CLASSICAL AND IMPULSE STOCHASTIC CONTROL ON THE OPTIMIZATION OF THE DIVIDENDS FOR THE TERMINAL BANKRUPTCY MODEL AND ITS APPLICATION

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

In this dissertation, I discuss the optimization of dividends of reinsurance companies with the terminal bankruptcy model, in which some money would be returned to shareholders at the state of terminal bankruptcy, meanwhile the tax rate and the fixed transaction cost for each dividend are considered. The mathematical problem of maximizing the summation of expected total discounted dividends before bankruptcy and expected discounted returned money at the state of terminal bankruptcy becomes a mixed classical-impulse stochastic control problem. In order to solve this problem, I reduce it to quasi-variational inequalities with nonzero boundary condition. The main contribution of this dissertation is to explicitly construct and verify solutions of these inequalities, and to consequently present the optimal policy. As an application, the solution of the optimization of dividends under the nonterminal bankruptcy model is provided in the end.

Chapter 1

Introduction

1.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the set of possible elementary outcomes ω, and $\mathbb{P}$ is a probability measure on the σ-field $\mathcal{F}$. Real-valued $\mathcal{F}$-measurable functions on Ω are called random variables, which can be viewed as functions whose values maybe uncertain but which depend on the elementary outcomes. If a set $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, then we say the event A occurs almost surely. The events of probability 0 are called null events.

1.1.1 Stochastic Processes and Brownian Motion

In finance market, the future price of a stock is uncertain, and it can be described by a random variable. In some models, however, the price of the stock at all moments of times in a certain interval is needed to be considered, which means that one needs to deal simultaneously with a set of random variables at continuous time t. This phenomena leads to a notion of a stochastic process.

A stochastic process $X(t), t \geq 0$, is a family of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On the time interval $[0, T]$, for any $s, t \in [0, T]$, $s \leq t$, if every set in the σ-algebra $\mathcal{F}(s)$ is also in the σ-algebra $\mathcal{F}(t)$, then the collection of σ-algebra $\mathcal{F}(t)$, $0 \leq t \leq T$ is a filtration. In general, a filtration $\mathcal{F}_t$ is a family of sub-σ-fields of $\mathcal{F}$ up to time t, such that the value of any observable process $X(t)$ is measurable with respect to $\mathcal{F}_t$. In this case, we say that $X(t)$ is adapted to the filtration $\mathcal{F}_t$.

In many recent finance models, the main source of uncertainty for stochastic process is modelled by a Brownian motion. In this paper, W_t is used to denote the process of Brownian motion, which is with respect to a filtration $\mathcal{F}_t$ if it is adapted to this filtration. Moreover, it satisfies

1. $W_0 = 0$.

2. W_t is continuous.

3. $W_t - W_s \sim N(0, t-s)$ for all $t > s \geq 0$, where $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2.

4. For any $0 = t_0 < t_1 < ... < t_m$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, ..., W_{t_m} - W_{t_{m-1}}$ are independent.

In addition, some important and useful properties of Brownian motion include:

1. W_t is a martingale, that is $\mathbb{E}(W_t|\mathcal{F}_s) = W_s$, $s \leq t$.

2. W_t is a Markov process, which means for any Borel-measurable function f, $s < t$, there exists another Borel-measurable function g, such that $\mathbb{E}(f(W_t)|\mathcal{F}_s) = g(W_s)$.

3. The quadratic variation $[W, W](T) = T$ for all $T \geq 0$ almost surely.

1.1.2 Itô Formula

Let $\Pi = \{t_0, t_1, t_2, ..., t_n\}$ be a partition of $[0, T]$ and $0 = t_0 < t_1 < ... < t_n = T$. For $t_k \leq t \leq t_{k+1}$, define Itô integral as follows:

$$I(t) := \int_0^t \Delta u dW_u = \sum_{j=0}^{k-1} \Delta(t_j)[W_{t_{j+1}} - W_{t_j}] + \Delta(t_k)[W_t - W_{t_k}].$$

It has been known that $dW_t \cdot dW_t = \Delta t$. Moreover, the Itô integral $I(t)$ has some properties:

1. $I(t)$ is $\mathcal{F}(t)$-measurable.
2. I_t is a martingale.
3. $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$.

For a function $f(t, x)$ we write $f \in C^{m,n}$ if f is m times continuously differentiable in t and n times continuously differentiable in x. By the property, $dW_t \cdot dW_t = \Delta t$, and Taylor series, for any function $f(t, W_t) \in C^{1,2}$, then Itô lemma can be provided as:

$$df(t, W_t) = (f_t + \frac{1}{2} f''_{xx})dt + f'_x dW_t. \tag{1.1.1}$$

If $f \in C^{1,2}$ and X is a trajectory of a Brownian semimartingale, then Itô lemma is:

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))d[X, X](t). \tag{1.1.2}$$

Integrating on both sides, then the Itô formula for (1.1.1) and (1.1.2) are as follows, respectively,

$$f(t, W_t) = f(0, W_0) + \int_0^t (f_s + \frac{1}{2} f''_{xx})ds + \int_0^t f'_x dW_s, \tag{1.1.3}$$

and

$$f(t, X(t)) = f(0, X(0)) + \int_0^t f_s(s, X(s))ds + \int_0^t f_x(s, X(s))dX(s) + \int_0^t \frac{1}{2} f_{xx}(s, X(s))d[X, X](s). \quad (1.1.4)$$

1.1.3 Stopping Times

A stopping time $\tau \geq 0$ is a random variable with respect to a filtration $\mathcal{F}_t$ satisfying

$$\{\tau \leq t\} \in \mathcal{F}_t, \ \ t \geq 0. \quad (1.1.5)$$

The σ-field $\mathcal{F}_\tau$ is defined as the collection of all events $A \in \mathcal{F}$ such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. For the case that the filtration $\mathcal{F}_t$ is generated by a stochastic process $X(\cdot)$, we will also call τ satisfying (1.1.5) a stopping time with respect to the process $X(\cdot)$.

In addition, it is well known that a martingale stopped at a stopping time is a martingale, which can be used to calculate the expectation of the first passage time of some process, such as drifted Brownian motion, etc.

1.1.4 Dynamic Programming

Dynamic programming is an elegant and powerful optimization technique and it refers to a collection of general methods developed to solve sequential or multi-stage decision problems. A dynamic programming problem can always be divided into stages with a decision required at each stage. So, the dynamic programming algorithm is basically a stage wise search method of optimization problems.

Principle of Optimality: For a problem, if the sub-solutions of an optimal solution of the problem are themselves optimal solutions for their subproblems, then this problem is said to satisfy the Principle of Optimality, which is the basis of the dynamic programming algorithm. Now, let's use an example to show how dynamic programming algorithm works in practical problems.

Assume that a finance problem has $N+1$ stages. At stage k, $0 \leq k \leq N-1$, x_k is the state of the problem at stage k, u_k is the decision chosen at stage k, and w_k is a random parameter, which is called disturbance. In the last stage N, the problem has a cost $g_N(x_N)$ depending only on x_N. In any other stage k, $0 \leq k \leq N-1$, the cost $g_k(x_k, u_k, w_k)$ depends on x_k, u_k and w_k. Then the total cost of this problem is: $g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)$. Our goal is to obtain the optimal policies for the decision sequences $\Pi^* = \{u_0^*, ..., u_{N-1}^*\}$, such that the expected cost reaches its minimum, that is

$$\mathrm{J}_{\Pi^*}(x_0) = \min_{\Pi} \mathbb{E}[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)]. \tag{1.1.6}$$

If the principle of optimality is verified, then the tail policy $\Pi_i^* = \{u_i^*, ..., u_{N-1}^*\}$ must be the optimal of the tail subproblem $E[g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, u_k, w_k)]$. To solve (1.1.6), by dynamic programming algorithm, one can start with $\mathrm{J}_N(x_N) = g_N(x_N)$, and then go backwards to minimize the cost function

$$\mathrm{J}_k(x_k) = \min_{u_k} \mathbb{E}_{w_k}[g_k(x_k, u_k, w_k) + \mathrm{J}_{k+1}(f(x_k, u_k, w_k))], \quad k = 0, ..., N-1,$$

where f is the state transition function satisfying $x_{k+1} = f(x_k, u_k, w_k)$. Then the last generated item from this algorithm, $\mathrm{J}_0(x_0)$, is the expected optimal cost.

1.1.5 Hamilton-Jacobi-Bellman Equation

Based on the dynamic programming principle of last section, in this section an extensible example in the stochastic control is illustrated and the Hamilton-Jacobi-Bellman Equation it satisfies are introduced and verified (see [16]).

Assume that a controlled process $X(t)$ is governed by the stochastic differential equation

$$dX(s) = \mu(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dw(s), \tag{1.1.7}$$

$$X(t) = x, \tag{1.1.8}$$

where $u(t)$ is an admissible control, which is an $\mathcal{F}_t$ adapted right-continuous process, $x \in R$, and $R \subseteq \mathbb{R}$ is an open bounded region. The optimal cost function $V(t, x)$ is defined as

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}} J_{t,x}(u(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^\tau h(s, X(s), u(s))ds + g(\tau, X(\tau))\right],$$

where $\mathcal{U}$ is denoted the set of all admissible controls, $h(s, X(s), u(s))$ is a running cost function, and $g(\tau, X(\tau))$ is the terminal cost function, .

The basic idea to solve this problem is that the first step is to find the equation the function $V(t, x)$ should satisfy and then to solve this equation. It turns out that under certain regularity conditions, such as twice continuously differentiability, the optimal cost function $V(t, x)$ satisfies a non-linear partial differential equation called Hamilton-Jacobi-Bellman equation(HJB).

To derive the Hamilton-Jacobi-Bellman equation, we start with the main tool, Dynamic Programming Principle, as follows.

Theorem 1.1. *(see [16]) The optimal cost function $V(t, x)$ satisfies the Dynamic Programming Principle (DPP), that is, for all stopping times $\theta \in [t, t_1]$*

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^{\theta\wedge\tau} h(s, X(s), u(s))ds + V(\theta\wedge\tau, X(\theta\wedge\tau))\right]. \tag{1.1.9}$$

Proof. Let $u(\cdot)$ be any control. Then

$$
\begin{aligned}
J_{t,x}(u) &= \mathbb{E}\left[\left(\int_t^\theta h(s,X(s),u(s))ds + \int_\theta^\tau h(s,X(s),u(s))ds + g(\tau,X(\tau))\right)1_{(\theta<\tau)}\right] \\
&\quad + \mathbb{E}\left[\left(\int_t^\tau h(s,X(s),u(s))ds + g(\tau,X(\tau)\right)1_{(\theta\geq\tau)}\right] \\
&= \mathbb{E}\left[\int_t^{\tau\wedge\theta} h(s,X(s),u(s))ds\right] \\
&\quad + \mathbb{E}\left[\left(\int_\theta^\tau h(s,X(s),u(s))ds 1_{(\theta<\tau)} + g(\tau,X(\tau))\right)\right].
\end{aligned}
$$

Conditioning on $\mathcal{F}_\theta$, by the strong Markov property it can be seen that the conditional expectation

$$
\mathbb{E}\left[\left(\int_\theta^\tau h(s,X(s),u(s))ds 1_{(\theta<\tau)} + g(\tau,X(\tau))\right)\middle|\mathcal{F}_\theta\right]
$$

on the set $\{\theta < \tau\}$ is equal to the cost function for a control problem in the time interval (θ,τ) with the initial values $(\theta, X(\theta))$ and will therefore be not smaller than the optimal value function $V(\theta, X(\theta))$. On the set $\{\theta \geq \tau\}$ it equals $g(\tau, X(\tau)) = V(\tau, X(\tau))$. Hence

$$
J_{t,x}(u) \geq \mathbb{E}_{t,x}\left[\int_t^{\tau\wedge\theta} h(s,X(s),u(s))ds + V(\theta\wedge\tau, X(\theta\wedge\tau))\right].
$$

This holds for all controls $u(\cdot)$ and equality can be obtained by taking infimum over all possible controls on both sides of the inequality. □

In the theorem above, the DPP actually states that the optimal cost function $V(t,x)$ will be the same for the original problem on the interval $[t,t_1]$ and for the problem on a smaller interval $[t,s_1]$, if the running cost function $h(s,x,u)$ remains the same while the terminal cost function at time s_1 is set to be equal to (the yet unknown function) $V(s_1,x)$.

The following is to apply the Itô formula to obtain an equation that must be satisfied by V. For each fixed $u \in A(t,x)$ define the operator

$$\mathcal{L}^u f(t,x) = \frac{\partial}{\partial t} f(t,x) + \mu(t,x,u)\frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2}{\partial x^2} f(t,x). \qquad (1.1.10)$$

Theorem 1.2. *(see [16]) Assume $V \in C^{1,2}$. Then V satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$\inf_{u \in A(t,x)} [\mathcal{L}^u V(t,x) + h(t,x,u)] = 0 \qquad (1.1.11)$$

with the terminal condition

$$\begin{aligned} &V(t_1, x) = g(t_1, x), \;\; x \in R \\ &V(t,x) = g(t,x), \;\; x \in \delta R, \;\; t \in [t_0, t_1). \end{aligned} \qquad (1.1.12)$$

Proof. For any admissible $u(\cdot)$ and $\varepsilon > 0$, let $\eta_u^\varepsilon = (\varepsilon + t) \wedge \inf\{s \geq t : X(s) \notin (x-\varepsilon, x+\varepsilon)\}$. Then $\eta_u^\varepsilon < \infty$ a.s. and $\eta_u^\varepsilon \to t$ a.s. as $\varepsilon \to 0$. Fix an arbitrary constant $u \in A$ and choose $u(\cdot)$ such that $u(s) = u$ for all s. Choose $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon]$ belongs to the interior of R. In this case $\eta_u^\varepsilon < \tau$. Substitute $\theta = \eta_u^\varepsilon$ into (1.1.9). By Ito's formula,

$$\mathbb{E}V(\eta_u^\varepsilon, X(\eta_u^\varepsilon)) = V(t,x) + \mathbb{E}\int_t^{\eta_u^\varepsilon} \mathcal{L}^u V(s, X(s))ds$$

Thus, from (1.1.9), it follows

$$V(t,x) \leq \mathbb{E}\int_t^{\eta_u^\varepsilon} h(s, X(s), u)ds + V(t,x) + \mathbb{E}\int_t^{\eta_u^\varepsilon} \mathcal{L}^u V(s, X(s))ds. \qquad (1.1.13)$$

Cancelling $V(t,x)$ on both sides and dividing by $\mathbb{E}[\eta_u^\varepsilon - t]$, one comes to

$$0 \leq \frac{1}{\mathbb{E}[\eta_u^\varepsilon - t]}\mathbb{E}\int_t^{\eta_u^\varepsilon} (h(s, X(s), u) + \mathcal{L}^u V(s, X(s)))ds. \qquad (1.1.14)$$

Taking into account the fact that $\eta_u^\varepsilon - t \le \varepsilon$ and $|X(s) - x| \le \varepsilon$, for each $s \in [t, \eta_u^\varepsilon]$, we have

$$h(s, X(s), u) + \mathcal{L}^u V(s, X(s)) = h(t, x, u) + \mathcal{L}^u V(t, x), +\phi(s),$$

where

$$|\phi(s)| \le \sup_{|x-y|,|t-s|\le\varepsilon} |h(t, x, u) - h(s, y, u)| + |\mathcal{L}^u V(t, x) - \mathcal{L}^u V(s, y)| \equiv \delta(\varepsilon) \to 0,$$

as $\varepsilon \to 0$. Therefore, the right hand side of (1.1.14) converges to $h(t, x, u) + \mathcal{L}^u V(t, x)$ as $\varepsilon \to 0$. Thus, for any u,

$$0 \le h(t, x, u) + \mathcal{L}^u V(t, x).$$

Arbitrariness of u yields

$$0 \le \inf_{u(\cdot)\in\mathcal{U}} [h(t, x, u) + \mathcal{L}^u V(t, x)] = \inf_{u\in A(t,x)} [h(t, x, u) + \mathcal{L}^u V(t, x)]. \tag{1.1.15}$$

Assume there exists control $u^*(\cdot)$ with $u^*(t) = u_0$ such that

$$V(t, x) = \mathbb{E}\left[\int_t^{\eta_{u^*}^\varepsilon} h(s, X^*(s), u^*(s))ds + V(\eta_{u^*}^\varepsilon, X^*(\eta_{u^*}^\varepsilon))\right]. \tag{1.1.16}$$

Then, by the same arguments as above

$$0 = [h(t, x, u_0) + \mathcal{L}^{u_0} V(t, x)]. \tag{1.1.17}$$

This shows that (1.1.15) holds with equality. □

1.2 Background of This Dissertation

In the past twenty years, there are many applications about diffusion models in financial mathematics, especially in (re)-insurance modelling (see Radner and Shepp 1996; Asmussen and Taksar 1997; Paulsen and Gjessing 1997; Boyle, Elliott, and Yang 1998; Hojgaard and Taksar 1998a,b; Taksar and Zhou 1998; Hubalek and Schachermayer 2004; Cadenillas, Choulli and Taksar 2006; Paulsen 2007; Paulsen 2008; etc). For these models, the liquid assets processes of the corporation are described by a Brownian motion with drift and diffusion coefficients. The drift term represents the expected profit per unit time, and the diffusion term is looked on as risk. By using diffusion models, many kinds of optimal dividend problems, such as in Asmussen and Taksar 1997; Cadenillas, Choulli and Taksar 2006; Paulsen 2007; Paulsen 2008; etc, are discussed and some optimal policies are presented in these papers. Especially, in some papers (see Jeanblanc-Picqué and Shiryaev 1995; Cadenillas, Choulli and Taksar 2006; Paulsen 2007; Paulsen 2008) authors discuss much more practical problems by considering a fixed transaction cost for each dividend paid out. In the paper, Cadenillas, Choulli and Taksar 2006, the optimal dividend problem without bankruptcy for insurance firms is considered under the assumption of constant tax rate and fixed cost incurred whenever the dividends take place. In Paulsen 2007; Paulsen 2008, the author considers the general income process, X_t, with drift term $\mu(X_t)$ and diffusion term $\sigma(X_t)$, and the case of bankruptcy. Moreover, numerical methods, such as Runge-Kutta method, are implemented to simulate the HJB equation, which is a nonlinear differential equation.

For some papers above, they just assume that the value function is zero when there is bankruptcy. But in real world, sometimes shareholders can get some money back when terminal bankruptcy happens. That means for this case the value function is not zero. So, it is very useful and necessary for us to discuss this kind of problem. In this dissertation, I suppose that the amount of money, shareholders can obtain for the terminal bankruptcy, is a positive constant, a. Moreover, I assume that the liquid assets, X_t, can be considered as a process with constant drift and diffusion coefficients. In the model of this dissertation, as that in Cadenillas, Choulli and Taksar 2006, the dividend distribution policy is given by a purely discontinuous increasing functional. The net amount of money received by shareholders is $k\xi_i - K$ for the i-th-dividends, where ξ_i is the amount of the dividend payments, $1-k$ is the tax rate the shareholder pays, and K is the fixed cost whenever the dividends are paid out. In addition, τ_i represents the moments of dividends. Based on these assumptions, I transform the value function, $V(x)$, into a quasi-variational inequalities, (QVI), then I list out a candidate solution $v(x)$ to (QVI) with positive boundary condition, $v(0) = a$. Subsequently, I check that the value function can be given by $v(x)$ and the optimal policy can be presented based on the solution $v(x)$. For some optimal policies, they may be reached when bankruptcy happens. A natural question is that how to point out whether there is a bankruptcy or not in order to get optimal policy under some conditions. To answer this question, some criteria are provided, which is an important frame of this dissertation. In addition, one difficulty of this dissertation is that the structure of the candidate solution is uncertain since the existing interval of it has unfixed endpoints, which depends on some unknown parameters. This phenomena is not appear in Cadenillas, Choulli and Taksar 2006. Enlightened by the derivatives of candidate solutions, I construct the integral $I(C)$ and then discuss it by several cases for μ, σ, k, K and a. For the model mentioned above, which is restricted to stay at the bankruptcy state, it is denoted by *terminal bankruptcy model* as in [14]. Based on the solution of it, the solution of so-called *nonterminal bankruptcy model* (or bankruptcy with recovery rate) is given out at the end of this dissertation.

The structure of this dissertation is as follows. In the next chapter, I provide a rigorous mathematical model for the optimal dividend problem. Further, the stochastic control problem is transformed to a quasi-variational inequality (QVI). Moreover, some definitions and an important verification are presented. Based on chapter 2, in the chapter 3, the detailed structure of candidate solutions is given under different situations. In chapter 4, the uniqueness of some unfixed parameters is verified, some formulae to calculate these parameters are proposed and some numerical examples are shown to support our theoretical results. In the chapter 5, I demonstrate that the candidate solutions satisfy (QVI), and then the optimal policy is obtained. Based on the results about the terminal bankruptcy model, the solution of the nonterminal bankruptcy model is discussed in chapter 6. In the last section, I summary our result and mention the work in the future.

Chapter 2

The Mathematical Model

2.1 Value Function

In this dissertation, I use standard notation as in [3]. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and $W = \{W_t; t \geq 0\}$ be a standard Brownian motion adapted to that filtration. Moreover, the reserve process $X = \{X(t); t \geq 0\}$ is a state variable, which denotes the liquid assets of the company. For an insurance company, in order to reduce risk, the risk control takes up the form of proportional reinsurance, which mathematically corresponds to decreasing the drift and diffusion coefficient by multiplying both quantities by the same factor $u(t) \in [0, 1]$. The time of dividends is described by a sequence of increasing stopping times $\{\tau_i; i = 1, 2, ...\}$ and the amounts of the dividends paid out to the shareholders, associated with the times, is represented by a sequence of random variables $\{\xi_i; i = 1, 2, ...\}$. Then the controlled state process $X(t)$ before bankruptcy is given by

$$X(t) = x + \int_0^t \mu u(s)ds + \int_0^t \sigma u(s)dW_s - \sum_{n=1}^{\infty} I_{(\tau_n<t)}\xi_n, \tag{2.1.1}$$

where $x \geq 0$ is the initial reserve and $I_{(\tau_n<t)}$ is an indicator function.

Let the time of bankruptcy be given by:

$$\tau := \inf\{t \geq 0 : X(t) = 0\}.$$

Definition 2.1.1. *Let $u : \Omega \times [0,\infty) \to [0,1]$ be an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process, τ_i, i=1,2,... be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, the random variable ξ_i, i=1,2,... be $\{\mathcal{F}_{\tau_i}\}$ measurable with $0 \leq \xi_i \leq X(\tau_i -)$, then*

$$\pi := (u, \mathcal{T}, \xi, \tau) = (u; \tau_1, \tau_2, ..., \tau_n, ...; \xi_1, \xi_2, ...\xi_n, ...; \tau) \tag{2.1.2}$$

is called an admissible control or an admissible policy. The class of all admissible controls is denoted by $\mathcal{A}(x)$.

In addition, I denote the net amount of money that shareholders receive by a function $g : [0,\infty) \to (-\infty,\infty)$ as:

$$g(\eta) = k\eta - K, \tag{2.1.3}$$

where the constant $K > 0$ is a fixed setup cost incurred each time that a dividend is paid out, and the constant $1 - k \in (0,1)$ is the tax rate at which the dividends are taxed, and η is the amount of liquid assets withdrawn.

A performance functional J with each admissible control π is defined by

$$J(x,\pi) := \mathbb{E}_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n} g(\xi_n) I_{(\tau_n<\tau)} + e^{-\lambda\tau} a\right],$$

which represents the total expected discounted value received by shareholders until the time of bankruptcy, where a is the known amount paid out to shareholders when the terminal bankruptcy happens.

Define the value function $V(x)$ by

$$V(x) := \sup_{\pi \in A(x)} \mathbb{E}_x \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{(\tau_n < \tau)} + e^{-\lambda \tau} a \right]. \tag{2.1.4}$$

Then the optimal control $\pi^* = (u^*, \mathcal{T}^*, \xi^*, \tau^*)$ is a policy for which the following equality can be satisfied:

$$V(x) = J(x, \pi^*). \tag{2.1.5}$$

2.2 Properties of The Value Function and Some Definitions

In this section, the quasi-variational inequalities(QVI) associated with this stochastic control problem is provided. Moreover, I derive some properties of the value function.

Proposition 2.2.1. *For every $x \in [0, \infty)$, the value function $V(x)$ in (2.1.4) satisfies*

$$V(x) \leq k(x - e^{-\lambda \tau} a + |\mu| / \lambda).$$

Proof. By the same method as in [3] and letting $X(\tau) = a$ instead of $X(\tau) = 0$, then the result can be obtained. □

Let g be given by (2.1.3), then for a function $\phi : [0, \infty) \to \mathbb{R}$, define the maximum utility operator M of it by

$$M\phi(x) := \sup_{\eta} \{ \phi(x - \eta) + g(\eta) : 0 < \eta \leq x \}. \tag{2.2.1}$$

Suppose that the payment of dividends occurs at time 0 and the amount of it equals η, then the reserve decreases from initial position x to $x - \eta$. After that, if the optimal policy is followed then the total expected utility is $k\eta - K + V(x - \eta)$. Consequently, under such a policy, the total maximal expected utility would be equal to $MV(x)$. On the other hand, for each initial position x, suppose that there exists an optimal policy, which is optimal for the whole domain. Then the expected utility associated with this optimal policy is $V(x)$, which is greater or equal to any expected utility associated with another different policy. So, it follows

$$V(x) \geq MV(x). \tag{2.2.2}$$

Now, define

$$\mathcal{L}^u v(x) = \frac{1}{2}\sigma^2 u^2 v''(x) + \mu u v'(x) - \lambda v(x), \tag{2.2.3}$$

then, by dynamic programming principle, we know that in the continuation region $V(x)$ satisfies:

$$\max_{u\in[0,1]} \mathcal{L}^u V(x) = 0. \tag{2.2.4}$$

These two arguments (2.2.2) and (2.2.4) give us an intuition for the following two definitions and one theorem.

Definition 2.2.1. *Assume that function $v(x) : [0,\infty) \to [0,\infty)$. For every $x \in [0,\infty)$ and $u \in [0,1]$, if we have*

$$\text{(QVI)} \begin{cases} v(x) \geq Mv(x), \\ \mathcal{L}^u v(x) \leq 0, \\ (v(x) - Mv(x))(\max\limits_{u\in[0,1]} \mathcal{L}^u v(x)) = 0, \\ v(0) = a, \end{cases} \tag{2.2.5}$$

then we claim that $v(x)$ satisfies the quasi-variational inequalities of the control problem.

Definition 2.2.2. *The control $\pi^v = (u^v, \mathcal{T}^v, \xi^v, \tau^v)$ is called the (QVI) control asso-*

ciated with v if

$$P\left\{u^v(t) \neq \arg\max_{u\in[0,1]} \mathcal{L}^u v(X_t^v),\ \ X_t^v \in \mathcal{C}\right\} = 0; \tag{2.2.6}$$

$$\tau_0^v = 0, \qquad \xi_0^v = 0;$$

$$\tau_1^v := \inf\{t \geq 0 : v(X^v(t)) = Mv(X^v(t))\}, \tag{2.2.7}$$

$$\xi_1^v := \arg \sup_{0<\eta\leq X^v(\tau_1^v)} \{v(X^v(\tau_1^v) - \eta) + g(\eta)\}; \tag{2.2.8}$$

and, for every $n \geq 2$

$$\tau_n^v := \inf\{t \geq \tau_{n-1} : v(X^v(t)) = Mv(X^v(t))\}, \tag{2.2.9}$$

$$\xi_n^v := \arg \sup_{0<\eta\leq X^v(\tau_n^v)} \{v(X^v(\tau_n^v) - \eta) + g(\eta)\}; \tag{2.2.10}$$

$$\tau^v := \inf\{t \geq 0 : X^v(t) = 0\}. \tag{2.2.11}$$

As in [3], we also have the following theorem.

Theorem 2.2.1. *Let $v \in C^1([0,\infty))$ be a solution of (QVI). Suppose that there exists $U > 0$ such that v is twice continuously differentiable on $[0, U)$ and v is linear on $[U,\infty)$. Then, for any $x \in [0,\infty)$*

$$V(x) \leq v(x). \tag{2.2.12}$$

Furthermore, if the (QVI) control $(u^v, \mathcal{T}^v, \xi^v, \tau^v)$ associated with v is admissible, then v coincides with the value function and the (QVI) control associated with v is the optimal policy, i.e.,

$$V(x) = v(x) = J(x; u^v, \mathcal{T}^v, \xi^v, \tau^v). \tag{2.2.13}$$

Proof. The idea of this proof is very similar to that of Theorem 3.4 in Cadenillas, Choulli, and Taksar(2006).

We notice that $v'(x)$ is bounded on $[0,\infty)$ since $v'(x)$ is a continuous function on $[0,\infty)$ and it is a constant on $[U,\infty)$. Moreover, $v(x)$ is also bounded on $[0,U)$ due to $v(x)\in C^1([0,\infty))$. Denote by $X = X^{(u,\mathcal{T},\xi,\tau)}$ the trajectory determined by an admissible control $(u,\mathcal{T},\xi,\tau)$. From the linearity of $v(x)$ on $[U,\infty)$ and the bound of $v(x)$ on $[0,U)$, then we have that

$$\lim_{T\to\infty} \mathbb{E}_x[e^{-\lambda T} v(X(T))] = 0. \tag{2.2.14}$$

In addition, from the bound of $v'(x)$ on $[0,\infty)$, it follows that

$$\mathbb{E}_x\left[\int_0^\infty \{e^{-\lambda t} v'(X(t))\}^2 dt\right] < \infty. \tag{2.2.15}$$

Then, for each $n \geq 1$, we have some equalities as follows

$$\begin{aligned}
&e^{-\lambda(t\wedge\tau_n)} v(X(t\wedge\tau_n)) - v(X(0)) \\
&= e^{-\lambda(t\wedge\tau_n)} v(X(t\wedge\tau_n)) - v(x) \\
&= \sum_{i=1}^{n} \left[e^{-\lambda(t\wedge\tau_i)} v(X(t\wedge\tau_i)) - e^{-\lambda(t\wedge\tau_{i-1})} v(X(t\wedge\tau_{i-1}))\right] \\
&= \sum_{i=1}^{n} \left[e^{-\lambda(t\wedge\tau_i)} v(X(t\wedge\tau_i-)) - e^{-\lambda(t\wedge\tau_{i-1})} v(X(t\wedge\tau_{i-1}))\right] \\
&+ \sum_{i=1}^{n} I_{\{\tau_i \leq t\}} e^{-\lambda\tau_i}[v(X(\tau_i)) - v(X(\tau_i-))].
\end{aligned} \tag{2.2.16}$$

Further, by using Itô's formula, it shows that

$$
\begin{aligned}
&e^{-\lambda(t\wedge\tau_i)}v(X(t\wedge\tau_i-)) - e^{-\lambda(t\wedge\tau_{i-1})}v(X(t\wedge\tau_{i-1}))\\
&= \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}\left[\frac{1}{2}\sigma^2 u_s^2 v''(X(s)) + \mu u_s v'(X(s)) - \lambda v(X(s))\right]ds\\
&+ \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X(s))\sigma u_s dW_s\\
&= \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}\mathcal{L}^{u_s}v(X(s))ds + \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X(s))\sigma u_s dW_s.
\end{aligned}
$$

In addition, from inequality $\mathcal{L}^{u_s}v(X(s)) \leq 0$, then we can conclude that

$$
\begin{aligned}
&e^{-\lambda(t\wedge\tau_i)}v(X(t\wedge\tau_i-)) - e^{-\lambda(t\wedge\tau_{i-1})}v(X(t\wedge\tau_{i-1}))\\
&\leq \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X(s))\sigma u_s dW_s. \qquad (2.2.17)
\end{aligned}
$$

From Definition 2.2.2, we notice that the inequality above becomes an equality for (QVI) control π^v associated with the function v.

From $v(x) \geq Mv(x)$, follows

$$
e^{-\lambda\tau_i}[v(X(\tau_i)) - v(X(\tau_i-))] \leq -e^{-\lambda\tau_i}g(\xi_i), \qquad (2.2.18)
$$

for $\tau_i \in [0, t)$. Moreover, this inequality also becomes an equality for the (QVI) control associated with $v(x)$.

Combining (2.2.16), (2.2.17) and (2.2.18) and taking expectations, we have that

$$
\begin{aligned}
&v(x) - \mathbb{E}_x[e^{-\lambda(t\wedge\tau_n)}v(X(t\wedge\tau_n))]\\
&\geq \mathbb{E}_x\left[\sum_{i=1}^{n}\left(I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i) - \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X(s))\sigma u_s dW_s\right)\right], \qquad (2.2.19)
\end{aligned}
$$

with the inequality being tight for the (QVI) control π^v.

From (2.2.14) and the property of Brownian motion, W_s, then

$$\mathbb{E}_x\left[\int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X(s))\sigma u_s dW_s\right] = 0.$$

Thus, it follows that

$$v(x) - \mathbb{E}_x[e^{-\lambda(t\wedge\tau_n)}v(X(t\wedge\tau_n))] \geq \mathbb{E}_x\left[\sum_{i=1}^{n} I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i)\right], \tag{2.2.20}$$

with the inequality again being tight for the (QVI) control π^v.

In addition, if the process $X(s)$ never reaches zero, then we have $\tau_n \to \infty$ a.s. as $n \to \infty$. For this case, $\tau = \infty$, which means no bankruptcy. On the other hand, if the process $X(s)$ reaches zero at $t = \tau < \infty$, which is bankruptcy, then there exists $N \in \mathbb{N}$, such that $\tau_{N-1} < \tau = \tau_N$. Let $\tau_{N+1} = \tau_{N+2} = \tau_{N+3} = ... = \tau$, then for both cases, we have $\lim_{n\to\infty} \tau_n = \tau$.

Therefore, for (2.2.20), let $n \to \infty$, it follows that

$$v(x) - \mathbb{E}_x[e^{-\lambda(t\wedge\tau)}v(X(t\wedge\tau))] \geq \mathbb{E}_x\left[\sum_{i=1}^{\infty} I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i)\right]. \tag{2.2.21}$$

Consequently, let $t \to \infty$, we have that

$$v(x) - \mathbb{E}_x[e^{-\lambda\tau}v(X(\tau))] \geq \mathbb{E}_x\left[\sum_{i=1}^{\infty} I_{\{\tau_i\leq \tau\}}e^{-\lambda\tau_i}g(\xi_i)\right]. \tag{2.2.22}$$

Then,

$$\begin{aligned} v(x) &\geq \mathbb{E}_x\left[\sum_{i=1}^{\infty} I_{\{\tau_i\leq \tau\}}e^{-\lambda\tau_i}g(\xi_i)\right] + \mathbb{E}_x[e^{-\lambda\tau}v(X(\tau))] \\ &= \mathbb{E}_x\left[\sum_{i=1}^{\infty} I_{\{\tau_i\leq \tau\}}e^{-\lambda\tau_i}g(\xi_i)\right] + \mathbb{E}_x[e^{-\lambda\tau}v(0)] \\ &= \mathbb{E}_x\left[\sum_{i=1}^{\infty} e^{-\lambda\tau_i}g(\xi_i)I_{\{\tau_i\leq \tau\}} + e^{-\lambda\tau}a\right]. \end{aligned} \tag{2.2.23}$$

From (2.2.23), we conclude that for every $(u, \mathcal{T}, \xi, \tau) \in \mathcal{A}(x)$, follows

$$v(x) \geq J(x; u, \mathcal{T}, \xi, \tau), \tag{2.2.24}$$

with (2.2.24) again being an equality for the (QVI) control $(u^v, \mathcal{T}^v, \xi^v, \tau^v)$ associated with v. $\square$

Chapter 3

Candidate Solution to The (QVI)

In this section, firstly I recall the zero boundary problem in [3], then by the similar method of this solved problem, I mainly discuss the smooth solutions of (QVI) properties.

3.1 Solution for The Problem with Zero Boundary Condition

Let's consider the similar problem of (QVI) as follows (see [3]):

$$(\text{QVI0}) \begin{cases} v_0(x) \geq Mv_0(x), \\ \mathcal{L}^u v_0(x) \leq 0, \\ (v_0(x) - Mv_0(x))(\max\limits_{u \in [0,1]} \mathcal{L}^u v_0(x)) = 0, \\ v_0(0) = 0, \end{cases} \tag{3.1.1}$$

Let γ be constant and

$$\gamma = \frac{\lambda}{\lambda + \frac{\mu^2}{2\sigma^2}}. \tag{3.1.2}$$

For the function $v_0(x)$ in (3.1.1), define

$$X_1 = \inf\{x \geq 0 : v_0(x) = Mv_0(x)\}, \tag{3.1.3}$$

and

$$X_0 := \frac{(1-\gamma)\sigma^2}{\mu}. \tag{3.1.4}$$

Then from [3], we can get the structure of the solution of (QVI0):

$$v_0(x) = \begin{cases} C_0 x^\gamma, & x \in [0, X_0), \\ C_0 a_1 e^{\theta_+(x-X_0)} + C_0 a_2 e^{\theta_-(x-X_0)}, & x \in [X_0, X_1), \\ v_0(\widetilde{X}) + k(x-\widetilde{X}) - K, & x \in [X_1, \infty), \end{cases} \tag{3.1.5}$$

where C_0 is a free constant, θ_+, θ_-, a_1, a_2 are given by

$$\theta_+ = \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}, \qquad \theta_- = \frac{-\mu - \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}, \tag{3.1.6}$$

$$a_1 = \frac{\gamma X_0^{\gamma-1} - \theta_- X_0^\gamma}{\theta_+ - \theta_-}, \qquad a_2 = \frac{\theta_+ X_0^\gamma - \gamma X_0^{\gamma-1}}{\theta_+ - \theta_-}, \tag{3.1.7}$$

and $\widetilde{X}$ is the unique solution of the following equation

$$v_0'(x) = k, \quad x \in (0, X_1). \tag{3.1.8}$$

In [3], it has shown that $a_1 > 0$ and $a_2 < 0$, which are important results would be used in the next section.

For (3.1.5), it is easy to get the derivative of $v_0(x)$ as follows

$$v_0'(x) = \begin{cases} C_0 \gamma x^{\gamma-1}, & x \in [0, X_0), \\ C_0 a_1 \theta_+ e^{\theta_+(x-X_0)} + C_0 a_2 \theta_- e^{\theta_-(x-X_0)}, & x \in [X_0, X_1), \\ k, & x \in [X_1, \infty). \end{cases} \tag{3.1.9}$$

In addition, from the equalities

$$v_0(X_1) - v_0(\widetilde{X}) = \int_{\widetilde{X}}^{X_1} v_0'(y)dy = k(X_1 - \widetilde{X}) - K,$$

we have that

$$\int_{\widetilde{X}}^{X_1} (k - v_0'(y))dy = K. \tag{3.1.10}$$

Define a function $H(x)$ by

$$H(x) := \begin{cases} \gamma x^{\gamma-1}, & x \in [0, X_0), \\ a_1\theta_+ e^{\theta_+(x-X_0)} + a_2\theta_- e^{\theta_-(x-X_0)}, & x \in [X_0, \infty). \end{cases} \tag{3.1.11}$$

Let

$$I_0(\widetilde{C}_0) := \int_{\widetilde{X}^{\widetilde{C}_0}}^{X_1^{\widetilde{C}_0}} (k - \widetilde{C}_0 H(x))dx, \tag{3.1.12}$$

then from [3], $I_0(\widetilde{C}_0)$ is a decreasing function of $\widetilde{C}_0$ with the range $[0, +\infty)$, and then there exist unique $\widetilde{C}_0$, $\widetilde{X}^{\widetilde{C}_0}$ and $X_1^{\widetilde{C}_0}$, such that $I_0(\widetilde{C}_0) = K$.

Comparing (3.1.10) with (3.1.12), then we have $C_0 = \widetilde{C}_0$, $\widetilde{X} = \widetilde{X}^{\widetilde{C}_0}$ and $X_1 = X_1^{\widetilde{C}_0}$. Further, in [3] it shows that the solution of (QVI0) can be given by (3.1.5) with $C_0 = \widetilde{C}_0$, $\widetilde{X} = \widetilde{X}^{\widetilde{C}_0}$ and $X_1 = X_1^{\widetilde{C}_0}$.

3.2 Smooth Solutions of (QVI)

The difference between (QVI) and (QVI0) is just the boundary condition. So, we guess that solutions of them may have some relations or similarity.

3.2.1 Smooth Solutions of (QVI) on $[0, x_1)$

First, as in (3.1.3), we define

$$x_1 = \inf\{x \geq 0 : v(x) = Mv(x)\}. \tag{3.2.13}$$

Notice that for $x = 0$, $Mv(x) = Mv(0) = v(0) - K < v(0)$. So, for the definition of (3.2.13), it follows that $x_1 > 0$.

Then, on the interval $(0, x_1)$, from (QVI) we have that

$$\max_{u \in [0,1]} \mathcal{L}^u v(x) = 0, \qquad 0 < x < x_1. \tag{3.2.14}$$

Let $u(x) \in \mathbb{R}$ be the maximizer of the expression on the left-hand side of (3.2.14), then

$$u(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}. \tag{3.2.15}$$

Put (3.2.15) into (3.2.14), then we can get

$$-\frac{\mu^2 [v'(x)]^2}{2\sigma^2 v''(x)} - \lambda v(x) = 0. \tag{3.2.16}$$

A general solution of (3.2.16) with boundary condition $v(0) = a$ is given by

$$v(x) = C\left[\left(\frac{a}{C}\right)^{\frac{1}{\gamma}} + x\right]^{\gamma}, \tag{3.2.17}$$

where C is a free constant and γ is also presented by (3.1.2).

From (3.2.15) and (3.2.17), it follows that

$$u(x) = \frac{\mu}{(1-\gamma)\sigma^2}\left[x + \left(\frac{a}{C}\right)^{\frac{1}{\gamma}}\right]. \tag{3.2.18}$$

Since $u(x)$ above is an increasing linear function, then $u(x) \leq 1$ if and only if

$x \le x_0$, where

$$x_0 := \frac{(1-\gamma)\sigma^2}{\mu} - \left(\frac{a}{C}\right)^{\frac{1}{\gamma}} = X_0 - \left(\frac{a}{C}\right)^{\frac{1}{\gamma}}. \tag{3.2.19}$$

Thus, if $0 < x_0 < x < x_1$, then it follows that $u(x) \ge 1$. But the range of $u(x)$ is $[0,1]$, then we must have that $u(x) = 1$ for $x \in (x_0, x_1)$, and (3.2.14) becomes

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - \lambda v(x) = 0, \quad x \in (x_0, x_1). \tag{3.2.20}$$

One general solution to (3.2.20) can be written by

$$v(x) = C_1 e^{\theta_+(x-x_0)} + C_2 e^{\theta_-(x-x_0)}, \; x \in (x_0, x_1), \tag{3.2.21}$$

where C_1 and C_2 are free constants, θ_+ and θ_- are given by (3.1.6). Continuity of the function $v(x)$ and its derivative $v'(x)$ at the point x_0 implies: $C_1 = Ca_1$, $C_2 = Ca_2$, where C is a free constant, a_1 and a_2 are defined by (3.1.7).

On the other hand, if $x_0 \le 0$, then $u(x) = 1$ for any $x \in [0, x_1)$, and (3.2.14) becomes

$$\begin{cases} \frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - \lambda v(x) = 0, & x \in [0, x_1), \\ v(0) = a. \end{cases} \tag{3.2.22}$$

Solving (3.2.22), we can get the general solution of (3.2.22) as follows.

$$v(x) = C_1 e^{\theta_+ x} + C_2 e^{\theta_- x}, \qquad x \in [0, x_1), \tag{3.2.23}$$

with $C_1 + C_2 = a$.

Now, let's summary the possible structure for the solution of (3.2.14) on $[0, x_1)$.

If $x_0 > 0$, then

$$v(x) = \begin{cases} C\left[\left(\frac{a}{C}\right)^{\frac{1}{\gamma}} + x\right]^{\gamma}, & x \in [0, x_0), \\ Ca_1 e^{\theta_+(x-x_0)} + Ca_2 e^{\theta_-(x-x_0)}, & x \in [x_0, x_1), \end{cases} \tag{3.2.24}$$

where C is a free constant.

If $x_0 \le 0$, then the structure of $v(x)$ on $[0, x_1)$ is given by (3.2.23).

Remark 3.2.1. *From (3.2.19), we notice that x_0 depends on the uncertain parameter C, which is needed to be estimated later. For different C, the sign of x_0 may be different. Moreover, it easily shows that the assumption, $x_0 > 0$, is equivalent to*

$$C > aX_0^{-\gamma}, \tag{3.2.25}$$

and $x_0 \to 0_+$ is equivalent to $C \to (aX_0^{-\gamma})_+$. In addition, for (3.2.23) and (3.2.24), they should be consistent with each other at $x_0 = 0$. That is, for any $x \in (0, x_1)$,

$$\begin{aligned} &\lim_{C \to (aX_0^{-\gamma})_+} \left(Ca_1 e^{\theta_+(x-x_0)} + Ca_2 e^{\theta_-(x-x_0)} \right) \\ &= aX_0^{-\gamma} a_1 e^{\theta_+ x} + aX_0^{-\gamma} a_2 e^{\theta_- x} \\ &= \lim_{C_1 \to (aX_0^{-\gamma} a_1)_-} \left(C_1 e^{\theta_+ x} + (a - C_1) e^{\theta_- x} \right). \end{aligned} \tag{3.2.26}$$

So, at $C = aX_0^{-\gamma}$, for the purpose of the consistency of two solutions, we must have that $C_1 = Ca_1$. In addition, we know that the condition of existence for (3.2.23) is $C \le aX_0^{-\gamma}$. Moreover, there is no conflict to denote C_1 by Ca_1 when $C \le aX_0^{-\gamma}$. So, we can let $C_1 = Ca_1$ for $C \le aX_0^{-\gamma}$. Then,

$$C_1 \le aX_0^{-\gamma} a_1, \tag{3.2.27}$$

which is the condition (3.2.23) satisfies.

Remark 3.2.2. *For $v(x)$ in (3.2.24), from $a_1 > 0$ and $a_2 < 0$, it is easy to show that $v'''(x) > 0$ on $[0, x_1)$, which implies that $v'(x)$ obtained from (3.2.24) have convexity on $[0, x_1)$.*

3.2.2 Smooth Solution of (QVI) at x_1

From the definition of x_1, we have that $v(x_1) = Mv(x_1)$. Then, by

$$\lim_{\eta\to 0}(v(x_1-\eta)+k\eta-K) = v(x_1)-K < v(x_1),$$

it shows that the maximizing sequence η for $v(x_1) = Mv(x_1)$ can't have zero as a limiting point. So, at x_1, the supremum in the right-hand side of (2.2.1) can be taken over $\eta \in [\epsilon, x_1]$ for some $\epsilon > 0$. Therefore, there exists $\eta(x_1) \in (0, x_1]$, such that

$$v(x_1) = v(x_1-\eta(x_1)) + k\eta(x_1) - K.$$

Let

$$\widetilde{x} = x_1 - \eta(x_1),$$

then, $0 \le \widetilde{x} < x_1$ and

$$v(x_1) = v(\widetilde{x}) + k(x_1-\widetilde{x}) - K. \tag{3.2.28}$$

From (3.2.28), it follows that

$$v(x_1)-v(\widetilde{x}) = \int_{\widetilde{x}}^{x_1} v'(x)dx = k(x_1-\widetilde{x}) - K,$$

then, we obtain that

$$\int_{\widetilde{x}}^{x_1}(k-v'(x))dx = K. \tag{3.2.29}$$

Remark 3.2.3. *If $\widetilde{x} = 0$, then $v(x_1) = v(0) + kx_1 - K = a + kx_1 - K$, which means the dividend happens once and then a bankruptcy follows. This is an important phenomena deserved to be discussed further in the later section.*

Chapter 4

Uniqueness of The Undetermined Parameter C

In chapter 3, some parameters, such as C, C_1 and C_2, are unfixed numbers. In this chapter, I mainly discuss the uniqueness of their corresponding parameters in two useful constructed integral functions, whose integrands can be used to get some solutions of (QVI).

In this chapter, as in Remark 3.2.1, I claim that Ca_1 can be used to denote C_1 for $C \leq aX_0^{-\gamma}$.

4.1 Definitions and Properties of Two Integral Functions

From Remark 3.2.1, it is known that different C would lead to two possible cases for x_0. The first case is, $x_0 > 0$. The second one is, $x_0 \leq 0$. In the following, we discuss these two cases by two constructed integral functions, respectively.

4.1.1 Two Cases of x_0 and Two Corresponding Integral Functions

Case I: $x_0 > 0$.

Let $H^C(x)$ be a function, with constant C, constructed by

$$H^C(x) := \begin{cases} \gamma[(\frac{a}{C})^{\frac{1}{\gamma}} + x]^{\gamma-1}, & x \in [0, x_0), \\ a_1\theta_+ e^{\theta_+(x-x_0)} + a_2\theta_- e^{\theta_-(x-x_0)}, & x \in [x_0, \infty), \end{cases} \tag{4.1.1}$$

where x_0 is also defined by (3.2.19) with $C > aX_0^{-\gamma}$.

Define

$$I_1(C) = \int_{\widetilde{x}^C \vee 0}^{x_1^C} (k - CH^C(x))dx, \tag{4.1.2}$$

where x_1^C and $\widetilde{x}^C$ are two nonnegative roots of the equation $k - CH^C(x) = 0$ with $\widetilde{x}^C < x_1^C$, and $\widetilde{x}^C \vee 0$ denotes $\max\{\widetilde{x}^C, 0\}$. If $\widetilde{x}^C$ doesn't exist on $[0, \infty)$, then let $\widetilde{x}^C \vee 0 = 0$.

From the definitions of $H^C(x)$ and $I_1(C)$, then obviously $CH^C(x)$ is a continuous function of C. Further, $\widetilde{x}^C \vee 0$ and x_1^C are also continuous functions of C if the existence of them is satisfied. So, $I_1(C)$ is continuous function.

Proposition 4.1.1. *Let $H^C(x)$ be defined by (4.1.1), then $CH^C(x)$ is an increasing function with respect to C. For $I_1(C)$, if the existence of it is satisfied on some subintervals of $(aX_0^{-\gamma}, \infty)$, then it is a strictly decreasing function on these subintervals.*

Proof. From (4.1.2), the derivative of $I_1(C)$ can be presented as follows.

$$I_1'(C) = -\int_{\widetilde{x}^C \vee 0}^{x_1^C} \left(H^C(x) - \frac{1}{\gamma}\left(\frac{a}{C}\right)^{\frac{1}{\gamma}} (H^C)'(x) \right) dx. \tag{4.1.3}$$

Let $f(x) = H^C(x) - \frac{1}{\gamma}(\frac{a}{C})^{\frac{1}{\gamma}}(H^C)'(x)$ on $[0,\infty)$, then from (4.1.1), we have that

$$f(x) = \begin{cases} \gamma\left(x + (\frac{a}{C})^{\frac{1}{\gamma}}\right)^{\gamma-1} + (1-\gamma)(\frac{a}{C})^{\frac{1}{\gamma}}\left(x + (\frac{a}{C})^{\frac{1}{\gamma}}\right)^{\gamma-2}, & 0 \le x < x_0, \\ a_1\theta_+(1 - \frac{1}{\gamma}\theta_+(\frac{a}{C})^{\frac{1}{\gamma}})e^{\theta_+(x-x_0)} + a_2\theta_-(1 - \frac{1}{\gamma}\theta_-(\frac{a}{C})^{\frac{1}{\gamma}})e^{\theta_-(x-x_0)}, & x \ge x_0. \end{cases} \tag{4.1.4}$$

From (3.1.2) and (3.1.6), it follows that $0 < \gamma < 1$, $1-\gamma > 0$, $\theta_+ > 0$ and $\theta_- < 0$. Consequently, $1 - \frac{1}{\gamma}\theta_+(\frac{a}{C})^{\frac{1}{\gamma}} > 0$ (see Appendix 1), and $1 - \frac{1}{\gamma}\theta_-(\frac{a}{C})^{\frac{1}{\gamma}} > 0$ due to $\theta_- < 0$. Hence, from (4.1.4), we have that $f(x) > 0$ on $[0,\infty)$, which means $CH^C(x)$ is an increasing function with respect to C.

Then from (4.1.3), $I_1'(C) < 0$. Therefore, $I_1(C)$ is a strictly decreasing function with respect to C. □

Remark 4.1.1. *Notice that x_1^C and $\widetilde{x}^C$ doesn't exist on $[0,\infty)$ when $CH^C(x) \to \infty$ as $C \to \infty$. So, for big enough C, $I_1(C)$ doesn't make sense. But for some other C on $(aX_0^{-\gamma},\infty)$, the existence of $I_1(C)$ is satisfied indeed, which can be shown in the next subsection. This is the reason why we make a statement about the existence of $I_1(C)$ in Proposition 4.1.1.*

Remark 4.1.2. *For the $H^C(x)$ defined by (4.1.1), we can have that $(H^C)''(x) > 0$, then $H^C(x)$ has convexity on $[0,\infty)$.*

Remark 4.1.3. *From the strict monotonicity of $I_1(C)$, we can conclude that if $I_1(aX_0^{-\gamma}) > 0$ and $K \in (0, I_1(aX_0^{-\gamma}))$, then there exists unique solution $C \in (aX_0^{-\gamma}, C^*)$, such that $I_1(C) = K$, where $(aX_0^{-\gamma}, C^*)$ is a domain of $I_1(C)$ and C^* is a point, such that $I_1(C^*) = 0$.*

Case II: $x_0 \le 0$.

Let

$$H^{C_1}(x) = C_1\theta_+e^{\theta_+x} + (a - C_1)\theta_-e^{\theta_-x}, \qquad x \in [0,\infty), \tag{4.1.5}$$

where $0 < C_1 \leq aX_0^{-\gamma} a_1$.

Define

$$I_2(C_1) := \int_{\widetilde{x}^{C_1} \vee 0}^{x_1^{C_1}} (k - H^{C_1}(x))dx, \tag{4.1.6}$$

where $x_1^{C_1}$ and $\widetilde{x}^{C_1}$ are two nonnegative roots of the equation $k - H^{C_1}(x) = 0$ with $\widetilde{x}^{C_1} < x_1^{C_1}$. If $\widetilde{x}^{C_1}$ doesn't exist on $[0, \infty)$, then let $\widetilde{x}^{C_1} \vee 0 = 0$.

From the definitions of $H^{C_1}(x)$ and $I_2(C_1)$, then $H^{C_1}(x)$ is a continuous function of C_1. Further, $\widetilde{x}^{C_1} \vee 0$ and $x_1^{C_1}$ are also continuous functions of C_1 if they exist. So, $I_2(C_1)$ is continuous function.

For (4.1.6), take the derivative of $I_2(C_1)$ with respect to C_1, then

$$I_2'(C_1) = \int_{\widetilde{x}^{C_1} \vee 0}^{x_1^{C_1}} \Big(-\theta_+ e^{\theta_+ x} + \theta_- e^{\theta_- x} \Big) dx. \tag{4.1.7}$$

Consequently, for any positive C_1, $I_2'(C_1) < 0$ due to the fact $\theta_+ > 0$ and $\theta_- < 0$. So, we have that

Proposition 4.1.2. *For $I_2(C_1)$, if the existence of it is satisfied on some subintervals of $(0, aX_0^{-\gamma} a_1)$, then it is a continuous and strictly decreasing function on these subintervals.*

Now, let's see what would happen for $x_1^{C_1}$ and $I_2(C_1)$ as $C_1 \to 0$.

Proposition 4.1.3. *For $I_2(C_1)$ defined by (4.1.6), we have that*

$$\lim_{C_1 \to 0} x_1^{C_1} = \infty, \tag{4.1.8}$$

and

$$\lim_{C_1 \to 0} I_2(C_1) = \infty. \tag{4.1.9}$$

Proof. Let D be a fixed number and $D = (\frac{k - a\theta_-}{\theta_+ - \theta_-}) \wedge (aX_0^{-\gamma} a_1)$, then we can see that for any $C_1 \in (0, D)$, $H^{C_1}(0) = C_1\theta_+ + (a - C_1)\theta_- < k$. On the other hand, for any such C_1,

$$H^{C_1}(x) = C_1\theta_+ e^{\theta_+ x} + (a - C_1)\theta_- e^{\theta_- x} \to \infty,$$

as $x \to \infty$. So, by continuity of $H^{C_1}(x)$, there exists $x_1^{C_1} \in (0, \infty)$, such that $H^{C_1}(x_1^{C_1}) = k$ and $H^{C_1}(x) < k$ for $x \in (0, x_1^{C_1})$. Then $I_2(C_1)$ makes sense for $0 < C_1 < D$. Moreover, we have $I_2(C_1) > 0$.

From above, for any $C_1 \in (0, D)$, it follows $x_1^{C_1} \in (0, \infty)$, which implies that

$$-a\theta_- > -a\theta_- e^{\theta_- x_1^{C_1}} > -(a - C_1)\theta_- e^{\theta_- x_1^{C_1}}.$$

Then, the following inequality can be satisfied.

$$C_1\theta_+ e^{\theta_+ x_1^{C_1}} + a\theta_- < C_1\theta_+ e^{\theta_+ x_1^{C_1}} + (a - C_1)\theta_- e^{\theta_- x_1^{C_1}} < C_1\theta_+ e^{\theta_+ x_1^{C_1}}. \quad (4.1.10)$$

Let $x_2^{C_1}$ and $x_3^{C_1}$ be two positive numbers, such that

$$C_1\theta_+ e^{\theta_+ x_3^{C_1}} + a\theta_- = C_1\theta_+ e^{\theta_+ x_1^{C_1}} + (a - C_1)\theta_- e^{\theta_- x_1^{C_1}} = C_1\theta_+ e^{\theta_+ x_2^{C_1}}. \quad (4.1.11)$$

Then, from (4.1.10) and (4.1.11), by monotonicity of exponential function, we have that

$$x_3^{C_1} > x_1^{C_1} > x_2^{C_1}. \quad (4.1.12)$$

If $C_1\theta_+ e^{\theta_+ x_3^{C_1}} + a\theta_- = k = C_1\theta_+ e^{\theta_+ x_2^{C_1}}$, then

$$x_3^{C_1} = \frac{1}{\theta_+}\ln(\frac{k - a\theta_-}{C_1\theta_+}), \qquad x_2^{C_1} = \frac{1}{\theta_+}\ln(\frac{k}{C_1\theta_+}).$$

Further, it follows that

$$\lim_{C_1 \to 0} x_3^{C_1} = \lim_{C_1 \to 0} \frac{1}{\theta_+}\ln(\frac{k - a\theta_-}{C_1\theta_+}) = \infty, \quad (4.1.13)$$

and

$$\lim_{C_1 \to 0} x_2^{C_1} = \lim_{C_1 \to 0} \frac{1}{\theta_+} \ln(\frac{k}{C_1 \theta_+}) = \infty. \tag{4.1.14}$$

From (4.1.12)-(4.1.14), we can get an important result: $\lim\limits_{C_1 \to 0} x_1^{C_1} = \infty$. So, the limit of $I_2(C_1)$ as $C_1 \to 0$ can be given by

$$\begin{aligned}
\lim_{C_1 \to 0} I_2(C_1) &= \lim_{C_1 \to 0} \int_0^{x_1^{C_1}} \Big[k - (C_1 \theta_+ e^{\theta_+ x} + (a - C_1)\theta_- e^{\theta_- x}) \Big] dx \\
&= \int_0^{\infty} (k - a\theta_- e^{\theta_- x}) dx \geq \int_0^{\infty} k dx = \infty. \qquad (4.1.15)
\end{aligned}$$

Therefore, $\lim\limits_{C_1 \to 0} I_2(C_1) = \infty$. □

Remark 4.1.4. *From the strict monotonicity of $I_2(C_1)$ and Proposition 4.1.3, we can conclude that if $K \in [I_2(C_1^*), \infty)$, then there exists unique solution $C_1 \in (0, C_1^*]$, such that $I_2(C_1) = K$, where $(0, C_1^*]$ is a domain of $I_2(C_1)$. In the next subsection, the determination of the points C_1^* is discussed.*

Remark 4.1.5. *Notice that for small enough and positive C_1, it is possible that $a - C_1 > 0$. Under this situation, $H^{C_1}(x)$ may have no convexity since $(H^{C_1})''(x) > 0$ may be not satisfied.*

4.1.2 Property of Two Integral Functions at $x_0 = 0$

Since $x_0 = 0$ is a joint point for both cases $x_0 > 0$ and $x_0 \leq 0$, then it is necessary for us to see some common properties of $I_1(C)$ and $I_2(Ca_1)$ at this special point.

At $x_0 = 0$, then it follows that $C_1 = aX_0^{-\gamma} a_1$ and $C = aX_0^{-\gamma}$. Denote $CH^C(x)$ at $C = aX_0^{-\gamma}$ by

$$B(x) = aX_0^{-\gamma} a_1 \theta_+ e^{\theta_+ x} + aX_0^{-\gamma} a_2 \theta_- e^{\theta_- x}.$$

From

$$a - C_1 = a - aX_0^{-\gamma}a_1 = a(1 - X_0^{-\gamma}a_1) = aX_0^{-\gamma}(X_0^{\gamma} - a_1) = aX_0^{-\gamma}a_2,$$

we obtain that $H^{C_1}(x) = B(x)$ at $C_1 = aX_0^{-\gamma}a_1$. Then, at $x_0 = 0$, the integrands of $I_1(C)$ and $I_2(C_1)$ are the same. Further, $x_1^{C_1} = x_1^C$ and $\widetilde{x}^{C_1} \vee 0 = \widetilde{x}^C \vee 0$ if $x_1^{C_1}$ exists. Consequently, we can conclude that if $x_1^{C_1}$ exists at $C_1 = aX_0^{-\gamma}a_1$, then

$$I_2(aX_0^{-\gamma}a_1) = I_1(aX_0^{-\gamma}). \tag{4.1.16}$$

In order to judge if $I_2(aX_0^{-\gamma}a_1) > 0$, we have the following result.

Proposition 4.1.4. *One equivalent condition of $I_2(aX_0^{-\gamma}a_1) > 0$ is:*

$$k > aM^*, \tag{4.1.17}$$

where M^* is given by

$$M^* = \left(-\frac{\theta_-}{\theta_+}\right)^{\frac{\theta_-}{\theta_+ - \theta_-}} \cdot \left(\frac{2\lambda}{\mu} - \theta_-\right). \tag{4.1.18}$$

Proof. Notice that $B''(x) > 0$. So, $B(x)$ has convexity. Combining with the definition of $I_2(C_1)$, then one equivalent condition of $I_2(aX_0^{-\gamma}a_1) > 0$ is that there exist two different nonnegative constants $\widetilde{x}^{C_1} \vee 0$ and $x_1^{C_1}$ satisfying $B(\widetilde{x}^{C_1} \vee 0) \leq k$ and $B(x_1^{C_1}) = k$, which implies $B(x) < k$ on $(\widetilde{x}^{C_1} \vee 0, x_1^{C_1})$. This condition is also equivalent to

$$\min_{x \in [0,\infty)} B(x) < k. \tag{4.1.19}$$

Solving $B'(x) = 0$, we have that

$$x = \frac{1}{\theta_+ - \theta_-} \ln\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right). \tag{4.1.20}$$

From Appendix 6, it shows that $\frac{-a_2\theta_-^2}{a_1\theta_+^2} > 1$, then it follows that $x > 0$ for (4.1.20).

Put (4.1.20) into $B(x)$, we can get

$$\begin{aligned} \min_{x\in[0,\infty)} B(x) &= aX_0^{-\gamma}\left[a_1\theta_+\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} + a_2\theta_-\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_-}{\theta_+-\theta_-}}\right] \\ &= aX_0^{-\gamma}\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}}\left[a_1\theta_+ - \frac{a_1\theta_+^2}{\theta_-}\right] \\ &= aX_0^{-\gamma}\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{a_1\theta_+(\theta_- - \theta_+)}{\theta_-}. \end{aligned} \tag{4.1.21}$$

From Appendix 8, we can obtain that

$$aX_0^{-\gamma}\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{a_1\theta_+(\theta_- - \theta_+)}{\theta_-} = a\left(-\frac{\theta_-}{\theta_+}\right)^{\frac{\theta_-}{\theta_+-\theta_-}} \cdot \left(\frac{2\lambda}{\mu} - \theta_-\right) = M^*.$$

Then, from (4.1.19), (4.1.21) and the equality above, it shows that one equivalent condition of $I_2(aX_0^{-\gamma}a_1) > 0$ is: $k > aM^*$. □

Remark 4.1.6. *In Appendix 9, an important inequality with respect to M^* is provided as follows,*

$$\frac{\lambda}{\mu} < M^* < \frac{2\lambda}{\mu}. \tag{4.1.22}$$

4.2 Integral Functions $I(C)$ under Different Conditions

In order to point out the conditions leading to bankruptcy, in this section we make some useful and necessary preparation by considering the important cases $\widetilde{x}^C \vee 0 = 0$

and $\widetilde{x}^{C_1} \vee 0 = 0$.

Notice that by the convexity of $H^C(x)$ on $[0, \infty)$, it shows that $CH^C(0) \leq k$, that is $C \leq a^{1-\gamma}(k/\gamma)^\gamma$, is equivalent to $\widetilde{x}^C \vee 0 = 0$. So, $a^{1-\gamma}(k/\gamma)^\gamma$ is a useful point to judge if $\widetilde{x}^C \vee 0 = 0$. But from the definition of $H^C(x)$, it is known that $C > aX_0^{-\gamma}$. Unfortunately, we can't conclude which one is bigger for $aX_0^{-\gamma}$ and $a^{1-\gamma}(k/\gamma)^\gamma$. Therefore, it is necessary for us to discuss two situations of them.

4.2.1 Case 1: $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$.

From Appendix 3, we have that $\frac{\gamma}{X_0} = \frac{2\lambda}{\mu}$. Then, the assumption $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$, that is $a\gamma < kX_0$, is equivalent to

$$k > \frac{2a\lambda}{\mu}. \tag{4.2.23}$$

For $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$, the parameter C has three possible situations: $C > a^{1-\gamma}(k/\gamma)^\gamma$, $a^{1-\gamma}(k/\gamma)^\gamma \geq C > aX_0^{-\gamma}$, and $C \leq aX_0^{-\gamma}$. If $C > a^{1-\gamma}(k/\gamma)^\gamma$, it can be verified that $x_0 > 0$, $CH^C(0) > k$, which means C is the parameter of $I_1(C)$ with $\widetilde{x}^C > 0$ if $\widetilde{x}^C > 0$ exists. In addition, if the parameter C satisfies $a^{1-\gamma}(k/\gamma)^\gamma \geq C > aX_0^{-\gamma}$, then it follows that $x_0 > 0$ and $\widetilde{x}^C \vee 0 = 0$. Then, we need to consider the integral $I_1(C)$ with $\widetilde{x}^C \vee 0 = 0$. If $C \leq aX_0^{-\gamma}$, then $x_0 \leq 0$, for which $I_2(C_1)$ will be used.

In the following, under the assumption $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$, firstly we discuss some properties of $I_1(C)$, and then based on these results, we present the corresponding suitable domains of $I_1(C)$ and $I_2(C_1)$. Consequently, by $I_1(C)$ and $I_2(C_1)$, an integral $I(C)$ is defined with the range, $(0, +\infty)$. Then, the parameter C can be calculated from the equation $I(C) = K$.

Now, let's see some properties of $I_1(C)$ under $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$. For $C = aX_0^{-\gamma}$, which implies $x_0 = 0$, then from Proposition 4.1.4, (4.2.23) and (4.1.22), we have

Proposition 4.2.1. *If* $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$*, then* $k > \frac{2a\lambda}{\mu} > aM^*$*, and*

$$I_1(aX_0^{-\gamma}) > 0. \tag{4.2.24}$$

In addition, at point $C = a^{1-\gamma}(k/\gamma)^\gamma$, from (4.1.1) we have that $CH^C(0) = k$ and $C(H^C)'(0) < 0$. Combining with the convexity of $H^C(x)$, then it follows that $x_1^C > 0$ and $k - CH^C(x) > 0$ on $(0, x_1^C)$. Then, the following result is verified.

Proposition 4.2.2. *Under the assumption* $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$*, at* $a^{1-\gamma}(k/\gamma)^\gamma$*, an important inequality* $I_1(a^{1-\gamma}(k/\gamma)^\gamma) > 0$ *is satisfied.*

Based on Proposition 4.2.1 and Proposition 4.2.2, the domains of $I_1(C)$ and $I_2(C_1)$ can be given as follows.

Remark 4.2.1. *If* $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$*, that is* $a\gamma < kX_0$*, then* $(0, aX_0^{-\gamma}a_1]$ *can be the domain of* $I_2(C_1)$*.*

Proof. From the proof of Proposition 4.1.3, it has been verified that $I_2(C_1) > 0$ for any $C_1 \in (0, D)$, where $D = (\frac{k-a\theta_-}{\theta_+-\theta_-}) \wedge (aX_0^{-\gamma}a_1)$. Fortunately, from Appendix 2, we have that $\frac{k-a\theta_-}{\theta_+-\theta_-} > aX_0^{-\gamma}a_1$ under the assumption $a\gamma < kX_0$. Then, $D = aX_0^{-\gamma}a_1$. So, for any $C_1 \in (0, aX_0^{-\gamma}a_1)$, $I_2(C_1) > 0$. Combining with (4.1.16) and (4.2.24), it follows that $I_2(C_1) > 0$ on $(0, aX_0^{-\gamma}a_1]$.

In addition, in the proof of Proposition 4.1.3, it has been shown that $I_2(C_1) \to \infty$ as $C_1 \to 0$. Therefore, zero only can be the right opening domain endpoint of $I_2(C_1)$.

So, it shows that $(0, aX_0^{-\gamma}a_1]$ can be the domain of $I_2(C_1)$. □

Remark 4.2.2. *If* $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$*, from Remark (4.2.1), it shows that* C_1^* *mentioned in Remark 4.1.4 can be given by* $aX_0^{-\gamma}a_1$*.*

From $I_2(aX_0^{-\gamma}a_1) = I_1(aX_0^{-\gamma}) > 0$, it shows that $aX_0^{-\gamma}$ can be the left domain endpoint for $I_1(C)$. On the other hand, from Proposition 4.2.2, $I_1(a^{1-\gamma}(k/\gamma)^\gamma) > 0$. Moreover, from Proposition 4.1.1, it is known that $I_1(C)$ is strictly decreasing function and $CH^C(x)$ is an increasing function with respect to C. Further, we notice that $CH^C(x) \to \infty$ as $C \to \infty$ and $I_1(C)$ is a continuous function of C. Then, there exists $C^* > a^{1-\gamma}(k/\gamma)^\gamma$, such that $I_1(C^*) = 0$ and $I_1(C) > 0$ for $aX_0^{-\gamma} < C < C^*$. So, it follows that

Remark 4.2.3. *If* $aX_0^{-\gamma} < a^{1-\gamma}(k/\gamma)^\gamma$*, then* $(aX_0^{-\gamma}, C^*)$ *can be the domain of* $I_1(C)$.

Remark 4.2.4. *In summary, from the discussion above, the following results can be given:*

(i) $I_1(C) \in (0, I_2(aX_0^{-\gamma}a_1))$, *for* $C \in (aX_0^{-\gamma}, C^*)$;

(ii) $I_2(Ca_1) \in [I_2(aX_0^{-\gamma}a_1), \infty)$, *for* $C \in (0, aX_0^{-\gamma}]$;

where $C^* > a^{1-\gamma}(k/\gamma)^\gamma$. *Moreover,* $I_1(C)$ *and* $I_2(Ca_1)$ *are both strictly decreasing functions on their corresponding domains.*

Based on the domains and the ranges of $I_1(C)$ and $I_2(C_1)$ above, then a useful integral function can be constructed as follows.

Define

$$I(C) := \begin{cases} I_2(Ca_1), & C \in (0, aX_0^{-\gamma}], \\ I_1(C), & C \in (aX_0^{-\gamma}, C^*). \end{cases} \tag{4.2.25}$$

Then, we have the following results.

Proposition 4.2.3. *From (4.2.25), it shows that* $I(C)$ *can be composed of* $I_1(C)$ *and* $I_2(Ca_1)$. *From Remark 4.2.4, it is known that* $I(C) \in (0, +\infty)$ *on the interval* $(0, C^*)$. *Moreover,* $I(C)$ *is a strictly decreasing and continuous function on the whole interval. Therefore, for any* $K \in (0, +\infty)$, *there exists unique* $C \in (0, C^*)$, *such that* $I(C) = K$.

4.2.2 Case 2: $aX_0^{-\gamma} \geq a^{1-\gamma}(k/\gamma)^{\gamma}$.

From Appendix 3, it follows that $\frac{\gamma}{X_0} = \frac{2\lambda}{\mu}$. Then, the assumption $aX_0^{-\gamma} \geq a^{1-\gamma}(k/\gamma)^{\gamma}$, that is $a\gamma \geq kX_0$, is equivalent to

$$k \leq \frac{2a\lambda}{\mu}. \tag{4.2.26}$$

Since $I_1(C)$ and $I_2(C_1)$ are both decreasing functions, then it is possible that their minimum may reach zero under some conditions. In the following, our major work is to find suitable domains of $I_1(C)$ and $I_2(C_1)$ under several situations of k. Then, based on these results, the integral function $I(C)$ is constructed.

Under the assumption $aX_0^{-\gamma} \geq a^{1-\gamma}(k/\gamma)^{\gamma}$, for the case $C > aX_0^{-\gamma}$, we have $x_0 > 0$ and $CH^C(0) > k$. Further, from the convexity of $H^C(x)$, it follows that $\widetilde{x}^C > 0$ if $\widetilde{x}^C$ exists. So, when $C > aX_0^{-\gamma}$, we can consider $I_1(C)$ if $I_1(C)$ exists. On the other hand, if $C \leq aX_0^{-\gamma}$, then $x_0 \leq 0$. So, we need to consider $I_2(C_1)$.

For $I_2(C_1)$, notice that the condition $H^{C_1}(0) \leq k$ is equivalent to

$$C_1 \leq \frac{k - a\theta_-}{\theta_+ - \theta_-}. \tag{4.2.27}$$

Then, we have that

Proposition 4.2.4. *If $k > \frac{a\lambda}{\mu}$, then $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) > 0$. If $k \leq \frac{a\lambda}{\mu}$, then $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) = 0$ and $I_2(C_1) > 0$ for any $C_1 \in (0, \frac{k-a\theta_-}{\theta_+-\theta_-})$.*

Proof. For $C_1 = \frac{k-a\theta_-}{\theta_+-\theta_-}$, denote $H^{C_1}(x)$ by

$$\widetilde{H}(x) = \frac{k - a\theta_-}{\theta_+ - \theta_-}\theta_+ e^{\theta_+ x} + \frac{a\theta_+ - k}{\theta_+ - \theta_-}\theta_- e^{\theta_- x},$$

then,

$$\widetilde{H}'(x) = \frac{k - a\theta_-}{\theta_+ - \theta_-}\theta_+^2 e^{\theta_+ x} + \frac{a\theta_+ - k}{\theta_+ - \theta_-}\theta_-^2 e^{\theta_- x}. \tag{4.2.28}$$

Now, let's see the situation $\widetilde{H}'(x) < 0$ for some $x > 0$, which is equivalent to

$$0 < x < \frac{1}{\theta_+ - \theta_-} \ln\left(\frac{\theta_-^2(k - a\theta_+)}{\theta_+^2(k - a\theta_-)}\right), \tag{4.2.29}$$

from (4.2.28). If (4.2.29) is satisfied, then the following condition is needed.

$$\frac{\theta_-^2(k - a\theta_+)}{\theta_+^2(k - a\theta_-)} > 1. \tag{4.2.30}$$

In Appendix 5, we show that $\frac{\theta_-^2(k-a\theta_+)}{\theta_+^2(k-a\theta_-)} > 1$ is equivalent to $k > \frac{a\lambda}{\mu}$.

From above, we have demonstrated that if $k > \frac{a\lambda}{\mu}$, then for x in (4.2.29), $\widetilde{H}'(x) < 0$. Combining with $\widetilde{H}(0) = k$ and $\widetilde{H}(x) \to \infty$ as $x \to \infty$, then there exists $x_1^{C_1} \in (0, \infty)$, such that $\widetilde{H}(x_1^{C_1}) = k$ and $\widetilde{H}(x) < k$ on $(0, x_1^{C_1})$. Therefore,

$$I_2(\frac{k - a\theta_-}{\theta_+ - \theta_-}) = \int_0^{x_1^{C_1}} (k - \widetilde{H}(x))dx > 0.$$

On the other hand, if $k \leq \frac{a\lambda}{\mu}$, then there is no $x \in (0, \infty)$, such that $\widetilde{H}'(x) < 0$. That means $\widetilde{H}'(x) > 0$ on $(0, \infty)$, then from $\widetilde{H}(0) = k$, we have that $\widetilde{H}(x) > k$ on $(0, \infty)$. That implies there is no $x_1^{C_1} \in (0, \infty)$, such that $\widetilde{H}(x_1^{C_1}) = k$. So, for such case, $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) = 0$.

But for any $C_1 \in (0, \frac{k-a\theta_-}{\theta_+-\theta_-})$, we can verify that $H^{C_1}(0) < k$ from (4.2.27). In addition, it is easy to see that $H^{C_1}(x) = C_1\theta_+e^{\theta_+ x} + (a - C_1)\theta_-e^{\theta_- x} \to \infty$ as $x \to \infty$. So, there exists $x_1^{C_1} \in (0, \infty)$ such that $H^{C_1}(x_1^{C_1}) = k$ and $H^{C_1}(x) < k$ on $(0, x_1^{C_1})$. Hence, $I_2(C_1) > 0$ for any $C_1 \in (0, \frac{k-a\theta_-}{\theta_+-\theta_-})$. □

Based on Proposition 4.2.4, the following result can be given.

Remark 4.2.5. *If $k \leq \frac{a\lambda}{\mu}$, then the domain of $I_2(C_1)$ can be given by $(0, \frac{k-a\theta_-}{\theta_+-\theta_-}]$. Moreover, the range of $I_2(C_1)$ is $[0, \infty)$. So, for any $K \in [0, \infty)$, there exists unique $C_1 \in (0, \frac{k-a\theta_-}{\theta_+-\theta_-}]$, such that $I_2(C_1) = K$.*

From Appendix 2, it is known that if $aX_0^{-\gamma} \geq a^{1-\gamma}(k/\gamma)^\gamma$, that is $k \leq \frac{2a\lambda}{\mu}$, then we have $aX_0^{-\gamma}a_1 \geq \frac{k-a\theta_-}{\theta_+-\theta_-}$.

If $k \leq aM^*$, from Proposition 4.1.4, it follows that $I_2(aX_0^{-\gamma}a_1) = 0$ or $I_2(aX_0^{-\gamma}a_1)$ is undefined. In addition, if $k > \frac{a\lambda}{\mu}$, from Proposition 4.2.4, we have $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) > 0$. Therefore, if $\frac{a\lambda}{\mu} < k \leq aM^*$, from the continuity and the monotonicity of $I_2(C_1)$, there exists $C^* \in (\frac{k-a\theta_-}{a_1(\theta_+-\theta_-)}, aX_0^{-\gamma}]$, such that $I_2(C^*a_1) = 0$, and $I_2(C_1) > 0$ for any $C_1 \in (\frac{k-a\theta_-}{\theta_+-\theta_-}, C^*a_1)$.

Remark 4.2.6. *If $\frac{a\lambda}{\mu} < k \leq aM^*$, then the domain of $I_2(Ca_1)$ can be given by $(0, C^*)$. Moreover, the range of $I_2(Ca_1)$ is $(0,\infty)$. So, for any $K \in (0,\infty)$, there exists unique $C \in (0, C^*)$, such that $I_2(Ca_1) = K$.*

If $aM^* < k \leq \frac{2a\lambda}{\mu}$, from (4.2.26) and Proposition 4.1.4, we have that $I_2(aX_0^{-\gamma}a_1) > 0$. Then, $I_1(aX_0^{-\gamma}) = I_2(aX_0^{-\gamma}a_1) > 0$. Further, for $I_1(C)$, as in section 5.2.1, there exists $C^* \in (aX_0^{-\gamma}, \infty)$, such that $I_1(C^*) = 0$ and $I_1(C) > 0$ for any $C \in (aX_0^{-\gamma}, C^*)$. Therefore,

Remark 4.2.7. *Under the assumption $aM^* < k \leq \frac{2a\lambda}{\mu}$, the domains of $I_2(Ca_1)$ and $I_1(C)$ can be given by $(0, aX_0^{-\gamma}]$ and $(aX_0^{-\gamma}, C^*)$, respectively.*

Based on Remark 4.2.5, Remark 4.2.6 and Remark 4.2.7, the following results can be summarized.

Remark 4.2.8. *If $aM^* < k \leq \frac{2a\lambda}{\mu}$, then*

(i) $I_1(C) \in (0, I_2(aX_0^{-\gamma}a_1))$, for $C \in (aX_0^{-\gamma}, C^*)$;

(ii) $I_2(Ca_1) \in [I_2(aX_0^{-\gamma}a_1), +\infty)$, for $C \in (0, aX_0^{-\gamma}]$.

where $C^* \in (aX_0^{-\gamma}, \infty)$. Moreover, $I_1(C)$ and $I_2(a_1C)$ are all strictly decreasing functions on their corresponding intervals.

Remark 4.2.9. *If $\frac{a\lambda}{\mu} < k \leq aM^*$, then we have*

(i) $I_2(Ca_1) \in (0, +\infty)$, for $C \in (0, C^*)$;

where $C^* \in (\frac{k-a\theta_-}{a_1(\theta_+-\theta_-)}, aX_0^{-\gamma}]$.

Remark 4.2.10. *If $k \le \frac{a\lambda}{\mu}$, let $C^* = \frac{k-a\theta_-}{\theta_+-\theta_-}$, then from Remark 4.2.5, we have*

(i) $I_2(Ca_1) \in (0,+\infty)$, *for* $C \in (0, C^*)$.

From above, we see that for different intervals of k, there always exists C^*, such that $I_1(C^*) = 0$ or $I_2(C^*a_1) = 0$. In the following definitions, according to the same condition of k, C^* is the same as that in Remark 4.2.8-4.2.10.

If $aM^* < k \le \frac{2a\lambda}{\mu}$ (see Fig 2), we define

$$I(C) := \begin{cases} I_2(Ca_1), & C \in (0, aX_0^{-\gamma}], \\ I_1(C), & C \in (aX_0^{-\gamma}, C^*). \end{cases} \tag{4.2.31}$$

If $\frac{a\lambda}{\mu} < k \le aM^*$ (see Fig 3), we define

$$I(C) := I_2(Ca_1), \qquad C \in (0, C^*). \tag{4.2.32}$$

If $k \le \frac{a\lambda}{\mu}$ (see Fig 4), we define

$$I(C) := I_2(Ca_1), \qquad C \in (0, C^*). \tag{4.2.33}$$

Then, we have the following proposition.

Proposition 4.2.5. *From (4.2.31)-(4.2.33), it shows that $I_1(C)$ and $I_2(Ca_1)$ can be used to construct $I(C)$. Moreover, all $I(C)$ are strictly decreasing and continuous functions on their corresponding intervals $(0, C^*)$ with ranges $(0, +\infty)$. Therefore, for any $K \in (0, +\infty)$, there exists unique C, such that $I(C) = K$.*

4.3 Compute Parameter C and Numerical Examples

In this subsection, firstly several steps are summarized to get the parameter, C. Secondly, under four different conditions, the graphes of $I(C)$ are plotted to support our theoretical results that all $I(C)$ are decreasing functions. Thirdly, by using the given steps to compute C, the results of several numerical examples, to get C, x_1, etc, are presented in a table.

4.3.1 Steps to Compute C

Notice that $I(C)$ in (4.2.25) and (4.2.31) have the same form. Then, in the process of calculating C, we can follow the same steps for these two cases. Now, let's list out these steps.

Step 1. Compare aM^* with k.

Step 2. If $k > aM^*$,

(i) Compute $I_2(aX_0^{-\gamma}a_1)$, and then compare $I_2(aX_0^{-\gamma}a_1)$ with K.

(ii) If $I_2(aX_0^{-\gamma}a_1) > K$, get C on $(aX_0^{-\gamma}, \infty)$, such that $I_1(C) = K$.

(iii) If $I_2(aX_0^{-\gamma}a_1) \leq K$, get C on $(0, aX_0^{-\gamma})$, such that $I_2(Ca_1) = K$.

Step 3. If $k \leq aM^*$, then compare k with $\frac{a\lambda}{\mu}$.

(i) If $k \leq \frac{a\lambda}{\mu}$, then solve $I(C) = K$ from (4.2.33).

(ii) If $\frac{a\lambda}{\mu} < k \leq aM^*$, then solve $I(C) = K$ from (4.2.32).

By these steps, the uncertain parameter, C, can be obtained. Further, we can get some other parameters, such as x_1^C, $x_1^{C_1}$, $\widetilde{x}^C$ and $\widetilde{x}^{C_1}$.

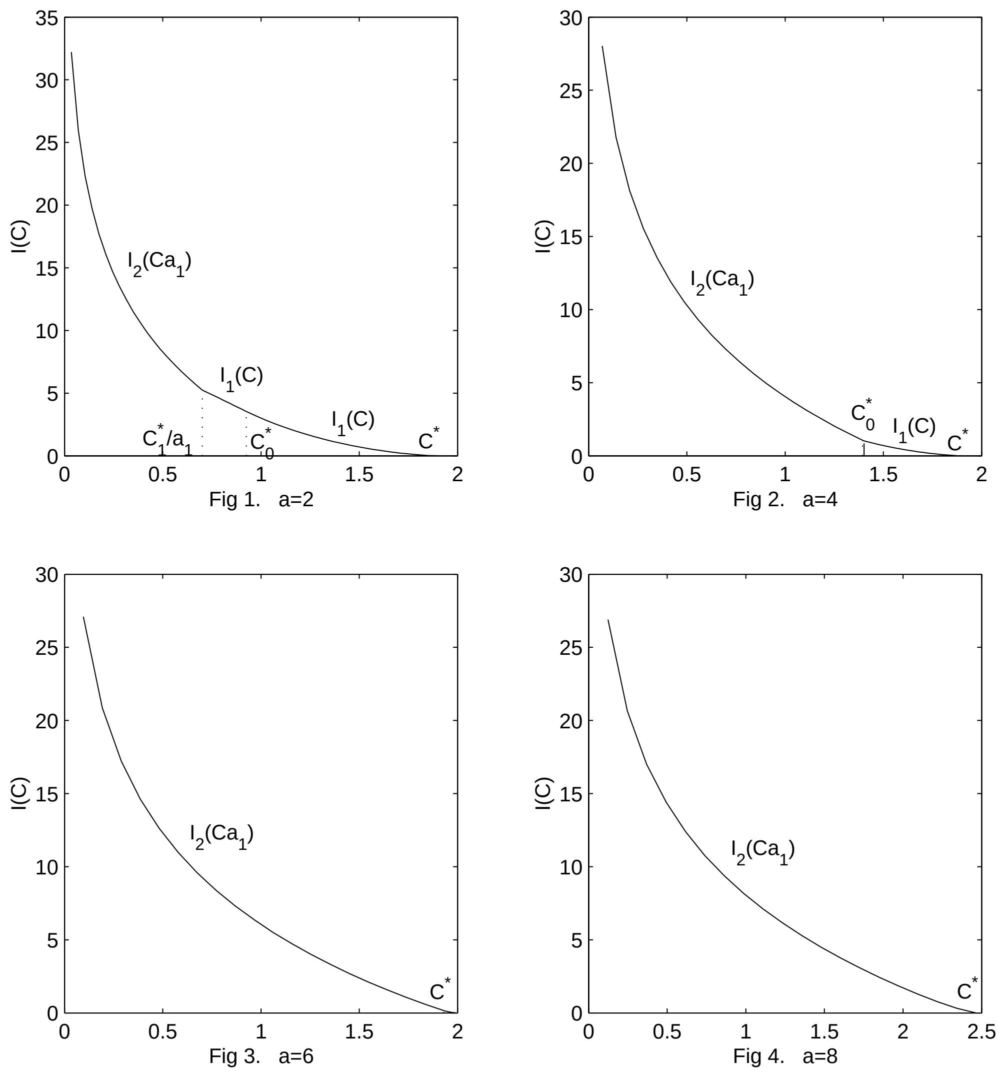

Figure 4.1: The graph of $I(C)$

4.3.2 Numerical Examples

In practical problems, $k \in (0, 1)$, but the restriction for a is just $a > 0$. So, in order to satisfy $0 < k < 1$, in convenience, we just choose different values for a, and never change the values of k, μ, λ, σ in the following numerical examples. The values of these parameters are selected as $\mu = 0.12$, $\lambda = 0.002$, $\sigma = 1.6$, $k = 0.13$ in our examples. For the value of a, we choose $a = 2$, $a = 4$, $a = 6$ and $a = 8$ to satisfy four different conditions, respectively. In Fig 1-4, the graphes of $I(C)$ are shown, and both forms of $I(C)$, $I_1(C)$ and $I_2(Ca_1)$, are listed out in their corresponding domains.

Moreover, the right domain endpoint C^* of $I(C)$ is also presented.

From Fig 1-4, it shows that the function $I(C)$ is always a decreasing function. In Fig 1, $a = 2$, which satisfies $k > \frac{2a\lambda}{\mu}$, C_1^*/a_1 denotes $aX_0^{-\gamma}$ and $C_0^* = a^{1-\gamma}(k/\gamma)^\gamma$. From Fig 1, we can see that $I(C)$ is composed of $I_2(Ca_1)$ for $C \in (0, C_1^*/a_1]$ and $I_1(C)$ for $C \in (C_1^*/a_1, C^*)$ with $I_2(C_1^*) = 5.2494 > 0$ and $I_1(C_0^*) = 3.5408 > 0$. This phenomena is consistent with Proposition 4.2.1-4.2.3.

In Fig 2, $a = 4$, which satisfies $aM^* < k \leq \frac{2a\lambda}{\mu}$, and C_0^* is given by $C_0^* = aX_0^{-\gamma}$. In this case, since $\frac{k-a\theta_-}{a_1(\theta_+-\theta_-)} = 1.3935$ is very closed to $C_0^* = 1.4018$, then we just draw the position of C_0^* in the graph. In fact, under the condition $aM^* < k \leq \frac{2a\lambda}{\mu}$, this phenomena happens almost on any other trials, for which we select different k, μ, λ, σ from those of this example. Anyway, from Fig 2, it is seen that $I_2(C_0^* a_1) > 0$, which supports Proposition 4.1.4 correctly.

For Fig 3, we choose $a = 6$, which satisfies $\frac{a\lambda}{\mu} < k \leq aM^*$. The function $I(C)$ in this graph is just composed of $I_2(Ca_1)$ with $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) = 0.1631 > 0$, which matches Remark 4.2.6 very well. Since the value 0.1631, compared with the numbers of y-axis, is so small, then it is not obvious in the graph. In addition, in Fig 4, $a = 8$, and then $k \leq \frac{a\lambda}{\mu}$ is verified. Moreover, for this case, by calculation we can obtain that $\frac{k-a\theta_-}{a_1(\theta_+-\theta_-)} = 2.4651$. On the other hand, from numerical simulation, we get that $C^* = 2.4651$. So, $C^* = \frac{k-a\theta_-}{a_1(\theta_+-\theta_-)}$ is satisfied. Therefore, the conclusion in Remark 4.2.5 is shown.

In the Table 1, for each case in Fig 1-4, we choose three different K to calculate some parameters, such as C, $x_0 \vee 0$, $\widetilde{x} \vee 0$ and x_1, from $I(C) = K$. For Fig 1, from $I_2(C_1^*) = 5.2494$ and $I_1(C_0^*) = 3.5408$, then three potential K are selected as $K = 0.15$, which is less than $I_1(C_0^*)$; $K = 4$, which is between $I_1(C_0^*)$ and $I_2(C_1^*)$; and $K = 8$, which is greater than $I_2(C_1^*)$. In addition, by the same selection method, for Fig 2, from $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) = 1.0669$ and $I_2(C_0^* a_1) = 1.0150$, then three trials for K are $K = 0.15$, $K = 1.05$ and $K = 4$. For Fig 3, $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) = 0.1631$, then three

Estimated parameters under $\mu = 0.12$, $\lambda = 0.002$, $k = 0.13$ and $\sigma^2 = 1.6$

Conditions	a	K	C	$\widetilde{x} \vee 0$	$x_0 \vee 0$	x_1	$I(C)$
$k > \frac{2a\lambda}{\mu}$	2	0.15	1.7489	18.4531	11.0865	40.0506	$I_1(C)$
$k > \frac{2a\lambda}{\mu}$	2	4	0.8648	0	4.9491	84.6289	$I_1(C)$
$k > \frac{2a\lambda}{\mu}$	2	8	0.5163	0	0	115.3948	$I_2(Ca_1)$
$aM^* < k \leq \frac{2a\lambda}{\mu}$	4	0.15	1.7490	12.5186	5.1482	34.1052	$I_1(C)$
$aM^* < k \leq \frac{2a\lambda}{\mu}$	4	1.05	1.3962	0.1815	0	46.2875	$I_2(Ca_1)$
$aM^* < k \leq \frac{2a\lambda}{\mu}$	4	4	1.0055	0	0	69.2676	$I_2(Ca_1)$
$\frac{a\lambda}{\mu} < k \leq aM^*$	6	0.15	1.9325	0.4645	0	22.0481	$I_2(Ca_1)$
$\frac{a\lambda}{\mu} < k \leq aM^*$	6	1.05	1.7409	0	0	31.7341	$I_2(Ca_1)$
$\frac{a\lambda}{\mu} < k \leq aM^*$	6	4	1.2550	0	0	54.2053	$I_2(Ca_1)$
$k \leq \frac{a\lambda}{\mu}$	8	0.15	2.3953	0	0	11.0653	$I_2(Ca_1)$
$k \leq \frac{a\lambda}{\mu}$	8	1.05	2.1493	0	0	21.6008	$I_2(Ca_1)$
$k \leq \frac{a\lambda}{\mu}$	8	4	1.5614	0	0	40.7502	$I_2(Ca_1)$

Table 4.1: Estimated parameters, C, $\widetilde{x} \vee 0$, $x_0 \vee 0$ and x_1, under different a and K.

comparable K are given by $K = 0.15$, $K = 1.05$, and $K = 4$. So does it for Fig 4.

From this table, it is seen that the result, about whether the values of $\widetilde{x} \vee 0$ and $x_0 \vee 0$ are zeroes or not, is totally consistent with the theoretical results in section 4.1 and 4.2. In addition, we notice that C, $\widetilde{x} \vee 0$ and $x_0 \vee 0$ are all decreasing when K is increasing under the condition of the same given a. But the phenomena for x_1 is different. It is increasing as K is increasing. On the other hand, under the same K, C is increasing and x_1 is decreasing as a is increasing.

Chapter 5

Solution of (QVI) and The Optimal Policy

Notice that all possible structures to calculate C are $I_1(C)$ and $I_2(Ca_1)$. Moreover, all integrands of these structures can be used to construct the candidate solutions of (QVI). In the following, firstly, I demonstrate all constructed candidate solutions are solutions of (QVI) under different conditions, and give several corresponding numerical examples by using the results of section 4.3.2. Secondly, I discuss the relationship between the solution of (QVI) and that of (QVI0). Thirdly, based on the solution of (QVI), I provide and verify the optimal policy.

5.1 Solution of (QVI) and Numerical Examples

5.1.1 Constructed Solution of (QVI)

Assume that we have gotten C, such that $I(C) = K$, by those steps provided in section 4.2.3. Then, other parameters, such as x_1^C, $x_1^{C_1}$, $\widetilde{x}^C \vee 0$ and $\widetilde{x}^{C_1} \vee 0$, can be also obtained. In the following of this paper, in simplicity, as before we still use the same denotations, such as C, C_1, x_1^C, $x_1^{C_1}$, $\widetilde{x}^C \vee 0$ and $\widetilde{x}^{C_1} \vee 0$, to represent all calculated parameters.

For C, such that $I_1(C) = K$, we define a function $T_1(x)$ by

$$T_1(x) = \begin{cases} CH^C(x), & x \in [0, x_1^C), \\ k & x \in [x_1^C, \infty), \end{cases} \tag{5.1.1}$$

then $C > aX_0^{-\gamma}$ and $T_1(x)$ satisfies $I_1(C) = K$.

Define

$$v_1(x) = a + \int_0^x T_1(y)dy, \tag{5.1.2}$$

then, we can obtain that

$$v_1(x) := \begin{cases} C[x + (\frac{a}{C})^{\frac{1}{\gamma}}]^\gamma, & x \in [0, x_0), \\ Ca_1 e^{\theta_+(x-x_0)} + Ca_2 e^{\theta_-(x-x_0)}, & x \in [x_0, x_1^C), \\ v(x_1^C) + k(x - x_1^C), & x \in [x_1^C, \infty), \end{cases} \tag{5.1.3}$$

where $x_0 = X_0 - (\frac{a}{C})^{\frac{1}{\gamma}}$.

In addition, for C, such that $I_2(Ca_1) = K$, define

$$T_2(x) = \begin{cases} H^{C_1}(x), & x \in [0, x_1^{C_1}), \\ k & x \in [x_1^{C_1}, \infty), \end{cases} \tag{5.1.4}$$

where $C_1 = Ca_1$. Then we have $C \leq aX_0^{-\gamma}$ and $T_2(x)$ satisfies $I_2(C_1) = K$.

Define

$$v_2(x) = a + \int_0^x T_2(y)dy, \tag{5.1.5}$$

then, it follows that

$$v_2(x) := \begin{cases} C_1 e^{\theta_+ x} + (a - C_1)e^{\theta_- x}, & x \in [0, x_1^{C_1}), \\ v(x_1^{C_1}) + k(x - x_1^{C_1}), & x \in [x_1^{C_1}, \infty). \end{cases} \tag{5.1.6}$$

In the following of this dissertation, for $v_1(x)$ and $v_2(x)$, in simplicity, we use the same notations, x_1, to denote both x_1^C and $x_1^{C_1}$, and $\widetilde{x}$, to denote both $\widetilde{x}^C$ and $\widetilde{x}^{C_1}$.

From the construction of $v_1(x)$ and $x_2(x)$, then we have

Theorem 5.1.1. *Both functions $v_1(x)$ and $v_2(x)$ given by (5.1.3) and (5.1.6) are continuously differentiable on $[0, \infty)$, and for each of v_1 and v_2, there exists $U > 0$ such that this function is twice continuously differentiable on $[0, U)$. Moreover, $v_1(x)$ and $v_2(x)$ provide a solution to (QVI).*

Proof. For $v_2(x)$, there is no x_0, then we define $x_0 \vee 0 = 0$. On intervals $[0, x_0 \vee 0]$, $[x_0 \vee 0, x_1)$ and $[x_1, \infty)$, we verify that $v_1(x)$ and $v_2(x)$ satisfy (QVI) as follows.

Step 1: on $[0, x_0 \vee 0]$, $v_1(x)$ satisfies $\max_{u\in[0,1]} \mathcal{L}^u v(x) = 0$.

For the case $x_0 > 0$, by construction, it is known that the function $v(x) = C[(\frac{a}{C})^{\frac{1}{\gamma}} + x]^\gamma$ satisfies (3.2.16) and this is a solution to

$$\max_{u\in[0,1]} \mathcal{L}^u v(x) = 0. \tag{5.1.7}$$

Step 2: on $[x_0 \vee 0, x_1)$, $v_1(x)$ and $v_2(x)$ satisfy $\max_{u\in[0,1]} \mathcal{L}^u v(x) = 0$.

Suppose that C is gotten from $I_2(Ca_1) = K$, and in (5.1.6), $C_1 > 0$ and $a - C_1 \geq 0$. Then it is easy to verify that $v_2''(x) > 0$. If $v_2'(x) < 0$, then we have $-\frac{\mu v_2'(x)}{\sigma^2 v_2''(x)} > 0$, which implies that the symmetric axis, with respect to u, of the quadratic function $\mathcal{L}^u v_2$ in (2.2.3) is positive. Moreover, from $v_2''(x) > 0$, it follows that $\mathcal{L}^u v_2$ has positive second order coefficient, $\frac{1}{2}\sigma^2 v_2''(x)$. So, for the case $v_2'(x) < 0$, we can conclude that $\mathcal{L}^u v_2$ reaches its maximum at $u = 0$, or $u = 1$. If $u = 0$, then $\mathcal{L}^u v_2 = -\lambda v_2 = 0$, which leads to $v_2 = 0$ on $[0, x_1)$. This is a contradiction with $v_2(0) = a$. So, we must have $u = 1$ is the maximizer of $\mathcal{L}^u v_2$. On the other hand, if $v_2'(x) > 0$, then $-\frac{\mu v_2'(x)}{\sigma^2 v_2''(x)} < 0$, which implies that the symmetric axis of the quadratic function $\mathcal{L}^u v_2$ is negative. Then, from $v_2''(x) > 0$, we also have that $\mathcal{L}^u v_2$ reaches its maximum at $u = 1$. Therefore, for this case, $v_2(x)$ satisfies (5.1.7) on $[0, x_1)$.

Assume that C is obtained from $I_2(Ca_1) = K$ with $C_1 > 0$ and $a - C_1 < 0$. Then $v_2'(x) > 0$, and $v_2'(x)$ has convexity on $[0, x_1)$ due to $v_2'''(x) > 0$. If there exists $\overline{x} \in [0, x_1)$, such that $v_2''(\overline{x}) = 0$, then define $x^* = \overline{x}$. Otherwise, we define $x^* = 0$. From the convexity of $v_2'(x)$, then $v_2'(x)$ is strictly increasing for $x \in (x^*, x_1)$. So, we have $v_2''(x) > 0$ on (x^*, x_1). Then, $-\frac{\mu v_2'(x)}{\sigma^2 v_2''(x)} < 0$, which means that the symmetric axis of the quadratic function $\mathcal{L}^u v_2$ with positive second order coefficient is negative. Thereby, from the property of quadratic function, we can see that $u = 1$ is the maximizer of $\mathcal{L}^u v_2$. On the other hand, on $[0, x^*)$, the function $v_2'(x)$ is strictly decreasing and therefore $v_2''(x) < 0$. Then, $-\frac{\mu v_2'(x)}{\sigma^2 v_2''(x)} > 0$. Let $U(x) = -\frac{\mu v_2'(x)}{\sigma^2 v_2''(x)}$. A direct inspection shows that $U'(x) > 0$, then $U(x)$ is an increasing function on $[0, x^*)$. So, $u = 1$ is the maximizer of $\mathcal{L}^u v_2$ since this property is also held at $x = 0$ from Appendix 10. Hence, $v_2(x)$ satisfies (5.1.7) on $[0, x^*)$. At $x = x^*$, it follows that $v_2''(x^*) = 0$. Then, $\mathcal{L}^u v_2$ becomes a linear function with first order coefficient $\mu v_2' > 0$. So, the maximizer of $\mathcal{L}^u v_2$ is also at $u = 1$. Hence, $v_2(x)$ is a solution of (5.1.7) on $[0, x_1)$.

If C is obtained from $I_1(C) = K$, then we have that $Ca_1 > 0$ and $Ca_2 < 0$. By the same method in the proof of $v_2(x)$ under $C_1 > 0$ and $a - C_1 < 0$, it follows that $v_1(x)$ defined by (5.1.2) is also the solution of (5.1.7).

Step 3: on $[0, x_1)$, $v_1(x)$ and $v_2(x)$ satisfy $v(x) > Mv(x)$.

From Appendix 11, it shows that for $\frac{a\lambda}{\mu} < k < \frac{2a\lambda}{\mu}$, $v_2'(x)$ has convexity for $C_1 \in (\frac{k - a\theta_-}{\theta_+ - \theta_-}, aX_0^{-\gamma} a_1)$. By Appendix 2 and Appendix 3, it is verified that if $k \geq \frac{2a\lambda}{\mu}$, then it follows that $aX_0^{-\gamma} a_1 \leq \frac{k - a\theta_-}{\theta_+ - \theta_-}$, which implies that for any $C_1 \in (0, aX_0^{-\gamma} a_1]$,

$v_2'(0) \leq k$ and $v_2'(x) < k$ on $(0, x_1)$ from (4.2.27). In addition, from Remark 4.2.5, if $k \leq \frac{a\lambda}{\mu}$, then the domain of $I_2(C_1)$ is $(0, \frac{k-a\theta_-}{\theta_+-\theta_-}]$, which also means $v_2'(0) \leq k$ and $v_2'(x) < k$ on $(0, x_1)$. From all these situations of $v_2(x)$, it is seen that we just need to discuss two cases of $v_2(x)$. The first one is $v_2'(x)$ have convexity on $[0, x_1)$ and the second one is that $v_2'(0) \leq k$ and $v_2'(x) < k$ on $(0, x_1)$.

For $v_2'(x)$, if it has convexity on $[0, x_1)$, then $v_2'(x) > k$ on $[0, \widetilde{x})$ for $\widetilde{x} > 0$. Therefore, $v_2(x-\eta)+k\eta-K$ is a decreasing function of η for $0 \leq x < \widetilde{x}$, which indicates that its supremum equals its limiting value at $0+$, that is $v_2(x) - K$. Obviously, we have $v_2(x) - K < v_2(x)$. For $x \in [\widetilde{x}, x_1)$, from $k - v_2'(x - \eta) = 0$, then $x - \eta = \widetilde{x}$ since the equation $v_2'(x) = k$ has only one root $\widetilde{x}$ smaller than x_1 by the convexity of $v_2(x)$. Therefore, $Mv_2(x) = v_2(\widetilde{x}) + k(x - \widetilde{x}) - K$. In addition, from the convexity of $v_2'(x)$, it follows that $v_2'(x) < k$ on $(\widetilde{x}, x_1)$. Consequently, the following inequality can be satisfied.

$$
\begin{aligned}
Mv_2(x) &= v_2(\widetilde{x}) + k(x - \widetilde{x}) - K \\
&= v_2(\widetilde{x}) + k(x_1 - \widetilde{x}) - K - k(x_1 - x) \\
&= v_2(x_1) - k(x_1 - x) < v_2(x).
\end{aligned} \tag{5.1.8}
$$

So, for $v_2'(x)$ with convexity, if $\widetilde{x} > 0$, then $v_2(x)$ satisfies $v(x) > Mv(x)$.

On the other hand, for $v_2(x)$, if $\widetilde{x} = 0$ or $\widetilde{x}$ doesn't exist, then $v_2'(x) < k$ on $(0, x_1)$. So, $v_2(x - \eta) + k\eta - K$ is an increasing function of η and then its supremum equals its limiting value at x, which implies $Mv_2(x) = v(0) + kx - K$. Combining with $v_2'(x) < k$ on $(0, x_1)$, we have that

$$
\begin{aligned}
Mv_2(x) &= v_2(0) + kx - K = v_2(0) + kx_1 - K - k(x_1 - x) \\
&= v_2(x_1) - k(x_1 - x) < v_2(x).
\end{aligned} \tag{5.1.9}
$$

From (5.1.8) and (5.1.9), it follows that $v_2(x)$ satisfies $v(x) > Mv(x)$ on $[0, x_1)$.

For $v_1(x)$, since $v_1'(x)$ always has convexity, then by repeating the same method of the proof of $v_2(x)$ with convexity, we can verify that $v_1(x)$ also satisfies $v(x) > Mv(x)$ on $[0, x_1)$.

Step 4: on $[x_1, \infty)$, $v_1(x)$ and $v_2(x)$ satisfy $v(x) = Mv(x)$.

For (5.1.6), taking the derivative of $v_2(x)$ on $[x_1, \infty)$, then

$$v_2'(x) = k, \quad x \geq x_1, \tag{5.1.10}$$

which shows that $v_2(x)$ is a straight line on $[x_1, \infty)$.

Then, if $\eta \leq x - x_1$, it follows that $v_2(x-\eta) + k\eta = v_2(x)$, thus $v_2(x-\eta) + k\eta - K < v_2(x)$. So, $Mv_2(x) < v_2(x)$ for $\eta \leq x - x_1$. On the other hand, if $x - x_1 < \eta \leq x$, then $v_2(x - \eta) = v_2(x_1 - (\eta - (x - x_1)))$ and

$$\begin{aligned} v_2(x-\eta) + k\eta - K &= v_2(x_1 - (\eta - (x - x_1))) + k(\eta - (x - x_1)) - K + k(x - x_1) \\ &\leq v_2(x_1) + k(x - x_1) = v_2(x), \end{aligned}$$

with equality achieved at $\eta = x - \widetilde{x}^{C_1} \vee 0$. Therefore, it is verified that $Mv_2(x) = v_2(x)$ for $x \in [x_1, +\infty)$.

Hence, $v_2(x)$ satisfies $v(x) = Mv(x)$ on $[x_1, \infty)$.

In addition, by the same method, we can show that $v_1(x)$ also satisfies $v(x) = Mv(x)$ on $[x_1, \infty)$.

Step 5: on $[x_1, \infty)$, $v_1(x)$ and $v_2(x)$ satisfy $\mathcal{L}^u v(x) \leq 0$.

From Step 2, we can see that $v_2''(x_1-) \geq 0$ for any $v_2(x)$ in (5.1.6) since $v_2'(x)$ is increasing in the small neighborhood of x_1. Therefore,

$$\mu uk - \lambda v_2(x_1) \leq \frac{1}{2}\sigma^2 u^2 v_2''(x_1-) + \mu uk - \lambda v_2(x_1) \leq 0. \tag{5.1.11}$$

Moreover, for $x > x_1$, since $v_2(x)$ is an increasing linear function, then $v_2''(x) = 0$ and

$$\frac{1}{2}\sigma^2 u^2 v_2''(x) + \mu uk - \lambda v_2(x) = \mu uk - \lambda v_2(x) < \mu uk - \lambda v_2(x_1). \tag{5.1.12}$$

So, combining (5.1.11) and (5.1.12), it shows that on $[x_1, \infty)$, the inequality $\mathcal{L}^u v_2(x) \leq 0$ holds.

By the same method, for $v_1(x)$, we also have that $\mathcal{L}^u v_1(x) \leq 0$.

So, on $[x_1, \infty)$, $v_1(x)$ and $v_2(x)$ satisfy $\mathcal{L}^u v(x) \leq 0$.

In summary, from Step 1-5 above, we conclude that both $v_1(x)$ in (5.1.3) and $v_2(x)$ in (5.1.6) satisfy (QVI). Then $v_1(x)$ and $v_2(x)$ provide a solution of (QVI). □

In addition, from the discussion of chapter 4, we have an important result as follows to make a decision with respect to whether there is a bankruptcy or not.

Remark 5.1.1. *From chapter 4, we can obtain the following results:*
(i) Under the assumption $k > \frac{2a\lambda}{\mu}$, if $I_1(a^{1-\gamma}(k/\gamma)^\gamma) \leq K$, then $\widetilde{x} \vee 0 = 0$ and there would be a bankruptcy. If $I_1(a^{1-\gamma}(k/\gamma)^\gamma) > K$, then $\widetilde{x} > 0$ and the optimal policy is continuation strategy.
(ii) Under the assumption $\frac{a\lambda}{\mu} < k \leq \frac{2a\lambda}{\mu}$, if $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) \leq K$, then $\widetilde{x} \vee 0 = 0$ and there would be a bankruptcy. If $I_2(\frac{k-a\theta_-}{\theta_+-\theta_-}) > K$, then $\widetilde{x} > 0$ and the optimal policy is continuation strategy.
(iii) If $k \leq \frac{a\lambda}{\mu}$, then $\widetilde{x} \vee 0 = 0$ and there would be a bankruptcy.

5.1.2 Numerical Examples for The Solutions of (QVI)

In this part, based on the calculated parameters, C, $x_0 \vee 0$ and x_1 of section 4.3.2, several numerical examples are provided to support our theoretical results.

In Fig 5-8, the graphes of $v(x)$ are shown out under $a = 2$, $a = 4$, $a = 6$ and $a = 8$, respectively. From Fig 5-8, it is seen that at the same value of a, the graphes

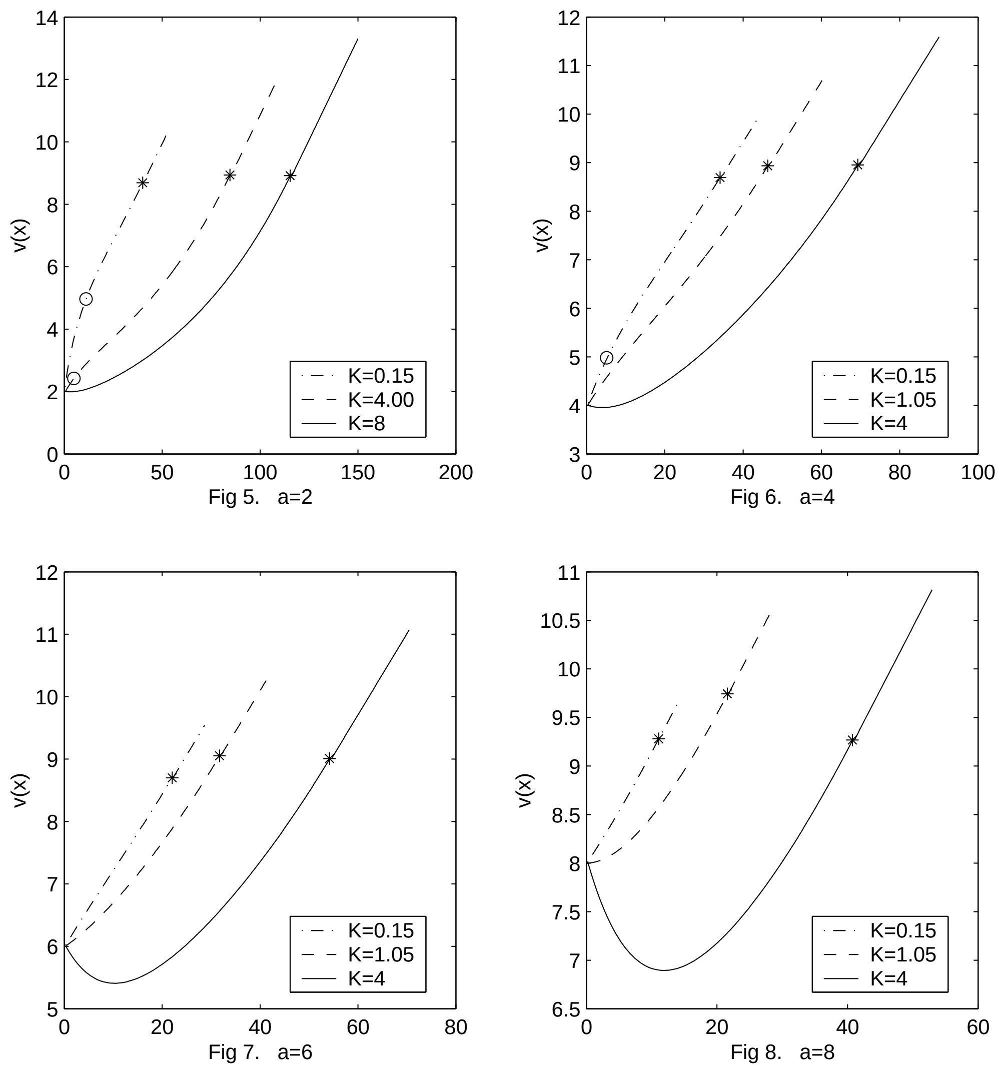

Figure 5.1: Note: In these graphes above, 'o' denotes the positions of x_0; '$*$' denotes the positions of x_1. For some curves, if there is no 'o' on them, then it means $x_0 \vee 0 = 0$.

of $v(x)$ with greater K are below those of $v(x)$ with smaller K. This phenomena means that the optimal dividends would be greater if the fixed costs become smaller, which accords with the actual situation in the real world.

In Fig 9-12, the corresponding derivatives of $v(x)$ in Fig 5-8 are plotted. For Fig 9 at $K = 0.15$, Fig 10 at $K = 0.15$ and $K = 1.05$, Fig 11 at $K = 0.15$, it shows that they have $v'(0) > k$ in corresponding graphes, which provides a strong support for our derivation, $\widetilde{x} > 0$, in Remark 5.1.1. In addition, for Fig 10 at $K = 1.05$, the graph of it has convexity, which is consistent with the conclusion of Appendix 11. Moreover,

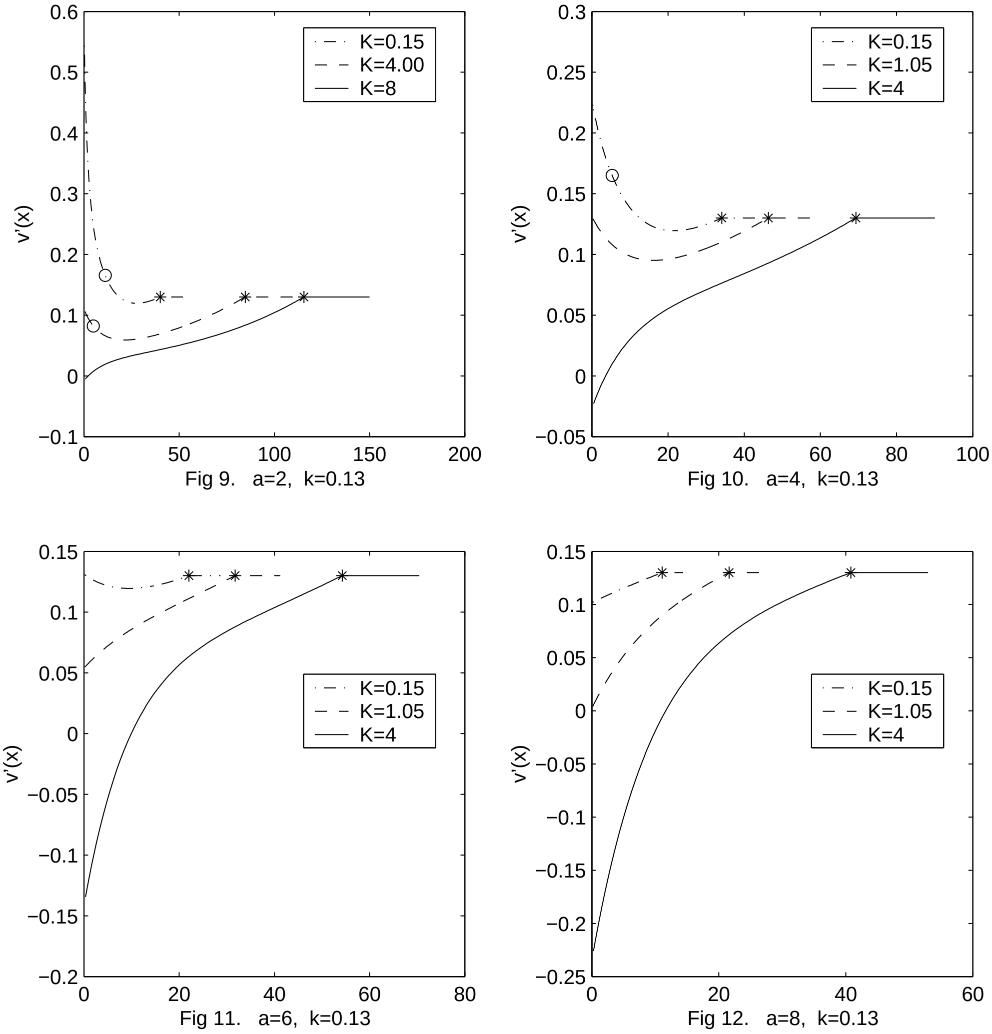

Figure 5.2: Note: In these graphes above, 'o' denotes the positions of x_0; '$*$' denotes the positions of x_1. For some curves, if there is no 'o' on them, then it means $x_0 \vee 0 = 0$.

for Fig 10 at $K = 4$, Fig 11 at $K = 1.05$ and $K = 4$, Fig 12 at $K = 0.15$, $K = 1.05$ and $K = 4$, all graphes of them are below the level line of k, which numerically verifies the summary results about $v_2'(x)$ in Step 3 of Theorem 5.1.1.

5.1.3 Relationship between Solutions of (QVI0) and (QVI)

From (5.1.3) and (3.1.5), we notice that if $C = C_0$ and $\widetilde{x} > 0$, $v_1(x)$ can be obtained by shifting $(\frac{a}{C})^{\frac{1}{\gamma}}$ units for $v_0(x)$. This phenomena enlightens us to see if there are some relationships between the solution of (QVI) and that of (QVI0).

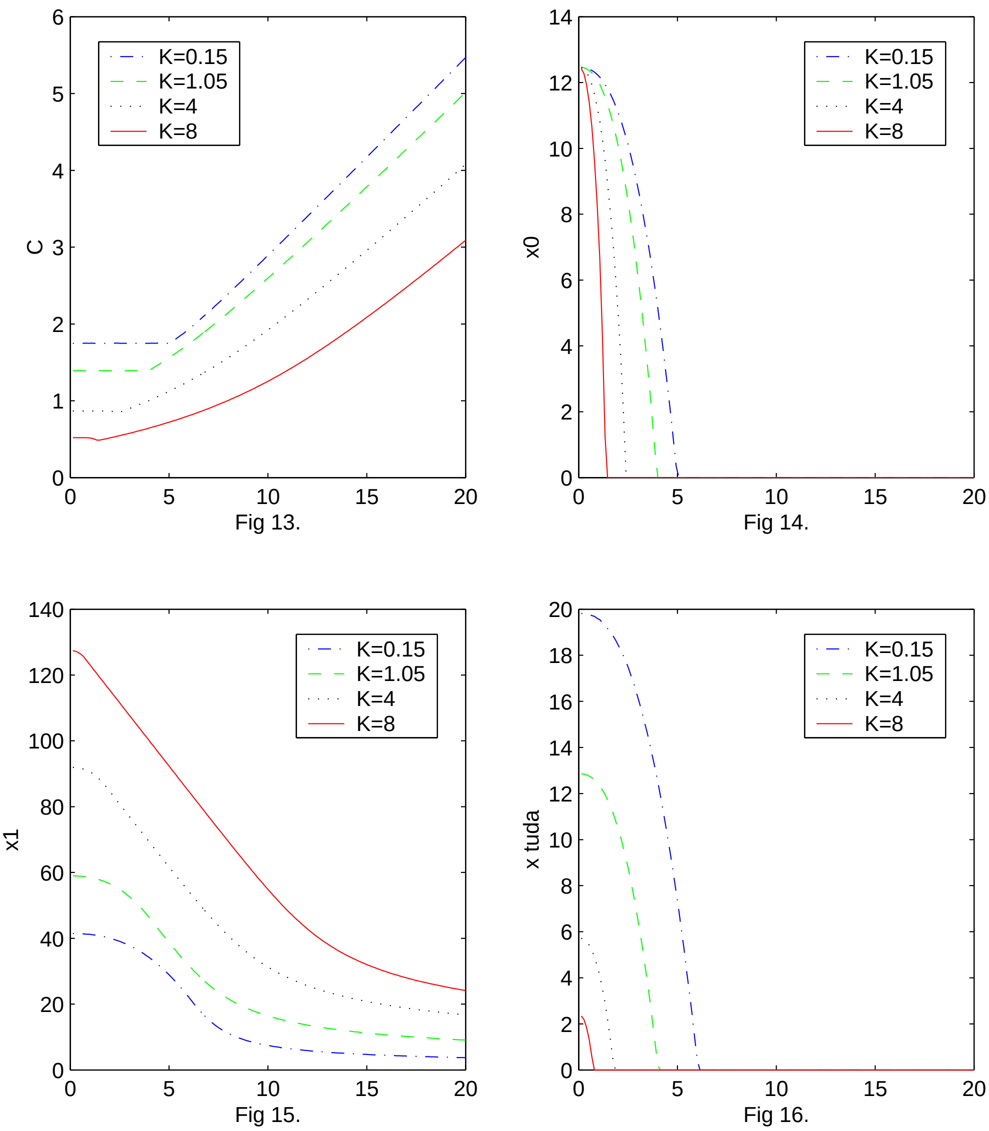

Figure 5.3: Note: In these graphes above, the x-axis is the variable $a \in [0, 20]$.

Let $v_0^{-1}(x)$ denote the inverse function of $v_0(x)$, and

$$\widehat{x}_1 = X_1 - v_0^{-1}(a), \;\; \widehat{\widetilde{x}} = \widetilde{X} - v_0^{-1}(a). \tag{5.1.13}$$

Then,

Proposition 5.1.1. *Suppose that $v_0(\widetilde{X}) \geq a$ and (QVI) has a unique solution. Let*

$$v(x) := v_0(x + v_0^{-1}(a)), \qquad x \in [0, \infty), \tag{5.1.14}$$

then, $v(x)$ is the solution of (QVI) on $[0,\infty)$.

Proof. The basic idea is to check that $v_0(x+v_0^{-1}(a))$ satisfies (QVI). From [3], it is known that $v_0(x)$ is a strictly increasing function. Then, from the assumption $v_0(\widetilde{X}) \geq a$ we can get that $\widetilde{X} \in [v_0^{-1}(a),\infty)$. Consequently, $\widehat{\widetilde{x}} = \widetilde{X} - v_0^{-1}(a) \geq 0$.

Put (5.1.13) into (5.1.14), we have that

$$v'(\widehat{\widetilde{x}}) = v_0'(\widehat{\widetilde{x}} + v_0^{-1}(a)) = v_0'(\widetilde{X}) = k,$$

and

$$v(\widehat{x}_1) = v_0(\widehat{x}_1 + v_0^{-1}(a)) = v_0(X_1) = Mv_0(X_1) = Mv_0(\widehat{x}_1 + v_0^{-1}(a)) = Mv(\widehat{x}_1).$$

So, $\widehat{\widetilde{x}}$ is the solution of $v'(x) = k$, and $\widehat{x}_1$ is the first point such that $v(x) = Mv(x)$. In addition, the function $\mathcal{L}^u v_0(x+v_0^{-1}(a))$ on $[0,\widehat{x}_1)$ and the function $\mathcal{L}^u v_0(x)$ on $[v_0^{-1}(a), X_1)$ are the same function. Then, $\mathcal{L}^u v_0(x+v_0^{-1}(a))$ satisfies the equation, $\lim\limits_{u\in[0,1]} \mathcal{L}^u v(x) = 0$, on $[0,\widehat{x}_1)$ as $\mathcal{L}^u v_0(x)$ satisfies on $[v_0^{-1}(a), X_1)$. Therefore, $v_0(x+v_0^{-1}(a))$ also satisfies the equation of (QVI) on $[0,\widehat{x}_1)$.

On $[\widehat{x}_1,\infty)$, from $\widehat{\widetilde{x}} \geq 0$, then we have

$$\begin{aligned} v(x) &= v_0(x+v_0^{-1}(a)) = v_0(\widetilde{X}) + k(x+v_0^{-1}(a) - \widetilde{X}) - K \\ &= v(\widehat{\widetilde{x}}) + k(x-\widehat{\widetilde{x}}) - K, \end{aligned} \tag{5.1.15}$$

which means $v(x)$ is also a linear function on $[\widehat{x}_1,\infty)$. In addition, from [3] it is known that $v_0(x+v_0^{-1}(a))$ satisfies $v_0(x+v_0^{-1}(a)) = Mv_0(x+v_0^{-1}(a))$ on $[\widehat{x}_1,\infty)$. So, $v(x) = v_0(x+v_0^{-1}(a))$ satisfies $v(x) = Mv(x)$ on $[\widehat{x}_1,\infty)$. Thereby, $v(x)$ is also the solution of (QVI) on $[\widehat{x}_1,\infty)$.

In addition, from $v(0) = v_0(0+v_0^{-1}(a)) = v_0(v_0^{-1}(a)) = a$, then $v(x) = v_0(x+v_0^{-1}(a))$ satisfies the boundary condition of (QVI).

From above, it shows that $v(x) = v_0(x + v_0^{-1}(a))$ is the solution of (QVI) on $[0, \infty)$. Moreover, x_1 and $\widetilde{x}$ can be given by $\widehat{x}_1$ and $\widehat{\widetilde{x}}$, respectively. □

Remark 5.1.2. *If $v_0(\widetilde{X}) < a$, then it follows that $\widehat{\widetilde{x}} = \widetilde{X} - v_0^{-1}(a) < 0$, which means that (5.1.15) can't be satisfied since there is no definition for $v(\widehat{\widetilde{x}})$ on $(-\infty, 0)$. So, for this case, we can't use shifting method to get the solution of (QVI).*

Remark 5.1.3. *Assume that $v_0(\widetilde{X}) \geq a$. If $v_0^{-1}(a) \leq X_0$, then it is not hard to get that $v_0^{-1}(a) = (\frac{a}{C_0})^{\frac{1}{\gamma}}$. So, $v(x) = v_0(x + (\frac{a}{C_0})^{\frac{1}{\gamma}})$, which is consistent with $v_1(x)$ given by (5.1.3). On the other hand, if $v_0^{-1}(a) > X_0$, then there exists $\widehat{X} \in (X_0, X_1)$, such that*

$$C_0 a_1 e^{\theta_+(\widehat{X} - X_0)} + C_0 a_2 e^{\theta_-(\widehat{X} - X_0)} = a.$$

Moreover, $v(x)$ on $[0, x_1)$ can be given by

$$v(x) = C_0 a_1 e^{\theta_+(x + \widehat{X} - X_0)} + C_0 a_2 e^{\theta_-(x + \widehat{X} - X_0)}, \quad x \in [0, x_1). \tag{5.1.16}$$

By comparing (5.1.16) with $v_2(x)$ in (5.1.6), we can obtain that

$$C_1 = C_0 a_1 e^{\theta_+(\widehat{X} - X_0)}, \qquad a - C_1 = C_0 a_2 e^{\theta_-(\widehat{X} - X_0)}.$$

5.2 Optimal Policy

In the following theorem, the optimal policy is presented and verified, and it is seen that the solution to (QVI) constructed above is the value function.

Theorem 5.2.1. *Let $\tau_0 = 0$, then the control $\pi^* = (u^*, \mathcal{T}^*, \xi^*, \tau^*)$ defined by*

$$u^*(t) := \begin{cases} \frac{\mu}{(1-\gamma)\sigma^2}[X^* + (\frac{a}{C})^{\frac{1}{\gamma}}], & if \;\; X^*(t) \in [0, x_0 \vee 0), \\ 1, & if \;\; X^*(t) \in [x_0 \vee 0, \infty); \end{cases} \tag{5.2.17}$$

$$\tau^* = \inf\{t \geq 0 : X^*(t) = 0\}, \tag{5.2.18}$$

$$\tau_1^* = \inf\{t \geq 0 : X^*(t) = x_1\}, \tag{5.2.19}$$

$$\xi_1^* = x_1 - \widetilde{x} \vee 0, \tag{5.2.20}$$

if $\widetilde{x} \vee 0 = 0$*, then let* $\tau_n^* = \infty$ *and* $\xi_n^* = 0$*; if* $\widetilde{x} \vee 0 > 0$*, then every* $n \geq 2$*,*

$$\tau_n^* = \inf\{t \geq \tau_{n-1} : X^*(t) = x_1\}, \tag{5.2.21}$$

$$\xi_n^* = x_1 - \widetilde{x}, \tag{5.2.22}$$

where X^* *is the solution to the stochastic differential equation:*

$$\begin{aligned} X^*(t) &= X^*(0) + \int_0^t \mu u^*(X^*(s))ds + \int_0^t \sigma u^*(X^*(s))dW_s \\ &\quad -(x_1 - \widetilde{x} \vee 0) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}}, \end{aligned} \tag{5.2.23}$$

is the (QVI) control associated with the functions v_1 *defined by (5.1.3) and* v_2 *defined by (5.1.6). This control is optimal and the functions,* $v_1(x)$ *and* $v_2(x)$*, coincides with the value function.*

Proof. In section 5.1, for any functions v_1 defined by (5.1.3) and v_2 defined by (5.1.6), we can see that $v_1(x)$ and $v_2(x)$ satisfies all conditions of Theorem 2.2.1. From Definition 2.2.1 and the discussion of section 3 and 4, we show that the control π^* defined in (5.2.17)-(5.2.22) is the control associated with v_1 and v_2. In addition, the control π^* is admissible from Definition 2.1.1. So, by Theorem 5.1.1, the claim that $v_1(x)$ and $v_2(x)$ are the value functions and π^* is the optimal policy is proved. $\square$

Chapter 6

Nonterminal Bankruptcy Model with Recovery Rate

In this section, based on the solution of (QVI), the model with respect to the bankruptcy with recovery will be considered, and the solution of it will be provided by solving (QVI) with a suitable boundary condition.

Assume that at time t the company is at the state of bankruptcy, and meanwhile the probability it receives an ϵ amount of wealth is ph, while with probability, $1-ph$, it remains bankruptcy. Thus, the reserve process $X(t+h)-X(t) = \triangle_t$ in distribution, where $\triangle_t$ $(t = 0, h, 2h, ...)$ are i.i.d. random variables with the distribution

$$\triangle_t = \begin{cases} 0, & \text{with probability } 1-ph, \\ \epsilon, & \text{with probability } ph. \end{cases} \tag{6.0.1}$$

So, we have that $\mathbb{E}\triangle_t = \eta h$, where $\eta = \epsilon p$. Again, we adopt the continuous analog,

$$dX(t) = \eta dt, \quad \text{at } X(t) = 0. \tag{6.0.2}$$

Combining with (2.1.1), then we can get

$$X(t) \;=\; x+\int_0^t \mu u(s)ds+\int_0^t \sigma u(s)dW_s-\sum_{n=1}^{\infty} I_{(\tau_n<t)}\xi_n+\zeta(t), \qquad (6.0.3)$$

where, as in [5], $\zeta(t)$ is a nondecreasing continuous process whose points of growth are the zeros of the process $X(t)$. Let Ω_0 be the set of zeros of $X(t)$, then the ordinary and stochastic integrals over Ω_0 will equal zero, and $dX(t) \geq 0$ for $t \in \Omega_0$ so that $d\zeta(t) \geq 0$ on Ω_0 due to $d\zeta(t) = dX(t) = \eta dt$ for all $t \in \Omega_0$.

From (6.0.2), it shows that the recovery rate η can be viewed as the rate at which the company can get wealth at the state of bankruptcy. As in [14], for the model considered in this part, we denote it by *nonterminal bankruptcy* (or bankruptcy with recovery). On the other hand, for the model, which is restricted to stay at the bankruptcy state, it is denoted by *terminal bankruptcy.* For the latter model, the boundary condition of the Hamilton-Jacobi-Bellmann (abbreviated by HJB) equation is $v(0) = a > 0$. For the former model, since the reserve process $X(t)$ will exit from the state of bankruptcy, then $a = 0$, which is the amount the shareholders get from the bankruptcy state. Consequently, we can see that the optimal value $V(x)$ becomes as follows

$$V(x) := \sup_{\pi\in A(x)} \mathbb{E}_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n} g(\xi_n) I_{(\tau_n<\tau)}\right]. \qquad (6.0.4)$$

Then, the HJB equation satisfied by (6.0.4) can be also given by $\max_{u\in[0,1]} \mathcal{L}^u V(x) = 0$, and furthermore the boundary condition of it should be related to recovery rate. Now, let's see the following results, which may give us some enlightenments about its boundary conditions.

Proposition 6.0.1. *Let $V(x)$ be given by (6.0.4). Suppose that $v_a(x)$ is the optimal value function for the problem with terminal bankruptcy. Let*

$$\eta = \frac{\lambda v_a(0)}{v_a'(0)}, \qquad (6.0.5)$$

then, $v_a(x)$ can present the optimal value function $V(x)$ for the nonterminal bankruptcy model with recovery rate η.

Proof. Repeating the argument used in the proof of Theorem 2.2.1, we can obtain that

$$v_a(x) - \mathbb{E}_x[e^{-\lambda\tau}v_a(X(\tau))] = \mathbb{E}_x\left[\sum_{i=1}^{N} I_{\{\tau_i\le\tau\}}e^{-\lambda\tau_i}g(\xi_i)\right], \tag{6.0.6}$$

where τ is the time of bankruptcy state, and N is an integer, such that $\tau_N \le \tau$ and $\tau_{N+1} > \tau$.

For the nonterminal model, $X(t)$ is a process with recovery rate, and the stopping time goes to infinite. That is, $\tau_n \to \infty$. So, the time, τ, for the bankruptcy state follows $\tau \to \infty$. Thereby,

$$v_a(x) = \mathbb{E}_x\left[\sum_{i=1}^{\infty} I_{\{\tau_i\le\tau\}}e^{-\lambda\tau_i}g(\xi_i)\right] = V(x). \tag{6.0.7}$$

Hence, $v_a(x)$ is also the optimal value function for the nonterminal bankruptcy model with recovery rate η. □

From chapter 5, it is known that $v_a(x)$ can be given by $v_1(x)$ in (5.1.3) or $v_2(x)$ in (5.1.6). Now, let's see the term, $\frac{v_a(0)}{v_a'(0)}$, from the point of view of these two solutions. From (5.1.3), it is easy to derive that

$$\frac{v_a(0)}{v_a'(0)} = \frac{1}{\gamma}(\frac{a}{C})^{\frac{1}{\gamma}}, \tag{6.0.8}$$

where $C > aX_0^{-\gamma}$, which implies that $\frac{a}{C} < X_0^\gamma$ and $\frac{v_a(0)}{v_a'(0)} < \frac{X_0}{\gamma}$. On the other hand, from (5.1.6), it follows that

$$\frac{v_a(0)}{v_a'(0)} = \frac{a}{Ca_1\theta_+ + (a - Ca_1)\theta_-} = \frac{1}{\theta_- + a_1(\theta_+ - \theta_-)/(\frac{a}{C})}, \tag{6.0.9}$$

where $C \le aX_0^{-\gamma}$, which implies that $\frac{a}{C} \ge X_0^\gamma$ and $\frac{v_a(0)}{v_a'(0)} \ge \frac{X_0}{\gamma}$.

For (6.0.9), notice that $\frac{v_a(0)}{v_a'(0)} \to \infty$ as $\frac{a}{C} \to (a_1(\theta_+ - \theta_-)/(-\theta_-))_-$. So, we obtain that $\frac{v_a(0)}{v_a'(0)} \in [\frac{X_0}{\gamma}, \infty)$ for $\frac{a}{C} \in [X_0^\gamma, a_1(\theta_+ - \theta_-)/(-\theta_-))$.

In addition, from the form of $v_1(x)$ and $v_2(x)$ in section 5, it is known that C continuously depends on a. Then, $\frac{a}{C}$ is a continuous function about a. Consequently, from (6.0.8) and (6.0.9), we can see that the ratio $\frac{a}{C}$ is an increasing function with respect to η. So, for any $\eta \in (0, \infty)$, there exists unique ratio $\frac{a}{C} \in (0, a_1(\theta_+ - \theta_-)/(-\theta_-))$, such that $\eta = \frac{\lambda v_a(0)}{v_a'(0)}$.

For the case $\eta \in (0, \frac{\lambda X_0}{\gamma})$, from (6.0.8) we have that $\eta = \frac{\lambda}{\gamma}(\frac{a}{C})^{\frac{1}{\gamma}}$. Then it follows that

$$\frac{a}{C} = (\frac{\eta\gamma}{\lambda})^\gamma, \tag{6.0.10}$$

which implies that $a = C(\frac{\eta\gamma}{\lambda})^\gamma$.

For $\eta \in [\frac{\lambda X_0}{\gamma}, \infty)$, from (6.0.9), then $\eta = \frac{\lambda}{\theta_- + a_1(\theta_+ - \theta_-)/(\frac{a}{C})}$. Consequently,

$$\frac{a}{C} = \frac{a_1(\theta_+ - \theta_-)}{\frac{\lambda}{\eta} - \theta_-}. \tag{6.0.11}$$

In simplicity, in the following, we denote $p_\eta = \frac{a_1(\theta_+ - \theta_-)}{\frac{\lambda}{\eta} - \theta_-}$.

Now, let's check the uniqueness of the pair (a, C) for any given $\eta \in (0, \infty)$. The basic ideal is that in the expressions of $I_1(C)$ and $I_2(Ca_1)$, we use $C(\frac{\eta\gamma}{\lambda})^\gamma$ and $C \cdot \frac{a_1(\theta_+ - \theta_-)}{\frac{\lambda}{\eta} - \theta_-}$ to replace the number a, respectively. Then the terms, $\frac{a}{C}$, in $I_1(C)$ and $I_2(Ca_1)$ will be constants. Consequently, $I_1(C)$ and $I_2(Ca_1)$ are only functions with respect to C.

Remark 6.0.1. *For any $K \in (0, \infty)$, there exists unique $C > 0$, such that $I_1(C) = K$, in which $a = C(\frac{\eta\gamma}{\lambda})^\gamma$.*

Proof. From $a = C(\frac{\eta\gamma}{\lambda})^\gamma$, then $C \to 0$ as $a \to 0$. In addition, since x_1^C and $\widetilde{x}^C$ are two roots of $k - CH(x + \eta\gamma/\lambda) = 0$ and $H(x + \eta\gamma/\lambda)$ has convexity, then it follows that $\lim_{C\to 0} x_1^C = \infty$ and $\lim_{C\to 0} \widetilde{x}^C = 0$. Consequently, we have that

$$\lim_{C\to 0} I_1(C) = \lim_{C\to 0} \int_{\widetilde{x}^C \vee 0}^{x_1^C} (k - CH(x+\eta\gamma/\lambda))dx = \int_0^\infty kdx = \infty.$$

On the other hand, from the convexity of $H(x+\eta\gamma/\lambda)$, it shows that there exists $C^* > 0$, such that $C^* \cdot \min_{x\in[0,\infty)} H(x+\eta\gamma/\lambda) = k$, which implies that $I_1(C^*) = 0$. Then, from the continuity of $I_1(C)$, it follows that $I_1(C) \in (0,\infty)$ on $(0, C^*)$. In addition, $I_1(C)$ is a strictly decreasing function from $I_1'(C) = -\int_{\widetilde{x}^C\vee 0}^{x_1^C} H(x+\eta\gamma/\lambda)dx < 0$. Thereby, for any $K > 0$, there exists unique $C \in (0, C^*)$, such that $I_1(C) = K$, in which $a = C(\frac{\eta\gamma}{\lambda})^\gamma$. □

By the similar method as in Remark 6.0.1, we can also get that

Remark 6.0.2. *For any $K \in (0,\infty)$, there exists unique $C > 0$, such that $I_2(Ca_1) = K$, in which $a = Cp_\eta$.*

Proof. For $a = Cp_\eta$, denote $H^{C_1}(x)$ in (4.1.5) by $CH_p(x)$, where $H_p(x) = a_1\theta_+ e^{\theta_+ x} + (p_\eta - a_1)\theta_- e^{\theta_- x}$. Then,

$$\lim_{C\to 0} I_2(Ca_1) = \lim_{C\to 0} \int_{\widetilde{x}^C \vee 0}^{x_1^C} (k - CH_p(x))dx = \int_0^\infty kdx = \infty.$$

On the other hand, from

$$p_\eta = \frac{a_1(\theta_+ - \theta_-)}{\frac{\lambda}{\eta} - \theta_-} < \frac{a_1(\theta_+ - \theta_-)}{-\theta_-},$$

then, we can get that $a_1(\theta_+ - \theta_-) + p_\eta\theta_- > 0$. That is, $H_p(0) > 0$. Therefore, if $p_\eta - a_1 \geq 0$, from $(H_p)'(x) > 0$ we have that $H_p(x) > H_p(0) > 0$ on $(0,\infty)$. In addition, if $p_\eta - a_1 < 0$, then $(p_\eta - a_1)\theta_- > 0$, which implies that $H_p(x) > 0$ on $(0,\infty)$. So, for any case of $p_\eta - a_1$ and any $x \in (0,\infty)$, $H_p(x) > 0$ is always satisfied. Consequently, there exists $C^* > 0$, such that $\min_{x\in[0,\infty)} C^* H_p(x) = k$, which means that

$I_2(C^*a_1) = 0$.

In addition, from $H_p(x) > 0$, we can derive that $I_2(Ca_1) = \int_{\bar{x}^C \vee 0}^{x_1^C} (k - CH_p(x))dx$ is a continuous and strictly decreasing function with respect to C. So, from above all, it shows that on $(0, C^*)$, $I_2(Ca_1) \in (0, \infty)$. Furthermore, for any $K \in (0, \infty)$, there exists unique $C > 0$, such that $I_2(Ca_1) = K$, in which $a = Cp_\eta$. □

Remark 6.0.3. *From Remark 6.0.1 and 6.0.2, it shows that for any* $\eta \in (0, \infty)$*, there exists unique pair* (a, C)*, satisfying* $\frac{a}{C} \in (0, a_1(\theta_+ - \theta_-)/(-\theta_-))$*, and* $\eta = \frac{\lambda v_a(0)}{v_a'(0)}$*.*

Based on Proposition 6.0.1 and Remark 6.0.1-6.0.2, the following result can be provided.

Proposition 6.0.2. *If* $\eta \in (0, \frac{\lambda X_0}{\gamma})$*, then the solution* $v(x)$ *of nonterminal bankruptcy model with recovery rate* η *is uniquely given by*

$$v(x) = \begin{cases} C(x + \eta\gamma/\lambda)^\gamma, & x \in [0, x_0), \\ Ca_1 e^{\theta_+(x - x_0)} + Ca_2 e^{\theta_-(x - x_0)}, & x \in [x_0, x_1), \\ v(x_1) + k(x - x_1), & x \in [x_1, \infty), \end{cases} \tag{6.0.12}$$

where $x_0 = X_0 - \frac{\eta\gamma}{\lambda}$*, and* C *is determined by* $I_1(C) = K$*, in which* $a = C(\frac{\eta\gamma}{\lambda})^\gamma$*.*

On the other hand, if $\eta \in [\frac{\lambda X_0}{\gamma}, \infty)$*, then the solution* $v(x)$ *of nonterminal bankruptcy model with recovery rate* η *is uniquely given by*

$$v(x) = \begin{cases} Ca_1 e^{\theta_+ x} + C(p_\eta - a_1)e^{\theta_- x}, & x \in [0, x_1), \\ v(x_1) + k(x - x_1), & x \in [x_1, \infty). \end{cases} \tag{6.0.13}$$

where C *is determined by* $I_2(Ca_1) = K$*, in which* $a = Cp_\eta$*.*

Proof. From proposition 6.0.1 and (6.0.10), for $\eta \in (0, \frac{\lambda X_0}{\gamma})$, $v(x)$ can be given by $v(x) = v_a(x) = v_1(x)$ with $a = C(\frac{\eta\gamma}{\lambda})^\gamma$. That is (6.0.12). In addition, from Remark

6.0.1, it is known that C can be uniquely determined by $I_1(C) = K$, in which $a = C(\frac{\eta\gamma}{\lambda})^\gamma$. So, the uniqueness of the solution for the nonterminal model is satisfied.

By the same method as the proof of the case $\eta \in (0, \frac{\lambda X_0}{\gamma})$, for $\eta \in [\frac{\lambda X_0}{\gamma}, \infty)$, we can verify that $v(x)$ can be uniquely given by (6.0.13) and C is determined by $I_2(Ca_1) = K$, in which $a = Cp_\eta$.

Thereby, this proposition is satisfied. □

Chapter 7

Summary and Concluding Remarks

In this dissertation, I mainly discuss the dividend optimization problem that some fixed money would be returned to shareholders at the state of terminal bankruptcy. In addition, for this problem, the optimal timing and the optimal amount of dividends paid out to the shareholders are provided. Moreover, I point out some conditions that there would be a terminal bankruptcy under optimal policy. The difficulty of this dissertation is that the structure of the candidate solution is not explicit since the existing interval of it has unfixed endpoints, which depends on some unknown parameters. I overcome this difficulty by using the integral, $I(C)$, and dividing it into several cases. In the end, based on the solution for the terminal bankruptcy model, I provide and verify the solution of the nonterminal bankruptcy model.

In the future, I may discuss some problems with general drift and diffusion terms for the case that shareholders can get some fixed money when there is terminal bankruptcy. For this kind of problem, generally there is no explicit solutions. Then some suitable numerical methods to simulate some nonlinear equations in the quasi-variational inequalities are needed to be selected. Further, two optimal policies derived from two choices, investing new money or terminating the business, can be calculated and compared. Consequently, the optimal policy will be found.

Appendix A

Appendix for Some Proofs

Appendix 1. Suppose that $C > aX_0^{-\gamma}$, then it follows that $1 - \frac{1}{\gamma}\theta_+(\frac{a}{C})^{\frac{1}{\gamma}} > 0$.

Proof. From $C > aX_0^{-\gamma}$, then we have $(\frac{a}{C})^{\frac{1}{\gamma}} < X_0$. Further, $\frac{1}{\gamma}\theta_+(\frac{a}{C})^{\frac{1}{\gamma}} < \frac{X_0\theta_+}{\gamma}$. On the other hand, since $\mu < \sqrt{\mu^2 + 2\lambda\sigma^2}$, then

$$\mu + \frac{2\lambda\sigma^2}{\mu} = \frac{\mu^2 + 2\lambda\sigma^2}{\mu} > \sqrt{\mu^2 + 2\lambda\sigma^2}.$$

So, the following inequality is satisfied.

$$\frac{\mu}{2\lambda\sigma^2}(-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}) < 1. \tag{1.0.1}$$

Hence, we have that

$$\begin{aligned} \frac{X_0\theta_+}{\gamma} &= \frac{(1-\gamma)\sigma^2}{\mu} \cdot \frac{1}{\gamma} \cdot \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \\ &= \frac{(1-\gamma)}{\mu\gamma} \cdot (-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}) \\ &= \frac{\mu}{2\lambda\sigma^2}(-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}). \end{aligned} \tag{1.0.2}$$

From (1.0.1) and (1.0.2), then

$$1 - \frac{1}{\gamma}\theta_+(\frac{a}{C})^{\frac{1}{\gamma}} > 1 - \frac{X_0\theta_+}{\gamma} > 0.$$

□

Appendix 2. The inequality $a\gamma < kX_0$ is equivalent to

$$aX_0^{-\gamma}a_1 < \frac{k - a\theta_-}{\theta_+ - \theta_-}.$$

Proof. Since $a_1 + a_2 = X_0^\gamma$ and $\theta_+ - \theta_- > 0$, then we only need to verify

$$aa_1(\theta_+ - \theta_-) < (k - a\theta_-)(a_1 + a_2),$$

which is equivalent to prove

$$a(a_1\theta_+ + a_2\theta_-) < k(a_1 + a_2).$$

In addition, from (3.1.6) and (3.1.7), we can obtain $a(a_1\theta_+ + a_2\theta_-) = a\gamma X_0^{\gamma-1}$. On the other hand, $k(a_1+a_2) = kX_0^\gamma$. So, if $a\gamma < kX_0$, then we can prove that $aX_0^{-\gamma}a_1 < \frac{k-a\theta_-}{\theta_+-\theta_-}$.

Conversely, if $aX_0^{-\gamma}a_1 < \frac{k-a\theta_-}{\theta_+-\theta_-}$, by the same method we also can prove that $a\gamma < kX_0$.

Therefore, $a\gamma < kX_0$ is equivalent to $aX_0^{-\gamma}a_1 < \frac{k-a\theta_-}{\theta_+-\theta_-}$. □

Appendix 3. The equality $\frac{\gamma}{X_0} = \frac{2\lambda}{\mu}$ is satisfied.

Proof. From (3.1.4) and (3.1.2), it follows that

$$\frac{\gamma}{X_0} = \frac{\mu\gamma}{(1-\gamma)\sigma^2} = \frac{\mu\lambda}{\frac{\mu^2}{2\sigma^2}\sigma^2} = \frac{2\lambda}{\mu}.$$

That is, $\frac{\gamma}{X_0} = \frac{2\lambda}{\mu}$. $\square$

Appendix 4. The inequality $\frac{\lambda}{\mu} > \theta_+$ is satisfied.

Proof. We need to verify that

$$\frac{\lambda}{\mu} > \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}.$$

That is,

$$\lambda\sigma^2 > -\mu^2 + \mu\sqrt{\mu^2 + 2\lambda\sigma^2},$$

which is equivalent to checking

$$\lambda\sigma^2 + \mu^2 > \mu\sqrt{\mu^2 + 2\lambda\sigma^2}.$$

Since

$$(\lambda\sigma^2 + \mu^2)^2 = \lambda^2\sigma^4 + \mu^4 + 2\lambda\sigma^2\mu^2 > \mu^2(\mu^2 + 2\lambda\sigma^2),$$

then

$$\lambda\sigma^2 + \mu^2 > \mu\sqrt{\mu^2 + 2\lambda\sigma^2}$$

is satisfied. Hence, we have $\frac{\lambda}{\mu} > \theta_+$. $\square$

Appendix 5. $\frac{\theta_-^2(k-a\theta_+)}{\theta_+^2(k-a\theta_-)} > 1$ is equivalent to $k > \frac{a\lambda}{\mu}$.

Proof. For

$$\frac{\theta_-^2(k - a\theta_+)}{\theta_+^2(k - a\theta_-)} > 1,$$

it is equivalent to

$$\theta_-^2(k - a\theta_+) > \theta_+^2(k - a\theta_-).$$

That is,

$$k(\theta_+^2 - \theta_-^2) < a\theta_+\theta_-(\theta_+ - \theta_-).$$

So, we only need to check $k(\theta_+ + \theta_-) < a\theta_+\theta_-$. From (3.1.6), it is equivalent to, $-2\mu k < -2a\lambda$. That is, $k > \frac{a\lambda}{\mu}$. □

Appendix 6. The inequality $\frac{-a_2\theta_-^2}{a_1\theta_+^2} > 1$ is satisfied.

Proof. Let's reduce the term, $\frac{-a_2\theta_-^2}{a_1\theta_+^2}$.

$$\frac{-a_2\theta_-^2}{a_1\theta_+^2} = \frac{-\frac{\theta_+ X_0^\gamma - \gamma X_0^{\gamma-1}}{\theta_+ - \theta_-}\theta_-^2}{\frac{\gamma X_0^{\gamma-1} - \theta_- X_0^\gamma}{\theta_+ - \theta_-}\theta_+^2} = \frac{(\gamma - \theta_+ X_0)\theta_-^2}{(\gamma - \theta_- X_0)\theta_+^2}. \tag{1.0.3}$$

In addition,

$$\begin{aligned} \gamma - \theta_+ X_0 &= \gamma - \theta_+ \frac{(1-\gamma)\sigma^2}{\mu} \\ &= \frac{\lambda}{\lambda + \frac{\mu^2}{2\sigma^2}} - \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \cdot \frac{\sigma^2}{\mu} \cdot \frac{\frac{\mu^2}{2\sigma^2}}{\lambda + \frac{\mu^2}{2\sigma^2}} \\ &= \frac{\lambda - \frac{\mu}{2\sigma^2}(-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2})}{\lambda + \frac{\mu^2}{2\sigma^2}}, \end{aligned}$$

and

$$\begin{aligned} \gamma - \theta_- X_0 &= \gamma - \theta_- \frac{(1-\gamma)\sigma^2}{\mu} \\ &= \frac{\lambda}{\lambda + \frac{\mu^2}{2\sigma^2}} + \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \cdot \frac{\sigma^2}{\mu} \cdot \frac{\frac{\mu^2}{2\sigma^2}}{\lambda + \frac{\mu^2}{2\sigma^2}} \\ &= \frac{\lambda + \frac{\mu}{2\sigma^2}(\mu + \sqrt{\mu^2 + 2\lambda\sigma^2})}{\lambda + \frac{\mu^2}{2\sigma^2}}, \end{aligned}$$

then, it follows that

$$\begin{aligned}\frac{\gamma-\theta_+X_0}{\gamma-\theta_-X_0} &= \frac{\lambda-\frac{\mu}{2\sigma^2}(-\mu+\sqrt{\mu^2+2\lambda\sigma^2})}{\lambda+\frac{\mu}{2\sigma^2}(\mu+\sqrt{\mu^2+2\lambda\sigma^2})}\\ &= \frac{2\lambda\sigma^2+\mu^2-\mu\sqrt{\mu^2+2\lambda\sigma^2}}{2\lambda\sigma^2+\mu^2+\mu\sqrt{\mu^2+2\lambda\sigma^2}}\\ &= \frac{\sqrt{2\lambda\sigma^2+\mu^2}-\mu}{\sqrt{2\lambda\sigma^2+\mu^2}+\mu}=\frac{\theta_+}{-\theta_-}.\end{aligned}$$

So, $\frac{-a_2\theta_-^2}{a_1\theta_+^2}=\frac{-\theta_-}{\theta_+}>1$. □

Appendix 7. Under the assumption, $a\gamma\geq kX_0$, then we have

$$aX_0^{-\gamma}(a_1\theta_++a_2\theta_-)\geq k.$$

Proof. Since

$$\begin{aligned}aX_0^{-\gamma}(a_1\theta_++a_2\theta_-) &= aX_0^{-\gamma}\left(\frac{\gamma X_0^{\gamma-1}-\theta_-X_0^{\gamma}}{\theta_+-\theta_-}\theta_++\frac{\theta_+X_0^{\gamma}-\gamma X_0^{\gamma-1}}{\theta_+-\theta_-}\theta_-\right)\\ &= aX_0^{-1}\left(\frac{\gamma-\theta_-X_0}{\theta_+-\theta_-}\theta_++\frac{\theta_+X_0-\gamma}{\theta_+-\theta_-}\theta_-\right)\\ &= a\gamma X_0^{-1},\end{aligned}$$

then, from $a\gamma\geq kX_0$, it follows that $aX_0^{-\gamma}(a_1\theta_++a_2\theta_-)\geq k$. □

Appendix 8. The following equality is satisfied.

$$aX_0^{-\gamma}\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}}\cdot\frac{a_1\theta_+(\theta_--\theta_+)}{\theta_-}=a\left(-\frac{\theta_-}{\theta_+}\right)^{\frac{\theta_-}{\theta_+-\theta_-}}\cdot(\frac{2\lambda}{\mu}-\theta_-).$$

Proof. From Appendix 6, we have $\frac{-a_2\theta_-^2}{a_1\theta_+^2} = \frac{-\theta_-}{\theta_+}$. Combining with Appendix 3, then

$$\begin{aligned}
& X_0^{-\gamma}\left(\frac{-a_2\theta_-^2}{a_1\theta_+^2}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{a_1\theta_+(\theta_- - \theta_+)}{\theta_-} \\
&= X_0^{-\gamma}\left(\frac{-\theta_-}{\theta_+}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{a_1\theta_+(\theta_- - \theta_+)}{\theta_-} \\
&= X_0^{-\gamma}\left(\frac{-\theta_-}{\theta_+}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{\theta_+(\theta_- - \theta_+)}{\theta_-} \cdot \frac{\gamma X_0^{\gamma-1} - \theta_- X_0^{\gamma}}{\theta_+ - \theta_-} \\
&= \left(\frac{-\theta_-}{\theta_+}\right)^{\frac{\theta_+}{\theta_+-\theta_-}} \cdot \frac{\theta_+(\theta_- - \theta_+)}{\theta_-} \cdot \frac{\gamma X_0^{-1} - \theta_-}{\theta_+ - \theta_-} \\
&= \left(-\frac{\theta_-}{\theta_+}\right)^{\frac{\theta_-}{\theta_+-\theta_-}} \cdot (\frac{2\lambda}{\mu} - \theta_-).
\end{aligned}$$

So, this appendix is verified. □

Appendix 9. Let M^* be defined by (4.1.18), then M^* satisfies

$$\frac{\lambda}{\mu} < M^* < \frac{2\lambda}{\mu}.$$

Proof. The basic idea is that, for fixed μ and λ, we look on M^* as a function with respect to σ^2, and then we check if it has monotonicity. If the answer is positive, then we take the infimum and the supremum of M^* with respect to σ^2.

For M^* given by (4.1.18), taking its logarithm, then we can get

$$\begin{aligned}
\ln M^* &= \frac{\theta_-}{\theta_+ - \theta_-} \cdot (\ln(-\theta_-) - \ln\theta_+) + \ln(\frac{2\lambda}{\mu} - \theta_-) \\
&= \frac{-\mu - \sqrt{\mu^2 + 2\lambda\sigma^2}}{2\sqrt{\mu^2 + 2\lambda\sigma^2}} \cdot \left[\ln\left(\frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}\right) - \ln\left(\frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}\right)\right] \\
&\quad + \ln(\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}) + \frac{1}{2}\ln(\mu^2 + 2\lambda\sigma^2) - \ln\mu - \ln\sigma^2 \qquad (1.0.4) \\
&= -\left(\frac{\mu}{2\sqrt{\mu^2 + 2\lambda\sigma^2}} + \frac{1}{2}\right) \cdot [\ln(\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}) - \ln(-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2})] \\
&\quad + \ln(\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}) + \frac{1}{2}\ln(\mu^2 + 2\lambda\sigma^2) - \ln\mu - \ln\sigma^2.
\end{aligned}$$

In addition, we have that $\frac{\partial \ln M^*}{\partial \sigma^2} > 0$, which is presented at the end of this proof. So, M^* is an increasing function of σ^2. From $\sigma^2 \in (0, \infty)$, then we can take limits as $\sigma^2 \to 0$ and $\sigma^2 \to \infty$ to get the infimum and the supremum of M^*. For (1.0.4), it follows that

$$\begin{aligned} \lim_{\sigma^2 \to 0} \ln M^* &= -\ln(2\mu) + \ln(2\mu) + \ln\mu - \ln\mu + \lim_{\sigma^2 \to 0} \ln(\frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}) \\ &= \ln(\frac{\lambda}{\mu}). \end{aligned} \tag{1.0.5}$$

So, $\lim\limits_{\sigma^2 \to 0} M^* = \frac{\lambda}{\mu}$, which means the infimum of M^* is $\frac{\lambda}{\mu}$.

On the other hand,

$$\begin{aligned} \lim_{\sigma^2 \to \infty} \ln M^* &= -\ln\mu + \lim_{\sigma^2 \to \infty} \ln\left(\frac{(-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2})\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}\right) \\ &= \ln(\frac{2\lambda}{\mu}). \end{aligned} \tag{1.0.6}$$

So, $\lim\limits_{\sigma^2 \to \infty} M^* = \frac{2\lambda}{\mu}$, which means the supremum of M^* is $\frac{2\lambda}{\mu}$.

Therefore, we can obtain that $\frac{\lambda}{\mu} < M^* < \frac{2\lambda}{\mu}$.

Now, let's check the derivative of M^* on σ^2 as follows.

$$\begin{aligned}
\frac{\partial \ln M^*}{\partial \sigma^2} &= -\frac{\mu}{2}\cdot(-\frac{1}{2})\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot 2\lambda\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) \\
&\quad -\left(\frac{\mu}{2\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{1}{2}\right)\cdot\left[\frac{\lambda(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}}}{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}-\frac{\lambda(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right] \\
&\quad +\frac{\lambda(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}}}{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{\lambda}{\mu^2+2\lambda\sigma^2}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) \\
&\quad +\left(\frac{\mu}{2\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{1}{2}\right)\cdot\frac{\mu}{\sigma^2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}} \\
&\quad +\frac{\lambda(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}}}{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{\lambda}{\mu^2+2\lambda\sigma^2}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) \\
&\quad +\frac{\mu(\mu+\sqrt{\mu^2+2\lambda\sigma^2})}{2\sigma^2(\mu^2+2\lambda\sigma^2)}+\frac{\lambda(\mu^2+2\lambda\sigma^2)^{-\frac{1}{2}}}{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{2\lambda\sigma^2}{2\sigma^2(\mu^2+2\lambda\sigma^2)}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) \\
&\quad +\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{2\sigma^2\sqrt{\mu^2+2\lambda\sigma^2}}+\frac{\lambda}{\sqrt{\mu^2+2\lambda\sigma^2}\cdot(\mu+\sqrt{\mu^2+2\lambda\sigma^2})}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) \\
&\quad +\frac{2(\mu^2+2\lambda\sigma^2+\mu\sqrt{\mu^2+2\lambda\sigma^2})}{2\sigma^2\sqrt{\mu^2+2\lambda\sigma^2}(\mu+\sqrt{\mu^2+2\lambda\sigma^2})}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right)+\frac{1}{\sigma^2}-\frac{1}{\sigma^2} \\
&= \frac{\lambda\mu}{2}\cdot(\mu^2+2\lambda\sigma^2)^{-\frac{3}{2}}\cdot\ln\left(\frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{-\mu+\sqrt{\mu^2+2\lambda\sigma^2}}\right) > 0.
\end{aligned}$$

□

Appendix 10. For $v_2(x)$, given by (5.1.6), if $v_2''(0) < 0$, then $u = 1$ is the maximizer of $\mathcal{L}^u v_2(x)$ at $x = 0$.

Proof. At $x = 0$, we have that

$$\mathcal{L}^u v_2(0) = \frac{1}{2}\sigma^2 u^2 [C_1\theta_+^2 + (a - C_1)\theta_-^2] + \mu u [C_1\theta_+ + (a - C_1)\theta_-] - \lambda a.$$

Since $v_2''(0) < 0$, and the symmetric axis of $\mathcal{L}^u v_2(0)$ with respect to u is

$$u(0) = -\frac{\mu[C_1\theta_+ + (a - C_1)\theta_-]}{\sigma^2[C_1\theta_+^2 + (a - C_1)\theta_-^2]},$$

then we need to prove $u(0) \geq 1$. That is,

$$C_1\theta_+ + (a - C_1)\theta_- \geq -\frac{\sigma^2}{\mu}[C_1\theta_+^2 + (a - C_1)\theta_-^2],$$

which is equivalent to

$$C_1(\theta_+ - \theta_-)[1 + \frac{\sigma^2}{\mu}(\theta_+ + \theta_-)] \geq -a\theta_-(1 + \frac{\sigma^2}{\mu}\theta_-). \tag{1.0.7}$$

Reducing (1.0.7), we can obtain that

$$C_1 \leq -a\theta_- \cdot \frac{\sigma^2}{\mu}.$$

In the following, we show that $a_1 X_0^{-\gamma} = -\theta_- \cdot \frac{\sigma^2}{\mu}$.

From

$$-\theta_- \cdot \frac{\sigma^2}{\mu} = \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{2\mu},$$

and

$$
\begin{aligned}
a_1 X_0^{-\gamma} &= \frac{\gamma X_0^{\gamma-1} - \theta_- X_0^{\gamma}}{(\theta_+ - \theta_-) X_0^{-\gamma}} = \frac{\gamma - \theta_- X_0}{(\theta_+ - \theta_-) X_0} \\
&= \frac{\frac{\lambda}{\lambda + \frac{\mu^2}{2\sigma^2}} + \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \frac{\sigma^2}{\mu} \frac{\frac{\mu^2}{2\sigma^2}}{\lambda + \frac{\mu^2}{2\sigma^2}}}{\frac{\sigma^2}{\mu} \frac{\frac{\mu^2}{2\sigma^2}}{\lambda + \frac{\mu^2}{2\sigma^2}} \frac{2\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}} \\
&= \frac{\lambda + \frac{\mu^2}{2\sigma^2} + \frac{\mu^2}{2\sigma^2} \sqrt{\mu^2 + 2\lambda\sigma^2}}{\frac{\mu}{\sigma^2} \sqrt{\mu^2 + 2\lambda\sigma^2}} \\
&= \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{2\mu},
\end{aligned}
$$

then, it follows that $a_1 X_0^{-\gamma} = -\theta_- \cdot \frac{\sigma^2}{\mu}$.

So, from the equality above, we have that $C_1 \leq a X_0^{-\gamma} a_1$, which is satisfied by the definition of $v_2(x)$. Therefore, $u(0) \geq 1$ is verified. Then, $u = 1$ is the maximizer of $\mathcal{L}^u v_2(x)$ at $x = 0$. ☐

Appendix 11. Assume that $\frac{a\lambda}{\mu} < k < \frac{2a\lambda}{\mu}$, then for any $C_1 \in (\frac{k - a\theta_-}{\theta_+ - \theta_-}, a X_0^{-\gamma} a_1)$, $H^{C_1}(x)$ has convexity.

Proof. From Appendix 2 and Appendix 3, it is known that if $k < \frac{2a\lambda}{\mu}$, then we have $a X_0^{-\gamma} a_1 > \frac{k - a\theta_-}{\theta_+ - \theta_-}$.

For any $C_1 \in (\frac{k - a\theta_-}{\theta_+ - \theta_-}, a X_0^{-\gamma} a_1)$, then

$$
a - a X_0^{-\gamma} a_1 < a - C_1 < a - \frac{k - a\theta_-}{\theta_+ - \theta_-},
$$

that is

$$
a X_0^{-\gamma} a_2 < a - C_1 < \frac{a\theta_+ - k}{\theta_+ - \theta_-}. \tag{1.0.8}
$$

From Appendix 4, we have that $\frac{\lambda}{\mu} > \theta_+$. Then, combining with the assumption $k > \frac{a\lambda}{\mu}$, we obtain $k > a\theta_+$. Further, from (1.0.8), it follows that $a - C_1 < 0$.

Therefore, $(H^{C_1})''(x) > 0$ is satisfied. So, we verify that on $(\frac{k-a\theta_-}{\theta_+-\theta_-}, aX_0^{-\gamma}a_1)$, $H^{C_1}(x)$ has convexity. □

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