VOLUMES OF LINE BUNDLES ON SCHEMES

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A mi mamá y a mi papá.

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ABSTRACT

The volume of a line bundle is an invariant defined in terms of a limit superior. It is a fundamental question whether this limit superior is a limit. It has been shown that this is always the case on generically reduced proper schemes over arbitrary fields. We show that volumes are limits in two classes of schemes that are not necessarily generically reduced: codimension one subschemes of projective varieties such that their components of maximal dimension contain normal points, and projective schemes whose nilradical squared equals zero.

Chapter 1

Introduction

Let X be a projective scheme of dimension d over a field k . Let $\mathcal{L}$ be a line bundle on X . The *volume* of $\mathcal{L}$ is defined as follows:

$$\mathrm{vol}(\mathcal{L}) := \limsup_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!},$$

where $h^0(X, \mathcal{L}^n) = \dim_k H^0(X, \mathcal{L}^n)$. This invariant has been intensively studied and many interesting results regarding its behaviour are known. We will now present some of its elementary properties. These properties are known at different levels of generality, so in order to keep the presentation simple we will assume for now that our projective scheme X is integral and our base field is algebraically closed. We refer the reader interested in a more detailed account of the properties of the volume function to the article [9], which gives an interesting survey of the general properties of this invariant; a more technical account with references can be found in [12, Section 2.2.C].

Let $N^1(X)_{\mathbb{R}}$ be the finite dimensional vector space of $\mathbb{R}$ -divisors modulo numerical equivalence. First, the volume of a line bundle descends to a continuous function

$$\mathrm{vol}: N^1(X)_{\mathbb{R}} \rightarrow \mathbb{R},$$

which is homogeneous of degree d . Moreover, this function is birationally invariant

in the sense that if $\pi: X' \rightarrow X$ is a birational map from a projective k -scheme, then

$$\mathrm{vol}(\pi^* \mathcal{L}) = \mathrm{vol}(\mathcal{L})$$

for any $\mathcal{L} \in N^1(X)_{\mathbb{R}}$. In fact, more is known. When the base field has characteristic zero, Boucksom, Favre, and Jonsson proved in [2] that this function is C^1 -differentiable in the big cone of X . They also showed that the directional derivative of the volume function can be interpreted in terms of intersections products. These results were later extended to projective varieties over arbitrary fields by Cutkosky in [4]. An example of a line bundle on a projective variety whose volume is an irrational number is given in Example 4 of Section 7 of [7].

To elucidate what sort of geometric information is encoded in the volume of a line bundle, let us consider an example.

Example 1.0.1. Let X be a nonsingular projective curve over an algebraically closed field. Let D be a divisor with $\deg(D) \geq 0$. Then

$$\mathrm{vol}(D) := \mathrm{vol}(\mathcal{O}_X(D)) = \deg(D).$$

Notice that in this example the line bundle $\mathcal{O}_X(D)$ is ample if and only if $\mathrm{vol}(\mathcal{O}_X(D)) > 0$. As it is frequently the case, the setting of nonsingular curves is much too restrictive to admit word-by-word generalizations to higher dimensions; however, it is fruitful to ask what geometric properties can be expected in general from a line bundle with nonzero volume.

Definition 1.0.2. A line bundle $\mathcal{L}$ on a d -dimensional projective scheme X over a field is said to be *big* if

$$\liminf_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!} > 0.$$

Big line bundles are birational analogs of ample line bundles. For example, high enough powers of big line bundles induce birational maps into closed subschemes of

projective space (see [8, Exercise 5 in Section 1.12]). It is clear from the definitions that

$$\mathcal{L} \text{ is big} \implies \text{vol}(\mathcal{L}) > 0.$$

It is however not known whether the reverse implication is true. A natural condition that would guarantee that bigness can be detected by the nonvanishing of the volume function is that the numbers

$$\liminf_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!}$$

agree in a common limit. Whenever this is the case, we will say that the volume of $\mathcal{L}$ exists as a limit. In this direction, we have the following theorem.

Theorem 1.0.3 ([3, Theorem 10.7]). *Suppose that X is a proper scheme of dimension d over a field k . Let $\mathcal{N}$ be the nilradical of X . Assume that $\dim(\text{Supp}(\mathcal{N})) < d$. Let $\mathcal{L}$ be a line bundle on X . Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k H^0(X, \mathcal{L}^n)}{n^d}$$

exists. That is, $\text{vol}(\mathcal{L})$ exists as a limit.

Theorem 1.0.3 has been proven by Lazarsfeld and Mustata [13] for big line bundles and by Kaveh and Khovanskii [11] for graded linear series (see definition below), both under the assumption that k is an algebraically closed field. These proofs use a method invented by Okounkov [18] which reduces the situation to a problem of counting points in an integral semigroup. In these proofs, the assumption that the base field is algebraically closed is necessary. The proof of Theorem 1.0.3 also uses Okounkov's technique.

Recall that X is a projective scheme over a field k . To any line bundle $\mathcal{L}$ on X we can associate a *section ring*

$$R(\mathcal{L}) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n).$$

Under the inclusion $k \subset \mathcal{O}_X(X)$, this is a graded k -algebra. One is naturally led to consider graded subalgebras of this section ring. For example, fix a line bundle $\mathcal{L}$ and let X_{red} be the reduced scheme associated to X . For all $n \in \mathbb{N}$, the natural morphism $\mathcal{O}_X \rightarrow \mathcal{O}_{X_{red}}$ can be tensored with $\mathcal{L}^n$ to induce maps of vector spaces

$$\Phi_n: H^0(X, \mathcal{L}^n) \rightarrow H^0(X_{red}, (\mathcal{L}|_{X_{red}})^n).$$

Since the global section functor is not right-exact, it is not in general true that the k -algebra

$$L := \bigoplus_{n \geq 0} \Phi_n(H^0(X, \mathcal{L}^n))$$

is equal to the section ring associated to $\mathcal{L}|_{X_{red}}$ on X_{red} . It is however the case that L is a k -subalgebra of $R(\mathcal{L}|_{X_{red}})$.

A *graded linear series* on X (often simply called a linear series) is a graded k -subalgebra $L = \bigoplus_{n \geq 0} L_n$ of a section ring $R(\mathcal{L})$ of some line bundle $\mathcal{L}$ on X . The linear series L is said to be *complete* if $L = R(\mathcal{L})$. We define the *volume* of a linear series L on X as follows:

$$\text{vol}(L) := \limsup_{n \rightarrow \infty} \frac{\dim_k L_n}{n^d/d!}.$$

The following theorem is a consequence of [3, Theorem 1.4] and [3, Theorem 10.3].

Theorem 1.0.4. *Suppose that X is a d -dimensional projective scheme over a field k with $d > 0$. Let $\mathcal{N}$ be the nilradical of X . Then, the following are equivalent:*

- 1) *For every graded linear series L on X there exists a positive integer r (depending on L) such that the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{rn}}{n^d}$$

exists.

2) $\dim(\text{Supp}(\mathcal{N})) < d$.

The implication 1) $\implies$ 2) of Theorem 1.0.4 is proven by constructing, on any given projective scheme not satisfying 2), a linear series L such that the limit in 1) does not exist (for any r). In particular, the volume $\text{vol}(L)$ does not exist as a limit. This linear series, however, is not a complete linear series. Therefore, the following question was raised in [6].

Question 1.0.5. Let X be a projective scheme over a field and let $\mathcal{L}$ be a line bundle on X . Does the volume of $\mathcal{L}$ exist as a limit?

This question has a positive answer whenever $\dim(\text{Supp}(\mathcal{N})) < d$ by Theorem 1.0.3. In [6], Question 1.0.5 was answered affirmatively for codimension 1 subschemes of nonsingular varieties. The main results of this thesis consist of positive answers to Question 1.0.5 in two different settings that we will describe precisely in the next section.

1.1 Main Results

The following are our main results as given in [17].

Theorem 1.1.1. *Let X be a $d+1$ -dimensional projective variety over an arbitrary field. Let $Y \subset X$ be a subscheme of codimension 1. Assume that no component of Y of maximal dimension is contained in the nonnormal locus of X . Let $\mathcal{L}$ be a line bundle on Y . Then, the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k H^0(Y, \mathcal{L}^n)}{n^d}$$

exists. That is, $\text{vol}(\mathcal{L})$ exists as a limit.

Theorem 1.1.2. *Let X be a d -dimensional projective scheme over an arbitrary field. Suppose that the nilradical $\mathcal{N}$ of X has the property that $\mathcal{N}^2 = 0$. Let $\mathcal{L}$ be a line bundle on X . Then, the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k H^0(X, \mathcal{L}^n)}{n^d}$$

exists. That is, $\text{vol}(\mathcal{L})$ exists as a limit.

Chapter 2

Nilradicals and Volumes

2.1 Notation and Terminology

Let X be a d -dimensional proper scheme over an arbitrary field. The scheme X is said to be a *variety* if it is integral. For any coherent $\mathcal{O}_X$ -module $\mathcal{F}$, we will use the following notation:

$$h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F}).$$

Let $\mathcal{L}$ and $\mathcal{F}$ represent a line bundle and a coherent sheaf on X , respectively. We define the *volume of $\mathcal{F}$ with respect to $\mathcal{L}$* to be

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) := \limsup_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F} \otimes \mathcal{L}^n)}{n^d/d!}.$$

We say that the volume of $\mathcal{F}$ with respect to $\mathcal{L}$ exists as a limit if the limsup above is actually a limit. Notice that $\mathrm{vol}_{\mathcal{L}}(\mathcal{O}_X) = \mathrm{vol}(\mathcal{L})$ is the volume of the line bundle $\mathcal{L}$ as defined in Chapter 1. In fact, Lemma 2.2.5 shows that if $\mathrm{vol}(\mathcal{L})$ exists as a limit, then $\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) = \mathrm{vol}(\mathcal{L})$ whenever $\mathcal{F}$ is invertible.

The *nilradical* of a scheme X is the kernel of the natural surjection $\mathcal{O}_X \rightarrow \mathcal{O}_{X_{\mathrm{red}}}$, where X_{red} is the reduced scheme associated to X . The nilradical of X will be denoted $\mathcal{N}(X)$ or simply $\mathcal{N}$ if there is no danger of confusion.

Following [15, Lecture 9], we say that a global section s of a line bundle $\mathcal{L}$

on a scheme X is a *non-zero-divisor* if the map $\mathcal{O}_X \rightarrow \mathcal{L}$ induced by s is injective. Locally, this means that for any point $p \in X$ and some (any) trivialization $\varphi: \mathcal{L}_p \rightarrow \mathcal{O}_{X,p}$, the element $\varphi(s)$ is a non-zero-divisor in the local ring $\mathcal{O}_{X,p}$.

Let L be a linear series on X . When X is a variety, we define the *index of L* to be natural number

$$m(L) := [\mathbb{Z} : G],$$

where G is the group generated by the semi-group $\{n \in \mathbb{N} : L_n \neq 0\}$. We define the *index* of a line bundle $\mathcal{L}$ on a variety to be the index of the complete linear series $R(\mathcal{L})$.

When X is a variety, we define the rank of any coherent $\mathcal{O}_X$ -module $\mathcal{F}$ to be the dimension of the $\mathcal{O}_{X,\xi}$ -vector space $\mathcal{F}_\xi$, where ξ is the generic point of X .

Throughout this thesis, the term “divisor” will always refer to a Cartier divisor. For divisors A and B , we will write $A \sim B$ to indicate linear equivalence.

We will use the terms “line bundle” and “invertible sheaves” interchangeably to refer to invertible sheaves.

2.2 Preliminary Results

2.2.1 Volumes and Exact Sequences

Lemma 2.2.1. *Let X be a proper scheme over a field k . Let $\mathcal{L}$ be an invertible sheaf. Let*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be an exact sequence of coherent $\mathcal{O}_X$ -modules. Let $\{1, 2, 3\} = \{i, j, k\}$. Suppose that the volume of $\mathcal{F}_i$ with respect to $\mathcal{L}$ exists as a limit and that $\mathcal{F}_k$ is supported on a closed subset of dimension strictly less than $\dim(X)$. Then the volume of $\mathcal{F}_j$ with respect to $\mathcal{L}$ exists as a limit as well. Moreover,

$$\text{vol}_{\mathcal{L}}(\mathcal{F}_i) = \text{vol}_{\mathcal{L}}(\mathcal{F}_j).$$

Proof. Fix $n \in \mathbb{N}$ and let $d = \dim(X)$. Tensoring the given exact sequence with $\mathcal{L}^n$ and taking global sections we obtain the following sequence of vector spaces:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n) &\longrightarrow H^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n) \xrightarrow{\delta_n} H^0(X, \mathcal{F}_3 \otimes \mathcal{L}^n) \\ &\xrightarrow{\epsilon_n} H^1(X, \mathcal{F}_1 \otimes \mathcal{L}^n). \end{aligned}$$

From this we get the following two exact sequences:

$$0 \rightarrow H^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n) \rightarrow H^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n) \rightarrow \text{image}(\delta_n) \rightarrow 0$$

and

$$0 \rightarrow H^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n) \rightarrow H^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n) \rightarrow H^0(X, \mathcal{F}_3 \otimes \mathcal{L}^n) \rightarrow \text{image}(\epsilon_n) \rightarrow 0.$$

By the additivity of dimension it follows that

$$h^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n) = h^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n) + \dim_k(\text{image}(\delta_n)) \quad (2.2.1)$$

and

$$\begin{aligned} h^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n) &= h^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n) + h^0(X, \mathcal{F}_3 \otimes \mathcal{L}^n) \\ &\quad - \dim_k(\text{image}(\epsilon_n)). \end{aligned} \quad (2.2.2)$$

Suppose first that $k = 3$ and $\{1, 2\} = \{i, j\}$. By [8, Proposition 1.31], since the support of $\mathcal{F}_3$ has dimension smaller than d , we get

$$\lim_{n \rightarrow \infty} \frac{\dim_k(\text{image}(\delta_n))}{n^d} \leq \lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_3 \otimes \mathcal{L}^n)}{n^d} = 0.$$

Thus, by (2.2.1), we see that

$$\lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n)}{n^d} = \lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n)}{n^d}.$$

We conclude that in this case the volume of $\mathcal{F}_1$ with respect to $\mathcal{L}$ exists as a limit if and only if the volume of $\mathcal{F}_2$ with respect to $\mathcal{L}$ exists as a limit. Moreover, when this happens, we have $\text{vol}_{\mathcal{L}}(\mathcal{F}_1) = \text{vol}_{\mathcal{L}}(\mathcal{F}_2)$.

Assume now that $k = 1$ and $\{2, 3\} = \{i, j\}$. By [8, Proposition 1.31], since the support of $\mathcal{F}_1$ has dimension smaller than d , it follows that

$$\lim_{n \rightarrow \infty} \frac{\dim_k(\text{image}(\epsilon_n))}{n^d} \leq \lim_{n \rightarrow \infty} \frac{h^1(X, \mathcal{F}_1 \otimes \mathcal{L}^n)}{n^d} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_1 \otimes \mathcal{L}^n)}{n^d} = 0.$$

By (2.2.2), we get that

$$\lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_2 \otimes \mathcal{L}^n)}{n^d} = \lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{F}_3 \otimes \mathcal{L}^n)}{n^d}.$$

Again, we conclude that in this case the volume of $\mathcal{F}_2$ with respect to $\mathcal{L}$ exists as a limit if and only if the volume of $\mathcal{F}_3$ with respect to $\mathcal{L}$ exists as a limit. Moreover, in this case $\text{vol}_{\mathcal{L}}(\mathcal{F}_2) = \text{vol}_{\mathcal{L}}(\mathcal{F}_3)$.

Finally suppose that $k = 2$; in other words, suppose that $\mathcal{F}_2$ is supported on a closed subscheme of dimension smaller than d . Then $\mathcal{F}_1$ and $\mathcal{F}_3$ are also supported on closed sets of dimension smaller than the dimension of X since $\text{supp}(\mathcal{F}_1) \cup \text{supp}(\mathcal{F}_3) \subset \text{supp}(\mathcal{F}_2)$. In this case, by [8, Proposition 1.31], all the limits in question equal zero and the lemma follows.

□

The following corollary shows that the existence of volumes as limits is preserved by morphisms of sheaves that are isomorphisms away from closed sets of dimension smaller than the dimension of the ambient scheme.

Corollary 2.2.2. *Let X be a proper scheme over a field k . Let $\mathcal{L}$ be an invertible sheaf. Let*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0$$

be an exact sequence of coherent $\mathcal{O}_X$ -modules. Suppose that $\mathcal{K}$ and $\mathcal{C}$ are supported on closed subsets of dimension strictly less than $\dim(X)$. Then the volume of $\mathcal{F}$ with respect to $\mathcal{L}$ exists as a limit if and only if the volume of $\mathcal{G}$ with respect to $\mathcal{L}$ exists as a limit and then

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{G}).$$

Proof. We break up the given exact sequence into the following two exact sequences:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{K} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{F}/\mathcal{K} \rightarrow \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0.$$

If the volume of $\mathcal{F}$ with respect to $\mathcal{L}$ exists as a limit, we apply Lemma 2.2.1 to the first sequence to conclude that the same is true for $\mathcal{F}/\mathcal{K}$ and that $\mathrm{vol}_{\mathcal{L}}(\mathcal{F}/\mathcal{K}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{F})$. Then, by Lemma 2.2.1 and the second sequence, we see that the volume of $\mathcal{G}$ with respect to $\mathcal{L}$ exists as a limit and $\mathrm{vol}_{\mathcal{L}}(\mathcal{G}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{F}/\mathcal{K}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{F})$.

An analogous argument shows that if the volume of $\mathcal{G}$ with respect to $\mathcal{L}$ exists as a limit, then the same is true for $\mathcal{F}$ and $\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{G})$.

□

2.2.2 Non-zero-divisors, Invertible Sheaves, and Cartier Divisors

Lemma 2.2.3. *Let X be a projective scheme over a field. Let $\mathcal{M}$ be an invertible sheaf on X and let $\mathcal{A}$ be an ample invertible sheaf. Then, there exists an $n_0 \in \mathbb{N}$ such that $H^0(X, \mathcal{A}^n \otimes \mathcal{M})$ contains a non-zero-divisor for $n \geq n_0$.*

Proof. Let $V_1, \dots, V_s$ be the associated subvarieties of X and choose closed points $x_i \in V_i$. Let $\mathcal{I} = \bigoplus_{i=1}^s \mathcal{I}_{x_i}$, where $\mathcal{I}_{x_i}$ is the ideal sheaf of the point x_i . We have a short exact sequence of $\mathcal{O}_X$ -modules

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{x_i} \rightarrow 0.$$

Tensor this exact sequence with $\mathcal{A}^n \otimes \mathcal{M}$ and take cohomology. By Serre's Vanishing Theorem, for $n \gg 0$, we have an exact sequence

$$H^0(X, \mathcal{A}^n \otimes \mathcal{M}) \rightarrow \bigoplus_{i=1}^s k(x_i) \rightarrow 0,$$

where $k(x_i)$ is the residue field of the point x_i . Thus, there exists a section $s \in H^0(X, \mathcal{A}^n \otimes \mathcal{M})$ which does not vanish at any of the x_i . It follows that s is a non-zero-divisor on $\mathcal{O}_X$ since it does not vanish along any of the V_i . □

For any scheme X , the association $D \rightarrow \mathcal{O}_X(D)$ gives an injective homomorphism from the group of Cartier divisors modulo linear equivalence to $\text{Pic}(X)$ (see [10, Corollary II.6.14]). Nakai [16] has shown that if X is a projective scheme over an infinite field, then this homomorphism is an isomorphism. We deduce the known fact that this is still the case for projective schemes over arbitrary fields.

Corollary 2.2.4. *Let X be a projective scheme over a field. Then, for any invertible sheaf $\mathcal{L}$, there exists a Cartier divisor D such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Moreover, under the identification described above, $\text{Pic}(X)$ is generated by effective divisors.*

Proof. Choose an ample line bundle $\mathcal{A}$ on X . After perhaps replacing $\mathcal{A}$ with a positive power of itself, we can use Lemma 2.2.3 with $\mathcal{M} = \mathcal{O}_X$ to find a non-zero-divisor $t \in H^0(X, \mathcal{A})$. Then $\mathcal{A} \simeq \mathcal{O}_X(H)$, where $H := \text{div}(t)$.

Again by Lemma 2.2.3, for some $n \in \mathbb{N}$, we can find a non-zero-divisor $s \in H^0(X, \mathcal{O}_X(nH) \otimes \mathcal{L})$. Thus, $\mathcal{O}_X(nH) \otimes \mathcal{L} \simeq \mathcal{O}_X(\text{div}(s))$. Setting $D = \text{div}(s) - nH$, we get that $\mathcal{L} \simeq \mathcal{O}_X(D)$.

For the last statement of the corollary, simply notice that D is a difference of effective divisors.

□

2.2.3 A Lemma on Volume

We now show that volumes are unaffected by tensoring with invertible sheaves.

Lemma 2.2.5. *Let X be a projective scheme over a field. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X . Suppose that the volume of $\mathcal{L}$ exists as a limit. Then, $\text{vol}_{\mathcal{L}}(\mathcal{M})$ also exists as a limit and*

$$\text{vol}_{\mathcal{L}}(\mathcal{M}) = \text{vol}(\mathcal{L}).$$

Proof. By Corollary 2.2.4, we can assume that $\mathcal{M} = \mathcal{O}_X(D)$ for some Cartier divisor D . Let us consider first the case where D is effective. We have a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

which, after tensoring with $\mathcal{O}_X(D)$, becomes

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_D \otimes \mathcal{O}_X(D) \rightarrow 0.$$

Now, $\text{vol}_{\mathcal{L}}(\mathcal{O}_X) = \text{vol}(\mathcal{L})$ exists as a limit by assumption. Also, the sheaf $\mathcal{O}_D \otimes \mathcal{O}_X(D)$ is supported on a closed set of dimension smaller than the dimension

of X . Thus, by Lemma 2.2.1, $\text{vol}_{\mathcal{L}}(\mathcal{O}_X(D))$ exists as a limit and $\text{vol}(\mathcal{L}) = \text{vol}_{\mathcal{L}}(\mathcal{O}_X(D))$.

Suppose now that D is an arbitrary Cartier divisor. Thanks to Corollary 2.2.4, we can write $D \sim A - B$ where both A and B are effective Cartier divisors. Since B is effective, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-B) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_B \rightarrow 0.$$

We can tensor the sequence above with $\mathcal{O}_X(A)$ and get

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(A) \rightarrow \mathcal{O}_B \otimes \mathcal{O}_X(A) \rightarrow 0.$$

Since A is effective, we proved above that $\text{vol}_{\mathcal{L}}(\mathcal{O}_X(A))$ exists as a limit and $\text{vol}(\mathcal{L}) = \text{vol}_{\mathcal{L}}(\mathcal{O}_X(A))$. Moreover, the sheaf $\mathcal{O}_B \otimes \mathcal{O}_X(A)$ is supported on a closed set of dimension smaller than the dimension of X . By Lemma 2.2.1, we conclude that $\text{vol}_{\mathcal{L}}(\mathcal{O}_X(D))$ exists as a limit and

$$\text{vol}_{\mathcal{L}}(\mathcal{O}_X(D)) = \text{vol}_{\mathcal{L}}(\mathcal{O}_X(A)) = \text{vol}(\mathcal{L}).$$

□

2.2.4 Volumes of generically free coherent sheaves with respect to line bundles

Lemma 2.2.6. *Let X be an irreducible projective scheme over a field and let $\mathcal{L}$ be a line bundle on X whose volume exists as a limit. Let $\mathcal{F}$ be a generically free coherent sheaf. Then the volume of $\mathcal{F}$ with respect to $\mathcal{L}$ exists as a limit and*

$$\text{vol}_{\mathcal{L}}(\mathcal{F}) = \text{rank}(\mathcal{F})\text{vol}(\mathcal{L}).$$

Proof. Let $\mathcal{A}$ be an ample line bundle such that $\mathcal{F} \otimes \mathcal{A}$ is generated by global sections. Let ξ be the generic point of X . Let $n = \text{rank}(\mathcal{F})$. Choose n global

sections $s_1, \dots, s_n$ such that they form a basis of $(\mathcal{F} \otimes \mathcal{A})_\xi$ over $\mathcal{O}_{X,\xi}$. These n sections induce a map $\bigoplus_{i=1}^n \mathcal{O}_X \rightarrow \mathcal{F} \otimes \mathcal{A}$, which is an isomorphism at the generic point. We have then a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \bigoplus_{i=1}^n \mathcal{O}_X \rightarrow \mathcal{F} \otimes \mathcal{A} \rightarrow \mathcal{C} \rightarrow 0,$$

where $\dim(\text{Supp}(\mathcal{K})) < \dim(X)$ and $\dim(\text{Supp}(\mathcal{C})) < \dim(X)$. Tensoring with $\mathcal{M} := \mathcal{A}^{-1}$ we get

$$0 \rightarrow \mathcal{K} \otimes \mathcal{M} \rightarrow \bigoplus_{i=1}^n \mathcal{M} \rightarrow \mathcal{F} \rightarrow \mathcal{C} \otimes \mathcal{M} \rightarrow 0.$$

Notice that

$$\text{vol}_{\mathcal{L}} \left(\bigoplus_{i=1}^n \mathcal{M} \right) = n \text{vol}_{\mathcal{L}}(\mathcal{M}) = \text{rank}(\mathcal{F}) \text{vol}_{\mathcal{L}}(\mathcal{M})$$

exists as a limit by Lemma 2.2.5. The result now follows from Corollary 2.2.2 applied to the second exact sequence. $\square$

2.3 Existence of Volumes of Linear Series on Varieties

In this section we will state a fundamental result regarding volumes of linear series. Our main theorems will depend crucially on this result. Theorem 2.3.1 was proven in [13] for certain linear series on projective varieties over an algebraically closed field and in [11] for arbitrary linear series on projective varieties over algebraically closed fields. The following theorem follows from [3, Theorem 8.1].

Theorem 2.3.1 (Cutkosky). *Let X be a d -dimensional proper variety over a field k . Let L be a graded linear series on X and let $m = m(L)$ be its index. Then,*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{mn}}{n^d}$$

exists.

Let $\mathcal{L}$ be a line bundle on a proper variety. It can be shown that if $\text{vol}(\mathcal{L}) > 0$, then the index of $\mathcal{L}$ is equal to one (the index of a line bundle is defined in Section 2.1). We can thus take $L = R(\mathcal{L})$ in Theorem 2.3.1 to show that the volume of any line bundle on a proper variety exists as a limit.

2.4 Nilradicals and Volumes

The following proposition refines [6, Proposition 4.1]. The first part of the proof is as in [6], but we repeat it here for the reader's convenience.

Proposition 2.4.1. *Let X be an irreducible projective scheme over a field k . Let $\mathcal{N}$ be the nilradical of X . Let $\mathcal{L}$ be an invertible sheaf on X . Suppose that $\text{vol}_{\mathcal{L}}(\mathcal{N})$ exists as a limit. Then, $\text{vol}(\mathcal{L})$ exists as a limit as well and we have the formula*

$$\text{vol}(\mathcal{L}) = \text{vol}(L) + \text{vol}_{\mathcal{L}}(\mathcal{N}),$$

where L is the linear series obtained by restricting the sections of $\mathcal{L}$ to X_{red} . Moreover, either $m(L) = 1$ or $\text{vol}(L) = 0$.

Proof. Let $d = \dim(X)$. Fix $n \in \mathbb{N}$. The short exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}} \rightarrow 0$$

can be tensored with $\mathcal{L}^n$ to get

$$0 \rightarrow \mathcal{N} \otimes \mathcal{L}^n \rightarrow \mathcal{L}^n \rightarrow (\mathcal{L}|_{X_{\text{red}}})^n \rightarrow 0.$$

Taking global sections and using the additivity of dimension we get that

$$\frac{h^0(X, \mathcal{L}^n)}{n^d/d!} = \frac{h^0(X, \mathcal{N} \otimes \mathcal{L}^n)}{n^d/d!} + \frac{\dim_k L_n}{n^d/d!}, \quad (2.4.1)$$

where $L_n = \text{image} \left(H^0(X, \mathcal{L}^n) \rightarrow H^0(X_{red}, (\mathcal{L}|_{X_{red}})^n) \right)$. The k -algebra $L = \bigoplus_n L_n$ is a graded linear series associated to $\mathcal{L}|_{X_{red}}$. Fix an ample divisor $\mathcal{A}$ such that there exist sections $\alpha \in H^0(X, \mathcal{A})$ and $\beta \in H^0(X, \mathcal{A} \otimes \mathcal{L})$ that are non-zero-divisors. The fact that this is possible is a consequence of Lemma 2.2.3. We have in particular that the restrictions $\alpha|_{X_{red}}$ and $\beta|_{X_{red}}$ to X_{red} are not zero. This is the case since α and β are non-zero-divisors and X_{red} is an associated subvariety of X . We consider two cases.

Case 1. Suppose that there exists $n_0 > 0$ such that the restriction map

$$H^0(X, \mathcal{A}^{-1} \otimes \mathcal{L}^{n_0}) \rightarrow H^0(X_{red}, \mathcal{A}^{-1} \otimes (\mathcal{L}|_{X_{red}})^{n_0})$$

is not zero. Choose $\gamma \in H^0(X, \mathcal{A}^{-1} \otimes \mathcal{L}^{n_0})$ such that $\gamma|_{X_{red}} \neq 0$. Since X_{red} is a variety,

$$0 \neq \alpha|_{X_{red}} \otimes \gamma|_{X_{red}} \in L_{n_0}$$

and

$$0 \neq \beta|_{X_{red}} \otimes \gamma|_{X_{red}} \in L_{n_0+1}.$$

It follows that the index of L is equal to 1. By Theorem 2.3.1, $\text{vol}(L)$ exists as a limit. Recall that $\text{vol}_{\mathcal{L}}(\mathcal{N})$ exists as a limit by assumption. Taking limits in (2.4.1) one gets that $\text{vol}(\mathcal{L})$ exists as a limit and

$$\text{vol}(\mathcal{L}) = \text{vol}(L) + \text{vol}_{\mathcal{L}}(\mathcal{N}).$$

Case 2. Suppose now that for all $n > 0$ the restriction map

$$H^0(X, \mathcal{A}^{-1} \otimes \mathcal{L}^n) \rightarrow H^0(X_{red}, \mathcal{A}^{-1} \otimes (\mathcal{L}|_{X_{red}})^n)$$

is the zero map. We begin by showing that $\text{vol}_{\mathcal{L}}(\mathcal{N} \otimes \mathcal{A}^{-1})$ exists as a limit and

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{N} \otimes \mathcal{A}^{-1}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{N}). \quad (2.4.2)$$

The global section α of $\mathcal{A}$ induces a short exact sequence

$$0 \rightarrow \mathcal{A}^{-1} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0, \quad (2.4.3)$$

where $H := \mathrm{div}(\alpha)$ is a closed subscheme of dimension smaller than the dimension of X . We can tensor this exact sequence with $\mathcal{N}$ to get

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A}^{-1} \otimes \mathcal{N} \rightarrow \mathcal{N} \rightarrow \mathcal{O}_H \otimes \mathcal{N} \rightarrow 0.$$

We will show that $\mathcal{K} = 0$. Let $p \in X$ and let $f \in \mathcal{O}_{X,p}$ be a local equation for α at p . We have that f is a non-zero-divisor on $\mathcal{O}_{X,p}$. Moreover, since $\mathcal{N}_p \subset \mathcal{O}_{X,p}$, then f is a non-zero-divisor on $\mathcal{N}_p$. Thus, $(\mathcal{A}^{-1} \otimes \mathcal{N})_p \rightarrow \mathcal{N}_p$ is injective and $\mathcal{K}_p = 0$. Since p was arbitrary, $\mathcal{K} = 0$.

Therefore, we have a short exact sequence

$$0 \rightarrow \mathcal{A}^{-1} \otimes \mathcal{N} \rightarrow \mathcal{N} \rightarrow \mathcal{O}_H \otimes \mathcal{N} \rightarrow 0.$$

Since $\mathrm{vol}_{\mathcal{L}}(\mathcal{N})$ exists as a limit by assumption, we can apply Lemma 2.2.1 to conclude that $\mathrm{vol}_{\mathcal{L}}(\mathcal{N} \otimes \mathcal{A}^{-1})$ exists as a limit and (2.4.2) holds.

It follows from the assumption of *Case 2* that

$$\mathrm{H}^0(X, \mathcal{N} \otimes \mathcal{A}^{-1} \otimes \mathcal{L}^n) = \mathrm{H}^0(X, \mathcal{A}^{-1} \otimes \mathcal{L}^n)$$

for all $n > 0$. This shows the $\mathrm{vol}_{\mathcal{L}}(\mathcal{A}^{-1})$ exists as a limit and

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{A}^{-1}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{N} \otimes \mathcal{A}^{-1}). \quad (2.4.4)$$

Finally, apply Lemma 2.2.1 to the short exact sequence (2.4.3) to get that $\mathrm{vol}(\mathcal{L})$ exists as a limit and

$$\mathrm{vol}(\mathcal{L}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{A}^{-1}). \tag{2.4.5}$$

Combining (2.4.2), (2.4.4), and (2.4.5), we get the relation

$$\mathrm{vol}(\mathcal{L}) = \mathrm{vol}_{\mathcal{L}}(\mathcal{N}).$$

Looking back at (2.4.1), this shows that $\mathrm{vol}(L) = 0$.

□

Chapter 3

Codimension One Case

As a first step towards proving Theorem 1.1.1, we address the case where X is normal and the subscheme Y has an invertible ideal sheaf. So, our first goal is to prove the following proposition.

Proposition 3.0.1. *Let X be a projective normal variety over an arbitrary field. Let $\mathcal{I}$ be a locally principal ideal sheaf with associated closed subscheme Y . Then, the volume of any invertible sheaf on Y exists as a limit.*

We will begin by proving a special case of this proposition, but before we need some preliminaries. Let X be a Noetherian scheme. Let $\mathcal{P} \subset \mathcal{O}_X$ be a prime ideal sheaf, that is, $\mathcal{P}$ is an ideal sheaf whose associated closed subscheme Y is integral. Let ξ be the generic point of Y . We define the n^{th} symbolic power of $\mathcal{P}$, denoted $\mathcal{P}^{(n)}$, to be the ideal sheaf given as the kernel of the natural map

$$\mathcal{O}_X \rightarrow \frac{\mathcal{O}_{X,\xi}}{m_\xi^n}.$$

We agree that the homomorphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,\xi}/m_\xi^n$ is the zero map whenever $\xi \notin U$. Thus, if $U \subset X$ is an open set not containing ξ , we have that $\mathcal{P}^{(n)}(U) = \mathcal{O}_X(U)$. On the other hand, for an affine open subset $U \subset X$ containing ξ , $\mathcal{P}^{(n)}(U)$ is equal to the n^{th} symbolic power of the prime ideal $\mathcal{P}(U)$ in the ring $\mathcal{O}_X(U)$.

Proposition 3.0.2. *Let X be a normal projective variety over an arbitrary field. Let $\mathcal{P}$ be a prime ideal sheaf with associated integral subscheme Y . Assume that Y*

has codimension one. Let m be a positive integer. Denote by $Y^{(m)}$ the subscheme associated to the ideal sheaf $\mathcal{P}^{(m)}$. Then, the volume of any invertible sheaf on $Y^{(m)}$ exists as a limit.

Proof. We argue by induction on m , where the case $m = 1$ follows from Theorem 2.3.1 ([3, Proposition 8.1]) since $Y = Y^{(1)}$ is a projective variety. Fix some $m > 1$. Let $\mathcal{N}$ be the nilradical of $Y^{(m)}$ and let $\mathcal{L}$ be an invertible sheaf on $Y^{(m)}$. By Proposition 2.4.1, since $Y^{(m)}$ is irreducible, it is enough to prove that $\text{vol}_{\mathcal{L}}(\mathcal{N})$ exists as a limit and we now proceed to do so.

We first show that $\mathcal{N}$ is generically free of rank one as an $\mathcal{O}_{Y^{(m-1)}}$ -module. Notice that $\mathcal{N} = \mathcal{P}/\mathcal{P}^{(m)}$ is a $\mathcal{O}_{Y^{(m-1)}} = \mathcal{O}_X/\mathcal{P}^{(m-1)}$ -module, since $\mathcal{P} \cdot \mathcal{P}^{(m-1)} \subset \mathcal{P}^{(m)}$. Let ξ be the generic point of Y . Since X is normal and Y has codimension one, X must be nonsingular at ξ . We then have that $\mathcal{P}_{\xi} = (f)$ for some $f \in \mathcal{O}_{X,\xi}$. Since $\mathcal{P}_{\xi}$ is principal, $\mathcal{P}_{\xi}^{(m)} = \mathcal{P}_{\xi}^m$ and then $\mathcal{N}_{\xi} = \mathcal{P}_{\xi}/\mathcal{P}_{\xi}^{(m)} = (f)/(f^m)$ is free of rank one as a $\mathcal{O}_{Y^{(m-1)},\xi}$ -module.

By induction, the volume of $\mathcal{L}|_{Y^{(m-1)}}$ exists as a limit. By Lemma 2.2.6, we have that

$$\text{vol}_{\mathcal{L}|_{Y^{(m-1)}}}(\mathcal{N}) = \text{vol}(\mathcal{L}|_{Y^{(m-1)}})$$

and these volumes exist as limits. Now, for any $n \in \mathbb{N}$, since $\mathcal{N}$ is both an $\mathcal{O}_{Y^{(m)}}$ -module and an $\mathcal{O}_{Y^{(m-1)}}$ -module, we have that

$$H^0(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n) = H^0(Y^{(m-1)}, \mathcal{N} \otimes (\mathcal{L}|_{Y^{(m-1)}})^n)$$

and hence the limit

$$\begin{aligned} \text{vol}_{\mathcal{L}}(\mathcal{N}) &= \lim_{n \rightarrow \infty} \frac{h^0(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n)}{n^d/d!} = \lim_{n \rightarrow \infty} \frac{h^0(Y^{(m-1)}, \mathcal{N} \otimes (\mathcal{L}|_{Y^{(m-1)}})^n)}{n^d/d!} \\ &= \text{vol}_{\mathcal{L}|_{Y^{(m-1)}}}(\mathcal{N}) \end{aligned}$$

exists, where $d = \dim(Y^{(m)}) = \dim(Y^{(m-1)})$. By Proposition 2.4.1, the proof is

complete.

□

Remark. Notice that we have also shown in the proof that

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{N}) = \mathrm{vol}(\mathcal{L}|_{Y^{(m-1)}}).$$

By Proposition 2.4.1, we obtain the relation

$$\mathrm{vol}(\mathcal{L}) = \mathrm{vol}(\mathcal{L}|_{Y^{(m-1)}}) + \mathrm{vol}(L), \quad (3.0.1)$$

where L is the linear series obtained by restricting the sections of $\mathcal{L}$ to Y . Moreover, either $m(L) = 1$ or $\mathrm{vol}(L) = 0$.

If we assume that $\mathcal{L}$ is a nef invertible sheaf, we can refine (3.0.1). Under this additional assumption, we claim that $\mathrm{vol}(\mathcal{L}|_Y) = \mathrm{vol}(L)$. Indeed, let $d = \dim(Y^{(m)})$ and take cohomology of the short exact sequence

$$0 \rightarrow \mathcal{N} \otimes \mathcal{L}^n \rightarrow \mathcal{L}^n \rightarrow \mathcal{O}_Y \otimes \mathcal{L}^n \rightarrow 0$$

to obtain

$$0 \rightarrow H^0(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n) \rightarrow H^0(Y^{(m)}, \mathcal{L}^n) \rightarrow H^0(Y, (\mathcal{L}|_Y)^n) \rightarrow V_n \rightarrow 0,$$

where V_n is the image of the map $H^0(Y, (\mathcal{L}|_Y)^n) \rightarrow H^1(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n)$. In particular, we have the relation

$$h^0(Y, (\mathcal{L}|_Y)^n) = h^0(Y^{(m)}, \mathcal{L}^n) - h^0(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n) + \dim_k(V_n). \quad (3.0.2)$$

Now, by [8, Proposition 1.31], since $\mathcal{L}$ is nef, we have that

$$\lim_{n \rightarrow \infty} \frac{\dim_k V_n}{n^d} \leq \lim_{n \rightarrow \infty} \frac{h^1(Y^{(m)}, \mathcal{N} \otimes \mathcal{L}^n)}{n^d} = 0.$$

Thus, dividing (3.0.2) by $n^d/d!$ and letting n go to infinity we see that

$$\mathrm{vol}(\mathcal{L}|_Y) = \mathrm{vol}(\mathcal{L}) - \mathrm{vol}_{\mathcal{L}}(\mathcal{N}),$$

and the quantity on the right is exactly $\mathrm{vol}(L)$. Therefore, (3.0.1) yields by induction the relation

$$\mathrm{vol}(\mathcal{L}) = m \cdot \mathrm{vol}(\mathcal{L}|_Y).$$

It is known [12, Section 2.2.C] that, when $\mathcal{L}$ is a nef invertible sheaf,

$$\mathrm{vol}(\mathcal{L}) = (\mathcal{L})^d,$$

where $(\mathcal{L})^d$ is the top self-intersection of $\mathcal{L}$. The equation above is then equivalent to

$$(\mathcal{L})^d = m \cdot (\mathcal{L}|_Y)^d.$$

Before giving the proof of Proposition 3.0.1, for lack of a precise reference, we state here as a lemma the following standard fact.

Lemma 3.0.1. *Let R be a Noetherian normal domain and let I be a proper nonzero principal ideal. Then, there exist height one prime ideals $P_1, \dots, P_t$ and positive integers $n_1, \dots, n_t$ such that*

$$I = P_1^{(n_1)} \cap \dots \cap P_t^{(n_t)}.$$

Proof. We have an irredundant primary decomposition

$$I = Q_1 \cap \dots \cap Q_t,$$

where the Q_i are primary ideals. By [14, Theorem 11.5], the prime ideals $P_i = \sqrt{Q_i}$ have height one. Thus, for any i , we can localize at P_i to get

$$I_{P_i} = (Q_i)_{P_i}, \quad (3.0.3)$$

thanks to [1, Proposition 4.8.(i)]. Since R is normal and P_i has height one, the ring R_{P_i} is a DVR. Thus, $I_{P_i} = (P_i)_{P_i}^{n_i}$ for some positive integer n_i . We can then contract (3.0.3) back to R and apply [1, Proposition 4.8.(ii)] to get that

$$P_i^{(n_i)} = Q_i.$$

□

Proof of Proposition 3.0.1. Since X is normal and $\mathcal{I}$ locally principal, there exist prime ideal sheaves $\mathcal{P}_1, \dots, \mathcal{P}_t$ such that

$$\mathcal{I} = \mathcal{P}_1^{(n_1)} \cap \dots \cap \mathcal{P}_t^{(n_t)}.$$

Moreover, the varieties associated to the ideal sheaves $\mathcal{P}_i$ have codimension one in X . This follows from Lemma 3.0.1 applied to an affine open cover of X . Let $Y_i^{(n_i)}$ be the subscheme associated to the ideal sheaf $\mathcal{P}_i^{(n_i)}$. This decomposition gives an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \prod_{i=1}^t \mathcal{O}_{Y_i^{(n_i)}} \rightarrow \mathcal{C} \rightarrow 0.$$

The sheaf $\mathcal{C}$ is supported on a closed subset of dimension smaller than the dimension of Y . Let $\mathcal{L}$ be an invertible sheaf on Y . The volume of $\prod_{i=1}^t \mathcal{O}_{Y_i^{(n_i)}}$ with respect to $\mathcal{L}$ exists as a limit by Proposition 3.0.2. By Lemma 2.2.1, the volume of $\mathcal{L}$ exists as a limit and

$$\text{vol}(\mathcal{L}) = \sum_{i=1}^t \text{vol}(\mathcal{L}|_{Y_i^{(n_i)}}).$$

□

We are now ready to give a proof of Theorem 1.1.1. We recall the notation for convenience. In the following, X will represent a variety over an arbitrary field and Y will denote a subscheme of X of codimension one. Moreover, we assume that no codimension one component of Y is contained in the nonnormal locus of X .

Proof of Theorem 1.1.1. Let V be the nonsingular locus of X and let Z be a codimension one component of Y . We claim that $V \cap Z \neq \emptyset$. To see this, first let W be the normal locus of X . By assumption, $W \cap Z$ is a closed codimension one subset of W . Since W is a normal open subscheme of X , its singular locus has codimension at least two and hence $W \cap Z$ must contain a nonsingular point of W . But this point is also nonsingular in X since W is open and hence $V \cap Z \neq \emptyset$ as we claimed.

Let $\pi: \tilde{X} \rightarrow X$ be the normalization of the blow-up of the ideal sheaf $\mathcal{I}_Y$ of Y in X . By [5, Lemma 6.11] (which is valid over arbitrary fields), the morphism π is an isomorphism over the open set

$$U := \{p \in X : p \text{ is normal and } \mathcal{I}_{Y,p} \text{ is a principal ideal}\}.$$

Notice that every codimension one component of Y intersects U . Indeed, every component of Y contains a nonsingular point of X , nonsingular points are normal, and the ideal sheaf $\mathcal{I}$ is principal at nonsingular points since Y is of codimension one.

Let $\tilde{Y}$ be the total transform of Y in $\tilde{X}$, that is, $\tilde{Y}$ is the closed subscheme of $\tilde{X}$ given by the invertible ideal sheaf $\mathcal{I}_Y \mathcal{O}_{\tilde{X}}$. The morphism π restricts to a morphism $\tilde{Y} \rightarrow Y$, which we also call π . We have that π is an isomorphism over $Y \cap U$. Moreover, $\pi_* \mathcal{O}_{\tilde{Y}}$ is a coherent $\mathcal{O}_Y$ -module since both $\tilde{X}$ and X are projective varieties (and hence π is a projective morphism). We have an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{C} \rightarrow 0,$$

where the sheaves $\mathcal{K}$ and $\mathcal{C}$ are supported on closed subsets of dimension smaller than the dimension of Y . Let $\mathcal{L}$ be an invertible sheaf on Y . By the projection formula, for any $n \in \mathbb{N}$, we have that $\pi_* \mathcal{O}_{\tilde{Y}} \otimes \mathcal{L}^n \cong \pi_* (\pi^* \mathcal{L})^n$ and hence $\text{vol}_{\mathcal{L}}(\pi_* \mathcal{O}_{\tilde{Y}})$ exists as a limit if and only if $\text{vol}(\pi^* \mathcal{L})$ exists as a limit. Now, notice that $\tilde{Y}$ is a closed subscheme of the normal projective variety $\tilde{X}$ given by an invertible ideal sheaf. By Proposition 3.0.1, the volume of $\pi^* \mathcal{L}$ exists as a limit. By Corollary 2.2.2 applied to the exact sequence above, the volume of $\mathcal{L}$ exists as a limit. Moreover,

$$\text{vol}(\mathcal{L}) = \text{vol}_{\mathcal{L}}(\pi_* \mathcal{O}_{\tilde{Y}}) = \text{vol}(\pi^* \mathcal{L}).$$

□

Chapter 4

Schemes with $\mathcal{N}^2 = 0$

4.1 Volumes of Coherent Sheaves on Reduced Schemes

The following proposition shows that the volume of a coherent $\mathcal{O}_X$ -module with respect to an invertible sheaf always exists on reduced schemes. Together with Proposition 2.4.1, this result is the main ingredient in the proof of Theorem 1.1.2

Proposition 4.1.1. *Let X be a reduced projective scheme over an arbitrary field. Let $X_1, \dots, X_r$ be the irreducible components of X of maximal dimension. Let $\mathcal{L}$ be an invertible sheaf on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. Then, the volume of $\mathcal{F}$ with respect to $\mathcal{L}$ exists as a limit and we have the formula*

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) = \sum_{j=1}^r \mathrm{rank}(\mathcal{F}|_{X_j}) \mathrm{vol}(\mathcal{L}|_{X_j}),$$

where the volumes $\mathrm{vol}(\mathcal{L}|_{X_j})$ exist as limits since the X_j are projective varieties.

This statement can be deduced with a little work from Lemma 2.2.6.

Proof of Proposition 4.1.1. Let us assume first that X is irreducible and hence a variety. By the comment after Theorem 2.3.1, we have that $\mathrm{vol}(\mathcal{L})$ exists as a limit. Let ξ be the generic point of X . If $\mathcal{F}_{\xi} = 0$, then $\mathcal{F}$ is supported on a closed set of dimension smaller than $\dim(X)$ and then the limit in question equals zero.

Thus, given that X is a variety, we can assume that $\mathcal{F}$ is generically free. By Lemma 2.2.6,

$$\mathrm{vol}_{\mathcal{L}}(\mathcal{F}) = \mathrm{rank}(\mathcal{F}) \mathrm{vol}(\mathcal{L})$$

exists as a limit.

If X is not irreducible, let $\xi_1, \dots, \xi_t$ be the generic points of the irreducible components $X_1, \dots, X_t$ of X and label them in such a way that, for some $r \leq t$, $X_1, \dots, X_r$ are all the components of maximal dimension. Since X is reduced, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \prod_{j=1}^t \mathcal{O}_{X_j} \rightarrow \mathcal{C} \rightarrow 0,$$

where $\mathcal{C}$ is supported on a closed subset of dimension less than the dimension of X . This exact sequence can be tensored with $\mathcal{F}$ to obtain

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \prod_{j=1}^t \mathcal{O}_{X_j} \otimes \mathcal{F} \rightarrow \mathcal{C} \otimes \mathcal{F} \rightarrow 0. \quad (4.1.1)$$

We claim that $\mathcal{K}$ is supported on a closed set of dimension smaller than the dimension of X . Indeed, for any j , since X is reduced, we have that $\mathcal{O}_{X_j, \xi_j} = \mathcal{O}_{X, \xi_j}$. This shows that the map $\mathcal{F} \rightarrow \prod_j \mathcal{O}_{X_j} \otimes \mathcal{F}$ is injective at ξ_j . Hence, $\mathcal{K}_{\xi_j} = 0$ for all $j \in \{1, \dots, t\}$ and this implies that $\dim(\mathrm{Supp}(\mathcal{K})) < \dim(X)$. Since we also have that $\dim(\mathrm{Supp}(\mathcal{C} \otimes \mathcal{F})) < \dim(X)$, we can apply Corollary 2.2.2 to (4.1.1) and the invertible sheaf $\mathcal{L}$ to obtain that

$$\begin{aligned} \mathrm{vol}_{\mathcal{L}}(\mathcal{F}) &= \sum_{j=1}^t \lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{O}_{X_j} \otimes \mathcal{F} \otimes \mathcal{L}^n)}{n^d/d!} \\ &= \sum_{j=1}^t \lim_{n \rightarrow \infty} \frac{h^0(X_j, \mathcal{F}|_{X_j} \otimes (\mathcal{L}|_{X_j})^n)}{n^d/d!}, \end{aligned}$$

where the limits in the second equality exist by the case treated above since the

X_j are projective varieties. Now, the limit

$$\lim_{n \rightarrow \infty} \frac{h^0(X_j, \mathcal{F}|_{X_j} \otimes (\mathcal{L}|_{X_j})^n)}{n^d/d!}$$

is equal to zero if $\dim(X_j) < d$, by [8, Proposition 1.31], and equal to $\text{vol}_{\mathcal{L}|_{X_j}}(\mathcal{F}|_{X_j})$ whenever $\dim(X_j) = d$. Thus, we can consider only the components of maximal dimension and obtain the formula

$$\text{vol}_{\mathcal{L}}(\mathcal{F}) = \sum_{j=1}^r \text{vol}_{\mathcal{L}|_{X_j}}(\mathcal{F}|_{X_j}) = \sum_{j=1}^r \text{rank}(\mathcal{F}|_{X_j}) \text{vol}(\mathcal{L}|_{X_j}).$$

□

4.2 Proof of Theorem 1.1.2

We recall for convenience the notation of Theorem 1.1.2. In what follows, X will denote a d -dimensional projective scheme over a field k . The nilradical of X will be represented by $\mathcal{N} = \mathcal{N}(X)$ and $\mathcal{L}$ will denote an invertible sheaf on X . We suppose that $\mathcal{N}^2 = 0$.

Proof of Theorem 1.1.2. Let us consider first the case where X is irreducible. Let X_{red} be the closed subscheme of X defined by the ideal sheaf $\mathcal{N}$. Notice that X_{red} is a variety. In order to apply Proposition 2.4.1, we have to show that $\text{vol}_{\mathcal{L}}(\mathcal{N})$ exists as a limit. Since $\mathcal{N}^2 = 0$, we have that $\mathcal{N} \subset \text{Ann}_{\mathcal{O}_X}(\mathcal{N})$ and, in consequence, $\mathcal{N}$ has the structure of an $\mathcal{O}_{X_{\text{red}}}$ -module. It follows that

$$H^0(X, \mathcal{N} \otimes \mathcal{L}^n) = H^0(X_{\text{red}}, \mathcal{N}|_{X_{\text{red}}} \otimes (\mathcal{L}|_{X_{\text{red}}})^n).$$

Thus, the limit

$$\text{vol}_{\mathcal{L}}(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{H^0(X_{\text{red}}, \mathcal{N}|_{X_{\text{red}}} \otimes (\mathcal{L}|_{X_{\text{red}}})^n)}{n^d/d!}$$

exists by Proposition 4.1.1 with $\mathcal{F} = \mathcal{N}|_{X_{\text{red}}}$. By Proposition 2.4.1, $\text{vol}(\mathcal{L})$ exists

as a limit.

If X is not irreducible, let $0 = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_t$ be an irredundant primary decomposition of the zero ideal sheaf with respect to a given projective embedding (this can be obtained by sheafifying a primary decomposition of the zero ideal in the coordinate ring of X with respect to the given embedding). Let Y_i be the irreducible closed subscheme of X determined by the ideal sheaf $\mathcal{Q}_i$. We have a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \prod_{i=1}^s \mathcal{O}_{Y_i} \rightarrow \mathcal{C} \rightarrow 0,$$

where the cokernel $\mathcal{C}$ is supported on a closed set of smaller dimension than X . By Lemma 2.2.1, in order to show that $\text{vol}(\mathcal{L})$ exists as a limit, it is enough to show that the limits

$$\lim_{n \rightarrow \infty} \frac{h^0(X, \mathcal{O}_{Y_i} \otimes \mathcal{L}^n)}{n^d} = \lim_{n \rightarrow \infty} \frac{h^0(Y_i, (\mathcal{L}|_{Y_i})^n)}{n^d}$$

exist for all i . By [8, Proposition 1.31], these limits will be equal to zero whenever Y_i is not a subscheme of maximal dimension. Thus, let us assume that Y_i has maximal dimension d . Notice that Y_i is irreducible. By the case considered at the beginning of the proof, it is enough to show that the nilradical $\mathcal{N}(Y_i)$ of Y_i satisfies $\mathcal{N}(Y_i)^2 = 0$. We prove this by working locally.

Let $U = \text{Spec}(R)$ be an affine open subset of X . If $U \cap Y_i = \emptyset$, then $\mathcal{N}(Y_i)^2|_U = 0$ trivially. We can hence assume that $U \cap Y_i \neq \emptyset$. Moreover, to simplify the notation, suppose that $U \cap Y_j \neq \emptyset$ for all other indices j as well. For otherwise, the ideal sheaves $\mathcal{Q}_j|_U = \mathcal{O}_U$ and we can disregard them in our considerations. We have $\mathcal{Q}_j = \widetilde{Q_j}$ for ideals Q_j of R . Let $P_j = \sqrt{Q_j}$. Then the ideals Q_j are P_j -primary. We have that $\mathcal{N}(X)|_U = \widetilde{N}$, where N is the nilradical of R and by assumption $N^2 = 0$. Also notice that $\mathcal{O}_{Y_i}|_U = \widetilde{R/Q_i}$ and $\mathcal{N}(Y_i)|_U = \widetilde{P_i/Q_i}$. Thus, it is enough to show that $P_i^2 \subset Q_i$ and we now proceed to do so.

In R we have the following relation:

$$Q_1 \cap \cdots \cap Q_s = 0 = N^2 = (P_1 \cap \cdots \cap P_s)^2.$$

Since Y_i has maximal dimension, P_i is a minimal prime. We can then localize the equation above at P_i and, with the help of [1, Proposition 4.8.(i)], obtain

$$(Q_i)_{P_i} = (P_i)_{P_i}^2 = (P_i^2)_{P_i}.$$

By [1, Proposition 4.8.(ii)], $Q_i = (P_i^2)_{P_i} \cap R \supset P_i^2$. Therefore, we get that $\mathcal{N}(Y_i)^2|_U = \widetilde{(P_i/Q_i)}^2 = \widetilde{(P_i^2 + Q_i)}/Q_i = 0$. Since U was arbitrary, $\mathcal{N}(Y_i)^2 = 0$ and the proof is complete. □

Remark. As in the previous proposition, let X be a projective scheme over a field with nilradical $\mathcal{N}$. Suppose that $\mathcal{N}^2 = 0$. Assume for simplicity that X is irreducible. Let $\mathcal{L}$ be an invertible sheaf on X . Theorem 1.1.2 shows that the volume of $\mathcal{L}$ exists as a limit. We now give a formula for this volume.

The nilradical $\mathcal{N}$ is an $\mathcal{O}_{X_{red}}$ -module as explained in the proof of Theorem 1.1.2. Then

$$\text{vol}_{\mathcal{L}}(\mathcal{N}) = \text{vol}_{\mathcal{L}|_{X_{red}}}(\mathcal{N}|_{X_{red}}) = \text{rank}(\mathcal{N}|_{X_{red}}) \text{vol}(\mathcal{L}|_{X_{red}}), \quad (4.2.1)$$

where the second equality follows from Proposition 4.1.1. By Proposition 2.4.1, we conclude that

$$\text{vol}(\mathcal{L}) = \text{vol}(L) + \text{rank}(\mathcal{N}|_{X_{red}}) \text{vol}(\mathcal{L}|_{X_{red}}),$$

where L is the linear series obtained by restricting the sections of $\mathcal{L}$ to X_{red} .

If $\mathcal{L}$ is a nef invertible sheaf, as shown in the remark after Proposition 3.0.2, $\text{vol}(L) = \text{vol}(\mathcal{L}|_{X_{red}})$ and we get the relation

$$\text{vol}(\mathcal{L}) = (\text{rank}(\mathcal{N}|_{X_{red}}) + 1) \text{vol}(\mathcal{L}|_{X_{red}}).$$

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VITA

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