

# THE VANISHING OF $\mathrm{Tor}_1^R(R^+, k)$ IMPLIES THAT $R$ IS REGULAR

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ABSTRACT. Let  $(R, \mathfrak{m}, k)$  be an excellent local ring of positive prime characteristic. We show that if  $\mathrm{Tor}_1^R(R^+, k) = 0$  then  $R$  is regular. This improves a result of Schoutens, in which the additional hypothesis that  $R$  was an isolated singularity was required for the proof.

Let  $R$  be an integral domain. Then we denote by  $R^+$  the integral closure of  $R$  in an algebraic closure of the fraction field of  $R$ . Under the assumption that  $R$  is a local excellent domain with positive prime characteristic  $p$ , the ring  $R^+$  is a balanced big Cohen-Macaulay algebra [3]. We assume for the rest of this paper that  $R$  is a commutative ring with positive prime characteristic  $p$ . Let  $F : R \rightarrow R$  be the Frobenius endomorphism given by  $r \mapsto r^p$ . It is a theorem of Kunz [6] that  $R$  is regular if and only if  $F$  is a flat map. From this theorem it is not difficult to show that  $R$  is regular if and only if  $R^+$  is flat over  $R$ . The more general question of whether  $\mathrm{Tor}_1^R(R^+, k) = 0$  implies that  $R$  is regular for a local ring  $(R, \mathfrak{m}, k)$  of positive characteristic is posed in the exercises in section 8 of [5] (when  $\mathrm{Tor}_1^R(S, k) = 0$  for a module-finite extension then Nakayama's lemma shows that  $S$  is flat over  $R$ , however,  $R^+$  is far from finitely generated over  $R$ ). Schoutens has shown that for an excellent local ring the condition  $\mathrm{Tor}_1^R(R^+, k) = 0$  implies that  $R$  is weakly  $F$ -regular, and if  $R$  has an isolated singularity then  $R$  is regular ([8], Theorems 1.3 and 1.1). We show here that, in fact, the vanishing of  $\mathrm{Tor}_1^R(R^+, k)$  suffices to imply regularity for excellent rings of positive prime characteristic.

Assume that  $(R, \mathfrak{m}, k)$  is a reduced excellent local ring.  $R$  is then approximately Gorenstein, so there is a sequence of irreducible  $\mathfrak{m}$ -primary ideals  $\{I_t\}$  cofinal with the powers of  $\mathfrak{m}$  (see [2]). By taking a subsequence we may assume that the sequence is non-increasing. Let  $u_t$  be an element of  $R$  representing the socle modulo  $I_t$ . Then the injective hull of the residue field is  $E = E_R(k) = \varinjlim R/I_t$  and the image of  $u_t$  in  $E$  is the socle element  $u$  of  $E$  for all  $t$ . Moreover, because the sequence is non-decreasing we may assume that for all  $t$  there is an injection  $R/I_t \hookrightarrow R/I_{t+1}$  sending  $u_t + I_t \mapsto u_{t+1} + I_{t+1}$ .

Recall that a ring  $R$  of positive prime characteristic is called *F-finite* if the Frobenius endomorphism is module-finite. Such rings are excellent [7], so if in addition  $R$  is reduced then it is approximately Gorenstein. Whenever  $R$  is reduced there is a well-defined ring of

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$q$ th roots of  $R$ , denoted  $R^{1/q}$ , which is a finitely generated  $R$ -module for some (equivalently, all)  $q$  precisely when  $R$  is  $F$ -finite. In this case we will write  $R^{1/q} \cong R^{a_q} \oplus_R M_q$ , where  $M_q$  is a module with no free  $R$  summands.

The characterization of the injective hull given above is very helpful in proving the next Lemma, which shows how to compute the values of  $a_q$  in a special case. By  $I^{[q]}$  we mean the ideal  $(i^q : i \in I)$ .

**Lemma 1.** *Let  $(R, \mathfrak{m}, k)$  be a reduced,  $F$ -finite ring with perfect residue field  $k$ . Then  $a_q = \lambda_R(R/(I_t^{[q]} : u_t^q))$  for all  $t \gg 0$ .*

*Proof.* This result is a special case of Corollary 2.8 of [1]. However, we give a proof here for the benefit of the reader. We will use the fact that over an approximately Gorenstein ring, a homomorphism  $f : R \rightarrow M$ , where  $M$  is finitely generated, has a splitting over  $R$  if and only if for all  $t$ ,  $f(u_t) \notin I_t M$  (see [2]).

Fix  $q$ , and write  $R^{1/q} \cong R^{a_q} \oplus_R M_q$  as above. We first claim that for  $t \gg 0$ ,  $u_t M_q \subseteq I_t M_q$ , since for any minimal generator of  $M_q$ , the map  $Rx \rightarrow M_q$  does not split, and hence,  $xu_t \in I_t M_q$ . The claim follows since  $M_q$  is a finitely generated  $R$ -module. We will also use the fact that if  $I$  is an  $\mathfrak{m}$ -primary ideal then  $\lambda_R(R/I^{[q]}) = \lambda_R(R^{1/q}/IR^{1/q})$ , since  $k$  is perfect.

Thus, for any  $t \gg 0$ , we have

$$\begin{aligned} \lambda(R/(I_t^{[q]} : u_t^q)) &= \lambda(R/I_t^{[q]}) - \lambda(R/(I_t, u_t)^{[q]}) = \lambda(R^{1/q}/I_t R^{1/q}) - \lambda(R^{1/q}/(I_t, u_t)R^{1/q}) \\ &= \lambda(R^{a_q}/I_t R^{a_q}) + \lambda(M_q/I_t M_q) - (\lambda(R^{a_q}/(I_t, u_t)R^{a_q}) + \lambda(M_q/(I_t, u_t)M_q)) \\ &= a_q \cdot 1 + \lambda(M_q/I_t M_q) - \lambda(M_q/(I_t, u_t)M_q) = a_q, \end{aligned}$$

since  $(I_t, u_t)M_q = I_t M_q$  (for  $t \gg 0$ ). □

We will need to pass to a  $\Gamma$  construction as described in [4], Section 6. We refer the reader to [4] for details. What we need to know is as follows. Let  $(R, \mathfrak{m}, k)$  be a complete ring of characteristic  $p$ . Then  $R \rightarrow R^\Gamma$  is a faithfully flat, purely inseparable extension, the maximal ideal of  $R^\Gamma$  is  $\mathfrak{m}R^\Gamma$ , and  $R^\Gamma$  is  $F$ -finite. Note that if  $I \subseteq R$  is an irreducible  $\mathfrak{m}$ -primary ideal of  $R$  then  $IR^\Gamma$  is also an irreducible  $\mathfrak{m}R^\Gamma$ -primary ideal of  $R^\Gamma$ . Moreover, if  $E_R(R/\mathfrak{m}) = \varinjlim_t R/I_t$ , then  $E_{R^\Gamma}(R^\Gamma/\mathfrak{m}R^\Gamma) = E_R(R/\mathfrak{m}) \otimes_R R^\Gamma = \varinjlim_t R^\Gamma/I_t R^\Gamma$ .

Our main theorem is

**Theorem 2.** *Let  $(R, \mathfrak{m}, k)$  be an excellent local domain of positive prime characteristic. Suppose that  $\mathrm{Tor}_1(R^+, k) = 0$ . Then  $R$  is regular.*

*Proof.* By [8], Theorem 1.2, the ring  $R$  is weakly  $F$ -regular, therefore a Cohen-Macaulay, normal domain. In particular,  $R$  is approximately Gorenstein. Also  $R \rightarrow R^+$  is cyclically pure. The assumption that  $\mathrm{Tor}_1(R^+, k) = 0$  and an induction on length shows that for any  $\mathfrak{m}$ -primary ideal  $I \subseteq R$  and element  $x$  we have  $IR^+ :_{R^+} x = (I :_R x)R^+$ .

We first claim that for all  $q$  and all  $t$ ,  $I_t^{[q]} :_R u_t^q \subseteq \mathfrak{m}^{[q]}$ . To see this suppose that  $vu_t^q \in I_t^{[q]}$ . Taking  $q$ th roots shows that  $v^{1/q} \in I_t R^+ :_{R^+} u_t = \mathfrak{m}R^+$ , and hence that  $v \in (\mathfrak{m}^{[q]})^+ = \mathfrak{m}^{[q]}$  (by

cyclic purity of  $R$  in  $R^+$ ). This shows that for all  $q$  and for all  $t$ ,  $\lambda(R/(I_t^{[q]} : u_t^q)) \geq \lambda(R/\mathfrak{m}^{[q]})$ , which is greater than or equal to  $q^d$  ([6]).

We consider  $R \rightarrow \widehat{R} \rightarrow (\widehat{R})^\Gamma = S$  for any Gamma extension of  $\widehat{R}$ . In particular we may take  $\Gamma$  to be the empty set, in which case the residue field of  $S$  is perfect. Then by faithful flatness and the fact that the maximal ideal of  $S$  is  $\mathfrak{m}S$ ,  $\lambda_R(R/(I_t^{[q]} :_R u_t^q)) = \lambda_S(S/(I_t S^{[q]} :_S u_t^q))$ . Since  $u_t^q \notin I_t S^{[q]}$  for all  $t$ , the ring  $S$  is  $F$ -pure, and hence reduced. Thus by Lemma 1, for large enough  $t$  (depending on  $q$ ),  $\lambda_R(R/(I_t^{[q]} :_R u_t^q)) = a_q(S)$  is the number of  $S$ -free summands in  $S^{1/q}$ . Since  $S$  has perfect residue field, the rank of  $S^{1/q}$  as an  $S$ -module is precisely  $q^d$ , hence  $a_q(S) \leq q^d$ . We have now shown that  $q^d \geq \lambda(R/(I_t^{[q]} : u_t^q)) \geq \lambda(R/\mathfrak{m}^{[q]}) \geq q^d$ . Thus  $\lambda(R/\mathfrak{m}^{[q]}) = q^d$  and  $R$  is regular [6].  $\square$

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