

Here q_n, $n=0,1,2,\ldots$, are constant parameters, $q_0=1$, and $P_n(y)$ and $Q_n(x)$ are, respectively, the Legendre polynomials and the associated Legendre functions of the second kind. Furthermore,

$$a_\pm=(x\pm1)^{\delta-1}[x(1-\lambda\mu)\pm(1+\lambda\mu)]\ ,$$
$$b_\pm=(x\pm1)^{\delta-1}[y(\lambda+\mu)\mp(\lambda-\mu)]\ , \quad (9)$$

with

$$\lambda=\alpha_1(x^2-1)^{1-\delta}(x+y)^{2\delta-2}\times\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^nq_n\beta_{n-}\right]\ , \quad (10)$$

$$\mu=\alpha_2(x^2-1)^{1-\delta}(x-y)^{2\delta-2}\times\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^nq_n\beta_{n+}\right]\ , \quad (11)$$

where α_1 and α_2 are constant parameters associated with two rank-zero HKX transformations and, for $n\geq1$,

$$\beta_{n\pm}=(\pm1)^n\left[\frac{1}{2}\ln\frac{(x\mp y)^2}{x^2-1}-Q_1\right]+P_nQ_{n-1}-\sum_{k=1}^{n-1}(\pm1)^kP_{n-k}(Q_{n-k+1}-Q_{n-k-1})\ . \quad (12)$$

Finally, the general expression for the function $\Gamma^{(mn)}(x,y)$ is cumbersome and will not be presented; however, this function has been given explicitly in Eq. (44) of Ref. 11. We have set $q_0=1$ for the sake of simplicity; this choice guarantees that the Schwarzschild solution is the simplest nontrivial static solution contained in Eqs. (1)–(12) as a special case. For the general charged solution, the exterior electromagnetic potential Φ, as defined by Ernst (cf. Ref. 8), is given in terms of the Ernst potential ξ by

$$\Phi=e/(1+\xi)\ , \quad (13)$$

where $\xi=\xi_0/\eta$ and ξ_0 is the Ernst potential of the seed vacuum solution:

$$\frac{\xi_0-1}{\xi_0+1}=\frac{a_-+ib_-}{a_++ib_+}\exp(-2\delta\hat{\psi})\ . \quad (14)$$

The electromagnetic field can be determined, in principle, from Eqs. (1)–(14); however, the explicit construction of the electromagnetic-field tensor is rather complicated.

In the general solution presented above, the Ehlers parameter τ has been set equal to zero for the sake of simplicity. In fact, the explicit form of the metric function ω would have become very complicated if τ had been included in the general solution. However, the possibility of an Ehlers transformation should be left open; this is especially useful for the discussion of the gravitational multipole moments of the general solution. The remaining parameters $\{\sigma,q_n,e,\alpha_1,\alpha_2,K_1,K_2\}$ should be chosen in such a way that any particular solution could describe the exterior field of a realistic axisymmetric body. It is therefore necessary to impose the conditions of asymptotic flatness and regularity of the axis of rotational symmetry. This can be achieved by restricting the values of the parameters α_1, α_2, K_1, and K_2. The set of parameters q_n remains totally arbitrary. To see an example of this in a general setting, let us calculate the multipole moments using the invariant definitions proposed by Geroch[12] and Hansen[13] for vacuum fields and generalized by Hoenselaers and Perjés[14] to include stationary Einstein-Maxwell fields. Using the procedure outlined in Ref. 11, we present here only the first two gravitoelectric (M_n) and gravitomagnetic (J_n) moments as well as the general electric (E_n) and magnetic (H_n) multipole moments:

$$M_0=\eta\sigma\left[\delta+2\frac{\alpha_1\alpha_2}{1-\alpha_1\alpha_2}\right] ,\quad M_1=\eta\sigma^2\left[-\frac{1}{3}\delta q_1+\frac{\alpha_2^2-\alpha_1^2}{(1-\alpha_1\alpha_2)^2}\right] , \tag{15}$$

etc.,

$$J_0=\eta\sigma\frac{\alpha_2-\alpha_1}{1-\alpha_1\alpha_2} ,\quad J_1=\eta\sigma^2\frac{\alpha_1+\alpha_2}{(1-\alpha_1\alpha_2)^2}[1-3\alpha_1\alpha_2-2\delta(1-\alpha_1\alpha_2)] , \tag{16}$$

etc.,

$$E_n=eM_n,\quad H_n=eJ_n . \tag{17}$$

It is interesting to note that the parameters K_1 and K_2, which appear in the metric functions as constants of integration for the Ernst potential, do not contribute to the gravitational moments. The proportionality between the electromagnetic and gravitational moments in Eq. (17) is a consequence of the Harrison transformation that results in the simple relationship between the electromagnetic and Ernst potentials given in Eq. (13). To ensure asymptotic flatness, it is *necessary* that the gravitomagnetic monopole moment should vanish. This can be achieved by setting $\alpha_1=\alpha_2$ in Eq. (16). It should be emphasized that it is also possible to have an asymptotically flat solution without the requirement that $\alpha_1=\alpha_2$. That is, one can have a vanishing gravitomagnetic monopole moment by introducing a parameter τ: An Ehlers transformation generates new moments M_n' and J_n', $n=0,1,2,\ldots$, via a rotation

$$M_n'=M_n\cos\tau-J_n\sin\tau ,\quad J_n'=M_n\sin\tau+J_n\cos\tau \text{ for } n=0,1,2,3 , \tag{18}$$

and by similar but more complicated transformations for $n>3$; the Ehlers parameter τ could then be so chosen as to ensure $J_0'=0$. This condition determines τ in terms of α_1 and α_2. Thus, for $\alpha_1\neq\alpha_2$, the Ehlers transformation will result in an asymptotically flat spacetime with new multipole moments M_n' and J_n' that depend upon $\tau=\tau(\alpha_1,\alpha_2)$. However, we choose $\alpha_1=\alpha_2$ for the sake of simplicity. For the special case $\alpha_1=\alpha_2=e=0$, all the gravitomagnetic and electromagnetic moments vanish and the arbitrary parameters q_n, $n=1,2,\ldots$, have a simple interpretation in terms of the Newtonian moments of the matter distribution.[11] Therefore, the parameters α_1 and α_2 characterize the rotation of the source. The arbitrariness of the set of parameters q_n implies that under certain conditions one may regard these constants as functions of other constant parameters in the solution. This possibility can be advantageous in some circumstances;[15] however, it can also lead to misconceptions.[16] For instance, q_n may be replaced by $p_n+\alpha_1u_n+\alpha_2v_n$, where p_n, u_n, and v_n are constants. One is then tempted to interpret p_n as representing the static gravitoelectric part of the stationary field, and u_n and v_n as determining the gravitomagnetic part of the field. This redefinition of q_n provides no advantage from a physical point of view, however, since the invariant moments depend upon $q_n=p_n+\alpha_1u_n+\alpha_2v_n$ and the actual determination of the constant parameters in the solution must come about as a result of a comparison of observations (e.g., investigation of the motion of test particles in the exterior field) with the theoretical predictions, which necessarily involve q_n.

In the general case ($e\neq0$), the physical significance of the constant e follows from the expressions for the electromagnetic moments given in Eq. (17). The constant e corresponds to the charge per unit mass of the source. Higher electric and magnetic moments are nonzero, but are determined uniquely by the constant e and the gravitational moments. This specific relationship between the multipole moments leads to the interesting property that the *magnetic-monopole moment vanishes identically for asymptotically flat spacetimes.* It should be noted that the gyromagnetic ratio of the source is equal to its specific charge. The same is true for an electron; however, the magnitude of the specific charge of an electron is $|e|\simeq2\times10^{21}$, while for the source under consideration in this paper $|e|<1$.

We will now investigate an important special case of the general solution (1)–(14), namely, a nonlinear superposition of the Kerr-Newman metric with the *generalized* Erez-Rosen metric. The choice of parameters $\delta=1$, $\sigma^2=(m/\eta)^2-a^2$,

$$\alpha=\alpha_1=\alpha_2=(\sigma-m/\eta)/a ,\quad K_1=\frac{4\sigma\alpha}{1-\alpha^2},\ \text{and}\ K_2=\frac{1}{(1-\alpha^2)^2} \tag{19}$$

leads to an asymptotically flat solution with a regular symmetry axis. This solution can be expressed in terms of the well-known Boyer-Lindquist coordinates by using the coordinate transformation

$$x=(r-m)/\sigma,\quad y=\cos\theta . \tag{20}$$

The corresponding multipole moments are $E_n=eM_n$ and $H_n=eJ_n$ with $M_0=m$, $J_0=0$,

$$M_1=-\tfrac{1}{3}\eta^{-1}q_1(m^2-a^2\eta^2) ,\quad M_2=-ma^2+\tfrac{2}{15}\eta^{-2}q_2(m^2-a^2\eta^2)^{3/2} , \tag{21}$$

etc.,

$$J_1 = ma, \quad J_2 = -\tfrac{2}{3}\eta^{-1} q_1 a (m^2 - a^2\eta^2) , \qquad (22)$$

etc., where $m \geq |a\eta|$. The Kerr parameter a is the specific angular momentum, i.e., the proper angular momentum per unit mass. Even in the presence of charge, the mass and angular momentum remain m and ma, respectively, and the electric monopole moment is $E_0 = me$. Thus all the solutions in this class must have $|a|/m \leq 1$; this ratio is $\simeq 0.2$ for the Sun[17] and $\simeq 10^3$ for the Earth. In the limiting case $q_n = 0$, $n = 1,2,3,\ldots$, this general solution reduces to the charged Kerr solution. The length scale $\sigma \geq 0$ is related to the total mass of the system, which is equal to m since we have set $q_0 = 1$. It is important to point out that the limitation $|a\eta| \leq m$ is a characteristic feature of the generalized charged Kerr solution; this property is not shared in general by other members of the general class of solutions under consideration in this paper. That is, the exterior field of rapidly rotating configurations can also be described by the solutions discussed in this work.

Some important properties of this *generalized Kerr-Newman spacetime* should be mentioned. In the limiting case $m^2 = a^2 + e^2 m^2$, the solution reduces to the extreme Kerr-Newman solution regardless of the values of q_n, $n = 1,2,3,\ldots$. In the general case, the Kerr-Newman horizon $(x = 1)$ becomes singular in conformity with the black-hole uniqueness theorem.[18] Outside this singular horizon, the metric is free of singularities. These particular properties may be of special importance in the study of the evolution of a real star into a black hole since the generalized solution contains the Kerr-Newman metric as a special case. The final state of gravitational collapse is described by the Kerr-Newman spacetime; therefore, the decay of the multipole moments of the initial configuration with time and the subsequent emission of gravitational and electromagnetic radiation must be taken into consideration. Hence the generalized Kerr-Newman spacetime can be considered as the first preliminary step in the investigation of the collapse of a realistic star. The next step would involve the development of an appropriate interior solution which should be smoothly matched to the general exterior solution. The continuity of the first and second fundamental forms at the (non-null) boundary of the star is expected to lead to relations connecting the multipole moments of the exterior spacetime with the distribution of matter.

Finally, it should be pointed out that the charged Kerr-Taub-NUT metric is also contained in our general solution as a special case. Let $\delta = 1$, $\sigma^2 = (m^2 + l^2)/\eta^2 - a^2$,

$$\alpha_1 = \frac{\sigma - m/\eta}{a + l/\eta}, \quad \alpha_2 = \frac{\sigma - m/\eta}{a - l/\eta} , \qquad (23)$$

and

$$K_1 = 2\sigma \frac{\alpha_1 + \alpha_2}{1 - \alpha_1\alpha_2}, \quad K_2 = (1 - \alpha_1\alpha_2)^{-2} , \qquad (24)$$

where l is the Taub-NUT parameter. Then the coordinate transformation (20) leads to the charged Kerr-Taub-NUT metric for $q_n = 0$, $n = 1,2,\ldots$. The total charge of the system is $E_0 = e(m^2 + l^2)^{1/2}$. The spacetime is not asymptotically flat in this case. The generalized Kerr-Newman spacetime is recovered as the Taub-NUT parameter l approaches zero. In the limit of vanishing σ, we recover the extreme Kerr-Taub-NUT metric regardless of the values of q_n, $n = 1,2,3,\ldots$.

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