ADAMS INEQUALITIES WITH EXACT GROWTH
CONDITION

ON $\mathbb{R}^n$ AND THE HEISENBERG GROUP

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ABSTRACT

In this thesis we prove sharp Adams inequality with exact growth condition for the Riesz potential as well as the more general strictly Riesz-like potentials on $\mathbb{R}^n$. Then we derive the Moser-Trudinger type inequality with exact growth condition for fractional Laplacians with arbitrary $0 < \alpha < n$, higher order gradients and homogeneous elliptic differential operators. Next we give an application to a quasilinear elliptic equation, and prove the existence of ground state solution of this equation. Lastly, we extend our result to the Heisenberg group. By applying the same technique used in $\mathbb{R}^n$, we derive a sharp Adams inequality with critical growth condition on $\mathbb{H}^n$ for integral operators whose kernels are strictly Riesz-like on $\mathbb{H}^n$. As a consequence we then derive the corresponding sharp Moser-Trudinger inequalities with exact growth condition for the powers of sublaplacian $(-\mathcal{L}_0)^{\alpha/2}$ when α is an even integer, and for the subgradient $\nabla_{\mathbb{H}^n}$.

Chapter 1

Introduction

1.1 Introduction

The Sobolev imbedding theorem tells us how we can estimate the size of a function by the size of its partial derivatives. However at the borderline case it is not true that in general $f \in L^\infty$. A natural condition that substitutes for the lack of boundedness is exponential integrability. Trudinger's paper in 1967 showed that it actually fills the gap in the Sobolev's theorem:

Theorem A ([Tr][Po][Yu]). *Let Ω be an open and bounded set in $\mathbb{R}^n$, $n > 1$. There exist constants β and C such that, if u belongs to the Sobolev space $W_0^{1,n}(\Omega)$ and*

$$\left(\int_{\Omega} |\nabla u|^n \right)^{\frac{1}{n}} \leq 1$$

then

$$\int_{\Omega} \exp(\beta |u|^{\frac{n}{n-1}}) dx \leq C. \tag{1.1}$$

Later in 1971 Moser ([Mo]) found the best possible value for β , that is, the inequality (1.1) holds at the critical value $\beta = n\omega_{n-1}^{1/(n-1)}$, but not for any larger values, where ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$. Two decades later, in

1988, Adams ([A1]) extended Moser's result to higher order derivatives:

Theorem B([A1]). *Let Ω be an open and bounded set in $\mathbb{R}^n$, $n > 1$ and let $m \in \mathbb{N}$ with $m < n$. There are constants $\beta(m, n)$ and C such that for all u in the Sobolev space $W_0^{m, \frac{n}{m}}(\Omega)$ and $\|\nabla^m u\|_{\frac{n}{m}} \leq 1$ we have*

$$\int_{\Omega} \exp \left[\beta(m, n) |u|^{\frac{n}{n-m}} \right] dx \leq C. \quad (1.2)$$

Here the constant

$$\beta(m, n) = \begin{cases} \frac{n}{\omega_{n-1}} c_m^{-\frac{n}{n-m}} & \text{if } m \text{ is even,} \\ \frac{n}{\omega_{n-1}} [(n-m-1)c_{m+1}]^{-\frac{n}{n-m}} & \text{if } m \text{ is odd} \end{cases}$$

appearing in (1.2) is sharp, that is, it cannot be replaced by a larger number. The constant c_m is the same as in (1.41). Also $\nabla^m u$ denotes the m -th order gradient, that is,

$$\nabla^m u = \begin{cases} (-\Delta)^{\frac{m}{2}} u & \text{if } m \text{ is even,} \\ \nabla (-\Delta)^{\frac{m-1}{2}} u & \text{if } m \text{ is odd} \end{cases}$$

where ∇ is the gradient operator and Δ is the Laplacian. The key to Adams' approach is to estimate u in terms of the Riesz potential of $\nabla^m u$ using the representation formulas

$$u = \begin{cases} c_m I_m \nabla^m u & \text{if } m \text{ is even,} \\ c_{m+1} \nabla I_{m+1} \nabla^m u & \text{if } m \text{ is odd,} \end{cases} \quad (1.3)$$

where the Riesz potential is defined by

$$I_{\alpha} f(x) = \int |x - y|^{\alpha-n} f(y) dy$$

for $\alpha \in (0, n)$. Then Adams established the following sharp inequality, from which

the above Theorem B follows immediately:

Theorem C ([A1]). *For $1 < p < \infty$, there is a constant $c_0 = c_0(p)$ such that for all $f \in L^p(\mathbb{R}^n)$ with support contained in Ω , $|\Omega| < \infty$,*

$$\int_{\Omega} \exp \left[\frac{n}{\omega_{n-1}} \left| \frac{I_{\alpha} f(x)}{\|f\|_p} \right|^{p'} \right] dx \leq c_0 \quad (1.4)$$

where $\alpha = n/p$ and $I_{\alpha} f(x) = \int |x - y|^{\alpha-n} f(y) dy$ is the Riesz potential of order α . Furthermore, no number greater than n/ω_{n-1} can replace this coefficient in (1.4) without forcing c_0 to depend on f as well as p .

Since Adams' paper, Moser-Trudinger and Adams inequalities has been widely applied in many fields including nonlinear PDEs, conformal Geometry, etc. Some of the works but not all are included in the list of references. They have also been studied in various settings. In [Fon] Fontana extended Adams' result on compact Riemannian manifolds. Cohn and Lu [CL 1,2] derived sharp Moser-Trudinger inequalities in the setting of spheres in complex space $\mathbb{C}^n$ as well as on the Heisenberg group $W_0^{1,Q}(\Omega)$. Branson, Fontana and Morpurgo [BFM] derived sharp Adams and Moser-Trudinger inequalities on the CR sphere, in order to obtain the sharp Beckner-Onofri's inequality. Later, the Adams inequality was completely worked out by Fontana and Morpurgo [FM1] on arbitrary measure spaces with finite measure and for general integral operators.

On the other hand, if Ω has infinite measure it is clear that the inequality (1.4) cannot hold. Therefore the exponential integral needs to be regularized, in other words, it needs to be modified in the following form:

$$\exp_N(t) = e^t - \sum_{k=0}^N \frac{t^k}{k!}$$

for some suitable integer N . So for $|\Omega| = \infty$ inequality (1.1) will become

$$\int_{\Omega} \exp_{[n-2]}(\beta|u|^{\frac{n}{n-1}})dx \leq C \quad (1.5)$$

and for general $\alpha \in (0, n)$,

$$\int_{\Omega} \exp_{[\frac{n}{\alpha}-2]}(\beta|u|^{\frac{n}{n-\alpha}})dx \leq C. \quad (1.6)$$

Adachi and Tanaka [AT] proved (1.5) in the following form

$$\int_{\mathbb{R}^n} \exp_{[n-2]} \left[\beta \left(\frac{|u(x)|}{\|\nabla u\|_n} \right)^{\frac{n}{n-1}} \right] dx \leq C \left(\frac{\|u\|_n}{\|\nabla u\|_n} \right)^n, \quad u \in W^{1,n}(\mathbb{R}^n) \setminus \{0\} \quad (1.7)$$

for $\beta < \beta(1, n) = n\omega_{n-1}^{\frac{1}{n-1}}$. Adachi and Tanaka also showed that (1.7) is false if $\beta \geq \beta_{1,n}$ by using the Moser sequence. The more general version of the Adachi-Tanaka type result was derived by Fontana and Morpurgo, as an immediate consequence of the main result in [FM2]. Fontana and Morpurgo [FM2] considered the family of Sobolev norms for $u \in W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$, $0 < \alpha < n$:

$$\begin{cases} \|u\|_{n/\alpha}^{pn/\alpha} + \|\nabla^\alpha u\|_{n/\alpha}^{pn/\alpha} \leq 1, & \text{if } p < \infty \\ \max\{\|u\|_{n/\alpha}, \|\nabla^\alpha u\|_{n/\alpha}\} \leq 1, & \text{if } p = +\infty \end{cases} \quad (1.8)$$

and proved, under the assumption (1.8), the sharp inequality

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{\alpha}-2]} \left[\theta \beta(\alpha, n) |u(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C(1 - \theta)^{-\frac{1}{p'}} \quad (1.9)$$

in the sense that the exponential integral cannot be uniformly bounded if $\theta = 1$.

If we consider the norm constraint

$$\|(I - \Delta)^{\frac{\alpha}{2}} u\|_{\frac{n}{\alpha}} \leq 1, \quad (1.10)$$

which is stronger than (1.8), then the sharp inequality in (1.6) with $\beta = \beta(\alpha, n)$ holds for any $\alpha \in (0, n)$. This result is in the paper by Adams [A1] for the case $n = 2\alpha$, who proved this by adopting the techniques from the proof of Theorem B ([A1]). The main idea is to use the Bessel potential representation for the function u . We also refer reader to Ruf-Sani's paper [RS] for the case α is an even integer, using comparison theorems for suitable Navier boundary problems, and Lam-Lu [LL2] for the case of any $\alpha \in (0, n)$ based on the Bessel potential approach.

Considering weaker norm conditions than (1.10), in [Ruf] Ruf showed that, in the context of $\mathbb{R}^2$, if the assumption on the Dirichlet norm $\|\nabla u\|_2 \leq 1$ is replaced by one using the standard Sobolev norm, i.e.

$$\|u\|_2^2 + \|\nabla u\|_2^2 \leq 1$$

then (1.5) holds with $|\Omega| = \mathbb{R}^2$, $\beta = \beta(1, 2)$. Later Li and Ruf [LR] extended this result to higher dimensions by proving that (1.5) holds at the critical constant $\beta = \beta(1, n)$ under the *Ruf condition*

$$\|u\|_n^n + \|\nabla u\|_n^n \leq 1.$$

For the second order derivatives, Lam and Lu [LL2] derived the inequality

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{2}-2]} \left[\beta(2, n) |u(x)|^{\frac{n}{n-2}} \right] dx \leq C$$

under the condition

$$\|u\|_{n/2}^{n/2} + \|\Delta u\|_{n/2}^{n/2} \leq 1.$$

Recently, Fontana and Morpurgo [FM2] proved the sharp Adams and Moser-Trudinger inequalities on $\mathbb{R}^n$ for the full range of $\alpha \in (0, n)$ under the Ruf conditions

$$\|f\|_{n/\alpha}^{n/\alpha} + \|I_\alpha f\|_{n/\alpha}^{n/\alpha} \leq 1 \tag{1.11}$$

$$||u||_{n/\alpha}^{n/\alpha} + ||\nabla^\alpha u||_{n/\alpha}^{n/\alpha} \leq 1. \quad (1.12)$$

In particular, in [FM2] Fontana and Morpurgo derived the Adams inequality

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{\alpha}-2]} \left[\frac{1}{|B_1|} |I_\alpha f|^{\frac{n}{n-\alpha}} \right] dx \leq C \quad (1.13)$$

for compactly supported f under the condition (1.11), and the Moser-Trudinger inequality

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{\alpha}-2]} \left[\beta(\alpha, n) |u(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C \quad (1.14)$$

for every $u \in W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$ under (1.12). Also, the exponential constants in (1.13) and (1.14) are sharp, that is, they cannot be replaced by any larger constants. In fact, in [FM2] the inequality (1.13) is proved for a class of convolutions kernels that are more general than the Riesz kernel, which they name *Riesz-like kernels* (see Definition 1 below).

In the setting of Heisenberg group, Lam and Lu [LL1] worked out the Li-Ruf version of the sharp Moser-Trudinger inequality, with an application to some subelliptic equations.

The literature dealing with Adams and Moser-Trudinger inequalities on the whole $\mathbb{R}^n$ mainly focuses on two approaches. One is to use the norm condition $||\nabla^\alpha u||_{\frac{n}{\alpha}} \leq 1$ and weaken the critical exponential constant $\beta(\alpha, n)$ as in the Adachi-Tanaka type result. Another approach is to put stronger norm conditions such as the Ruf condition or (1.10), and keep the critical constant. All of those results are all sharp in their formulations. In [IMN] Ibrahim, Masmoudi and Nakanishi proposed to keep both the constraint $||\nabla^\alpha u||_{\frac{n}{\alpha}} \leq 1$ and the critical exponential constant $\beta(\alpha, n)$ by weakening slightly the exponential inside (1.14). Specifically, they proved that when $\alpha = 1$ and $n = 2$ (in which case $\beta(1, 2) = 4\pi$) there is C such that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi|u|^2} - 1}{1 + |u|^2} dx \leq C ||u||_2^2 \quad \text{for all } u \in W^{1,2}(\mathbb{R}^2), \quad ||\nabla u||_2 \leq 1, \quad (1.15)$$

and that the inequality is sharp in the sense that the constant 4π cannot be replaced by a higher one, and the power 2 in the denominator cannot be replaced by a smaller one. Such an inequality has been named *Adams inequality with exact growth condition*. Masmound and Sani later extended this result to $\alpha = 2$ in dimension 4 [MS1], and to $\alpha = 1$ in any dimension [MS2], while Lu-Tang derived the case $\alpha = 2$ in any dimension in [LT2]. Finally, Masmoudi-Sani proved the general case in [MS3]:

$$\int_{\mathbb{R}^n} \frac{\exp[\frac{n}{\alpha}-2] \left[\beta(\alpha, n) |u|^{\frac{n}{n-\alpha}} \right]}{1 + |u|^{\frac{n}{n-\alpha}}} dx \leq C_n \|u\|_{n/\alpha}^{n/\alpha} \quad \text{for all } u \in W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n), \quad \|\nabla^\alpha u\|_{\frac{n}{\alpha}} \leq 1 \quad (1.16)$$

where α is an integer with $2 < \alpha < n$. It is important to point out (as shown in in [IMN], [MS1], [MS2]) that the validity of such inequality with exact growth implies the validity of the corresponding Moser-Trudinger inequality under the Ruf condition (1.12), so in this sense it is a stronger inequality.

We mention that the known results on sharp Adams and Moser-Trudinger inequalities with exact growth condition in the literature [IMN],[MS1,2,3] [LT] in the end rely on the symmetrization results such as the Pólya Szegő inequality and and Talenti's comparison theorem, in order to reduce the problems to the radial case. However, this approach is not suitable for proving Adams inequalities for arbitrary orders of $\alpha \in (0, n)$.

Our initial goal was to derive the sharp Adams inequality with exact growth condition for the Riesz potential, that is

$$\int_{\mathbb{R}^n} \frac{\exp[\frac{n}{\alpha}-2] \left[\frac{1}{|B_1|} |I_\alpha f|^{\frac{n}{n-\alpha}} \right]}{1 + |I_\alpha f|^{\frac{n}{n-\alpha}}} dx \leq C \|I_\alpha f\|_{n/\alpha}^{n/\alpha}, \quad \|f\|_{\frac{n}{\alpha}} \leq 1 \quad (1.17)$$

which includes Fontana-Morpurgo's result (1.13) under the Ruf condition (1.11), as a special case, and which implies (1.16) for arbitrary $\alpha \in (0, n)$.

In this thesis we prove that (1.17) is true, and not only for the Riesz kernel but for a subclass of the Riesz-like kernels introduced by Fontana-Morpurgo, which we call *strictly Riesz-like kernels*.

To better describe our result let us recall the definition given in [FM2]:

Definition 1. *A measurable function $K : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ is a Riesz-like kernel of order $\alpha \in (0, n)$ if it satisfies the following properties:*

$$K(x) = g(x^*)|x|^{\alpha-n} + O(|x|^{\alpha-n+\delta_1}) , \quad x^* = \frac{x}{|x|}, \quad 0 < |x| \leq B \quad (\text{A1})$$

$$|K(x)| \leq H_1|x|^{\alpha-n} \quad (\text{A2})$$

$$|K(z_1) - K(z_2)| \leq H_2|z_1 - z_2| \max\{|z_1|^{\alpha-n-1}, |z_2|^{\alpha-n-1}\}, \quad z_1, z_2 \neq 0 \quad (\text{A3})$$

where $g : S^{n-1} \rightarrow \mathbb{R}$ is a measurable function and δ, H_1, H_2, B are positive constants.

If we add an additional condition (A4) as below, we have more restrictive control of the kernel K when $|x|$ is large:

Definition 2. *A measurable function $K : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ is a strictly Riesz-like kernel of order $\alpha \in (0, n)$ if it is Riesz-like and satisfies the following property:*

$$|K(x)| \leq |g(x^*)||x|^{\alpha-n} + O(|x|^{\alpha-n-\delta_2}) , \quad |x| > B \quad (\text{A4})$$

where $g : S^{n-1} \rightarrow \mathbb{R}$ is a measurable function and $B, \delta_1, \delta_2, H_1, H_2$ are positive constants.

Here the “big O” notation in (A1) means that $|O(|x|^{\alpha-n+\delta_1})| \leq C|x|^{\alpha-n+\delta_1}$ for all x such that $0 < |x| \leq B$. And the same notation in (A4) means that $|O(|x|^{\alpha-n-\delta_2})| \leq C|x|^{\alpha-n-\delta_2}$. It is clear that (A3) implies that g is Lipschitz. Also, (A1), (A3) and (A4) imply (A2).

Clearly, any kernel of type $g(x^*)|x|^{\alpha-n}$ with g Lipschitz on the sphere, provides an example of strictly Riesz-like kernel.

To state our main result in terms of strictly Riesz-like kernel, we introduce some notations. For $m \in \mathbb{N}$, a kernel K is called *m-regular* if $K \in C^m(\mathbb{R}^n \setminus 0)$ and

$$|D_h^j K(x)| \leq C|x|^{\alpha-n-j}, \quad x \neq 0, \quad j = 1, \dots, m$$

with h is a multi-index with $|h| = j$ and $D_h^j K$ denotes the j -th derivatives of K of the multi-index h . Clearly the Riesz kernel is m -regular for all m , and any 1-regular K satisfies condition (A3).

Let us denote T the convolution operator with kernel K :

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$

For vector valued functions

$$K = (K_1, \dots, K_m), \quad f = (f_1, \dots, f_m)$$

we define Tf in the same way with

$$Kf = K_1 f_1 + \dots + K_m f_m, \quad |f| = (f_1^2 + \dots + f_m^2)^{1/2}.$$

The results and proofs in this paper apply to both scalar and vector cases, so we will not distinguish between these two cases, except in the proof of sharpness.

In [FM2, Theorem 5] the authors proved that if K is a Riesz-like kernel, then under the Ruf condition

$$||f||_{n/\alpha}^{n/\alpha} + ||Tf||_{n/\alpha}^{n/\alpha} \leq 1 \tag{1.18}$$

the following Adams inequality holds

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{\alpha}-2]} \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C \quad (1.19)$$

where

$$A_g = \frac{1}{n} \int_{S^{n-1}} |g(x^*)|^{\frac{n}{n-\alpha}} dx^*,$$

and where dx^* is the surface measure of the unit sphere and A_g^{-1} is a sharp exponential constant if the kernel is n -regular.

As pointed out in [FM2] Adams type estimates involving an integral of the regularized exponential over the whole space, have equivalent “local” formulations in terms of the standard exponential. In particular, estimate (1.19) is equivalent to

$$\int_E \exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C(1 + |E|) \quad (1.20)$$

valid for every f satisfying (1.18) and for every measurable set E with finite measure. This is a simple consequence of the “Exponential Regularization Lemma” (see Lemma B in section 1.2.4).

In this thesis, we prove that for strictly Riesz-like kernels a stronger result holds, namely the following Masmoudi-Sani type inequality:

Theorem 1. *Let $0 < \alpha < n$, and K is strictly Riesz-like. There exists $C = C(n, \alpha, K) > 0$ such that for all compactly supported f with*

$$||f||_{\frac{n}{\alpha}} \leq 1$$

we have

$$\int_{\mathbb{R}^n} \frac{\exp_{[\frac{n}{\alpha}-2]} \left[\frac{1}{A_g} |Tf|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{n-\alpha}}} dx \leq C ||Tf||_{\frac{n}{\alpha}}^{n/\alpha}, \quad (1.21)$$

where

$$A_g = \frac{1}{n} \int_{S^{n-1}} |g(x^*)|^{\frac{n}{n-\alpha}} dx^*. \quad (1.22)$$

If K is n -regular, then the exponential constant A_g^{-1} in (1.21) cannot be replaced by a larger number. Furthermore, if K is n -regular, then (1.21) cannot hold if the power $\frac{n}{n-\alpha}$ in the denominator is replaced by any smaller power.

Via the exponential regularization lemma (Lemma B in section 1.2.4) estimate (1.21) is equivalent to the following local version

$$\int_E \frac{\exp \left[\frac{1}{A_g} |Tf|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{n-\alpha}}} dx \leq C(|E| + \|Tf\|_{n/\alpha}^{n/\alpha}) \quad (1.23)$$

valid for all measurable E with finite measure, and under $\|f\|_{\frac{n}{\alpha}} \leq 1$.

We mention that inequality (1.21) still holds if we have “ $\leq$ ” instead of “ $=$ ” in condition (A1), that is, if (A1) is replaced by

$$|K(x)| \leq |g(x^*)| |x|^{\alpha-n} + |O(|x|^{\alpha-n+\delta_1})|, \quad 0 < |x| \leq B, \quad (\text{A1}')$$

but in order to have sharpness in the exponential constant A_g^{-1} we need to assume the condition (A1).

Next, we point out that for Theorem 1 to hold it is not enough to assume that K be only Riesz-like. It is relatively easy to find an example of a Riesz-like kernel such that the inequality in Theorem 1 cannot hold, but the one in [FM2, Theorem 5] holds. In section 2.5 Remark 1, we will address this example, which indicates that it is necessary for us to strengthen our assumption for large $|x|$, i.e. (A4).

As a consequence of Theorem 1 we derive the following general Adachi-Tanaka type inequality:

Corollary 1. *If K is a strictly Riesz-like kernel, then for any $\theta \in (0, 1)$ there exists*

C independent of θ such that for all compactly supported f with

$$\|f\|_{n/\alpha} \leq 1, \quad (1.24)$$

we have

$$\int_{\mathbb{R}^n} \exp_{[\frac{n}{\alpha}-2]} \left[\frac{\theta}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq \frac{C}{1-\theta} \|Tf\|_{n/\alpha}^{n/\alpha} \quad (1.25)$$

where A_g is the same as in (1.22). If K is n -regular and $K \notin L^{\frac{n}{n-\alpha}}(|x| \geq 1)$ then inequality (1.25) is sharp, in the sense that the exponential integrals cannot be uniformly bounded if $\theta = 1$.

Estimate (1.25) improves the one obtained in [FM2, Theorem 6], which does not have $\|Tf\|_{n/\alpha}$ on the right hand side, and which has $(1-\theta)^{-1}$ only in the case K homogeneous.

In view of Theorem 1, we want to mention that we have two different types of sharpness. In terms of the exponential constant the inequality (1.21) is sharp. And also in terms of the power in the denominator one cannot have the inequality (1.21) if the power of the denominator is smaller than $\frac{n}{n-\alpha}$. In [MS1]-[MS3] the authors addressed this point that the inequality (1.16) is false for any smaller power of the denominators. However, we find that if we lower both the power of denominator and of the norm on the RHS, we have the following more general inequality:

Theorem 2. *Let $0 < \alpha < n$, $\beta > 0$ and K satisfies (A1)-(A4). There exists $C = C(n, \alpha, K) > 0$ such that for all compactly supported f with*

$$\|f\|_{\frac{n}{\alpha}} \leq 1 \quad \text{and} \quad \beta \left(\frac{n}{\alpha} \right) \geq 1 \quad (1.26)$$

we have

$$\int_{\mathbb{R}^n} \frac{\exp_{[\frac{\beta n}{\alpha}-\beta-1]} \left[\frac{1}{A_g} |Tf|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{\beta n}{n-\alpha}}} dx \leq C \|Tf\|_{\beta n/\alpha}^{\beta n/\alpha} \quad (1.27)$$

where A_g is as in (1.22).

If K is n -regular, then the exponential constant A_g^{-1} in (1.27) is sharp, i.e. it cannot be replaced by a larger number. Moreover, (1.27) cannot hold if the power $\frac{\beta n}{n-\alpha}$ in the denominator is replaced by any smaller power.

Our next goal is to derive the inequality (1.5) for arbitrary $\alpha \in (0, n)$. Now recall that a homogeneous elliptic differential operator of even order $\alpha < n$ with real constant coefficients has the form

$$Pu = \sum_{|k|=\alpha} a_k D^k u \quad (1.28)$$

for $u \in C_c^\infty(\mathbb{R}^n)$, with homogeneous principal symbol

$$p_\alpha(\xi) = P(2\pi i\xi) = (2\pi)^\alpha (-1)^{\alpha/2} \sum_{|k|=\alpha} a_k \xi^k, \quad |p_\alpha(\xi)| \geq c_0 |\xi|^\alpha, \quad \xi \in \mathbb{R}^n$$

for some $c_0 > 0$. The fundamental solution of P is given by a convolution operator with kernel g_P given as

$$g_P(x) = \int_{\mathbb{R}^n} \frac{e^{-2\pi i x \cdot \xi}}{p_\alpha(\xi)} d\xi \quad (1.29)$$

in the sense of distributions. Since p_α is homogeneous of order α , the kernel g_P is also homogeneous of order $\alpha - n$.

Theorem 3. For $0 < \alpha < n$, let P be either $(-\Delta)^{\frac{\alpha}{2}}$, $\nabla(-\Delta)^{\frac{\alpha-1}{2}}$ for α odd, or a homogeneous elliptic operator of even order $\alpha < n$ with constant coefficients. Then there exists $C = C(\alpha, n, P) > 0$ such that for every $u \in W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$ with

$$||Pu||_{n/\alpha}^{n/\alpha} \leq 1 \quad (1.30)$$

we have

$$\int_{\mathbb{R}^n} \frac{\exp[\frac{n}{\alpha}-2] [\gamma(P)|u(x)|^{\frac{n}{n-\alpha}}]}{1 + |u(x)|^{\frac{n}{n-\alpha}}} dx \leq C ||u||_{n/\alpha}^{n/\alpha} \quad (1.31)$$

where

$$\gamma(P) = \begin{cases} \frac{c_\alpha^{-\frac{n}{n-\alpha}}}{|B_1|}, & \text{if } P = (-\Delta)^{\frac{\alpha}{2}} \\ \frac{((n-\alpha-1)c_{\alpha+1})^{-\frac{n}{n-\alpha}}}{|B_1|}, & \text{if } P = \nabla(-\Delta)^{\frac{\alpha-1}{2}} \text{ and } \alpha \text{ odd.} \end{cases} \quad (1.32)$$

and where

$$\gamma(P) = \frac{n}{\int_{S^{n-1}} |g_P(x^*)|^{\frac{n}{n-\alpha}} dx^*}$$

if P elliptic and α even. The exponential constant $\gamma(P)$ is sharp. Moreover, the above inequality (1.31) cannot hold if the power $\frac{n}{n-\alpha}$ in the denominator is replaced by any smaller power.

As an immediate consequences of Theorem 3, we have the following Corollary:

Corollary 2. *Let Ω be a bounded and open set in $\mathbb{R}^n$, $0 < \alpha < n$ an integer. There exists $C > 0$ such that for all $u \in W_0^{\alpha, \frac{n}{\alpha}}(\Omega)$ with $\|\nabla^\alpha u\|_{\frac{n}{\alpha}} \leq 1$ we have*

$$\int_{\Omega} \frac{\exp[\frac{n}{\alpha}-2] [\gamma(P)|u(x)|^{\frac{n}{n-\alpha}}]}{1 + |u(x)|^{\frac{n}{n-\alpha}}} dx \leq C \|u\|_{\frac{n}{\alpha}}^{n/\alpha}. \quad (1.33)$$

The exponential constant $\gamma(P)$ is sharp. Furthermore, the above inequality (1.33) cannot hold if the power $\frac{n}{n-\alpha}$ in the denominator is replaced by any smaller power.

Although the proof of (1.33) uses Adams inequality on Ω , it is still not an easy direct consequence from Adams (1.4). We also mention that the above inequality (1.33) is different than (1.2) in [A1] because of the norm of u in the RHS. For example, take $\alpha = 1$, $n = 2$, then by Corollary 2 we have

$$\int_{\Omega} \frac{e^{4\pi u^2} - 1}{1 + u^2} dx \leq C \|u\|_2^2 \quad (1.34)$$

and [A1] showed

$$\int_{\Omega} e^{4\pi u^2} dx \leq C |\Omega|. \quad (1.35)$$

So for fixed Ω , we can see that if $\|u\|_2^2$ becomes very small then so is the LHS of (1.34), but this point may not be reflected by the second inequality (1.35).

Motivated by Theorem 2, we incorporate the parameter β into (1.31) and get the following sharp inequality as a Corollary of Theorem 3 for integer α :

Corollary 3. *Let $0 < \alpha < n$. If α is an integer and $\beta\left(\frac{n}{\alpha}\right) \geq 1$, then there exists $C > 0$ such that for every $u : \mathbb{R}^n \rightarrow \mathbb{R}$ with*

$$\|u\|_{\beta n/\alpha}^{\beta n/\alpha} < \infty \quad \text{and} \quad \|\nabla^\alpha u\|_{\frac{n}{\alpha}} \leq 1, \quad (1.36)$$

we have

$$\int_{\mathbb{R}^n} \frac{\exp\left[\frac{\beta n}{\alpha} - \beta - 1\right] \left[\gamma(P)|u(x)|^{\frac{n}{n-\alpha}}\right]}{1 + |u(x)|^{\frac{\beta n}{n-\alpha}}} dx \leq C \|u\|_{\beta n/\alpha}^{\beta n/\alpha}. \quad (1.37)$$

The exponential constant $\gamma(P)$ is sharp. Moreover, the above inequality (1.37) cannot hold if the power $\frac{\beta n}{n-\alpha}$ in the denominator is replaced by any smaller power.

Later in section 2.9, we give an application of the above Corollary to the following quasilinear elliptic equation

$$-\Delta_n u + c|u|^{p-2}u = f(u), \quad \|\nabla u\|_n < \infty, \quad \|u\|_p < \infty,$$

where $c > 0$, $p > 1$, $n \geq 2$, and where

$$\Delta_n u = \operatorname{div}(|\nabla u|^{n-2} \nabla u)$$

is the n -Laplacian operator. Under suitable exponential growth of f , the existence of ground state solutions for the case $p = n$ was proved by Masmoudi and Sani [MS2], by using their inequality (1.15). In this thesis we will show the existence of ground state solutions of the above equation for all $p > 1$ by applying inequality (1.37).

The last section, Chapter 3, is devoted to Adams and Moser-Trudinger inequalities

with exact growth in the context of Heisenberg group $\mathbb{H}^n$.

The analogues of Adams' original results (1.2), (1.4) for functions supported on a given open set $\Omega \subseteq \mathbb{H}^n$ were given first by Cohn and Lu in [CL1] in the case of the horizontal gradient $\nabla_{\mathbb{H}^n}$ and for homogeneous “Riesz-like” potentials. The case of higher integer powers of the sublaplacian $\mathcal{L}_0$ is not explicitly studied in [CL1], however it is an immediate consequence of the explicit formula for the fundamental solution of $(-\mathcal{L}_0)^{\alpha/2}$ (see (3.24) in section 3.2) and the Adams inequality in [CL1].

In the case of inequalities in the whole of $\mathbb{H}^n$, i.e. for function with arbitrary support, the only result available, as far as we know, is the one due to Lam and Lu in [LL1], who proved the sharp version of Li-Ruf's result (1.5) in the context of $\mathbb{H}^n$, i.e.

$$\int_{\mathbb{H}^n} \exp_{[Q-2]} \left[\frac{1}{A_Q} |u|^{\frac{Q}{Q-1}} \right] dzdt \leq C, \quad \|u\|_Q + \|\nabla_{\mathbb{H}^n} u\|_Q \leq 1 \quad (1.38)$$

where $Q = 2n + 2$ denotes the homogeneous dimension, $dzdt$ the volume element, and where the constant A_Q^{-1} is sharp and explicit (see (3.31) in Section 3.2). The result proved in [LL1] is in fact slightly more general than (1.38), in that they allow a “singular” measure in the exponential integral. The proof given in [LL1] is direct, and works well when dealing with the subgradient, i.e. first order derivatives. To date there is no analogue on $\mathbb{H}^n$ of the Adams inequality or the Moser-Trudinger inequalities for higher order gradients under Ruf condition, which were derived on $\mathbb{R}^n$ by Fontana-Morpurgo, i.e. (1.13) and (1.14).

In Section 3.1 we will apply the same technique used in $\mathbb{R}^n$ to derive a sharp Adams inequality with critical growth condition on $\mathbb{H}^n$, namely the inequality (see Theorem 5)

$$\int_{\mathbb{H}^n} \frac{\exp_{[\frac{Q}{\alpha}-2]} \left[\frac{1}{A_g} |Tf|^{\frac{Q}{Q-\alpha}} \right]}{1 + |Tf|^{\frac{Q}{Q-\alpha}}} dzdt \leq C \|Tf\|_{Q/\alpha}^{Q/\alpha}, \quad \|f\|_{\frac{Q}{\alpha}} \leq 1, \quad (1.39)$$

for integral operators T whose kernels are strictly Riesz-like on $\mathbb{H}^n$, that is, they satisfy the conditions (H1)-(H4) stated in Section 3.1.

As a consequence of the above inequality, we will then derive the corresponding sharp Moser-Trudinger inequalities with exact growth condition for the powers of sublaplacian $(-\mathcal{L}_0)^{\alpha/2}$ when α is an even integer, and for the subgradient $\nabla_{\mathbb{H}^n}$ (see Theorems 6, 7). As in the case of $\mathbb{R}^n$ such inequalities will in turn allow us to derive higher order versions of Li-Ruf type results on $\mathbb{H}^n$ (see Corollary 4).

1.2 Background material

1.2.1 L^p spaces and Sobolev spaces

Given a measure space (M, μ) , if f is a measurable function on M and $0 < p < \infty$, we define

$$\|f\|_p = \left[\int |f|^p d\mu \right]^{1/p}$$

if $p = \infty$, define

$$\|f\|_\infty = \inf \{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\} = \operatorname{ess\,sup}_{x \in M} |f(x)|.$$

We now define

$$L^p(M) = \{f : M \rightarrow \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty\}.$$

Also, we denote p' be the number such that $p^{-1} + (p')^{-1} = 1$, i.e. p' is the conjugate exponent of p . Let $C_c^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, with compact support in M , the test functions.

Suppose $u, v \in L_{loc}^1(\mathbb{R}^n)$ and α is a multiindex. We say that v is the α -th weak

partial derivative of u , written $D^\alpha u = v$, provided

$$\int_{\mathbb{R}^n} u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \phi \, dx$$

for all test functions $\phi \in C_c^\infty$. Fix $1 \leq p \leq \infty$ and let α be a nonnegative integer. We now define Sobolev spaces, whose elements have weak derivatives of various orders in various L^p spaces.

Definition 3. *The Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ consists of all locally summable functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for each multiindex k with $|k| \leq \alpha$, $D^k u$ exists in the weak sense and belongs to $L^p(\mathbb{R}^n)$.*

Definition 4. *If $u \in W^{\alpha,p}(\mathbb{R}^n)$, we define*

$$\|u\|_{W^{\alpha,p}(\mathbb{R}^n)} := \begin{cases} \left(\sum_{|k| \leq \alpha} \int_{\mathbb{R}^n} |D^k u|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sum_{|k| \leq \alpha} \operatorname{ess\,sup}_{\mathbb{R}^n} |D^k u| & \text{if } p = \infty. \end{cases}$$

An equivalent characterization of the functions in $W^{\alpha,p}(\mathbb{R}^n)$, which does not explicitly use the notion of the weak derivative is given as follows:

Theorem ([S1, Chapter V Proposition 1]). *Let $1 \leq p < \infty$. Then $f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if there exists a sequence $\{f_m\}$ so that*

- (a) *each $f_m \in C_c^\infty$*
- (b) *$\|f - f_m\|_p \rightarrow 0$*
- (c) *For each $|k| \leq \alpha$, $\left\{ \frac{\partial^k f_m}{\partial x^k} \right\}$ converges in L^p norm.*

For non integer α , the corresponding Sobolev spaces $W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$ is the Bessel potential spaces whose definition is given later in (1.42).

1.2.2 Riesz potential and Bessel potential

We shall denote $\mathcal{S}$ to be the Schwartz space consisting of C^∞ function which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. That is,

$$\mathcal{S} = \{f \in C^\infty : \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)| < \infty \text{ for all } N, \alpha\},$$

where N is any nonnegative integer and α is any multi-index. The space of tempered distributions will be denoted by $\mathcal{S}'$, that is, the space of continuous linear functionals on $\mathcal{S}$.

For $f \in L^1(\mathbb{R}^n)$, define the Fourier transform of f by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx.$$

Then $\mathcal{F}$ maps the Schwartz class $\mathcal{S}$ continuously into itself. If $f \in L^1$, define the inverse Fourier transform

$$\mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

If both $f, \hat{f} \in L^1$, the the inversion formula gives

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

Let $0 < \alpha < n$. If Δ denotes the Laplace operator, we define $(-\Delta)^{\frac{\alpha}{2}} f$ for arbitrary $\alpha > 0$ as the inverse Fourier transform of $(2\pi|\xi|)^\alpha \hat{f}$, for any $f \in \mathcal{S}$. That is,

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} f) = (2\pi|\xi|)^\alpha \hat{f}(\xi).$$

To extend the above operator to negative powers, we denote

$$c_\alpha I_\alpha(f) = (-\Delta)^{-\frac{\alpha}{2}}(f), \quad 0 < \alpha < n$$

where the *Riesz potential* is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y) dy \quad (1.40)$$

with

$$c_\alpha = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{n/2} \Gamma(\frac{\alpha}{2})}. \quad (1.41)$$

The identity ([S1, Chapter V, Lemma 1])

$$\mathcal{F}(I_\alpha(f)) = (2\pi|\xi|)^{-\alpha} \widehat{f}(\xi)$$

holds in the sense of distribution, and the inverse of $c_\alpha I_\alpha$ is $(-\Delta)^{\frac{\alpha}{2}}$. Let us recall the Hardy-Littlewood-Sobolev theorem for fractional integration.

Theorem D ([S1]) *Let $0 < \alpha < n$, $1 \leq p < q < \infty$, $1/q = 1/p - \alpha/n$.*

(a) *If $f \in L^p(\mathbb{R}^n)$, then the integral (1.40), defining $I_\alpha(f)$, converges absolutely for almost every x .*

(b) *If in addition, $p > 1$, then there is a constant $A_{p,q}$ such that for all f ,*

$$\|I_\alpha(f)\|_q \leq A_{p,q} \|f\|_p.$$

(c) *If $f \in L^1(\mathbb{R}^n)$, then $m\{x : |I_\alpha| > \lambda\} \leq (\frac{A\|f\|_1}{\lambda})^q$, for all λ . That is, the mapping $f \rightarrow I_\alpha(f)$ is of weak-type $(1, q)$, ($1/q = 1 - \alpha/n$).*

Next, we denote G_α the kernel of the Bessel potential $(I - \Delta)^{-\alpha/2}$, that is

$$\widehat{G_\alpha}(x) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

When α is an integer and $1 < p < \infty$, the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ can be characterized in terms of Bessel potentials

$$W^{\alpha,p}(\mathbb{R}^n) = \{u \in \mathcal{S}' : (I - \Delta)^{\alpha/2} u \in L^p(\mathbb{R}^n)\} = \{G_\alpha * f, f \in L^p(\mathbb{R}^n)\}. \quad (1.42)$$

If α is an arbitrary positive number, we will still use (1.42) to denote the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, with the understanding that it is really the Bessel potential space.

1.2.3 Symmetric decreasing rearrangements and O'Neil's

lemma

Suppose that (M, μ) and (N, ν) are measure spaces. Given a measurable function $f : M \rightarrow [-\infty, \infty]$ its distribution function will be denoted by

$$m_f(s) = m(f, s) = \mu(\{x \in M : |f(x)| > s\}), \quad s \geq 0.$$

Assume that the distribution function of f is finite for $s \geq 0$. Here are some basic properties of $m_f(s)$:

- a. $m_f(s)$ is decreasing and right continuous.
- b. If $|f| \leq |g|$, then $m_f \leq m_g$.
- c. If $|f_n|$ increases to $|f|$, then m_{f_n} increases to m_f .
- d. If $f = g + h$, then $m_f(s) \leq m_g(\frac{1}{2}s) + m_h(\frac{1}{2}s)$.

If $0 < p < \infty$, we also have the following formula

$$\int_M |f|^p d\mu = p \int_0^\infty \alpha^{p-1} m_f(\alpha) d\alpha.$$

The decreasing rearrangement of f will be denoted by

$$f^*(t) = \inf\{s \geq 0 : m(f, s) \leq t\}, \quad t > 0$$

and we define

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0$$

which is sometimes called the *maximal function* of f^* .

Given a $\nu \times \mu$ -measurable function $k : N \times M \rightarrow [-\infty, \infty]$, assume that the level sets of $k(x, \cdot)$ and $k(\cdot, y)$ have finite measure for all $x \in N$ and all $y \in M$. Let $k_1^*(x, t)$ and $k_2^*(y, t)$ be the nonincreasing rearrangement of $k(x, y)$ with respect to the variable y (resp. x) for fixed x (resp. y), and define

$$k_1^*(t) = \sup_{x \in N} k_1^*(x, t),$$

$$k_2^*(t) = \sup_{y \in M} k_2^*(y, t).$$

Here are some further properties of rearrangements from [FM1]:

Fact 1. $m(f, s) = m(f^*, s)$ and $m(f^*, s) \leq m(g^*, s)$ for all $s > s_0$ (some $s_0 > 0$) if and only if $f^*(t) \leq g^*(t)$ for all $t < t_0$ (some $t_0 > 0$).

Fact 2. Given a measurable $k(x, y)$ on $N \times M$, if $\tilde{m}(k, s) = \sup_x m(k_1^*(x, t), s) = \sup_x m(k(x, \cdot), s)$, then $m(\tilde{k}, s) = \tilde{m}(k, s)$ and $\tilde{k}(t) = \sup_x k_1^*(x, t)$.

Fact 3: The following are equivalent ($A, B, \gamma > 0$):

a) $m(f^*, s) \leq As^{-q'}(1 + C \log^{-\gamma} s)$, for all $s > s_0 > 1$.

b) $f^*(t) \leq A^{1/q'} s^{-1/q'}(1 + C' |\log t|^{-\gamma})$, for all $t < t_0 < 1$.

Likewise, the following are equivalent:

a') $m(f^*, s) \geq As^{-q'}(1 - C \log^{-\gamma} s) > 0$, for all $s > s_0 > 1$.

b') $f^*(t) \geq A^{1/q'} s^{-1/q'}(1 - C' |\log t|^{-\gamma}) > 0$, for all $t < t_0 < 1$.

The following lemma from [FM1] will be needed later:

Lemma A ([FM1, Lemma 9]). *Suppose that K is as in Theorem 1, satisfying (A1) with g bounded and measurable. Then for $s > 0$,*

$$\sup_{x \in \mathbb{R}^n} |\{y \in \mathbb{R}^n : |K(x - y)| > s\}| \leq A_g s^{-q'} + O(s^{-q' - \delta'}) \quad (1.43)$$

for suitable $\delta > 0$, with A_g as in Theorem 1. Moreover,

$$\sup_{y \in \mathbb{R}^n} |\{x \in \mathbb{R}^n : |K(x - y)| > s\}| \leq C s^{-q'}. \quad (1.44)$$

In particular, we have

$$\sup_{y \in \mathbb{R}^n} |\{x \in E_\tau : |K(x - y)| > s\}| \leq C s^{-q'} \quad (1.45)$$

$$k_1^*(t) \leq A_g^{1/q'} t^{-1/q'} (1 + O(t^\epsilon)), \quad k_2^*(t) \leq C t^{-1/q'}. \quad (1.46)$$

Proof of Lemma A: The hypothesis implies

$$|K(x, y)| \leq |g((y - x)^*)| |x - y|^{\alpha - n} + C |x - y|^{\alpha - n + \delta_1}$$

so for any $x \in \mathbb{R}^n$

$$m_x(s) := |\{y \in \mathbb{R}^n : |K(x, y)| > s\}| \leq |\{y \in \mathbb{R}^n : |g(y^*)| |y|^{\alpha - n} + C |y|^{\alpha - n + \delta_1} > s\}|$$

and since

$$|g(y^*)| |y|^{\alpha - n} + C |y|^{\alpha - n + \delta_1} > s \implies |y| \leq s^{-q'/n} (|g(y^*)| + C |y|^\epsilon)^{q'/n} \leq C s^{-q'/n}$$

then

$$m_x(s) \leq \frac{s^{-q'}}{n} \int_{S^{n-1}} (|g(y^*)| + C s^{-\epsilon q'/n})^{q'} dy^*$$

which implies (1.43). And (1.44), (1.45) are simple consequences of (A1) and the fact that

$$\sup_{y \in \mathbb{R}^n} |\{x \in E_\tau : |K(x - y)| > s\}| \leq \sup_{y \in \mathbb{R}^n} |\{x \in \mathbb{R}^n : |K(x - y)| > s\}|.$$

Finally, by the properties **Fact 3** of rearrangements, we have (1.46). $\square$

Lastly, let T be an integral operator defined as

$$Tf(x) = \int_M k(x, y)f(y)d\mu(y). \quad (1.47)$$

One of the main tools used in the proof is the following version of O'Neil's lemma.

Its proof is based on the proof of lemma 9 in [FM1] with some small modifications.

Lemma 1. (Improved O'Neil's lemma)

Let $k : N \times M \rightarrow [-\infty, \infty]$ be measurable and

$$k_1^*(t) \leq Dt^{-\frac{1}{q'}}, \quad k_2^*(t) \leq Bt^{-\frac{1}{q'}}, \quad t > 0 \quad (1.48)$$

with $q' > 1$. Let $f(x, y) : N \times M \rightarrow \mathbb{R}$ be a measurable function on $N \times M$. For each $x \in N$, let $f_x : M \rightarrow \mathbb{R}$ be defined as $f_x(y) = f(x, y)$ on M and assume $f_x \in L^q(M)$. $(q')^{-1} + q^{-1} = 1$. Suppose there is a measurable function $\bar{f} : M \rightarrow [-\infty, \infty]$, $\bar{f} \in L^q(M)$ such that

$$\text{For all } x \in N, \quad \bar{f}(y) \geq |f_x(y)| \quad \mu - a.e. \ y \in M \quad (1.49)$$

Let

$$T'f(x) = Tf_x(x) = \int_M k(x, y)f_x(y)d\mu(y), \quad (1.50)$$

then there is a constant $C_0 = C_0(D, B, q)$ such that

$$(T'f)^{**}(t) \leq C_0 \max\{\tau^{-\frac{1}{q'}}, t^{-\frac{1}{q'}}\} \int_0^\tau \bar{f}^*(u)du + \text{ess sup}_{x \in N} \int_\tau^\infty f_x^*(u)k_1^*(x, u)du, \quad \forall t, \tau > 0. \quad (1.51)$$

Note that in this Lemma we consider the rearrangement of $Tf_x(x)$, where the function f_x depends on x . This makes it different from the other improved O'Neil

lemma in [FM3], which estimated the rearrangement of Tf for a fixed function f .

Proof. Without loss of generality we can assume that both f, f_x and k are nonnegative. Let us also assume the following claim proved in [FM3]:

Claim 1. ([FM3]) *Let $\text{supp}(f) = \{y \in M : f(y) \neq 0\}$. If $\mu(\text{supp}f) = z$, and $0 \leq f(z) \leq a$, and if k_1^*, k_2^* satisfy condition (1.48), then*

$$Tf(x) \leq azk_1^{**}(x, z) \quad \forall x \in N, \quad (1.52)$$

$$(Tf)^{**}(t) \leq C azt^{-\frac{1}{q'}} \quad \forall t > 0. \quad (1.53)$$

Fix $t, \tau > 0$, pick a sequence $\{y_n\}_{-\infty}^{\infty}$ such that $y_0 = \bar{f}^*(\tau)$, $y_n \leq y_{n+1}$, $y_n \rightarrow \infty$ as $n \rightarrow \infty$, and $y_n \rightarrow 0$ as $n \rightarrow -\infty$. Then for all $x \in N$,

$$f_x(y) = \sum_{-\infty}^{\infty} f_{x,n}(y) \quad \text{where} \quad f_{x,n}(y) = \begin{cases} 0 & \text{if } f_x(y) \leq y_{n-1} \\ f_x(y) - y_{n-1} & \text{if } y_{n-1} < f_x(y) \leq y_n \\ y_n - y_{n-1} & \text{if } y_n < f_x(y) \end{cases}$$

with $\text{supp}f_{x,n} \subseteq E_{x,n} := \{y \in M : f_{x,n}(y) > y_{n-1}\}$, $\mu(E_{x,n}) = m(f_x, y_{n-1})$ and $0 \leq f_{x,n}(y) \leq y_n - y_{n-1}$. Furthermore, we can write

$$f_x = \sum_{-\infty}^0 f_{x,n} + \sum_1^{\infty} f_{x,n} = g_{x,1} + g_{x,2}. \quad (1.54)$$

Similarly,

$$\bar{f}(y) = \sum_{-\infty}^{\infty} \bar{f}_n(y) \quad \text{where} \quad \bar{f}_n(y) = \begin{cases} 0 & \text{if } \bar{f}(y) \leq y_{n-1} \\ \bar{f}(y) - y_{n-1} & \text{if } y_{n-1} < \bar{f}(y) \leq y_n \\ y_n - y_{n-1} & \text{if } y_n < \bar{f}(y) \end{cases}$$

with $\text{supp } \bar{f}_n \subseteq E_n := \{y \in M : \bar{f}_n(y) > y_{n-1}\}$, $\mu(E_n) = m(\bar{f}, y_{n-1})$, $0 \leq \bar{f}_n(y) \leq y_n - y_{n-1}$ and write

$$\bar{f} = \sum_{-\infty}^0 \bar{f}_n + \sum_1^{\infty} \bar{f}_n = g_1 + g_2. \quad (1.55)$$

By (1.49) and definitions of $f_{x,n}, \bar{f}_n$, it is clear that $f_{x,n} \leq \bar{f}_n$ $\mu - a.e.$ and $\forall x \in N$, hence

$$g_{x,2} \leq g_2 \quad \mu - a.e. \quad \text{and } \forall x \in N. \quad (1.56)$$

Let

$$T'g_2(x) = Tg_{x,2}(x) = \int_M k(x, y)g_{x,2}(y)dy$$

and

$$Tg_2(x) = \int_M k(x, y)g_2(y)dy$$

then we have

$$T'g_2(x) \leq Tg_2(x) \quad \forall x \in M. \quad (1.57)$$

Therefore, by (1.57) and the subadditivity of $(\cdot)^{**}$, using (1.53) we have

$$(T'g_2)^{**}(t) \leq (Tg_2)^{**}(t) \leq \sum_1^{\infty} (T\bar{f}_n)^{**}(t) \leq Ct^{-\frac{1}{q'}} \sum_1^{\infty} (y_n - y_{n-1})m(\bar{f}, y_{n-1}). \quad (1.58)$$

Taking the inf over all such $\{y_n\}$ we get

$$\begin{aligned} (Tg_2)^{**}(t) &\leq Ct^{-\frac{1}{q'}} \int_{\bar{f}^*(\tau)}^{\infty} m(\bar{f}, y)dy = -Ct^{-\frac{1}{q'}} \int_0^{\tau} m(\bar{f}, \bar{f}^*(u))d\bar{f}^*(u) \\ &\leq -Ct^{-\frac{1}{q'}} \int_0^{\tau} u d\bar{f}^*(u) \\ &= Ct^{-\frac{1}{q'}} \left(-u\bar{f}^*(u) \Big|_0^{\tau} + \int_0^{\tau} \bar{f}^*(u)du \right) \\ &= Ct^{-\frac{1}{q'}} \left(-\tau\bar{f}^*(\tau) + \int_0^{\tau} \bar{f}^*(u)du \right) \leq Ct^{-\frac{1}{q'}} \int_0^{\tau} \bar{f}^*(u)du \end{aligned} \quad (1.59)$$

where the second-last equality follows since $\bar{f} \in L^q$ implies $u^{\frac{1}{q}}\bar{f}^*(u) \rightarrow 0$, as $u \rightarrow 0$.

Then use (1.52), we obtain

$$Tg_{x,1}(x) \leq \sum_{-\infty}^0 Tf_{x,n}(x) \leq \sum_{-\infty}^0 (y_n - y_{n-1})m(f_{x,n}, y)k_1^{**}(x, m(f_{x,n}, y)). \quad (1.60)$$

Given $x \in N$, let $\tau_x > 0$ be a number such that $\tau_x \leq \tau$ and $f_x^*(\tau_x) = \bar{f}^*(\tau)$ if it exists.

If such τ_x does not exist, then $f_x \leq \bar{f}^*(\tau)$ for $\mu - a.e.$ x , and then let $\tau_x = 0$ in this case. Hence

$$\begin{aligned} Tg_{x,1}(x) &\leq \int_0^{\bar{f}^*(\tau)} m(f_{x,n}, y)k_1^{**}(x, m(f_{x,n}, y))dy \\ &= - \int_{\tau_x}^{\infty} m(f_{x,n}, f_x^*(u))k_1^{**}(x, m(f_{x,n}, f_x^*(u)))df_x^*(u) \\ &= - \int_{\tau_x}^{\infty} uk_1^{**}(x, u)df_x^*(u) = -uk_1^{**}(x, u)f_x^*(u) \Big|_{\tau_x}^{\infty} + \int_{\tau_x}^{\infty} f_x^*(u)k_1^*(x, u)du \\ &\leq \tau_x k_1^{**}(x, \tau_x)\bar{f}^*(\tau) + \int_{\tau_x}^{\tau} f_x^*(u)k_1^*(x, u)du + \int_{\tau}^{\infty} f_x^*(u)k_1^*(x, u)du \\ &\leq \int_0^{\tau_x} k_1^*(x, u)du \frac{1}{\tau} \int_0^{\tau} \bar{f}^*(u)du + \bar{f}^*(\tau) \int_{\tau_x}^{\tau} k_1^*(x, u)du + \int_{\tau}^{\infty} f_x^*(u)k_1^*(x, u)du \\ &\leq C\tau^{-\frac{1}{q'}} \int_0^{\tau} \bar{f}^*(u)du + \int_{\tau}^{\infty} f_x^*(u)k_1^*(x, u)du. \end{aligned} \quad (1.61)$$

Recall that

$$T'f(x) = Tg_{x,2}(x) + Tg_{x,1}(x) = T'g_2(x) + Tg_{x,1}(x)$$

so by (1.58) and (1.61) and take ess sup over $x \in M$, we get

$$\begin{aligned} (T'f)^{**}(t) &\leq (T'g_2)^{**}(t) + \text{ess sup}_{x \in M} Tg_{x,1}(x) \\ &\leq C_0 \max\{\tau^{-\frac{1}{q'}}, t^{-\frac{1}{q'}}\} \int_0^{\tau} \bar{f}^*(u)du + \text{ess sup}_{x \in N} \int_{\tau}^{\infty} f_x^*(u)k_1^*(x, u)du. \end{aligned} \quad (1.62)$$

Proof of Claim 1: Estimate (1.53) is an immediate consequence of the following

special case of a weak-type estimate due to Adams [A2]: if f, k are nonnegative with

$$\sup_{x \in N} m(k(x, \cdot), s) \leq Ds^{-q'}, \quad \sup_{y \in M} m(k(\cdot, y), s) \leq Bs^{-q'},$$

which are equivalent to (1.48), then for $s > 0$

$$(Tf)^*(t) \leq Ct^{-1/q'} \|f\|_1, \quad \forall t > 0. \quad (1.63)$$

To prove (1.52), let $r > 0$ and set

$$k_r(x, y) = \begin{cases} k(x, y) & \text{if } k(x, y) \leq r \\ r & \text{otherwise,} \end{cases} \quad k(x, y) = k_r(x, y) + k^r(x, y)$$

so that

$$Tf(x) = \int_M k_r(x, y) f(y) d\mu(y) + \int_M k^r(x, y) f(y) d\mu(y) = h_1(x) + h_2(x).$$

Assume that k_1^* is integrable. Then, for every $x \in N$,

$$h_2(x) \leq \|f\|_\infty \int_M k^r(x, y) d\mu(y) \leq a \int_r^\infty m(k_1^*(x, \cdot), s) ds \quad (1.64)$$

$$h_1(x) \leq \|f\|_1 \sup_y k_r(x, y) \leq azr, \quad (1.65)$$

then letting $r = k_1^*(x, z)$ in (1.64) and (1.65) gives us

$$Tf(x) \leq a \int_{k_1^*(x, z)}^\infty m(k_1^*(x, \cdot), s) ds + azk_1^*(x, z) = a \int_0^z k_1^*(x, s) ds = azk_1^{**}(x, z), \quad (1.66)$$

which is (1.52).

In order to apply the above Lemma 1 to the proof of our main theorem, we also

need the following lemma regarding the rearrangement of the sum of two functions whose supports are mutually disjoint.

Lemma 2. *Let $f_1, f_2 : M \rightarrow [-\infty, \infty]$ be measurable functions. Suppose that the supports of f_1 and f_2 are mutually disjoint, $\mu(\text{supp} f_1) = z$ and*

$$f_1 \geq \|f_2\|_\infty \quad \mu - a.e. \ x \in \{x : f_1(x) \neq 0\},$$

then we have

$$(f_1 + f_2)^*(u) = \begin{cases} f_1^*(u) & \text{if } 0 < u < z \\ f_2^*(u - z) & \text{if } u \geq z. \end{cases} \quad (1.67)$$

Proof of lemma 2: Given the assumptions on f_1, f_2 we have

$$\begin{cases} \lambda_{f_1}(s) = t_1 & \text{if } s < (f_1 + f_2)^*(z) \\ \lambda_{f_1}(s) \leq z & \text{if } s \geq (f_1 + f_2)^*(z) \\ \lambda_{f_2}(s) = 0 & \text{if } s \geq (f_1 + f_2)^*(z) \end{cases} \quad (1.68)$$

and therefore

$$\lambda_{f_1+f_2}(s) = \begin{cases} \lambda_{f_2}(s) + z & \text{if } s < (f_1 + f_2)^*(z) \\ \lambda_{f_1}(s) & \text{if } s \geq (f_1 + f_2)^*(z). \end{cases}$$

Therefore, using (1.68) we get that for $u \geq z$

$$\begin{aligned} (f_1 + f_2)^*(u) &= \inf\{s > 0 : \lambda_{f_1+f_2}(s) \leq u\} = \inf\{s < (f_1 + f_2)^*(z) : \lambda_{f_2}(s) \leq u - z\} \\ &= \inf\{s > 0 : \lambda_{f_2}(s) \leq u - z\} = f_2^*(u - z). \end{aligned}$$

For $u < z$, we have that $\lambda_{f_1}(s) \leq u$ implies $s \geq (f_1 + f_2)^*(z)$, but

$$\lambda_{f_2}(s) = 0 \quad \text{if } s \geq (f_1 + f_2)^*(z)$$

so $\lambda_{f_1}(s) \leq u$ if and only if $\lambda_{f_1}(s) + \lambda_{f_2}(s) \leq u$. Therefore we get

$$\begin{aligned} (f_1 + f_2)^*(u) &= \inf\{s > 0 : \lambda_{f_1+f_2}(s) \leq u\} = \inf\{s > 0 : \lambda_{f_1}(s) + \lambda_{f_2}(s) \leq u\} \\ &= \inf\{s > 0 : \lambda_{f_1}(s) \leq u\} = f_1^*(u). \end{aligned}$$

□

1.2.4 O'Neil's operator and Adams inequality

It is clear that as a consequence of Lemma 1, if $f_x = f$ for all $x \in N$, the O'Neil estimate takes the following form:

$$Tf^{**}(t) \leq C_0 t^{-\frac{1}{q'}} \int_0^t f^*(u) du + \int_t^\infty k_1^*(u) f^*(u) du.$$

Under the hypothesis that $m(f, s) < \infty$ for $s \geq 0$, we are able to define the O'Neil operator U as follows:

$$Uf(t) = C_0 t^{-\frac{1}{q'}} \int_0^t f^*(u) du + \int_t^\infty k_1^*(u) f^*(u) du$$

where C_0 is the constant in the improved O'Neil's lemma 1.

We state an Adams inequality due to Fontana and Morpurgo ([FM3, Corollary 2]) in terms of the O'Neil operator. Although the original theorem in their paper is not stated in this form, it is clear from the proof in [FM3] that everything also works for the O'Neil operator instead of the original operator. This result plays a crucial role in the proof of our main result.

Theorem E ([FM3, Corollary 2]) *Suppose $\nu(N) < \infty$, $\mu(M) < \infty$, and that*

$$k_1^*(t) \leq A^{\frac{1}{q'}} t^{-\frac{1}{q'}} (1 + H(1 + |\log t|)^{-\gamma}), \quad 0 < t \leq \mu(M), \quad (1.69)$$

$$k_2^*(t) \leq B t^{-\frac{1}{q'}}. \quad 0 < t \leq \nu(N). \quad (1.70)$$

Then there exists a constant $C = C(q, \gamma, A, B, H)$ such that for each $f \in L^q(M)$ with $\|f\|_q \leq 1$

$$\int_0^{\nu(N)} \exp \left[\frac{1}{A} (Uf(t))^{q'} \right] dt \leq C(\nu(N) + \mu(M)). \quad (1.71)$$

Let us also state following lemma from [FM2] for later use, in order to clarify the equivalence between exponential inequalities on sets $\{x \in \mathbb{R}^n : |Tf(x)| \geq 1\}$ and regularized exponential inequalities over $\mathbb{R}^n$.

Lemma B ([FM2, Lemma 9]) *Let (N, ν) be a measure space and $1 < p < \infty$, $a > 0$. Then for every $u \in L^p(N)$ we have*

$$\int_{\{|u| \geq 1\}} e^{a|u|^{p'}} dx - e^a \|u\|_p^p \leq \int_N \left(e^{a|u|^{p'}} - \sum_{k=0}^{[p-2]} \frac{a^k |u|^{kp'}}{k!} \right) dx \leq \int_{\{|u| \geq 1\}} e^{a|u|^{p'}} dx + e^a \|u\|_p^p \quad (1.72)$$

and also

$$\int_{\{|u| \geq 1\}} \frac{e^{a|u|^{p'}}}{1 + |u|^{p'}} dx - e^a \|u\|_p^p \leq \int_N \frac{e^{a|u|^{p'}} - \sum_{k=0}^{[p-2]} \frac{a^k |u|^{kp'}}{k!}}{1 + |u|^{p'}} dx \leq \int_{\{|u| \geq 1\}} \frac{e^{a|u|^{p'}}}{1 + |u|^{p'}} dx + e^a \|u\|_p^p. \quad (1.73)$$

In particular, the following three inequalities are equivalent:

(a)

$$\int_N \frac{\exp_{[p-2]} [a|u|^{p'}]}{1 + |u|^{p'}} dx \leq C \|u\|_p^p, \quad (1.74)$$

(b)

$$\int_{\{|u| \geq 1\}} \frac{e^{a|u|^{p'}}}{1 + |u|^{p'}} dx \leq C \|u\|_p^p, \quad (1.75)$$

(c)

$$\int_E \frac{e^{a|u|^{p'}}}{1 + |u|^{p'}} dx \leq C(\|u\|_p^p + |E|) \quad (1.76)$$

for every measurable set E with finite measure.

Proof of Lemma B: Recall that $[p-2]$ is the smallest integer greater than or equal

$p - 2$. Observe that

$$e^{a|u|^{p'}} - e^a|u|^p \leq e^{a|u|^{p'}} - \sum_{k=0}^{[p-2]} \frac{a^k|u|^{kp'}}{k!} \leq e^{a|u|^{p'}}, \quad \text{if } |u| \geq 1$$

and

$$0 \leq e^{a|u|^{p'}} - \sum_{k=0}^{[p-2]} \frac{a^k|u|^{kp'}}{k!} \leq e^a|u|^p. \quad \text{if } |u| < 1$$

Then split the middle integral in (1.72) and (1.73) according to

$$N = \{|u| \geq 1\} \cup \{|u| < 1\}.$$

Lastly, the proof of equivalence of (a), (b), (c) is straightforward and is omitted here.

□

Chapter 2

Proof of results

2.1 Proof of the inequalities in Theorem 1

In view of Lemma B, in order to prove (1.21), it is enough to show that

$$\int_{\{|Tf| \geq 1\}} \frac{\exp \left[\frac{1}{A_g} |Tf|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{n-\alpha}}} dx \leq C \|Tf\|_{n/\alpha}^{n/\alpha}. \quad (2.1)$$

Let

$$t_0 = \sup\{t > 0 : (Tf)^*(t) \geq 1\}$$

so that (2.1) is equivalent to

$$\int_0^{t_0} \frac{\exp \left[\frac{1}{A_g} ((Tf)^*(t))^{\frac{n}{n-\alpha}} \right]}{1 + ((Tf)^*(t))^{\frac{n}{n-\alpha}}} dt \leq C \|Tf\|_{n/\alpha}^{n/\alpha}. \quad (2.2)$$

To estimate $(Tf)^*$, we first define E_τ , F_τ as follows:

For $\tau > 0$, let E_τ be the set such that

$$\begin{cases} |E_\tau| = \tau \\ \{x : |Tf(x)| > (Tf)^*(\tau)\} \subseteq E_\tau \subseteq \{x : |Tf(x)| \geq (Tf)^*(\tau)\}. \end{cases} \quad (2.3)$$

In order to show that such E_τ exists, we denote $V_1 = \{x : |Tf(x)| > (Tf)^*(\tau)\}$ and $V_2 = \{x : |Tf(x)| \geq (Tf)^*(\tau)\}$. By definition of rearrangement, we have that $\mu(V_1) \leq \tau$ and $\mu(V_2) \geq \tau$. If $\mu(V_2) = \tau$, we take $E_\tau = V_2$. Otherwise, consider the continuous function $g(r) = \mu(V_1) + \mu(B_r \cap (V_2 \setminus V_1))$ for $r \geq 0$, where $B_r = B(0, r)$ is the ball centered at 0 with radius r . It is clear that $g(0) = \mu(V_1) \leq \tau$, and $g(r) \rightarrow \mu(V_2)$ as $r \rightarrow \infty$. Since $\mu(V_2) > \tau$, there exists a r such that $g(r) = \tau$, and $E_\tau = V_1 \cup (B_r \cap V_2 \setminus V_1)$ is a measurable set that satisfies the condition (2.3).

Similarly, let F_τ to be

$$\begin{cases} |F_\tau| = \tau \\ \{x : |f(x)| > f^*(\tau)\} \subseteq F_\tau \subseteq \{x : |f(x)| \geq f^*(\tau)\}. \end{cases} \quad (2.4)$$

Let

$$f_\tau = f\chi_{F_\tau}, \quad f'_\tau = f\chi_{F_\tau^c}$$

and $r(\tau) = (\tau/|B_1|)^{1/n}$ so that

$$|B(0, r(\tau))| = \tau. \quad (2.5)$$

Next, for all $x \in E_\tau$ define

$$W(\tau, x) = \int_\tau^{2\tau} k_1^*(u) (f'_\tau \chi_{B(x, r(\tau))})^*(u - \tau) du \quad (2.6)$$

$$M(\tau, x) = |T(f'_\tau \chi_{B^c(x, r(\tau))})(x)|. \quad (2.7)$$

Lastly, for fixed $\tau > 0$, take the essential supremum in (2.6) and (2.7), and let

$$W_\tau = \operatorname{ess\,sup}_{x \in E_\tau} W(\tau, x) \quad (2.8)$$

$$M_\tau = \operatorname{ess\,sup}_{x \in E_\tau} M(\tau, x). \quad (2.9)$$

We want to point out that for all $\tau > 0$, we have $W_\tau < \infty$ and $M_\tau < \infty$ by the fact that f is compactly supported and $\|f\|_{n/\alpha} \leq 1$. Also, note that for each x and τ ,

$$f = f_\tau + f'_\tau = f_\tau + f'_\tau \chi_{B(x, r(\tau))} + f'_\tau \chi_{B^c(x, r(\tau))}$$

and

$$Tf(x) = Tf_\tau(x) + T(f'_\tau \chi_{B(x, r(\tau))})(x) + T(f'_\tau \chi_{B^c(x, r(\tau))})(x). \quad (2.10)$$

From now on we will use the following notation:

$$q = \frac{n}{\alpha}, \quad q' = \frac{n}{n - \alpha}.$$

Recall that the O'Neil operator is defined as

$$Uf(t) = C_0 t^{-\frac{1}{q'}} \int_0^t f^*(u) du + \int_t^\infty k_1^*(u) f^*(u) du.$$

Our first step toward a proof of (2.2) is to prove the following estimate:

$$(Tf)^*(t) \leq (Tf)^{**}(t) \leq Uf_\tau(t) + W_\tau + M_\tau \quad \text{for } 0 < t \leq \tau. \quad (2.11)$$

In what follows we will denote the rearrangement of Tf with respect to a measurable set E as $(Tf)_E^*$ and its corresponding maximal function as $(Tf)_E^{**}$. Clearly

$$(Tf)_E^*(t) = ((Tf)\chi_E)^*(t), \quad 0 < t \leq |E|.$$

The definition of E_τ implies that

$$|(Tf)\chi_{E_\tau}(x)| \geq \|(Tf)\chi_{E_\tau^c}\|_\infty, \quad \text{for a.e. } x \in E_\tau,$$

hence we can apply Lemma 2 to get

$$(Tf)_{E_\tau}^*(t) = (Tf)^*(t), \quad \text{for } 0 < t \leq \tau. \quad (2.12)$$

Let

$$f_{x,\tau} = f_\tau + f'_\tau \chi_{B(x,r(\tau))},$$

note that by the definition of M_τ in (2.9) and the decomposition of Tf in (2.10),

$$|Tf(x)| \leq |Tf_{x,\tau}(x)| + M_\tau, \quad \text{for } x \in E_\tau.$$

Due to subadditivity of $(\cdot)^{**}$ and (2.12), we have

$$(Tf)^*(t) = (Tf)_{E_\tau}^{**}(t) \leq (Tf_{x,\tau})_{E_\tau}^{**}(t) + M_\tau, \quad 0 < t \leq \tau. \quad (2.13)$$

Therefore in order to prove (2.11) it is enough to show the following:

$$(Tf_{x,\tau})_{E_\tau}^{**}(t) \leq Uf_\tau(t) + W_\tau \quad \text{for } 0 < t \leq \tau, \quad (2.14)$$

in other words, we only need to show the rearrangement of $Tf_{x,\tau}(x)$ over the set E_τ satisfies (2.14). Let us apply the improved O'Neil Lemma 1 with $N = E_\tau$, $M = \mathbb{R}^n$, $q = n/\alpha$, $f_x = f_{x,\tau}$, and

$$\bar{f} = |f_\tau + f'_\tau| = |f| \quad (2.15)$$

so that (1.49) holds. For $x \in E_\tau$, let

$$T'f(x) = Tf_{x,\tau}(x) = \int_{\mathbb{R}^n} K(x-y)(f_\tau + f'_\tau \chi_{B(x,r(\tau))})(y)dy. \quad (2.16)$$

We still need to show the condition (1.48) in lemma 1 is satisfied. In other words, we need to estimate the distribution function of the kernel K . But this is the consequences of (1.46) in Lemma A.

From now on we will use k^* to denote k_1^* since $k(x, y) = k(x - y)$ is a convolution kernel, and $k^*(t) = k_1^*(t)$. We obtain

$$\begin{aligned} (T'f)_{E_\tau}^{**}(t) &\leq C_0 t^{-\frac{1}{q'}} \int_0^t f^*(u)du + \operatorname{ess\,sup}_{x \in E_\tau} \int_t^\infty k_1^*(x, u) f_{x,\tau}^*(u)du \\ &= C_0 t^{-\frac{1}{q'}} \int_0^t f^*(u)du + \operatorname{ess\,sup}_{x \in E_\tau} \int_t^\infty k^*(u) f_{x,\tau}^*(u)du. \end{aligned} \quad (2.17)$$

By definition of F_τ, f_τ and (2.15), we apply Lemma 2 to get

$$f^*(u) = (f_\tau + f'_\tau)^*(u) = f_\tau^*(u) \quad \text{if } 0 < u < \tau \quad (2.18)$$

and

$$f_{x,\tau}^*(u) = (f_\tau + f'_\tau \chi_{B(x,r(\tau))})^*(u) = \begin{cases} f_\tau^*(u) & \text{if } 0 < u < \tau \\ (f'_\tau \chi_{B(x,r(\tau))})^*(u - \tau) & \text{if } u \geq \tau. \end{cases} \quad (2.19)$$

Therefore, (2.17) can be rewritten as

$$\begin{aligned} (T'f)_{E_\tau}^{**}(t) &\leq C_0 t^{-\frac{1}{q'}} \int_0^t f_\tau^*(u)du + \int_t^\tau k^*(u) f_\tau^*(u)du \\ &\quad + \operatorname{ess\,sup}_{x \in E_\tau} \int_\tau^{2\tau} k^*(u) (f'_\tau \chi_{B(x,r(\tau))})^*(u - \tau)du \\ &= Uf_\tau(t) + W_\tau. \end{aligned} \quad (2.20)$$

Hence (2.14) is proved and (2.11) follows.

Next we use estimation (2.11) and apply the following inequality

$$(a + b)^p \leq \lambda^{1-p} a^p + (1 - \lambda)^{1-p} b^p \quad a, b \geq 0, 0 < \lambda < 1, p > 1 \quad (2.21)$$

to the integrand in (2.2), with

$$p = q', \quad a = (Uf_\tau)^*(t), \quad \text{and } b = W_\tau + M_\tau.$$

That is

$$\begin{aligned} \frac{\exp \left[\frac{1}{A_g} ((Tf)^*(t))^{q'} \right]}{1 + ((Tf)^*(t))^{q'}} &\leq \frac{\exp \left[\frac{1}{A_g} (Uf_\tau(t) + W_\tau + M_\tau)^{q'} \right]}{1 + (Uf_\tau(t) + W_\tau + M_\tau)^{q'}} \\ &\leq \frac{\exp \left[\frac{(1 - \lambda)^{1-q'}}{A_g} (W_\tau + M_\tau)^{q'} \right]}{1 + (W_\tau + M_\tau)^{q'}} \cdot \exp \left[\frac{\lambda^{1-q'}}{A_g} (Uf_\tau(t))^{q'} \right]. \end{aligned} \quad (2.22)$$

Let $t_1 > 0$ be the number such that

$$\frac{\int_0^{t_1} f^*(u)^q du}{\|f\|_q^q} = \frac{1}{4}$$

and

$$\epsilon_\tau = \min \left\{ \frac{1}{4}, \frac{\int_0^\tau f^*(u)^q du}{\|f\|_q^q} \right\}.$$

We estimate (2.22) using the following two lemmas. The first one is an integral estimate (essentially the Adams inequality):

Lemma 3. *If we define*

$$I_2(\tau, t, \lambda) = \exp \left[\frac{\lambda^{1-q'}}{A_g} (Uf_\tau(t))^{q'} \right],$$

then

$$\int_0^\tau I_2(\tau, t, \epsilon_\tau) dt \leq C\tau, \quad 0 < \tau \leq t_1 \quad (2.23)$$

Proof of Lemma 3: First note that when $\tau \leq t_1$, we have

$$\epsilon_\tau = \frac{\int_0^\tau f^*(u)^q du}{\|f\|_q^q}.$$

If we let

$$\tilde{f} := \frac{f_\tau}{\epsilon_\tau^{1/q}},$$

then we have that $\tilde{f}$ has measure of support $\mu(\text{supp } \tilde{f}) \leq \tau$ with

$$\|\tilde{f}\|_q \leq 1,$$

also assumption (A1) implies the estimate (1.46) on k^* by Lemma A. That is, conditions (1.69) and (1.70) are satisfied. Therefore by Theorem E, the Adams inequality for the O'Neil's operator, we obtain (2.23). $\square$

The estimation for $I_1(\tau, \lambda)$ which is stated in the following lemma is essential for the rest of the proof. Let us assume the lemma for now, and its proof will be given in sections 3.2-3.3.

Lemma 4. *Let $0 < \tau \leq t_0$. Then there exists constant $C > 0$ such that for all $x_1, x_2 \in E_z$*

$$I_1(\tau, \epsilon_\tau) = \frac{\exp \left[\frac{(1 - \epsilon_\tau)^{1-q'}}{A_g} (W_\tau + M_\tau)^{q'} \right]}{1 + (W_\tau + M_\tau)^{q'}} \leq \frac{C}{\tau} \|Tf\|_q^q \quad (2.24)$$

where $C = C(n, \alpha, K)$.

Assuming Lemma 4, let $\tau_0 = \min\{t_0, t_1\}$. To prove (2.2) it is enough to show that

$$\int_0^{\tau_0} \frac{\exp\left[\frac{1}{A_g}((Tf)^*(t))^{q'}\right]}{1 + ((Tf)^*(t))^{q'}} dt \leq C \|Tf\|_q^q \quad (2.25)$$

and then show that if $t_1 < t_0$,

$$\int_{t_1}^{t_0} \frac{\exp\left[\frac{1}{A_g}((Tf)^*(t))^{q'}\right]}{1 + ((Tf)^*(t))^{q'}} dt \leq C \|Tf\|_q^q. \quad (2.26)$$

To prove (2.25), we take $\tau = \tau_0$ in (2.24) and (2.23) to get

$$I_1(\tau_0, \epsilon_{\tau_0}) \leq \frac{C}{\tau_0} \|Tf\|_q^q \quad \text{and} \quad \int_0^{\tau_0} I_2(\tau_0, t, \epsilon_{\tau_0}) dt \leq C \tau_0. \quad (2.27)$$

Therefore, using (2.22) it is immediate that

$$\int_0^{\tau_0} \frac{\exp\left[\frac{1}{A_g}((Tf)^*(t))^{q'}\right]}{1 + ((Tf)^*(t))^{q'}} dt \leq \int_0^{\tau_0} I_1(\tau_0, \epsilon_{\tau_0}) I_2(\tau_0, t, \epsilon_{\tau_0}) dt \leq C \|Tf\|_q^q \quad (2.28)$$

and (2.25) follows.

Next, to show (2.26), we take $\tau = t$ for $t_1 \leq t \leq t_0$, and $\lambda = \frac{1}{8}$ in the definition of I_2 in Lemma 3. Then by definition of O'Neil's operator and the fact that the support f_t has measure less than or equal t ,

$$Uf_t(t) = C_0 t^{-\frac{1}{q'}} \int_0^t f_t^*(u) du \leq C \|f_t\|_q \leq C.$$

So we have

$$I_2\left(t, t, \frac{1}{8}\right) \leq C. \quad (2.29)$$

Since $t_1 \leq t \leq t_0$ and $\epsilon_{t_1} = \frac{1}{4}$, by definition $\epsilon_t = \frac{1}{4}$. Take $\theta = (6/7)^{\frac{\alpha}{n-\alpha}} < 1$. Hence

$$\begin{aligned}
I_1\left(t, \frac{1}{8}\right) &= \frac{\exp\left[\frac{(7/8)^{1-q'}}{A_g}(W_t + M_t)^{q'}\right]}{1 + (W_t + M_t)^{q'}} = \frac{\exp\left[(7/6)^{1-q'}\frac{(3/4)^{1-q'}}{A_g}(W_t + M_t)^{q'}\right]}{1 + (W_t + M_t)^{q'}} \\
&= \frac{\left(\exp\left[\frac{(3/4)^{1-q'}}{A_g}(W_t + M_t)^{q'}\right]\right)^\theta}{1 + (W_t + M_t)^{q'}} \leq \left(\frac{\exp\left[\frac{(3/4)^{1-q'}}{A_g}(W_t + M_t)^{q'}\right]}{1 + (W_t + M_t)^{q'}}\right)^\theta \\
&= I_1^\theta\left(t, \frac{1}{4}\right) = I_1^\theta(t, \epsilon_t) \leq \frac{C}{t^\theta} \|Tf\|_q^{\theta q}.
\end{aligned} \tag{2.30}$$

Now integrate $I_1 I_2$ from t_1 to t_0 using (2.29), (2.30) to get

$$\begin{aligned}
\int_{t_1}^{t_0} I_1\left(t, \frac{1}{8}\right) I_2\left(t, \frac{1}{8}\right) dt &\leq C \int_{t_1}^{t_0} \frac{1}{t^\theta} \|Tf\|_q^{\theta q} dt \leq C t_0^{1-\theta} \|Tf\|_q^{\theta q} \\
&\leq C \|Tf\|_q^{(1-\theta)q} \|Tf\|_q^{\theta q} = C \|Tf\|_q^q.
\end{aligned} \tag{2.31}$$

Where the last inequality is by the fact that

$$\|Tf\|_q^q \geq t_0 \tag{2.32}$$

since $(Tf)^*(t) \geq 1$ for all $t < t_0$, by the definition of t_0 . So (2.26) is proved. Lastly, to complete the proof of Theorem 1, we are left to prove Lemma 4.

2.2 Proof of Lemma 4

It is enough to show that

$$\frac{\exp\left[\frac{(1 - \epsilon_\tau)^{-\frac{\alpha}{n-\alpha}}}{A_g}(W(\tau, x_2) + M(\tau, x_1))^{\frac{n}{n-\alpha}}\right]}{1 + (W(\tau, x_2) + M(\tau, x_1))^{\frac{n}{n-\alpha}}} \leq C \frac{\|Tf\|_q^q}{\tau} \tag{2.33}$$

for all $x_1, x_2 \in E_\tau$. Now let us state the following more general version of the key lemma in [MS1]-[MS3], [LT], whose proof will be given in the Appendix.

Lemma 5. *Given any sequence $a = \{a_k\}_{k \geq 0}$. Let $q > 1$, $\beta > 0$ and*

$$\|a\|_1 = \sum_{k=0}^{\infty} |a_k|, \quad \|a\|_q = \left(\sum_{k=0}^{\infty} |a_k|^q \right)^{1/q} \quad (2.34)$$

define

$$\mu_d(h) = \inf \left\{ \sum_{k=0}^{\infty} |a_k|^{\beta q} e^{qk} : \|a\|_1 = h, \|a\|_q \leq 1 \right\}.$$

Then for $h > 1$, we have

$$C_1(q, \beta) \frac{\exp[qh^{q'}]}{h^{\beta q'}} \leq \mu_d(h) \leq C_2(q, \beta) \frac{\exp[qh^{q'}]}{h^{\beta q'}}. \quad (2.35)$$

As a consequence of the above optimal growth lemma, we deduce that for any $q > 1$ and any $\mu > 0$, $h > 1$ there is $C = C(q)$ such that for any sequence $\{a_k\}$ satisfying

$$\sum_{k=0}^{\infty} |a_k| = h \quad \sum_{k=0}^{\infty} |a_k|^q \leq \mu \quad (2.36)$$

we have

$$\frac{\exp[q\mu^{1-q'}h^{q'}]}{h^{q'}} \leq C\mu^{-q'} \sum_{k=0}^{\infty} |a_k|^q e^{qk}. \quad (2.37)$$

The next task is to find a number h_1 , depending on f and x_1 , and a sequence $a = \{a_k\}$, also depending on f and x_1 , such that

$$q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) \leq h_1 \quad (2.38)$$

$$\sum_{k=0}^{\infty} |a_k| = h_1, \quad \sum_{k=0}^{\infty} |a_k|^q \leq 1 - \epsilon_\tau, \quad \sum_{k=0}^{\infty} |a_k|^q e^{qk} \leq \frac{C}{\tau} \|Tf\|_q^q. \quad (2.39)$$

Clearly (2.33) follows from (2.36)-(2.39), with $\mu = 1 - \epsilon_\tau \geq \frac{3}{4}$ and $h = h_1$.

From now on, throughout the proof of Lemma 4, we fix $0 < \tau \leq t_0$, and $x_1, x_2 \in E_\tau$ as defined in (2.3). First let us introduce some notation. Recall that $r(\tau)$ is the number

such that $|B(0, r(\tau))| = \tau$. Define for each $j = 0, 1, 2, \dots$

$$r_j = r(\tau)e^{\frac{q}{n}j}, \quad D_j = B(x_1, r_j)$$

$$\alpha_j = \|f'_\tau \chi_{D_{j+1} \setminus D_j}\|_q, \quad \alpha_{-1} = \|f'_\tau \chi_{D_0}\|_q$$

$$\bar{\alpha}_j = \max\{\alpha_{-1}, \alpha_0, \dots, \alpha_j\}, \quad \beta_j = \|f'_\tau \chi_{D_j^c}\|_q.$$

Notice that for any j

$$\alpha_j \leq \beta_j \leq 1.$$

Clearly β_j is decreasing, and it vanishes for all j large enough, since f has compact support. In particular, there is an integer N so that

$$\text{supp } f \subseteq D_N = B(x_1, r_N).$$

Now we are ready to state the main estimates on $M(\tau, x_2)$ and $W(\tau, x_1)$:

Proposition 1. *There exist constants C_2, C_3 independent of f and an integer J such that*

$$q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) \leq \sum_{j=0}^J \alpha_j + C_2 \bar{\alpha}_J + C_2 \beta_J \quad (2.40)$$

and

$$\sum_{j=0}^J \alpha_j^q e^{qj} + \beta_J^q e^{qJ} \leq \frac{C_3}{\tau} \|Tf\|_q^q. \quad (2.41)$$

The proof of the above proposition 1 will be given in section 2.3. Assuming the proposition, we now show how to derive (2.24), and hence finish the proof of Lemma 4, using (2.36)-(2.39) together with Proposition 1. Our goal is to find a number h_1 and a sequence $a = \{a_k\}$ that satisfies (2.38) and (2.39). Let

$$h_1 = \sum_{j=0}^J \alpha_j + C_2 \bar{\alpha}_J + C_2 \beta_J. \quad (2.42)$$

Clearly, by proposition 1, we have

$$q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) \leq h_1. \quad (2.43)$$

Let J^* be the smallest integer such that $\alpha_{J^*} = \bar{\alpha}_J$. It is clear that $J^* \leq J$. To construct the sequence a that satisfies (2.39), let us first define N_i , $i = 1, \dots, 4$ as follows:

$$\begin{aligned} N_1 &= J^* \\ N_2 &= N_1 + [(1 + C_2)^{q'}] \\ N_3 &= N_2 + J - 1 - J^* \\ N_4 &= N_3 + [(1 + C_2)^{q'}] \end{aligned} \quad (2.44)$$

where $[x]$ denotes the ceiling of x , i.e. the smallest integer greater or equal x . Let $a = \{a_k\}$ be the following:

$$a_k = \begin{cases} \alpha_{k-1} \text{ if } J^* \neq -1; 0 \text{ if } J^* = -1 & \text{for } k = 0, \dots, N_1 \\ \frac{(1 + C_2)\bar{\alpha}_J}{N_2 - N_1} \text{ if } J^* \neq -1; \frac{C_2\bar{\alpha}_J}{N_2 - N_1} \text{ if } J^* = -1 & \text{for } k = N_1 + 1, \dots, N_2 \\ \alpha_{k-N_2+J^*} & \text{for } k = N_2 + 1, \dots, N_3 \\ \frac{\alpha_J + C_2\beta_J}{N_4 - N_3} & \text{for } k = N_3 + 1, \dots, N_4. \end{cases} \quad (2.45)$$

With this definition of a_k we have

$$\|a\|_1 = \sum_{k=0}^{N_4} |a_k| = \sum_{k=0}^{N_4} a_k = h_1. \quad (2.46)$$

If $J^* \neq -1$,

$$\begin{aligned}
\sum_{k=0}^{N_4} |a_k|^q &= \sum_{k=0}^{N_1} \alpha_{k-1}^q + \sum_{k=N_1+1}^{N_2} \left(\frac{(1+C_2)\bar{\alpha}_J}{N_2-N_1} \right)^q + \sum_{k=N_2+1}^{N_3} \alpha_{k-N_2+J^*}^q \\
&\quad + \sum_{k=N_3+1}^{N_4} \left(\frac{\alpha_J + C_2\beta_J}{N_4-N_3} \right)^q \\
&\leq \sum_{k=0}^{J^*-1} \alpha_{k-1}^q + \bar{\alpha}_J^q + \sum_{k=J^*+1}^{J-1} \alpha_k^q + \beta_J^q \\
&= \|f'_\tau\|_q^q \leq (1 - \epsilon_\tau) \|f\|_q^q \leq 1 - \epsilon_\tau.
\end{aligned} \tag{2.47}$$

Likewise for $J^* = -1$,

$$\begin{aligned}
\sum_{k=0}^{N_4} |a_k|^q &= \sum_{k=N_1+1}^{N_2} \left(\frac{C_2\bar{\alpha}_J}{N_2-N_1} \right)^q + \sum_{k=N_2+1}^{N_3} \alpha_{k-N_2+J^*}^q \\
&\quad + \sum_{k=N_3+1}^{N_4} \left(\frac{\alpha_J + C_2\beta_J}{N_4-N_3} \right)^q \\
&\leq \alpha_{-1}^q + \sum_{k=0}^{J-1} \alpha_k^q + \beta_J^q \\
&= \|f'_\tau\|_q^q \leq (1 - \epsilon_\tau) \|f\|_q^q \leq 1 - \epsilon_\tau.
\end{aligned} \tag{2.48}$$

And using (2.41) in Proposition 1, we also have, if $J^* \neq -1$

$$\begin{aligned}
\sum_{k=0}^{N_4} |a_k|^q e^{qk} &= \sum_{k=0}^{N_1} \alpha_{k-1}^q e^{qk} + \sum_{k=N_1+1}^{N_2} \left(\frac{(1+C_2)\bar{\alpha}_J}{N_2-N_1} \right)^q e^{qk} + \sum_{k=N_2+1}^{N_3} \alpha_{k-N_2+J^*}^q e^{qk} \\
&\quad + \sum_{k=N_3+1}^{N_4} \left(\frac{\alpha_J + C_2\beta_J}{N_4-N_3} \right)^q e^{qk} \\
&\leq \sum_{k=0}^{J^*-1} \alpha_{k-1}^q e^{qk} + C\bar{\alpha}_J^q e^{q(J^*+C_4)} + \sum_{k=J^*+1}^{J-1} \alpha_k^q e^{q(k+C_4)} + \beta_J^q e^{q(J+2C_4)} \\
&\leq C e^{2C_4} \left(\alpha_{-1}^q + \sum_{k=0}^J \alpha_k^q e^{qk} + \beta_J^q e^{qJ} \right) \leq C e^{2C_4} \left(C + \sum_{k=0}^J \alpha_k^q e^{qk} + \beta_J^q e^{qJ} \right) \\
&= C e^{2C_4} \left(C \frac{\tau}{\tau} + \sum_{k=0}^J \alpha_k^q e^{qk} + \beta_J^q e^{qJ} \right) \leq \frac{C}{\tau} \|Tf\|_q^q
\end{aligned} \tag{2.49}$$

where C_4 in the above inequality is $C_4 = [(1 + C_2)^{q'}]$, and in the last inequality we used the fact that $\tau \leq \|Tf\|_q^q$ since $(Tf)^*(t) \geq 1$ for $0 < t \leq \tau \leq t_0$. Similarly for $J^* = -1$, we also have

$$\sum_{k=0}^{N_4} |a_k|^q e^{qk} \leq C e^{2C_4} \left(\alpha_{-1}^q + \sum_{k=0}^J \alpha_k^q e^{qk} + \beta_J^q e^{qJ} \right) \leq \frac{C}{\tau} \|Tf\|_q^q. \quad (2.50)$$

Finally, (2.47)-(2.50) shows that the sequence a satisfies (2.38) and (2.39). Hence (2.24) follows and the proof is concluded. $\square$

2.3 Proof of Proposition 1

In the following proof we will set for any measurable function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$

$$S_j \phi = \phi \chi_{D_j^c} = \phi \chi_{\{|y-x_1| \geq r_j\}}.$$

With this notation we then have

$$(S_j - S_{j+1})f'_\tau = f'_\tau \chi_{D_{j+1} \setminus D_j} = f'_\tau \chi_{\{r_j \leq |y-x_1| < r_{j+1}\}}$$

and

$$\alpha_j = \|(S_j - S_{j+1})f'_\tau\|_q, \quad \beta_j = \|S_j f'_\tau\|_q.$$

Also note that

$$f'_\tau = f'_\tau \chi_{B(x_1, r(\tau))} + f'_\tau \chi_{B^c(x_1, r(\tau))} = f'_\tau \chi_{D_0} + S_0 f'_\tau. \quad (2.51)$$

For the rest of the proof in sections 3.2-3.3 we assume that

$$TS_0 f'_\tau(x_1) \geq 0. \quad (2.52)$$

If, on the other hand,

$$TS_0 f'_\tau(x_1) < 0, \quad (2.53)$$

we replace T by $-T$, and the proof is exactly the same.

We first give some preliminary estimates on $W(\tau, x_2)$ and $M(\tau, x_1)$. We have that

$$\begin{aligned} W(\tau, x_2) &= \int_\tau^{2\tau} k_1^*(u) (f'_\tau \chi_{B(x_2, r(\tau))})^*(u - \tau) du \\ &\leq C \int_\tau^{2\tau} u^{-\frac{1}{q'}} (f'_\tau \chi_{B(x_2, r(\tau))})^*(u - \tau) du \leq C \| (f'_\tau \chi_{B(x_2, r(\tau))})^* \|_q \\ &= C \| f'_\tau \chi_{B(x_2, r(\tau))} \|_q. \end{aligned} \quad (2.54)$$

Since f is supported in D_N , we also have that

$$\text{supp } f'_\tau \chi_{B(x_2, r(\tau))} \subseteq D_N = \bigcup_{j=0}^{N-1} (D_{j+1} \setminus D_j) \cup D_0$$

by the definition of D_j it is clear that $B(x_2, r(\tau))$ can only have nonempty intersection with at most two elements in the set

$$\{D_0, D_{j+1} \setminus D_j, \text{ for } j = 0, 1, \dots, N-1\},$$

therefore we have

$$\|f'_\tau \chi_{B(x_2, r(\tau))}\|_q \leq \alpha_{j_1} + \alpha_{j_2} \quad (2.55)$$

for some $j_1, j_2 \in \{-1, 0, 1, \dots, N\}$. Then by the definitions of $\alpha, \bar{\alpha}, \beta$, we have for any $J \in \{0, 1, \dots, N\}$

$$\begin{cases} \alpha_j \leq \bar{\alpha}_J & \text{if } J \geq j \\ \alpha_j \leq \beta_J & \text{if } J \leq j \end{cases} \quad j = j_1, j_2 \quad (2.56)$$

so that by combining (2.54), (2.55) and (2.56), we have

$$W(\tau, x_2) \leq C \bar{\alpha}_J + C \beta_J \quad (2.57)$$

where $C = C(n, \alpha, K)$. Next, recall that

$$M(\tau, x) = |T(f'_\tau \chi_{B^c(x, r(\tau))})(x)|. \quad (2.58)$$

By (2.51) and (2.52), we can write, for any $J \in \{0, 1, \dots, N\}$

$$\begin{aligned} M(\tau, x_1) &= |TS_0 f'_\tau(x_1)| = TS_0 f'_\tau(x_1) \\ &= \sum_{j=0}^J (TS_j f'_\tau(x_1) - TS_{j+1} f'_\tau(x_1)) + TS_{J+1} f'_\tau(x_1) \\ &= \sum_{j=0}^J T(S_j f'_\tau - S_{j+1} f'_\tau)(x_1) + TS_{J+1} f'_\tau(x_1). \end{aligned}$$

Next, for any integer j , we have the estimate

$$\begin{aligned} T(S_j f'_\tau - S_{j+1} f'_\tau)(x_1) &\leq |T(S_j f'_\tau - S_{j+1} f'_\tau)(x_1)| \\ &\leq \left(\int_{r_j \leq |y| < r_{j+1}} |K(y)|^{q'} dy \right)^{1/q'} \|S_j f'_\tau - S_{j+1} f'_\tau\|_q. \end{aligned} \quad (2.59)$$

Using (A1), (A4) and the inequality $(a + b)^\beta \leq a^\beta + \beta 2^{\beta-1} (a^{\beta-1} b + b^\beta)$ for $\beta > 1$ we get

$$|K(y)|^{q'} \leq |g(y^*)|^{q'} |y|^{-n} + C \min\{|y|^{-n+\delta_1}, |y|^{-n-\delta_2}\} \quad (2.60)$$

for some $C > 0$, $C = C(n, \alpha, H_1, H_2, B, \delta_1, \delta_2)$. Since $r_{j+1} = e^{\frac{\alpha}{n}} r_j$,

$$\begin{aligned} T(S_j f'_\tau - S_{j+1} f'_\tau)(x_1) &\leq (qA_g + C \min\{r_j^{\delta_1}, r_j^{-\delta_2}\})^{1/q'} \alpha_j \\ &\leq \left(q^{\frac{1}{q'}} A_g^{\frac{1}{q'}} + C \min\{r_j^{\delta_1/q'}, r_j^{-\delta_2/q'}\} \right) \alpha_j. \end{aligned} \quad (2.61)$$

Using (2.61), we then get that

$$\begin{aligned}
M(\tau, x_1) &\leq \sum_{j=0}^J \left(q^{\frac{1}{q'}} A_g^{\frac{1}{q'}} + C \min\{r_j^{\delta_1/q'}, r_j^{-\delta_2/q'}\} \right) \alpha_j + TS_{J+1} f'_\tau(x_1) \\
&= q^{\frac{1}{q'}} A_g^{\frac{1}{q'}} \sum_{j=0}^J \alpha_j + C \bar{\alpha}_J \sum_{j=0}^{\infty} \min\{r_j^{\delta_1/q'}, r_j^{-\delta_2/q'}\} + TS_{J+1} f'_\tau(x_1) \\
&\leq q^{\frac{1}{q'}} A_g^{\frac{1}{q'}} \sum_{j=0}^J \alpha_j + C \bar{\alpha}_J \sum_{j=0}^{\infty} (e^{-\frac{q}{n} \frac{\delta_1}{q'} j} + e^{-\frac{q}{n} \frac{\delta_2}{q'} j}) + TS_{J+1} f'_\tau(x_1) \\
&= q^{\frac{1}{q'}} A_g^{\frac{1}{q'}} \sum_{j=0}^J \alpha_j + C \bar{\alpha}_J + TS_{J+1} f'_\tau(x_1).
\end{aligned} \tag{2.62}$$

Note that (2.56) and (2.62) are true for any $J \in \{0, 1, \dots, N\}$. The main task now is prove that there exists $J \in \{0, 1, \dots, N\}$ such that

$$TS_{J+1} f'_\tau(x_1) \leq C \bar{\alpha}_J + C \beta_J. \tag{2.63}$$

This will be effected by a double stopping time argument, which will simultaneously yield (2.41) in Proposition 1.

Recall that N is an integer such that $\text{supp} f \subseteq D_N$. Let $J_1 \in \{1, \dots, N\}$ be such that

$$\left\{ \beta_{j+1}^q \leq \left(\int_{D_{j+1} \setminus D_j} |Tf(x)| dx \right)^q \quad \text{for } j = 0, \dots, J_1 - 1 \right. \tag{2.64}$$

$$\left. \left\{ \beta_{J_1+1}^q > \left(\int_{D_{J_1+1} \setminus D_{J_1}} |Tf(x)| dx \right)^q \right. \right. \tag{2.65}$$

If condition (2.64) is never satisfied we let $J_1 = 0$, and if (2.65) is never satisfied let $J_1 = N + 1$. Next, let $J_2 \in \{1, \dots, N\}$ be such that

$$\left\{ TS_{j+1} f'_\tau(x_1) \geq \left(\frac{e^{q-1} + 1}{2e^{q-1}} \right) TS_j f'_\tau(x_1) \quad \text{for } j = 0, \dots, J_2 - 1 \right. \tag{2.66}$$

$$\left. \left\{ TS_{J_2+1} f'_\tau(x_1) < \left(\frac{e^{q-1} + 1}{2e^{q-1}} \right) TS_{J_2} f'_\tau(x_1) \right. \right. \tag{2.67}$$

As in the definition of J_1 , we let $J_2 = 0$ if condition (2.66) is never satisfied, and let $J_2 = N + 1$ if (2.67) is never satisfied.

We will first prove (2.63), and hence (2.40), in three cases depending on J_1, J_2 , then we will show that (2.41) holds with the chosen J in each case.

Case 1: $J_2 \leq J_1 \leq N + 1$ and $J_2 \neq N + 1$.

Case 2: $J_2 \geq J_1 + 1$.

Case 3: $J_1 = J_2 = N + 1$.

Proof of (2.63) in the case $J_2 \leq J_1 \leq N + 1$ and $J_2 \neq N + 1$:

In this case, by (2.67) we have

$$\begin{aligned}
TS_{J_2+1}f'_\tau(x_1) &< \left(\frac{e^{q-1}+1}{2e^{q-1}}\right) TS_{J_2}f'_\tau(x_1) \\
&= \left(\frac{e^{q-1}+1}{2e^{q-1}}\right) (TS_{J_2+1}f'_\tau(x_1) + T(S_{J_2} - S_{J_2+1})f'_\tau(x_1)) \\
&\leq \left(\frac{e^{q-1}+1}{2e^{q-1}}\right) (TS_{J_2+1}f'_\tau(x_1) + |T(S_{J_2} - S_{J_2+1})f'_\tau(x_1)|) \\
&\leq \left(\frac{e^{q-1}+1}{2e^{q-1}}\right) (TS_{J_2+1}f'_\tau(x_1) + C\alpha_{J_2})
\end{aligned} \tag{2.68}$$

where the last inequality is by (2.61). So we have

$$TS_{J_2+1}f'_\tau(x_1) < C \left(\frac{e^{q-1}+1}{2e^{q-1}}\right) \alpha_{J_2} = C\alpha_{J_2}. \tag{2.69}$$

Hence by taking $J = J_2$ in (2.62), we obtain

$$\begin{aligned}
q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) &\leq \sum_{j=0}^{J_2} \alpha_j + C\bar{\alpha}_{J_2} + C\beta_{J_2} + \tilde{T}S_{J_2+1}f'_\tau(x_1) \\
&\leq \sum_{j=0}^{J_2} \alpha_j + C\bar{\alpha}_{J_2} + C\beta_{J_2} + C\alpha_{J_2} \\
&\leq \sum_{j=0}^{J_2} \alpha_j + C\bar{\alpha}_{J_2} + C\beta_{J_2}.
\end{aligned} \tag{2.70}$$

Therefore we get (2.40) with $J = J_2$ and $C_2 = C$ in the above inequality.

Proof of (2.63) in the case $J_2 \geq J_1 + 1$:

We will need the following lemma to handle this case. Let us state it here, and its proof will be postponed to the Appendix.

Lemma 6. *There is a constant $C_1 = C_1(n, \alpha, K)$ such that for any $J \leq N - 1$*

$$\int_{D_{J+1} \setminus D_J} |Tf_\tau(x)| dx \leq C_1 \left(\frac{1}{e^{q-1}} \right)^J, \quad (2.71)$$

$$\left| \int_{D_{J+1} \setminus D_J} TS_{J+2} f'_\tau(x) dx - TS_{J+1} f'_\tau(x_1) \right| \leq C_1 \beta_{J+1}, \quad (2.72)$$

$$\left| \int_{D_{J+1} \setminus D_J} T(S_0 - S_{J+2}) f'_\tau(x) dx \right| \leq C_1 \bar{\alpha}_{J+1}, \quad (2.73)$$

and

$$\left| \int_{D_{J+1} \setminus D_J} T(f'_\tau \chi_{D_0})(x) dx \right| \leq C_1 \alpha_{-1}. \quad (2.74)$$

Assuming Lemma 6, let us first make a reduction. Recall that $0 < \tau \leq t_0$. We will assume that

$$M(\tau, x_1) \geq \max\{4C_1, 1\}. \quad (2.75)$$

where C_1 is the constant which is defined in Lemma 6. If the above is not true, then we have that $M(\tau, x_1) \leq C$, and on the other hand, by (2.54)

$$W(\tau, x_2) \leq C \|f'_\tau \chi_{B(x_2, r(\tau))}\|_q \leq C. \quad (2.76)$$

Therefore, $W(\tau, x_2) + M(\tau, x_1) \leq C$, and hence

$$\frac{\exp \left[\frac{(1 - \epsilon_\tau)^{-\frac{\alpha}{n-\alpha}}}{A_g} (W(\tau, x_2) + M(\tau, x_1))^{\frac{n}{n-\alpha}} \right]}{1 + (W(\tau, x_2) + M(\tau, x_1))^{\frac{n}{n-\alpha}}} \leq C = C \frac{\tau}{\tau} \leq C \frac{\|Tf\|_q^q}{\tau} \quad (2.77)$$

which is (2.33), and the last inequality is by (2.32).

By (2.72) in Lemma 6 and recalling that

$$f = f_\tau + f'_\tau = f_\tau + f'_\tau \chi_{D_0} + f'_\tau \chi_{D_{J_1+2} \setminus D_0} + f'_\tau \chi_{D_{J_1+2}^c},$$

we have

$$\begin{aligned} TS_{J_1+1} f'_\tau(x_1) &\leq \int_{D_{J_1+1} \setminus D_{J_1}} TS_{J_1+2} f'_\tau(x) dx + C\beta_{J_1+1} \\ &\leq \left| \int_{D_{J_1+1} \setminus D_{J_1}} TS_{J_1+2} f'_\tau(x) dx \right| + C\beta_{J_1+1} \\ &= \left| \int_{D_{J_1+1} \setminus D_{J_1}} (Tf - Tf_\tau - T(f'_\tau \chi_{D_0}) - T(S_0 - S_{J_1+2})f'_\tau)(x) dx \right| + C\beta_{J_1+1} \\ &\leq \left| \int_{D_{J_1+1} \setminus D_{J_1}} Tf(x) dx \right| + \left| \int_{D_{J_1+1} \setminus D_{J_1}} Tf_\tau(x) dx \right| + \left| \int_{D_{J_1+1} \setminus D_{J_1}} T(f'_\tau \chi_{D_0})(x) dx \right| \\ &\quad + \left| \int_{D_{J_1+1} \setminus D_{J_1}} T(S_0 - S_{J_1+2})f'_\tau(x) dx \right| + C\beta_{J_1+1} \\ &\leq \int_{D_{J_1+1} \setminus D_{J_1}} |Tf(x)| dx + \int_{D_{J_1+1} \setminus D_{J_1}} |Tf_\tau(x)| dx + C\alpha_{-1} + C\bar{\alpha}_{J_1+1} + C\beta_{J_1+1} \end{aligned} \quad (2.78)$$

where the second last inequality is by Lemma 6 (2.73), (2.74). To estimate the second integral, note first that by reduction (2.75) we have

$$M(\tau, x_1) = TS_0 f'_\tau(x_1) \geq 4C_1. \quad (2.79)$$

Using (2.71) in Lemma 6, and condition (2.66), we get

$$\begin{aligned}
\int_{D_{J_1+1} \setminus D_{J_1}} |Tf_\tau(x)| dx &\leq C_1 \left(\frac{1}{e^{q-1}} \right)^{J_1} \leq \frac{1}{4} \left(\frac{1}{e^{q-1}} \right)^{J_1} TS_0 f'_\tau(x_1) \\
&\leq \frac{1}{4} \left(\frac{1}{e^{q-1}} \right)^{J_1} \left(\frac{2e^{q-1}}{e^{q-1} + 1} \right)^{J_1+1} TS_{J_1+1} f'_\tau(x_1) \\
&= \frac{1}{4} \left(\frac{2}{e^{q-1} + 1} \right)^{J_1} \left(\frac{2e^{q-1}}{e^{q-1} + 1} \right) TS_{J_1+1} f'_\tau(x_1) \leq \frac{1}{2} TS_{J_1+1} f'_\tau(x_1).
\end{aligned} \tag{2.80}$$

Hence we have

$$TS_{J_1+1} f'_\tau(x_1) \leq \int_{D_{J_1+1} \setminus D_{J_1}} |Tf(x)| dx + \frac{1}{2} TS_{J_1+1} f'_\tau(x_1) + C\bar{\alpha}_{J_1+1} + C\beta_{J_1+1}.$$

So the above inequality along with the condition (2.65) give us

$$\begin{aligned}
TS_{J_1+1} f'_\tau(x_1) &\leq 2 \int_{D_{J_1+1} \setminus D_{J_1}} |Tf(x)| dx + 2C\bar{\alpha}_{J_1+1} + 2C\beta_{J_1+1} \\
&\leq 2C\bar{\alpha}_{J_1+1} + (2C + 2)\beta_{J_1+1}.
\end{aligned} \tag{2.81}$$

By taking $J = J_1$ in (2.62), we get

$$\begin{aligned}
q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) &\leq \sum_{j=0}^{J_1} \alpha_j + C\bar{\alpha}_{J_1} + C\beta_{J_1} + TS_{J_1+1} f'_\tau(x_1) \\
&\leq \sum_{j=0}^{J_1} \alpha_j + C\bar{\alpha}_{J_1+1} + C\beta_{J_1+1} \\
&\leq \sum_{j=0}^{J_1} \alpha_j + C\bar{\alpha}_{J_1} + C\beta_{J_1}
\end{aligned} \tag{2.82}$$

where the last inequality is by the fact that $\bar{\alpha}_{J+1} \leq \bar{\alpha}_J + \alpha_{J+1} \leq \bar{\alpha}_J + \beta_{J+1} \leq \bar{\alpha}_J + \beta_J$.

Therefore we get (2.40) with $J = J_1$.

Proof of (2.63) in the case $J_1 = J_2 = N + 1$:

In this case, we will simply write the entire series, that is, we will take $J = N$.

Since $TS_{N+1}f'_z(x_1) = 0$ we have

$$\begin{aligned} q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) &\leq \sum_{j=0}^N \alpha_j + C\bar{\alpha}_N + C\beta_N + TS_{N+1}f'_\tau(x_1) \\ &\leq \sum_{j=0}^N \alpha_j + C\bar{\alpha}_N. \end{aligned} \quad (2.83)$$

To check (2.41), note that we take $J = J_2$ in case 1, $J = J_1$ in case 2 and $J = N$ in case 3. Assume first that $J_1 \neq 0$. Then in all the cases we have that (2.64) is true for all $j \leq J_1 - 1$, so

$$\begin{aligned} \sum_{j=0}^J \alpha_j^q e^{qj} + \beta_J^q e^{qJ} &\leq 3e^{2q} \sum_{j=0}^{J_1-1} \beta_j^q e^{qj} \leq C \sum_{j=0}^{J_1-1} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)| dx \right)^q e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} \left(\frac{1}{r_j^n} \int_{D_{j+1} \setminus D_j} |Tf(x)| dx \right)^q e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} r_j^{-qn} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)|^q dx \right) \left(\int_{D_{j+1} \setminus D_j} dx \right)^{q/q'} e^{qj} \\ &= C \sum_{j=0}^{J_1-1} r_j^{-qn} r_j^{(n-\alpha)q} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)|^q dx \right) e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} r_j^{-n} e^{qj} \int_{D_{j+1} \setminus D_j} |Tf(x)|^q dx \leq \frac{C}{\tau} \|Tf\|_q^q \end{aligned} \quad (2.84)$$

where in the last inequality we used the fact that $r_j^n = r_0^n e^{qj}$ and $\tau = |B(x_1, r_0)|$, so $\tau = Cr_0^n$.

If $J_1 = 0$, then we just need to check (2.41) for $J = 0$:

$$\alpha_0^q + \beta_0^q \leq 2\beta_0^q \leq C = C \frac{\tau}{\tau} \leq \frac{C}{\tau} \|Tf\|_q^q, \quad (2.85)$$

where the last inequality is by (2.32). Proposition 1 is proved. $\square$

2.4 Proof of the inequalities in Theorem 2

First notice that we choose $[\beta q - \beta - 1]$, where $q = n/\alpha$, in our regularized exponential integral (1.27). So by the same argument as in Lemma B, we have that (1.27) holds if and only if the following is true:

$$\int_{|Tf| \geq 1} \frac{\exp \left[\frac{1}{A_g} |Tf|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{\beta n}{n-\alpha}}} dx \leq C \|Tf\|_{\beta n/\alpha}^{\beta n/\alpha}. \quad (2.86)$$

Then the rest of the proof is essentially the same as the proof of Theorem 1, except that we need to show that Lemma 4 still holds for $\beta \neq 1$. That is, we want to show the following

$$\frac{\exp \left[\frac{(1 - \epsilon_\tau)^{-\frac{\alpha}{n-\alpha}}}{A_g} (W(\tau, x_2) + M(\tau, x_1))^{\frac{n}{n-\alpha}} \right]}{1 + (W(\tau, x_2) + M(\tau, x_1))^{\frac{\beta n}{n-\alpha}}} \leq \frac{C}{\tau} \|Tf\|_{\beta n/\alpha}^{\beta n/\alpha}. \quad (2.87)$$

But only small variations will be made, we will point out the places that needs to be modified.

From the optimal growth Lemma 5, we conclude that for any $q > 1$ and any $\mu > 0$, $h > 1$ there is $C = C(q)$ such that for any sequence $\{a_k\}$ satisfying

$$\sum_{k=0}^{\infty} |a_k| = h \quad \sum_{k=0}^{\infty} |a_k|^q \leq \mu \quad (2.88)$$

we have

$$\frac{\exp [q\mu^{1-q'} h^{q'}]}{h^{\beta q'}} \leq C \mu^{-\beta q'} \sum_{k=0}^{\infty} |a_k|^{\beta q} e^{qk}. \quad (2.89)$$

The next task is to find a number h_1 , depending on f , and a sequence $a = \{a_k\}$, also depending on f , such that

$$q^{-\frac{1}{q'}} A_g^{-\frac{1}{q'}} (W(\tau, x_2) + M(\tau, x_1)) \leq h_1 \quad (2.90)$$

$$\sum_{k=0}^{\infty} |a_k| = h_1, \quad \sum_{k=0}^{\infty} |a_k|^q \leq 1 - \epsilon_\tau, \quad \sum_{k=0}^{\infty} |a_k|^{\beta q} e^{qk} \leq \frac{C}{\tau} \|Tf\|_{\beta q}^{\beta q}. \quad (2.91)$$

Clearly (2.87) follows from (2.88)-(2.91), with $\mu = 1 - \epsilon_\tau \geq \frac{3}{4}$ and $h = h_1$. The choice of the sequence a as well as the integer N_i is the same, we just need to use the βq -th norms for Tf and a_k , whenever they appear in (2.49), (2.50).

The analogue of Proposition 1 still hold, after following the same set up we just need to make slight changes to (2.84) and (2.85). All those modifications are completely similar, we simply use βq , and $(\beta q)'$ instead of q, q' when applying Holder's inequality. To make it more clear, we modify (2.84) as follows:

$$\begin{aligned} \sum_{j=0}^J \alpha_j^{\beta q} e^{qj} + \beta_j^{\beta q} e^{qJ} &\leq 3e^{2\beta q} \sum_{j=0}^{J_1-1} \beta_j^{\beta q} e^{qj} \leq C \sum_{j=0}^{J_1-1} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)| dx \right)^{\beta q} e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} \left(\frac{1}{r_j^n} \int_{D_{j+1} \setminus D_j} |Tf(x)| dx \right)^{\beta q} e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} r_j^{-\beta q n} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)|^{\beta q} dx \right) \left(\int_{D_{j+1} \setminus D_j} dx \right)^{\beta q - 1} e^{qj} \\ &= C \sum_{j=0}^{J_1-1} r_j^{-\beta q n} r_j^{(\beta q - 1)n} \left(\int_{D_{j+1} \setminus D_j} |Tf(x)|^{\beta q} dx \right) e^{qj} \\ &\leq C \sum_{j=0}^{J_1-1} r_j^{-n} e^{qj} \int_{D_{j+1} \setminus D_j} |Tf(x)|^{\beta q} dx \leq \frac{C}{\tau} \|Tf\|_{\beta q}^{\beta q}. \end{aligned} \quad (2.92)$$

2.5 Proof of sharpness statements in Theorems 1 and 2.

Since Theorem 2 includes Theorem 1 as a special case, we only need to prove the sharpness of Theorem 2. We will use the same family of extremal functions as in [FM2, Section 6], with some suitable modifications. Notice that we have two types of sharpness, so firstly let us clarify what we want to show for each case. In the sense of

exponential constant, we will show that the constant A_g is best possible in inequality (1.27) of Theorem 2. By the exponential regularization Lemma B, it would be enough to find a family of compactly supported functions $\psi_{\epsilon,r} \in L^q(\mathbb{R}^n)$ with

$$|T\psi_{\epsilon,r}(x)| \geq 1 \quad \text{for } x \in B(0, \epsilon r/2) =: B_{\epsilon r/2} \quad (2.93)$$

such that:

For Theorem 2 (in terms of exponential constant):

$$\max\{\|\psi_{\epsilon,r}\|_q^q, \|T\psi_{\epsilon,r}\|_{\beta q}^{\beta q}\} \leq 1 \quad (2.94)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \int_{B_{\epsilon r/2}} \frac{\exp\left[\frac{\theta}{A_g} |T\psi_{\epsilon,r}(x)|^{\frac{n}{n-\alpha}}\right]}{1 + |T\psi_{\epsilon,r}(x)|^{\frac{\beta n}{n-\alpha}}} dx = \infty \quad (2.95)$$

if $\theta > 1$.

For Theorem 2 (in terms of power of denominator):

$$\|\psi_{\epsilon,r}\|_q^q \leq 1 \quad (2.96)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \|T\psi_{\epsilon,r}\|_{\beta q}^{-\beta q} \int_{B_{\epsilon r/2}} \frac{\exp\left[\frac{1}{A_g} |T\psi_{\epsilon,r}(x)|^{\frac{n}{n-\alpha}}\right]}{1 + |T\psi_{\epsilon,r}(x)|^{\frac{\theta \beta n}{n-\alpha}}} dx = \infty \quad (2.97)$$

if $\theta < 1$.

For clarity we will consider the scalar case first, then the modifications in the vector-valued case will be shown after the proof of scalar case. To begin with, assume that K is n -regular and $|y| \leq \frac{1}{2}|x|$, $|x| \neq 0$. From Taylor's formula centered at x , for any positive integer $m \leq n$, there exists $\theta = \theta(x, y, m) \in (0, 1)$ such that

$$K(x - y) = \sum_{j=0}^{m-1} p_j(x, y) + R_m(x, y) \quad (2.98)$$

where

$$\begin{cases} p_j(x, y) = \frac{1}{j!} d^j K(x, -y) & j = 0, 1, \dots, m-1 \\ R_m(x, y) = \frac{1}{m!} d^m K(x - \theta y, -y) \end{cases}$$

and where $d^j K(x, y)$ denotes the j -th differential of K at x with increment y , i.e.

$$d^j K(x, y) = \sum_{j_1 + \dots + j_n = j} \frac{j!}{j_1! \dots j_n!} \frac{\partial^j K}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x) y_1^{j_1} \dots y_n^{j_n}.$$

Note that $p_j(x, y)$ is a polynomial of order j in the y variable for $j \leq m-1$. For $x \neq 0$ we have

$$\begin{cases} |p_j(x, y)| \leq C|x|^{\alpha-n-j}|y|^j & \text{for } j \leq m-1, y \in \mathbb{R}^n, \\ |R_m(x, y)| \leq C|x|^{\alpha-n-m}|y|^m & \text{for } |y| \leq \frac{1}{2}|x| \end{cases} \quad (2.99)$$

for some constant C independent of y , x in the given range.

Now let $\mathcal{P}_m$ be the space of polynomials of degree up to m in the unit ball B_1 of $\mathbb{R}^n$.

Let $\{v_1, \dots, v_N\}$ be an orthonormal basis of $\mathcal{P}_m$, with $v_1 = |B_1|^{-1/2}$. If P_m denotes the projection of $L^2(B_1)$ onto $\mathcal{P}_m$, then P_m has the integral kernel

$$P_m(y, z) = \chi_{B_1}(y) \sum_{k=1}^N v_k(y) v_k(z). \quad (2.100)$$

For $r > 0$ the projection on the space of polynomials of degree up to m on the ball $B_r = B(0, r)$ is given by the integral kernel

$$P_m^r(y, z) = r^{-n} P_m\left(\frac{y}{r}, \frac{z}{r}\right).$$

For $0 \leq \epsilon r < 1 \leq r$ let

$$\phi_{\epsilon,r}(y) = \begin{cases} K(-y)|K(-y)|^{\frac{\alpha}{n-\alpha}-1} & \text{if } \epsilon r < |y| \leq r \\ 0 & \text{otherwise} \end{cases} \quad (2.101)$$

and consider the functions on B_r

$$\tilde{\phi}_{\epsilon,r} = \phi_{\epsilon,r} - P_n^r \phi_{\epsilon,r} \quad (2.102)$$

which are orthogonal to every polynomial of order up to n on the ball B_r . By assumption (A3), we have $\|\phi_{\epsilon,r}\|_1 \leq Cr^{n-\alpha}$, so $|P_n^r \phi_{\epsilon,r}| \leq Cr^{-\alpha}$ and

$$|\tilde{\phi}_{\epsilon,r}| \leq |K(y)|^{\frac{\alpha}{n-\alpha}} + Cr^{-\alpha}, \quad \epsilon r < |y| \leq r.$$

By the inequality $(a+b)^p \beta a^\beta + \beta 2^{\beta-1}(a^{\beta-1}b + b^\beta)$ we have

$$|\tilde{\phi}_{\epsilon,r}|^q \leq |K(y)|^{q'} + Cr^{-\alpha}|K(y)| + Cr^{-n}, \quad \epsilon r < |y| \leq r.$$

When $\epsilon r < |y| \leq 1$ we use (A1) and (A4) to obtain

$$\begin{aligned} |\tilde{\phi}_{\epsilon,r}|^q &\leq |K(y)|^{q'} + C|K(y)| + C \\ &\leq |y|^{-n}(|g(y^*)| + C|y|^\delta)^{q'} + C|y|^{\alpha-n}(|g(y^*)| + C|y|^\delta) + C \\ &\leq |g(y^*)|^{q'}|y|^{-n} + C(|y|^{-n+\delta} + |y|^{-n+q'\delta} + |y|^{\alpha-n} + 1) \\ &= |g(y^*)|^{q'}|y|^{-n} + \Phi_0(y) \end{aligned}$$

where $\Phi_0 \in L^1(|y| \leq 1)$. Hence

$$\begin{aligned} \|\tilde{\phi}_{\epsilon,r}\|_q^q &\leq \int_{\epsilon r < |y| \leq 1} (|g(y^*)|^{q'}|y|^{-n} + \Phi_0(y)) dy + \int_{1 \leq |y| \leq r} |K(y)|^{q'} dy \\ &\quad + C \int_{1 \leq |y| \leq r} (r^{-\alpha}|y|^{\alpha-n} + r^{-n}) dy = A_g \log \frac{1}{(\epsilon r)^n} + b_r + C \end{aligned} \quad (2.103)$$

where

$$b_r := \int_{1 \leq |y| \leq r} |K(y)|^{q'} dy.$$

Note that under the assumptions (A1), (A4),

$$b_r \leq A_g \log r^n + C. \quad (2.104)$$

On the other hand, applying the inequality

$$|a - b|^d \geq |a|^d - d|a|^{d-1}|b|, \quad \forall a, b \in \mathbb{R}, \quad d \geq 1$$

to

$$|K(y)|^{q'} \geq ||K_0(y)| - |K(y) - K_0(y)||^{q'},$$

with $K_0(y) = g(y^*)|y|^{\alpha-n}$, we get

$$|K(y)|^{q'} \geq |g(y^*)|^{q'} |y|^{-n} - \Psi_0(y), \quad \epsilon r < |y| \leq 1$$

where $\Psi_0(y) \in L^1(|y| \leq 1)$. Therefore we also have the estimate from below

$$\int_{\epsilon r \leq |y| \leq r} |K(y)|^{q'} dy = \int_{\epsilon r \leq |y| \leq 1} |K(y)|^{q'} dy + b_r \geq A_g \log \frac{1}{(\epsilon r)^n} - C + b_r. \quad (2.105)$$

Next let us estimate $||T\tilde{\phi}_{\epsilon,r}||_{\beta q}$. If $|x| \geq 2r$, since $\tilde{\phi}_\epsilon$ is orthogonal to p_j for $j \leq n-1$, by (2.98) we get

$$T\tilde{\phi}_{\epsilon,r}(x) = \int_{B_r} \tilde{\phi}_{\epsilon,r}(y) R_n(x, y) dy.$$

Using (2.99) and $\beta q \geq 1$ we have

$$|T\tilde{\phi}_{\epsilon,r}(x)| \leq \int_{B_r} C|y|^{-\alpha} |x|^{\alpha-2n} |y|^n dy \leq C|x|^{\alpha-2n} r^{2n-\alpha}, \quad (2.106)$$

and therefore

$$\int_{|x| \geq 2r} |T\tilde{\phi}_{\epsilon,r}(x)|^{\beta q} \leq Cr^{(2n-\alpha)\beta q} \int_{|x| \geq 2r} |x|^{(\alpha-2n)\beta q} \leq Cr^{(2n-\alpha)\beta q} r^{(\alpha-2n)\beta q+n} = Cr^n. \quad (2.107)$$

For the case $|x| \leq 2r$ we have

$$|TP_n^r \phi_{\epsilon,r}(x)| \leq Cr^{-\alpha} \int_{|z-x| \leq r} |z|^{\alpha-n} dz \leq Cr^{-\alpha} \int_{|z| \leq 4r} |z|^{\alpha-n} dx = C$$

$$|T\tilde{\phi}_{\epsilon,r}(x)| \leq \int_{|y| \leq r} |x-y|^{\alpha-n} |y|^{-\alpha} + C = \int_{|y| \leq 1} \left| \frac{x}{r} - y \right|^{\alpha-n} |y|^{-\alpha} + C := I_\alpha \phi_0 \left(\frac{x}{r} \right) + C.$$

where $\phi_0(y) = |y|^{-\alpha} \chi_{|y| \leq r}$ is in $L^\tau(\mathbb{R}^n)$ if $1 < \tau < q$. Take $\tau > 1$ such that

$$\frac{q\beta}{\beta+1} \leq \tau < q$$

so that $(\tau^{-1} - q^{-1})^{-1} > \beta q$, which gives us $I_\alpha \phi_0 \in L^{\beta q}(|x| \leq 2)$ by the Hardy-Littlewood-Sobolev theorem. Hence

$$\int_{|x| \leq 2r} |T\tilde{\phi}_{\epsilon,r}(x)|^{\beta q} dx \leq Cr^n \int_{|x| \leq 2} |I_\alpha \phi_0(x) + C|^{\beta q} dx = Cr^n$$

And

$$\|T\tilde{\phi}_{\epsilon,r}\|_{\beta q}^{\beta q} \leq C_4 r^n \quad (2.108)$$

with C_4 independent of ϵ, r . Note that by taking $\beta = 1$ in (2.108), we have

$$\|T\tilde{\phi}_{\epsilon,r}\|_q^q \leq C_4 r^n. \quad (2.109)$$

Let

$$b_{\epsilon,r} = A_g \log \frac{1}{(\epsilon r)^n} + b_r \quad (2.110)$$

and

$$1 \leq r^{n/\beta} \leq \frac{A_g}{2C_4^{1/\beta}} \left(\log \frac{1}{\epsilon^n} \right). \quad (2.111)$$

Note that for ϵ small enough, by the choice of r above, we have

$$b_{\epsilon,r} \geq \frac{1}{2} A_g \log \frac{1}{\epsilon^n}, \quad (2.112)$$

so by the choice of r in (2.111) we have

$$\max\{\|\tilde{\phi}_{\epsilon,r}\|_q, \|T\tilde{\phi}_{\epsilon,r}\|_{\beta q}\} \leq (\max\{b_{\epsilon,r} + C, C_4 r^{n/\beta}\})^{1/q} \leq b_{\epsilon,r}^{1/q} \left(1 + \frac{C}{\log \frac{1}{\epsilon^n}} \right). \quad (2.113)$$

For $|x| \leq \epsilon r/2$ we estimate the potential from below using (2.105),

$$\begin{aligned} & |T\tilde{\phi}_{\epsilon,r}(x)| \\ &= \left| \int_{\epsilon r \leq |y| \leq r} |K(-y)|^{q'} dy + \int_{\epsilon r \leq |y| \leq r} (K(x-y) - K(-y))K(-y)|K(-y)|^{\frac{\alpha}{n-\alpha}-1} dy \right| \\ &\geq \int_{\epsilon r \leq |y| \leq r} - \int_{\epsilon r \leq |y| \leq r} |K(x-y) - K(-y)| |K(-y)|^{\frac{\alpha}{n-\alpha}} dy \\ &\geq A_g \log \frac{1}{(\epsilon r)^n} - C + b_r - C \int_{|y| \geq \epsilon r} |x||y|^{-n-1} dy \geq b_{\epsilon,r} - C \geq b_{\epsilon,r} \left(1 - \frac{C}{\log \frac{1}{\epsilon^n}} \right). \end{aligned} \quad (2.114)$$

Now we can define

$$\psi_{\epsilon,r} = \frac{\tilde{\phi}_{\epsilon,r}}{\max\{\|\tilde{\phi}_{\epsilon,r}\|_q, \|T\tilde{\phi}_{\epsilon,r}\|_{\beta q}\}}, \quad (2.115)$$

and combining (2.113) and (2.114) we get for $|x| \leq \epsilon r/2$,

$$\begin{aligned} |T\psi_{\epsilon,r}(x)|^{q'} &\geq \left(\frac{b_{\epsilon,r}(1 - C/\log \frac{1}{\epsilon^n})}{b_{\epsilon,r}^{1/q}(1 + C/\log \frac{1}{\epsilon^n})} \right)^{q'} \geq b_{\epsilon,r} \left(1 - \frac{C}{\log \frac{1}{\epsilon^n}} \right) \\ &= A_g \log \frac{1}{(\epsilon r)^n} + b_r \left(1 - \frac{C}{\log \frac{1}{\epsilon^n}} \right) - C. \end{aligned} \quad (2.116)$$

By (2.105) and (2.112) we also have

$$||\tilde{\phi}_{\epsilon,r}||_q^q \geq Cb_{\epsilon,r} - C \geq C \log \frac{1}{\epsilon^n},$$

hence

$$||T\psi_{\epsilon,r}||_{\beta q}^{\beta q} \leq Cr^n (\log \frac{1}{\epsilon^n})^{-\beta}. \quad (2.117)$$

Note that the condition (2.93) is satisfied since ϵ is small.

For Theorem 2 (exponential constant):

Take $\theta > 1$ and choose

$$r^{n/\beta} = \frac{A_g}{2C_4^{1/\beta}} \left(\log \frac{1}{\epsilon^n} \right).$$

We estimate the integral in (2.95) to get

$$\begin{aligned} \int_{B_{\epsilon r/2}} \frac{\exp \left[\frac{\theta}{A_g} |T\psi_{\epsilon,r}(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |T\psi_{\epsilon,r}(x)|^{\frac{\beta n}{n-\alpha}}} dx &\geq |B_{\epsilon r/2}| \frac{\exp \left[\theta \log \frac{1}{(\epsilon r)^n} + \frac{\theta b_r}{A_g} \left(1 - \frac{C}{\log \frac{1}{\epsilon^n}} \right) - C \right]}{1 + (A_g \log \frac{1}{(\epsilon r)^n} + Cb_r - C)^\beta} \\ &\geq \frac{C(\epsilon r)^{-(\theta-1)n}}{\left(\log \frac{1}{(\epsilon r)^n} \right)^\beta} \rightarrow \infty \end{aligned} \quad (2.118)$$

where the last inequality is by the estimate of b_r in (2.104).

For Theorem 2 (power of denominator):

Taking $r = 1$, we have $b_r = 0$. Let $\theta < 1$, we have that the integral

$$\begin{aligned} \int_{B_{\epsilon/2}} \frac{\exp \left[\frac{1}{A_g} |T\psi_{\epsilon,1}(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |T\psi_{\epsilon,1}(x)|^{\frac{\theta \beta n}{n-\alpha}}} dx &\geq C\epsilon^n \frac{\exp \left[\log \frac{1}{\epsilon^n} - C \right]}{1 + (A_g \log \frac{1}{\epsilon^n} - C)^{\theta \beta}} \\ &\geq C \frac{1}{1 + (\log \frac{1}{\epsilon^n})^{\theta \beta}} \geq C (\log \frac{1}{\epsilon^n})^{-\theta \beta}. \end{aligned} \quad (2.119)$$

Therefore by the estimation (2.117) on the βq -th norm of $T\psi_{\epsilon,1}$, we have (2.97) goes

to infinity as $\epsilon \rightarrow 0^+$.

For the vector case the proof of sharpness is almost the same. First write the Taylor expansion as in (2.98) but each p_j is a vector-valued polynomial, with each components correspond to Taylor formula of each $K_i(x - y)$, $1 \leq i \leq m$. Then define the functions $\phi_{\epsilon,r}$, $\tilde{\phi}_{\epsilon,r}$ and $\psi_{\epsilon,r}$ in the same way as (2.101), (2.102) and (2.115). Note that the projection P_n^r acts pointwise in this case. Then the rest of the proof still works and hence the sharpness follows.

Remark 1. *An example where the inequality in Theorem 1 fails but (1.19) holds*

For an example that Theorem 1 cannot hold merely under the assumption that K is a Riesz-like kernel, we can take $0 < \alpha < \frac{n}{2}$ and let $K \in C^1(\mathbb{R}^n \setminus \{0\})$ be such that

$$K(x) = \begin{cases} |x|^{\alpha-n} & \text{if } |x| \leq 1 \\ 2|x|^{\alpha-n} & \text{if } |x| \geq 2 \end{cases}$$

and define the functions $\phi_{\epsilon,r}$, $\tilde{\phi}_{\epsilon,r}$ and $\psi_{\epsilon,r}$ in the same way as above. Note that we have

$$2^{q'}|B_1|\log r^n - C \leq b_r \leq 2^{q'}|B_1|\log r^n + C.$$

Take $\beta = 1$ and choose

$$r^n = \frac{A_g}{2C_4} \left(\log \frac{1}{\epsilon^n} \right), \quad (2.120)$$

which satisfies (2.111). Hence we have

$$\begin{aligned} \int_{B_{\epsilon r/2}} \frac{\exp \left[\frac{1}{|B_1|} |T\psi_\epsilon(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |T\psi_\epsilon(x)|^{\frac{n}{n-\alpha}}} dx &\geq |B_{\epsilon r/2}| \frac{\exp \left[\log \frac{1}{(\epsilon r)^n} + \frac{b_r}{|B_1|} \left(1 - \frac{C}{\log \frac{1}{\epsilon^n}} \right) - C \right]}{1 + C \log \frac{1}{(\epsilon r)^n} + b_r - C} \\ &\geq C \frac{r^{2q'n}}{1 + Cr^n} \rightarrow \infty \end{aligned} \quad (2.121)$$

as $\epsilon \rightarrow 0^+$.

On the other hand, since K is a Riesz-like kernel, the inequality (1.19) [FM2, Theorem 5] will hold under the Ruf condition.

2.6 Proof of Corollary 1

Assume that

$$\|f\|_{n/\alpha} \leq 1. \quad (2.122)$$

Let $q = n/\alpha$. It is enough to show that

$$\int_{\{|Tf| \geq 1\}} \exp \left[\frac{\theta}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq \frac{C}{1-\theta} \|Tf\|_q^q \quad (2.123)$$

since (1.25) is then a direct consequence of the exponential regularization Lemma B.

To show (2.123), write

$$\begin{aligned} & \exp \left[\frac{\theta}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] \\ &= \frac{\exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{n-\alpha}}} \frac{1 + |Tf|^{\frac{n}{n-\alpha}}}{\exp \left[\frac{1-\theta}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right]}. \end{aligned} \quad (2.124)$$

Observe that

$$\frac{1+y}{e^{(1-\theta)y/A_g}} \leq \frac{C}{1-\theta}, \quad \text{for } y \geq 0$$

So by Theorem 1,

$$\begin{aligned} \int_{\{|Tf| \geq 1\}} \exp \left[\frac{\theta}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx &\leq \frac{C}{1-\theta} \int_{\{|Tf| \geq 1\}} \frac{\exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{n-\alpha}}} dx \\ &\leq \frac{C}{1-\theta} \|Tf\|_q^q. \end{aligned} \quad (2.125)$$

Obviously (2.125) also follows under the more restrictive condition

$$||f||_{n/\alpha}^{pn/\alpha} + ||Tf||_{n/\alpha}^{pn/\alpha} \leq 1, \quad p < \infty.$$

The proof of sharpness is the same as in [FM2]. We use the family of functions $\psi_{\epsilon,r}$ in section 2.5, and choose

$$r^n = \frac{A_g}{2C_4} \left(\log \frac{1}{\epsilon^n} \right).$$

2.7 Proof that Theorem 1 implies (1.19) and (1.20)

It is enough to show that

$$\int_{\{|Tf| \geq 1\}} \exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C \quad (2.126)$$

under the Ruf condition

$$||f||_{n/\alpha}^{n/\alpha} + ||Tf||_{n/\alpha}^{n/\alpha} \leq 1.$$

Let $\tau = ||Tf||_q^q$. Clearly we can assume that $\tau \in (0, 1)$. We consider two cases:

Case 1: $\tau \geq 1 - (\frac{2}{3})^{q-1}$.

Case 2: $\tau < 1 - (\frac{2}{3})^{q-1}$.

Proof of (2.126) in case 1: In this case,

$$||f||_q^q \leq 1 - \tau \leq \left(\frac{2}{3} \right)^{q-1}, \quad (2.127)$$

so letting $\tilde{f} = f / (\frac{2}{3})^{\frac{q-1}{q}} = (\frac{3}{2})^{\frac{q-1}{q}} f$ gives $||\tilde{f}||_q^q \leq 1$. We can write

$$\int_{\{|Tf| \geq 1\}} \exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx = \int_{\{|Tf| \geq 1\}} \exp \left[\frac{2}{3A_g} |T\tilde{f}(x)|^{\frac{n}{n-\alpha}} \right] dx. \quad (2.128)$$

So by taking $\theta = \frac{2}{3}$ in Adachi-Tanaka result, we have

$$\int_{\{|Tf| \geq 1\}} \exp \left[\frac{2}{3A_g} |T\tilde{f}(x)|^{\frac{n}{n-\alpha}} \right] dx \leq \frac{C}{1-2/3} \|T\tilde{f}\|_q^q = 3\left(\frac{3}{2}\right)^{q-1} C \|Tf\|_q^q \leq C. \quad (2.129)$$

Combining (2.128) and (2.129) finishes the proof in case 1.

Proof of (2.126) in case 2: In this case,

$$\|f\|_q^q \leq 1 - \tau \in \left(\left(\frac{2}{3}\right)^{q-1}, 1 \right). \quad (2.130)$$

Let $p > 1$ be such that

$$p(1 - \tau)^{\frac{\alpha}{n-\alpha}} = 1.$$

Rewrite (2.126) and apply Holder's inequality,

$$\begin{aligned} \int_{\{|Tf| \geq 1\}} \exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx &= \int_{\{|Tf| \geq 1\}} \frac{\exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{(n-\alpha)p}}} \left(1 + |Tf|^{\frac{n}{(n-\alpha)p}} \right) dx \\ &\leq \left(\int_{\{|Tf| \geq 1\}} \left(\frac{\exp \left[\frac{1}{A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |Tf|^{\frac{n}{(n-\alpha)p}}} \right)^p dx \right)^{\frac{1}{p}} \left(\int_{\{|Tf| \geq 1\}} \left(1 + |Tf|^{\frac{n}{(n-\alpha)p}} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &= I' \cdot I''. \end{aligned} \quad (2.131)$$

Let

$$\tilde{f} = \frac{f}{(1 - \tau)^{1/q}}$$

so that $||\tilde{f}||_q^q \leq 1$. Applying Theorem 1 gives

$$\begin{aligned}
I' &\leq \frac{C}{(1-\tau)^{1/q}} \left(\int_{\{|Tf| \geq 1\}} \left(\frac{\exp \left[\frac{(1-\tau)^{\frac{\alpha}{n-\alpha}} |T\tilde{f}(x)|^{\frac{n}{n-\alpha}}}{A_g} \right]}{1 + |T\tilde{f}|^{\frac{n}{(n-\alpha)p}}} \right)^p dx \right)^{\frac{1}{p}} \\
&\leq C \left(\int_{\{|Tf| \geq 1\}} \frac{\exp \left[\frac{1}{A_g} |T\tilde{f}(x)|^{\frac{n}{n-\alpha}} \right]}{1 + |T\tilde{f}|^{\frac{n}{(n-\alpha)}}} dx \right)^{\frac{1}{p}} \\
&\leq C ||T\tilde{f}||_q^{\frac{q}{p}} \leq \frac{C}{1-\tau} ||Tf||_q^{\frac{q}{p}} \leq C ||Tf||_q^{\frac{q}{p}}.
\end{aligned} \tag{2.132}$$

Next, to estimate I'' we start with the following Adachi-Tanaka inequality:

$$\int_{\{|Tf| \geq 1\}} \exp \left[\frac{1}{2A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \leq C ||Tf||_q^q. \tag{2.133}$$

Let

$$P = \frac{1}{p-1}, \quad P_0 = \left\lfloor \frac{1}{p-1} \right\rfloor - 1, \quad P_1 = \left\lfloor \frac{1}{p-1} \right\rfloor$$

and define

$$F(x) := \begin{cases} Tf(x) & \text{if } |Tf(x)| \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.134}$$

By the power series expansion of the exponential function, we have that the inequality

(2.133) implies that for any integer $N \geq 1$,

$$\begin{aligned}
\int_{\{|Tf| \geq 1\}} |Tf(x)|^{\frac{nN}{n-\alpha}} &= ||F||_{q'_N}^{q'_N} \leq (2A_g)^N N! \int_{\{|Tf| \geq 1\}} \exp \left[\frac{1}{2A_g} |Tf(x)|^{\frac{n}{n-\alpha}} \right] dx \\
&\leq C(2A_g)^N N! ||Tf||_q^q \leq C(2A_g)^N N^N ||Tf||_q^q.
\end{aligned} \tag{2.135}$$

By (2.130), we have $P_0, P_1 \geq 1$, hence

$$\begin{aligned} \|F\|_{q^{P_0}}^{q'} &\leq 2A_g C^{1/P_0} P_0 \|Tf\|_q^{q/P_0} \leq C P_0 \|Tf\|_q^{q/P_0} \\ \|F\|_{q^{P_1}}^{q'} &\leq 2A_g C^{1/P_1} P_1 \|Tf\|_q^{q/P_1} \leq C P_1 \|Tf\|_q^{q/P_1}. \end{aligned}$$

Let $a \in [0, 1]$ be the number such that

$$\frac{1}{P} = \frac{a}{P_0} + \frac{1-a}{P_1}.$$

By interpolation [Fol, Proposition 6.10] we have

$$\begin{aligned} \|F\|_{q^P}^{q'} &\leq \|F\|_{q^{P_0}}^{q'a} \|F\|_{q^{P_1}}^{q'(1-a)} \leq C P_0^a P_1^{1-a} \|Tf\|_q^{q(a/P_0)} \|Tf\|_q^{q(1-a)/P_1} \\ &= C P_0^a P_1^{1-a} \|Tf\|_q^{q/P} \leq C P \|Tf\|_q^{q/P}. \end{aligned} \tag{2.136}$$

Hence, since $p > 1$

$$\begin{aligned} I'' &= \left(\int_{\{|Tf| \geq 1\}} |Tf(x)|^{\frac{n}{n-\alpha} \frac{1}{p-1}} dx \right)^{\frac{p-1}{p}} = \|F\|_{q^P}^{q'/p} \leq C \left(\frac{1}{p-1} \right)^{1/p} \|Tf\|_q^{q(\frac{p-1}{p})} \\ &\leq \frac{C}{p-1} \|Tf\|_q^{q(\frac{p-1}{p})}. \end{aligned} \tag{2.137}$$

So combining (2.132) and (2.137), we get

$$I' \cdot I'' \leq C \frac{1}{p-1} \|Tf\|_q^q = C \frac{\tau}{p-1} \leq C \frac{\tau}{1 - (1-\tau)^{\frac{\alpha}{n-\alpha}}} \leq C \tag{2.138}$$

where the second inequality is by (2.130).

2.8 Proof of Theorem 3 and Corollaries 2, 3

In Theorem 1 we assume that the functions f are compactly supported, with both f and Tf in the space $L^q(\mathbb{R}^n)$. We denote this space of functions by D_0 :

$$D_0 := \{f \in L^q(\mathbb{R}^n) : \text{supp } f \text{ is compact and } Tf \in L^q(\mathbb{R}^n)\}.$$

In the following Theorem [FM2, Theorem 7] we see that T has a smallest closed extension, which enables us to extend Theorem 1 to all functions in the domain of the extension. In particular, Theorem 3 is a consequence of this Theorem.

Theorem F ([FM2, Theorem 7]). *If K is a Riesz-like kernel, then the operator $T : D_0(T) \rightarrow L^q(\mathbb{R}^n)$ is closable, and its smallest closed extension (still denoted T) has domain*

$$D(T) = \{f \in L^q(\mathbb{R}^n) : \exists \{f_k\} \subseteq D_0(T), \exists h \in L^q(\mathbb{R}^n) \text{ with } f_k \xrightarrow{L^q} f, Tf_k \xrightarrow{L^q} h\} \quad (2.139)$$

and

$$Tf = h.$$

In the case of Riesz potential we have

$$W^{\alpha,q}(\mathbb{R}^n) = \{I_\alpha f, f \in D(I_\alpha)\} \quad (2.140)$$

and the operator $(-\Delta)^{\frac{\alpha}{2}}$ is a bijection between $W^{\alpha,q}(\mathbb{R}^n)$ and $D(I_\alpha)$, with inverse $c_\alpha I_\alpha$.

By using the above Theorem F and Fatou's lemma, we easily deduce that Theorem 1 is still valid for all functions f in $D(T)$. Also (2.140) tells us that the Riesz potential for all functions f in the extended domain $D(I_\alpha)$ is the space $W^{\alpha,q}(\mathbb{R}^n)$. Therefore by the fact that the inverse of $(-\Delta)^{\frac{\alpha}{2}}$ is $c_\alpha I_\alpha$, we have (1.31).

In the case of elliptic operator, by the formula (1.29), the kernel of the integral operator is homogeneous of order $\alpha - n$, therefore we have (1.31).

It is enough to assume $u \in C_c^\infty(\mathbb{R}^n)$ since α is an integer for the remaining cases. For $P = \nabla(-\Delta)^{\frac{\alpha-1}{2}}$ and α is an odd integer, since $u = c_{\alpha+1}I_{\alpha+1}(-\Delta)^{\frac{\alpha+1}{2}}u$, we can write

$$u(x) = \int_{\mathbb{R}^n} c_{\alpha+1}(n - \alpha - 1)|x - y|^{\alpha-n-1}(x - y) \cdot f(y), \quad f = \nabla(-\Delta)^{\frac{\alpha-1}{2}}u. \quad (2.141)$$

Clearly the kernel in the above formula satisfies our assumptions (A1)-(A4), so (1.31) follows.

For the proof of Corollary 2, it is clear that the inequality (1.33) is a direct consequence of (1.31) since $\Omega \subseteq \mathbb{R}^n$.

Lastly, we derive (1.37) in Corollary 3 in exactly the same way as the above proof for (1.31) in the case of α is an integer, except that we use the more general inequality (1.27) in Theorem 2.

Proof of sharpness: To prove the sharpness, let $\psi_{\epsilon,r}$ be the function constructed in the proof of sharpness (2.115) with $\beta = 1$. If $P = (-\nabla)^{\frac{\alpha}{2}}$, consider the functions

$$u_{\epsilon,r} = c_\alpha I_\alpha \psi_{\epsilon,r}.$$

Similarly, for P a an elliptic operator, let $u_{\epsilon,r} = g_P * \psi_{\epsilon,r}$.

Lastly, we construct the extremal family of functions that proves sharpness for the case $P = \nabla(-\Delta)^{\frac{\alpha-1}{2}}$ in Theorem 3, as well as sharpness for Corollaries 2, 3. Note that in all these cases α is an integer. We use the same extremal functions as in Adams ([A1], see also [FM1], [FM2], [MS2]). Let $\varphi \in C^\infty([0, 1])$ such that $\varphi^{(k)}(0) = 0$ for $0 \leq k \leq \alpha - 1$, and $\varphi(1) = \varphi'(1) = 1$, $\varphi^{(k)}(1) = 0$ for $2 \leq k \leq m - 1$. Let ϵ be

small enough, define

$$v_\epsilon(y) = \begin{cases} 0 & \text{for } |y| \geq \frac{3}{4} \\ \varphi(\log \frac{1}{|y|}) & \text{for } \frac{1}{2} \leq |y| \leq \frac{3}{4} \\ \log \frac{1}{|y|} & \text{for } 2\epsilon \leq |y| \leq \frac{1}{2} \\ \log \frac{1}{\epsilon} - \varphi(\log \frac{|y|}{\epsilon}) & \text{for } \epsilon \leq |y| \leq 2\epsilon \\ \log \frac{1}{\epsilon} & \text{for } |y| \leq \epsilon. \end{cases}$$

Then we have that

$$\|v_\epsilon\|_{\beta q} \leq C, \quad \|\nabla^\alpha v_\epsilon\|_q^{q'} = \frac{\gamma(P)}{n} (\log \frac{1}{\epsilon})^{q'-1} + O(1).$$

Let

$$u_\epsilon = \frac{v_\epsilon}{\|\nabla^\alpha v_\epsilon\|_q},$$

it is clear that

$$\|\nabla u_\epsilon\|_q \leq 1, \quad \|u_\epsilon\|_{\beta q}^{\beta q} \leq C (\log \frac{1}{\epsilon})^{-\beta}, \quad (2.142)$$

and

$$|u_\epsilon|^{q'} \geq \gamma(P)^{-1} \log \frac{1}{\epsilon^n}, \quad |y| \leq \epsilon.$$

For the sharpness of the exponential constant, we take $\theta > 1$ and estimate

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\exp[\frac{n}{\alpha}-2] [\theta \gamma(P) |u_\epsilon|^{\frac{n}{n-\alpha}}]}{1 + |u_\epsilon|^{\frac{\beta n}{n-\alpha}}} dy &\geq \int_{|y| \leq \epsilon} \frac{\exp [\theta \gamma(P) |u_\epsilon|^{\frac{n}{n-\alpha}}]}{1 + |u_\epsilon|^{\frac{\beta n}{n-\alpha}}} dy \geq C \epsilon^n \frac{\exp[\theta \log \frac{1}{\epsilon^n} + C]}{1 + C (\log \frac{1}{\epsilon})^\beta} \\ &= C \frac{\epsilon^{(1-\theta)n}}{1 + C (\log \frac{1}{\epsilon})^\beta} \rightarrow \infty \end{aligned}$$

as $\epsilon \rightarrow 0^+$.

For the sharpness of the power of the denominator, we take $\theta < 1$ and get

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\exp[\frac{n}{\alpha}-2] [\gamma(P)|u_\epsilon|^{\frac{n}{n-\alpha}}]}{1 + |u_\epsilon|^{\theta\beta\frac{n}{n-\alpha}}} dy &\geq \int_{|y| \leq \epsilon} \frac{\exp[\gamma(P)|u_\epsilon|^{\frac{n}{n-\alpha}}]}{1 + |u_\epsilon|^{\theta\beta\frac{n}{n-\alpha}}} dy \geq C\epsilon^n \frac{\exp[\log \frac{1}{\epsilon^n} + C]}{1 + C(\log \frac{1}{\epsilon})^{\theta\beta}} \\ &\geq C(\log \frac{1}{\epsilon})^{-\theta\beta}. \end{aligned}$$

Hence by (2.142) we have that the quotient of the above integral over the norm $\|u_\epsilon\|_{\beta q}^{\beta q}$ goes to infinity as $\epsilon \rightarrow 0^+$, so the sharpness follows.

2.9 Application to existence of ground state solutions for quasilinear equations

In [MS2] Masmoudi and Sani showed that, their inequality (1.15) leads to the existence of ground state solutions for the following quasilinear equation in $\mathbb{R}^n$, $n \geq 2$:

$$-\Delta_n u + c|u|^{n-2}u = f(u) \quad (2.143)$$

where

$$\Delta_n u = \operatorname{div}(|\nabla u|^{n-2} \nabla u)$$

is the n -Laplacian operator, c is a positive constant, and the allowed maximal growth on the nonlinear term f is exponential.

In this section, by making use of the inequality (1.37) in Corollary 3, we prove an existence result for the following more general quasilinear elliptic equation:

$$\begin{cases} -\Delta_n u + c|u|^{p-2}u = f(u), & \text{in } \mathbb{R}^n, \ p > 1, \ n \geq 2 \\ \|\nabla u\|_n < \infty, \ \|u\|_p < \infty. \end{cases} \quad (2.144)$$

Let $\mathcal{S} := \{u : \|\nabla u\|_n < \infty, \|u\|_p < \infty\}$. Note that if $1 < p \leq q = n$, then $\mathcal{S}$ is a Banach subspace of $W^{1,n}(\mathbb{R}^n)$. In fact, if $u \in \mathcal{S}$, then so is its nonincreasing radial rearrangement u^* by Pólya Szegő's symmetrization inequality. Since $\|u^*\|_p = \|u\|_p < \infty$, there is a $R > 0$ such that $u^*(x) \leq 1$ if $|x| \geq R$. It is clear that $\|u^* \chi_{|x| \geq R}\|_n \leq \|u^* \chi_{|x| \geq R}\|_p < \infty$. On the other hand, by Poincaré inequality for the ball $B(0, R)$, we have that

$$\|u^* \chi_{|x| \leq R}\|_n \leq \|\nabla(u^* \chi_{|x| \leq R})\|_n < \infty.$$

Therefore, $\|u\|_n < \infty$ and $u \in W^{1,n}(\mathbb{R}^n)$. It remains to show that $\mathcal{S}$ is complete, but it is obvious by the fact that $L^p(\mathbb{R}^n)$ and $W^{1,n}(\mathbb{R}^n)$ are complete.

Now we state the following result:

Theorem 4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $f(0) = 0$ and*

$$\text{There exists } \mu > p \text{ such that } 0 < \mu F(t) := \mu \int_0^t f(s) ds \leq t f(t) \quad \forall t \in \mathbb{R} \setminus \{0\}, \quad (f_1)$$

$$\text{There exist } t_0, M_0 > 0 \text{ such that } F(t) \leq M_0 f(t) \quad \forall t \geq t_0, \quad (f_2)$$

$$\text{There exist } \gamma_0 \geq 0 \text{ such that } \lim_{t \rightarrow \infty} \exp[-\gamma t^{n/(n-1)}] f(t) = \begin{cases} 0 & \text{if } \gamma > \gamma_0, \\ \infty & \text{if } \gamma < \gamma_0. \end{cases} \quad (f_3)$$

If furthermore

$$\lim_{t \rightarrow \infty} t^{p/(n-1)} \exp[-\gamma_0 t^{n/(n-1)}] F(t) = \infty \quad (f_4)$$

then for each $c > 0$, (2.144) has a positive radial solution $u \in \mathcal{S}$ which is also a ground state solution.

According to the condition (f_4) , we can see that for different values of p , we have different assumptions on the growth rate of F at infinity. For example, if $p < n$, then F must grow a little faster at infinity than the one for $p = n$ in [MS2].

To better understand (f_4) , we also mention that using l'Hospital's rule, we easily see that a sufficient condition for it to hold is

$$\lim_{t \rightarrow \infty} t^{(p-1)/(n-1)} \exp[-\gamma_0 t^{n/(n-1)}] f(t) = \infty. \quad (f_5)$$

Given any $\gamma_0 > 0$, an example of function f that satisfies the above (f_5) as well as (f_1) - (f_3) is simply

$$f = \exp[\gamma_0 t^{n/(n-1)}] \chi_{t \geq t_0},$$

for which $t_0 = (p(n-1)/\gamma_0 n)^{\frac{n-1}{n}} + 1$, $\mu = p + 1$, and $M_0 = \frac{1}{p} + 1$.

The proof for the above Theorem 4 is essentially the same as the proof of [MS2], with some slight changes. So we focus on the main argument, and highlight the differences with an emphasis on the use of the crucial inequality in Corollary 3.

The functional $I(u)$ for the variational approach is

$$I(u) = \frac{1}{n} (\|\nabla u\|_n^n + \frac{cn}{p} \|u\|_p^p) - \int_{\mathbb{R}^n} F(u) dx,$$

in the sense that a critical point of $I(u)$ is a solution to the quasilinear equation (2.144).

A solution u is called a *ground state solution* if it satisfies

$$I_c(u) = m_c := \inf\{I(u) : u \text{ is a solution of (2.144)}\},$$

i.e. a solution with the least energy among all solutions of (2.144). Define the functional

$$G_c(u) = c\|u\|_p^p - p \int_{\mathbb{R}^n} F(u) dx.$$

It can be shown that if u is solution of (2.144), then it must satisfy the condition $G_c(u) = 0$, which is the Pohozaev identity for the equation (2.144). The proof of the

Pohozaev identity for general solution u can be found in [BL, Proposition 1], where the authors proved a special case of the operator, but their proof also works in our case. Here we will only check the case of radial solution u here. Given that $u(r)$ is a solution of (2.144). We multiply this equation by $r^n u'(r)$ and integrate to get

$$\begin{aligned}
& \int_0^\infty (-\Delta_n u(r)) r^n u'(r) + r^n u'(r) c |u|^{p-2} u - r^n u'(r) f(u) dr \\
&= \frac{n-1}{n} \int_0^\infty ((ru')^n)' dr + \frac{c}{p} \|u\|_p^p - \int_{\mathbb{R}^n} F(u) dx \\
&= \frac{c}{p} \|u\|_p^p - \int_{\mathbb{R}^n} F(u) dx = 0,
\end{aligned} \tag{2.145}$$

where the second equality is by the computation $\Delta_n u(r) = (n-1)(u')^{n-2} u'' + \frac{(n-1)}{r} (u')^{n-1}$. For the last equality, we use the fact that radial solution u is in $C^2(\mathbb{R}^n)$ and has exponential decay, i.e.

$$|D^k u(x)| \leq C e^{-\delta|x|}, \quad x \in \mathbb{R}^n,$$

for some $C, \delta > 0$, $|k| \leq 2$, which was proved in [BL, Lemmas 1, 2] for a similar equation. But the proof also works for equation (2.144).

Hence it is enough to show the existence of u in the following constrained minimization problem

$$\begin{aligned}
A_c &:= \inf \left\{ \frac{1}{n} \|\nabla u\|_n^n : \|\nabla u\|_n < \infty, \|u\|_p < \infty, u > 0, G_c(u) = 0 \right\} \\
&= \inf \{ I(u) : \|\nabla u\|_n < \infty, \|u\|_p < \infty, u > 0, G_c(u) = 0 \} \leq m_c.
\end{aligned}$$

It is worth noting that the set

$$\mathcal{P} = \{ \|\nabla u\|_n < \infty, \|u\|_p < \infty, u > 0, G_c(u) = 0 \} \tag{2.146}$$

is not empty. In fact, let $u \neq 0$ be any compactly supported function in $\mathcal{S}$ and

define $h(s) := G_c(su) = cs^p \|u\|_p^p - p \int_{\mathbb{R}^n} F(su) dx$. Since condition (f_2) gives us that $F'(t)/F(t) \geq 1/M_0$, by integrating both sides we get $F(t) \geq Ce^{t/M_0}$ when t large enough. Hence if s is large enough, we have that $h(s) < 0$. On the other hand, by using similar argument, we have that condition (f_1) implies (2.149) below, therefore $h(s) > 0$ for $s > 0$ small enough. So there exists an s_0 so that $h(s_0 u) = G_c(s_0 u) = 0$, which means that $s_0 u$ is an element of $\mathcal{P}$.

Moreover, if the infimum A_c is attained by some $u \in \mathcal{P}$, then there exists a Lagrange multiplier $\theta \in \mathbb{R}$ that satisfies

$$-\Delta_n u + c|u|^{p-2}u - f(u) = \theta(cp|u|^{p-2}u - pf(u)) \quad \text{for } x \in \mathbb{R}^n,$$

which can be written as

$$-\Delta_n u = (p\theta - 1)(c|u|^{p-2}u - f(u)).$$

Also, by the condition (f_1) and the fact that $u \in \mathcal{P}$,

$$\begin{aligned} \int_{\mathbb{R}^n} (c|u|^{p-2}u - f(u))u dx &= c\|u\|_p^p - \int_{\mathbb{R}^n} (uf(u) + pF(u) - pF(u)) dx \\ &= - \int_{\mathbb{R}^n} (uf(u) - pF(u)) dx < 0. \end{aligned}$$

But $\int_{\mathbb{R}^n} u \Delta_n u dx < 0$, so we must have that $p\theta - 1 < 0$. Hence the ground state solution is

$$\tilde{u}(x) = u \left(\frac{x}{(1 - p\theta)^{1/n}} \right).$$

Now we are left to show that A_c is attained by some $u \in \mathcal{P}$. First recall that $\gamma_n = n\omega_{n-1}^{1/(n-1)}$ is the sharp exponential constant as in (1.37) Corollary 3, with $P = \nabla$. Along the same line as Masmoudi-Sani in [MS2], we define

$$Q_D(F) := \sup \left\{ \frac{p}{\|u\|_p^p} \int_{\mathbb{R}^n} F(u) dx : \|\nabla u\|_n \leq D, \|u\|_p < \infty, u \neq 0 \right\}$$

$$C(F) := \sup\{D > 0 : Q_D(F) < \infty\}.$$

Assume that $\gamma_0 > 0$, and the case of $\gamma_0 = 0$ will be considered later. We first show that, by applying Corollary 3, we have

$$C(F) = \left(\frac{\gamma_n}{\gamma_0}\right)^{\frac{n-1}{n}}. \quad (2.147)$$

To see this, note that the assumptions (f_1) and (f_3) imply

$$\lim_{t \rightarrow \infty} t^{p/(n-1)} \exp[-\gamma t^{n/(n-1)}] F(t) = \begin{cases} 0 & \text{if } \gamma > \gamma_0, \\ \infty & \text{if } \gamma < \gamma_0. \end{cases} \quad (2.148)$$

The above (2.148) is also true if $t^{p/(n-1)}$ is replaced by t^a for any $a > 0$. And from condition (f_1) we get

$$\limsup_{t \rightarrow 0} t^{-p} F(t) = 0. \quad (2.149)$$

Since $F(t)$ is continuous, by (2.148), there is a constant C depending on F such that

$$F(t) \leq C \min\{t^p, t^{-\frac{p}{n-1}}\} \exp[\gamma t^{n/(n-1)}], \quad \text{for } \gamma > \gamma_0.$$

Hence from the improved sharp Adams inequality with exact growth condition (1.37), that is

$$\int_{|u| \geq 1} \frac{\exp[\frac{\gamma_n}{D^{n/(n-1)}} |u|^{n/(n-1)}]}{|u|^{p/(n-1)}} dx \leq C \|u\|_p^p, \quad \text{for } \|\nabla u\|_n \leq D \quad \|u\|_p < \infty,$$

we can deduce that

$$Q_D(F) < \infty \quad \text{for } D < \left(\frac{\gamma_n}{\gamma_0}\right)^{\frac{n-1}{n}}. \quad (2.150)$$

If $D \geq \left(\frac{\gamma_n}{\gamma_0}\right)^{\frac{n-1}{n}}$, then by the assumption (f_4) , (2.148) and using the same extremal

family of functions $u_{\epsilon,r}$ as in the proof of sharpness for (1.37), we have that

$$\frac{p}{\|u_{\epsilon,r}\|_p^p} \int_{\mathbb{R}^n} F(u_{\epsilon,r}) dx \rightarrow \infty \quad \text{as } \epsilon \rightarrow 0^+$$

while $\|u_{\epsilon,r}\|_p \leq 1$ and $\|\nabla u_{\epsilon,r}\|_n \leq D$.

Therefore

$$Q_D(F) = \infty \tag{2.151}$$

when $D \geq \left(\frac{\gamma_n}{\gamma_0}\right)^{\frac{n-1}{n}}$, and hence we have (2.147).

It turns out that a sufficient condition for A_c to be attained is

$$A_c < \frac{(C(F))^n}{n}, \quad \forall c > 0. \tag{2.152}$$

and hence the Theorem 4 follows if (2.152) is proved. The sufficiency of (2.152) can be proved as in [MS2], and we refer the readers to Proposition 7.1 in [MS2], with the conditions (7.3), (7.4) replaced by (2.148) and (2.149).

To prove (2.152), first note that since $Q_{C(F)} = \infty$, then for any $c > 0$, there exists $u_0 \neq 0$ with $\|u_0\|_p < \infty$ and $\|\nabla u\|_n \leq C(F)$ such that

$$c < \frac{p}{\|u_0\|_p^p} \int_{\mathbb{R}^n} F(u_0) dx, \tag{2.153}$$

which implies $G_c(u_0) < 0$. Recall that $h(s) = G_c(su_0)$. It is clear that $h(1) < 0$. And by (2.149), we have $h(s) > 0$ for $0 < s < 1$ small enough. Hence there is an s_0 such that $h(s_0) = G_c(s_0u_0) = 0$, so $s_0u_0 \in \mathcal{P}$. Therefore,

$$A_c \leq \frac{1}{n} \|\nabla(s_0u_0)\|_n^n = \frac{1}{n} s_0^n \|\nabla u_0\|_n^n \leq s_0^n \frac{(C(F))^n}{n} < \frac{(C(F))^n}{n}, \tag{2.154}$$

which proves (2.152) and hence A_c is attained, in the case $\gamma_0 > 0$.

If $\gamma_0 = 0$, then

$$\lim_{t \rightarrow \infty} \exp[-\gamma t^{n/(n-1)}] f(t) = 0, \quad \forall \gamma > 0$$

which implies that $Q_D(F) < \infty$ for any $D > 0$. Hence $C(F) = \infty$ and A_c is always attained for any $c > 0$. □

Chapter 3

Results on the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is defined as the set

$$\mathbb{C}^n \times \mathbb{R} = \{(z, t) : z \in \mathbb{C}^n, t \in \mathbb{R}\} \cong \mathbb{R}^{2n+1}$$

with the multiplication law

$$(z, t)(z', t') = (z_1 + z_2, t_1 + t_2 + 2\text{Im}(z \cdot \overline{z'})),$$

where $z \cdot \overline{z'}$ is the Hermitian inner product of the vectors $z, z' \in \mathbb{C}^n$. We can see that the additive identity is the origin $(0, 0)$ and the inverse is given by $(z, t)^{-1} = (-z, -t)$.

For elements $z \in \mathbb{C}^n$ we will use the notation

$$z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n), \quad x_j, y_j \in \mathbb{R}$$

and (with some slight abuse of notation) elements of $\mathbb{H}^n$ will also be denoted by x, y .

The Haar measure of $\mathbb{H}^n$ will be denoted as dx and it coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$:

$$dx = dzdt = dx_1 \dots dx_n dy_1 \dots dy_n dt.$$

The dilation on $\mathbb{H}^n$ is given for any $\delta > 0$ as

$$\delta(z, t) = (\delta z, \delta^2 t).$$

The Jacobian determinant of the dilation is δ^Q , where

$$Q = 2n + 2$$

is the homogeneous dimension of $\mathbb{H}^n$.

The norm on $\mathbb{H}^n$ is defined as

$$|(z, t)| = (|z|^4 + t^2)^{\frac{1}{4}}$$

and we can define the Heisenberg ball centered at x with radius r as

$$B(x, r) = \{y \in \mathbb{H}^n : |y^{-1}x| < r\}$$

with volume $C_Q r^Q$ where C_Q is a constant depending only on Q .

We will let Σ denote the unit sphere on Heisenberg group, i.e. $\Sigma = \{x \in \mathbb{H}^n : |x| = 1\}$. Given any $x \in \mathbb{H}^n$ let

$$x^* = (z^*, t^*) = \frac{x}{|x|} = \left(\frac{z}{|x|}, \frac{t}{|x|^2} \right) \in \Sigma.$$

For a measurable function f on $\mathbb{H}^n$, we denote

$$\|f\|_p = \left(\int_{\mathbb{H}^n} |f(x)|^p dx \right)^{1/p} = \left(\int_{\mathbb{H}^n} |f(z, t)|^p dz dt \right)^{1/p},$$

and $L^p(\mathbb{H}^n)$ be the space of functions on $\mathbb{H}^n$ such that $\|f\|_p < \infty$.

The Lie algebra of the Heisenberg group is a vector space defined as follows.

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

for $j = 1, \dots, n$. The above $2n + 1$ vector fields form a basis for the vector space of left-invariant vector fields on $\mathbb{H}^n$. Note that we have the commutation relations

$$[Y_j, X_k] = 4\delta_{jk}T$$

and all other commutators vanish.

The sublaplacian on $\mathbb{H}^n$ is

$$\mathcal{L}_0 = -\frac{1}{4} \sum_{j=1}^n (X_j^2 + Y_j^2),$$

and the horizontal gradient is the $2n$ dimensional vector

$$\nabla_{\mathbb{H}^n} u = (X_1 u, Y_1 u, \dots, X_n u, Y_n u).$$

Let $1 < p < \infty$, $\alpha \geq 0$, the Sobolev space on Heisenberg group is

$$W^{\alpha,p}(\mathbb{H}^n) = \{f \in L^p(\mathbb{H}^n) : (-\mathcal{L}_0)^{\frac{\alpha}{2}} f \in L^p(\mathbb{H}^n)\},$$

with the graph norm $\|f\|_p + \|(-\mathcal{L}_0)^{\frac{\alpha}{2}} f\|_p$, or equivalently, the norm $\|(I - \mathcal{L}_0)^{\frac{\alpha}{2}}\|_p$.

3.1 Adams inequality with exact growth condition on Heisenberg group

In this section we consider kernels that satisfy the following conditions

$$K(x) = g(x^*)|x|^{\alpha-Q} + O(|x|^{\alpha-Q+\delta_1}), \quad 0 < |x| \leq B \quad (\text{H1})$$

$$|K(x)| \leq |g(x^*)||x|^{\alpha-Q} + |O(|x|^{\alpha-Q-\delta_2})|, \quad |x| > B \quad (\text{H2})$$

$$|K(x)| \leq H_3|x|^{\alpha-Q} \quad (\text{H3})$$

$$|K(x) - K(y)| \leq H_4|y|^{-1}|x| \max\{|x|^{\alpha-Q-1}, |y|^{\alpha-Q-1}\}, \quad x, y \neq 0, \quad (\text{H4})$$

which are analogous to the strictly Riesz-like kernels on $\mathbb{R}^n$.

Let $m \in \mathbb{N}$, we say that $K(z, t) : \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is m -regular if $K \in C^m(\mathbb{C}^n \times \mathbb{R} \setminus \{0\})$, and

$$|D_z^h D_t^\ell K(x)| \leq C|x|^{\alpha-Q-j}, \quad u \neq 0, \quad j = 1, \dots, m, \quad |h| + 2\ell = j, \quad (3.1)$$

where $h = (h_1, \dots, h_{2n})$ is a multi-index, ℓ is a nonnegative integer, $|h| = h_1 + \dots + h_{2n}$, and where

$$D_z^h = \frac{\partial^{|h|}}{\partial x_1^{h_1} \dots \partial x_n^{h_n} \partial y_1^{h_{n+1}} \dots \partial y_n^{h_{2n}}}, \quad D_t^\ell = \frac{\partial^\ell}{\partial t^\ell}.$$

If $g : \Sigma \rightarrow \mathbb{R}$ is in $C^m(\Sigma)$, then the standard kernel $K(x) = g(x^*)|x|^{\alpha-Q}$ is homogeneous of order $\alpha - Q$, i.e. $K(\delta x) = K(\delta z, \delta^2 t) = \delta^{\alpha-Q} K(z, t)$, and it is in $C^m(\mathbb{H}^n \setminus \{0\})$. By induction we have that $D_z^h D_t^\ell K$ is homogeneous of order $\alpha - Q - |h| - 2\ell$. Since $K \in C^m(\mathbb{H}^n \setminus \{0\})$, all partial derivatives of K up to order m are continuous and hence bounded on the Heisenberg sphere. Therefore we have, for $|h| + 2\ell \leq m$

$$D_z^h D_t^\ell K(x) = D_z^h D_t^\ell K(x^*|x|) = D_z^h D_t^\ell K(x^*)|x|^{\alpha-Q-|h|-2\ell} \leq C|x|^{\alpha-Q-|h|-2\ell}.$$

Now we state our main result on Heisenberg group.

Theorem 5. *Let $0 < \alpha < Q$, $B > 0$, $\delta_1, \delta_2, H_1, H_2 > 0$ and $K_i : \mathbb{H}^n \setminus 0 \rightarrow \mathbb{R}$ for $1 \leq i \leq m$. Let $K = (K_1, \dots, K_m)$ satisfies (H1)-(H4). Let $f = (f_1, \dots, f_m)$, $|f| = (f_1^2 + \dots + f_m^2)^{\frac{1}{2}}$, $Kf = K_1 f_1 + \dots K_m f_m$, and*

$$Tf(x) = \int_{\mathbb{H}^n} K(y^{-1}x)f(y)dy.$$

There exists $C = C(Q, \alpha, K) > 0$ such that for all compactly supported f with

$$\|f\|_{\frac{Q}{\alpha}} \leq 1$$

we have

$$\int_{\mathbb{H}^n} \frac{\exp_{[\frac{Q}{\alpha}-2]} \left[\frac{1}{A_g} |Tf|^{\frac{Q}{Q-\alpha}} \right]}{1 + |Tf|^{\frac{Q}{Q-\alpha}}} dx \leq C \|Tf\|_{\frac{Q}{\alpha}}^{Q/\alpha} \quad (3.2)$$

where

$$A_g = \frac{1}{Q} \int_{\Sigma} |g(x^*)|^{\frac{Q}{Q-\alpha}} dx^*, \quad (3.3)$$

and where dx^ is the surface measure on the Heisenberg sphere Σ .*

If K is Q -regular, then the exponential constant A_g^{-1} in (1.21) is sharp, i.e. it cannot be replaced by a larger number. Moreover, (3.2) cannot hold if the power $\frac{Q}{Q-\alpha}$ is replaced by any smaller power.

Proof of Theorem 5: The proof is quite similar to the proof in $\mathbb{R}^n$. First note that by Lemma B in section 1.2.4 with $N = \mathbb{H}^n$, we can reduce the proof of (3.2) to the proof of

$$\int_{\{|Tf| \geq 1\}} \frac{\exp \left[\frac{1}{A_g} |Tf|^{\frac{Q}{Q-\alpha}} \right]}{1 + |Tf|^{\frac{Q}{Q-\alpha}}} dx \leq C \|Tf\|_{\frac{Q}{\alpha}}^{Q/\alpha}. \quad (3.4)$$

From now on let $q = Q/\alpha$ and $q' = Q/(Q - \alpha)$. Also, let $(Tf)^*$, f^* be the rearrange-

ment of Tf and f respectively, and let

$$w_0 = \sup\{w > 0 : (Tf)^*(w) \geq 1\}$$

Then (3.4) is equivalent to

$$\int_0^{w_0} \frac{\exp\left[\frac{1}{A_g}((Tf)^*(w))^{q'}\right]}{1 + ((Tf)^*(w))^{q'}} dw \leq C \|Tf\|_q^q. \quad (3.5)$$

Fix $\tau > 0$, we define the sets E_τ, F_τ and split the function f into f_τ, f'_τ the same as in (2.3), (2.4). Note that Lemma A and O'Neil's Lemma 1 in section 1.2.3 still hold on Heisenberg group $\mathbb{H}^n$, so we obtain the key estimate for $Tf^*(w)$

$$(Tf)^*(w) \leq (Tf)^{**}(w) \leq Uf_\tau^*(w) + W_\tau + M_\tau. \quad \text{for } 0 < w \leq \tau, \quad (3.6)$$

where W_τ, M_τ are defined the same way as in (2.8) and (2.9). Then as in the case of $\mathbb{R}^n$, we need the following Lemma, which is analogous to Lemma 4.

Lemma 7. *Let $0 < \tau \leq w_0$ and $\epsilon_\tau = \min\left\{\frac{1}{4}, \frac{\int_0^\tau f^*(w)^q dw}{\|f\|_q^q}\right\}$. Then there exists constant $C = C(Q, \alpha, K) > 0$ such that for all $x_1, x_2 \in E_\tau$*

$$\frac{\exp\left[\frac{(1 - \epsilon_\tau)^{1-q'}}{A_g}(W(\tau, x_2) + M(\tau, x_1))^{q'}\right]}{1 + (W(\tau, x_2) + M(\tau, x_1))^{q'}} \leq \frac{C}{\tau} \|Tf\|_q^q. \quad (3.7)$$

Proof of Lemma 7: The key for the proof is to establish the analogous of Proposition 1 and Lemma 6 on the Heisenberg group. To begin with, let $r(\tau) = (\tau/C_Q)^{1/Q}$ where B_1 is the Heisenberg ball of radius 1. And for each $j = 0, 1, 2, \dots$

$$r_j = r(\tau)e^{\frac{2}{Q}j}, \quad D_j = B(x_1, r_j)$$

$$\alpha_j = \|f'_\tau \chi_{D_{j+1} \setminus D_j}\|_q, \quad \alpha_{-1} = \|f'_\tau \chi_{D_0}\|_q$$

$$\bar{\alpha}_j = \max \{\alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_j\}, \quad \beta_j = \|f'_\tau \chi_{D_j^c}\|_q.$$

And we set for any measurable function $\phi : \mathbb{H}^n \rightarrow \mathbb{R}$

$$S_j \phi = \phi \chi_{D_j^c} = \phi \chi_{\{|y^{-1}x_1| \geq r_j\}}.$$

Also suppose that

$$\text{supp } f \subseteq D_N = B(x_1, r_N)$$

for some integer $N \in \mathbb{N}$. Similar to the proof in sections 3.2-3.3 we assume that

$$TS_0 f'_\tau(x_1) \geq 0.$$

But the proofs are completely similar to those in $\mathbb{R}^n$, just use $y^{-1}x$ instead of $x - y$, and $Q = 2n + 2$ instead of n . $\square$

Now we are able to follow all the steps in the rest of the proof as in $\mathbb{R}^n$, with only some slight modifications to make. The last step of applying optimal growth lemma is the same since it only involves one dimensional estimates.

Proof of sharpness for Theorem 5:

The construction of the extremal family of functions ψ_ϵ is essentially the same as in the previous proof of sharpness on $\mathbb{R}^n$. Let us only highlight the differences occurring in the context of Heisenberg group.

Let $x = (z, t)$, $y = (z', t')$. If K is Q -regular, by applying the Taylor formula centered at $x \neq 0$ to the function $K(y^{-1}x)$ on $\mathbb{R}^{2n} \times \mathbb{R} \setminus \{x\}$ for positive integer $m \leq Q$,

there exists $\theta = \theta(x, y, m) \in (0, 1)$ such that

$$K(y^{-1}x) = \sum_{j=0}^{m-1} p_j(x, y) + R_m(x, y) \quad (3.8)$$

where

$$p_j(x, y) = \sum_{|h|+2\ell=j} \frac{1}{h!\ell!} D_z^h D_t^\ell K(x) [z']^h (t')^\ell,$$

for $j = 0, 1, \dots, m-1$, and

$$R_m(x, y) = \sum_{\substack{|h|+2\ell \geq m \\ |h|+\ell \leq m-1}} \frac{1}{h!\ell!} D_z^h D_t^\ell K(x) [z']^h (t')^\ell + \sum_{|h|+\ell=m} \frac{1}{h!\ell!} D_z^h D_t^\ell K((\theta y)^{-1}x) [z']^h (t')^\ell,$$

and where $h! = h_1! \dots h_n!$, and where we set

$$[z']^h = (x'_1)^{h_1} \dots (x'_n)^{h_n} (y'_1)^{h_{n+1}} \dots (y'_n)^{h_{2n}}, \quad z' = (x'_1 + iy'_1, \dots, x'_n + iy'_n).$$

Note that $p_j(x, y)$ is a homogeneous polynomial of order j in the y variable, for $j \leq m-1$. For $x \neq 0$ we have

$$|p_j(x, y)| \leq C |x|^{\alpha-Q-j} |y|^j \quad j \leq m-1, \quad y \in \mathbb{C}^n \times \mathbb{R},$$

and for $|y| \leq \frac{1}{2}|x|$

$$\begin{aligned} |R_m(x, y)| &\leq C \sum_{\substack{|h|+2\ell \geq m \\ |h|+\ell \leq m}} |x|^{\alpha-|h|-2\ell} |[z']^h| |(t')|^\ell \leq \sum_{m \leq |h|+2\ell \leq 2m} |x|^{\alpha-|h|-2\ell} |[z']^h| |(t')|^\ell \\ &\leq C \sum_{i=m}^{2m} |x|^{\alpha-Q-i} |y|^i \leq C |x|^{\alpha-Q-m} |y|^m \end{aligned} \quad (3.9)$$

for some constant C independent of y , x in the given range.

For $r > 0$ we denote

$$P_m(x, y) = \chi_{B_1}(x) \sum_{k=1}^N v_k(x) v_k(y) \quad (3.10)$$

to be the integral kernel of the projection on the space of polynomials of degree up to m on the ball $B_r = B(0, r)$. For $0 \leq \epsilon r < 1 \leq r$ let

$$\phi_{\epsilon, r}(y) = \begin{cases} K(y^{-1}) |K(y^{-1})|^{\frac{\alpha}{Q-\alpha}-1} & \text{if } \epsilon r < |y| \leq r \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

and consider the functions on B_r

$$\tilde{\phi}_{\epsilon, r} = \phi_{\epsilon, r} - P_Q^r \phi_{\epsilon, r} \quad (3.12)$$

which are orthogonal to every polynomial of order up to n on the ball B_r .

Then the estimations up to (2.105) still work, by replacing the homogeneous dimension n by Q . In particular, we have

$$b_r = \int_{1 \leq |y| \leq r} |K(y)|^{q'} dy \leq A_g \log r^Q + C, \quad (3.13)$$

and

$$A_g \log \frac{1}{(\epsilon r)^Q} + b_r - C \leq \|\tilde{\phi}_{\epsilon, r}\|_q^q \leq A_g \log \frac{1}{(\epsilon r)^Q} + b_r + C. \quad (3.14)$$

Next we estimate $\|T\tilde{\phi}_{\epsilon, r}\|_q$. If $|x| \geq 2r$, using the same argument, since $\tilde{\phi}_{\epsilon}$ is orthogonal to p_j for $j \leq Q-1$, by (3.8) we get

$$T\tilde{\phi}_{\epsilon, r}(x) = \int_{B_r} \tilde{\phi}_{\epsilon, r}(y) R_Q(x, y) dy.$$

Arguing as in (2.106), by using (3.9) we get $|T\tilde{\phi}_{\epsilon,r}(x)| \leq C|x|^{\alpha-2Q}r^{2Q-\alpha}$, hence

$$\int_{|x| \geq 2r} |T\tilde{\phi}_{\epsilon,r}(x)|^q dx \leq Cr^{2Q-\alpha} \int_{|x| \geq 2r} |x|^{\alpha-2Q} dx \leq Cr^{2Q-\alpha} r^{\alpha-2Q+Q} = Cr^Q. \quad (3.15)$$

For the case $|x| \leq 2r$ we have

$$|TP_Q^r \phi_{\epsilon,r}(x)| \leq Cr^{-\alpha} \int_{|y^{-1}x| \leq r} |y|^{\alpha-Q} dy \leq Cr^{-\alpha} \int_{|y| \leq 4r} |y|^{\alpha-Q} dy = C$$

$$|T\tilde{\phi}_{\epsilon,r}(x)| \leq \int_{|y| \leq r} |y^{-1}x|^{\alpha-Q} |y|^{-\alpha} dy + C = \int_{|y| \leq 1} \left| y^{-1} \left(\frac{x}{r} \right) \right|^{\alpha-Q} |y|^{-\alpha} dy + C := I_\alpha \phi_0 \left(\frac{x}{r} \right) + C.$$

where $\phi_0(y) = |y|^{-\alpha} \chi_{|y| \leq r}$ is in $L^\tau(\mathbb{H}^n)$ if $1 < \tau < q$. Since we have $|x/r| \leq 2$ and $I_\alpha \in L^1(|x| \leq 2)$, so

$$m\{x : |I_\alpha \phi_0(x)| > 1\} \leq \|I_\alpha \phi_0\|_\tau^\tau \leq \|\phi_0\|_\tau^\tau \|I_\alpha\|_1^\tau \leq C \quad (3.16)$$

where the second inequality is by Young's inequality. Therefore

$$\int_{|x| \leq 2} |I_\alpha \phi_0(x) + C|^q dx \leq C$$

and hence

$$\int_{|x| \leq 2r} |T\tilde{\phi}_{\epsilon,r}(x)|^q dx \leq Cr^Q \int_{|x| \leq 2} |I_\alpha \phi_0(x) + C|^q dx = Cr^Q$$

So we have this similar estimation as before

$$\|T\tilde{\phi}_{\epsilon,r}\|_q^q \leq C_5 r^Q \quad (3.17)$$

Choose $r \geq 1$ such that

$$r^Q \leq \frac{A_g}{2C_5} \log \frac{1}{\epsilon^Q},$$

and define $b_{\epsilon,r} = A_g \log \frac{1}{\epsilon^Q} + b_r$. Then we have

$$\max\{||\tilde{\phi}_{\epsilon,r}||_q, ||T\tilde{\phi}_{\epsilon,r}||_q\} \leq (\max\{b_{\epsilon,r} + C, C_4 r^Q\})^{1/q} \leq b_{\epsilon,r}^{1/q} \left(1 + \frac{C}{\log \frac{1}{\epsilon^Q}}\right). \quad (3.18)$$

For $|x| \leq \epsilon r/2$ we estimate the potential from below and get

$$|T\tilde{\phi}_{\epsilon,r}| \geq b_{\epsilon,r} \left(1 - \frac{C}{\log \frac{1}{\epsilon^Q}}\right). \quad (3.19)$$

We define

$$\psi_{\epsilon,r} = \frac{\tilde{\phi}_{\epsilon,r}}{\max\{||\tilde{\phi}_{\epsilon,r}||_q, ||T\tilde{\phi}_{\epsilon,r}||_q\}} \quad (3.20)$$

so that we have for $|x| \leq \epsilon r/2$,

$$|T\psi_{\epsilon,r}(x)|^{q'} \geq A_g \log \frac{1}{(\epsilon r)^Q} + b_r \left(1 - \frac{C}{\log \frac{1}{\epsilon^Q}}\right) - C. \quad (3.21)$$

For the sharpness in the exponential constant, we take $\theta > 1$ and estimate the integral in (3.2) to get

$$\int_{E \cap B_{\epsilon r/2}} \frac{\exp \left[\frac{\theta}{A_g} |T\psi_{\epsilon,r}|^{q'} \right]}{1 + |T\psi_{\epsilon,r}|^{q'}} dx \geq C(\epsilon r)^Q \frac{\exp \left[\theta \log \frac{1}{(\epsilon r)^Q} - C \right]}{1 + C \log \frac{1}{(\epsilon r)^Q}} \geq C \frac{(\epsilon r)^{-(\theta-1)Q}}{1 + \log \frac{1}{(\epsilon r)^Q}} \rightarrow \infty \quad (3.22)$$

as $\epsilon \rightarrow 0^+$.

For the sharpness in the power of the denominator, take $r = 1$ and $\theta < 1$. Then it is clear that $b_r = 0$. By (3.14) and the definition of $\psi_{\epsilon,1}$ we deduce that

$$||T\psi_{\epsilon,1}||_q^q \leq C \left(\log \frac{1}{\epsilon^Q} \right)^{-1}.$$

On the other hand, we have

$$\begin{aligned} \int_{E \cap B_{\epsilon/2}} \frac{\exp \left[\frac{1}{A_g} |T\psi_{\epsilon,1}|^{q'} \right]}{1 + |T\psi_{\epsilon,1}|^{\theta q'}} dx &\geq C \epsilon^Q \frac{\exp \left[\log \frac{1}{\epsilon^Q} - C \right]}{1 + A_g \log \frac{1}{\epsilon^Q} - C} \\ &\geq C (\log \frac{1}{\epsilon^Q})^{-\theta}. \end{aligned} \quad (3.23)$$

Therefore the quotient of the above integral over $\|T\psi_{\epsilon,1}\|_q^q$ goes to infinity as $\epsilon \rightarrow 0^+$.

The vector case is completely analogue to the scalar case. So the sharpness of (3.2) is proved.

3.2 Moser-Trudinger inequality with exact growth condition on Heisenberg group

If $\alpha < Q$ is an even integer, we have the explicit formula for the fundamental solution of the powers of the $\mathbb{H}^n$ sublaplacian, which was presented in [BFM], [BDR], [CT]:

$$\mathcal{L}_0^{-\alpha/2}(x, 0) = \frac{1}{2} G_\alpha(\theta) |x|^{\alpha-Q} \quad (3.24)$$

$$G_\alpha(\theta) = \frac{2^{n+1} \Gamma(\frac{Q-\alpha}{2})}{\pi^{n+1} \Gamma(\frac{\alpha}{2})} \operatorname{Re} \left\{ e^{i \frac{Q-\alpha}{2} \theta} \int_0^\infty \left(\frac{s}{1 - e^{-2s}} \right)^{\frac{\alpha}{2}-1} \frac{e^{-ns}}{(e^{2i\theta} + e^{-2s})^{(\frac{Q-\alpha}{2})}} ds \right\} \quad (3.25)$$

where

$$\theta = \theta(u) = \arg \frac{|z|^2 + it}{|x|^2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \quad \text{for } x = (z, t) \in \mathbb{H}^n.$$

Therefore, as a direct consequence of Theorem 5 with $K(x) = \frac{1}{2} G_\alpha(\theta) |x|^{\alpha-Q}$, we have the following:

Theorem 6. *Let $0 < \alpha < Q$ be an even integer. Then there exists $C = C(Q, \alpha)$ such*

that for every $u \in W^{\alpha, \frac{Q}{\alpha}}(\mathbb{H}^n)$ with

$$\|(-\mathcal{L}_0)^{\frac{\alpha}{2}} u\|_{Q/\alpha} \leq 1 \quad (3.26)$$

we have

$$\int_{\mathbb{H}^n} \frac{\exp_{[\frac{Q}{\alpha}-2]} \left[\frac{1}{A_G} |u|^{\frac{Q}{Q-\alpha}} \right]}{1 + |u|^{\frac{Q}{Q-\alpha}}} dx \leq C \|u\|_{Q/\alpha}^{Q/\alpha} \quad (3.27)$$

where

$$A_G = \frac{1}{Q} \int_{\Sigma} \left| \frac{1}{2} G_{\alpha}(\theta) \right|^{\frac{Q}{Q-\alpha}} dx^*, \quad (3.28)$$

and the exponential constant A_G^{-1} is sharp, i.e. it cannot be replaced by any larger number. Furthermore, the inequality (3.27) cannot hold if the power $\frac{Q}{Q-\alpha}$ in the denominator is replaced by any smaller number.

Note that the relation between the surface measure dx^* in the above (3.28) and θ is as follows [BFM, (2.4)]

$$\int_{\Sigma} \phi dx^* = \omega_{2n-1} \int_{-\pi/2}^{\pi/2} \phi(\theta) (\cos \theta)^{n-1} d\theta,$$

where $\phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}$ is a measurable function.

In the case of subgradient, we have the following sharp inequality:

Theorem 7. *There exists $C = C(Q, \alpha)$ such that for every $u \in W^{1, Q}(\mathbb{H}^n)$ with*

$$\|\nabla_{\mathbb{H}^n} u\|_Q \leq 1 \quad (3.29)$$

we have

$$\int_{\mathbb{H}^n} \frac{\exp_{[Q-2]} \left[\frac{1}{A_Q} |u|^{\frac{Q}{Q-1}} \right]}{1 + |u|^{\frac{Q}{Q-1}}} dx \leq C \|u\|_Q^Q \quad (3.30)$$

where

$$A_Q = \frac{1}{Q} \left(\int_{\Sigma} |z^*|^Q dx^* \right)^{-\frac{1}{Q-1}} = \frac{1}{Q} \left(\frac{\Gamma(Q/2)\Gamma(n)}{2\pi^{n+1/2}\Gamma((Q-1)/2)} \right)^{\frac{1}{Q-1}}, \quad (3.31)$$

and the exponential constant A_Q^{-1} is sharp, i.e. it cannot be replaced by any larger number. Furthermore, the inequality (3.30) cannot hold if the power $\frac{Q}{Q-1}$ in the denominator is replaced by any smaller number.

For the computation of the constant A_Q in (3.31), see [CL1].

Proof: Let $x = (z, t)$, $y = (z', t')$. In the case of subgradient, the following representation formula was derived by [CL1]:

$$u(x) = \frac{-(c_\beta)^{-1}}{4} \int_{\mathbb{H}^n} \frac{|z'|^{\beta-2}}{|y|^{Q+\beta}} \nabla_{\mathbb{H}^n}(|y|^4) \cdot \nabla_{\mathbb{H}^n} F(y^{-1}x) dy \quad (3.32)$$

where “ $\cdot$ ” denotes the standard inner product of two vectors, $\beta > -2n + 1$, $c_\beta = \int_{\Sigma} |z^*|^\beta dx^*$ and $u \in C_0^\infty(\mathbb{H}^n)$. Take $\beta = Q$ and

$$K(x) = \frac{-(c_Q)^{-1}}{4} \frac{|z|^{Q-2}}{|x|^{2Q}} \nabla_{\mathbb{H}^n}(|x|^4), \quad x \neq 0,$$

we will verify that it satisfies the conditions of Theorem 5. By [CL1, Lemma 2.1] we have $|\nabla_{\mathbb{H}^n}(|x|^4)| = 4|z||x|^2$. Therefore

$$|K(x)| = c_Q^{-1} |z|^{Q-2} |x|^{2-2Q} = c_Q^{-1} |z^*|^{Q-2} |x|^{1-Q} := g(x^*) |x|^{1-Q}.$$

Therefore (3.30) holds with

$$A_Q = \frac{1}{Q} \int_{\Sigma} |g(x^*)|^{\frac{Q}{Q-1}} dx^* = \frac{1}{Q} (c_Q)^{-\frac{Q}{Q-1}} \int_{\Sigma} |z^*|^{(Q-1)\frac{Q}{Q-1}} dx^* = \frac{1}{Q} (c_Q)^{-\frac{1}{Q-1}}.$$

To prove the sharpness, we use the same extremal family of functions as in [CL1].

We consider the function

$$v_\epsilon(y) = \begin{cases} \log \frac{1}{|y|} & \text{for } \epsilon \leq |y| \leq 1 \\ \log \frac{1}{|\epsilon|} & \text{for } |y| \leq \epsilon. \end{cases}$$

Then [CL1] computed that

$$\|\nabla_{\mathbb{H}^n} v_\epsilon\|_Q^{\frac{Q}{Q-1}} = \left(\log \frac{1}{\epsilon}\right)^{\frac{1}{Q-1}} c_Q^{\frac{1}{Q-1}}. \quad (3.33)$$

Also, it is clear that $\|v_\epsilon\|_Q \leq C$. Let

$$u_\epsilon = \frac{v_\epsilon}{\|\nabla_{\mathbb{H}^n} v_\epsilon\|_Q},$$

so that

$$\|\nabla_{\mathbb{H}^n} u_\epsilon\|_Q \leq 1, \quad \|u_\epsilon\|_Q^Q \leq C \left(\log \frac{1}{\epsilon}\right)^{-1},$$

and

$$|u_\epsilon|^{\frac{Q}{Q-1}} \geq c_Q^{\frac{1}{Q-1}} \log \frac{1}{\epsilon} \quad \text{for } |y| \leq \epsilon.$$

Similarly to the case in $\mathbb{R}^n$, for the sharpness of the exponential constant, we take $\theta > 1$ and estimate

$$\begin{aligned} \int_{\mathbb{H}^n} \frac{\exp\left[\frac{n}{\alpha}-2\right] \left[\theta A_Q^{-1} |u_\epsilon|^{\frac{Q}{Q-1}}\right]}{1 + |u_\epsilon|^{\frac{Q}{Q-1}}} dy &\geq \int_{|y| \leq \epsilon} \frac{\exp\left[\theta A_Q^{-1} |u_\epsilon|^{\frac{Q}{Q-1}}\right]}{1 + |u_\epsilon|^{\frac{Q}{Q-1}}} dy \geq C \epsilon^Q \frac{\exp\left[\theta \log \frac{1}{\epsilon^Q} + C\right]}{1 + C \log \frac{1}{\epsilon}} \\ &= C \frac{\epsilon^{(1-\theta)Q}}{1 + C \log \frac{1}{\epsilon}} \rightarrow \infty \end{aligned}$$

as $\epsilon \rightarrow 0^+$.

For the sharpness of the power of the denominator, we take $\theta < 1$ and get

$$\begin{aligned} \int_{\mathbb{H}^n} \frac{\exp[\frac{n}{\alpha}-2] [A_Q^{-1}|u_\epsilon|^{\frac{Q}{Q-1}}]}{1 + |u_\epsilon|^{\frac{\theta Q}{Q-1}}} dy &\geq \int_{|y| \leq \epsilon} \frac{\exp[A_Q^{-1}|u_\epsilon|^{\frac{Q}{Q-1}}]}{1 + |u_\epsilon|^{\frac{\theta Q}{Q-1}}} dy \geq C\epsilon^Q \frac{\exp\left[\log \frac{1}{\epsilon^Q} + C\right]}{1 + C(\log \frac{1}{\epsilon})^\theta} \\ &\geq C(\log \frac{1}{\epsilon})^{-\theta}. \end{aligned}$$

Hence we have that the quotient of the above integral over the norm $\|u_\epsilon\|_Q^Q$ goes to infinity as $\epsilon \rightarrow 0^+$, so the sharpness follows. $\square$

Lastly, we mention that in the same way as we did in $\mathbb{R}^n$, the above Theorem 7 implies the sharp Moser-Trudinger inequality for the subgradient with Ruf condition on $\mathbb{H}^n$, which was derived by Lam and Lu [LL1].

Similarly, from Theorem 6 we can obtain the sharp Moser-Trudinger inequalities for the powers of sublaplacian under Ruf conditions on $\mathbb{H}^n$:

Corollary 4. *Let $0 < \alpha < Q$ be an even integer. There exists $C = C(Q, \alpha)$ such that for every $u \in W^{\alpha, \frac{Q}{\alpha}}(\mathbb{H}^n)$ with*

$$\|u\|_{Q/\alpha} + \|(-\mathcal{L}_0)^{\frac{\alpha}{2}} u\|_{Q/\alpha} \leq 1 \quad (3.34)$$

we have

$$\int_{\mathbb{H}^n} \exp[\frac{Q}{\alpha}-2] \left[\frac{1}{A_G} |u|^{\frac{Q}{Q-\alpha}} \right] dx \leq C \quad (3.35)$$

where A_G is the same as in (3.28), and A_G^{-1} is sharp.

The proof of sharpness is completely similar to the one in $\mathbb{R}^n$, so we omit it here. The corresponding Adachi-Tanaka type inequalities can also be derived from Theorems 6, 7.

Appendix

A. Proof of Lemma 5

Fix $1 > h_0 > 0$. The above lemma is also true for all $1 \geq h > h_0$. Since $h \leq 1$, we have $a_k \leq 1$ for all k . So

$$\|a\|_1 > \|a\|_q$$

suppose we have

$$\|a\|_1 = h, \quad \|a\|_q \leq \|a\|_1 \leq h \leq 1.$$

then we can take $a' = a/h$, to have

$$\|a'\|_1 = 1, \quad \|a'\|_q \leq 1$$

hence

$$\mu_d(h) = \frac{1}{h^{\beta q}} \mu_d(1) \leq \frac{c}{h^{\beta q}}$$

where the last inequality is by taking $a = \{a_1\} = \{1\}$ and then using the definition of $\mu_d(1)$.

Notice that $\mu_d(h)$ increases as $h > 0$ increases, it is enough to show that , for any integer $N \geq 2$, the following holds

$$C_1(q, \beta) \frac{e^{qN}}{N^\beta} \leq \mu_d(N^{1/q'}) \leq C_2(q, \beta) \frac{e^{qN}}{N^\beta}. \quad (3.36)$$

WLOG, we will assume a sequence a is nonnegative throughout the whole proof. Let $\|a\|_{(e)} = \left(\sum_{k=0}^{\infty} |a_k|^{\beta k} e^{qk} \right)^{1/q}$. Consider the sequence $a_k = N^{-1/q}$ for $k = 0, \dots, N-1$, and $a_k = 0$ otherwise. Then $\|a\|_1 = N^{1/q'}$,

$\|a\|_q = 1$ and

$$\mu_d(N^{1/q'}) \leq \|a\|_{(e)}^q = N^{-\beta} \frac{e^{Nq} - e^q}{e^q - 1} \leq C \frac{e^{Nq}}{N^\beta}$$

So we are left to prove

$$C_1(q, \beta) \frac{e^{qN}}{N^\beta} \leq \mu_d(N^{1/q'}). \quad (3.37)$$

We consider two cases:

Case 1: $q \geq 2$

Let a be a sequence such that

$$\|a\|_1 = N^{1/q'} \quad \|a\|_q = 1 \quad (3.38)$$

Let $a' = a\chi_{k \leq N-1}$, then

$$\|a'\|_1 = \|a\|_1 - \sum_{k=N}^{\infty} |a_k| = N^{1/q'} - \sum_{k=N}^{\infty} |a_k|$$

By the inequality $(1-x)^p \geq 1-px$ for $p > 1$, $0 \leq x < 1$, we have

$$\|a'\|_1^{q'} = \left(N^{1/q'} - \sum_{k=N}^{\infty} |a_k| \right)^{q'} \geq N \left(1 - q' \frac{\sum_{k=N}^{\infty} |a_k|}{N^{1/q'}} \right) = N - q' N^{1/q} \sum_{k=N}^{\infty} |a_k| \quad (3.39)$$

Also, we have the following upper bound (see [LTZ](4.7)):

$$\|a'\|_1^{q'} \leq N - \frac{q'}{2} \frac{\sum_{0 \leq j, k \leq N-1} \frac{1}{2} (a_j - a_k)^2}{N^{1-2/q}} \quad (3.40)$$

Combining (3.39) and (3.40) we get

$$\sum_{0 \leq j, k \leq N-1} \frac{1}{2} (a_j - a_k)^2 \leq 2N^{1/q'} \sum_{k \geq N}^{\infty} |a_k|$$

Take $a_m := \min\{|a_k| : k \leq N-1\}$,

$$\begin{aligned} \|a'\|_1 - N|a_m| &\leq \|a_k - a_m\|_{l^1} \leq N^{1/2} \|a_k - a_m\|_{l^2} \\ &\leq N^{1/2} \left(2N^{1/q'} \sum_{k \geq N}^{\infty} |a_k| \right)^{1/2} = 2N^{1-1/(2q)} \left(\sum_{k \geq N}^{\infty} |a_k| \right)^{1/2} \end{aligned} \quad (3.41)$$

Therefore

$$\begin{aligned} |a_m| &\geq N^{-1} \|a'\|_1 - 2N^{-1/(2q)} \left(\sum_{k \geq N}^{\infty} |a_k| \right)^{1/2} \\ &= N^{-1/q} \left(1 - N^{-1} (N^{1/q} \sum_{k \geq N}^{\infty} |a_k|) - 2 (N^{1/q} \sum_{k \geq N}^{\infty} |a_k|)^{1/2} \right) \end{aligned} \quad (3.42)$$

From definition of $\|a\|_{(e)}$, it is clear that $|a_k| \leq \|a\|_{(e)}^{1/\beta} e^{-k/\beta}$, and hence

$$\sum_{k=N}^{\infty} |a_k| \leq \|a\|_{(e)}^{1/\beta} \sum_{k=N}^{\infty} e^{-k/\beta} = \|a\|_{(e)}^{1/\beta} \frac{e^{-N/\beta}}{1 - e^{-1/\beta}} =: C_\beta \frac{J^{1/(\beta q)}}{N^{1/q}} \quad (3.43)$$

where we define

$$C_\beta = (1 - e^{-1/\beta})^{-1}, \quad J = \|a\|_{(e)}^q \frac{N^\beta}{e^{Nq}} \quad (3.44)$$

So (3.42) becomes

$$|a_m| \geq N^{-1/q} \left(1 - N^{-1} C_\beta J^{1/(\beta q)} - 2 (C_\beta J^{1/(\beta q)})^{1/2} \right)$$

Assume that

$$C_\beta J^{1/(\beta q)} = \delta^2 < 1,$$

we have

$$\begin{aligned}
||a||_{(e)}^q &\geq \sum_{k=0}^{N-1} |a_m|^{q\beta} e^{qk} = |a_m|^{q\beta} \frac{e^{qN} - 1}{e^q - 1} \\
&\geq \frac{e^{qN}}{e^q} |a_m|^{q\beta} \geq \frac{e^{qN}}{e^q} \left(N^{-1/q} (1 - N^{-1} C_\beta J^{1/(\beta q)} - 2(C_\beta J^{1/(\beta q)})^{1/2}) \right)^{q\beta} \\
&= \frac{1}{e^q} \frac{e^{qN}}{N^\beta} (1 - N^{-1} \delta^2 - 2\delta)^{q\beta} \geq \frac{1}{e^q} \frac{e^{qN}}{N^\beta} (1 - 3\delta)^{q\beta}
\end{aligned}$$

that is,

$$e^q J = e^q (\delta^2 / C_\beta)^{q\beta} \geq (1 - 3\delta)^{q\beta}$$

Since $C_\beta > 1$, we have

$$e^{1/\beta} \delta^2 \geq 1 - 3\delta \quad (3.45)$$

But the above inequality (3.45) is false if $\delta \leq \frac{1}{4e^{1/\beta}}$. Hence we have

$$\delta^2 > \frac{1}{16e^{2/\beta}}$$

which is equivalent to, by definition,

$$J = ||a||_{(e)}^q \frac{N^\beta}{e^{Nq}} > \left(\frac{1 - e^{-1/\beta}}{16e^{2/\beta}} \right)^{q\beta} \quad (3.46)$$

So (3.37) is proved in this case.

Case 2: $1 < q < 2$

Similarly to case 1, let a be a sequence that satisfies (3.38), and $a' = a\chi_{k \leq N-1}$. By (3.43) and with the same definition of C_β , J as in case 1, we have

$$\sum_{k=N}^{\infty} |a_k| \leq C_\beta \frac{J^{1/(\beta q)}}{N^{1/q}}$$

and

$$\sum_{k=N}^{\infty} |a_k|^q \leq ||a||_{(e)}^{q/\beta} \sum_{k=N}^{\infty} e^{-qk/\beta} = ||a||_{(e)}^{q/\beta} \frac{e^{-qN/\beta}}{1 - e^{-q/\beta}} \leq C_\beta^q \frac{J^{1/\beta}}{N}.$$

Combining with (3.38), it implies

$$\sum_{k=0}^{N-1} |a_k| \geq N^{1/q'} - C_\beta \frac{J^{1/(\beta q)}}{N^{1/q}}, \quad \sum_{k=0}^{N-1} |a_k|^q \geq 1 - C_\beta^q \frac{J^{1/\beta}}{N}$$

Rewrite as

$$0 \leq \sum_{k=0}^{N-1} (1 - N^{1/q} a_k) \leq C_\beta J^{1/(\beta q)}, \quad 0 \leq \sum_{k=0}^{N-1} (1 - (N^{1/q} |a_k|)^q) \leq C_\beta^q J^{1/\beta}$$

Let $b_k = 1 - N^{1/q} a_k$, then the above inequalities is the same as

$$0 \leq \sum_{k=0}^{N-1} b_k \leq C_\beta J^{1/(\beta q)}, \quad 0 \leq \sum_{k=0}^{N-1} (1 - (1 - b_k)^q) \leq C_\beta^q J^{1/\beta} \quad (3.47)$$

Applying Jensen's inequality, we have by (3.47)

$$N \geq \sum_{k=0}^{N-1} (1 - b_k)^q \geq N \left(\frac{N - \sum_{k=0}^{N-1} b_k}{N} \right)^q \geq N \left(1 - \frac{C_\beta J^{1/(\beta q)}}{N} \right)^q \geq N - q C_\beta J^{1/(\beta q)}$$

Note that by definition of b_k , we have $1 - b_k \geq 0$. Hence if we assume that

$$C_\beta J^{1/(\beta q)} = \delta^2 \ll 1$$

we will have $b_k \leq c \ll 1$, which enable us to apply Tayler's formula centered at 0 as follows for $k \leq N - 1$:

$$1 - (1 - b_k)^q = 1 - (1 - qb_k + \frac{q(q-1)}{2} b_k^2 + cb_k^2)$$

with $|c| < \frac{q(q-1)}{4}$. So

$$\frac{4}{3q(q-1)} (qb_k - (1 - (1 - b_k)^q)) \leq b_k^2 \leq \frac{4}{q(q-1)} (qb_k - (1 - (1 - b_k)^q))$$

Take summation of the above estimation and use (3.47) we have

$$\left| \sum_{k=0}^{N-1} (1 - (N^{1/q} a_k)^2) \right| = \left| \sum_{k=0}^{N-1} b_k^2 - 2 \sum_{k=0}^{N-1} b_k \right| \leq c_q \delta^2 \quad (3.48)$$

where $c_q = \frac{4}{q(q-1)}$. Now we go back to a_k and by (3.48) we have

$$N^{1-2/q} - c_q N^{-2/q} \delta^2 \leq \sum_{k=0}^{N-1} a_k^2 \leq N^{1-2/q} + c_q N^{-2/q} \delta^2 \quad (3.49)$$

On the other hand, similar to (3.40) in case 1 we have, by (3.49)

$$\|a'\|_1^{q'} \leq N - \frac{\sum_{0 \leq j, k \leq N-1} \frac{1}{2} (a_j - a_k)^2}{N^{1-2/q}} \left(1 + \frac{c_q \delta^2}{N}\right) + c_q \delta^2$$

By (3.39) and (3.43),

$$N - \frac{\sum_{0 \leq j, k \leq N-1} \frac{1}{2} (a_j - a_k)^2}{N^{1-2/q}} \left(1 + \frac{c_q \delta^2}{N}\right) + c_q \delta^2 > N - q' \delta^2$$

which gives

$$\sum_{0 \leq j, k \leq N-1} \frac{1}{2} (a_j - a_k)^2 \leq 2c_q \delta^2 N^{1-2/q}$$

Again, let $a_m := \min\{|a_k| : k \leq N-1\}$ and estimate

$$\|a'\|_1 - N|a_m| \leq \|a_k - a_m\|_{l^1} \leq N^{1/2} \|a_k - a_m\|_{l^2} \leq N^{1/2} (c_q \delta^2 N^{1-2/q})^{1/2} = c_q^{1/2} N^{1-1/q} \delta \quad (3.50)$$

Hence, we have

$$|a_m| \geq N^{-1/q} \left(1 - N^{-1} \delta^2 - c_q^{1/2} \delta\right)$$

and

$$\|a\|_{(e)}^q \geq \frac{1}{e^q} \frac{e^{qN}}{N^\beta} (1 - N^{-1} \delta^2 - c_q^{1/2} \delta)^{q\beta} \geq \frac{1}{e^q} \frac{e^{qN}}{N^\beta} (1 - 2c_q^{1/2} \delta)^{q\beta}$$

So

$$e^{1/\beta} \geq 1 - 2c_q^{1/2}\delta \quad (3.51)$$

It is clear that the above inequality is false if

$$0 < \delta \leq c_{q,\beta} < \frac{-c_q^{1/2} + \sqrt{c_q + e^{1/\beta}}}{e^{1/\beta}}$$

So we have $\delta^2 > c_{q,b}^2$, which gives us

$$J > \left(\frac{c_{q,\beta}}{C_\beta} \right)^{q\beta} \quad (3.52)$$

B. Proof of Lemma 6

Proof of (2.71): Using (A3) we get

$$\begin{aligned} \int_{D_{J+1} \setminus D_J} |Tf_\tau(x)| dx &\leq \frac{C}{|D_{J+1} \setminus D_J|} \int_{\mathbb{R}^n} |f_\tau(y)| \int_{D_{J+1} \setminus D_J} |x-y|^{\alpha-n} dx dy \\ &\leq \frac{C}{r_J^n} \int_{\mathbb{R}^n} |f_\tau(y)| r_J^\alpha dy. \end{aligned} \quad (3.53)$$

Here the second inequality above is by the straightforward computation:

$$\begin{aligned} &\int_{D_{J+1} \setminus D_J} |x-y|^{\alpha-n} dx \\ &= \int_{\{|x-y| \leq r_J\} \cap (D_{J+1} \setminus D_J)} |x-y|^{\alpha-n} dx + \int_{\{|x-y| > r_J\} \cap (D_{J+1} \setminus D_J)} |x-y|^{\alpha-n} dx \\ &\leq \int_{\{|x| \leq r_J\}} |x|^{\alpha-n} dx + \int_{D_{J+1} \setminus D_J} r_J^{\alpha-n} dx \leq Cr_J^\alpha + Cr_J^{\alpha-n} r_J^n = Cr_J^\alpha. \end{aligned} \quad (3.54)$$

Recall that $|F_\tau| = \tau = |D_0|$, we have

$$\begin{aligned} \frac{C}{r_J^n} \int_{\mathbb{R}^n} |f_\tau(y)| r_J^\alpha dy &= Cr_J^{\alpha-n} \int_{\mathbb{R}^n} |f_\tau(y)| dy = Cr_J^{\alpha-n} \int_{\mathbb{R}^n} |f| \chi_{F_\tau} dy \\ &\leq Cr_J^{\alpha-n} |F_\tau|^{1/q'} \|f\|_q \leq Cr_J^{\alpha-n} r_0^{n-\alpha} = C_1 \left(\frac{r_0}{r_J} \right)^{n-\alpha} = C_1 \left(\frac{1}{e^{q-1}} \right)^J. \end{aligned} \quad (3.55)$$

Proof of (2.72): First write

$$TS_{J+2}f'_\tau(x) - TS_{J+1}f'_\tau(x_1) = TS_{J+2}f'_\tau(x) - TS_{J+2}f'_\tau(x_1) - T(S_{J+1} - S_{J+2})f'_\tau(x_1)$$

so

$$\begin{aligned} &|TS_{J+2}f'_\tau(x) - TS_{J+1}f'_\tau(x_1)| \\ &\leq |TS_{J+2}f'_\tau(x) - TS_{J+2}f'_\tau(x_1)| + |T(S_{J+1} - S_{J+2})f'_\tau(x_1)|. \end{aligned} \quad (3.56)$$

Use similar argument as in (2.59), we get

$$\begin{aligned}
|T(S_{J+1} - S_{J+2})f'_\tau(x_1)| &= \left| \int_{D_{J+2} \setminus D_{J+1}} K(x_1 - y) f'_\tau(y) dy \right| \\
&\leq C \left| \int_{D_{J+2} \setminus D_{J+1}} |x_1 - y|^{\alpha-n} f'_\tau(y) dy \right| \\
&\leq Cr_{J+1}^{\alpha-n} \int_{D_{J+2} \setminus D_{J+1}} |f'_\tau(y)| dy \leq Cr_{J+1}^{\alpha-n} r_{J+1}^{n-\alpha} \alpha_{J+1} \leq C\beta_{J+1}.
\end{aligned} \tag{3.57}$$

Next, by the regularity assumption (A3), since $x_1 \in D_0$, we have for $x \in D_{J+1} \setminus D_J$ and $y \in D_{J+2}^c$

$$|K(x - y) - K(x_1 - y)| \leq C|x - x_1|(e^{q/n})^{n+1-\alpha}|x_1 - y|^{\alpha-n-1}.$$

Hence

$$\begin{aligned}
|TS_{J+2}f'_\tau(x) - TS_{J+2}f'_\tau(x_1)| &= \left| \int_{D_{J+2}^c} (K(x - y) - K(x_1 - y)) f'_\tau(y) dy \right| \\
&\leq C|x - x_1| \int_{D_{J+2}^c} |f'_\tau(y)| |x_1 - y|^{\alpha-n-1} dy \\
&\leq C|x - x_1| \beta_{J+2} \left(\int_{D_{J+2}^c} |x_1 - y|^{-n-\frac{n}{n-\alpha}} dy \right)^{\frac{n-\alpha}{n}} \\
&\leq Cr_{J+1} \beta_{J+2} \frac{C}{r_{J+2}} \leq C\beta_{J+2} \leq C\beta_{J+1}.
\end{aligned} \tag{3.58}$$

So we have (2.72) by (3.56)-(3.58).

Proof of (2.73) and (2.74): Let $j \in \{0, 1, \dots, J\}$, then

$$\begin{aligned}
\left| \int_{D_{J+1} \setminus D_J} T(S_j - S_{j+1}) f'_\tau(x) dx \right| &= \left| \int_{D_{J+1} \setminus D_J} \int_{D_{j+1} \setminus D_j} K(x-y) f'_\tau(y) dy dx \right| \\
&\leq C \int_{D_{J+1} \setminus D_J} \int_{D_{j+1} \setminus D_j} |x-y|^{\alpha-n} |f'_\tau(y)| dy dx \\
&= C \int_{D_{j+1} \setminus D_j} |f'_\tau(y)| \int_{D_{J+1} \setminus D_J} |x-y|^{\alpha-n} dx dy \\
&\leq C r_J^{\alpha-n} \int_{D_{j+1} \setminus D_j} |f'_\tau(y)| dy \leq C r_J^{\alpha-n} r_j^{n-\alpha} \alpha_j \\
&= C e^{\frac{q}{n}(j-J)} \alpha_j
\end{aligned} \tag{3.59}$$

where the second inequality is by (3.54). Therefore,

$$\begin{aligned}
\left| \int_{D_{J+1} \setminus D_J} T(S_0 - S_{J+2}) f'_\tau(x) dx \right| &= \left| \int_{D_{J+1} \setminus D_J} \left(\sum_{j=0}^{J+1} T(S_j - S_{j+1}) f'_\tau(x) \right) dx \right| \\
&\leq \sum_{j=0}^{J+1} \left| \int_{D_{J+1} \setminus D_J} T(S_j - S_{j+1}) f'_\tau(x) dx \right| \\
&\leq C \sum_{j=0}^{J+1} e^{\frac{q}{n}(j-J)} \alpha_j \leq C \bar{\alpha}_{J+1}.
\end{aligned} \tag{3.60}$$

So we have (2.73). For (2.74), by similar calculations as in (3.59), we have

$$\begin{aligned}
\left| \int_{D_{J+1} \setminus D_J} T(f'_\tau \chi_{D_0})(x) dx \right| &= \left| \int_{D_{J+1} \setminus D_J} \int_{D_0} K(x-y) f'_\tau(y) dy dx \right| \\
&\leq C \int_{D_{J+1} \setminus D_J} \int_{D_0} |x-y|^{\alpha-n} |f'_\tau(y)| dy dx \\
&= C \int_{D_0} |f'_\tau(y)| \int_{D_{J+1} \setminus D_J} |x-y|^{\alpha-n} dx dy \\
&\leq C r_J^{\alpha-n} \int_{D_0} |f'_\tau(y)| dy \leq C r_J^{\alpha-n} r_0^{n-\alpha} \alpha_{-1} \\
&\leq C \alpha_{-1}.
\end{aligned} \tag{3.61}$$

□

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