Endogenous Credit Cycles*

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Abstract

We study models of credit with limited commitment, which implies endogenous borrowing constraints. We show that there are multiple stationary equilibria, as well as nonstationary equilibria, including some that display deterministic cyclic and chaotic dynamics. There are also stochastic (sunspot) equilibria, in which credit conditions change randomly over time, even though fundamentals are deterministic and stationary. We show this can occur when the terms of trade are determined by Walrasian pricing or by Nash bargaining. The results illustrate how it is possible to generate equilibria with credit cycles (crunches, freezes, crises) in theory, and as recently observed in actual economies.

JEL classification: E2

Key words: credit, commitment, dynamics, cycles

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"As we have seen, all financial institutions are at the mercy of our innate inclination to veer from euphoria to despondency." Niall Ferguson, *The Ascent of Money*.

1 Introduction

Without doubt, recent events have put credit markets front and center in economics and finance. As Akerlof and Shiller (2009) put it, "the overwhelming threat to the current economy is the credit crunch." In the *Wall Street Journal* (January 27, 2009) Shiller goes on to say: "To a great extent these traders borrowed short term at low interest rates against collateral of asset-backed securities, of which residential mortgage-backed securities would be just one example. What enabled them to do that? It was the animal spirits. Those who loaned short to the shadow banking sector were confident. They thought they would be repaid...They were trusting. But as soon as these lenders lost their confidence they were no longer trusting. It was like a classic bank run, but this time not on the formal banking sector." Are credit markets susceptible to animal spirits, or extrinsic uncertainty, and why? There is much work on fluctuations in credit markets driven by the fundamentals, by which we mean preferences, technologies and policies (see e.g. Kiyotaki and Moore 1997, or the survey by Gertler and Kiyotaki 2010). The goal here is to construct a model in which movements in credit markets are driven by beliefs, not fundamentals. And we want a theory based on explicit microeconomic foundations – not just a story.

Obviously such a theory will have to contain *frictions* of one form or another, since frictionless models like Arrow-Debreu, where the first welfare theorem says that equilibria are efficient, cannot generate endogenous (self-fulfilling) fluctuations in credit conditions.[1] While our model allows for several frictions, including imperfections in monitoring and collateral, we take a stand on limited commitment being the key to understanding credit markets.

[1] To be more precise, if not pedantic, one might add the adjectives finite-dimensional and convex to frictionless, but that is a detail.

Limited commitment naturally leads to endogenous borrowing limits, which may or may not be binding, in equilibrium. We prove that, if one sets up the model carefully, there can be multiple steady state equilibria with different credit limits and different allocations, as well as dynamic equilibria where credit limits and allocations vary over time even if fundamentals do not. These exotic dynamics include deterministic cyclic and chaotic equilibria, as well as stochastic equilibria, in which credit conditions change randomly over time even though fundamentals are deterministic and time invariant. In these equilibria, credit conditions change for no reason other than beliefs. We think these results are instructive about credit cycles, crunches, freezes, crises, or whatever one likes to call them in actual economies; they are certainly instructive about what can happen in economic theory.

The model contains infinitely-lived agents, where different types may want to borrow or lend at different points in time. Limited commitment means they are free to renege on debts whenever they like. Hence we need some way to punish those who behave badly, or reward those who behave well. As is standard, if agents are caught deviating (not honoring their obligations) they are punished by exclusion from access to future credit, but we allow deviators to be caught only probabilistically (imperfect monitoring). In our environment, collateral mitigates the commitment problem but may not completely solve it. Different mechanisms are considered for determining the terms of trade. We show how to generate endogenous credit market dynamics using Walrasian price taking and generalized Nash bargaining, and show that this is not possible using some other mechanisms, including proportional bargaining or take-it-or-leave-it offers. When endogenous dynamics do arise, the economic forces different under Nash bargaining and Walrasian pricing: in the former case, results hinge on the property of Nash bargaining that agents' individual payoffs need not increase monotonically as the bargaining set expands; in the latter case, they hinge on the fact that payoffs in Walrasian markets need not increase monotonically as we relax quantity restrictions.

There is a large literature on limited commitment and endogenous borrowing constraints.[2] These papers typically consider only Walrasian pricing and do not incorporate the frictions considered here, other than limits to commitment (also much of that literature uses pure exchange, while we have production, but this is a detail). Generally, at least some version of the welfare theorems hold in those models, and they cannot generate fluctuations except through changes in fundamentals. We deliver genuinely endogenous fluctuations. This is similar to the literature on dynamics in monetary economies.[3] Once we reduce our model to a dynamical system, the techniques used to study it are similar to those in monetary theory, which is fine since our objective is to develop an economic model, not mathematical tools. One thing we learn from this is that credit models with limited commitment behave in some ways like monetary models – they can have complicated sets of equilibrium, including some displaying exotic dynamics.[4] In any case, we think the framework has some nice features: it is tractable, yet it generates a variety of interesting outcomes. Moreover, by contrast with other models, endogenous dynamics arise here for very reasonable parameters values, including those measuring risk aversion and discounting.

The presentation is organized as follows. Section 2 lays out the environment. Section 3 defines equilibrium. Section 4 analyzes stationary equilibria. Section 5 analyzes dynamics, including cyclic, chaotic and stochastic equilibria. Section 6 discusses the economics behind the results. Section 7 concludes. All proofs are relegated to an Appendix.

2 Environment

Time is discrete and continues forever. Each period is divided into two subperiods. There are two types of agents of equal measure in the economy: type 1 agents consume good 1

[2] Our set up is close to Kehoe and Levine (1993,2001) and especially Alvarez and Jermann (2000). See also Azariadis and Kass (2007,2008), Lorenzoni (2008), and Hellwig and Lorrenzoni (2009).

[3] We refer readers to Azariadis (1993) for a textbook treatment and references to the original contributions.

[4] Sanchez and Williamson (2010) also highlight the relation between credit and money. There are too many others, so we refer to surveys by Nosal and Rocheteau (2010) and Williamson and Wright (2010a,b).

and produce good 2; type 2 agents consume good 2 and produce good 1. Both goods are produced in the first subperiod, but while good 1 is consumed in the first subperiod, good 2 is consumed in the second. Type 2 thus produce before they consume while type 1 consume before they produce. Moreover, only the producer of good 2 can store or invest it across subperiods. This generates a natural if stylized role for collateralized debt: type 1 gets to consume in the first subperiod in exchange for a promise to deliver goods in the second subperiod out the returns on his investments. When the time to deliver rolls around, type 1 has less of an incentive to renege on his obligation than if he had to produce on the spot, since the cost is sunk. However, so that collateral does not work too well, we assume type 1 can liquidate his investments, by consuming the proceeds himself, say, so that there is an opportunity cost if not a production cost to making good on one's promises.[5]

Agents of different types meet in the first subperiod, and can enter into credit contracts, described as follows. Suppose in the first subperiod type 2 produces x for type 1 to consume, while type 1 produces y, invests it, and delivers the proceeds, say ρy, for type 2 to consume in the second subperiod. The utility from this exchange is $U^1(x, y)$ for type 1 and $U^2(\rho y, x)$ for type 2. It should be clear that we can reduce notation by normalizing $\rho = 1$, with no loss in generality. Both utility functions are strictly increasing in consumption and decreasing in production, strictly concave, twice differentiable, and $U^j(0, 0) = 0$. We also assume normal goods for some results.[6] Once x is produced, type 2 has no reason not to hand it over to type 1 in the first subperiod. But in the second subperiod, type 1 can liquidate (consume) the output y from the previous subperiod, for a payoff of λy over and above the utility from the original consumption of x and production of y. The parameter λ measures the temptation

[5] This captures the idea that, as Ferguson (2008) puts it, "Collateral is, after all, only good if a creditor can get his hands on it." Also, note that technology does not let type 2 store or otherwise invest y himself, or, more generally, at least not as efficiently as type 1, as that eliminates the need for credit. Nor does it allow for goods to be carried across periods, only across subperiods.

[6] That is, at the solution to $\max U^j$ s.t. a standard budget equation, x is increasing and y is decreasing in wealth for type $j = 1$, and vice versa for type $j = 2$. This assumption is only used to clarify certain aspects of the presentation, and is quite mild, especially since many of the points are made by example.

to deviate by reneging on one's obligations, and hence the degree to which collateralized borrowing ameliorates the commitment problem: if $\lambda = 0$, collateral works perfectly. We assume $U^1(x, y) + \lambda y < 0$ for all $x, y \geq 0$, so that it is never efficient ex ante for type 1 to produce and invest for his own consumption. By design, liquidation of collateral may potentially occur ex post only for opportunistic reasons.

Since there is no commitment, credit contracts have to be self enforcing. Therefore, we have to guarantee that (x, y) makes both agents no worse off than walking away, without trading, which gives them a (normalized) payoff of 0 that period, and we have to ensure type 1 does not want to renege in the second subperiod by liquidating y. As is standard, for type 1 the incentive to honor his obligations comes from the threat of exclusion from future credit, which is equivalent here to living in autarky with a payoff of 0. However, we allow imperfect monitoring: a deviant type 1 can only be punished with autarky if he gets caught, and this happens with probability π. Of course the impact of any future punishment depends on the discount rate $\beta \in (0, 1)$, where without loss in generality we assume agents discount across periods but not across subperiods. For many purposes, the discount rate β, the monitoring probability π, and the liquidation parameter λ play a similar role, but it is useful to include all three in the specification for the economic interpretation and for constructing examples.[7]

In terms of the market structure by which agents meet and trade, we consider two scenarios. In the first, we assume they meet bilaterally, where the matching technology is such that each period every type 1 agent matches with a type 2 agent, and vice versa, with probability 1 (it is straightforward to consider more general matching technologies). In each bilateral meeting the agents negotiate the terms of trade (x, y) according to some protocol that we

[7] The environment here, including the liquidation option and imperfect monitoring, is close to the one studied in Mattesini, Monnet and Wright (2010), but the method and application are different – in that paper mechanism design is used to analyze banking. It also bears a resemblance to Lagos and Wright (2005), with its alternating subperiods, but this is superficial – we do not not use the subperiod strucutre to simplify the distribution of money holdings (since there is no money in our model), and we do not need quasi-linear utility, as is required in monetary models. It does turn out, however, that *if* we assume quasi-linear utility, the bargaining solutions are similar (see below) in the two models (see fn. 9 below).

take as a primitive (i.e. the mechanism is part of the environment, not subject to choice). With bilateral meetings, we usually use generalized Nash bargaining, but we also discuss alternative bargaining solutions. In the second scenario, each period agents are randomly assigned to one of a large number of spatially distinct Walrasian markets, in each of which there are enough agents that it makes sense to assume they take as given the price that clears the market.[8] In either case, we can assume agents cannot enter into long-term contracts because they never meet again (see e.g. Aliprantis, Camera and Puzzello 2006, 2007). Also, to avoid issues concerning renegotiation, or the incentive compatibility of punishments, when we say deviators are excluded from future markets we really mean they are excluded – they not only lose access to credit, they do not even participate in the matching process.

3 Equilibrium

Let V_t^j be a type j agent's lifetime expected discounted utility when at date t he enters into the credit arrangement (x_t, y_t). Since we focus on symmetric equilibria, V_t^j does not depend on the individual, only his type $j = 1, 2$. If credit contracts are honored, we have:

$$V_t^1 = U^1(x_t, y_t) + \beta V_{t+1}^1 \quad (1)$$

$$V_t^2 = U^2(y_t, x_t) + \beta V_{t+1}^2 \quad (2)$$

A feasible contract at t must satisfy the *participation constraints* in the first subperiod,

$$U^1(x_t, y_t) \geq 0 \text{ and } U^2(y_t, x_t) \geq 0, \quad (3)$$

as well as the *repayment constraint* for type 1 in the second subperiod,

$$\lambda y_t + (1-\pi)\beta V_{t+1}^1 \leq \beta V_{t+1}^1. \quad (4)$$

[8]If it helps, one may compare these two scenarios to what is done in search models of the labor market: the first corresponds to Mortensen and Pissarides (1996), with bilateral matching and bargaining; the second corresponds to Lucas and Prescott (1974), with price taking in spatially separated competitive markets. See Rocheteau and Wright (2005) for more disucssion.

Condition (4) says type 1 does not want to renege when it comes time to deliver the goods. The LHS is the instantaneous payoff to liquidating y_t, plus the expected continuation value, since with probability π he is caught and excluded from future markets, while with probability $1-\pi$ he gets away with it and continues in good standing.

By defining

$$\phi_t \equiv \frac{\beta\pi}{\lambda} V^1_{t+1} \tag{5}$$

we can rewrite the repayment constraint as

$$y_t \leq \phi_t. \tag{6}$$

Feasible credit arrangements at t cannot specify that type 1 repay more than ϕ_t. Of course, this credit limit is endogenous, and depends on credit conditions in the future. Using (5) and (1), it is useful to express this relationship recursively as

$$\phi_{t-1} = \frac{\beta\pi}{\lambda} U^1\left(x_t, y_t\right) + \beta\phi_t, \tag{7}$$

As (7) indicates, credit limits in one period depend on credit limits in the next period.

3.1 Nash

Agents decide a contract (x_t, y_t) when they meet at t, taking as given what happens in other meetings, at t and in the future. Here we use the generalized Nash bargaining solution to determine (x_t, y_t), where the type 1 agent has bargaining power θ and threat points are given by continuation values. Since the continuation values and threat points cancel, the bargaining outcome solves the following problem:

$$\max_{(x_t, y_t)} U^1\left(x_t, y_t\right)^{\theta} U^2\left(y_t, x_t\right)^{1-\theta} \text{ s.t. } (3) \text{ and } (6). \tag{8}$$

Since it is obvious that (3) is always satisfied, in this problem, we can ignore the participation constraints and focus on the repayment constraint (6).

Let $\left(x^{N*}, y^{N*}\right)$ solve the Nash bargaining problem (8) without the repayment constraint. The necessary and sufficient first-order conditions are

$$\theta U_x^1 \left(x_t, y_t\right) U^2 \left(y_t, x_t\right) + (1-\theta) U^1 \left(x_t, y_t\right) U_x^2 \left(y_t, x_t\right) = 0 \tag{9}$$

$$\theta U_y^1 \left(x_t, y_t\right) U^2 \left(y_t, x_t\right) + (1-\theta) U^1 \left(x_t, y_t\right) U_y^2 \left(y_t, x_t\right) = 0. \tag{10}$$

Given $\phi_t \geq y^{N*}$, we can implement the unconstrained credit contract, where type 1 consumes $x_t = x^{N*}$ in the first subperiod and repays $y_t = y^{N*}$ in the second. But if $\phi_t < y^{N*}$, the unconstrained outcome is not implementable. In this case, we substitute the constraint at equality $y_t = \phi_t$ into (9), the solution to which defines $x_t = h^N(\phi_t)$.[9]

Noting that $x^{N*} = h^N \left(y^{N*}\right)$, we can express the arrangement emerging from bargaining with limited commitment as follows:

$$\begin{array}{ll} \text{if } \phi_t < y^{N*} \text{ then} & x_t = h^N \left(\phi_t\right) \text{ and } y_t = \phi_t \\ \text{if } \phi_t \geq y^{N*} \text{ then} & x_t = h^N \left(y^{N*}\right) \text{ and } y_t = y^{N*} \end{array} \tag{11}$$

Since they are useful in developing economic intuition, we highlight some results about how this contract depends on $\phi_t < y^{N*}$, which for now we take as given. First, we have

$$\frac{\partial x}{\partial \phi} = \frac{-\theta \left(U_{xy}^1 U^2 + U_x^1 U_y^2\right) - (1-\theta) \left(U_y^1 U_x^2 + U^1 U_{xy}^2\right)}{\theta \left(U_{xx}^1 U^2 + U_x^1 U_x^2\right) + (1-\theta) \left(U_x^1 U_x^2 + U^1 U_{xx}^2\right)}. \tag{12}$$

Since the denominator is negative, but the sign of the numerator is ambiguous, consumption by type 1 is not necessarily increasing in his credit limit. We can also derive his payoff:

$$\frac{\partial U^1 \left(x, y\right)}{\partial \phi} = \frac{\theta U^2 \left(U_{xx}^1 U_y^1 - U_{xy}^1 U_x^1\right) + (1-\theta) U^1 \left(U_y^1 U_{xx}^2 - U_x^1 U_{xy}^2\right)}{\theta \left(U_{xx}^1 U^2 + U_x^1 U_x^2\right) + (1-\theta) \left(U_x^1 U_x^2 + U^1 U_{xx}^2\right)}. \tag{13}$$

This numerator is also ambiguous, and as we show below, $\partial U^1 / \partial \phi_t < 0$ is not only possible but inevitable for some values of ϕ_t.

[9] In the special case of quasi-linear utilities, $U^1 = u(x) - y$ and $U^2 = y - v(x)$, $x_t = h^N(\phi_t)$ can be written explicitly as $x_t = g^{-1}(\phi_t)$, where

$$g(x) \equiv \frac{\theta u'(x) v(x) + (1-\theta) u(x) v'(x)}{\theta u'(x) + (1-\theta) v'(x)}.$$

Those who know monetary theory might recognize this as the bargaining solution in Lagos and Wright (2005), with the credit limit ϕ here replacing real balances m/p.

Since $\partial U^1/\partial \phi_t < 0$ is important for understanding the dynamics below, we say a little more about it. This result comes from the well-known property of Nash bargaining that one's payoff does not necessarily increase monotonically as the bargaining set expands (see e.g. Aruoba, Rocheteau and Waller 2007). When ϕ increases, the borrower can get a bigger loan, but perhaps at terms that reduce his payoff. It is easy to check that this cannot happen under take-it-or-leave-it offers, $\theta = 1$, or under some alternative approaches to bargaining, such as Kalai's (1977) proportional solution, both of which imply an agent's surplus is monotonic in the total surplus.

3.2 Walras

Now suppose that agents meet in large groups, where they act as price takers, in the Walrasian sense. Normalizing the price of good y to 1, type 1 maximizes utility subject to his budget and credit constraints:

$$\max_{x_t, y_t} U^1\left(x_t, y_t\right) \text{ s.t. } p_t x_t = y_t \text{ and } (6) \tag{14}$$

Meanwhile, type 2, who has no repayment problem, solves

$$\max_{x_t, y_t} U^2\left(y_t, x_t\right) \text{ s.t. } p_t x_t = y_t \tag{15}$$

Let $\left(x^{W*}, y^{W*}\right)$ denote equilibrium ignoring the repayment constraint, the solution to

$$U_x^1\left(x_t, y_t\right) x_t + U_y^2\left(x_t, y_t\right) y_t = 0 \tag{16}$$

$$U_x^2\left(y_t, x_t\right) x_t + U_y^2\left(y_t, x_t\right) y_t = 0. \tag{17}$$

As in the previous case, write (17) as $x_t = h^W\left(y_t\right)$.

If $\phi_t \geq y^{W*}$, we can implement the unconstrained allocation. If $\phi_t < y^{W*}$, we substitute $y_t = \phi_t$, and solve type 2's problem to get $x_t = h^W\left(\phi_t\right)$. Noting that $x^{W*} = h^W\left(y^{W*}\right)$, the equilibrium arrangement under price taking is:

$$\begin{array}{lll} \text{if } \phi_t < y^{W*} \text{ then} & x_t = h^W\left(\phi_t\right) \text{ and } y_t = \phi_t \\ \text{if } \phi_t \geq y^{W*} \text{ then} & x_t = h^W\left(y^{W*}\right) \text{ and } y_t = y^{W*} \end{array} \tag{18}$$

Again, when $\phi_t < y^{W*}$ consumption by type 1 is not necessarily increasing in ϕ,

$$\frac{\partial x}{\partial \phi} = -\frac{U_y^2 + y\left(U_{yy}^2 - \frac{U_y^2}{U_x^2} U_{xy}^2\right)}{U_x^2 + x\left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2\right)}, \tag{19}$$

since the numerator is ambiguous. Also

$$\frac{\partial U^1(x,y)}{\partial \phi} = \frac{U_y^1 U_x^2 - U_x^1 U_y^2 - y U_x^1 \left(U_{yy}^2 - \frac{U_y^2}{U_x^2} U_{xy}^2\right) + x U_y^1 \left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2\right)}{U_x^2 + x\left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2\right)}, \tag{20}$$

and as we show below, $\partial U^1 / \partial \phi_t < 0$ is again not only possible but inevitable for some ϕ_t.

Hence, a borrower's payoff can decrease with his credit limit in Walrasian markets, just as it can under Nash bargaining. In this case, the effect is due to moving the allocation away from the competitive outcome and closer to the monopsony outcome – not because of self-control problems or other exotica.[10]

3.3 Equilibrium

For convenience, in what follows, we use $h(\phi_t)$ to denote either $h^N(\phi_t)$ or $h^W(\phi_t)$, and y^* to denote either y^{N*} or y^{W*}, depending on the pricing mechanism under consideration. Now note that in any feasible allocation payoffs must be bounded, and hence we can bound the credit limit ϕ_t, analogous to the way one rules our "explosive bubbles" in monetary theory. In particular, as in Alvarez and Jermann (2000), we define equilibria in such a way that ϕ_t gives the *exact* credit limit at every t, even if it is not binding. For instance, imagine a case where the credit constraint is never binding, so that we can implement (x^*, y^*) at every t. There are unbounded sequences for ϕ_t satisfying (7) with the property that $\phi_t \geq y^*$ for all t. But we want ϕ_t to have the property that if a type 1 agent ever found himself off the equilibrium path owing $\phi_t + \varepsilon$, for $\varepsilon > 0$, he would renege.

[10] There are papers where limits can help borrowers who cannot control themselves (Laibson et al. 2000), or even more interestingly cannot control their spouses (Bertaut and Haliassos 2002; Haliassos and Reiter 2003). Speaking of which, we are reminded of the anecdote in which someone told a friend that a theif stole the family credit cards, and was spending \$5,000 a week. When asked why he didn't report it, he said that this was less than his wife used to spend. This is *not* what is going on here.

Given this, ϕ_t must be bounded. We can also bound $x_t \in [0, x^*]$ and $y \in [0, y^*]$ with no loss in generality. Hence we have the following:

Definition 1 An equilibrium is given by nonnegative and bounded sequences of credit limits $\{\phi_t\}_{t=1}^{\infty}$ and contracts $\{x_t, y_t\}_{t=1}^{\infty}$ such that, for all t:

(i) (x_t, y_t) solves (11) or (18) given ϕ_t;

(ii) ϕ_t solves (7).

We can collapse the two equilibrium conditions in Definition 1 into one, by combining (7) with either (11) or (18), depending on the pricing mechanism. This leads to:

$$\phi_{t-1} = f(\phi_t) \equiv \begin{cases} \dfrac{\beta\pi}{\lambda} U^1 [h(\phi_t), \phi_t] + \beta\phi_t & \text{if } \phi_t < y^* \\ \\ \dfrac{\beta\pi}{\lambda} U^1 (x^*, y^*) + \beta\phi_t & \text{otherwise} \end{cases} \tag{21}$$

By eliminating (x_t, y_t), the dynamical system (21) describes the evolution of the credit limit in terms of itself. Equilibria are characterized by nonnegative bounded solutions $\{\phi_t\}$ to (21), from which one can back out the contracts from (11) or (18).

4 Stationary Equilibria

Although we are primarily interested in dynamics, we begin with stationary equilibria, or fixed points (steady states) of the dynamical system, $f(\phi) = \phi$. Obviously $\phi = 0$ is one such point, and it is associated with the degenerate credit contract $(x, y) = (0, 0)$. Intuitively, if there is to be no credit in the future, you have nothing to lose by reneging on debts, so no one will extend you credit, today.[11] We are more interested in nondegenerate equilibria, where $\phi^s > 0$ solves $f(\phi^s) = \phi^s$ and credit is extended. For this not to be vacuous, we adopt the mild assumption $U_x^1(0,0)\, h'(0) + U_y^1(0,0) > \lambda(1-\beta)/\beta\pi$, which guarantees:

[11] This is obviously reminescent of nonmonetary equilibrium in a monetary model, which is one way in which models of money and credit are similar. More on this in Section 5.

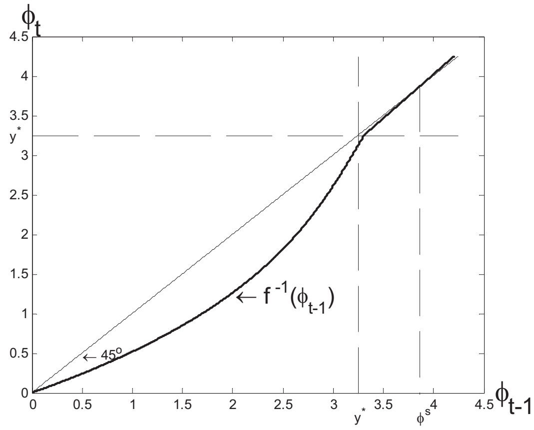

Figure 1-1 Example with $\phi^s > y^*$

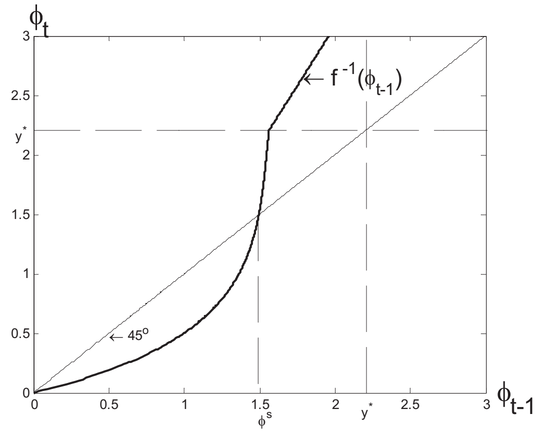

Figure 1-2 Example with $\phi^s < y^*$

Proposition 1 There exists at least one solution to $f(\phi^s) = \phi^s$ with $\phi^s > 0$. If $f(y^*) < y^*$, the repayment constraint is binding at any such solution.

All proofs are in the Appendix, but the result is easy to understand from Figures 1-1 and 1-2, illustrating the two possible cases, in which $\phi^s > y^*$ and $\phi^s < y^*$, respectively. Note that $f(\phi)$ is linearly increasing for $\phi > y^*$, but is not necessarily monotone for $\phi \in (0, y^*)$, so we cannot guarantee uniqueness in general. For most of the rest of the paper, however, we concentrate on cases where ϕ^s is unique.

Following Sanches and Williamson (2010), consider a planner restricted to stationary allocations and respecting limited commitment. One could perhaps interpret the stationarity restriction as an implication of anonymity or a lack of record keeping; here we simply impose it. Stationarity reduces the repayment constraint to $y \leq \frac{\beta\pi}{(1-\beta)\lambda} U^1(x,y)$. This can be written $y \leq \eta(x)$, for the appropriately defined $\eta(y)$, which is simply a clockwise rotation of $U^1(x,y) = 0$ about the origin (see Figure 2-1 and 2-2 below). Therefore, given some value for $U^2 \geq 0$, the planner's problem is

$$\max_{x,y} U^1(x,y) \text{ s.t. } U^2(y,x) = U^2,\ U^1(x,y) \geq 0,\ y \leq \eta(x). \tag{22}$$

Now let $\mathcal{P}$ denote the contract curve from elementary microeconomics

$$\mathcal{P} = \left\{ (x, y) \,|\, \frac{U_x^1 (x, y)}{-U_y^1 (x, y)} = \frac{-U_x^2 (y, x)}{U_y^2 (y, x)} \right\},$$

and let and $\mathcal{C} \subset \mathcal{P}$ denote the core

$$\mathcal{C} = \left\{ (x, y) \,|\, (x, y) \in \mathcal{P},\, U^1 (x, y) \geq 0 \text{ and } U^2 (y, x) \geq 0 \right\}.$$

It is easy to verify that the graphs of $\mathcal{C}$ and $\mathcal{P}$ are downward sloping in (x, y) space under the assumption of normal goods.

Points in $\mathcal{C}$ are efficient with commitment, but may not satisfy the repayment constraint. In order to characterize constrained efficient allocations we proceed using the standard approach: increase U^2, starting from $U^2 = 0$, and the solution to (22) traces out what we call the *constrained core*,

$$\bar{\mathcal{C}} = \left\{ (x, y) \,|\, (x, y) \text{ solves (22) for } U^2 \geq 0 \right\}.$$

In Figures 2-1 and 2-2, the curve from a to d is in the core $\mathcal{C}$, and c is the point of tangency between type 2's indifference curve and the repayment constraint. When the repayment constraint is not too tight, as shown in Figure 2-1, as we increase U^2 we trace out the core below the repayment constraint, then move along the constraint but only as far as c, since moving between c and the origin reduces type 2's payoff. Hence, the indifference curve through c gives an upper bound on U^2. In Figure 2-2, no allocation in $\mathcal{C}$ is achievable, so $\bar{\mathcal{C}}$ lies entirely on the repayment constraint between b and c.[12]

[12] As Christian Hellwig emphasized to us, our model differs from much of the literature on limited commitment in endowment economies. In those models, agents can renege on promises to deliver goods out of their endowment, which leads to $\bar{\mathcal{C}} \subset \mathcal{C}$ (i.e., constrained efficient allocations still entail the tangency of indifference curves). We alternatively give type 1 agents the option to produce, invest, and then inefficiently liquidate. This generates points between b and c in the Figures that are in $\bar{\mathcal{C}}$ but not $\mathcal{C}$ (e.g. point c is a tangency between type 2's indifference curve and the repayment constraint, which is a rotation of type 1's indifference curve through the origin). Hence, in our model, it can be efficient to sacrifice ex ante gains from trade in the interest of ex post incentives.

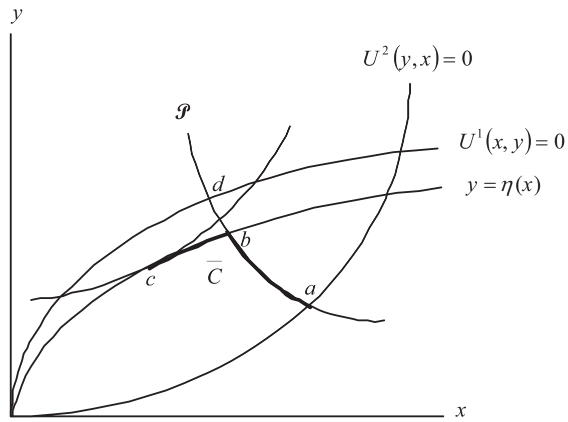

Figure 2-1: Repayment loose

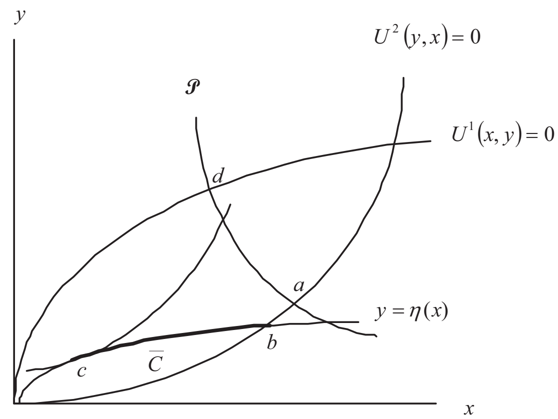

Figure 2-2: Repayment tight

4.1 Nash

Although we are more interested in equilibrium dynamics than welfare economics, we can compare efficient stationary allocations and equilibria. Starting with bargaining, stationary equilibrium is characterized by $x = h^N(y)$ and

$$\frac{U_x^1(x,y)}{-U_y^1(x,y)} = \frac{-U_x^2(y,x)}{U_y^2(y,x)}, \text{ if } y < \eta(x);\ y = \eta(x), \text{ otherwise.} \tag{23}$$

As we vary the parameter $\theta \in [0,1]$, we get different stationary equilibria in the set

$$\mathcal{N} = \left\{ (x,y) \,|\, (x,y) \in \bar{\mathcal{C}} \text{ and } y < \eta(x); \text{ or } \frac{U_x^1(x,y)}{-U_y^1(x,y)} > \frac{-U_x^2(y,x)}{U_y^2(y,x)} \text{ and } y = \eta(x) \right\}.$$

As shown in Figures 3-1 and 3-2, $\mathcal{N}$ includes all allocations on the repayment constraint below the core and (when available) those in the core below the repayment constraint. We already know stationary equilibrium exists, by Proposition 1. The following shows that something like the second welfare theorem holds but the first does not:

Proposition 2 Assume Nash bargaining. For all $(x,y) \in \mathcal{N}$, $\exists\ \theta \in [0,1]$ such that (x,y) is a stationary equilibrium. Since $\bar{\mathcal{C}} \subset \mathcal{N}$, for all $(x,y) \in \bar{\mathcal{C}}$, $\exists\ \theta \in [0,1]$ such that (x,y) is an equilibrium. Since $\mathcal{N} \neq \bar{\mathcal{C}}$ there are equilibria for some θ such that $(x,y) \notin \bar{\mathcal{C}}$.

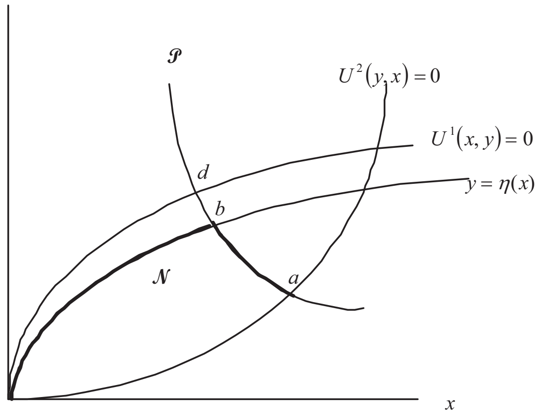

Figure 3-1: Repayment loose

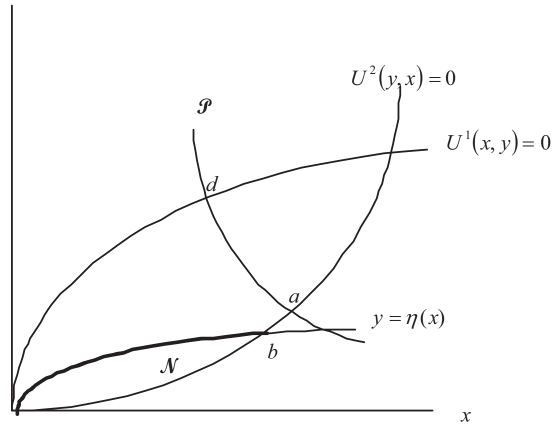

Figure 3-2: Repayment tight

4.2 Walras

Stationary equilibrium under price taking is characterized by $x = h^W(y)$ and

$$\frac{U_x^1(x,y)}{-U_y^1(x,y)} = \frac{-U_x^2(y,x)}{U_y^2(y,x)}, \text{ if } y < \eta(x); \; y = \eta(x), \text{ otherwise}$$

Notice $x = h^W(y)$ is agent 2's *offer curve*. The following is our version of the first welfare theorem (restricting attention to stationary allocations and equilibria):

Proposition 3 Assume Walrasian pricing. In stationary equilibrium, $(x, y) \in \bar{\mathcal{C}}$.

5 Dynamics

Figures 4-1 and 4-2 depict the dynamical system for two examples. Proposition 5 below is reminiscent of what one finds in monetary theory, where there exist multiple monetary equilibria, some of which converge to the autarkic (nonmonetary) steady state. As shown in the Figures, there are also multiple credit equilibria here, some of which converge to the autarkic (no credit) steady state. In Figure 4-1 f is monotonically increasing, while in Figure 4-2 it is not (note that with ϕ_{t-1} on the horizontal and ϕ_t the vertical axis the curve in the graph should be read as f^{-1}). The difference is important: in the first case, once we pick an initial credit limit $\phi_0 \in (0, \phi^s)$ the sequence $\{\phi_t\}$ is pinned down; in the second case, over some range we can pick ϕ_0 and have multiple choices for how to continue $\{\phi_t\}$. This latter case is even consistent with a perfect foresight equilibrium starting and staying at ϕ^s for any number of periods, then dropping to the lower branch of f^{-1} and heading off to autarky – a credit collapse if you ever saw one.

Proposition 4 Suppose there is a unique stationary equilibrium with $\phi^s > 0$. Let $\tilde{\phi} = \arg\max f(\phi_t)$ s.t. $\phi_t \in [0, \phi^s]$. Staring from any $\phi_0 < \tilde{\phi}$, we can construct a nonstationary equilibrium, and possibly more than one, in which $\phi_t \to 0$.

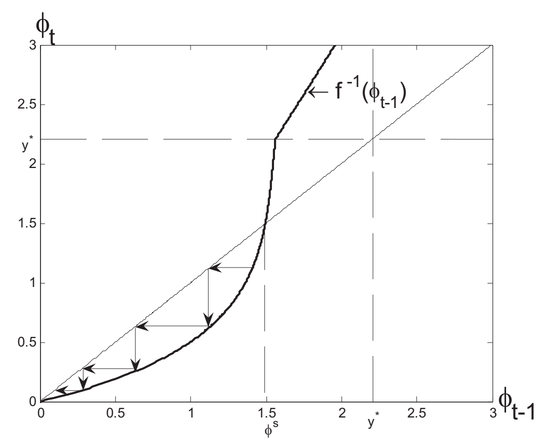

Figure 4-1: Nonstationary equilibria

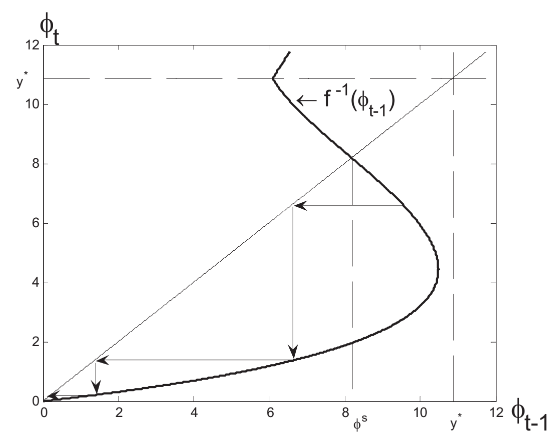

Figure 4-2: Nonstationary equilibria

5.1 Cycles

In this Section, we are interested in deterministic cycles, where credit limits and allocations fluctuate over time purely as self-fulfilling prophecies. Starting with a two-period cycle, let ϕ^1 and $\phi^2 > \phi^1$ denote its periodic points. Then, following textbook methods (e.g. Azariadis 1993), we have:

Proposition 5 Suppose there is a unique stationary equilibrium ϕ^s with $\phi^s > 0$. If $f'(\phi^s) < -1$, there exists a two-period cycle, where $\phi^1 < \phi^s < \phi^2$.

We illustrate the result by way of examples. The examples all use

$$U^1(x,y) = \frac{(x+b)^{1-\alpha} - b^{1-\alpha}}{1-\alpha} - y \text{ and } U^2(y,x) = y - \frac{Ax^{1+\gamma}}{1+\gamma}.$$

Notice the parameter b forces U^1 through the origin, which is useful in some applications (although it is not especially important here). Examples 1 and 2 use Nash bargaining; Examples 3 and 4 use Walrasian pricing

Example 1 Let $\alpha = 2$, $b = 0.082$, $A = 1.5$, $\beta = 0.6$, $\pi/\lambda = 40/3$, $\theta = 0.01$, $\gamma = 0$. Then $\phi^s = 8.96$, $y^* = 10.87$, and there is a two-cycle with $\phi^1 = 7.50 < y^*$ and $\phi^2 = 10.56 < y^*$. See Figure 5-1.

Example 2 Same as Example 1 except $A = 1.1$. Now $\phi^s = 9.35$, $y^* = 11.04$, and there is a two-cycle $\phi^1 = 7.79 < y^*$ and $\phi^2 = 11.63 > y^*$. See Figure 5-2.

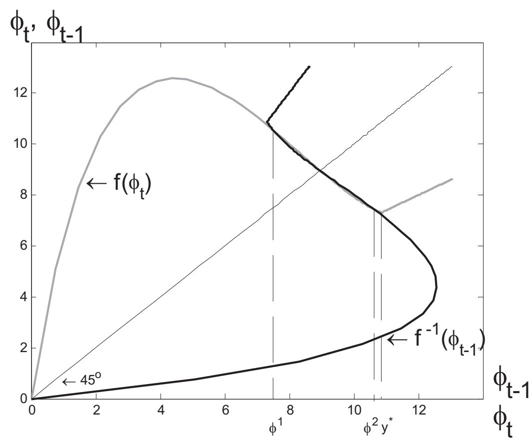

Figure 5-1: Nash cycle, $\phi^1, \phi^2 < y^*$

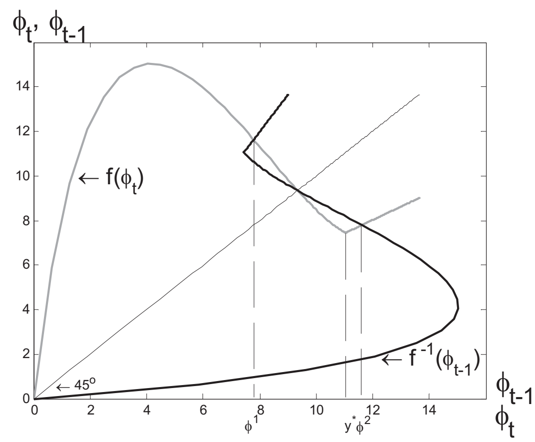

Figure 5-2: Nash cycle, $\phi^1 < y^* < \phi^2$

Example 3 Let $\alpha = 0.01$, $b = 0.2$, $A = 1.5$, $\gamma = 2.5$, $\beta = 0.4$, $\pi/\lambda = 50/9$. Now $\phi^s = 0.61$, $y^* = 0.85$, and there is a two-cycle with $\phi^1 = 0.44 < y^*$ and $\phi^2 = 0.78 < y^*$. See Figure 6-1.

Example 4 Same as Example 3 except $\gamma = 5$. Then $\phi^s = 0.70$ and $y^* = 0.92$, and there is a two-cycle with $\phi^1 = 0.43 < y^*$ and $\phi^2 = 1.04 > y^*$. See Figure 6-2.

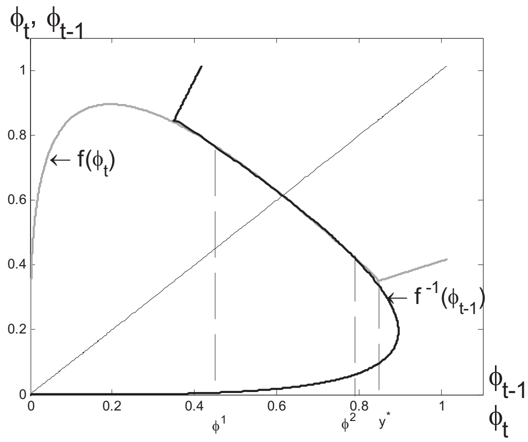

Figure 6-1: Walras cycle, $\phi^1, \phi^2 < y^*$

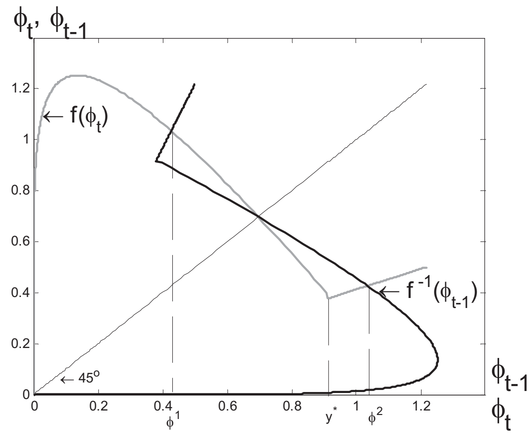

Figure 6-2: Walras cycle, $\phi^1 < y^* < \phi^2$

5.2 Chaos

Before discussing the economics behind results, we point out that our dynamical system can also generate higher-order cycles. Example 5 below shows a three-period cycle. The existence of a three-cycle implies the existence of cycles of all orders (Sarkovskii theorem) and chaotic dynamics (Li-Yorke theorem). Chaos is observationally equivalent to a stochastic process. Thus, credit limits and allocations can appear random, even though they are obviously deterministic in this perfect foresight economy. Proposition 6 below says that in any cycle at least some periodic points are below y^*, so the credit limit must bind at some point over the cycle, although not necessarily all the time. Example 5 is a case in which $\phi < y^*$ in two periods followed by one period in which $\phi > y^*$.

Example 5 Let $\alpha = 2.25$, $b = 0.082$, $A = 1.3$, $\beta = 0.81$, $\pi/\lambda = 40/3$, $\theta = 0.01$, $\gamma = 0$. Now $\phi^s = 16.65$, $y^* = 17.14$, and there is a three-cycle with $\phi^1 = 15.73 < y^*$, $\phi^2 = 17.09 < y^*$ and $\phi^3 = 18.93 > y^*$. See Figures 7 and 8.

Proposition 6 Suppose there is a unique stationary equilibrium with $\phi^s > 0$. In any n-period cycle, at least one periodic point is binding, $\phi_t < y^*$.

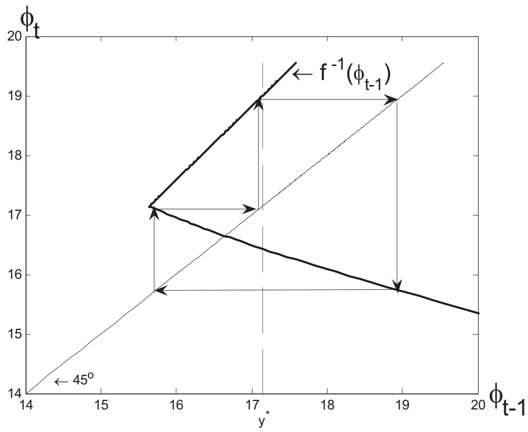

Figure 7: A three-period cycle

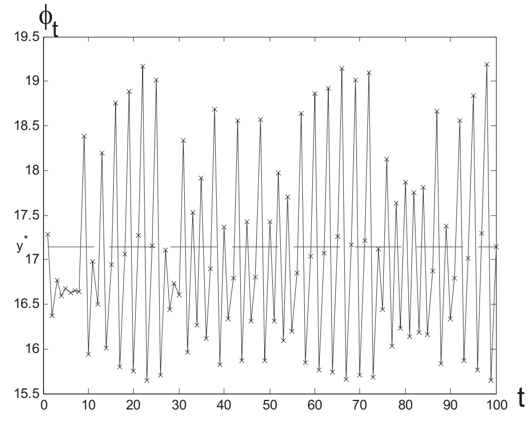

Figure 8: Chaotic dynamics

5.3 Sunspots

The model can also generate stochastic cycles, in which credit limits and allocations fluctuate randomly over time, even though fundamentals (preferences, technologies and government policies) are deterministic and time invariant. If this is not animal spirits, we don't know what is. To illustrate we introduce a Markov sunspot variable $s_t \in \{1, 2\}$ for each t. The sunspot does not affect fundamentals, but as we show, it can still affect equilibrium. Let $\Pr(s_{t+1} = 1 | s_t = 1) = \sigma_1$ and $\Pr(s_{t+1} = 2 | s_t = 2) = \sigma_2$. The economy is in state s if $s_t = s$. Let V_s^j be type j's value function in state s, and let (x_s, y_s) be the allocation.

Agents now trade state-contingent credit contracts $(x_{s,t}, y_{s,t})$, and we can write

$$V_{s,t}^1 = U^1(x_{s,t}, y_{s,t}) + \beta\left[\sigma_s V_{s,t+1}^1 + (1-\sigma_s) V_{-s,t+1}^1\right] \quad (24)$$

$$V_{s,t}^2 = U^2(y_{s,t}, x_{s,t}) + \beta\left[\sigma_s V_{s,t+1}^2 + (1-\sigma_s) V_{-s,t+1}^2\right]. \quad (25)$$

Contracts must satisfy the generalized participation conditions

$$U^1(x_{s,t}, y_{s,t}) \geq 0 \text{ and } U^2(y_{s,t}, x_{s,t}) \geq 0, \quad (26)$$

plus the repayment constraint

$$\lambda y_{s,t} \leq \phi_{s,t} \equiv \beta\pi\left[\sigma_s V_{s,t+1}^1 + (1-\sigma_s) V_{-s,t+1}^1\right]. \quad (27)$$

The relevant recursive representation is now

$$\phi_{s,t-1} = \sigma_s \left[\frac{\beta\pi}{\lambda} U^1 \left(x_{s,t}, y_{s,t}\right) + \beta\phi_{s,t} \right] + (1 - \sigma_s) \left[\frac{\beta\pi}{\lambda} U^1 \left(x_{-s,t}, y_{-s,t}\right) + \beta\phi_{-s,t} \right]. \tag{28}$$

Under Nash bargaining, in state s at t we maximize $U^1 \left(x_{s,t}, y_{s,t}\right)^\theta U^2 \left(y_{s,t}, x_{s,t}\right)^{1-\theta}$, subject to the state-contingent repayment constraint. Equilibrium in state s at date t is then given by:

$$\begin{array}{ll} \text{if } \phi_{s,t} < y^{N*} \text{ then} & x_{s,t} = h^N \left(\phi_{s,t}\right) \text{ and } y_{s,t} = \phi_{s,t} \\ \text{if } \phi_{s,t} \geq y^{N*} \text{ then} & x_{s,t} = h^N \left(y^{N*}\right) \text{ and } y_{s,t} = y^{N*} \end{array} \tag{29}$$

Under Walrasian pricing, agents maximize $U^1 \left(x_{s,t}, y_{s,t}\right)$ and $U^2 \left(y_{s,t}, x_{s,t}\right)$, subject to budget and repayment constraints. Equilibrium in state s at date t is:

$$\begin{array}{ll} \text{if } \phi_{s,t} < y^{W*} \text{ then} & x_{s,t} = h^W \left(\phi_{s,t}\right) \text{ and } y_{s,t} = \phi_{s,t} \\ \text{if } \phi_{s,t} \geq y^{W*} \text{ then} & x_{s,t} = h^W \left(y^{W*}\right) \text{ and } y_{s,t} = y^{W*} \end{array} \tag{30}$$

Definition 2 A sunspot equilibrium is given by nonnegative and bounded sequences of credit limits $\left\{\phi_{s,t}\right\}_{t=1,s=0,1}^{\infty}$ and contracts $\left\{x_{s,t}, y_{s,t}\right\}_{t=1,s=0,1}^{\infty}$ contingent on the state such that, for for all t and s:

(i) $\left(x_{s,t}, y_{s,t}\right)$ solves (29) or (30) given ϕ_t^s;

(ii) $\phi_{s,t}$ solves (28).

Using either of the pricing mechanisms, rewrite (28) as

$$\phi_{s,t-1} = \sigma_s f\left(\phi_{s,t}\right) + (1 - \sigma_s) f\left(\phi_{-s,t}\right), \tag{31}$$

where f is defined as in the benchmark case. The economy is in a *proper* sunspot equilibrium if $\phi_{s,t} \neq \phi_{-s,t}$ for some t. Consider equilibria that depend only on state, not the date, and assume $\phi_2 > \phi_1$. Then the repayment constraint is binding in state 1 (otherwise, we have $x^s = x^*$ and $y^s = y^*$ for both states, which implies $\phi_1 = \phi_2$). Following one standard methods (see e.g. Azariadis 1981), the next result shows that proper sunspot equilibria exist for some parameters.

Proposition 7 Suppose there is a unique stationary equilibrium with $\phi^s > 0$. If $f'(\phi^s) < -1$ then there exist (σ_1, σ_2), $\sigma_1 + \sigma_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of ϕ^s.

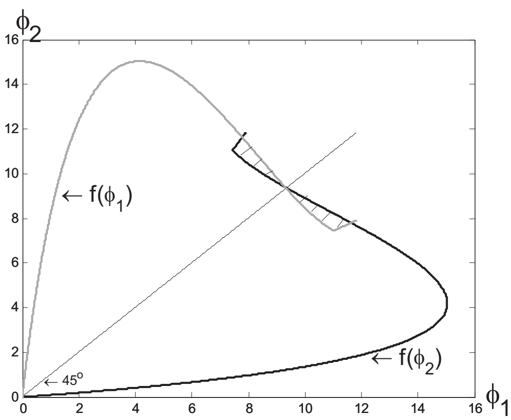

Figure 9-1 Nash sunspot equilibria

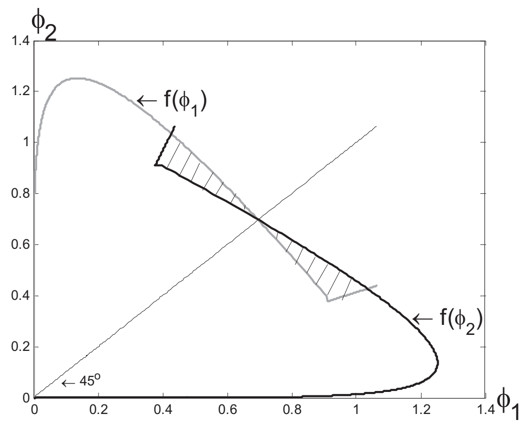

Figure 9-2: Walras sunspot equilibria

The condition $f'(\phi^s) < -1$ in Proposition 7 is the same as the condition used for two-period deterministic cycles. Hence, our previous examples of two-cycles also have sunspot cycles. In Figure 9-1 and 9-2, the shaded area surrounded by $f(\phi_1)$ and $f(\phi_2)$ depicts the set of (ϕ_1, ϕ_2) that can be supported as sunspot equilibria for some σ_1 and σ_2, as stated in the proposition.

6 Economics

The existence of equilibria with deterministic or stochastic cycles relies on the nonmonotonicity of $f(\phi_t)$. To understand this, recall the dynamical system (21), which for convenience we reproduce as

$$\phi_t = f\left(\phi_{t+1}\right) = \frac{\beta\pi}{\lambda} U^1\left(x_{t+1}, y_{t+1}\right) + \beta\phi_{t+1}. \tag{32}$$

An increase in credit limit at $t+1$ influences the economy at t in two ways. First it directly raises ϕ_t through the term $\beta\phi_{t+1}$ on the RHS of (32). This effect in isolation means that when credit is easier tomorrow, agents will have more to gain from access to credit, so they will be less inclined to renege today and hence we can allow them more credit today. But there is a second effect, since ϕ_{t+1} also affects $U^1(x_{t+1}, y_{t+1})$ on the RHS of (32). This effect is ambiguous, in general, but as we mentioned earlier, it is negative when ϕ_{t+1} is near y^* under the mild assumptions of normal goods. In this case, easier credit tomorrow make borrowers worse off, which makes them less inclined to honor their obligations today and hence we can allow them less credit today. When this second, negative, effect is big enough to dominate the first effect, f is nonmonotone.

Proposition 8 If y is a normal good for type 1 and type 2 then in Nash equilibrium $\partial U^1(x, y)/\partial\phi < 0$ for $\phi = y^* - \varepsilon$ for some $\varepsilon > 0$.

Proposition 9 If y is a normal good for type 2 then in Walrasian equilibrium $\partial U^1(x, y)/\partial\phi < 0$ for $\phi = y^* - \varepsilon$ for some $\varepsilon > 0$.

These results should not be too surprising. As remarked earlier, it is known that with nonlinear utility the Nash bargaining solution is not monotone: the surplus of an individual does not necessarily increase with the total surplus. As discussed in Aruoba et al. (2007), this manifests itself in monetary theory with Nash bargaining by buyers being worse off when they have enough money to buy the analog of y^* than they would be if they had just enough to buy $y^* - \varepsilon$ (even when monetary policy is optimal, which means it is given by the Friedman rule). Buyers are better off when the constraint that they cannot spend more money than they have binds slightly. Similarly, our borrowers are better off when the constraint that they cannot borrow more than their credit limit binds slightly. If we set $\theta = 1$, or if we use the proportional bargaining solution of Kalai instead of Nash, since these imply agents' surpluses are monotone in the total surplus we cannot get this effect. Hence, with those bargaining solutions f is monotone and there are no endogenous dynamics.

Lest one is suspicious of results arising from nonmonotonicity or other curious properties of particular bargaining solutions, let us turn to Walrasian pricing. In this case, when the credit limit is relaxed around $y^* - \varepsilon$, the supply of y increases, which means relaxes type 1's credit constraint. This makes him better off at fixed prices, but for small ε this has only a second-order effect on utility (the envelope theorem). The dominant effect is that the terms of trade turn against him: when he is able to promise a bigger y, he may get more x, but even if he does it is not enough to compensate for the bigger repayment. To put it another way, in Walrasian equilibrium a buyer of good y is always better off under the restriction $y \leq y^* - \varepsilon$ for some $\varepsilon > 0$, for the same reason that monopolists produce less than perfectly competitive suppliers. In our Walrasian equilibrium agents are perfectly competitive, so they cannot unilaterally impose quantity restrictions to move prices in their favor, but endogenous credit limits based on limited commitment can get the the job done for them.[13]

While credit constraints can make borrowers better off, they cannot make everyone better off. Propositions 2 and 3 imply $(x, y) \in \bar{\mathcal{C}}$ with Walrasian pricing, and with Nash bargaining at least if θ is not too high. When ϕ is reduced around $y^* - \varepsilon$, someone has to lose, which has to be type 2, the lenders in our economy. Of course, when credit limits are too tight they make everyone worse off (consider $\phi = 0$), but they make borrowers better off if not too tight. As we said above, when credit limits are not too tight, loosening them tomorrow makes borrows worse off tomorrow, and hence more inclined to renege today, which imposes stricter credit constraints today. Notice in (32) that this negative effect of ϕ_{t+1} on ϕ_t is amplified by π/λ, so if π/λ is large then $f(\phi)$ is decreasing around y^*. By choosing $\beta\pi/(1-\beta)\lambda$ appropriately – i.e., close to $y^*/U^1(x^*, y^*)$ – we can ensure that stationary equilibrium is near y^*, which makes it easy to guarantee the critical condition $f'(\phi^s) < -1$ for endogenous

[13] By analogy, in a competitive labor market no individual worker can cause a wage increase by restricting his own labor, but a union can do so by restricting everyone's labor. Our endogenous credit limit similarly gives our borrowers some market power.

dynamics. And because we have some freedom in choosing π, λ, our results do not dependent critically on β or the curvature of the utility function as they do in other models; risk aversion and discounting in our examples are quite reasonable.

In cyclic equilibria, welfare as measured by V^j for type j varies over the cycle. It turns out that sometimes cycles Pareto dominate stationary equilibria; sometimes stationary equilibria dominate cycles; and sometimes they are noncomparable.

Example 6 (continuation of example 4). In stationary equilibrium, $V^1 = 0.28$ and $V^2 = 0.73$. In the two-cycle starting at $\phi^1 = 0.43$, $V^1 = 0.47$ and $V^2 = 0.79$, which dominates the stationary equilibrium. But starting with $\phi^2 = 1.04$, $V^1 = 0.19$ and $V^2 = 1.08$, which is not comparable with the stationary equilibrium.

Example 7 (continuation of example 5). In stationary equilibrium, $V^1 = 1.54$ and $V^2 = 83.51$. In the two-cycle starting at $\phi^1 = 16.17$, $V^1 = 1.65$ and $V^2 = 83.08$, which is not comparable with the stationary equilibrium. But starting with $\phi^2 = 17.79$, $V^1 = 1.50$ and $V^2 = 83.39$, which is dominated by the stationary equilibrium.

Figures 10 and 11 show how in Examples 1 and 3 ϕ affects x, y, U^1, U^2 and $U = U^1 + U^2$ (note that summing utilities makes sense, as the examples are quasi-linear). They also show how ϕ affects the terms of trade, or the interest rate $R = y/x$.[14] These are "partial equilibrium" experiments, showing how certain endogenous variables depend on another endogenous variable ϕ, but dynamic equilibria can be interpreted as moving along the curves. In the examples, both x and y increase until ϕ hits y^*. The payoff of the lender U^2 increases with ϕ, while the payoff of the borrower U^1 first increases then decreases.[15] Note that U^1

[14] In Walrasian equilibrium, in general, the budget equation is $p_x x + y = p_x \bar{x} + \bar{y}$ where $\bar{x}$ and $\bar{y}$ are endowments and we normalize $p_y = 1$ (note that x and p_x could be vectors here). Represent this recursively as $x = \bar{x} - s$ and $y = \bar{y} + sR$, where s is saving and R is the gross interest rate. Eliminating s implies $x = \bar{x} - (y - \bar{y})/R$. Hence, $R = p_x$, and in our economy $p_x = y/x$ by type 2's budget equation.

[15] One can show U^2 increasing in ϕ for ϕ near y^*, and U^j increasing for ϕ near 0. Indeed, with Walrasian pricing, U^2 is globally increasing in ϕ.

not only decreases near y^*, as guaranteed by Propositions 8 and 9, it decreases over a wide range of ϕ. A difference between the Figures is the behavior of R. With Walrasian pricing, Figure 9 shows R increasing with ϕ, which as we discussed is the reason U^1 decreases. With Nash bargaining, Figure 8 shows R decreasing in ϕ, and in this case U^1 falls with ϕ for a different reason, the nonmonotonicity of the Nash solution.

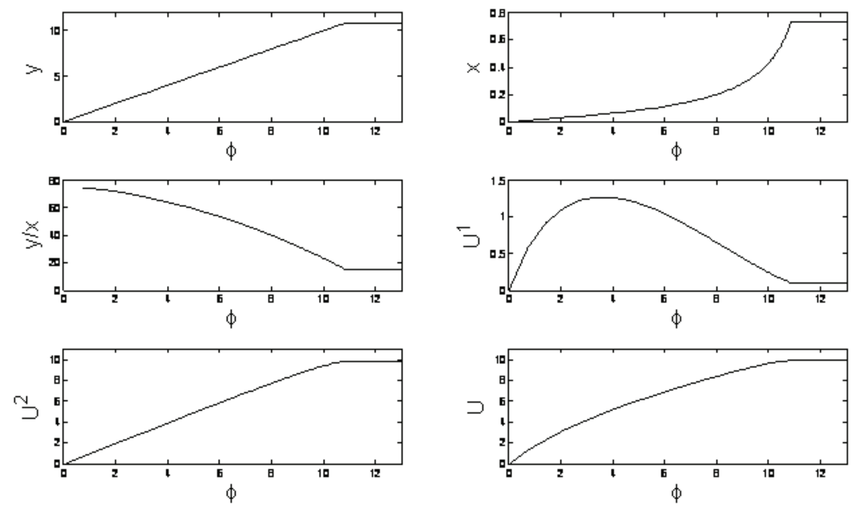

Figure 10: Example 1 continued

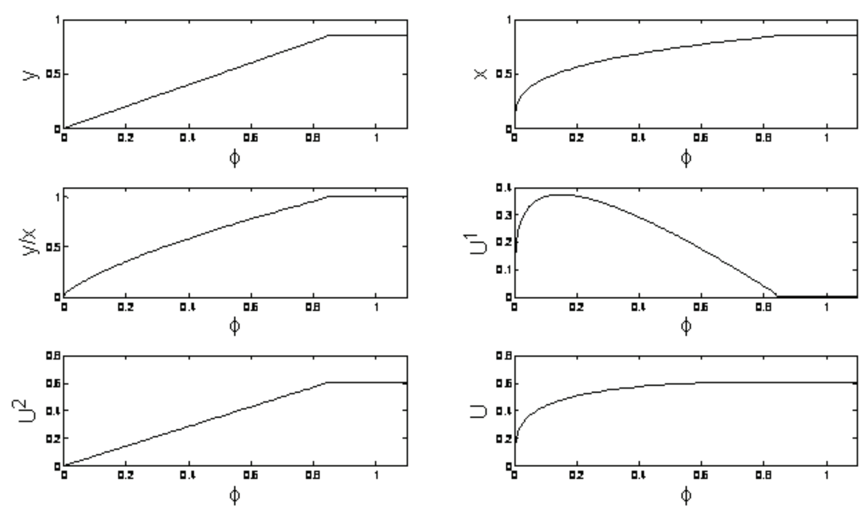

Figure 11: Example 3 continued

Finally, Figure 12 shows time series for the endogenous variables in the example with chaotic dynamics. These series are consistent with the idea that the economy fluctuates between normal times, when credit is easy in the sense that ϕ is high and R low, and crunch time, when the opposite is true. This is driven exclusively by beliefs. While some agents (borrowers in this example) are better off in a credit crunch, others (lenders) are worse off, and since quasi-linear utility allows us to measure total welfare, we can meaningfully say that the economy as a whole is worse off in a crunch. This example may be too "regular" to match actual data – but it is, after all, only an example. Still, a message one might take away from this is that it can be hard to explain actual data purely with animal spirits, at least in a model as simple as this. This suggests that it may be useful to combine self-fulfilling beliefs with changes in fundamentals. Once it is understood that beliefs can generate credit cycles, with no change in preferences, technologies or policies, it must be acknowledged that they can also amplify or propagate shocks to fundamentals.

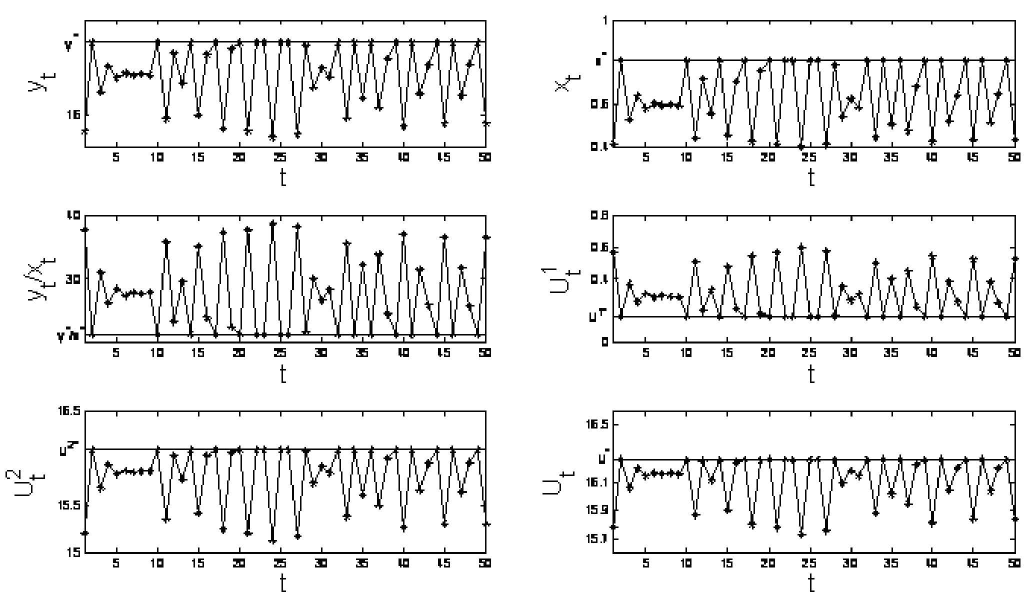

Figure 12: Example 5 continued

7 Conclusion

We developed a framework to study credit market dynamics. There is no fundamental uncertainty, although in principle this would be easy to add. Still there exist multiple equilibria, including deterministic, chaotic, and stochastic credit cycles. This illustrates how agents' beliefs – animal spirits or extrinsic uncertainty – can play an important role in credit markets. Our model contains ingredients that we think are particularly relevant for recent events, including imperfect collateral and imperfect monitoring. Even with these features in the framework, it is still quite tractable. Moreover, in the examples presented, the existence of endogenous credit market dynamics does not depend on unrealistic parameter values. Perhaps endogenous credit cycles are more pervasive than we used to think.

Appendix A

Proof of Proposition 1: Define $T(\phi) = f(\phi) - \phi$. Our parametric assumption implies $T'(0) > 0$. Also, $T'(\phi) < 0$ for $\phi > y^*$. By the continuity and monotonicity of $T(\phi)$ for $\phi > y^*$, it is easy to see the following: if $T(y^*) \geq 0$ then there exists $\phi^s > y^*$ such that $T(\phi^s) = 0$; and if $T(y^*) < 0$ then there exists at least one ϕ^s in $(0, y^*)$ such that $T(\phi^s) = 0$. In the latter case, there is no stationary equilibria in which $\phi^s > y^*$, because $T(\phi)$ is strictly decreasing for $\phi > y^*$. ■

Proof of Proposition 2: When $\theta = 1$, bargaining equilibrium is the same as the planner's allocation with $U^2(x, y) = 0$. When $\theta = 0$, $(x, y) = (0, 0)$ is the equilibrium.

Case 1: The repayment constraint is not binding at $\theta = 1$. The unconstrained equilibrium $(x, y) \in \bar{\mathcal{C}}$ is continuous in θ and has the property that $\dfrac{\partial x}{\partial \theta} > 0$ and $\dfrac{\partial y}{\partial \theta} < 0$. The repayment constraint becomes binding at some $\hat{\theta} \in (0, 1)$. Denote the equilibrium at $\hat{\theta}$ by $(\hat{x}, \hat{y})$ .For $\theta < \hat{\theta}$, the repayment constraint is binding. Equilibrium is characterized by

$$\theta U_x^1(x, \eta(x)) U^2(\eta(x), x) + (1 - \theta) U^1(x, \eta(x)) U_x^2(\eta(x), x) = 0. \tag{33}$$

If $\theta = 0$ then $x = 0$, and if $\theta = \hat{\theta}$ then $x = \hat{x}$. Because x is continuous in θ, for any $x \in [0, \hat{x}]$ there exists $\theta \in \left[0, \hat{\theta}\right]$ s.t. (33) is satisfied. Because the allocation is below the core $\frac{U_x^1(x,y)}{-U_y^1(x,y)} > \frac{-U_x^2(y,x)}{U_y^2(y,x)}$.

Case 2: The repayment constraint in binding at $\theta = 1$. By repeating the argument in case 1 for $\theta < \hat{\theta}$, we conclude the set of equilibrium allocations for $\theta \in [0, 1]$ is that part of the repayment constraint below the core. ■

Proof of Proposition 3: Let (x^s, y^s) be a stationary equilibrium allocation. There two cases.

Case 1: $y^s < \eta(x^s)$. The equilibrium is on the contract curve and in the constrained core.

Case 2: $y^s = \eta(x^s)$. We show x^s is to the right of point c (the tangency point of type

2's indifference curve and the repayment constraint) in Figures 2-1 and 2-2. Denote the allocation at c by $(\tilde{x}, \tilde{y})$. The equilibrium (x^s, y^s) solves $x = h^W(y)$ and $y = \eta(x)$. The slope of type 2 agent's indifference curve at the equilibrium (x^s, y^s) is

$$\frac{-U_x^2(y^s, x^s)}{U_y^2(y^s, x^s)} = \frac{y^s}{x^s} = \frac{\eta(x^s)}{x^s} = \frac{\frac{\beta\pi}{\lambda}U^1(x^s, y^s)}{x^s}. \tag{34}$$

The first equality is simply $x = h^W(y)$; the second follows from $y^s = \eta(x^s)$; the last from the definition of $\eta(x)$. The slope of $\eta(x^s)$ is

$$\eta'(x^s) = \frac{\frac{\beta\pi}{\lambda}U_x^1(x^s, y^s)}{1 - \frac{\beta\pi}{\lambda}U_y^1(x^s, y^s)}. \tag{35}$$

Because $U^1(x, y)$ is concave in x, we have

$$\frac{\frac{\beta\pi}{\lambda}U^1(x^s, y^s)}{x^s} > \frac{\beta\pi}{\lambda}U_x^1(x^s, y^s). \tag{36}$$

By the fact that $U_y^1 < 0$, we have

$$\frac{\beta\pi}{\lambda}U_x^1(x^s, y^s) > \frac{\frac{\beta\pi}{\lambda}U_x^1(x^s, y^s)}{1 - \frac{\beta\pi}{\lambda}U_y^1(x^s, y^s)}. \tag{37}$$

Combining (34) – (37), $\frac{-U_x^2(y^s, x^s)}{U_y^2(y^s, x^s)} > \eta'(x^e)$. Therefore, type 2's indifference curve at x^s intersects $\eta(x)$ from below. The planner's allocation $(\tilde{x}, \tilde{y})$ satisfies $\eta'(\tilde{x}) = \frac{-U_x^2(\tilde{y}, \tilde{x})}{U_y^2(\tilde{y}, \tilde{x})}$. By the concavity of $\eta(x)$ and convexity of type 2's indifference curve, $x^s > \tilde{x}$. ■

Proof of Proposition 4: Because $f(\phi_t)$ is continuous, ϕ_{t-1} covers the interval $\left[0, \tilde{\phi}\right]$ for $\phi_t \in [0, \phi^s]$. Since there is a unique positive stationary equilibrium $f(\phi_t) > \phi_t$ for $\phi_t \in (0, \phi^s)$ and $f(\phi_t) < \phi_t$ for $\phi_t \in (\phi^s, \infty)$. That is $\phi_{t-1} > \phi_t$ for $\phi_t \in (0, \phi^s)$ and $\phi_{t-1} < \phi_t$ for $\phi_t \in (\phi^s, \infty)$. Given $\phi_0 < \tilde{\phi}$, there is a ϕ_1 such that $\phi_1 \in (0, \phi^s)$ and $\phi_1 < \phi_0$, which implies a $\phi_2 \in (0, \phi^s)$ with $\phi_2 < \phi_1$, and so on. This decreasing sequence $\{\phi_t\}_0^\infty$ converges to 0. ■

Proof of Proposition 5: Let $f^2(\phi) = f \circ f(\phi)$. Because ϕ^s is the unique positive stationary equilibrium $f(\phi) > \phi$ for $\phi < \phi^s$ and $f(\phi) < \phi$ for $\phi > \phi^s$. Because $f(\phi)$ is linearly increasing for $\phi > y^*$, there exists a $\tilde{\phi} > y^*$ such that $f\left(\tilde{\phi}\right) > y^*$. By the uniqueness of the positive stationary equilibrium, $f^2\left(\tilde{\phi}\right) < f\left(\tilde{\phi}\right) < \tilde{\phi}$. The slope of $f^2(\phi^s)$ is

$$\frac{df^2(\phi^s)}{d\phi^s} = f'[f(\phi^s)]\, f'(\phi^s) = f'(\phi^s)\, f'(\phi^s) = [f'(\phi^s)]^2 > 1.$$

The last inequality follows from $f'(\phi^s) < -1$. Similarly, $f^2(0) = [f'(0)]^2 > 0$. By continuity, f^2 must cross the 45 degree line in $(0, \phi^s)$. Because f^2 lies below the diagonal at $\tilde{\phi}$, it crosses it at least once in $\left(\phi^s, \tilde{\phi}\right)$. Therefore, there are two more fixed points (in addition to 0 and ϕ^s) such that $0 < \phi^1 < \phi^s < \phi^2$ for $f^2(\phi)$. ■

Proof of Proposition 6: Let ϕ^1, ϕ^2,..., ϕ^n be the periodic points of a n-period. We prove the proposition in two steps.

Step 1: At least one periodic point is less than ϕ^s.

Prove by contradiction. Suppose instead all periodic points are larger than ϕ^s. By the definition of a $n-$period cycle,

$$\phi^1 = f(\phi^n) < \phi^n$$

The inequality follows from the fact that $f(\phi) < \phi$ for $\phi > \phi^s$ by the uniqueness of the positive stationary equilibrium. Repeat the procedure starting from ϕ^n to get

$$\phi^n = f\left(\phi^{n-1}\right) < \phi^{n-1} = f\left(\phi^{n-2}\right) < \phi^{n-2} .... < \phi^1.$$

A contradiction.

Step 2: There does not exist a cycle if $\phi^s > y^*$.

Prove by contradiction. Suppose instead there is a cycle and $\phi^s > y^*$. By step 1, there exist at least one periodic point larger than ϕ^s. Let $\phi^1 > \phi^s$. The periodic point $\phi^2 > \phi^s$ because

$$\phi^2 = f\left(\phi^1\right) > f(\phi^s) = \phi^s$$

The inequality follows from the fact the f is strictly increasing for $\phi > y^*$. Repeat the procedure to get $\phi^i > \phi^s$, $i = 1, ..., n$, which is a contradiction to step 1.

We conclude from steps 1 and 2 that if there exists a cycle, ϕ must be binding in some, if not all, periods. ■

Proof of Proposition 7: Because f is decreasing around ϕ^s, there exists an interval $[\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]$, $\varepsilon_1, \varepsilon_2 > 0$, such that $f(\phi_1) > f(\phi_2)$ for $\phi_1 \in [\phi^s - \varepsilon_1, \phi^s)$, $\phi_2 \in (\phi^s, \phi^s + \varepsilon_2]$. By definition (ϕ_1, ϕ_2), $\phi_1 \neq \phi_2$, is a proper sunspot equilibrium if there exist (σ_1, σ_2), $\sigma_1, \sigma_2 < 1$, such that

$$\phi_1 = \sigma_1 f(\phi_1) + (1 - \sigma_1) f(\phi_2) \tag{38}$$

$$\phi_2 = (1 - \sigma_2) f(\phi_1) + \sigma_2 f(\phi_2). \tag{39}$$

Because ϕ_1 and ϕ_2 are weighted average of $f(\phi_1)$ and $f(\phi_2)$, and $f(\phi_1) > \phi_1$ and $f(\phi_2) < \phi_2$ by the uniqueness of the positive stationary equilibrium, necessary and sufficient conditions for (38)and (39) are

$$f(\phi_2) < \phi_1 < f(\phi_1), \tag{40}$$

$$f(\phi_2) < \phi_2 < f(\phi_1). \tag{41}$$

Because $\phi_1 < \phi_2$ we can reduce (40) and (41) to

$$\phi^2 < f\left(\phi^1\right), \tag{42}$$

$$\phi^1 > f\left(\phi^2\right). \tag{43}$$

Expanding $f(\phi_1)$ and $f(\phi_2)$ around ϕ^s and using $f(\phi^s) = \phi^s$, (42) – (43) are equivalent to

$$\frac{\phi_2 - \phi^s}{\phi^s - \phi_1} < -f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s},$$

Because $-f'(\phi^s) > 1$, $\frac{\phi_2 - \phi^s}{\phi^s - \phi_1} < -f'(\phi^s)$ is redundant if $-f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}$. Now we have two unknowns (ϕ_1, ϕ_2) and only one inequality $-f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}$ to solve. It is straightforward that multiple solutions exist on $[\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]$. To show $\sigma_1 + \sigma_2 < 1$, rewrite (38)and (39) as

$$\sigma_1 + \sigma_2 = \frac{\phi_1 - f(\phi_2) - \phi_2 + f(\phi_1)}{f(\phi_1) - f(\phi_2)} = \frac{\phi_1 - \phi_2}{f(\phi_1) - f(\phi_2)} + 1 < 1,$$

because $\frac{\phi_1 - \phi_2}{f(\phi_1) - f(\phi_2)}$ is negative. ■

Proof of Proposition 8: If $\phi = y^*$, the equilibrium is on the contract curve and $\frac{U_x^1}{-U_y^1} = \frac{-U_x^2}{U_y^2}$. Thus, (13) evaluated as $\phi \to y_-^*$ is

$$\left.\frac{\partial U^1(x,y)}{\partial \phi}\right|_{\phi \to y_-^*} \approx \frac{\theta U^2 U_y^1 \left(U_{xx}^1 - \frac{U_x^1}{U_y^1} U_{xy}^1\right) + (1-\theta) U^1 U_y^1 \left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2\right)}{\theta \left(U_{xx}^1 U^2 + U_x^1 U_x^2\right) + (1-\theta)\left(U_x^1 U_x^2 + U^1 U_{xx}^2\right)}$$

The denominator is negative. The numerator is positive if y is normal for type 1 and type 2. ■

Proof of Proposition 9: If $\phi = y^*$, equilibrium is on the contract curve and $\frac{U_x^1}{-U_y^1} = \frac{-U_x^2}{U_y^2} = \frac{y}{x}$. Thus, (20) evaluated as $\phi \to y_-^*$ is

$$\left.\frac{\partial U^1(x,y)}{\partial \phi}\right|_{\phi \to y_-^*} = \frac{U_y^1}{x}\left[\frac{x^2 U_{xx}^2 + 2xyU_{xy}^2 + y^2 U_{yy}^2}{U_x^2 + x\left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2\right)}\right].$$

The term outside the brackets is negative. The term in brackets is positive as long as y is normal for type 2. ■

Appendix B

Consider a stationary allocation $(x, y) \in \bar{\mathcal{C}}$. We show it can be dominated by a time-varying allocation if the stationary repayment constraint is binding, $y = \eta(x)$, but type 2's participation constraint is not, $U^2(y, x) > 0$. For this we use quasi-linear preferences, $U^1(x, y) = u(x) - y$ and $U^2(y, x) = y - v(x)$.

Since $y = \eta(x)$ binds, $x < x^*$. Consider an alternative allocation $(x_1, y_1) = (x, y + \varepsilon_1)$, $(x_2, y_2) = (x + \delta, y - \varepsilon_2)$, and $(x_t, y_t) = (x, y)$ for $t \geq 3$. We claim there exists $(\varepsilon_1, \varepsilon_2, \delta)$ such that this dominates the original allocation. The difference in payoffs in the two original allocations for type 1 is $\Delta V^1 = \beta[u(x+\delta) - u(x)] - \varepsilon_1 + \beta\varepsilon_2$, and for type 2 $\Delta V^2 = \varepsilon_1 - \beta\varepsilon_2 + \beta[v(x) - v(x+\delta)]$. Set $\Delta V^2 = 0$, so $\Delta V^1 = \beta[u(x+\delta) - v(x+\delta)] - \beta[u(x) - v(x)]$. Because $x < x^*$, we can find δ such that $\Delta V^1 > 0$.

Next, we show (x_1, y_1) and (x_2, y_2) are feasible for some $(\varepsilon_1, \varepsilon_2, \delta)$. By construction, the repayment constraint at $t = 2$, all participation constraints for type 1, and the participation constraints for type 2 at $t = 1$ hold. It remains to check 2's participation constraint at $t = 2$,

$$V_2^2 = y - \varepsilon_2 - v(x+\delta) + \frac{\beta}{1-\beta} U^2(y, x) \geq 0, \tag{44}$$

and the repayment constraints at $t = 1$,

$$\beta \frac{\pi}{\lambda} U^1(x_2, y_2) + \frac{\beta^2}{1-\beta} \frac{\pi}{\lambda} U^1(x, y) \geq y_1. \tag{45}$$

Rewrite (44) to get

$$\frac{1}{1-\beta} U^2(y, x) + v(x) - v(x+\delta) - \varepsilon_2 \geq 0 \tag{46}$$

Because $U^2(y, x) > 0$, we can find ε_2 and δ to satisfy (46). Using $\frac{\beta}{1-\beta}\frac{\pi}{\lambda} U^1(x, y) = y$ to rewrite (45), we get

$$\beta \frac{\pi}{\lambda} [u(x+\delta) - u(x) + \varepsilon_2] \geq \varepsilon_1 \tag{47}$$

By setting ε_1 small (47) is satisfied. ■

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