A TRANSITION MATRIX FOR TWO BASES OF THE INTEGRAL COHOMOLOGY OF THE HILBERT SCHEME OF POINTS IN THE PROJECTIVE PLANE

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RYAN MURPHY

Dr. ZHENBO QIN, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Ryan Murphy,

a candidate for the degree of Doctor of Philosophy and hereby certify that in their opinion it is worthy of acceptance.

Professor Dan Edidin

Professor Qi Zhang

Professor Shuguang Wang

(Department of Economics) Professor Xinghe Wang

(Supervisor) Professor Zhenbo Qin

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TABLE OF CONTENTS

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ABSTRACT

This work is devoted to comparing two integral bases for the integral cohomology of the Hilbert scheme of points in the projective plane. Let X be a smooth complex projective surface. One of the more interesting moduli spaces parameterizing objects associated with X is the Hilbert scheme of points, denoted $X^{[n]}$, which parameterizes all 0-dimensional closed subschemes of length n in X. W. Wang, Z. Qin and W.P. Li used Heisenberg algebra operators to construct an integral basis of the integral cohomology of $X^{[n]}$ whenever X is a smooth projective surface with vanishing odd Betti numbers. On the other hand, a work by G. Ellingsrud and S.A. Strømme gives a cellular decomposition of the Hilbert scheme of points on the projective plane. From this work, we have a second integral basis for the integral cohomology of $X^{[n]}$ when $X = \mathbb{P}^2$. We compare the elements of these two bases and ultimately give the upper triangular transition matrix from one basis to the other.

Chapter 1

INTRODUCTION

Suppose X is a complex projective scheme. The Hilbert scheme of points, which we will call $X^{[n]}$, paramaterizes all 0-dimensional closed subschemes of length n in X. As with most moduli spaces, the Hilbert scheme of points inherits structures from X. This follows from the construction of Hilbert schemes, which we will provide in Chapter 2. If X is a complex projective scheme, we see that $X^{[n]}$ is also a complex projective scheme. Beauville showed that if X has a holomorphic symplectic form, then $X^{[n]}$ also has one ([1]). However, $X^{[n]}$ is particularly interesting as it is a moduli space which has newly arising structures. These structures appear when we consider the components of $X^{[n]}$ all together.

In 1990, Göttsche provided the Betti numbers of $X^{[n]}$ for an aribtrary surface X ([11]). Nakajima was able to construct a Heisenberg algebra which acts irreducibly on the direct sum of the rational cohomology of the Hilbert schemes $X^{[n]}$ for all n ([20],[12]). A resulting corollary gave a linear basis for the rational cohomology of the Hilbert schemes $X^{[n]}$ in terms of these Heisenberg operators.

In 2004, using Heisenberg operators, Qin and Wang ([22]) were able to find an integral basis for the integral cohomology of $X^{[n]}$ whenever X was a projective surface such that $H^1(X;\mathcal{O}_X) = H^2(X;\mathcal{O}_X) = 0$ ([22]). In 2008, Li and Qin ([16]) improved upon this result, now only requiring that X have vanishing odd Betti numbers ([16]).

In the 1970s and 1980s, significant work was done in the case $X = \mathbb{P}^2$. Fogarty calculated the Picard group of $(\mathbb{P}^2)^{[n]}$ ([8]). In 1984, Hirschowitz computed the homology groups of $(\mathbb{P}^2)^{[3]}$ ([14]). Finally, in 1987, Ellingsrud and Strømme applied results of Bialynicki-Birula to give an integral basis for the integral cohomology of $(\mathbb{P}^2)^{[n]}$, using a cellular decomposition defined by the natural action of a maximal torus of $SL(3)$ on $(\mathbb{P}^2)^{[n]}$ ([6]).

The purpose of this paper is to compare the basis for $(\mathbb{P}^2)^{[n]}$ that Ellingsrud and Strømme established with the general integral basis described in [22] and [16]. We will see we can eventually describe the transistion matrix between these two bases. In Chapter 2, we provide some important results on the Hilbert scheme of points $X^{[n]}$ where X is a complex smooth projective surface. We will define the creation Heisenberg algebra operators $\mathfrak{a}_{-r}(1)$ and $\mathfrak{a}_{-r}(x)$ associated to the identity cohomology class 1 and the point cohomology class x, resp.

Suppose a partition $\lambda = (1^{m_1}, 2^{m_2}, ...)$ where m_r is the number of parts equal to r, $|\lambda| = \sum_{r\geq 1} rm_r$ and $l(\lambda) = \sum_{r\geq 1} m_r$. Then define

$$\mathfrak{z}_\lambda = \prod_{r\geq 1} r^{m_r} m_r!.$$

Also, for a class $\alpha \in H^*(X)$ define

$$\mathfrak{a}_{-\lambda}(\alpha) = \prod_{r\geq 1} \mathfrak{a}_{-r}(\alpha)^{m_r}.$$

From here, we will define anothe operator $\mathfrak{m}_{\lambda,C}$ which we can describe as $\mathfrak{a}_{[L^\lambda C]}$, where $[L^\lambda C] \subset X^{[n]}$ is defined in [20].

We then look at the special case $X = \mathbb{P}^2$. Following [6], we define cellular decomposition and provide one for $(\mathbb{P}^2)^{[n]}$, denoting the cells of this decomposition as $C_{\lambda,\mu,\nu}$ where λ, μ and ν are partitions such that $|\lambda| + |\mu| + |\nu| = n$. Let us fix a system of homogeneous coordinates T_0, T_1, T_2 on $\mathbb{P}^2$. Then we will define a line $\mathfrak{L} = \{T_2 = 0\}$. From here, we can get an integral basis for the integral homology (and thus cohomology) for $(\mathbb{P}^2)^{[n]}$.

It also becomes clear in this chapter that describing $C_{\lambda,\mu,\nu}$ reduces to describing cells of the form $C_{\lambda,0,0}$, $C_{0,\mu,0}$, and $C_{0,0,\nu}$. Thus, in Chapter 3, we study the special cases of cells of these forms.

In Chapter 4, we provide our main theorem which is the following.

Theorem 1.0.1. *Let λ,μ, and ν be partitions such that $|\lambda| + |\mu| + |\nu| = n$. Then*

$$[\overline{C}_{\lambda,\mu,\nu}] = (-1)^{|\nu|-l(\nu)} \frac{1}{\mathfrak{z}_\nu} \mathfrak{a}_{-\nu}(1)\mathfrak{m}_{\mu,\mathfrak{L}}\mathfrak{a}_{-\lambda}(x)|0\rangle + \sum_{\substack{\lambda',\mu',\nu' \\ |\lambda'|+|\mu'|+|\nu'|=n}} e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \frac{1}{\mathfrak{z}_{\nu'}} \mathfrak{a}_{-\nu'}(1)\mathfrak{m}_{\mu',\mathfrak{L}}\mathfrak{a}_{-\lambda'}(x)|0\rangle$$

where $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \in \mathbb{Z}$ and one of the following is true about the triple (λ', μ', ν'):

(i) $|\nu'| < |\nu|$

(ii) $\nu' = \nu$ and $|\mu'| < |\mu|$

(iii) $\nu' = \nu$, $\mu' = \mu$, $\quad l(\lambda') = l(\lambda)$, and $\lambda < \lambda'$.

With the appropriate ordering of basis elements, this theorem allows us to give a tranisition matrix between these two bases such that this matrix is upper triangular with diagonal entries of ± 1. In Chapter 5, we will look closely at the cases in which $n = 2$ and $n = 3$.

While we can determine quite a bit about the transition matrices, we are not certain of every entry. Thus, our result suggests one might be able to find the exact entries for transition matrices between these two bases. Future work might be done to test cases, such as the $n = 2$ and $n = 3$ cases started in Chapter 5 of this work.

Chapter 2

The Hilbert Scheme of Points

2.1 Important Results on the Hilbert Schemes of Points

First, we would like to give the precise definition of a Hilbert scheme. We follow Nakajima's definition from [20], but we note Grothendieck originally constructed this definition in [13].

Definition 2.1.1. ([13],[20]) Let X be a projective scheme over an algebraically closed field k and $\mathcal{O}_X(1)$ an ample line bundle on X. Now consider the contravariant functor $Hilb_X$ from the category of complex schemes to the category of sets

$$Hilb_X : [\text{Schemes}] \to [\text{Sets}],$$

defined by

$$Hilb_X(U) = \{Z \subset X \times U | Z \text{ is a closed subscheme and flat over } U\}.$$

Note that $Hilb_X$ associates a complex scheme U with a set of families of closed subschemes in X parametrized by U. Let $\pi : Z \to U$ be the projection. Then for

$u \in U$, the Hilbert polynomial in u is defined by

$$P_u(m) = \chi(\mathcal{O}_{Z_u} \otimes \mathcal{O}_X(m)),$$

where $Z_u = \pi^{-1}(u)$.

Let $Hilb_X^P$ be the subfunctor of $Hilb_X$ which associates U with a set of families of closed subschemes in X parameterized by U which has P as its Hilbert polynomial. In 1960, Grothendieck ([13]) proved the functor $Hilb_X^P$ is representible by a projective scheme Hilb_X^P. Now, suppose P is the constant polynomial $P(m) = n$ for $m \in \mathbb{Z}$, where n is an integer such that $n > 0$. Then, we denote

$$X^{[n]} = \mathrm{Hilb}_X^P$$

and call it the Hilbert scheme of n points in X.

Now we would like to highlight three results which are important in the study of Hilbert schemes of points. Let $S^n X$ be the nth symmetric product of X. We can define the Hilbert-Chow morphism

$$\pi_n : X^{[n]} \to S^n X$$

by sending an element in $X^{[n]}$ to its support in $S^n X$. Our first result gives a few properties of $X^{[n]}$ which Fogarty ([7]) proved in 1968.

Theorem 2.1.2. *(Fogarty) Suppose X is a smooth complex projective surface. Then the following hold:*

i) $X^{[n]}$ is smooth and has dimension $2n$.

ii) $\pi_n : X^{[n]} \to S^n X$ is a resolution of singularities.

Recall that $X^{[n]}$ parameterizes 0-dimensional closed subschemes of length n in X. We would like to define a punctured Hilbert scheme.

Definition 2.1.3. Suppose $x \in X$. Then we define $M_n(x)$ to be the closed subscheme of $X^{[n]}$ parameterizing 0-dimensional closed subscemes of length n supported at x. We will call this the *punctured Hilbert scheme* at x.

In 1977, Briancon ([4]) provided an important result on punctured Hilbert schemes.

Theorem 2.1.4. *(Briancon) $M_n(x)$ is irreducible of dimension $n-1$.*

This result can also be seen as a result of Theorem 2 in [10] and a corollary of Theorem 1.1 in [6].

Finally, in 1990, Göttsche([11]) used the Weil conjecture to prove the following formula.

Theorem 2.1.5. *(Göttsche) Let $P_t(X^{[n]})$ be the Poincaré polynomial of $X^{[n]}$. The generating function of these polynomials is given by*

$$\sum_{n=0}^{\infty} q^n P_t(X^{[n]}) = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}}$$

where $b_i(X)$ is the ith Betti number of X.

This formula allows us to find the Betti numbers of $X^{[n]}$ whenever X is a quasi-projective nonsingular surface. In 2000, a more elementary and natural proof of this theorem was devised by Cataldo and Migliorini ([5]).

2.2 Heisenberg Algebra Integral Operators

Suppose X is a smooth complex projective surface. As we have mentioned, the the Hilbert scheme of points on X, $X^{[n]}$, parameterizes all length-n 0-dimensional closed subschemes of X. Let

$$H^*(X^{[n]}) = \bigoplus_{i=0}^{4n} H^i(X^{[n]})$$

be the total cohomology of $X^{[n]}$ with $\mathbb{Q}$-coefficients. Set

$$\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}).$$

There is a non-degenerate super-symmetric bilinear form $\langle -, - \rangle$ on $\mathbb{H}_X$ induced from the standard one on $H^*(X^{[n]})$ defined by

$$\langle \alpha, \beta \rangle = \int_{X^{[n]}} \alpha\beta, \quad \alpha, \beta \in H^*(X^{[n]}). \tag{2.2.1}$$

In this section, we will see we can find a basis for $H^*(X^{[n]}; \mathbb{Z})/Tor$ using integral operators defined in [22]. We first introduce some terms and operators defined in Nakajima ([20]).

We now want to define the Heisenberg algebra action on $\mathbb{H}_X$ as given in Nakajima ([20]) and Grojnowski ([12]). Let I_ξ be the corresponding sheaf of ideals for a length-n 0-dimensional closed subscheme ξ of X. For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and $Q^{[m+n,m]}$ be the closed subscheme of $X^{[m+n]} \times X \times X^{[m]}$ defined set-theoretically by

$$\left\{ (\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} | \xi \supset \eta \text{ and } \mathrm{Supp}(I_\eta / I_\xi) = \{x\} \right\}.$$

Let $\tilde{p_1}, \tilde{\rho}, \tilde{p_2}$ be the projections from $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively.

Definition 2.2.1. Let $n > 0$. Then for $Z \in H^*(X^{[m]})$ and $\alpha \in H^*(X)$, we define the linear operator $\mathfrak{a}_{-n}(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ by

$$\mathfrak{a}_{-n}(\alpha)(Z) = \tilde{p}_{1*} \left([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* Z \right).$$

We define $\mathfrak{a}_n(\alpha) \in \mathrm{End}(\mathbb{H}_\mathrm{X})$ to be $(-1)^n$ times the operator in the definition of $\mathfrak{a}_{-n}(\alpha)$ by interchanging the roles of $\tilde{p}_1$ and $\tilde{p}_2$. We call $\mathfrak{a}_{-n}(\alpha)$ (resp. $\mathfrak{a}_n(\alpha)$) the *creation* (resp. *annihilation*) operator. Also, we define $\mathfrak{a}_0(\alpha) = 0$.

From (2.2.1) we can define the *adjoint* $f^T \in \mathrm{End}(\mathbb{H}_X)$ for $f \in \mathrm{End}(\mathbb{H}_X)$. Thus

$$\mathfrak{a}_n(\alpha) = (-1)^n \cdot \mathfrak{a}_{-n}(\alpha)^T.$$

We provide the following theorem from [21] and [12].

Theorem 2.2.2. *(Nakajima, Grojnowski) The operators $\mathfrak{a}_n(\alpha) \in End(\mathbb{H}_X)$ with $\alpha, \beta \in H^*(X)$ and $n \in \mathbb{Z}$ satisfy the following Heisenberg algebra commutation relation:*

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot Id_{\mathbb{H}_X}.$$

Note that $\mathbb{H}_X$ is an irreducible module over the Heisenberg algebra generated by the operators $\mathfrak{a}_n(\alpha)$ with a highest weight vector

$$|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{Q}.$$

Thus $\mathbb{H}_X$ is spanned by the Heisenberg monomial classes

$$\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle$$

where $k \geq 0$, $n_1, \ldots, n_k > 0$, and $\alpha_1, \ldots, \alpha_k$ run over a linear basis of $H^*(X)$.

Now, let $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$ be a partition of $n = \sum_{r\geq 1} rm_r$, i.e. the r part of λ has multiplicity m_r. Note the length of λ, denoted by $l(\lambda)$, is $\sum_{r\geq 1} m_r$. Suppose $\alpha \in H^*(X)$. Then define

$$\begin{aligned} \mathfrak{z}_\lambda &= \prod_{r\geq 1} r^{m_r} m_r! \\ \mathfrak{a}_{-\lambda}(\alpha) &= \prod_{r\geq 1} \mathfrak{a}_{-r}(\alpha)^{m_r}. \end{aligned}$$

2.2.1 The Operator $\mathfrak{m}_{\lambda,\alpha}$

Suppose C is a smooth irreducible curve in the surface X. Abusing notation, we will also use C to denote its corresponding divisor and cohomology class. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of n (we may write this $\lambda \vdash n$). Define

$$S^n_\lambda C = \left\{ \sum_i \lambda_i x_i | x_i \in C, x_i \neq x_j \text{ for } i \neq j \right\}.$$

From Section 9.3 in [20], we have the following definition:

Definition 2.2.3. Let $\pi_n : X^{[n]} \to S^n(X)$ be the Hilbert-Chow morphism. Then

$$L^\lambda C = \overline{(\pi_n)^{-1}(S^n_\lambda C)}.$$

For $n \geq 0$, let $\mathbb{H}_{n,C}$ be the $\mathbb{Q}$-linear span of the classes $\mathfrak{a}_{-\lambda}(C)|0\rangle$ where λ is a partition of n. Theorem 9.14 in [20] tells us the fundamental class

$$[L^\lambda C] \in H^{2n}(X^{[n]}; \mathbb{Z})$$

is contained in $\mathbb{H}_{n,C} \subset H^*(X^{[n]})$. Define

$$\mathbb{H}_C = \bigoplus_{n=0}^{\infty} \mathbb{H}_{n,C}. \tag{2.2.2}$$

Let Λ be the ring of symmetric functions in infinitely many variables (see [19]) and $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. For a partition λ, let p_λ and m_λ denote the power-sum symmetric function and the monomial symmetric function, respectively. Nakajima defines the following linear isomorphism

$$\Phi_C : \Lambda_{\mathbb{Q}} \to \mathbb{H}_{n,C}$$

satisfying the following two properties

$$\Phi_C(p_\lambda) = \mathfrak{a}_{-\lambda}(C)|0\rangle, \quad \Phi_C(m_\lambda) = [L^\lambda C].$$

Definition 2.2.4. Let $\xi \in \mathbb{H}_X$. We define $\mathfrak{a}_\xi$ be the unique polynomial of creation operators such that we can write ξ as $\mathfrak{a}_\xi|0\rangle$.

Now for a partition λ, let us denote

$$\mathfrak{m}_{\lambda,C} = \mathfrak{a}_{[L^\lambda C]} \in \mathrm{End}(\mathbb{H}_C).$$

Note that $[L^\lambda C] \in \mathbb{H}_{|\lambda|,C}$. By the definition of $\mathbb{H}_{n,C}$, we see the operator $\mathfrak{m}_{\lambda,C}$ is a polynomial of creation operators of the form $\mathfrak{a}_{-i}(C)$, $i > 0$. We can extend this idea to an arbitrary class $\alpha \in H^2(X)$.

Definition 2.2.5. Let $\alpha \in H^2(X)$ and λ be a partition. If we replace the creation operators $\mathfrak{a}_{-i}(C)$ in $\mathfrak{m}_{\lambda,C}$ by the corresponding operators $\mathfrak{a}_{-i}(\alpha)$, then we can define

$$\mathfrak{m}_{\lambda,\alpha} \in \mathrm{End}(\mathbb{H}_X).$$

We define the class $[L^\lambda \alpha] \in H^*(X^{[|\lambda|]})$ by

$$[L^\lambda \alpha] = \mathfrak{m}_{\lambda,\alpha}|0\rangle.$$

Note that by replacing C with arbitrary $\alpha \in H^2(X)$, we can define subspaces $\mathbb{H}_{n,\alpha}$ and $\mathbb{H}_\alpha$ of $\mathbb{H}_X$ corresponding to $\mathbb{H}_{n,C}$ and $\mathbb{H}_C$, respectively.

2.2.2 An Integral Basis for $H^*(X^{[n]}; \mathbb{Z})/Tor$ using Integral Operators

Here we state some results from Qin and Wang ([22]), which were later improved by Li and Qin ([16]). From this work, we have our first basis of $H^*(X^{[n]}; \mathbb{Z})/Tor$, which we will later compare to the basis we describe below. First, we need the following definition.

Definition 2.2.6. (i) A class $Z \in H^*(X^{[n]})$ is *integral* if it is contained in

$$H^*(X^{[n]};\mathbb{Z})/Tor \subset H^*(X^{[n]}).$$

(ii) A linear basis of $H^*(X^{[n]})$ is *integral* if its members are integral classes and form a $\mathbb{Z}$-basis of the lattice $H^*(X^{[n]};\mathbb{Z})/Tor$.

(iii) A linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$ is *integral* if $\mathfrak{f}(Z) \in \mathbb{H}_X$ is an integral class whenever $Z \in \mathbb{H}_X$ is an integral cohomology class.

Lemma 2.2.7. *(Qin, Wang) Suppose $\alpha \in H^*(X)$. Then we have the following*

(i) The operators $\mathfrak{a}_n(\alpha)$, $n \in \mathbb{Z}$, are integral operators if $\alpha \in H^*(X)$ is integral.

(ii) The operators $1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1)$ are integral operators for all partitions λ.

(iii) For every divisor α on X and every partition λ, $\mathfrak{m}_{\lambda,\alpha}$ is an integral operator.

In [16], part (iii) of Lemma 2.2.7 was improved as it was shown $\mathfrak{m}_{\lambda,\alpha}$ is integral whenever $\alpha \in H^2(X)$ is integral. Thus, for integral $\alpha_1, \ldots, \alpha_k \in H^2(X)$, we see the following are integral operators:

$$1/\mathfrak{z}_\lambda \cdot \mathfrak{a}_{-\lambda}(1), \qquad \mathfrak{m}_{\mu_i,\alpha_i}, \qquad \mathfrak{a}_{-\nu}(x)$$

where λ, μ_i, ν are partitions.

Theorem 2.2.8. *(Li, Qin) Let X be a smooth projective surface with vanishing odd Betti numbers. Let $\alpha_1, \ldots, \alpha_k$ be an integral basis for $H^2(X;\mathbb{Z})/Tor$. Then the classes*

$$\frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)\mathfrak{m}_{\mu_1,\alpha_1}\cdots\mathfrak{m}_{\mu_k,\alpha_k}\mathfrak{a}_{-\nu}(x)|0\rangle, \quad |\lambda| + |\nu| + \sum_{i=1}^{k}|\mu_i| = n \tag{2.2.3}$$

are integral, and furthermore, they form an integral basis for $H^*(X^{[n]};\mathbb{Z})/Tor$.

2.3 A Cellular Decomposition for $(\mathbb{P}^2)^{[n]}$

Now we look specifically at the case of $X = \mathbb{P}^2$. By following the work of Ellingsrud and Strømme ([6]), we can find a cellular decomposition of $(\mathbb{P}^2)^{[n]}$. We will then use this decomposition to find a basis for the integral cohomology of $H^*((\mathbb{P}^2)^{[n]}; \mathbb{Z})$.

2.3.1 Definition of Cellular Decomposition

Definition 2.3.1. A scheme X has a *cellular decomposition* if there is a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes where each $X_i - X_{i-1}$ is a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. The U_{ij} are called the *cells* of the decomposition.

Now suppose X is a variety with an action of $\mathbb{C}^*$. Then we have the following result, proved in [2] and [3].

Theorem 2.3.2. *(Bialynicki-Birula). Let X be a smooth projective variety with an action of $\mathbb{C}^*$. Suppose the fixpoint set $\{x_1, \ldots, x_n\}$ is finite and let*

$$X_i = \{x \in X | \lim_{t \to 0} tx = x_i\}.$$

Then X has a cellular decomposition with cells X_i.

Suppose X is a smooth projective variety with an action of a torus G with a finite fixpoint set. By Remark 1.7 in [6], we see if $\phi : \mathbb{C}^* \to G$ is a "general" one-parameter subgroup, then we get an action of $\mathbb{C}^*$ on X with the same set of fixpoints as G. Thus, if G has a finite set of fixpoints, we may use Theorem 2.3.2 to show X has a cellular decomposition.

Now, let X be $\mathbb{P}^2$. We can introduce torus action on $\mathbb{P}^2$ and apply Theorem 2.3.2 to achieve a cellular decomposition for $\mathbb{P}^2$, and subsequently, for $(\mathbb{P}^2)^{[n]}$. Fix a system of homogeneous coordinates T_0, T_1, T_2 on $\mathbb{P}^2$ and let $G \subset SL(3, \mathbb{C})$ be the maximal torus consisting of all diagonal matrices. Suppose $g = \mathrm{diag}(g_0, g_1, g_2) \in G$. Then G acts on $\mathbb{P}^2$ by

$$g(T_i) = g_i T_i.$$

For points $(a_0, a_1, a_2) \in \mathbb{P}^2$, we have

$$g(a_0, a_1, a_2) = (g_0^{-1} a_0, g_1^{-1} a_1, g_2^{-1} a_2).$$

Note the fixpoints of G are $P_0 = (1, 0, 0)$, $P_1 = (0, 1, 0)$, and $P_2 = (0, 0, 1)$.

Recall $\mathfrak{L} = \{T_2 = 0\} \subset \mathbb{P}^2$. Set

$$F_0 = \{P_0\}$$

$$F_1 = \mathfrak{L} - \{P_0\}$$

and

$$F_2 = \mathbb{P}^2 - \mathfrak{L}.$$

Consider the one-parameter subgroups $\phi : \mathbb{C}^* \to G$ of the type

$$\phi(t) = \mathrm{diag}(t^{w_0}, t^{w_1}, t^{w_2})$$

such that $w_0 < w_1 < w_2$ and $w_0 + w_1 + w_2 = 0$. We have a cellular decomposition of $\mathbb{P}^2$

$$\mathbb{P}^2 \supset \mathfrak{L} \supset \{P_0\}$$

and F_0, F_1 and F_2 are the cells.

We see in [6] that G naturally induces an action on $(\mathbb{P}^2)^{[n]}$ and that this action has finitely many fixpoints (Lemma 2.1 in [6]). Since $(\mathbb{P}^2)^{[n]}$ is smooth and projective, we have a cellular decomposition of $(\mathbb{P}^2)^{[n]}$ by Theorem 2.3.2.

2.3.2 Cells of the Decomposition

Suppose $\xi \subset \mathbb{P}^2$ is of finite length n. For any such ξ, we can write

$$\xi = \xi_0 \cup \xi_1 \cup \xi_2$$

where ξ_i is a closed subscheme of $\mathbb{P}^2$ supported in F_i for $i = 0, 1, 2$. Now define $n_i(\xi) = \text{length}(\xi_i)$. Suppose we have non-negative integers n_0, n_1, n_2 such that $n_0 + n_1 + n_2 = n$. Then we can define $W(n_0, n_1, n_2)$ to be the locally closed subset of $(\mathbb{P}^2)^{[n]}$ corresponding to the set of subschemes

$$\{\xi \subset \mathbb{P}^2 | n_i(\xi) = n_i \text{ for } i = 0, 1, 2\}.$$

It is clear

$$(\mathbb{P}^2)^{[n]} = \bigcup_{n_0+n_1+n_2=n} W(n_0, n_1, n_2). \tag{2.3.1}$$

Recall that the one-parameter subgroup ϕ induces a cellular decomposition of $(\mathbb{P}^2)^{[n]}$. Then $W(n_0, n_1, n_2)$ is a union of cells from this decomposition. Now suppose $\xi \in W(n_0, n_1, n_2)$. If we write $\xi = \xi_0 \cup \xi_1 \cup \xi_2$, then we see $\lim_{t\to 0} \phi(t)(\xi_i)$ is a subscheme supported in P_i. Thus, $W(n_0, n_1, n_2)$ has a cellular decomposition. Note

$$W(n_0, n_1, n_2) \cong W(n_0, 0, 0) \times W(0, n_1, 0) \times W(0, 0, n_2).$$

As mentioned in [6], this allows us to use different one-parameter subgroups

$$\phi : \mathbb{C}^* \to G$$

on the different W's as long as they leave the W's invariant.

Suppose $n > 0$. The cells contained in $W(n,0,0)$ correspond to fixpoints supported in P_0. From Section 3 of [6], we see that this means we are reduced to the study of ideals of $\mathbb{C}[u,v]$ of finite colength, invariant under the action of a two-dimensional torus. These in turn are in one-to-one correspondence with the partitions λ of n. Of course, the same can be said for $W(0,n,0)$ and $W(0,0,n)$. Thus, we can give a description of the cells of these subsets.

Since they are in correspondence with partitions of n, let us call the fixpoints of G

$$\xi_{\lambda,0,0}, \qquad \xi_{0,\mu,0}, \qquad \xi_{0,0,\nu}$$

for partitions λ, μ and ν of n. Note these are supported at P_0, P_1 and P_2, respecitvely. Then the cells of $W(n,0,0), W(0,n,0)$ and $W(0,0,n)$ are of the form

$$C_{\lambda,0,0} = \left\{\xi \in (\mathbb{P}^2)^{[n]} | \lim_{t\to 0} \phi(t)\xi = \xi_{\lambda,0,0}\right\} \tag{2.3.2}$$

$$C_{0,\mu,0} = \left\{\xi \in (\mathbb{P}^2)^{[n]} | \lim_{t\to 0} \phi(t)\xi = \xi_{0,\mu,0}\right\} \tag{2.3.3}$$

$$C_{0,0,\nu} = \left\{\xi \in (\mathbb{P}^2)^{[n]} | \lim_{t\to 0} \phi(t)\xi = \xi_{0,0,\nu}\right\} \tag{2.3.4}$$

respectively. Suppose

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_j)$$

$$\mu = (\mu_1 \geq \cdots \geq \mu_k)$$

and

$$\nu = (\nu_1 \geq \cdots \geq \nu_l).$$

Then, we see from [6], the corresponding ideals of $\mathbb{C}[u,v]$ are of the form

$$I_{\xi_{\lambda,0,0}} = \left\langle v^{\lambda_1}, uv^{\lambda_2}, \ldots, u^{j-1}v^{\lambda_j}, u^j \right\rangle \tag{2.3.5}$$

$$I_{\xi_{0,\mu,0}} = \left\langle v^{\mu_1}, uv^{\mu_2}, \ldots, u^{k-1}v^{\mu_k}, u^k \right\rangle \tag{2.3.6}$$

$$I_{\xi_{0,0,\nu}} = \left\langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^{l-1}v^{\nu_l}, u^l \right\rangle \tag{2.3.7}$$

respectively. Thus, for a general cell $C \subset W(n_0, n_1, n_2)$, we see that for each $\xi \in C$, we can write $\xi = \xi_0 \cup \xi_1 \cup \xi_2$ such that for some $\lambda \vdash n_0$, $\mu \vdash n_1$, and $\nu \vdash n_2$, we have

$$\begin{aligned} &\xi_0 \in C_{\lambda,0,0} \quad \text{and} \quad \mathrm{Supp}(\xi_0) \in F_0 = \{P_0\} \\ &\xi_1 \in C_{0,\mu,0} \quad \text{and} \quad \mathrm{Supp}(\xi_1) \in F_1 = \mathfrak{L} - \{P_0\} \\ &\xi_2 \in C_{0,0,\nu} \quad \text{and} \quad \mathrm{Supp}(\xi_2) \in F_2 = \mathbb{P}^2 - \mathfrak{L}. \end{aligned}$$

Let us then give this cell C the name $C_{\lambda,\mu,\nu}$. Clearly now, we see

$$W(n_0, n_1, n_2) = \bigcup_{|\lambda|=n_0, |\mu|=n_1, |\nu|=n_2} C_{\lambda,\mu,\nu}.$$

2.3.3 An Integral Basis for $H^*((\mathbb{P}^2)^{[n]}; \mathbb{Z})$ using Cells

In Chapter 19.1 of [9], the following proposition was proven.

Proposition 2.3.3. *(Fulton) Let X be a scheme with a cellular decomposition. Let $A_*(X)$ be the Chow group. Then for $0 \leq i \leq \dim X$*

(i) $H_{2i+1}(X) = 0$

(ii) $H_{2i}(X)$ is a $\mathbb{Z}$-module freely generated by the classes of the closure of the i-dimensional cells.

(iii) The cycle map $cl\colon A_*(X) \to H_*(X)$ is an isomorphism.

Note from part (ii) of this proposition that $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$ is torsion-free. Since $\mathbb{P}^2$ is a smooth projective surface with vanishing odd Betti numbers, we have one basis for $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$ by Theorem 2.2.8. Now that we have a description of the cells of the cellular decomposition of $(\mathbb{P}^2)^{[n]}$, we can find another basis for $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$. Since we have a cellular decomposition of $(\mathbb{P}^2)^{[n]}$, we can apply this proposition to $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$ with the Poincare Duality. Thus, we have a basis for $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$ with baisis elements

$$[\overline{C}_{\lambda,\mu,\nu}], \quad |\lambda| + |\mu| + |\nu| = n.$$

Chapter 3

Three Special Cases

In the above sections, we discovered two bases for the integral cohomology of $(\mathbb{P}^2)^{[n]}$. We found a basis for $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$ by looking at a cellular decomposition of $(\mathbb{P}^2)^{[n]}$. This is given by

$$\mathcal{A} = \left\{ [\overline{C}_{\lambda,\mu,\nu}] \;\;|\;\; |\lambda| + |\mu| + |\nu| = n \right\}. \tag{3.0.1}$$

Now, recall the line $\mathfrak{L} = \{T_2 = 0\}$ of $\mathbb{P}^2$ we described above. Since the class $[\mathfrak{L}]$ is an integral basis for $H^2(\mathbb{P}^2;\mathbb{Z})$, by Theorem 2.2.8 we have

$$\mathcal{B} = \left\{ \frac{1}{\mathfrak{z}_\lambda} \mathfrak{a}_{-\lambda}(1)\mathfrak{m}_{\mu,[\mathfrak{L}]}\mathfrak{a}_{-\nu}(x)|0\rangle \;\;|\;\; |\lambda| + |\mu| + |\nu| = n \right\} \tag{3.0.2}$$

is also an integral basis for $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$.

Our goal is to compare these two bases and eventually describe the transition matrix from $\mathcal{A}$ to $\mathcal{B}$. In this chapter, we look specifically at classes of the form $[\overline{C}_{\lambda,0,0}]$, $[\overline{C}_{0,\mu,0}]$ and $[\overline{C}_{0,0,\nu}]$.

3.1 Classes of the form $[\overline{C}_{0,\mu,0}]$

Here we want to describe classes of the form $[\overline{C}_{0,\mu,0}]$ where $\mu \vdash n$, and

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k).$$

Note we assume that $l(\mu) = k$. We intend to show $[\overline{C}_{0,\mu,0}] = \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$ where $\mathfrak{L} = \{T_2 = 0\}$. Now, we recall by (2.3.3),

$$C_{0,\mu,0} = \left\{ \xi \in (\mathbb{P}^2)^{[n]} | \lim_{t \to 0} \phi(t)\xi = \xi_{0,\mu,0} \right\}.$$

Since the subschemes corresponding to points in $W(0, n, 0)$ are contained in the affine plane Spec $\mathbb{C}[\frac{T_0}{T_1}, \frac{T_2}{T_1}]$, we may set $u = \frac{T_0}{T_1}$ and $v = \frac{T_2}{T_1}$. Then the invariant ideal of $\mathbb{C}[u, v]$ corresponding to the cell $C_{0,\mu,0}$, which we recall from (2.3.6), is

$$I_{\xi_{0,\mu,0}} = \left\langle v^{\mu_1}, uv^{\mu_2}, \ldots, u^{k-1}v^{\mu_k}, u^k \right\rangle.$$

Now, if we look at the operator $\mathfrak{m}_{\mu,\mathfrak{L}}$ we recall from Definition 2.2.5 that

$$\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle = [L^{\mu}\mathfrak{L}].$$

Then $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$ is the closure of the set of subschemes of the form

$$\xi_{x_1} + \xi_{x_2} + \cdots + \xi_{x_k}$$

where $x_i \in \mathfrak{L}$ are mutually distinct, $l(\xi_{x_i}) = \mu_i$ and $\mathrm{Supp}(\xi_{x_i}) = \{x_i\}$.

Next, we construct an open subset of $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$. Let $a_1, \ldots, a_k \in \mathbb{C}$ be distinct. Let

$$x_i = (a_i, 0)$$

where $i = 1, \ldots, k$. Define the subscheme ξ_{x_i} by the ideal in $\mathbb{C}[u, v]$

$$I_{\xi_{x_i}} = \left\langle (u - a_i) + \sum_{j=1}^{\mu_i - 1} b_{i,j}v^j, \quad v^{\mu_i} \right\rangle$$

where $b_{i,j} \in \mathbb{C}$. Define $\xi = \xi_{x_1} + \xi_{x_2} + \cdots + \xi_{x_k}$. Then the ideal corresponding to ξ is

$$I_{\xi} = \prod_{j=1}^{k} I_{\xi_{x_j}}. \tag{3.1.1}$$

and $\xi \in \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$. Let U be the subset of $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$ consisting of all such ξ. Then the dimension of U is $n = |\mu|$, and U is an open subset of $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$.

By [6], we know the dimension of $\overline{C}_{0,\mu,0}$ is n. Thus, $\overline{C}_{0,\mu,0}$ and $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$ have the same dimension and are both irreducible. Therefore, to show $[\overline{C}_{0,\mu,0}] = \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$, it suffices to show $U \subset \overline{C}_{0,\mu,0}$. Of course, this amounts to showing for any ξ in U we have $\xi \in C_{0,\mu,0}$, i.e.

$$\lim_{t\to 0} \phi(t)\xi = \xi_{0,\mu,0}.$$

We will show this by proving an equivalent statement:

$$\lim_{t\to 0} \phi(t) I_\xi = I_{\xi_{0,\mu,0}}.$$

First, however, we need the following lemma.

Lemma 3.1.1. *Let $a_{j+1}, \ldots, a_k \in \mathbb{C}$ be mutually distinct. Then the vector space $\{c_0 + c_1 u + \cdots + c_{k-j-1}u^{k-j-1} | c_i \in \mathbb{C}\}$ has a basis*

$$\left\{ \prod_{i=j+1, i\neq s}^{k} (u - a_i) \;\;|\;\; j+1 \leq s \leq k \right\}.$$

Proof. Clearly we have $k - j$ elements, so we must only check these are linearly indepedent. Suppose there exist $\lambda_{j+1}, \ldots, \lambda_k \in \mathbb{C}$ such that

$$\sum_{s=j+1}^{k} \lambda_s \prod_{i=j+1, i\neq s}^{k} (u - a_i) = 0.$$

For each $j + 1 \leq l \leq k$, let $u = a_l$. Then we have

$$\lambda_l \prod_{i=j+1, i\neq l}^{k} (a_l - a_i) = 0$$

Since the a_i are distinct, we must have $\lambda_l = 0$. ∎

Proposition 3.1.2. $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle = [\overline{C}_{0,\mu,0}]$

Proof. Let $I = \lim_{t\to 0}\phi(t)I_\xi$. As we stated above, our intention is to show $I = I_{\xi_{0,\mu,0}}$. We will prove the generators of $I_{\xi_{0,\mu,0}}$ are in $\lim_{t\to 0}\phi(t)I_\xi$. First, we show

$$u^k \in I.$$

From (3.1.1), we know

$$\prod_{i=1}^{k}\left[(u-a_i)+\sum_{j=1}^{\mu_i-1} b_{i,j}v^j\right] = u^k + f(u,v) \in I_\xi$$

where $f(u,v)$ consists of terms where the degree of u is at most $k-1$. We remark that when f consists of terms where the degree of u is less than k, we will write $\deg_u(f) < k$. Now we note $\phi(t)(u) = t^{w_0-w_1}u$ and $\phi(t)(v) = t^{w_2-w_1}v$. Thus,

$$t^{k(w_0-w_1)}u^k + f(t^{w_0-w_1}u, t^{w_2-w_1}v) \in \phi(t)I_\xi.$$

We can multiply by $t^{k(w_1-w_0)}$ to see

$$u^k + t^{k(w_1-w_0)}f(t^{w_0-w_1}u, t^{w_2-w_1}v) \in \phi(t)I_\xi.$$

Since $w_1 - w_0 > 0, w_2 - w_0 > 0, w_2 - w_1 > 0$ and the degree of u in $f(u,v)$ is less than k, we see the above polynomial will approach u^k as $t \to 0$. Thus, $u^k \in I$.

We finish proving Proposition 3.1.2 by showing

$$v^{\mu_{j+1}}u^j \in I \quad \text{for } 0 \le j \le k-1.$$

Let $0 \le j \le k-1$. Then, for each $j+1 \le s \le k$, we know the product

$$\prod_{i=1}^{j}\left[(u-a_i)+\sum_{l=1}^{\mu_i-1} b_{i,l}v^l\right]\prod_{i=j+1}^{s-1}\left[(u-a_i)+\sum_{l=1}^{\mu_i-1} b_{i,l}v^l\right]\cdot$$
$$v^{\mu_s}\prod_{i=s+1}^{k}\left[(u-a_i)+\sum_{l=1}^{\mu_i-1} b_{i,l}v^l\right]$$

is contained in I_ξ. If we expand out, we can rewrite these polynomials as

$$p_s(u,v)=\left[\prod_{i=1}^{j}(u-a_i)\prod_{\substack{i=j+1\\ i\neq s}}^{k}(u-a_i)+v\cdot f_s(u,v)\right]v^{\mu_s}$$

for some polynomial $f_s(u,v)$. By Lemma 3.1.1 there must exsit

$$\lambda_{j+1},\ldots,\lambda_k\in\mathbb{C}$$

such that $\displaystyle\sum_{s=j+1}^{k}\lambda_s\prod_{i=j+1,i\neq s}^{k}(u-a_i)=1$. Now we see

$$\begin{aligned} p(u,v):=\sum_{s=j+1}^{k}\lambda_s v^{\mu_{j+1}-\mu_s}p_s(u,v) \qquad &= \left[\prod_{i=1}^{j}(u-a_i)+v\cdot f(u,v)\right]v^{\mu_{j+1}}\\ &= \left[u^j+g(u)+v\cdot f(u,v)\right]v^{\mu_{j+1}}\in I_\xi \end{aligned}$$

where $f(u,v)=\displaystyle\sum_{s=j+1}^{k}\lambda_s f_s(u,v)$ and $g(u)$ has degree less than j.

Applying $\phi(t)$ to $p(u,v)$ and then multiplying by $t^{j(w_1-w_0)+\mu_{j+1}(w_0-w_2)}$, we have

$$t^{j(w_1-w_0)+\mu_{j+1}(w_0-w_2)}\phi(t)p(u,v)=$$

$$\left[u^j+t^{j(w_1-w_0)}g(t^{w_0-w_1}u)+t^{j(w_1-w_0)+(w_2-w_0)}v\cdot f(t^{w_0-w_1}u,t^{w_2-w_0}v)\right]v^{\mu_{j+1}}$$

Since $g(u)$ has degree less than j, $t^{j(w_1-w_0)}g(t^{w_0-w_1}u)$ approaches 0 as $t\to 0$. Assuming $w_2-w_0>>w_1-w_0$, we see that $t^{j(w_1-w_0)+(w_2-w_0)}v\cdot f(t^{w_0-w_1}u,t^{w_2-w_0}v)$ approaches 0 as $t\to 0$. Thus we have shown

$$u^jv^{\mu_{j+1}}\in\lim_{t\to 0}\phi(t)I_\xi.$$

□

3.2 Classes of the form $[\overline{C}_{0,0,\nu}]$

In this section, we want to describe classes of the form $[\overline{C}_{0,0,\nu}]$ where $\nu \vdash n$, and

$$\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k).$$

While we cannot determine the class $[\overline{C}_{0,0,\nu}]$ exactly in terms one operator, we can show

$$[\overline{C}_{0,0,\nu}] \equiv \frac{(-1)^{|\nu|+l(\nu)}\tilde{f}_\nu}{\mathfrak{z}_\nu}\mathfrak{a}_{-\nu}(1)|0\rangle \quad (\bmod \mathcal{I}^{[n]}).$$

where $\mathcal{I}^{[n]}$ will be defined below and $\tilde{f}_\nu$ is a positive integer. Now, we recall by (2.3.4),

$$C_{0,0,\nu} = \left\{\xi \in (\mathbb{P}^2)^{[n]} | \lim_{t\to 0}\phi(t)\xi = \xi_{0,0,\nu}\right\}.$$

Since the subschemes corresponding to points in $W(0,0,n)$ are contained in the affine plane Spec $\mathbb{C}[\frac{T_1}{T_2}, \frac{T_0}{T_2}]$, we may set $u = \frac{T_1}{T_2}$ and $v = \frac{T_0}{T_2}$. Then the ideal of $\mathbb{C}[u,v]$ corresponding to the cell $C_{0,0,\nu}$, which we recall from (3.2.1), is

$$I_{\xi_{0,0,\nu}} = \left\langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^{k-1}v^{\nu_k}, u^k\right\rangle.$$

Our first goal is to show what generic elements of $\overline{C}_{0,0,\nu}$ look like. Consider subschemes $\xi \in X^{[n]}$ of the form

$$\xi = \sum_{i=1}^{k}\sum_{j=0}^{\nu_i}(a_i, b_{i,j}) \tag{3.2.1}$$

where $a_i, b_{i,j}$ are all distinct. For any such ξ, we see the corresponding ideal is given by

$$I_\xi = \prod_{i=1}^{k}\left\langle (u-a_i), \prod_{j=1}^{\nu_i}(v-b_{i,j})\right\rangle$$

We will see the subschemes of (3.2.1) will be the generic elements of $\overline{C}_{0,0,\nu}$. First, of course, we would like to show these subschemes are in $C_{0,0,\nu}$. We accomplish this with the following lemma.

Lemma 3.2.1. $\lim_{t\to 0}\phi(t)I_\xi = I_{\xi_{0,0,\nu}}$.

Proof. We proceed in a similar fashion as Lemma 3.1.2. Now we note

$$\phi(t)(u) = t^{w_1-w_2}u \quad \text{and} \quad \phi(t)(v) = t^{w_0-w_2}v.$$

First we show $u^k \in \lim_{t\to 0}\phi(t)I_\xi$. We know

$$\prod_{i=1}^{k}(u-a_i) = u^k + f(u) \in I_\xi$$

where $\deg_u(f) < k$. Thus,

$$t^{k(w_1-w_2)}u^k + f(t^{w_1-w_2}u) \in \phi(t)I_\xi.$$

We can multiply by $t^{k(w_2-w_1)}$ to see

$$u^k + t^{k(w_2-w_1)}f(t^{w_1-w_2}u) \in \phi(t)I_\xi.$$

Since $w_2 - w_1 > 0$ and the degree of u in $f(u,v)$ is less than k, we see the above polynomial will approach u^k as $t\to 0$. Thus, $u^k \in \lim_{t\to 0}\phi(t)I_\xi$.

We finish proving Lemma 3.2.1 by showing

$$v^{\nu_{j+1}}u^j \in \lim_{t\to 0}\phi(t)I_\xi$$

for $0 \le j \le k-1$. Let $0 \le j \le k-1$. Then, for each $j+1 \le s \le k$, we have

$$\prod_{i=1}^{j}(u-a_i)\prod_{i=j+1}^{s-1}(u-a_i)\prod_{i=1}^{\nu_s}(v-b_{i,s})\prod_{i=s+1}^{k}(u-a_i)\in I_\xi.$$

If we expand out, we can rewrite these polynomials as

$$p_s(u,v)=\left[\prod_{i=1}^{j}(u-a_i)\prod_{i=j+1}^{s-1}(u-a_i)\prod_{i=s+1}^{k}(u-a_i)\right][v^{\nu_s}+g_s(v)]$$

where $g_s(v)$ has degree at most ν_s-1. By Lemma 3.1.1, there must exsit

$$\lambda_{j+1},\ldots,\lambda_k\in\mathbb{C}$$

such that $\sum_{s=j+1}^{k}\lambda_s\prod_{i=j+1,i\neq s}^{k}(u-a_i)=1$. Recalling $\nu_{j+1}\geq\nu_{j+2}\geq\ldots\geq\nu_k$, we see

$$p(u,v):=\sum_{s=j+1}^{k}\lambda_s v^{\nu_{j+1}-\nu_s}p_s(u,v)=[u^j+f(u)]\,[v^{\nu_{j+1}}+g(u,v)]\in I_\xi$$

where $g(u,v)=\displaystyle\sum_{s=j+1}^{k}\left[\prod_{i=j+1}^{s-1}(u-a_i)\prod_{i=s+1}^{k}(u-a_i)\lambda_s v^{\nu_{j+1}-\nu_s}g_s(v)\right]$ and thus $g(u,v)$ consists of terms where the degree of v is at most $\nu_{j+1}-1$. Also note $f(u)$ has degree at most $j-1$.

Applying $\phi(t)$ to $p(u,v)$ and then multiplying by $t^{j(w_2-w_1)+\nu_{j+1}(w_2-w_0)}$, we have

$$\left[u^j+t^{j(w_2-w_1)}f(t^{w_1-w_2}u)\right]\left[v^{\nu_{j+1}}+t^{\nu_{j+1}(w_2-w_0)}g(t^{w_1-w_2}u,t^{w_0-w_2}v)\right]\in\phi(t)I_\xi$$

Now, as $t\to0$, clearly $t^{j(w_2-w_1)}f(t^{w_1-w_2}u)\to0$. Recall, we have assumed that $w_0<w_1<w_2$. Thus, we can choose w_0,w_1,w_2 such that $w_2-w_0>>w_1-w_2$, then we assure $t^{\nu_{j+1}(w_2-w_0)}g(t^{w_1-w_2}u,t^{w_0-w_2}v)\to0$ as $t\to0$. Thus, $v^{\nu_{j+1}}u^j\in\lim_{t\to0}\phi(t)I_\xi$. □

Now we are prepared to present a proof of our earlier claim that subschemes of the form (3.2.1) are the generic elements of $\overline{C}_{0,0,\nu}$.

Proposition 3.2.2. *Suppose $\nu = (\nu_1 \geq \cdots \geq \nu_k)$ is a partition of n. Then generic elements in $\overline{C}_{0,0,\nu}$ are of the form (3.2.1), i.e.*

$$\sum_{i=1}^{\nu_1}(a_1, b_{i,1}) + \cdots + \sum_{i=1}^{\nu_k}(a_k, b_{i,k}) \in (\mathbb{C}^2)^{[n]} \subset (\mathbb{P}^2)^{[n]} \tag{3.2.2}$$

where the $a_1, b_{1,1}, \ldots, a_k, b_{1,k}, \ldots, b_{\nu_k,k}$ are all distinct.

Proof. By Proposition 4.1 in [6], we know the dimension of $C_{0,0,\nu}$ is $|\nu| + l(\nu)$. As a cell, $C_{0,0,\nu}$ is isomorphic to $\mathbb{C}^{|\nu|+l(\nu)}$, and thus $C_{0,0,\nu}$ is irreducible. In addition, Lemma 3.2.1 showed that elements of the form 3.2.2 are contained in $C_{0,0,\nu}$. Now, one will notice there are $l(\nu)$ distinct a_i and $|\nu|$ distinct $b_{i,j}$. Thus, the subvariety made up of subschemes in the form of (3.2.2) has dimension $|\nu| + l(\nu)$. Thus, we conclude the generic elements of $\overline{C}_{0,0,\nu}$ are of the form (3.2.2). □

It remains to show the claim we made at the beginning of the section. For this, we will need the help of several lemmas. First, let us encapsulate our progress. By Theorem 2.2.8, we can write $\left[\overline{C}_{0,0,\nu}\right]$ as an integral linear combination of

$$\frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\overline{\nu}}(x)\mathfrak{m}_{\mu,[\mathfrak{L}]}|0\rangle$$

where $\lambda, \overline{\nu}, \mu$ are partitions such that $|\lambda| + |\overline{\nu}| + |\mu| = n$. Let P_ν be the part of the linear combination in which either $\overline{\nu} \neq \emptyset$ or $\mu \neq \emptyset$. Then

$$\left[\overline{C}_{0,0,\nu}\right] = \sum_{\lambda, |\lambda| = |\nu|} f_\lambda \frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)|0\rangle + P_\nu \tag{3.2.3}$$

for some $f_\lambda \in \mathbb{Z}$. Then we have the following lemma.

Lemma 3.2.3. *Suppose $f_\lambda \in \mathbb{Z}$ are the coefficients in (3.2.3). If $f_\lambda \neq 0$, then $l(\lambda) = l(\nu)$.*

Proof. Again, by [6], the dimension of $\overline{C}_{0,0,\nu}$ is $|\nu| + l(\nu)$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$, then

$$\begin{aligned}\dim \mathfrak{a}_{-\lambda}(1)|0\rangle &= \sum_{i=1}^{r} (2 + (\lambda_i - 1)) \\ &= \sum_{i=1}^{r} (\lambda_i + 1) \\ &= |\lambda| + r\end{aligned}$$

since $\lambda_i - 1$ is the dimension of the punctured Hilbert scheme $M_{\lambda_i}(x)$. Now ν and λ are both partitions of n, so if $f_\lambda \neq 0$, then we must have $r = l(\nu)$. □

We will also need the following lemma concerning commutation relations of the Heisenberg operators. We will see this is useful in subsequent lemmas.

Lemma 3.2.4. *Suppose* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ *is a partition of* n. *Then*

$$\left\langle \mathfrak{a}_{-\lambda}(x)|0\rangle, \quad \frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)|0\rangle \right\rangle = (-1)^{n+k}.$$

Proof. We procede by induction on k. When $k = 1$, $\lambda = (\lambda_1) = (n)$ and $\mathfrak{z}_\lambda = n$. We compute the inner product.

$$\begin{aligned}\left\langle \mathfrak{a}_{-n}(x)|0\rangle, \quad \frac{1}{n}\mathfrak{a}_{-n}(1)|0\rangle \right\rangle &= \frac{(-1)^n}{n} \langle |0\rangle, \quad \mathfrak{a}_n(x)\mathfrak{a}_{-n}(1)|0\rangle\rangle \\ &= \frac{(-1)^n}{n} \langle |0\rangle, \quad ((-n) \cdot \mathrm{Id}_{\mathbb{H}_X} + \mathfrak{a}_{-n}(1)\mathfrak{a}_n(x))|0\rangle\rangle \\ &= (-1)^{n+1}.\end{aligned}$$

Fix $k > 1$. Suppose we have $\lambda_1 = \lambda_2 = \ldots = \lambda_r > \lambda_{r+1}$. Let $\tilde{\lambda} = (\lambda_{r+1}, \ldots, \lambda_k)$. Then $\mathfrak{z}_\lambda = \mathfrak{z}_{\tilde{\lambda}}(\lambda_1)^r r!$. Then we have

$$\left\langle [\mathfrak{a}_{-\lambda_1}(x)]^r \mathfrak{a}_{-\tilde{\lambda}}(x)|0\rangle, \quad \frac{1}{\mathfrak{z}_{\tilde{\lambda}}(\lambda_1)^r r!}[\mathfrak{a}_{-\lambda_1}(1)]^r \mathfrak{a}_{-\tilde{\lambda}}(1)|0\rangle \right\rangle$$

which we can rewrite as

$$\frac{(-1)^{\lambda_1}}{\lambda_1 r}\left\langle [\mathfrak{a}_{-\lambda_1}(x)]^{r-1} \mathfrak{a}_{-\tilde{\lambda}}(x)|0\rangle, \quad \frac{1}{\mathfrak{z}_{\tilde{\lambda}}(\lambda_1)^{r-1}(r-1)!}\mathfrak{a}_{\lambda_1}(x)[\mathfrak{a}_{-\lambda_1}(1)]^r \mathfrak{a}_{-\tilde{\lambda}}(1)|0\rangle \right\rangle .$$

After r iterations of the commutation relation, we have

$$\frac{(-1)^{\lambda_1}}{\lambda_1 r}\left\langle [\mathfrak{a}_{-\lambda_1}(x)]^{r-1} \mathfrak{a}_{-\tilde{\lambda}}(x)|0\rangle, \quad -r\lambda_1 \cdot \frac{1}{\mathfrak{z}_{\tilde{\lambda}}(\lambda_1)^{r-1}(r-1)!}[\mathfrak{a}_{-\lambda_1}(1)]^{r-1} \mathfrak{a}_{-\tilde{\lambda}}(1)|0\rangle \right\rangle$$

which, by induction, is

$$(-1)^{\lambda_1+1} \cdot (-1)^{(n-\lambda_1)+(k-1)} = (-1)^{n+k}.$$

□

Let us now define $\mathcal{I}^{[n]}$ which we referred to earlier this section.

Definition 3.2.5. Let us define $\mathcal{I}^{[n]} \subset H^*((P^2)^{[n]};\mathbb{Z})$ to be the subgroup generated by basis elements of the form

$$\frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\nu}(x)\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$$

where $|\lambda| + |\nu| + |\mu| = n$ and either $\nu \neq \emptyset$ or $\mu \neq \emptyset$.

We can show that this subgroup $\mathcal{I}^{[n]}$ is, in fact, an ideal of $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$.

Lemma 3.2.6. *$\mathcal{I}^{[n]}$ is an ideal of $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$.*

Proof. We have the inclusion map $\iota : \mathbb{C}^2 \to \mathbb{P}^2$ and this induces the map

$$\iota_n : (\mathbb{C}^2)^{[n]} \to (\mathbb{P}^2)^{[n]}$$

and the ring homomorphism

$$\iota_n^* : H^*((\mathbb{P}^2)^{[n]}; \mathbb{Z}) \to H^*((\mathbb{C}^2)^{[n]}; \mathbb{Z})$$

for $n \geq 0$. We complete the proof by showing $\mathcal{I}^{[n]} = \ker \iota_n^*$, and thus an ideal of $H^*((\mathbb{P}^2)^{[n]}; \mathbb{Z})$.

Let

$$\iota^* : H^*((\mathbb{P}^2); \mathbb{Z}) \to H^*((\mathbb{C}^2); \mathbb{Z})$$

be the ring homomorphism induced from ι. By Lemma 4.1 from [17], we know

$$\iota_n^*(\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle) = \mathfrak{a}_{-n_1}(\iota^*\alpha_1) \cdots \mathfrak{a}_{-n_k}(\iota^*\alpha_k)|0\rangle$$

where $k \geq 0, n_1, \ldots, n_k > 0, n_1 + \cdots + n_k = n$ and $\alpha_1, \ldots, \alpha_k \in H^*(\mathbb{P}^2; \mathbb{Z})$.

Since $H^*(\mathbb{C}^2; \mathbb{Z}) \cong H^0(\mathbb{C}^2; \mathbb{Z}) \cong \mathbb{Z}$, we see the basis elements of $H^*(\mathbb{P}^2; \mathbb{Z})$, $1_{\mathbb{P}^2}, [\mathfrak{L}], [x]$ have images of $1_{\mathbb{C}^2}, 0, 0$, respectively, under ι^*. Thus, by the paragraph above, $\mathcal{I}^{[n]} \subset \ker \iota_n^*$. Also we see

$$\iota_n^* \left(\frac{1}{\mathfrak{z}_\lambda} \mathfrak{a}_{-\lambda}(1)|0\rangle \right) = \frac{1}{\mathfrak{z}_\lambda} \mathfrak{a}_{-\lambda}(1_{\mathbb{C}^2})|0\rangle \in H^*((\mathbb{C}^2)^{[n]}; \mathbb{Z}).$$

Now, we see in [15] that $\left\{ \frac{1}{\mathfrak{z}_\lambda} \mathfrak{a}_{-\lambda}(1_{\mathbb{C}^2})|0\rangle \big| |\lambda| = n \right\}$ is a basis for $H^*((\mathbb{C}^2)^{[n]}; \mathbb{Z})$. Thus, we must have $\mathcal{I}^{[n]} = \ker \iota_n^*$. □

Note that ι_n^* is surjective and thus, $H^*((\mathbb{P}^2)^{[n]}; \mathbb{Z})/\mathcal{I}^{[n]} \cong H^*((\mathbb{C}^2)^{[n]}; \mathbb{Z})$. With one final lemma, we will have the necessary facts in place to give our main result for this subsection.

Lemma 3.2.7. *Suppose ν and λ are partitions of n. Then we have the following two results.*

(i) If the intersection pairing between the cohomology classes $\mathfrak{a}_{-\lambda}(x)|0\rangle$ and $[\overline{C}_{0,0,\nu}]$ is nonzero, then $\lambda = \nu$.

(ii) The intersection pairing between $\mathfrak{a}_{-\nu}(x)|0\rangle$ and $[\overline{C}_{0,0,\nu}]$ is a positive integer.

Proof. (i) Recall $P_{\overline{\nu},\mu}$ is the part of the linear combination in (3.2.3) in which either $\overline{\nu} \neq \emptyset$ or $\mu \neq \emptyset$. From (3.2.3) and Lemma 3.2.4, we can calculate the intersection pairing.

$$\begin{aligned} \left\langle \mathfrak{a}_{-\lambda}(x)|0\rangle, \left[\overline{C}_{0,0,\nu}\right] \right\rangle &= \left\langle \mathfrak{a}_{-\lambda}(x)|0\rangle, \sum_{\tilde{\lambda}, |\tilde{\lambda}|=|\nu|} f_{\tilde{\lambda}} \frac{1}{\mathfrak{z}_{\tilde{\lambda}}} \mathfrak{a}_{-\tilde{\lambda}}(1)|0\rangle + P_{\overline{\nu},\mu} \right\rangle \\ &= f_\lambda \left\langle \mathfrak{a}_{-\lambda}(x)|0\rangle, \frac{1}{\mathfrak{z}_\lambda} \mathfrak{a}_{-\lambda}(1)|0\rangle \right\rangle \\ &= (-1)^{|\lambda|+l(\lambda)} f_\lambda. \end{aligned} \quad (3.2.4)$$

Suppose $\left\langle \mathfrak{a}_{-\lambda}(x)|0\rangle, \left[\overline{C}_{0,0,\nu}\right] \right\rangle \neq 0$. Then $f_\lambda \neq 0$ and by Lemma 3.2.3, we have $l(\nu) = l(\lambda)$. Let $k = l(\nu) = l(\lambda)$. Thus, we can write

$$\nu = (\nu_1 \geq \cdots \geq \nu_k) \text{ and } \lambda = (\lambda_1 \geq \cdots \geq \lambda_k).$$

Now, let $\xi \in \mathfrak{a}_{-\lambda}(x)|0\rangle \cap \overline{C}_{0,0,\nu}$. Fix $x_1 = (a_1, b_1), \ldots, x_r = (a_k, b_k) \in \mathbb{C}^2$ with the a_i distinct. Since $\xi \in \mathfrak{a}_{-\lambda}(x)|0\rangle$ and

$$\mathfrak{a}_{-\lambda}(x)|0\rangle = \mathfrak{a}_{-\lambda_1}(x_1)\cdots\mathfrak{a}_{-\lambda_r}(x_r)|0\rangle = \{\xi_1 + \cdots + \xi_k | \xi_i \in M_{\lambda_i}(x_i)\}$$

we can write $\xi = \xi_1 + \cdots + \xi_k$ where $\xi_i \in M_{\lambda_i}(x_i)$. On the other hand, since $\xi \in \overline{C}_{0,0,\nu}$, then $\xi = \lim_{t\to 0} \eta(t)$ where

$$\eta(t) = \eta_1(t) + \cdots + \eta_k(t)$$

each $\eta_i(t)$ being on the line $x = a_i(t)$.

For each $a_i(t)$ we must have $\lim_{t\to 0} a_i(t) = a_j$ for some a_j in $a_1, \ldots, a_k$, and thus $\lim_{t\to 0} \eta_i(t) = \xi_j$. By rearranging indices, we can say $\lim_{t\to 0} \eta_i(t) = \xi_i$ for $i = 1, \ldots, k$. Since λ and ν are both partitions of n, we must have $\lambda = \nu$. This proves (i).

(ii) Without loss of generality, we may assume $l(\nu) = 1$ (so $\nu = (n)$) and $x = (a, 0)$. Then $\mathfrak{a}_{-\nu}(x)|0\rangle = M_n(x)$.

Let ξ_0 be defined by the ideal $\langle u - a, v^n\rangle$. Clearly, $\xi_0 \in M_n(x)$. Notice that

$$\begin{aligned}\lim_{t\to 0} \phi(t) I_{\xi_0} &= \lim_{t\to 0} \langle t^{w_1 - w_2} u - a, t^{(w_0 - w_2)^n} v^n\rangle \\ &= \lim_{t\to 0} \langle u - t^{w_2 - w_1} a, v^n\rangle \\ &= I_{\xi_{0,0,\nu}}.\end{aligned}$$

Thus, $\xi_0 \in \overline{C}_{0,0,\nu}$ from which we conclude $\xi_0 \in M_n(x) \cap \overline{C}_{0,0,\nu}$.

Finally, we show ξ_0 is the only element in the intersection. Suppose

$$\xi \in M_n(x) \cap \overline{C}_{0,0,\nu}.$$

Then $\mathrm{supp}(\xi) = \{x\} = \{(a, 0)\}$. By Proposition 3.2.2, $\xi = \lim_{t\to 0} \xi(t)$ where $\xi(t) \in C_{0,0,\nu}$ is given by the ideal

$$I_{\xi(t)} = \left\langle u - a(t), \prod_{i=1}^{n} (v - b_i(t)) \right\rangle$$

and $a(t), b_1(t), \ldots, b_n(t)$ are distinct. Since $\mathrm{supp}(\xi) = \{(a, 0)\}$, we have

$$\lim_{t\to 0} a(t) = a \quad \text{and} \quad \lim_{t\to 0} b_i(t) = 0$$

for $i = 1 \ldots n$, and thus $\langle u - a, v^n\rangle \subset \lim_{t\to 0} I_{\xi(t)}$. But we also know $\langle u - a, v^n\rangle$ has length n. Therefore,

$$I_{\xi_0} = \langle u - a, v^n\rangle = \lim_{t\to 0} I_{\xi(t)} = I_\xi.$$

This shows the intersection of $M_n(x)$ and $\overline{C}_{0,0,\nu}$ is $\{\xi_0\}$ and so their intersection number is a positive integer. $\square$

Proposition 3.2.8. *For some positive integer $\tilde{f}_\nu$, we have*

$$[\overline{C}_{0,0,\nu}] \equiv \frac{(-1)^{|\nu|+l(\nu)}\tilde{f}_\nu}{\mathfrak{z}_\nu}\mathfrak{a}_{-\nu}(1)|0\rangle \pmod{\mathcal{I}^{[\mathrm{n}]}}.$$

Proof. Recall from (3.2.3)

$$[\overline{C}_{0,0,\nu}] = \sum_{\lambda,|\lambda|=|\nu|} f_\lambda \frac{1}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)|0\rangle + P_\nu.$$

By Lemma 3.2.7 (i), $f_\lambda = 0$ if $\lambda \neq \nu$. Also, by Lemma 3.2.7 (ii),

$$f_\nu = (-1)^{|\nu|+l(\nu)}\tilde{f}_\nu$$

where $\tilde{f}_\nu$ is a positive integer. $\square$

3.3 Classes of the form $[\overline{C}_{\lambda,0,0}]$

In this section, we want to describe classes of the form $[\overline{C}_{\lambda,0,0}]$ where $\lambda \vdash n$, and

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k).$$

Note we assume that $l(\lambda) = k$.

We remind the reader of some facts. Recall that by (2.3.2),

$$C_{\lambda,0,0} = \left\{\xi \in (\mathbb{P}^2)^{[n]} | \lim_{t\to 0}\phi(t)\xi = \xi_{\lambda,0,0}\right\}.$$

Since the subschemes corresponding to points in $W(n,0,0)$ are contained in the affine plane Spec $\mathbb{C}[\frac{T_1}{T_0},\frac{T_2}{T_0}]$, we may set $u = \frac{T_1}{T_0}$ and $v = \frac{T_2}{T_0}$. Then the invariant ideal of $\mathbb{C}[u,v]$ corresponding to the cell $C_{\lambda,0,0}$, which we recall from (2.3.5), is

$$I_{\xi_{\lambda,0,0}} = \left\langle v^{\lambda_1}, uv^{\lambda_2}, \ldots, u^{k-1}v^{\lambda_k}, u^k\right\rangle.$$

Once again, we cannot determine the class $[\overline{C}_{\lambda,0,0}]$ exactly. However, we can show $[\overline{C}_{\lambda,0,0}]$ will be a linear combination of classes of the form $\mathfrak{a}_{-\nu}(x)|0\rangle$ where $|\nu| = |\lambda|$. In fact, we can improve upon this. First, let us state a definition so that we may order partitions. The following definition comes from [19].

Definition 3.3.1. Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ and $\tilde{\lambda} = (\tilde{\lambda_1} \geq \tilde{\lambda_2} \geq \cdots \geq \tilde{\lambda_l})$ are two partitions. Then $\lambda < \tilde{\lambda}$ if one of the three following conditions hold:

(i) $|\lambda| < |\tilde{\lambda}|$

(ii) $|\lambda| = |\tilde{\lambda}|$ and $l(\lambda) < l(\tilde{\lambda})$

(iii) $|\lambda| = |\tilde{\lambda}|$, $l(\lambda) = l(\tilde{\lambda})$, and $\lambda_r < \tilde{\lambda_r}$ but $\lambda_i = \tilde{\lambda_i}$ for all $i = r+1, \ldots, k$.

Note that this gives a total ordering on the set of partitions.

We will eventually show

$$[\overline{C}_{\lambda,0,0}] = \mathfrak{a}_{-\lambda}(x)|0\rangle + \sum_{\substack{l(\nu)=l(\lambda) \\ |\nu|=|\lambda| \\ \nu>\lambda}} e_{\lambda,\nu}\mathfrak{a}_{-\nu}(x)|0\rangle$$

where $e_{\nu,\lambda} \in \mathbb{Z}$.

Following Theorem 2.2.8, we can write $[\overline{C}_{\lambda,0,0}]$ as an integral linear combination of

$$\frac{1}{\mathfrak{z}_{\overline{\lambda}}}\mathfrak{a}_{-\overline{\lambda}}(1)\mathfrak{a}_{-\nu}(x)\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$$

where $\overline{\lambda}, \nu, \mu$ are partitions such that $|\overline{\lambda}| + |\nu| + |\mu| = n$. As we did in the previous section, let us call P_λ be the part of the linear combination in which either $\overline{\lambda} \neq \emptyset$ or $\mu \neq \emptyset$. Then

$$[\overline{C}_{\lambda,0,0}] = \sum_{\nu,|\nu|=|\lambda|} g_\nu \mathfrak{a}_{-\nu}(x)|0\rangle + P_\lambda \tag{3.3.1}$$

for some $g_\nu \in \mathbb{Z}$. Then we have the following observation which we will state as a lemma.

Lemma 3.3.2. $P_\lambda = 0$.

Proof. Any element in $C_{\lambda,0,0}$ is supported at a point. However, elements contained in $\frac{1}{\mathfrak{z}_{\overline{\lambda}}}\mathfrak{a}_{-\overline{\lambda}}(1)|0\rangle$ are supported on all of $\mathbb{P}^2$. Elements contained in $\mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$ are supported on a line. Thus, we must have $P_\lambda = 0$. □

In addition, we can cut out some more terms from the linear combination with the following observation. By [6], $\dim C_{\lambda,0,0} = |\lambda| - l(\lambda)$. Also, we notice that

$$\dim \mathfrak{a}_{-\nu}(x) = \sum_{i}^{l(\nu)} (\nu_i - 1) = |\nu| - l(\nu).$$

Thus, if $g_\nu \neq 0$, we must have $l(\nu) = l(\lambda)$. With Lemma 3.3.2, we can now say

$$[\overline{C}_{\lambda,0,0}] = \sum_{\substack{\nu,|\nu|=|\lambda| \\ l(\nu)=l(\lambda)}} g_\nu \mathfrak{a}_{-\nu}(x)|0\rangle.$$

Since we will eventually want to consider all λ such that $|\lambda| = n$, let us say $g_\nu = e_{\lambda,\nu}$. Thus we have

$$[\overline{C}_{\lambda,0,0}] = \sum_{\substack{\nu,|\nu|=|\lambda| \\ l(\nu)=l(\lambda)}} e_{\lambda,\nu} \mathfrak{a}_{-\nu}(x)|0\rangle. \tag{3.3.2}$$

Notice that we, seemingly, have very little left to prove. We must show only $e_{\lambda,\nu} = 0$ when $\lambda > \nu$ and $e_{\lambda,\lambda} = 1$.

The strategy is to find these $e_{\lambda,\nu}$ by intersecting the classes $[\overline{C}_{\lambda,0,0}]$ with the cohomology classes of a constructed subspace in the hopes we can isolate each coefficient. Let us define these subspaces here.

Definition 3.3.3. Suppose $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$ is a partition of n. Then let us define the subspace

$$W_\nu = \left\{ \sum_{i=1}^{\nu_1} (a_1, b_{1,i}) + \cdots + \sum_{i=1}^{\nu_k} (a_k, b_{k,i}) \right\} \subset (\mathbb{C}^2)^{[n]} \subset (\mathbb{P}^2)^{[n]} \tag{3.3.3}$$

where $a_1, \ldots, a_k, b_{1,1}, \ldots, b_{k,\nu_k}$ are all distinct. In this definition, we note that $\mathbb{C}^2$ is centered at P_0.

Remark 3.3.4. We recall from Proposition 3.2.2 that the elements of W_ν look like the generic elements of $\overline{C}_{0,0,\nu}$. Of course, the only difference between the two is that in the case of W_ν, $\mathbb{C}^2$ is centered at P_0 and in the case of $C_{0,0,\nu}$, $\mathbb{C}^2$ is centered at P_2. Now, it should be clear that

$$\overline{W}_\nu \cong \overline{C}_{0,0,\nu}.$$

Thus, by Proposition 3.2.8, we have

$$[\overline{W}_\nu] \equiv \frac{(-1)^{|\nu|+l(\nu)} \tilde{f}_\nu}{\mathfrak{z}_\nu} \mathfrak{a}_{-\nu}(1)|0\rangle \pmod{\mathcal{I}^{[\mathrm{n}]}} \tag{3.3.4}$$

where $\tilde{f}_\nu$ is a positive integer.

Claim 3.3.5. Suppose $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ and

$$\lim_{t \to 0} \phi(t)\xi = \xi_{\nu,0,0}$$

where λ and ν are both partitions of n. Then $\lambda < \nu$.

Recall here, of course, that $\xi_{\nu,0,0}$ is the subscheme associated with the partiton ν and the ideal $I_{\xi_{\nu,0,0}}$. In addition, we recall the meaning of $\lambda < \nu$ is explained in Definition 3.3.1. To prove this claim, we must go through a bit of backstory, including some rather messy techinical proofs. To be sure, the method which we use again and again is of some interest. However, the actual tedium of the proofs can probably be foregone after the reader acquaints himself with said method.

Lemma 3.3.6. *Suppose ξ is supported at $(0,0)$. Then there exist $n_1, n_2 > 0$ such that $u^{n_1}, v^{n_2} \in I_\xi \subset \mathbb{C}[u,v]$.*

Proof. If ξ is supported at $(0,0)$, then $Z(I_\xi) = \{(0,0)\}$. From Nullstellensatz, we know $I(Z(I_\xi)) = \sqrt{I_\xi}$. Also, we have $I(\{(0,0)\}) = \langle u, v\rangle$. Thus, $u, v \in \sqrt{I_\xi}$ and so there exist some $n_1, n_2 > 0$ such that $u^{n_1}, v^{n_2} \in I_\xi$. □

Lemma 3.3.7. *Suppose $f(u,v) = \left[u^k + vh(u,v) + u^{k+1}g(u)\right] v^l$ for some $k, l \geq 0$. Then*

$$\lim_{t\to 0} t^{-k(w_1-w_0)-l(w_2-w_0)}\phi(t)f(u,v) = u^k v^l.$$

Proof. We see that $t^{-k(w_1-w_0)-l(w_2-w_0)}\phi(t)f(u,v)$ is equal to

$$\left[u^k + t^{(w_2-w_0)-k(w_1-w_0)}vh(t^{w_1-w_0}u, t^{w_2-w_0}v) + t^{w_1-w_0}u^{k+1}g(t^{w_1-w_0}u)\right] v^l.$$

As $t \to 0$, we see $t^{w_1-w_0}u^{k+1}g(t^{w_1-w_0}u) \to 0$ since $w_1 > w_0$. Furthermore, we can assume $w_2 - w_0 >> w_1 - w_0$. Thus $t^{(w_2-w_0)-k(w_1-w_0)}vh(t^{w_1-w_0}u, t^{w_2-w_0}v) \to 0$ as $t \to 0$. This shows $\lim_{t\to 0} t^{-k(w_1-w_0)-l(w_2-w_0)}\phi(t)f(u,v) = u^k v^l$. □

Here, let us recall that if we have some $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$, then we must have some subschemes ξ_w such that $\xi_w \in C_{\lambda,0,0}$ and

$$\lim_{w\to 0} \xi_w = \xi.$$

Lemma 3.3.8. *Let $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ and ξ_w be the subschemes described above so that $\xi_w \in C_{\lambda,0,0}$ and $\lim_{w\to 0} \xi_w = \xi$. Let $\lim_{t\to 0} \phi(t)\xi = \xi_{\nu,0,0}$. If $l(\lambda) = l(\nu) = k$, then the ideal associated to the subscheme ξ_w, I_{ξ_w}, will contain a polynomial*

$$P(u, v; w) = u^k + vf(u, v; w) \in \mathbb{C}[u, v]$$

where the degree of u in $f(u, v; w)$ is less than k and the $P(u, v; w)$ converge as $w \to 0$.

Proof. Suppose $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$. From (2.3.5), we have

$$I_{\xi_{\nu,0,0}} = \langle v^{\nu_1}, v^{\nu_2}u, \ldots, v^{\nu_k}u^{k-1}, u^k\rangle.$$

Therefore $u^k \in I_{\xi_{\nu,0,0}}$. Since $\lim_{t\to 0} \phi(t)I_\xi = I_{\xi_{\nu,0,0}}$, we have a polynomial of the form

$$Q(u, v) = u^k + vh(u, v) + u^{k+1}g(u) \in I_\xi$$

by Lemma 3.3.7. Since $\lim_{w\to 0} I_{\xi_w} = I_\xi$, for w sufficiently close to 0, there must exist polynomials

$$Q(u, v; w) = \rho(w)u^k + vh(u, v; w) + u^{k+1}g(u; w) + \sum_{i<k} u^i c_i(w) \in I_{\xi_w}$$

where, as $w \to 0$, we have

$$\begin{aligned} \rho(w) &\to 1 \\ h(u, v; w) &\to h(u, v) \\ g(u; w) &\to g(u) \\ c_i(w) &\to 0. \end{aligned}$$

Now we claim we actually have $\sum_{i<k} u^i c_i(w) = 0$ for all such $Q(u, v; w)$. Let l be the smallest integer such that $c_l(w) \neq 0$ and $l < k$. Then $u^l \in \lim_{t\to 0} \phi(t)I_{\xi_w} = I_{\xi_{\lambda,0,0}}$ by

Lemma 3.3.7. But this is a contradiction, since k is the smallest possible power for a monomial of the form u^i in $I_{\xi_{\lambda,0,0}}$. Thus $\sum_{i<k} u^i c_i(w) = 0$.

For each w, let us define

$$\begin{aligned} Q_0(u,v;w) &= \rho(w)^{-1}Q(u,v;w) \\ &= u^k + \rho(w)^{-1}vh(u,v;w) + \rho(w)^{-1}u^{k+1}g(u;w). \end{aligned}$$

We now notice we can write

$$Q_0(u,v;w) = u^k + vh_0(u,v;w) + u^{k+1}g_0(u;w)$$

where $h_0(u,v;w)$ and $g_0(u;w)$ are the obvious adjustments of $h(u,v;w)$ and $g(u;w)$ respectively. We will make this maneuver, without explicitly saying so, often to keep notation somewhat under control. Now, let us define

$$\begin{aligned} Q_1(u,v;w) &= Q_0(u,v;w) - ug_0(u;w)Q_0(u,v;w) \\ &= u^k + vh_1(u,v;w) + u^{k+2}g_1(u;w) \in I_{\xi_w}. \end{aligned}$$

Note, for example in this case, we assume

$$h_1(u,v;w) = h_0(u,v;w) - ug_0(u;w)h_0(u,v;w)$$

and

$$g_1(u;w) = -g_0^2(u;w).$$

We can continue this process and define for any $i > 1$

$$\begin{aligned} Q_i(u,v;w) &= Q_{i-1}(u,v;w) - u^i g_{i-1}(u;w)Q_{i-1}(u,v;w) \\ &= u^k + vh_i(u,v;w) + u^{k+i+1}g_i(u;w) \in I_{\xi_w}. \end{aligned}$$

Of course, ξ_w is supported at $(0,0)$. Thus, we may apply Lemma 3.3.6, which asserts that there exist n_1, n_2 such that $u^{n_1}, v^{n_2} \in I_{\xi_w}$. Then

$$\begin{aligned} Q_{n_1-k-1}(u,v;w) &= Q_{n_1-k-2}(u,v;w) - u^i g_{n_1-k-2}(u;w) Q_{n_1-k-2}(u,v;w) \\ &= u^k + v h_{n_1-k-1}(u,v;w) + u^{n_1} g_{n_1-k-1}(u;w) \in I_{\xi_w}. \end{aligned}$$

Since $u^{n_1} \in I_{\xi_w}$, we have $u^k + v h_{n_1-k-1}(u,v;w) \in I_{\xi_w}$. Let us give this polynomial the name $P_0(u,v;w)$ and note that we can write it as

$$P_0(u,v;w) = u^k + v f_0(u,v;w) + v u^k R_0(u,v;w) \in I_{\xi_w}$$

breaking up $h_{n_1-k-1}(u,v;w)$ such that the degree of u in $f_0(u,v;w)$ is less than k. Now define

$$\begin{aligned} P_1(u,v;w) &= P_0(u,v;w) - v R_0(u,v;w) P_0(u,v;w) \\ &= u^k + v f_1(u,v;w) + v^2 u^k R_1(u,v;w) \in I_{\xi_w} \end{aligned}$$

where we have again adjusted so that the degree of u in $f_1(u,v;w)$ is less than k. Noticing again that we can continue this process, we define for $i > 1$

$$\begin{aligned} P_i(u,v;w) &= P_{i-1}(u,v;w) - v^i R_{i-1}(u,v;w) P_{i-1}(u,v;w) \\ &= u^k + v f_i(u,v;w) + v^{i+1} u^k R_i(u,v;w) \in I_{\xi_w}. \end{aligned}$$

We note that for each step, we adjust the terms so that the degree of u in $f_i(u,v;w)$ is less than k. Then we see

$$\begin{aligned} P_{n_2-1}(u,v;w) &= P_{n_2-2}(u,v;w) - v^{n_2-1} R_{n_2-2}(u,v;w) P_{n_2-2}(u,v;w) \\ &= u^k + v f_{n_2-1}(u,v;w) + v^{n_2} u^k R_{n_2-1}(u,v;w) \in I_{\xi_w}. \end{aligned}$$

Let us define $f(u, v; w) = f_{n_2-1}(u, v; w)$ and let us define $P(u, v; w)$ to be given by

$$P(u, v; w) = u^k + vf(u, v; w).$$

Since $v^{n_2} \in I_{\xi_w}$, we see $P(u, v; w) \in I_{\xi_w}$ and the degree of u in $f(u, v; w)$ is less than k. Finally, since $f(u, v; w)$ is convergent, $P(u, v; w)$ converges as $w \to 0$. □

In Lemma 3.3.8, we have essentially proven the base case of an argument by induction. Suppose $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$. Then certainly there exists some partition ν such that

$$\lim_{t \to 0} \phi(t)\xi = \xi_{\nu,0,0}$$

where, once again, we realize $\xi_{\nu,0,0}$ is the subscheme associated with the ideal $I_{\xi_{\nu,0,0}}$.

Now we provide the more general result.

Lemma 3.3.9. *Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of n and $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ where $\lim_{t\to 0} \phi(t)\xi = \xi_{\nu,0,0}$ for some partition ν of n. Let $\xi = \lim_{w \to 0} \xi_w$ where $\xi_w \in C_{\lambda,0,0}$. Assume the following:*

(i) $l(\nu) = l(\lambda) = k$

(ii) $\lambda_s \neq \nu_s$ for some s with $1 \leq s \leq k$

(iii) $\lambda_r = \nu_r$ for $s < r \leq k$

(iv) $\lambda_s > \lambda_{s+1}$.

Then for $s < r \leq k + 1$, I_{ξ_w} contains a polynomial of the form

$$P_{r-1}(u, v; w) = \left[u^{r-1} + vf_{r-1}(u, v; w)\right] v^{\lambda_r}$$

where the degree of u in $f_{r-1}(u, v; w)$ is less than $r-1$ and the $P_{r-1}(u, v; w)$ converge as $w \to 0$.

Proof. We will prove the lemma by induction. By Lemma 3.3.8, we have already shown the case in which $r = k + 1$. Now fix r_0 such that $s < r_0 \leq k$ and assume our statement holds for all $r_0 < r \leq k + 1$. First, note there exists $r_1 \leq r_0$ such that $\lambda_{r_1} = \cdots = \lambda_{r_0}$ and $\lambda_{r_1 - 1} > \lambda_{r_1}$. If $r_1 \leq s$, then

$$\lambda_{r_1} \geq \lambda_s > \lambda_{s+1} \geq \lambda_{r_0}$$

which would be a contradiction. Thus, we must have $s < r_1$.

We know by (2.3.5) that $u^{r_1 - 1} v^{\nu_{r_1}} \in I_{\xi_{\nu,0,0}}$. Now, since $\lambda_{r_1} = \nu_{r_1}$, we have

$$u^{r_1 - 1} v^{\lambda_{r_1}} = u^{r_1 - 1} v^{\nu_{r_1}} \in I_{\xi_{\nu,0,0}} = \lim_{t \to 0} \phi(t) I_\xi.$$

Then by Lemma 3.3.7, we must have a polynomial of the form

$$Q(u, v) = \left[u^{r_1 - 1} + v h(u, v) + u^{r_1} g(u)\right] v^{\lambda_{r_1}} \in I_\xi.$$

Since $\lim_{w \to 0} I_{\xi_w} = I_\xi$, for w sufficiently close to 0, there must exist polynomials

$$Q(u, v; w) = \left[u^{r_1 - 1} + v h(u, v; w) + u^{r_1} g(u; w)\right] v^{\lambda_{r_1}} + \sum_{i,j} a_{i,j}(w) u^i v^j \in I_{\xi_w}$$

where as $w \to 0$, we have

$$\begin{aligned} h(u, v; w) &\to h(u, v) \\ g(u; w) &\to g(u) \\ a_{i,j}(w) &\to 0. \end{aligned}$$

Note that in the tail summation, all (i, j) should be such that either $j < \lambda_{r_1}$ or $j = \lambda_{r_1}$ and $i < r_1 - 1$ since terms that do not fit those restrictions can be included in either polynomials $h(u, v; w)$ or $g(u; w)$.

As our notation grows more and more cumbersome, we may find it necessary to abuse notation. We will strive to keep an explicit nature to each newly-introduced function and symbol. However, the reader should be aware that we may neglect to explain each and every step. Let us begin by defining

$$Q(u,v;w) = A(u,v;w) + B(u,v;w) + C(u,v;w)$$

where

$$\begin{aligned} A(u,v;w) &= \left[u^{r_1-1} + vh(u,v;w) + u^{r_1}g(u;w)\right] v^{\lambda_{r_1}} \\ B(u,v;w) &= \sum_{\substack{i<r_0 \\ j}} a_{i,j}(w)u^i v^j + \sum_{i=r_0}^{k-1} \sum_{j<\lambda_{i+1}} a_{i,j}(w)u^i v^j \end{aligned}$$

and

$$C(u,v;w) = \sum_{i=r_0}^{k} u^i v^{\lambda_{i+1}} h_i(u,v;w)$$

for some polynomials $h_i(u,v;w), \ldots, h_k(u,v;w)$.

We first show $A(u,v;w) + B(u,v;w) \in I_{\xi_w}$ (here $A(u,v;w)$ and $B(u,v;w)$ will be of the same form as above, but might not be the same polynomials as above). By our induction assumption, we know there exist some

$$D_{r-1}(u,v;w) = \left[u^{r-1} + vf_{r-1}(u,v;w)\right] v^{\lambda_r} \in I_{\xi_w}$$

for $r_0 < r \leq k+1$. By using tactics similar to those used in our proof of Lemma 3.3.8, we can "cut off" part of the tail. For example, set

$$\begin{aligned} Q_0(u,v;w) &= Q(u,v;w) - h_k(u,v;w)D_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) \\ &- vh_k(u,v;w)f_k(u,v;w). \end{aligned}$$

To see how our abuse of notation will work, we will explicitly detail it here. Notice we can write

$$vh_k(u,v;w)f_k(u,v;w) = \sum_{i=0}^{k-1} vu^i g_i(u,v;w) + \sum_{i=k}^{\infty} vu^i g_i(u,v;w)$$

for some polynomials $g_0(u,v;w),\ldots,g_k(u,v;w),\ldots$ where infinitely many $g_i(u,v;w)$ may be 0. Of course, we realize we can rewrite $A(u,v;w), B(u,v;w)$ and

$$\sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w)$$

from above to include $vg_0(u,v;w) + vug_1(u,v;w) + \cdots + vu^{k-1}g_{k-1}(u,v;w)$. Thus, when we abuse the notation, we are foregoing changing the names of these polynomials when we do this type of inclusion. So, as we will do many more times, we will simply say that

$$Q_0(u,v;w) = A(u,v;w) + B(u,v;w) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + vu^k h_k(u,v;w).$$

Now if we set

$$\begin{aligned} Q_1(u,v;w) &= Q_0(u,v;w) - vh_k(u,v;w)D_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^2u^k h_k(u,v;w) \end{aligned}$$

and in general

$$\begin{aligned} Q_i(u,v;w) &= Q_{i-1}(u,v;w) - v^i h_k(u,v;w)D_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^{i+1}u^k h_k(u,v;w). \end{aligned}$$

By Lemma 3.3.6, there exists some n_2 such that $v^{n_2} \in I_{\xi_w}$. Thus, we see

$$\begin{aligned} Q_{n_2-1}(u,v;w) &= Q_{n_2-2}(u,v;w) - v^{n_2-1} h_k(u,v;w)D_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^{n_2}u^k h_k(u,v;w). \end{aligned}$$

Since $v^{n_2} \in I_{\xi_w}$, we have

$$A_w(u,v) + B_w(u,v) + \sum_{i=r_0}^{k-1} u^i v^{\lambda_{i+1}} h^{(i)}(u,v) \in I_{\xi_w}.$$

It should be clear we can repeat this process another $k - r_0$ times, using each polynomial $D_{r-1}(u,v;w)$ with $r_0 < r \leq k$ at some point to "cut off" another piece of the tail. After this is completed, we are left with $A(u,v;w) + B(u,v;w) \in I_{\xi_w}$.

Our next step is to show we actually have $B(u,v;w) = 0$. Recall

$$B(u,v;w) = \sum_{\substack{i<r_0 \\ j}} a_{i,j}(w) u^i v^j + \sum_{i=r_0}^{k-1} \sum_{j<\lambda_{i+1}} a_{i,j}(w) u^i v^j$$

Suppose $a_{i,j}(w) \neq 0$ for some (i,j). Since $a_{i,j}(w)$ is in one of the summations, we can say either $i < r_0$ or $r_0 \leq i \leq k-1$ and $j < \lambda_{i+1}$. However, we also must recall either $j < \lambda_{r_1}$ or $j = \lambda_{r_1}$ and $i < r_1 - 1$. Now, if $a_{i,j}(w) \neq 0$, then $u^i v^j \in \lim_{t \to 0} \phi(t) I_{\xi_w} = I_{\xi_{\lambda,0,0}}$ and thus we also must have $j \geq \lambda_{i+1}$.

Suppose $i < r_0$ or equivalently $i + 1 \leq r_0$. Then we see

$$j \geq \lambda_{i+1} \geq \lambda_{r_0} = \lambda_{r_1} \geq j.$$

So we must have $j = \lambda_{r_1}$. But in this case, $i < r_1 - 1$, which implies

$$j \geq \lambda_{i+1} \geq \lambda_{r_1 - 1} > \lambda_{r_1}$$

by assumption. This is a contradiction. Next assume $r_0 \leq i \leq k-1$ and $j < \lambda_{i+1}$. Then we have

$$j < \lambda_{i+1} \leq j$$

which is absurd. Since both cases lead to contradicitons, we must conclude $a_{i,j}(w) = 0$ for all (i,j). Therefore

$$A(u,v;w) = \left[u^{r_1-1} + v h(u,v;w) + u^{r_1} g(u;w)\right] v^{\lambda_{r_1}} \in I_{\xi_w}.$$

Now set

$$\begin{aligned} A_1(u,v;w) &= A(u,v;w) - ug(u;w)A(u,v;w) \\ &= \left[u^{r_1-1} + vh(u,v;w) + u^{r_1+1}g(u;w)\right] v^{\lambda_{r_1}}. \end{aligned}$$

In general, set

$$\begin{aligned} A_i(u,v;w) &= A_{i-1}(u,v;w) - ug(u;w)A_{i-1}(u,v;w) \\ &= \left[u^{r_1-1} + vh(u,v;w) + u^{r_1+i}g(u;w)\right] v^{\lambda_{r_1}}. \end{aligned}$$

By Lemma 3.3.6, there exists some n_1 such that $u^{n_1} \in I_{\xi_w}$. Thus, we see

$$\begin{aligned} A_{n_1-r_1}(u,v;w) &= A_{n_1-r_1-1}(u,v;w) - ug(u;w)A_{n_1-r_1-1}(u,v;w) \\ &= \left[u^{r_1-1} + vh(u,v;w) + u^{n_1}g(u;w)\right] v^{\lambda_{r_1}}. \end{aligned}$$

Thus, we have whittled our way down to show that

$$\left[u^{r_1-1} + vh(u,v;w)\right] v^{\lambda_{r_1}} \in I_{\xi_w} \tag{3.3.5}$$

since $u^{n_1}g(u;w)v^{\lambda_{r_1}} \in I_{\xi_w}$.

Finally, let rewrite our polynomial from above as

$$P_0(u,v;w) = \left[u^{r_1-1} + vf(u,v;w) + vu^{r_1-1}R(u,v;w)\right] v^{\lambda_{r_1}}$$

where we have split $h(u,v;w)$ from (3.3.5) so that the degree of u in $f(u,v;w)$ is less than $r_1 - 1$. As before, define

$$\begin{aligned} P_1(u,v;w) &= P_0(u,v;w) - vR(u,v;w)P_0(u,v;w) \\ &= \left[u^{r_1-1} + vf(u,v;w) + v^2u^{r_1-1}R(u;w)\right] v^{\lambda_{r_1}}. \end{aligned}$$

Again, following similar procedures as last time, we adjust the terms so that the degree of u in $f_w(u,v)$ is still less than $r_1 - 1$. Thus, in general,

$$\begin{aligned} P_i(u,v;w) &= P_{i-1}(u,v;w) - vR(u,v;w)P_{i-1}(u,v;w) \\ &= \left[u^{r_1-1} + vf(u,v;w) + v^{i+1}u^{r_1-1}R(u;w)\right] v^{\lambda_{r_1}} \end{aligned}$$

and

$$\begin{aligned} P_{n_2-1}(u,v;w) &= P_{n_2-2}(u,v;w) - vR(u,v;w)P_{n_2-2}(u,v;w) \\ &= \left[u^{r_1-1} + vf(u,v;w) + v^{n_2}u^{r_1-1}R(u;w)\right] v^{\lambda_{r_1}}. \end{aligned}$$

Since $v^{n_2} \in I_{\xi_w}$, we have $[u^{r_1-1} + vf(u,v;w)]\, v^{\lambda_{r_1}} \in I_{\xi_w}$ and the degree of u in $f(u,v;w)$ is less than $r_1 - 1$. To complete the proof, we now finally set

$$\begin{aligned} P(u,v;w) &= u^{r_0-r_1}\left[u^{r_1-1} + vf(u,v;w)\right] v^{\lambda_{r_1}} \\ &= \left[u^{r_0-1} + vf(u,v;w)\right] v^{\lambda_{r_0}}. \end{aligned}$$

Note that $\lambda_{r_1} = \lambda_{r_0}$ and that now the degree of u in $f(u,v;w)$ is less than $r_0 - 1$. We also note that the $P(u,v;w)$ converge as $w \to 0$ since the $f(u,v;w)$ are convergent. Thus, by induction, the proof is complete.

□

Remark 3.3.10. Recall if

$$\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$$

then there exist $\xi_w \in C_{\lambda,0,0}$ such that

$$\lim_{w\to 0} \xi_w = \xi.$$

Of course then, we can say that the ideal I_ξ is the flat limit of the ideals I_{ξ_w} as $w \to 0$. Now, suppose $f(u,v;w) \in I_{\xi_w}$ such that it converges to a polynomail $f(u,v) \in I_\xi$ as $w \to 0$. Suppose some monomial of the form u^i is a term of $f(u,v;w)$. If $l(\lambda) = k$, then we claim for sufficiently small w, we must have $i \geq k$. Assume $1 \leq i \leq k-1$. Then $u^i \in \lim_{t\to 0} \phi(t) I_{\xi_w} = I_{\xi_{\lambda,0,0}}$ by Lemma 3.3.7, since $\xi_w \in C_{\lambda,0,0}$. But this is a contradiction since k is the smallest possible power of a monomial of the form u^i in $I_{\xi_{\lambda,0,0}}$. Now, if we let $w \to 0$, we see $f(u,v)$ similarly will only have a monomial u^i as a term if $i \geq k$.

We can now prove Claim 3.3.5.

Proposition 3.3.11. *Suppose λ is a partition of n and $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ where $\lim_{t\to 0} \phi(t)\xi = \xi_{\nu,0,0}$ for some partition ν of n. Then $\lambda < \nu$.*

Proof. Suppose $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ and $\nu = (\nu_1 \geq \cdots \geq \nu_l)$. We first note $\lambda \neq \nu$ since $\xi \notin C_{\lambda,0,0}$. By Remark 3.3.10, if $u^j \in \lim\limits_{t\to 0} \phi(t) I_\xi = I_{\xi_{\nu,0,0}}$, then we must have $j \geq k$. Since $u^l \in I_{\xi_{\nu,0,0}}$, we have $l \geq k$. Of course, if $l > k$, we are done by Definition 3.3.1 since we would have $\lambda < \nu$. Thus, we will assume $l = k$. Since $\lambda \neq \nu$, there exists some $1 \leq s \leq k$ such that $\lambda_s \neq \nu_s$ and $\lambda_i = \nu_i$ for $s < i \leq k$. Note if $\lambda_s = \lambda_{s+1}$, then $\lambda_s = \nu_{s+1} \leq \nu_s$. Since $\lambda_s \neq \nu_s$, we would conclude $\lambda_s < \nu_s$, which would again give us $\lambda < \nu$. Thus, let us assume $\lambda_s > \lambda_{s+1}$.

We know $u^{s-1}v^{\nu_s} \in I_{\xi_{\nu,0,0}} = \lim_{t\to 0} \phi(t) I_\xi$. By Lemma 3.3.7, there exists a polynomial of the form

$$Q(u,v) = \left[u^{s-1} + vh(u,v) + u^s g(u)\right] v^{\nu_s} \in I_\xi$$

and as before we see there exists a polynomial of the form

$$Q(u,v;w) = \left[u^{s-1} + vh(u,v;w) + u^s g(u;w)\right] v^{\nu_s} + \sum_{i,j} a_{i,j}(w) u^i v^j \in I_{\xi_w}$$

where

$$\begin{aligned} h(u,v;w) &\to h(u,v) \\ g(u;w) &\to g(u) \\ a_{i,j}(w) &\to 0 \end{aligned}$$

as $w \to 0$. Also, note in the summation tail we have either $j < \nu_s$ or $j = \nu_s$ and $i < s-1$, since any terms not fitting those restrictions can be included in the polynomials $h(u,v;w)$ or $g(u;w)$.

As in the proof of Lemma 3.3.9, we can define

$$Q(u,v;w) = A(u,v;w) + B(u,v;w) + C(u,v;w)$$

where

$$\begin{aligned} A(u,v;w) &= \left[u^{s-1} + vh(u,v;w) + u^s g(u;w)\right] v^{\nu_s} \\ B(u,v;w) &= \sum_{\substack{i<s \\ j}} a_{i,j}(w) u^i v^j + \sum_{i=s}^{k-1} \sum_{j<\nu_{i+1}} a_{i,j}(w) u^i v^j \end{aligned}$$

and

$$C(u,v;w) = \sum_{i=s}^{k} u^i v^{\nu_{i+1}} h_i(u,v;w)$$

for some polynomials $h_1(u,v;w), \ldots, h_k(u,v;w)$. Note that we have met all the assumptions of Lemma 3.3.9. Thus, we know there exist some

$$P_{r-1}(u,v;w) = \left[u^{r-1} + vf_{r-1}(u,v;w)\right] v^{\lambda_r} \in I_{\xi_w}$$

for $s < r \leq k+1$. Note that for $s < r \leq k+1$, we have $\lambda_r = \nu_r$. By using tactics similar to those used in our proof of Lemma 3.3.8, we can "cut off" part of the tail. For example, set

$$\begin{aligned} Q_0(u,v;w) &= Q(u,v;w) - h_k(u,v;w)P_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=s}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + vu^k h_k(u,v;w). \end{aligned}$$

Again, be aware we have abused notation and adjusted our polynomials. Now if we set

$$\begin{aligned} Q_1(u,v;w) &= Q_0(u,v;w) - vh_k(u,v;w)P_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=s}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^2 u^k h_k(u,v;w) \end{aligned}$$

and in general

$$\begin{aligned} Q_i(u,v;w) &= Q_{i-1}(u,v;w) - v^i h_k(u,v;w)P_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=s}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^{i+1} u^k h_k(u,v;w) \end{aligned}$$

then, keeping in mind there exists some n_2 such that $v^{n_2} \in I_{\xi_w}$ by Lemma 3.3.6, we see

$$\begin{aligned} Q_{n_2-1}(u,v;w) &= Q_{n_2-2}(u,v;w) - v^{n_2-1} h_k(u,v;w)P_k(u,v;w) \\ &= A(u,v;w) + B(u,v;w) + \sum_{i=s}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) + v^{n_2} u^k h_k(u,v;w). \end{aligned}$$

Since $v^{n_2} \in I_{\xi_w}$, we have

$$A(u,v;w) + B(u,v;w) + \sum_{i=s}^{k-1} u^i v^{\lambda_{i+1}} h_i(u,v;w) \in I_{\xi_w}.$$

It should be clear we can repeat this process another $k - s$ times, using each polynomial $P_{r-1}(u, v; w)$ with $s < r \leq k$ at some point to "cut off" another piece of the tail. After this is completed, we are left with $A(u, v; w) + B(u, v; w) \in I_{\xi_w}$.

Now suppose $a_{i_0,j_0} \neq 0$ for some a_{i_0,j_0} in the sums of $B(u, v; w)$. Then

$$u^{i_0} v^{j_0} \in \lim_{t \to 0} \phi(t) I_{\xi_w} = I_{\xi_{\lambda,0,0}}$$

where either $i_0 < s$ or $s \leq i_0 \leq k - 1$ and $j_0 < \nu_{i_0+1}$.

Suppose we have the latter case. Since $u^{i_0} v^{j_0} \in I_{\xi_{\lambda,0,0}}$, we must have $j_0 \geq \lambda_{i_0+1}$. Thus $\lambda_{i_0+1} \leq j_0 < \nu_{i_0+1}$, which is a contradiction since $\lambda_{i+1} = \nu_{i+1}$ for all $s \leq i \leq k - 1$. Now suppose $i_0 < s$. Recall from before we must have $j_0 \leq \nu_s$. Then we see $\lambda_s \leq \lambda_{i_0+1} \leq j_0 \leq \nu_s$. But, of course, $\lambda_s \neq \nu_s$. Thus, in this case we would have $\lambda_s < \nu_s$, and by Definition 3.3.1, we would be done.

Therefore, let us assume $a_{i,j} = 0$ for all $a_{i,j}$ in $B(u, v; w)$ (i.e. we assume $B(u, v; w) = 0$). Then I_{ξ_w} contains the polynomial

$$A(u, v; w) = \left[u^{s-1} + vh(u, v; w) + u^s g(u; w)\right] v^{\nu_s}$$

Thus, $u^{s-1} v^{\nu_s} \in \lim_{t \to 0} \phi(t) I_{\xi_w} = I_{\xi_{\lambda,0,0}}$. Therefore we must have $\lambda_s \leq \nu_s$. Once again, we assumed $\lambda_s \neq \nu_s$, so we conclude $\lambda_s < \nu_s$. □

We can illustrate Proposition 3.3.11 in the following examples.

Example 3.3.12. Suppose $\lambda = (3, 1, 1)$. Let ξ be the flat limit of ξ_w where $I_{\xi_w} = \langle v^3, wuv + v^2, u^2 v, u^3 \rangle$. Note here that

$$\lim_{t \to 0} \phi(t) I_{\xi_w} = \langle v^3, uv, u^2 v, u^3 \rangle = I_{\xi_{\lambda,0,0}}$$

so we confirm $\xi_w \in C_{\lambda,0,0}$ and $\xi \in \overline{C}_{\lambda,0,0}$. Also, we can see $I_\xi = \lim_{w\to 0} I_{\xi_w} = \langle v^3, v^2, u^2v, u^3\rangle$. Of course, applying the action to this ideal and taking the limit as $t \to 0$ will yield the same ideal, so we see

$$\lim_{t\to 0} \phi(t) I_\xi = \langle v^3, v^2, u^2v, u^3\rangle = \langle v^2, uv^2, u^2v, u^3\rangle = I_{\xi_{\nu,0,0}}$$

where $\nu = (2,2,1)$. Thus, $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ and as we expect, $\lambda < \nu$.

Example 3.3.13. Suppose $\lambda = (3,1)$. Let ξ be the flat limit of ξ_w where

$$I_{\xi_w} = \langle v^3, wuv + v^2, u^2\rangle.$$

Note here that

$$\lim_{t\to 0} \phi(t) I_{\xi_w} = \langle v^3, uv, u^2\rangle = I_{\xi_{\lambda,0,0}}$$

so we confirm $\xi_w \in C_{\lambda,0,0}$ and $\xi \in \overline{C}_{\lambda,0,0}$. Also, we can see $I_\xi = \lim_{w\to 0} I_{\xi_w} = \langle v^3, v^2, u^2\rangle$. Of course, applying the action to this ideal and taking the limit as $t \to 0$ will yield the same ideal, so we see

$$\lim_{t\to 0} \phi(t) I_\xi = \langle v^3, v^2, u^2\rangle = \langle v^2, uv^2, u^2\rangle = I_{\xi_{\nu,0,0}}$$

where $\nu = (2,2)$. Thus, $\xi \in \overline{C}_{\lambda,0,0} - C_{\lambda,0,0}$ and as we expect, $\lambda < \nu$.

Now, recall from Definition 3.3.3 that

$$W_\nu = \left\{ \sum_{i=1}^{\nu_1} (a_1, b_{1,i}) + \cdots + \sum_{i=1}^{\nu_k} (a_k, b_{k,i}) \right\}.$$

The idea is to understand the intersection pairing

$$\left\langle \overline{C}_{\lambda,0,0}, \overline{W}_\nu \right\rangle.$$

Proposition 3.3.11 will help to do this, as will the following lemma.

Lemma 3.3.14. *Let $|\nu| = n$. Suppose $\xi \in \overline{W}_\nu \cap M_n(P_0)$ and λ is a partition of n such that $\lim_{t\to 0} \phi(t)\xi = \xi_{\lambda,0,0}$. Then $\lambda \leq \nu$. Furthermore, if $\lambda = \nu$, then $\xi = \xi_{\lambda,0,0}$.*

Proof. Suppose $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$. Let I_ξ be the ideal corresponding to ξ. Since $\xi \in \overline{W}_\nu$, I_ξ is the flat limit, as $w \to 0$, of some ideals I_{ξ_w} corresponding to some $\xi_w \in W_\nu$. From our definition of W_ν, we have

$$I_{\xi_w} = \prod_{i=1}^{k} \left\langle u - a_i(w), \prod_{j=1}^{\nu_i} (v - b_{i,j}(w)) \right\rangle$$

for w close to 0. Since I_{ξ_w} is concentrated at P_0, we must have

$$\lim_{w\to 0} a_i(w) = \lim_{w\to 0} b_{i,j}(w) = 0.$$

Now, we see $\prod_{i=1}^{k}(u - a_i(w)) \in I_{\xi_w}$, and thus $u^k \in I_\xi$. Since

$$\lim_{t\to 0} \phi(t) I_\xi = I_{\xi_{\lambda,0,0}}$$

we have $u^k \in I_{\xi_{\lambda,0,0}}$. Thus, $l(\lambda) \leq k = l(\nu)$. Notice if $l(\lambda) < l(\nu)$, we are done. Thus, let us assume $l(\lambda) = l(\nu)$.

Fix some $0 \leq r \leq k-1$. We intend to show I_{ξ_w} contains some polynomial

$$P_r(u, v; w) = \prod_{i=1}^{r}(u - a_i(w))(v^{\nu_{r+1}} + f_r(u, v; w))$$

where $deg_u f_r < k - r$ and $deg_v f_r < \nu_{r+1}$. We employ a previous tactic. Let $r + 1 \leq s \leq k$. Then the following polynomials are contained in I_{ξ_w}

$$p_s(u, v; w) = \prod_{i=1}^{r}(u - a_i(w)) \prod_{i=r+1}^{s-1} (u - a_i(w)) \prod_{j=1}^{\nu_s}(v - b_{s,j}(w)) \prod_{i=s+1}^{k} (u - a_i(w)).$$

Note we can rewrite this as

$$p_s(u, v; w) = \prod_{i=1}^{r}(u - a_i(w)) \prod_{\substack{i=r+1 \\ i\neq s}}^{k} (u - a_i(w))(v^{\nu_s} - f_s(v; w))$$

where $deg_v f_s < \nu_s$. Now by Lemma 3.1.1, there exist some $\beta_{r+1}(w), \ldots, \beta_k(w) \in \mathbb{C}$ such that

$$\sum_{s=r+1}^{k} \beta_s(w) \prod_{\substack{i=r+1 \\ i \neq s}}^{k} (u - a_i(w)) = 1.$$

Therefore, I_{ξ_w} contains

$$\begin{aligned} P_r(u, v; w) &:= \sum_{s=r+1}^{k} \beta_s(w) v^{\nu_{r+1} - \nu_s} p_s(u, v; w) \\ &= \prod_{i=1}^{r} (u - a_i(w))(v^{\nu_{r+1}} + f_r(u, v; w)). \end{aligned}$$

Note that, as we claimed before, that $deg_u f_r < k - r$ and $deg_v f_r < \nu_{r+1}$. Thus, let us write

$$f_r(u, v; w) = \sum_{\substack{0 \leq p < k-r \\ 0 \leq q < \nu_{r+1}}} d_{p,q,r}(w) u^p v^q.$$

Next, we intend to show $d_{p,q,r}(w) = 0$ whenever $q \geq \nu_{p+r+1}$. For $r = k - 1$ there is no work to be done, since $q < \nu_k$ and $p = 0$ in the sum. Suppose $r = k - 2$. We can write

$$\begin{aligned} P_{k-2}(u, v; w) &= \prod_{i=1}^{k-2} (u - a_i(w)) \left(v^{\nu_{k-1}} + \sum_{\substack{0 \leq p < 2 \\ 0 \leq q < \nu_{k-1}}} d_{p,q,r}(w) u^p v^q \right) \\ &= \prod_{i=1}^{k-2} (u - a_i(w)) \left(v^{\nu_{k-1}} + \sum_{\substack{0 \leq p < 2 \\ 0 \leq q < \nu_{p+k-1}}} d_{p,q,r}(w) u^p v^q + u v^{\nu_k} g_1(v; w) \right) \end{aligned}$$

where $deg_v g_1(v; w) \leq \nu_{k-1} - \nu_k - 1$. Note $P_{k-2}(u, v; w) - g_1(v; w) P_{k-1}(u, v; w) \in I_{\xi_w}$. If we simplify this difference, we have

$$\prod_{i=1}^{k-2} (u - a_i(w)) \left[v^{\nu_{k-1}} + \sum_{\substack{0 \leq p < 2 \\ 0 \leq q < \nu_{p+k-1}}} d_{p,q,r}(w) u^p v^q + A(u, v; w) \right]$$

where

$$A(u,v;w) = a_{k-1}(w)v^{\nu_k}g_1(v;w) + \sum_{0\le q<\nu_k} d_{p,q,r}(w)uv^q g_1(v;w) + \sum_{0\le q<\nu_k} d_{p,q,r}(w)a_{k-1}(w)v^q g_1(v;w).$$

Notice the degree of v in $a_{k-1}(w)v^{\nu_k}g_1(v;w)$ and $\sum_{0\le q<\nu_k} d_{p,q,r}(w)a_{k-1}(w)v^q g_1(v;w)$ is less than ν_{k-1} and thus they may be absorbed into the sum $\sum_{\substack{0\le p<2\\0\le q<\nu_{p+k-1}}} d_{p,q,r}(w)u^p v^q$. We also note that in the sum $\sum_{0\le q<\nu_k} d_{p,q,r}(w)uv^q g_1(v;w)$, the degree of v is less than $\nu_{k-1}-1$. Thus, we can write $P_{k-2}(u,v;w) - g_1(v;w)P_{k-1}(u,v;w)$ as

$$\prod_{i=1}^{k-2}(u-a_i(w))\left(v^{\nu_{k-1}} + \sum_{\substack{0\le p<2\\0\le q<\nu_{p+k-1}}} d_{p,q,r}(w)u^p v^q + uv^{\nu_k}g_2(v;w)\right)$$

where $deg_v g_2(v;w) \le \nu_{k-1} - \nu_k - 2$. If we abuse notation by calling this difference $P_{k-2}(u,v;w)$ we see we have dropped the degree of $g_1(v;w)$ by 1. After $\nu_{k-1}-\nu_k$ iterations of this process, we will have some polynomial

$$P_{k-2}(u,v;w) = \prod_{i=1}^{k-2}(u-a_i(w))\left(v^{\nu_{k-1}} + \sum_{\substack{0\le p<2\\0\le q<\nu_{p+k-1}}} d_{p,q,r}(w)u^p v^q\right) \in I_{\xi_w}.$$

Suppose $0 \le r \le k-3$. Using the same reasoning as above, we can subtract suitable multiples of $P_{r+1}(u,v;w),\ldots,P_{k-1}(u,v;w)$ from $P_r(u,v;w)$ so that we may safely assume $d_{p,q,r}(w) = 0$ whenever $q \ge \nu_{p+r+1}$, and thus,

$$f_r(u,v;w) = \sum_{\substack{0\le p<k-r\\0\le q<\nu_{p+r+1}}} d_{p,q,r}(w)u^p v^q.$$

Note that these $d_{p,q,r}(w)$ are analytic functions in w. Thus, we can factor $d_{p,q,r}(w)$ such that we have

$$d_{p,q,r}(w) = w^{e_{p,q,r}}\tilde{d}_{p,q,r}(w)$$

where $e_{p,q,r} \in \mathbb{Z}$ and $\tilde{d}_{p,q,r}(0) \neq 0$. Now define $e_r = \min\{e_{p,q,r}, 0\}$. Then for all $0 \leq r \leq k-1$, we have

$$\lim_{w \to 0} w^{-e_r} P_r(u, v; w) = u^r \left(d_{q,r} v^{\nu_{r+1}} + \sum_{\substack{0 \leq p < k-r \\ 0 \leq q < \nu_{p+r+1}}} d_{p,q,r} u^p v^q \right) \in I_\xi$$

where $d_{q,r}, d_{p,q,r} \in \mathbb{C}$ and at least one of them is nonzero.

This leaves us with two cases. First, suppose $d_{p,q,r} = 0$ for all p, q, r. Then $u^r v^{\nu_{r+1}} \in I_\xi$ for all r. This implies $\langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^k \rangle \subset I_\xi$. Since both ideals have colengths of n, we have $\langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^k \rangle = I_\xi$. Thus,

$$I_{\xi_{\lambda,0,0}} = \lim_{t \to 0} \phi(t) I_\xi = I_{\xi_{\nu,0,0}}$$

In this case, we have $\lambda = \nu$. Since $I_\xi = I_{\xi_{\lambda,0,0}}$ in this case, we have $\xi = \xi_{\lambda,0,0}$.

Now assume we have $d_{p,q,r} \neq 0$ for some p, q, r. Here we may assume $d_{p,q,s} = 0$ for all $r < s \leq k-1$, $0 \leq p < k-r$, and $0 \leq q < \nu_{p+r+1}$. Then $u^s v^{\nu_{s+1}} \in \lim_{t \to 0} \phi(t) I_\xi = I_{\xi_{\lambda,0,0}}$. Thus, for $r < s \leq k-1$, we have $\lambda_{s+1} \leq \nu_{s+1}$. Also, for some $0 \leq p < k-r$ and $0 \leq q < \nu_{p+r+1}$ we have $u^{r+p} v^q \in \lim_{t \to 0} \phi(t) I_\xi = I_{\xi_{\lambda,0,0}}$. Now, if we let $s_0 = r + p$, then $r \leq s_0 < k$ and $u^{s_0} v^{\lambda_{s_0+1}} \in I_{\xi_{\lambda,0,0}}$. Combining these facts we have

$$\lambda_{s_0+1} \leq q < \nu_{s_0+1}$$

Hence, in this case, we see $\lambda < \nu$.

□

Remark 3.3.15. Suppose we let $I_{\xi_w} = \prod_{i=1}^{k} \langle u - a_i(w), v^{\nu_i} \rangle$, i.e. we let all $b_{i,j}(w) = 0$ for all i, j in the proof of Lemma 3.3.14. Then the I_{ξ_w} define elements in $\overline{W}_\nu$. Now, we notice $\lim_{t \to 0} \phi(t) I_{\xi_w} = \langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^{k-1} v^{\nu_k}, u^k \rangle$. To see this, we refer to the method used in the proof. Using the same notation as in the proof, we see for these I_{ξ_w}, we have

$$p_s(u,v;w) = \prod_{i=1}^{r}(u-a_i(w)) \prod_{\substack{i=r+1\\ i\neq s}}^{k} (u-a_i(w))v^{\nu_s} \in I_{\xi_w}$$

and so

$$\begin{aligned} P_r(u,v;w) &= \sum_{s=r+1}^{k} \beta_s(w)v^{\nu_{r+1}-\nu_s}p_s(u,v;w) \\ &= \prod_{i=1}^{r}(u-a_i(w))v^{\nu_{r+1}} \in I_{\xi_w}. \end{aligned}$$

As we let $w \to 0$, we see $\lim_{t\to 0}\phi(t)I_{\xi_w} \supset \langle v^{\nu_1}, uv^{\nu_2}, \ldots, u^{k-1}v^{\nu_k}, u^k\rangle$ and since these both have codimensions of n, we have equality. We conclude that in this special case, we have $\xi_{\nu,0,0} = \lim_{t\to 0}\phi(t)\xi_w \in \overline{W}_\nu$.

We provide an example which expands upon Example 3.3.13.

Example 3.3.16. Suppose $\lambda = (3,1)$, $\nu = (2,2)$, and we define $I_{w,\tilde{w}}$ to be $\langle u, v(v-w\tilde{w})\rangle\langle u-\tilde{w}, v^2\rangle$ where $w, \tilde{w} \in \mathbb{C}^*$. By definition, these $I_{w,\tilde{w}}$ correspond to elements in $\overline{W}_\nu \subset (\mathbb{C}^2)^{[n]}$. Now, set $I_{\xi_w} = \lim_{\tilde{w}\to 0} I_{w,\tilde{w}}$. Then we see $\xi_w \in \overline{W}_\nu$. We can also show

$$I_{\xi_w} = \lim_{\tilde{w}\to 0} I_{w,\tilde{w}} = \langle v^3, wuv + v^2, u^2\rangle.$$

Indeed, since $u^2 - u\tilde{w} \in I_{w,\tilde{w}}$, we see $u^2 \in I_{\xi_w}$. Also, we have uv^2 and $uv^2 - uvw\tilde{w} - v^2\tilde{w} - vw(\tilde{w})^2 \in I_{w,\tilde{w}}$. Thus, their difference, $uvw\tilde{w} + v^2\tilde{w} + vw(\tilde{w})^2$ is in the ideal. By dividing through by $\tilde{w}$ and then letting $\tilde{w} \to 0$, we see $uvw + v^2 \in I_{\xi_w}$. Finally, having just shown $uvw\tilde{w} + v^2\tilde{w} + vw(\tilde{w})^2 \in I_{w,\tilde{w}}$, we observe

$$uv^2w - \left[v(uvw\tilde{w} + v^2\tilde{w} + vw(\tilde{w})^2)\right] = v^3 + v^2w\tilde{w} \in I_{w,\tilde{w}}.$$

Thus $v^3 \in I_{\xi_w}$. By Example 3.3.13, we have $\xi_w \in \overline{C}_{\lambda,0,0}$ for $w \in \mathbb{C}^*$, and thus from this example, we can conclude $\dim\left(\overline{W}_\nu \cap \overline{C}_{\lambda,0,0}\right) \geq 1$.

Now, we can finally make some claims about the intersection of $\overline{W}_\nu$ and $\overline{C}_{\lambda,0,0}$. Note that Proposition 3.3.11 and Lemma 3.3.14 are essential to this proof.

Lemma 3.3.17. *Suppose λ and ν are partitions of n.*

(i) If $\langle [\overline{C}_{\lambda,0,0}], [\overline{W}_\nu] \rangle$ is nonzero, then $l(\lambda) = l(\nu)$. Moreover, $\lambda \leq \nu$.

(ii) $\langle [\overline{C}_{\lambda,0,0}], [\overline{W}_\lambda] \rangle$ is a positive integer.

Proof. Assume $\langle [\overline{C}_{\lambda,0,0}], [\overline{W}_\nu] \rangle$ is nonzero. Then we must have $\dim \overline{C}_{\lambda,0,0} + \dim \overline{W}_\nu = 2n$. Now, recall that by [6], we have

$$\dim \overline{C}_{\lambda,0,0} = \dim C_{\lambda,0,0} = n - l(\lambda)$$

$$\dim \overline{W}_\nu = \dim W_\nu = \dim C_{0,0,\nu} = n + l(\nu).$$

Thus $l(\lambda) = l(\nu)$.

By assumption, we know $\overline{C}_{\lambda,0,0} \cap \overline{W}_\nu \neq \emptyset$. Let $\xi \in \overline{C}_{\lambda,0,0} \cap \overline{W}_\nu$. Then $\lim_{t\to 0} \phi(t)\xi = \xi_{\lambda',0,0}$ for some partition λ' of n. Since $\xi \in \overline{C}_{\lambda,0,0}$, we have $\lambda \leq \lambda'$ by Proposition 3.3.11. Since $\xi \in \overline{W}_\nu$ and $\xi \in M_n(P_0)$ (since $\overline{C}_{\lambda,0,0} \subset M_n(P_0)$) , we have $\lambda' \leq \nu$ by Lemma 3.3.14. Thus, $\lambda \leq \lambda' \leq \nu$ and we have shown part (i).

Now we claim $\overline{C}_{\lambda,0,0} \cap \overline{W}_\lambda = \{\xi_{\lambda,0,0}\}$. Note that $\xi_{\lambda,0,0} \in \overline{C}_{\lambda,0,0}$ by definition, and by Remark 3.3.15 we have $\xi_{\lambda,0,0} \in \overline{W}_\lambda$. Thus $\xi_{\lambda,0,0} \in \overline{C}_{\lambda,0,0} \cap \overline{W}_\lambda$. Now suppose $\xi \in \overline{C}_{\lambda,0,0} \cap \overline{W}_\lambda$. Then $\lim_{t\to 0} \phi(t)\xi = \xi_{\lambda',0,0}$ for some partition λ' of n. By part (i), we have $\lambda \leq \lambda' \leq \lambda$, and so $\lambda' = \lambda$. By Lemma 3.3.14, $\xi = \xi_{\lambda,0,0}$. □

Recall from (3.3.2), we established that

$$[\overline{C}_{\lambda,0,0}] = \sum_{\substack{l(\nu)=l(\lambda) \\ |\nu|=|\lambda|}} e_{\lambda,\nu}\mathfrak{a}_{-\nu}(x)|0\rangle.$$

Now we finally have enough information to prove our initial claim.

Theorem 3.3.18. $[\overline{C}_{\lambda,0,0}] = e_{\lambda,\lambda}\mathfrak{a}_{-\lambda}(x)|0\rangle + \sum_{\substack{l(\nu)=l(\lambda) \\ |\nu|=|\lambda| \\ \nu>\lambda}} e_{\lambda,\nu}\mathfrak{a}_{-\nu}(x)|0\rangle$ *where* $e_{\nu,\lambda} \in \mathbb{Z}$ *and* $e_{\lambda,\lambda}$ *is a positive integer.*

Proof. Fix ν_0 be a partition such that $l(\nu_0) = l(\lambda)$ and $|\nu_0| = |\lambda| = n$. From Proposition 3.2.8, we recall

$$[\overline{W}_{\nu_0}] = [\overline{C}_{0,0,\nu_0}] \equiv \frac{(-1)^{n+l(\nu_0)}}{\mathfrak{z}_{\nu_0}}\mathfrak{a}_{-\nu_0}(1)|0\rangle \pmod{\mathcal{I}^{[\mathrm{n}]}}.$$

Now consider the intersection of both sides of (3.3.2) with $[\overline{W}_{\nu_0}]$. The intersection between $\mathfrak{a}_{-\nu}(x)|0\rangle$ and anything in $\mathcal{I}^{[n]}$ is 0. Thus by Lemma 3.2.4, we have

$$[\overline{C}_{\lambda,0,0}] \cdot [\overline{W}_{\nu_0}] = e_{\lambda,\nu_0}.$$

If $e_{\lambda,\nu_0} = 0$, then its corresponding term in the expansion of $[\overline{C}_{\lambda,0,0}]$ is 0. Otherwise, by part (i) of Lemma 3.3.17, we must have $\lambda \leq \nu_0$. Finally

$$e_{\lambda,\lambda} = [\overline{C}_{\lambda,0,0}] \cdot [\overline{W}_{\lambda}].$$

By Lemma 3.3.17 (ii), $e_{\lambda,\lambda}$ is a positive integer.

□

Chapter 4

Main Theorem

Recall from the definition of $C_{\lambda,\mu,\nu}$ we can write $\xi \in C_{\lambda,\mu,\nu}$ as $\xi = \xi_0 + \xi_1 + \xi_2$ where

$$\xi_0 \in C_{\lambda,0,0}, \quad \mathrm{supp}(\xi_0) \in F_0 = \{P_0\}$$

$$\xi_1 \in C_{0,\mu,0}, \quad \mathrm{supp}(\xi_1) \in F_1 = \mathfrak{L} - \{P_0\}$$

$$\xi_2 \in C_{0,0,\nu}, \quad \mathrm{supp}(\xi_2) \in F_2 = \mathbb{P}^2 - \mathfrak{L}.$$

Definition 4.0.19. Fix λ', μ', and ν' such that $|\lambda'|+|\mu'|+|\nu'| = n$. Choose $\tilde{\mu}$ such that $|\tilde{\mu}| = |\mu'|$. Let $\tilde{\mathfrak{L}} \subset \mathbb{P}^2$ be a line different from $\mathfrak{L}$. Now fix $x_1, \ldots, x_{l(\nu')} \in \mathbb{P}^2$ not lying on $\tilde{\mathfrak{L}} \cup \mathfrak{L}$. Then we define $Z_{\lambda',\tilde{\mu},\nu'} \subset (\mathbb{P}^2)^{[n]}$ to be the subspace consisting of elements of the form

$$\sum_{i=1}^{l(\lambda')} \eta_{1,i} + \eta_2 + \sum_{i=1}^{l(\nu')} \eta_{3,i}$$

where $\eta_{3,i} \in M_{\nu'_i}(x_i)$, $\eta_2 \in L^{\tilde{\mu}}\tilde{\mathfrak{L}}$, and $\eta_{1,i} \in M_{\lambda'_i}(y_i)$ where $y_1, \ldots, y_{l(\lambda')}$ are distinct points not in $\{x_1, \ldots, x_{l(\nu')}\} \cup \tilde{\mathfrak{L}}$.

Remark 4.0.20. Let $\lambda' = (1^{m_1(\lambda')}, 2^{m_2(\lambda')}, \ldots, r^{m_r(\lambda')}, \ldots)$. By Proposition 3.2 in ([18]),

$$[\overline{Z}_{\lambda',\tilde{\mu},\nu'}] = \left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')}\right) \cdot \frac{1}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)\mathfrak{m}_{\tilde{\mu},\tilde{\mathfrak{L}}}\mathfrak{a}_{-\nu'}(x)|0\rangle. \tag{4.0.1}$$

Note $\mathfrak{m}_{\tilde{\mu},\tilde{\mathfrak{L}}} = \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}$. Now, assume $|\lambda| + |\mu| + |\nu| = n$. By Theorem 2.2.8, we can write

$$[\overline{C}_{\lambda,\mu,\nu}] = \sum_{\substack{\lambda',\mu',\nu' \\ |\lambda'|+|\mu'|+|\nu'|=n}} e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \frac{1}{\mathfrak{z}_{\nu'}} \mathfrak{a}_{-\nu'}(1)\mathfrak{m}_{\mu',\mathfrak{L}}\mathfrak{a}_{-\lambda'}(x)|0\rangle \tag{4.0.2}$$

$$= \sum_{\lambda',\nu'} \sum_{\substack{\mu' \\ |\mu'|=n-|\lambda'|-|\nu'|}} e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \frac{1}{\mathfrak{z}_{\nu'}} \mathfrak{a}_{-\nu'}(1)\mathfrak{m}_{\mu',\mathfrak{L}}\mathfrak{a}_{-\lambda'}(x)|0\rangle \tag{4.0.3}$$

where $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \in \mathbb{Z}$. Thus, we can calculate $\left\langle [\overline{C}_{\lambda,\mu,\nu}], [\overline{Z}_{\lambda',\tilde{\mu},\nu'}] \right\rangle$ and see from (4.0.1) that we get

$$\left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')} \right) \cdot (-1)^{|\lambda'|+l(\lambda')+|\nu'|+l(\nu')} \left\langle \sum_{|\overline{\mu}|=n-|\lambda'|-|\nu'|} e^{\lambda,\mu,\nu}_{\lambda',\overline{\mu},\nu'} \mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle \right\rangle.$$

Let us define $S_{\lambda',\nu'}$ such that

$$S_{\lambda',\nu'} = \sum_{|\overline{\mu}|=n-|\lambda'|-|\nu'|} e^{\lambda,\mu,\nu}_{\lambda',\overline{\mu},\nu'} \mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle$$

and we can write

$$\left\langle [\overline{C}_{\lambda,\mu,\nu}], [\overline{Z}_{\lambda',\tilde{\mu},\nu'}] \right\rangle = \left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')} \right) \cdot (-1)^{|\lambda'|+l(\lambda')+|\nu'|+l(\nu')} \left\langle S_{\lambda',\nu'}, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle \right\rangle. \tag{4.0.4}$$

We will need this theorem from [22].

Theorem 4.0.21. *Let $\alpha_1, \ldots, \alpha_k \in H^2(X)$ be linearly independent classes, and let $M_{\underline{\alpha}}$ be the intersection matrix of $\alpha_1, \ldots, \alpha_k$. Fix a positive integer n. Let $M_{n,\underline{\alpha}}$ be the intersection matrix of the classes in $H^{2n}(X^{[n]})$:*

$$\mathfrak{m}_{\lambda_1,\alpha_1} \cdots \mathfrak{m}_{\lambda_k,\alpha_k}|0\rangle, \quad |\lambda_1| + \cdots + |\lambda_k| = n.$$

If det $M_{\underline{\alpha}} = \pm 1$, then we have det $M_{n,\underline{\alpha}} = \pm 1$ as well.

Lemma 4.0.22. *Let* λ,μ, *and* ν *be partitions such that* $|\lambda|+|\mu|+|\nu|=n$. *Then*

$$[\overline{C}_{\lambda,\mu,\nu}] = (-1)^{|\nu|+l(\nu)}\frac{e_{\lambda,\lambda}\tilde{f}_\nu}{\mathfrak{z}_\nu}\mathfrak{a}_{-\nu}(1)\mathfrak{m}_{\mu,\mathfrak{L}}\mathfrak{a}_{-\lambda}(x)|0\rangle + \sum_{\substack{\lambda',\mu',\nu' \\ |\lambda'|+|\mu'|+|\nu'|=n}} e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}\frac{1}{\mathfrak{z}_{\nu'}}\mathfrak{a}_{-\nu'}(1)\mathfrak{m}_{\mu',\mathfrak{L}}\mathfrak{a}_{-\lambda'}(x)|0\rangle$$

where $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}\in\mathbb{Z}$ *and one of the following is true for the triple* (λ',μ',ν')*:*

(i) $|\nu'|<|\nu|$

(ii) $\nu'=\nu$ *and* $|\mu'|<|\mu|$

(iii) $\nu'=\nu$, $\mu'=\mu$, $\quad l(\lambda')=l(\lambda)$, *and* $\lambda<\lambda'$.

Proof. We assume from (4.0.2) that $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}\neq 0$ for some partitions λ',μ',ν' where $|\lambda'|+|\mu'|+|\nu'|=n$. By Theorem 4.0.21, the classes $\mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle$ with $|\overline{\mu}|=n-|\lambda'|-|\nu'|$ are linearly independent. Thus,

$$\sum_{|\overline{\mu}|=n-|\lambda'|-|\nu'|} e^{\lambda,\mu,\nu}_{\lambda',\overline{\mu},\nu'}\mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle\neq 0.$$

Therefore, there must exist some $\tilde{\mu}$ such that $|\tilde{\mu}|=n-|\lambda'|-|\nu'|$ and

$$\left\langle \sum_{|\overline{\mu}|=n-|\lambda'|-|\nu'|} e^{\lambda,\mu,\nu}_{\lambda',\overline{\mu},\nu'}\mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle\right\rangle\neq 0. \qquad (4.0.5)$$

By (4.0.4), $\overline{C}_{\lambda,\mu,\nu}\cap\overline{Z}_{\lambda',\tilde{\mu},\nu'}$ is nonempty.

Let $\xi\in\overline{C}_{\lambda,\mu,\nu}\cap\overline{Z}_{\lambda',\tilde{\mu},\nu'}$. We can write $\xi=\xi_1+\xi_2$ where $\mathrm{supp}(\xi_1)\subset\mathfrak{L}$ and $\mathrm{supp}(\xi_2)\cap\mathfrak{L}=\emptyset$. Thus, we must have $l(\xi_1)\geq|\lambda|+|\mu|$ and $l(\xi_2)\geq|\nu'|$. But, since we also have $l(\xi_2)=l(\xi)-l(\xi_1)=(|\lambda|+|\mu|+|\nu|)-l(\xi_1)\leq|\nu|$, we get

$$|\nu|\geq l(\xi_2)\geq|\nu'|.$$

If $|\nu|>|\nu'|$, we have case (i) and we are done.

Assume $|\nu| = |\nu'|$. Then $l(\xi_2) = |\nu|$, so by definition we have

$$\xi_2 \in \overline{C}_{0,0,\nu} \cap \prod_{i=1}^{l(\nu')} M_{\nu'_i}(x_i).$$

Therefore $\overline{C}_{0,0,\nu} \cap \mathfrak{a}_{-\nu'}(x)|0\rangle \neq \emptyset$. By our proof of Lemma 3.2.7 (i), we must have $\nu' = \nu$.

Since locally we can split $(\mathbb{P}^2)^{[n]}$ into $(\mathbb{P}^2)^{[|\lambda|+|\mu|]} \times (\mathbb{P}^2)^{[|\nu|]}$, we can have

$$\left\langle [\overline{C}_{\lambda,\mu,\nu}], [\overline{Z}_{\lambda',\tilde{\mu},\nu}] \right\rangle = \left\langle [\overline{C}_{\lambda,\mu,0}], [\overline{Z}_{\lambda',\tilde{\mu},0}] \right\rangle \cdot \left\langle [\overline{C}_{0,0,\nu}], [\overline{Z}_{0,0,\nu}] \right\rangle \tag{4.0.6}$$

But $\left\langle [\overline{C}_{0,0,\nu}], [\overline{Z}_{0,0,\nu}] \right\rangle = \left\langle [\overline{C}_{0,0,\nu}], \mathfrak{a}_{-\nu}(x)|0\rangle \right\rangle = \tilde{f}_\nu$ by Proposition 3.2.8.

Recalling (4.0.4), we see $\overline{C}_{\lambda,\mu,0} \cap \overline{Z}_{\lambda',\tilde{\mu},0}$ is nonempty. Let $\xi \in \overline{C}_{\lambda,\mu,0} \cap \overline{Z}_{\lambda',\tilde{\mu},0}$. Write $\xi = \xi_3 + \xi_4$ where $\text{supp}(\xi_3) = \{P_0\}$ and

$$\text{supp}(\xi_4) \subset \mathfrak{L} - \{P_0\}.$$

By definition, we have $l(\xi_3) \geq |\lambda|$ and $l(\xi_4) \geq |\tilde{\mu}|$ so that

$$\begin{aligned} |\tilde{\mu}| &\leq l(\xi_4) \\ &= l(\xi) - l(\xi_3) \\ &= (|\lambda| + |\mu|) - l(\xi_3) \\ &\leq |\mu|. \end{aligned}$$

If we recall that $|\tilde{\mu}| = |\mu'|$, we see $|\mu'| \leq |\mu|$. Now, if $|\mu'| < |\mu|$ we have case (ii) and we are done.

Assume $\nu = \nu'$ and $|\mu'| = |\mu|$. Then $l(\xi_4) = |\mu|$ and consequently, $l(\xi_3) = |\lambda|$. Thus, $\xi_3 \in \overline{C}_{\lambda,0,0} \cap \overline{Z}_{\lambda',0,0}$ and $\xi_4 \in \overline{C}_{0,\mu,0} \cap \overline{Z}_{0,\tilde{\mu},0}$, i.e., $\xi_4 \in \overline{C}_{0,\mu,0} \cap L^{\tilde{\mu}}\tilde{\mathfrak{L}}$. By similar logic as above, we have

$$\left\langle [\overline{C}_{\lambda,\mu,0}], [\overline{Z}_{\lambda',\tilde{\mu},0}] \right\rangle = \left\langle [\overline{C}_{\lambda,0,0}], [\overline{Z}_{\lambda',0,0}] \right\rangle \cdot \left\langle [\overline{C}_{0,\mu,0}], [\overline{Z}_{0,\tilde{\mu},0}] \right\rangle. \tag{4.0.7}$$

Note that this is true for every $\tilde{\mu}$ with $|\tilde{\mu}| = |\mu|$. Now recall, by (4.0.1)

$$[\overline{Z}_{\lambda',0,0}] = \left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')}\right) \cdot \frac{1}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)|0\rangle \quad (4.0.8)$$

$$\left[\overline{Z}_{0,\tilde{\mu},0}\right] = \mathfrak{m}_{\tilde{\mu},\tilde{\mathfrak{L}}}|0\rangle = \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle \quad (4.0.9)$$

and by Proposition 3.1.2,

$$[\overline{C}_{0,\mu,0}] = \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle. \quad (4.0.10)$$

Thus, $\left\langle[\overline{C}_{\lambda,\mu,0}], [\overline{Z}_{\lambda',\tilde{\mu},0}]\right\rangle$ is equal to

$$\left\langle [\overline{C}_{\lambda,0,0}], \left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')}\right) \cdot \frac{1}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)|0\rangle \right\rangle \cdot \left\langle \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle\right\rangle . \quad (4.0.11)$$

By (4.0.4) and (4.0.6), we have

$$\tilde{f}_\nu \left\langle[\overline{C}_{\lambda,\mu,0}], [\overline{Z}_{\lambda',\tilde{\mu},0}]\right\rangle = \left(\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')}\right) \cdot (-1)^{|\lambda'|+l(\lambda')+|\nu|+l(\nu)} \left\langle S_{\lambda',\nu}, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle\right\rangle (4.0.12)$$

Thus, combining (4.0.11) and (4.0.12), and dividing out $\displaystyle\prod_{i=1}^{l(\lambda')} i^{m_i(\lambda')}$, we have

$$\left\langle \left\langle [\overline{C}_{\lambda,0,0}], \frac{\tilde{f}_\nu}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)|0\rangle \right\rangle \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle \right\rangle = (-1)^{|\lambda'|+l(\lambda')+|\nu|+l(\nu)} \left\langle S_{\lambda',\nu}, \mathfrak{m}_{\tilde{\mu},\mathfrak{L}}|0\rangle\right\rangle .$$

Now, let us define $\mathbb{L}$ to be the $\mathbb{Z}$-linear span in $H^*((\mathbb{P}^2)^{[n]}, \mathbb{Z})$ of classes of the form

$$\mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle, \quad |\overline{\mu}| = n - |\lambda'| - |\nu|.$$

By Theorem 4.0.21, these $\mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle$ form a basis. Thus, we have

$$\begin{aligned} \left\langle [\overline{C}_{\lambda,0,0}], \frac{\tilde{f}_\nu}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)|0\rangle \right\rangle \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle &= (-1)^{|\lambda'|+l(\lambda')+|\nu|+l(\nu)} S_{\lambda',\nu} \\ &= (-1)^{|\lambda'|+l(\lambda')+|\nu|+l(\nu)} \sum_{|\overline{\mu}|=n-|\lambda'|-|\nu|} e^{\lambda,\mu,\nu}_{\lambda',\overline{\mu},\nu} \mathfrak{m}_{\overline{\mu},\mathfrak{L}}|0\rangle. \end{aligned}$$

Since $|\mu'| = n - |\lambda'| - |\nu|$ and $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu} \neq 0$, we must have $\mu = \mu'$ and

$$\left\langle [\overline{C}_{\lambda,0,0}], \frac{\tilde{f}_\nu}{\mathfrak{z}_{\lambda'}} \mathfrak{a}_{-\lambda'}(1)|0\rangle \right\rangle = (-1)^{|\lambda'|+l(\lambda')+|\nu|+l(\nu)} e^{\lambda,\mu,\nu}_{\lambda',\mu,\nu}. \tag{4.0.13}$$

Since $\nu = \nu'$ and $\mu = \mu'$, $|\lambda| = |\lambda'|$. Now, since $e^{\lambda,\mu,\nu}_{\lambda',\mu,\nu} \neq 0$, by Theorem 3.3.18 and (4.0.13), we have $l(\lambda) = l(\lambda')$ and $\lambda' \geq \lambda$. If $\lambda' = \lambda$, then the inner product in (4.0.13) becomes $\langle e_{\lambda,\lambda}\mathfrak{a}_{-\lambda}(x)|0\rangle, \frac{\tilde{f}_\nu}{\mathfrak{z}_\lambda}\mathfrak{a}_{-\lambda}(1)|0\rangle\rangle$ by Theorem 3.3.18 which is equal to $(-1)^{|\lambda|+l(\lambda)} e_{\lambda,\lambda}\tilde{f}_\nu$ by Lemma 3.2.4. Thus,

$$e^{\lambda,\mu,\nu}_{\lambda,\mu,\nu} = (-1)^{|\nu|+l(\nu)} \cdot e_{\lambda,\lambda} \cdot \tilde{f}_\nu.$$

□

Now recall that Definition 3.3.1 gives a total ordering on partitions, and thus, our following definition also gives a total ordering.

Definition 4.0.23. Suppose we have 3-tuples of partitions (λ, μ, ν) and (λ', μ', ν') such that $|\lambda| + |\mu| + |\nu| = |\lambda'| + |\mu'| + |\nu'|$. Then $(\lambda, \mu, \nu) > (\lambda', \mu', \nu')$ if one of the following is true:

(i) $\nu > \nu'$,

(ii) $\nu = \nu'$ and $\mu > \mu'$, or

(iii) $\nu = \nu', \mu = \mu'$ and $\lambda' > \lambda$.

We now prove our main theorem.

Theorem 4.0.24. *Let λ, μ, and ν be partitions such that $|\lambda| + |\mu| + |\nu| = n$. Then*

$$[\overline{C}_{\lambda,\mu,\nu}] = (-1)^{|\nu|+l(\nu)}\frac{1}{\mathfrak{z}_\nu}\mathfrak{a}_{-\nu}(1)\mathfrak{m}_{\mu,\mathfrak{L}}\mathfrak{a}_{-\lambda}(x)|0\rangle + \sum_{\substack{\lambda',\mu',\nu' \\ |\lambda'|+|\mu'|+|\nu'|=n}} e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}\frac{1}{\mathfrak{z}_{\nu'}}\mathfrak{a}_{-\nu'}(1)\mathfrak{m}_{\mu',\mathfrak{L}}\mathfrak{a}_{-\lambda'}(x)|0\rangle$$

where $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'} \in \mathbb{Z}$ *and one of the following is true about the triple* (λ', μ', ν')*:*

(i) $|\nu'| < |\nu|$

(ii) $\nu' = \nu$ *and* $|\mu'| < |\mu|$

(iii) $\nu' = \nu$, $\mu' = \mu$, $\quad l(\lambda') = l(\lambda)$, *and* $\lambda < \lambda'$.

Proof. By Lemma 4.0.22, it remains to show that

$$e_{\lambda,\lambda} = \tilde{f}_\nu = 1$$

where $\tilde{f}_\nu$ is the positive integer from Proposition 3.2.8 and $e_{\lambda,\lambda}$ is the positive integer from Theorem 3.3.18.

Consider the transition matrix T between the two bases of $H^*((\mathbb{P}^2)^{[n]};\mathbb{Z})$

$$\mathcal{A} = \left\{[\overline{C}_{\lambda,\mu,\nu}]\,\middle|\,|\lambda| + |\mu| + |\nu| = n\right\} \tag{4.0.14}$$

$$\mathcal{B} = \left\{\frac{1}{\mathfrak{z}_\nu}\mathfrak{a}_{-\nu}(1)\mathfrak{m}_{\mu,\mathfrak{L}}\mathfrak{a}_{-\lambda}(x)|0\rangle\,\middle|\,|\lambda| + |\mu| + |\nu| = n\right\}. \tag{4.0.15}$$

Let P be the set of 3-tuples of the form (λ, μ, ν) such that $|\lambda| + |\mu| + |\nu| = n$. Then we have $|P|$ basis elements for $\mathcal{A}$ and $|P|$ basis elements for $\mathcal{B}$. Let us order these using Definition 4.0.23. Let $(\lambda_1, \mu_1, \nu_1)$ be the 3-tuple where $\lambda_1 = \mu_1 = 0$ and $\nu_1 = (\overbrace{1, \ldots, 1}^{n}) = (1^n)$.

Now, for $i = 2, \dots, |P|$, we can inductively define the 3-tuple $(\lambda_i, \mu_i, \nu_i)$ to be the "next" 3-tuple after $(\lambda_{i-1}, \mu_{i-1}, \nu_{i-1})$, i.e. $(\lambda_i, \mu_i, \nu_i) < (\lambda_{i-1}, \mu_{i-1}, \nu_{i-1})$ and there exists no $(\lambda, \mu, \nu) \in P$ such that $(\lambda_i, \mu_i, \nu_i) < (\lambda, \mu, \nu) < (\lambda_{i-1}, \mu_{i-1}, \nu_{i-1})$. Since the ordering from Definition 4.0.23 is a total ordering, this definition makes sense.

We order the basis elements of $\mathcal{A}$ and $\mathcal{B}$ by putting

$$A_i = [\overline{C}_{\lambda_i,\mu_i,\nu_i}] \quad \text{and} \quad B_i = \frac{1}{\mathfrak{z}_{\nu_i}} \mathfrak{a}_{-\nu_i}(1)\mathfrak{m}_{\mu_i,\mathfrak{L}}\mathfrak{a}_{-\lambda_i}(x)|0\rangle.$$

With these orderings the transition matrix T between $\mathcal{A}$ and $\mathcal{B}$ is an upper triangular matrix. Moreover, by Proposition 3.2.8 and Theorem 3.3.18, the positive integers $e_{\lambda,\lambda}$ and $\tilde{f}_\nu$ appear on the diagonal of T. Since $\det(T) = \pm 1$, we must have

$$e_{\lambda,\lambda} = \tilde{f}_\nu = 1.$$

□

Chapter 5

Examples

By Theorem 4.0.24, we now have our correlation between our two bases of

$$H^*((\mathbb{P}^2)^{[n]};\mathbb{Z}).$$

In this chapter, we will explicitly work out the cases of $n = 2$ and $n = 3$.

5.1 Transition matrix for $n = 2$

Let $n = 2$. There are nine basis elements in this case. Let us list the basis elements for both bases $\mathcal{A}$ and $\mathcal{B}$, arranging them decreasing in the ordering we defined above:

$$\begin{array}{ll}
A_1 = [\overline{C}_{0,0,(1,1)}] & B_1 = \frac{1}{2}\mathfrak{a}_{-(1,1)}(1)|0\rangle \\
A_2 = [\overline{C}_{0,0,(2)}] & B_2 = \frac{1}{2}\mathfrak{a}_{-(2)}(1)|0\rangle \\
A_3 = [\overline{C}_{0,(1),(1)}] & B_3 = \mathfrak{a}_{-(1)}(1)\mathfrak{m}_{(1),\mathfrak{L}}|0\rangle \\
A_4 = [\overline{C}_{(1),0,(1)}] & B_4 = \mathfrak{a}_{-(1)}(1)\mathfrak{a}_{-(1)}(x)|0\rangle \\
A_5 = [\overline{C}_{0,(1,1),0}] & B_5 = \mathfrak{m}_{(1,1),\mathfrak{L}}|0\rangle \\
A_6 = [\overline{C}_{0,(2),0}] & B_6 = \mathfrak{m}_{(2),\mathfrak{L}}|0\rangle \\
A_7 = [\overline{C}_{(1),(1),0}] & B_7 = \mathfrak{m}_{(1),\mathfrak{L}}\mathfrak{a}_{-(1)}(x)|0\rangle \\
A_8 = [\overline{C}_{(1,1),0,0}] & B_8 = \mathfrak{a}_{-(1,1)}(x)|0\rangle \\
A_9 = [\overline{C}_{(2),0,0}] & B_9 = \mathfrak{a}_{-(2)}(x)|0\rangle.
\end{array}$$

Now applying several of the theorems above, we can write the elements of basis

$\mathcal{A}$ as linear combinations of elements in $\mathcal{B}$. Since the $n = 2$ case is relatively small, let us explicitly write down these linear combinations:

$$A_1 = B_1 + \sum_{j=3}^{9} d_{1,j} B_j \quad (5.1.1)$$

$$A_2 = -B_2 + \sum_{j=3}^{9} d_{2,j} B_j \quad (5.1.2)$$

$$A_3 = B_3 + \sum_{j=4}^{9} d_{3,j} B_j \quad (5.1.3)$$

$$A_4 = B_4 + \sum_{j=5}^{9} d_{4,j} B_j \quad (5.1.4)$$

$$A_5 = B_5 \quad (5.1.5)$$

$$A_6 = B_6 \quad (5.1.6)$$

$$A_7 = B_7 + \sum_{j=8}^{9} d_{7,j} B_j \quad (5.1.7)$$

$$A_8 = B_8 \quad (5.1.8)$$

$$A_9 = B_9. \quad (5.1.9)$$

where $d_{i,j} \in \mathbb{Z}$. In fact, one will note that these $d_{i,j}$ are exactly the $e^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}$ from Theorem 4.0.24.

Note (5.1.5) and (5.1.6) come directly from Proposition 3.1.2. Likewise, we get (5.1.8) and (5.1.9) from Theorem 3.3.18, since in the case of $n = 2$ the tail in Theorem 3.3.18 would be an empty sum. In the remaining linear combinations, we cannot say exactly what all the integral coefficients are. However, one will notice that by the stipulations of Theorem 4.0.24, for each A_i we have the coefficients must be 0 on any B_j for $j < i$.

It should be obvious that the ordering described in Definition 4.0.23 allows us to have this nice arrangement. For example, in (5.1.3), the coefficient for B_1 must be 0 since $|(1,1)| > |(1)|$ and $(1,1) \neq (1)$. The same can be said for B_2 since $|(2)| > |(1)|$ and $(2) \neq (1)$. In other words, these are the cases where $|\nu'| > |\nu|$. Now, if we look at (5.1.4), again coefficients for B_1 and B_2 must be 0, but also for B_3. Although $\nu' = \nu$, we have $\mu' = (1)$ and $\mu = 0$. Thus, the coefficient on B_3 must be 0.

From Theorem 4.0.24, we can also see for each A_i we get this leading coefficient of ± 1 on B_i. As we hinted at before, the following transition matrix is upper triangular with ± 1 for the diagonal entries. The unknown entries are marked by *.

$$\begin{bmatrix} 1 & 0 & * & * & * & * & * & * & * \\ 0 & -1 & * & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2 Transition matrix for $n = 3$

If $n = 3$, we have 22 elements in each basis. By using the ordering outlined in Definition 4.0.23 and in the proof of Theorem 4.0.24, we can again produce an ordered list of the basis elements for each basis and linear combinations as we did in the previous section. Because of the cumbersome amount of elements, we omit these lists and instead simply provide the transition matrix from $\mathcal{A}$ to $\mathcal{B}$ in the $n = 3$ case. The matrix is

$$
\begin{bmatrix}
1 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & -1 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 1 & * & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & -1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Once again, we note that we have an upper triangular matrix with a determinant of ± 1. Just as in the $n = 2$, there are instances in which previous results allow us to know exactly certain rows. For example, by Proposition 3.1.2, we know

$$[\overline{C}_{0,\mu,0}] = \mathfrak{m}_{\mu,\mathfrak{L}}|0\rangle$$

where $|\mu| = 3$. Thus, we get rows 13-15.

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VITA

Ryan Murphy was born May 20, 1983 in St. Louis, Missouri. He attended high school at De Smet Jesuit High School in St. Louis and graduated from Lindenwood University (St. Charles, MO) with a B.A. in Mathematics. In 2007, he received his M.A. in Mathematics from the University of Missouri - Columbia. He studied Algebraic Geometry under Dr. Zhenbo Qin at UMC and will graduate with his Ph.D. in Mathematics in December 2010.