WHITTAKER-FOURIER COEFFICIENTS OF METAPLECTIC EISENSTEIN SERIES

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The occurrence of quadratic L-functions in the Fourier coefficients of Eisenstein series of half-integral weight was first discovered in 1937 by Maass [M]. His result is an analog for Eisenstein series of a phenomenon later discovered by Waldspurger [Wa], who showed that the Fourier coefficients of holomorphic cusp forms of half-integral weight are (essentially) square roots of quadratic twists of L-functions attached to cusp forms on $GL(2)$. The Maass phenomenon was further investigated by Siegel [S], by Goldfeld and Hoffstein [GH], and by Goldfeld, Hoffstein, and Patterson [GHP].

In particular, the paper of Siegel foreshadowed more recent work that studies (double) Dirichlet series formed with the quadratic twists of certain L-functions (cf. the survey article of Bump, Friedberg and Hoffstein [BFH]). From this point of view, the paper of Goldfeld and Hoffstein gave applications of the Maass phenomenon to analytic number theory by providing new estimates for the mean values of

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Dirichlet L-functions summed over quadratic twists. Specifically, they estimated

$$\sum_{\substack{|n|<x \\ n \text{ squarefree}}} L(s,\chi_n), \qquad \mathrm{Re}(s) \geq 1/2,$$

and obtained results including:

$$\sum_{\substack{|n|<x \\ n \text{ squarefree}}} L(1,\chi_n) = c_1 x + O(x^{\frac{1}{2}+\epsilon}),$$
$$\sum_{\substack{|n|<x \\ n \text{ squarefree}}} L(1/2,\chi_n) = c_2 x \log x + c_3 x + O(x^{\frac{19}{32}+\epsilon}),$$

where c_1, c_2 and c_3 are certain (computable) constants.

The possibility of a generalization to higher order twists was demonstrated by Bump and Hoffstein [BH1], who (following the related work of Proskurin [P]) established that on the 3-fold metaplectic cover of $GL(3)$, the Whittaker-Fourier coefficients of a certain Eisenstein series contain cubic L-functions. The Eisenstein series they considered are those induced from the generalized theta series on the 3-fold cover of $GL(2)$. Lieman [L], and also Farmer, Hoffstein, and Lieman [FHL], have given applications of this phenomenon to analytic number theory similar to those obtained in the quadratic setting.

Once this result for the cubic case is known, it becomes natural to conjecture that a similar phenomenon occurs for every $n \geq 2$. That is, one expects that n-th order Hecke L-functions will be contained in the Whittaker-Fourier coefficients of an Eisenstein series on the n-fold cover of $GL(n)$ induced from the generalized theta series on the n-fold cover of $GL(n-1)$. We will refer to this as the *L-function conjecture*. This conjecture will undoubtedly have many applications in analytic number theory and is the subject of our paper.

Kazhdan and Patterson [KP] showed that the "exceptional" representations corresponding to the generalized theta series on the n-fold metaplectic covers of $GL(n)$

and of $GL(n-1)$ (taking $c = -1$ in their notation in the latter case) are special in that they have unique Whittaker models. This remarkable fact helps explain why we consider Eisenstein series on the n-fold cover of $GL(n)$ constructed with the theta function on the n-fold cover of $GL(n-1)$. There seems to exist a peculiar "resonance" between the rank of the group and the degree of its cover.

One aspect of this resonance is the uniqueness of Whittaker models for the induced representations corresponding to these Eisenstein series. These are not exactly Whittaker models in the usual sense but are models for the subgroup obtained by extending the maximal unipotent in the n-fold cover of $GL(n)$ by the full preimage in the metaplectic group of the center of $GL(n)$, which is abelian but not central. The uniqueness of these models was proved by Gelbart, Howe and Piatetski-Shapiro [GHP-S] when $n = 2$ and by Bump and Lieman [BL] in general. See also Theorem 3.1 below.

This uniqueness, which is a purely local result, underlies the L-function conjecture, for it implies that the Whittaker integrals of the Eisenstein series are Euler products, just as the uniqueness of Whittaker models for (nonmetaplectic) $GL(n)$ implies that the global Whittaker model is Eulerian. See Proposition 9.2 of Jacquet and Langlands [JL] or Theorem 3.5.4 of Bump [B] for this standard argument.

Evaluation of these Euler products is therefore an essentially local matter. Results of Kazhdan and Patterson reduce the proof of the L-function conjecture to a combinatorial problem involving identities among n-th order Gauss sums. Nevertheless, the combinatorial difficulties involved are quite substantial.

In this paper, we prove local results leading to a proof of the L-function conjecture over any global field that contains the n-th roots of unity. We also prove in this paper a generalization of the L-function conjecture that includes twists of these L-series by arbitrary Hecke characters.

Another proof of the L-function conjecture can also be found in the important and difficult paper of T. Suzuki [Su], whose work we now discuss.

Bump and Hoffstein [BH2] also made a more general conjecture concerning Fourier coefficients of Eisenstein series on the metaplectic group. If f is an automorphic form on the n-fold cover of $GL(r)$, and if $k < n$, then $GL(n+r-k)$ has a parabolic subgroup whose Levi factor is $GL(r) \times GL(n-k)$, and one may form an Eisenstein series induced from f and the theta function on the n-fold cover of $GL(n-k)$. Bump and Hoffstein conjectured that a Whittaker-Fourier coefficient of this Eisenstein series is equal to a Rankin-Selberg integral involving f and a theta function on the n-fold cover of $GL(k)$. In the special case where $r = k = 1$, the corresponding L-function is simply an n-th order Hecke L-function; the L-function conjecture described above is therefore a special case of the general conjectures of Bump and Hoffstein.

The difficulty in establishing the Bump-Hoffstein conjectures in full generality is more than combinatorial, since the methods of Kazhdan and Patterson [KP] yield only partial information about the Whittaker-Fourier coefficients on the n-fold cover of $GL(k)$ if $k \neq n, n-1$. The most that can be said is that the information one is able to obtain is compatible with the conjectures.

Suzuki [Su] managed to overcome these obstacles and prove the general conjectures of Bump and Hoffstein over a function field in which -1 is an n-th power. To do this, he had the insight to use the Rankin-Selberg method in a novel way in order to overcome the apparent incompleteness of the information available on the Whittaker models. For technical reasons, most of his results are stated only in the function field case. An exception, which he states in the case of an arbitrary global field, is his result of Section 7.5 (not in Section 6.4 as stated in his introduction) which is essentially the L-function conjecture.

Because Suzuki relies on the Kazhdan-Patterson cocycle, which is incorrect if -1 is not an n-th power (see [BLS]), the reader approaching his paper should assume that the ground field contains the $2n$-th roots of unity.

In view of the importance of the conjecture, we feel that an independent treatment of the theorem is not superfluous. Our proof relies on a correct cocycle and we do not need to assume that -1 is an n-th power. Interestingly, in the case where -1 is *not* an n-th power in the underlying field, we observe a surprising dichotomy: the L-functions that arise in the Whittaker-Fourier coefficients of the Eisenstein series are either twisted by a certain quadratic Hecke character or they are untwisted, depending only on the residue class of n mod 8. This result is new.

We now turn to a more precise description of our results. Fix once and for all an integer $n \geq 2$, and let k be a global field in which the group μ_n of n-th roots of unity in $k^\times$ has cardinality n. Let $\mathbb{A}$ be the ring of adeles of k. For every $r \geq 1$ and $c \in \mathbb{Z}/n\mathbb{Z}$, the n-fold c-twisted metaplectic group $\widetilde{GL}_r^{(c)}(\mathbb{A})$ is a nontrivial central extension of $GL_r(\mathbb{A})$ by μ_n that is constructed by means of the n-th order (global) Hilbert symbol $(\cdot,\cdot)_{\mathbb{A}} : \mathbb{A}^\times \times \mathbb{A}^\times \to \mu_n$ (cf. [We2]). For any Hecke character $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^\times$, one constructs a theta representation θ_χ of the group $\widetilde{G}'(\mathbb{A}) := \widetilde{GL}_{n-1}^{(-1)}(\mathbb{A})$ as in [KP]. Let P be the standard (maximal) parabolic subgroup in GL_n of type $(n-1,1)$, and let $\widetilde{P}(\mathbb{A})$ be the preimage of $P(\mathbb{A})$ in $\widetilde{G}(\mathbb{A}) := \widetilde{GL}_n^{(0)}(\mathbb{A})$. By means of the embedding:

$$\iota : \widetilde{G}'(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A}), \qquad (g,\xi) \mapsto \left(\begin{pmatrix} g & \\ & \det g^{-1} \end{pmatrix}, \xi \right),$$

the representation θ_χ can be extended to a representation of $\widetilde{P}_n(\mathbb{A})$, the metaplectic preimage of the subgroup $P_n(\mathbb{A})$ consisting of elements of $P(\mathbb{A})$ whose determinants are n-th powers in $\mathbb{A}^\times$. Since θ_χ is automorphic, there exists a nonzero $G(k)$-invariant linear functional Λ on the space of θ_χ. Taking f_s to lie in the induced series $\mathrm{Ind}_{\widetilde{P}_n(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}(\theta_\chi \otimes \delta_P^s)$, where $\delta_P : \widetilde{P}(\mathbb{A}) \to \mathbb{C}^\times$ is the modular character of $P(\mathbb{A})$, we form the metaplectic Eisenstein series:

$$E(g, f_s) := \sum_{\gamma \in P_n(k)\backslash G(k)} \Lambda f_s(\gamma g), \qquad \text{for all } g \in \widetilde{G}(\mathbb{A}).$$

Here $G(k) := GL_n(k)$ is embedded in $\widetilde{G}(\mathbb{A})$ under the canonical splitting [KP]. Let $\psi : \mathbb{A}/k \to \mathbb{C}^\times$ be a fixed nontrivial additive character. For any $a \in k^\times$, the "a-th Whittaker-Fourier coefficient" $W_{s,a}(g)$ of $E(f_s, g)$ is defined by:

$$W_{s,a}(g) := \int_{N(k)\backslash N(\mathbb{A})} E(f_s, ng)\, \overline{\psi}_a(n)\, dn$$

Here N is the standard unipotent subgroup of GL_n, and $\psi_a : N(\mathbb{A})/N(k) \to \mathbb{C}^\times$ is the character given by:

$$\psi_a(n) = \psi(an_{1,2})\psi(n_{2,3}) \dots \psi(n_{n-2,n-1}), \qquad \text{for all } n \in N(\mathbb{A}).$$

Theorem. *If n is odd, or $n \equiv 2$ or $4 \pmod 8$, then the a-th Whittaker-Fourier coefficient of the metaplectic Eisenstein series $E(f_s, g)$ can be expressed as an Euler product:*

$$W_{s,a}(g) = \prod_{v \in S} W^v_{s,a}(g_v) \cdot \prod_{v \notin S} \frac{L_v\big(ns, \chi_v\,(\,\cdot\,, a)_v\big)}{L_v(n^2 s, \chi_v^n)}.$$

If $n \equiv 0$ or $6 \pmod 8$, then:

$$W_{s,a}(g) = \prod_{v \in S} W^v_{s,a}(g_v) \cdot \prod_{v \notin S} \frac{L_v\big(ns, \chi_v\,(\,\cdot\,, -a)_v\big)}{L_v(n^2 s, \chi_v^n)}.$$

The notation may be explained as follows. The set S is any finite collection of places of the global field k that contains every archimedean place and all nonarchimedean places v for which $v(2n) \neq 0$. If $v \notin S$, the local n-th order Hilbert symbol $(\cdot, \cdot)_v : k_v^\times \times k_v^\times \to \mu_n$ is unramified, as is the quadratic Hilbert symbol

$(\cdot,\cdot)_{2,v} : k_v^\times \times k_v^\times \to \{\pm 1\}$. If $g = (g_v)$, we also include in S those places such that the local component g_v does not lie in the canonical lift K_v^* of the standard maximal compact subgroup K_v of $GL_n(k_v)$. We may assume that f_s has the form of a metaplectic tensor product $\widetilde{\otimes} f_{s,v}$, where each $f_{s,v}$ lies in a local induced representation $\mathrm{Ind}_{\widetilde{P}_n(k_v)}^{\widetilde{G}(k_v)}(\theta_{\chi,v} \otimes \delta_{P,v}^s)$, and we include in S those places for which $f_{s,v}$ is not the normalized K_v^*-fixed vector $\phi_{s,v}$ (cf. §3). For each place $v \in S$, $W_{s,a}^v$ is a Whittaker function for the local induced representation. Finally, the local L-functions occurring in the product over $v \notin S$ are defined in the usual way (cf. Theorem 3.2 for a precise definition).

To prove this theorem, one unfolds the integral to write $W_{s,a}(g)$ as

$$\sum_{\gamma \in P(k)\backslash G(k)/N(k)} \int_{N_\gamma(k)\backslash N(\mathbb{A})} \Lambda f_s(\gamma n g)\, \overline{\psi}_a(n)\, dn,$$

where $N_\gamma = N \cap \gamma^{-1} P \gamma$. There are n double cosets in $P\backslash G/N$ with representatives

$$\gamma = \begin{pmatrix} & I_{n-r} \\ I_r & \end{pmatrix}.$$

Only $r = 1$ contributes since otherwise γ conjugates a simple root into the unipotent radical of P and the term vanishes. When $r = 1$ the resulting global integral factorizes into local integrals (3.1) computed in Theorem 3.2. More precisely one splits the integration into $\int_{N_\gamma(k)\backslash N_\gamma(\mathbb{A})}$ and $\int_{N_\gamma(\mathbb{A})\backslash N(\mathbb{A})}$. The first integral produces the Whittaker functional on the theta representation of $\widetilde{GL}(n-1)$, and the second gives the integral (3.1) at every place.

Our theorem asserts that the Whittaker-Fourier coefficients of metaplectic Eisenstein series are essentially quotients of standard (completed) Hecke L-functions:

$$\frac{L\big(ns, \chi\,(\cdot, \pm a)_{\mathbb{A}}\big)}{L(n^2 s, \chi^n)} = \prod_v \frac{L_v\big(ns, \chi_v\,(\,\cdot\,, \pm a)_v\big)}{L_v(n^2 s, \chi_v^n)}.$$

Though we have not attempted to do so here, a more thorough analysis would entail a proof of the nonvanishing of the local Whittaker functions $W_{s,a}^v$ for an appropriately chosen $f_s = \widetilde{\otimes} f_{s,v}$. When $n \geq 3$, this can certainly be accomplished using standard techniques, since the n-fold metaplectic cover splits over $GL_n(\mathbb{C})$ in this case. For nonarchimedean v, the nonvanishing of $W^v_{s,a}$ can certainly be shown if $f_{s,v}$ has sufficiently small support.

As alluded to earlier, the proof of our theorem rests primarily on the calculation of local Euler factors for the metaplectic Eisenstein series at "good" places; the bulk of our work is devoted to this calculation. The paper is organized as follows. In §1, we recall the construction of local metaplectic groups and describe the metaplectic cocycles from [BLS] in a form suitable for calculations. In §2, we review the construction of the (local) exceptional representations on the n-fold -1-twisted cover of $GL(n-1)$; these were first considered in [KP]. The main result in this section (Theorem 2.1) gives an explicit evaluation of the normalized Whittaker function W_θ on certain diagonal elements $\mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})$ in the local metaplectic group. We remark that these are essentially the only elements for which W_θ can be easily evaluated, and it is a fortunate circumstance that we do not need to know the other values of W_θ. In §3, we review the construction of the induced series corresponding to θ, which live on the n-fold 0-twisted cover of $GL(n)$; these are the local representations corresponding to our metaplectic Eisenstein series. The main result in this section (Theorem 3.2) gives an explicit evaluation of the normalized Whittaker function $W_{s,a}$ at the identity; the theorem stated above follows from this result in the manner previously described.

§1. Preliminary notation

Let n be a fixed positive integer, and let $\mathbb{F}$ be a nonarchimedean local field such that the group μ_n of n-th roots of unity in $\mathbb{F}^\times$ has cardinality n. Once and for all, we will fix an embedding $\mu_n \hookrightarrow \mathbb{C}^\times$ and identify μ_n with the group of n-th roots of unity in $\mathbb{C}^\times$.

Let $\mathcal{O}$ denote the ring of integers in $\mathbb{F}$, $\wp$ the unique maximal ideal in $\mathcal{O}$, and q the cardinality of the residue field $\mathcal{O}/\wp$. Let $|\cdot|_{\mathbb{F}}$ denote the absolute value symbol on $\mathbb{F}$, and let $v : \mathbb{F} \to \mathbb{Z} \cup \{\infty\}$ be the corresponding normalized discrete valuation. Then $|x|_{\mathbb{F}} = q^{-v(x)}$ for all $x \in \mathbb{F}$. We fix a prime element $\varpi \in \mathbb{F}$ with $v(\varpi) = 1$.

Let $(\cdot,\cdot)_{\mathbb{F}} : \mathbb{F}^\times \times \mathbb{F}^\times \to \mu_n$ be the *n-th order Hilbert symbol* on $\mathbb{F}$ (cf. [We2] XIII-5); it is a map that satisfies:

$$\begin{aligned}
(xx', y)_{\mathbb{F}} &= (x,y)_{\mathbb{F}}\,(x',y)_{\mathbb{F}},\\
(x, yy')_{\mathbb{F}} &= (x,y)_{\mathbb{F}}\,(x,y')_{\mathbb{F}},\\
(x,y)_{\mathbb{F}}^{-1} &= (y,x)_{\mathbb{F}},\\
(x,-x)_{\mathbb{F}} &= 1,
\end{aligned}$$

for all $x, x', y, y' \in \mathbb{F}^\times$. Also:

$$\left\{x \in \mathbb{F}^\times \,\middle|\, (x,y)_{\mathbb{F}} = 1 \text{ for all } y \in \mathbb{F}^\times\right\} = \mathbb{F}^{\times n},$$

where:

$$\mathbb{F}^{\times n} := \left\{x \in \mathbb{F}^\times \,\middle|\, x = y^n \text{ for some } y \in \mathbb{F}^\times\right\}.$$

In the sequel, we will often assume that the Hilbert symbol is *unramified*, i.e., that $(x,y)_{\mathbb{F}} = 1$ for all $x, y \in \mathcal{O}^\times$. This is equivalent to the condition that $|n|_{\mathbb{F}} = 1$.

For every positive integer r and every $c \in \mathbb{Z}/n\mathbb{Z}$, let $\widetilde{GL}_r^{(c)}(\mathbb{F})$ denote the *n-fold c-twisted metaplectic cover* of $GL_r(\mathbb{F})$; it is a central extension of $GL_r(\mathbb{F})$ by μ_n:

$$1 \to \mu_n \to \widetilde{GL}_r^{(c)}(\mathbb{F}) \xrightarrow{\mathbf{p}} GL_r(\mathbb{F}) \to 1.$$

With r and c fixed for the moment, put $G := GL_r(\mathbb{F})$, and $\widetilde{G} := \widetilde{GL}_r^{(c)}(\mathbb{F})$. Then we may regard $\widetilde{G}$ as the set $G \times \mu_n$ equipped with a multiplication law given by:

$$(g,\xi)(g',\xi') = \big(gg', \xi\xi'\sigma(g,g')\big), \qquad \text{for all } g,g' \in G,\ \xi,\xi' \in \mu_n.$$

Here $\sigma : G \times G \to \mu_n$ is a certain 2-cocycle in $Z^2(G;\mu_n)$ whose properties are described below. The natural projection $\mathbf{p} : \widetilde{G} \to G$ is defined by $(g,\xi) \mapsto g$, and we identify μ_n with the subgroup $\ker(\mathbf{p})$ of $\widetilde{G}$ via the map $\xi \mapsto (I,\xi)$, where I denotes the identity matrix in G. Since $\sigma(g,I) = \sigma(I,g) = 1$ for all $g \in G$ (see below), it follows that μ_n is contained in the center of $\widetilde{G}$. Let $\mathbf{s} : G \to \widetilde{G}$ be the $\mathbf{p}$-section given by $g \mapsto (g,1)$. Then:

$$\mathbf{s}(g)\mathbf{s}(g') = \mathbf{s}(gg')\sigma(g,g'),$$
$$\mathbf{s}(g)\xi = \xi\mathbf{s}(g),$$

for all $g,g' \in G$, $\xi \in \mu_n$.

We will now summarize the properties of $\sigma = \sigma_r^{(c)}$ that are needed for our calculations. First of all, the c-twisted cocycle $\sigma_r^{(c)}$ is obtained from the *untwisted* (i.e., 0-twisted) cocycle $\sigma_r := \sigma_r^{(0)}$ by the relation:

$$\sigma_r^{(c)}(g,g') = \sigma_r(g,g')(\det g, \det g')_{\mathbb{F}}^c, \qquad \text{for all } g,g' \in G. \tag{1.1}$$

The particular cocycle $\sigma_r \in Z^2(G;\mu_n)$ that is used in this paper was constructed in [BLS] from the (bilinear) *Steinberg symbol* $(\cdot,\cdot)_{\mathbb{F}}^{-1}$; for proofs of the basic properties of σ_r, we refer the reader to [BLS].

If $r = 1$, then $G = GL_1(\mathbb{F}) = \mathbb{F}^\times$, and σ_1 is *trivial*, i.e., $\sigma_1(g,g') = 1$ for all $g,g' \in G$ (cf. [BLS] §3 Corollary 8). Note that $\sigma_1^{(c)} = (\cdot,\cdot)_{\mathbb{F}}^c$ for all $c \in \mathbb{Z}/n\mathbb{Z}$.

If $r = 2$, then $G = GL_2(\mathbb{F})$, and σ_2 is the *Kubota cocycle* in $Z^2(G;\mu_n)$ that is defined by:

$$\sigma_2(g,g') := \left(\frac{\mathbf{x}(gg')}{\mathbf{x}(g)}, \frac{\mathbf{x}(gg')}{\mathbf{x}(g')\det g}\right)_{\mathbb{F}}, \qquad \text{for all } g,g' \in G,$$

where for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$:

$$\mathbf{x}(g) := \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

Almost all of the cocycle calculations of this paper can be performed using only the properties of σ_1 and σ_2 stated above, together with the fact that the system of cocycles $\{\sigma_r \,|\, r \geq 1\}$ is *block-compatible* in the following sense.

THEOREM 1.1 *For every standard Levi subgroup of $GL_r(\mathbb{F})$, the following block formula holds:*

$$\sigma_r\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{pmatrix}\right) = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g_i') \prod_{1 \leq i < j \leq k} (\det g_i, \det g_j')_{\mathbb{F}},$$

where $r = r_1 + \ldots + r_k$ with every $r_i \geq 1$, and $g_i, g_i' \in GL_{r_i}(\mathbb{F})$ for $1 \leq i \leq k$. □

This is [BLS] §3 Theorem 11. In particular, if T is the subgroup of diagonal matrices in $G := GL_r(\mathbb{F})$, then the restriction of σ_r to $T \times T$ is given by:

$$(1.2) \qquad \sigma_r(t, t') = \prod_{1 \leq i < j \leq r} (t_i, t_j')_{\mathbb{F}}, \qquad \text{for all } t = \operatorname{diag}(t_i),\ t' = \operatorname{diag}(t_i') \in T.$$

For the remainder of this section, we will assume that $r \geq 2$. We now introduce some notation to be used throughout the sequel. Consider the $(r-1)$ embeddings $\{\iota_i \,|\, 1 \leq i \leq r-1\}$ of $GL_2(\mathbb{F})$ along the diagonal in G:

$$\iota_i : GL_2(\mathbb{F}) \hookrightarrow G, \qquad g \mapsto \begin{pmatrix} I_{i-1} & & \\ & g & \\ & & I_{r-1-i} \end{pmatrix}, \qquad \text{for all } g \in GL_2(\mathbb{F}),$$

where I_k denotes the $(k \times k)$ identity matrix. For each i, let G_i denote the image of ι_i, and observe that the subgroups $\{G_i \,|\, 1 \leq i \leq r-1\}$ generate the group G. As generators for the subgroup $\iota_i\big(SL_2(\mathbb{F})\big)$ of G_i, we take:

$$\begin{aligned}
h_i(x) &:= \iota_i \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}, && \text{for all } x \in \mathbb{F}^\times, \\
n_i(x) &:= \iota_i \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, && \text{for all } x \in \mathbb{F}, \\
w_i &:= \iota_i \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.
\end{aligned}$$

These elements, together with:

$$\begin{aligned}
t_i(x,y) &:= \iota_i \begin{pmatrix} x & \\ & y \end{pmatrix}, && \text{for all } x, y \in \mathbb{F}^\times, \\
s_i &:= \iota_i \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},
\end{aligned}$$

clearly generate the group G_i. By Theorem 1.1 above, it follows that there are $(r-1)$ canonical embeddings $\big\{\tilde{\iota}_i \,\big|\, 1 \le i \le r-1\big\}$ of $\widetilde{GL}_2^{(0)}(\mathbb{F})$ into $\widetilde{G} := \widetilde{GL}_r^{(0)}(\mathbb{F})$ given by:

$$\tilde{\iota}_i : \widetilde{GL}_2^{(0)}(\mathbb{F}) \hookrightarrow \widetilde{G}, \qquad (g,\xi) \mapsto \big(\iota_i(g), \xi\big), \qquad \text{for all } g \in GL_2(\mathbb{F}),\ \xi \in \mu_n.$$

Let $\widetilde{G}_i$ denote the image of $\tilde{\iota}_i$, and note that $\widetilde{G}_i$ is generated by μ_n together with the elements:

$$\begin{aligned}
\tilde{h}_i(x) &:= \mathbf{s}\big(h_i(x)\big), && \text{for all } x \in \mathbb{F}^\times, \\
\tilde{n}_i(x) &:= \mathbf{s}\big(n_i(x)\big), && \text{for all } x \in \mathbb{F}, \\
\tilde{w}_i &:= \mathbf{s}(w_i), \\
\tilde{t}_i(x,y) &:= \mathbf{s}\big(t_i(x,y)\big), && \text{for all } x, y \in \mathbb{F}^\times, \\
\tilde{s}_i &:= \mathbf{s}(s_i).
\end{aligned}$$

In order to describe the cocycle σ_r in a form suitable for calculations, we next recall the characterization of σ_r given in [BLS].

Let N be the standard maximal unipotent subgroup of G, i.e., the set of all upper triangular matrices with 1's along the diagonal. The group N is generated by the collection $\{n_i(x) \,|\, x \in \mathbb{F},\ 1 \le i \le r-1\}$. The metaplectic group $\widetilde{G}$ splits canonically over N via the section $\mathbf{s}$, hence $N^* := \mathbf{s}(N)$ is isomorphic to N. This follows immediately from the fact that σ_r is trivial on $N \times N$. Moreover:

$$\begin{aligned} \sigma_r(g,n) &= \sigma_r(n,g) = 1, \\ \sigma_r(ng, g'n') &= \sigma_r(g,g'), \\ \sigma_r(gn,g') &= \sigma_r(g,ng'), \end{aligned} \tag{1.3}$$

for all $n, n' \in N,\ g, g' \in G$.

Next, let W be the Weyl group of permutation matrices in G, i.e., the collection of matrices with a single 1 in every row and column, and 0's elsewhere. The group W is generated by the simple reflections $\{s_i \,|\, 1 \le i \le r-1\}$. For any $w \in W$, the *length* of w is the smallest integer $\ell = \ell(w)$ such that w can be expressed as a product of ℓ simple reflections: $w = s_{i_1} \ldots s_{i_\ell}$. For any such reduced expression, we form the element $\eta(w) := w_{i_1} \ldots w_{i_\ell}$ (by [Ma] Lemme 6.2, the map $w \mapsto \eta(w)$ is well-defined). Then our cocycle σ_r satisfies:

$$\begin{aligned} \sigma_r\big(t, \eta(w)\big) &= 1, \qquad \text{for all } t \in T,\ w \in W, \\ \sigma_r\big(\eta(w), \eta(w')\big) &= 1, \qquad \text{for all } w, w' \in W \text{ with } \ell(ww') = \ell(w) + \ell(w'). \end{aligned} \tag{1.4}$$

Now let Φ be the set of roots of G relative to T, which can be identified with the collection of ordered pairs $\{(i,j) \,|\, 1 \le i,j \le r,\ i \ne j\}$:

$$t^\alpha := t_i/t_j, \qquad \text{for all } t = \mathrm{diag}(t_i) \in T,\ \alpha = (i,j) \in \Phi.$$

A root $\alpha = (i,j)$ is *positive* [resp. *negative*] if $i < j$ [resp. $i > j$]. The group W acts on T by conjugation:

$$t^w := w^{-1} t w, \qquad \text{for all } t \in T,\ w \in W,$$

hence W also acts on Φ:

$$t^{(w\alpha)} := (t^w)^\alpha, \qquad \text{for all } t \in T,\ w \in W,\ \alpha \in \Phi.$$

The cocycle σ_r satisfies:

$$\sigma_r\big(\eta(w), t\big) = \prod_{\substack{\alpha=(i,j)>0 \\ w\alpha<0}} (-t_j, t_i)_{\mathbb{F}}, \qquad \text{for all } w \in W,\ t = \operatorname{diag}(t_i) \in T. \tag{1.5}$$

Finally, for all $x \in \mathbb{F}$ and $1 \le i \le r-1$, we have that:

$$\sigma_r\big(w_i, n_i(x) w_i\big) = \begin{cases} (x, x)_{\mathbb{F}} & \text{if } x \neq 0, \\ (-1, -1)_{\mathbb{F}} & \text{if } x = 0, \end{cases} \tag{1.6}$$

as is easily verified using Theorem 1.1 and the definition of the Kubota cocycle. The following characterization of σ_r is proved in [BLS] §3 Theorem 7.

Theorem 1.2 *The cocycle σ_r is the unique element of $Z^2(G; \mu_n)$ that satisfies all of the properties in* (1.2) *through* (1.6) *above.* □

For the remainder of this section, we assume that $(\cdot,\cdot)_{\mathbb{F}}$ is unramified. In this situation, the metaplectic group $\widetilde{G}$ splits canonically over the maximal compact subgroup $K := GL_r(\mathcal{O})$ of G (cf. [KP] Proposition 0.1.2). Let $\mathbf{k} : K \to \widetilde{G}$ denote the splitting. By [KP] Proposition 0.1.3, the map $\mathbf{k}$ satisfies:

$$\begin{aligned} \mathbf{k}\big|_{T\cap K} &= \mathbf{s}\big|_{T\cap K}, \\ \mathbf{k}\big|_{W} &= \mathbf{s}\big|_{W}, \\ \mathbf{k}\big|_{N\cap K} &= \mathbf{s}\big|_{N\cap K}, \end{aligned}$$

and these relations determine $\mathbf{k}$ uniquely. Let $K^* := \mathbf{k}(K)$, and for every $m \ge 0$, let $K_m^* := \mathbf{k}(K_m)$, where $K_m := \big\{k \in K \,\big|\, k \equiv I \,(\operatorname{mod} \wp^m)\big\}$. Then the collection $\big\{K_m^* \,\big|\, m \ge 0\big\}$ is a basis of open compact neighborhoods of the identity element $\widetilde{I} := \mathbf{s}(I)$ of $\widetilde{G}$.

This completes our review of the metaplectic groups $\big\{\widetilde{GL}_r^{(c)}(\mathbb{F})\big\}$ and their associated cocycles $\{\sigma_r^{(c)}\}$.

To conclude this section, we recall the definition and some elementary properties of Gauss sums. Let $(\cdot,\cdot)_{2,\mathbb{F}} : \mathbb{F}^\times \times \mathbb{F}^\times \to \{\pm 1\}$ be the quadratic Hilbert symbol on $\mathbb{F}$. We will assume that $(\cdot,\cdot)_{2,\mathbb{F}}$ is also unramified, i.e., that $(x,y)_{2,\mathbb{F}} = 1$ for all $x, y \in \mathcal{O}^\times$. This is equivalent to the assertion that q is odd. Let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be a nontrivial additive character whose conductor is $\mathcal{O}$, and for every $i \in \mathbb{Z}/n\mathbb{Z}$, let $\mathfrak{g}_\psi^{(i)}$ denote the *unnormalized n-th order Gauss sum*:

$$\mathfrak{g}_\psi^{(i)} := q \int_{x \in \mathcal{O}^\times} (\varpi, x)_{\mathbb{F}}^i \, \psi(x/\varpi)\, dx. \tag{1.7}$$

Here dx is the unique additive Haar measure on $\mathbb{F}$ such that $\mathrm{Vol}(\mathcal{O}; dx) = 1$. It is well-known that $\mathfrak{g}_\psi^{(i)}\, \mathfrak{g}_\psi^{(-i)} = q\, (\varpi,\varpi)_{\mathbb{F}}^i$ and $\left|\mathfrak{g}_\psi^{(i)}\right| = \sqrt{q}$ if $i \not\equiv 0 \,(\mathrm{mod}\ n)$. Now let $\widehat{\mathfrak{g}}_\psi$ denote the *normalized quadratic Gauss sum*:

$$\widehat{\mathfrak{g}}_\psi := \sqrt{q} \int_{x \in \mathcal{O}^\times} (\varpi, x)_{2,\mathbb{F}} \, \psi(x/\varpi)\, dx. \tag{1.8}$$

Then $\left|\widehat{\mathfrak{g}}_\psi\right| = 1$. Since $\mathfrak{g}_\psi^{(n/2)} = q^{1/2}\, \widehat{\mathfrak{g}}_\psi$ if n is even, and $(\varpi,\varpi)_{\mathbb{F}} = 1$ if n is odd, it follows that:

$$\prod_{i=1}^{n-1} \mathfrak{g}_\psi^{(i)} = \begin{cases} q^{(n-1)/2} (\varpi,\varpi)_{\mathbb{F}}^{n(n-2)/8}\, \widehat{\mathfrak{g}}_\psi & \text{if } n \text{ is even,} \\ q^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases} \tag{1.9}$$

This relation will be used in §2.

§2. The Whittaker function for the theta representation

We continue to use the notation of §1. Throughout this section, we will assume that $n \geq 2$, $|n|_{\mathbb{F}} = 1$, and q is odd. Let:

$$\mathbb{F}_* := \left\{ x \in \mathbb{F}^\times \,\middle|\, v(x) \equiv 0 \,(\mathrm{mod}\ n) \right\} = \varpi^{n\mathbb{Z}}\mathcal{O}^\times.$$

Since $|n|_{\mathbb{F}} = 1$, $(\cdot,\cdot)_{\mathbb{F}}$ is unramified, and it follows that:

$$\left\{ x \in \mathbb{F}^\times \,\middle|\, (x,y)_{\mathbb{F}} = 1 \text{ for all } y \in \mathbb{F}_* \right\} = \mathbb{F}_*. \tag{2.1}$$

In other words, $\mathbb{F}_*$ is *maximal isotropic* with respect to pairing determined by the Hilbert symbol.

Now let $G := GL_{n-1}(\mathbb{F})$, let $\widetilde{G} := \widetilde{GL}_{n-1}^{(-1)}(\mathbb{F})$, and let $\sigma := \sigma_{n-1}^{(-1)}$ (cf. §1). Let T be the subgroup of diagonal matrices in G. For any $t \in T$ and $1 \leq i \leq n-1$, we denote by t_i the i-th entry of t along the diagonal. Then by (1.1) and (1.2):

$$\mathbf{s}(t)\,\mathbf{s}(t') = \mathbf{s}(t\,t') \cdot \prod_{i<j} (t_i, t'_j)_{\mathbb{F}} \cdot (\det\, t, \det\, t')_{\mathbb{F}}^{-1}, \qquad \text{for all } t, t' \in T. \tag{2.2}$$

Consequently:

$$\mathbf{s}(t)\,\mathbf{s}(t')\,\mathbf{s}(t)^{-1}\mathbf{s}(t')^{-1} = \prod_i (t_i, t'_i)_{\mathbb{F}}^{-1} \cdot (\det\, t, \det\, t')_{\mathbb{F}}^{-1}. \tag{2.3}$$

We define:

$$T_n := \left\{ t \in T \,\middle|\, t_i/t_j \in \mathbb{F}^{\times n} \text{ for all } i,j \right\},$$

$$T_* := \left\{ t \in T \,\middle|\, t_i/t_j \in \mathbb{F}_* \text{ for all } i,j \right\}.$$

By (2.3), it follows that $\widetilde{T}_n := \mathbf{p}^{-1}(T_n)$ is the center of $\widetilde{T}$, and $\widetilde{T}_* := \mathbf{p}^{-1}(T_*)$ is a maximal abelian subgroup of $\widetilde{T}$. Note that if Z is the center of G (i.e., the scalar matrices), then $\widetilde{Z} := \mathbf{p}^{-1}(Z)$ is the center of $\widetilde{G}$.

Recall that if $H \subset G$, $\widetilde{H} := \mathbf{p}^{-1}(H)$, and X is any set on which μ_n acts, then a function $f : \widetilde{H} \to X$ is said to be *genuine* if $f(\xi h) = \xi f(h)$ for all $\xi \in \mu_n$, $h \in \widetilde{H}$.

For the remainder of this section, let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be a fixed nontrivial additive character whose conductor is $\mathcal{O}$, and let $\chi : \mathbb{F}^\times \to \mathbb{C}^\times$ be an unramified quasicharacter. Using ψ and χ, we will next construct a certain *exceptional* representation of the metaplectic group $\widetilde{G}$ (cf. [KP] §I.2). To do this, we first define a genuine quasicharacter $\omega_\theta : \widetilde{T}_* \to \mathbb{C}^\times$ as follows. Let:

$$T'_* := \big\{ t \in T \,\big|\, t_i \in \mathbb{F}_* \text{ for all } i, \text{ and } t_{n-1} = 1 \big\}.$$

By (2.1) and (2.2), it follows that σ is trivial on $T'_* \times T'_*$, hence $\mathbf{s}(T'_*) \cong T'_*$. Since $\widetilde{T}_* = \widetilde{Z} \cdot \mathbf{s}(T'_*)$ with $\widetilde{Z} \cap \mathbf{s}(T'_*) = \{\widetilde{I}\}$, the group $\widetilde{T}_*$ is the direct product of $\widetilde{Z}$ and $\mathbf{s}(T'_*)$. On $\mathbf{s}(T'_*)$, we define ω_θ by:

$$\omega_\theta\big(\mathbf{s}(t)\big) := \chi(\det t)\, \delta_B(t)^{1/2n}, \qquad \text{for all } t \in T'_*. \tag{2.4}$$

Here δ_B denotes the modular character of the Borel subgroup $B := TN$ in G. To define ω_θ on $\widetilde{Z}$, we first observe that by (2.2):

$$\mathbf{s}(x{\cdot}I)\, \mathbf{s}(y{\cdot}I) = \mathbf{s}(x\, y{\cdot}I)(x, y)_{\mathbb{F}}^{n(n-3)/2}, \qquad \text{for all } x, y \in \mathbb{F}^\times.$$

As in §1, let $(\cdot,\cdot)_{2,\mathbb{F}} : \mathbb{F}^\times \times \mathbb{F}^\times \to \{\pm 1\}$ denote the quadratic Hilbert symbol on $\mathbb{F}$. Note that $(\cdot,\cdot)_{2,\mathbb{F}}$ is unramified since q is odd. Let:

$$\varepsilon_2 := \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Then:

$$\mathbf{s}(x{\cdot}I)\mathbf{s}(y{\cdot}I) = \mathbf{s}(xy{\cdot}I)(x, y)_{2,\mathbb{F}}^{\varepsilon_2}, \qquad \text{for all } x, y \in \mathbb{F}^\times. \tag{2.5}$$

Following the ideas of Weil (cf. [We1]), we define $\gamma_\psi : \mathbb{F}^\times \to \{\pm 1, \pm i\}$ to be the map given by:

$$\gamma_\psi(\varpi^k x) := (\varpi, x)_{2,\mathbb{F}}^k (\varpi, \varpi)_{2,\mathbb{F}}^{k(k-1)/2}\, \widehat{\mathfrak{g}}_\psi^k, \qquad \text{for all } k \in \mathbb{Z},\ x \in \mathcal{O}^\times, \tag{2.6}$$

Then it is easily verified that:

$$\gamma_\psi(x)\gamma_\psi(y) = \gamma_\psi(xy)\,(x,y)_{2,\mathbb{F}}, \qquad \text{for all } x,y \in \mathbb{F}^\times. \tag{2.7}$$

Now let:

$$\omega_\theta\big(\xi\,\mathbf{s}(x\cdot I)\big) := \xi\,\chi(x)^{n-1}\gamma_\psi(x)^{\varepsilon_2}, \qquad \text{for all } \xi \in \mu_n,\ x \in \mathbb{F}^\times. \tag{2.8}$$

By (2.5) and (2.7), it follows that $\omega_\theta : \widetilde{Z} \to \mathbb{C}^\times$ is a genuine quasicharacter. Clearly, there exists a unique genuine quasicharacter $\omega_\theta : \widetilde{T}_* \to \mathbb{C}^\times$ that satisfies both (2.4) and (2.8), and after a brief calculation, we obtain the explicit formula:

$$\omega_\theta\big(\xi\,\mathbf{s}(t)\big) = \xi\,\chi(\det t)\,\delta_B(t)^{1/2n}\gamma_\psi(t_{n-1})^{\varepsilon_2}(t_{n-1},t_{n-1})_{\mathbb{F}}^{\varepsilon_4}\prod_{i=1}^{n-2}(t_i,t_{n-1})_{\mathbb{F}}^{i}, \tag{2.9}$$

which is valid for all $\xi \in \mu_n$, $t \in T_*$. Here:

$$\varepsilon_4 := \begin{cases} 1 & \text{if } 4\,\big|\,n, \\ 0 & \text{otherwise.} \end{cases}$$

To establish (2.9), we have used the fact that if n is odd, $(x,x)_{\mathbb{F}} = 1$ for all $x \in \mathbb{F}^\times$. Note that ω_θ is *unramified*, i.e., ω_θ is trivial on $\mathbf{s}(T\cap K)$. Moreover, ω_θ is *exceptional* in the sense of [KP] §I.2:

$$\omega_\theta\big(\tilde{h}_i(x^n)\big) = |x|_{\mathbb{F}}, \qquad \text{for all } x \in \mathbb{F}^\times,\ 1 \le i \le n-2.$$

Now for *any* genuine quasicharacter $\omega : \widetilde{T}_* \to \mathbb{C}^\times$, we extend ω to a quasicharacter of $\widetilde{B}_* := \widetilde{T}_* N^*$ that is trivial on N^*, and let $V(\omega)$ denote the space of the (normalized) induced representation $\mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{G}}(\omega)$ (cf. [KP] §I.2):

$$V(\omega) := \big\{ f \in C^\infty(\widetilde{G}) \,\big|\, f(bg) = (\delta_B^{1/2}\omega)(b)f(g) \text{ for all } b \in \widetilde{B}_*,\ g \in \widetilde{G} \big\}.$$

Here δ_B is regarded as a quasicharacter of $\widetilde{B} := \widetilde{T}N^*$ that is trivial on μ_n, and $(\delta_B^{1/2}\omega)(b) := \delta_B(b)^{1/2}\omega(b)$ for all $b \in \widetilde{B}_*$. The group $\widetilde{G}$ acts on $V(\omega)$ by right translation.

Let (θ, V_θ) be the *exceptional representation* defined as follows. Let w_0 denote the long element of the Weyl group W, let $\tilde{w}_0 := \mathbf{s}(w_0)$, and let $\omega_\theta' : \widetilde{T}_* \to \mathbb{C}^\times$ be the genuine quasicharacter given by:

$$\omega_\theta'(t) := \omega_\theta(\tilde{w}_0^{-1} t \tilde{w}_0), \qquad \text{for all } t \in \widetilde{T}_*.$$

Since ω_θ is *dominant* (cf. [KP] §I.1), we can define the standard intertwining operator $I_{w_0} : V(\omega_\theta) \to V(\omega_\theta')$ by the absolutely convergent integrals:

$$I_{w_0} f(g) := \int\limits_{n \in N^*} f(\tilde{w}_0^{-1} n g)\, dn, \qquad \text{for all } f \in V(\omega_\theta),\ g \in \widetilde{G}.$$

Here dn is the unique Haar measure for N^* such that $\mathrm{Vol}(N^* \cap K^*; dn) = 1$. By [KP] Theorem I.2.9, the image V_θ of I_{w_0} is the unique irreducible subrepresentation of $V(\omega_\theta')$, and V_θ is isomorphic to the unique irreducible subquotient of $V(\omega_\theta)$. Let θ denote the action of $\widetilde{G}$ on V_θ by right translation: $\theta(g)f(g') := f(g'g)$ for all $g, g' \in \widetilde{G}$, $f \in V_\theta$.

The main goal of this section is to calculate special values of the *normalized Whittaker function* W_θ for use in §3. To define W_θ, first observe that since ω_θ is unramified, V_θ contains a unique *normalized* K^**-fixed vector*. That is, there exists a unique vector $\phi_\theta \in V_\theta$ such that $\theta(k)\phi_\theta = \phi_\theta$ for all $k \in K^*$, and $\phi_\theta(\widetilde{I}) = 1$ (cf. [KP] Lemma I.1.3). Next, given the character ψ on $\mathbb{F}$, let ψ also denote the unique character on N^* that satisfies:

$$\psi\big(\tilde{n}_i(x)\big) := \psi(x), \qquad \text{for all } x \in \mathbb{F},\ 1 \le i \le n-2.$$

A *ψ-Whittaker functional for θ* is a linear functional $\lambda : V_\theta \to \mathbb{C}$ such that $\lambda\big(\theta(n)f\big) = \psi(n)\lambda(f)$ for all $n \in N^*$, $f \in V_\theta$. By [KP] Corollary I.3.6, the space of such functionals is one-dimensional, hence there exists a unique ψ-Whittaker functional λ_θ such that $\lambda_\theta(\phi_\theta) = 1$. The normalized Whittaker function is then defined by:

$$W_\theta(g) := \lambda_\theta\bigl(\theta(g)\phi_\theta\bigr), \qquad \text{for all } g \in \widetilde{G}.$$

Note that $W_\theta(\widetilde{I}\,) = 1$, and for all $\xi \in \mu_n$, $z \in \widetilde{Z}$, $n \in N^*$, $g \in \widetilde{G}$, $k \in K^*$:

$$W_\theta(\xi zngk) = \xi\omega_\theta(z)\psi(n)W_\theta(g).$$

Consequently, W_θ is determined by its values on elements of the form $\mathbf{s}(\varpi^{\mathfrak{f}})$, where:

$$\varpi^{\mathfrak{f}} := \operatorname{diag}(\varpi^{\mathfrak{f}_i}), \qquad \text{for all } \mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_{n-1}) \in \mathbb{Z}^{n-1}.$$

The main result of this section is the following theorem.

Theorem 2.1 *For all $0 \le k \le n-1$, let $\mathfrak{f}^{(k)} = \bigl(\mathfrak{f}_1^{(k)}, \ldots, \mathfrak{f}_{n-1}^{(k)}\bigr) \in \mathbb{Z}^{n-1}$, where:*

$$\mathfrak{f}_i^{(k)} := \begin{cases} 1 & \text{if } i \le k, \\ 0 & \text{if } i > k. \end{cases}$$

Then $W_\theta\bigl(\mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})\bigr)$ is equal to:

$$\chi(\varpi)^k q^{-k(n-k-2)/2}(\varpi,\varpi)_{\mathbb{F}}^{k(k+1)/2}(\varpi,\varpi)_{\mathbb{F}}^{\varepsilon_2 kn(n-2)/8}(\varpi,\varpi)_{2,\mathbb{F}}^{\varepsilon_2 k}\prod_{i=1}^{k}\bigl(\mathfrak{g}_{\overline{\psi}}^{(-i)}\bigr)^{-1},$$

where $\mathfrak{g}_{\overline{\psi}}^{(-i)}$ is the complex conjugate of the Gauss sum $\mathfrak{g}_{\psi}^{(i)}$ defined by (1.7)*.*

Proof: Let $\mathrm{Wh}\bigl(V(\omega_\theta)\bigr)$ denote the space of ψ-Whittaker functionals for $V(\omega_\theta)$. For every $t \in \widetilde{T}$, let $\lambda_t \in \mathrm{Wh}\bigl(V(\omega_\theta)\bigr)$ be defined by the absolutely convergent integrals:

$$\lambda_t(f) := \int_{N^*} f(t\tilde{w}_0 n)\overline{\psi}(n)\,dn, \qquad \text{for all } f \in V(\omega_\theta).$$

Note that:

$$\lambda_{t't}(f) = (\delta_B^{1/2}\omega_\theta)(t')\lambda_t, \qquad \text{for all } t' \in \widetilde{T}_*,\ t \in \widetilde{T}.$$

Since $\{\lambda_t \,|\, t \in \widetilde{T}_* \backslash \widetilde{T}\}$ is a basis for $\mathrm{Wh}\big(V(\omega_\theta)\big)$ (cf. [KP] Lemma I.3.2), and the composition $\lambda_\theta \, I_{w_0}$ lies in $\mathrm{Wh}\big(V(\omega_\theta)\big)$, we have:

$$\lambda_\theta \, I_{w_0} = \sum_{t \in \widetilde{T}_* \backslash \widetilde{T}} \mathbf{c}(t) \lambda_t,$$

where $\mathbf{c} : \widetilde{T} \to \mathbb{C}$ is a uniquely determined function that satisfies:

$$\mathbf{c}(t't) = (\delta_B^{1/2} \omega_\theta)(t')^{-1} \mathbf{c}(t), \qquad \text{for all } t' \in \widetilde{T}_*,\ t \in \widetilde{T}. \tag{2.10}$$

According to [KP] Theorem I.4.2:

$$W_\theta\big(\mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})\big) = \delta_B(\varpi^{\mathfrak{f}^{(k)}}) \mathbf{c}\big(\tilde{w}_0^{-1} \mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})^{-1} \tilde{w}_0\big). \tag{2.11}$$

By a straightforward (though tedious) calculation, we have that:

$$\tilde{w}_0^{-1} \mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})^{-1} \tilde{w}_0 = \mathbf{s}(\varpi^{-1} \cdot I\,) \mathbf{s}(\varpi^{\mathfrak{f}^{(n-k-1)}}) (\varpi, \varpi)_{\mathbb{F}}^{(n-k)(k+1)/2}. \tag{2.12}$$

Using (2.6), (2.8), (2.10), (2.11) and (2.12), we obtain the following relation:

$$W_\theta\big(\mathbf{s}(\varpi^{\mathfrak{f}^{(k)}})\big) = \chi(\varpi)^{n-1} q^{-k(n-k-1)} (\varpi, \varpi)_{\mathbb{F}}^{k(k+1)/2} (\varpi, \varpi)_{2,\mathbb{F}}^{\varepsilon_2 k} \, \widehat{\mathfrak{g}}_\psi^{\varepsilon_2} \, \mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}^{(n-k-1)}})\big). \tag{2.13}$$

Thus, to prove the theorem, it will suffice to compute $\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}^{(n-k-1)}})\big)$.

For the moment, we will turn to the study of $\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)$ for arbitrary $\mathfrak{f} \in \mathbb{Z}^{n-1}$.

LEMMA 2.2 *For all* $\mathfrak{f} \in \mathbb{Z}^{n-1}$ *such that* $\mathfrak{f}_i \equiv \mathfrak{f}_j \pmod n$ *for all* i, j*:*

$$\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big) = (\delta_B^{1/2} \omega_\theta)\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)^{-1}.$$

PROOF: By the relation (2.10):

$$\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big) = (\delta_B^{1/2} \omega_\theta)\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)^{-1} \mathbf{c}(\widetilde{I}\,).$$

On the other hand, if we take $k = 0$ in (2.13), then:

$$1 = W_\theta(\widetilde{I}\,) = \chi(\varpi)^{n-1}\,\widehat{\mathfrak{g}}_\psi^{\varepsilon_2}\,\mathbf{c}\big(\mathbf{s}(\varpi{\cdot}I)\big) = \mathbf{c}(\widetilde{I}\,).$$

These statements imply the lemma. □

To describe the next result, we study the *local coefficients* $\{\tau_w(\omega, \mathfrak{f}, \mathfrak{f}')\}$ that are defined as follows. For any genuine unramified quasicharacter $\omega : \widetilde{T}_* \to \mathbb{C}^\times$, let $V(\omega)$ be the induced representation constructed earlier, and let $\mathrm{Wh}\big(V(\omega)\big)$ be the space of ψ-Whittaker functionals for $V(\omega)$. As before, we can define $\lambda_t \in \mathrm{Wh}\big(V(\omega)\big)$ by:

$$\lambda_t(f) := \int\limits_{N^*} f(t\tilde{w}_0 n)\overline{\psi}(n)\,dn, \qquad \text{for all } t \in \widetilde{T},\ f \in V(\omega).$$

Here the integrals are understood to be "regularized" if ω is not dominant in the sense of [KP] §I.1. For any $w \in W$, let ${}^w\omega : \widetilde{T}_* \to \mathbb{C}^\times$ be the genuine unramified quasicharacter given by:

$${}^w\omega(t) := \omega(\tilde{w}^{-1}t\tilde{w}), \qquad \text{for all } t \in \widetilde{T}_*,$$

where $\tilde{w} := \mathbf{s}(w)$. If $I_w : V(\omega) \to V({}^w\omega)$ is the standard (regularized) intertwining operator, then the local coefficients are defined by the relation:

$$\lambda_{\mathbf{s}(\varpi^{\mathfrak{f}})} I_w = \sum_{\mathfrak{f}' \in (\mathbb{Z}/n\mathbb{Z})^{n-1}} \tau_w(\omega, \mathfrak{f}, \mathfrak{f}')\lambda_{\mathbf{s}(\varpi^{\mathfrak{f}'})}.$$

Note that for all $w_1, w_2 \in W$ such that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$:

$$\tau_{w_1w_2}(\omega, \mathfrak{f}, \mathfrak{f}') = \sum_{\mathfrak{f}'' \in (\mathbb{Z}/n\mathbb{Z})^{n-1}} \tau_{w_1}({}^{w_2}\omega, \mathfrak{f}, \mathfrak{f}'')\tau_{w_2}(\omega, \mathfrak{f}'', \mathfrak{f}').$$

Hence, in studying the local coefficients, we can reduce to the case where w is a simple reflection s_i with $1 \le i \le n-2$.

Now consider the action of W on $\mathbb{Z}^{n-1}$ that is defined as follows. Let $\mathfrak{f}^\delta$ denote the special element $(0,1,2,\ldots,n-2)$ in $\mathbb{Z}^{n-1}$. For any $w \in W$, $\mathfrak{f} \in \mathbb{Z}^{n-1}$, we define $w[\mathfrak{f}]$ to be the unique element of $\mathbb{Z}^{n-1}$ such that $\varpi^{w[\mathfrak{f}]} = w\varpi^{\mathfrak{f}-\mathfrak{f}^\delta}w^{-1}\varpi^{\mathfrak{f}^\delta}$.

Proposition 2.3 *Let $\omega : \widetilde{T}_* \to \mathbb{C}^\times$ be a genuine unramified quasicharacter. Then for all $\mathfrak{f} \in \mathbb{Z}^{n-1}$ and every simple reflection s_i:*

$$\tau_{s_i}(\omega,\mathfrak{f},\mathfrak{f}) = \big(1-\omega\big(\tilde{h}_i(\varpi^n)\big)\big)^{-1}(1-q^{-1})\omega\big(\tilde{h}_i\big(\varpi^{-n[(\mathfrak{f}_i-\mathfrak{f}_{i+1})/n]}\big)\big),$$
$$\tau_{s_i}\big(\omega,\mathfrak{f},s_i[\mathfrak{f}]\big) = q^{\mathfrak{f}_{i+1}-\mathfrak{f}_i-2}\mathfrak{g}_{\overline{\psi}}^{(\mathfrak{f}_i-\mathfrak{f}_{i+1}+1)}(\varpi,\varpi)_{\mathbb{F}}^{\mathfrak{f}_i\mathfrak{f}_{i+1}}.$$

Moreover, $\tau_{s_i}(\omega,\mathfrak{f},\mathfrak{f}') = 0$ if $\mathfrak{f}' \not\equiv \mathfrak{f}$ or $s_i[\mathfrak{f}]$ in $(\mathbb{Z}/n\mathbb{Z})^{n-1}$.

Proof: This is essentially the content of [KP] Lemma I.3.3. To verify this result, we have corrected some minor typographical errors that occurred in the original proof (cf. [KP] pp. 80–85). Moreover, our calculations were performed using the cocycle $\sigma := \sigma_{n-1}^{(-1)}$ described in §1, which differs slightly from the cocycle used by Kazhdan and Patterson. We omit the details of the calculation. □

Corollary 2.4 *For all $\mathfrak{f} \in \mathbb{Z}^{n-1}$ and every simple reflection s_i, we have:*

$$\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big) = q^{\mathfrak{f}_i-\mathfrak{f}_{i+1}+1+[(\mathfrak{f}_i-\mathfrak{f}_{i+1})/n]}\mathfrak{g}_{\overline{\psi}}^{(\mathfrak{f}_{i+1}-\mathfrak{f}_i-1)}(\varpi,\varpi)_{\mathbb{F}}^{(\mathfrak{f}_i+1)(\mathfrak{f}_{i+1}-1)}\mathbf{c}\big(\mathbf{s}(\varpi^{s_i[\mathfrak{f}]})\big).$$

Proof: Applying Proposition 2.3 to the exceptional quasicharacter ω_θ, we obtain:

$$\tau_{s_i}({}^{s_i}\omega_\theta,\mathfrak{f},\mathfrak{f}) = -q^{-1-[(\mathfrak{f}_i-\mathfrak{f}_{i+1})/n]},$$
$$\tau_{s_i}\big({}^{s_i}\omega_\theta,s_i[\mathfrak{f}],\mathfrak{f}\big) = q^{\mathfrak{f}_i-\mathfrak{f}_{i+1}}\mathfrak{g}_{\overline{\psi}}^{(\mathfrak{f}_{i+1}-\mathfrak{f}_i-1)}(\varpi,\varpi)_{\mathbb{F}}^{(\mathfrak{f}_i+1)(\mathfrak{f}_{i+1}-1)}, \tag{2.14}$$

since ${}^{s_i}\omega_\theta\big(\tilde{h}_i(x^n)\big) = |x|_{\mathbb{F}}^{-1}$ for all $x \in \mathbb{F}^\times$, $\big(s_i[\mathfrak{f}]\big)_i = \mathfrak{f}_{i+1}-1$, and $\big(s_i[\mathfrak{f}]\big)_{i+1} = \mathfrak{f}_i+1$. Again, since ω_θ is exceptional, we have for every $\mathfrak{f}' \in \mathbb{Z}^{n-1}$ (cf. [KP] §I.3):

$$\sum_{\mathfrak{f}\in(\mathbb{Z}/n\mathbb{Z})^{n-1}} \mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)\tau_{s_i}({}^{s_i}\omega_\theta,\mathfrak{f},\mathfrak{f}') = 0.$$

If we set $\mathfrak{f}' := \mathfrak{f}$, then by (2.14) and the last statement of Proposition 2.3:

$$q^{\mathfrak{f}_i-\mathfrak{f}_{i+1}}\mathfrak{g}_{\overline{\psi}}^{(\mathfrak{f}_{i+1}-\mathfrak{f}_i-1)}(\varpi,\varpi)_{\mathbb{F}}^{(\mathfrak{f}_i+1)(\mathfrak{f}_{i+1}-1)}\mathbf{c}\big(\mathbf{s}(\varpi^{s_i[\mathfrak{f}]})\big) - q^{-1-[(\mathfrak{f}_i-\mathfrak{f}_{i+1})/n]}\mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)=0.$$

The corollary follows immediately. □

We are now in a position to complete the proof of Theorem 2.1. The cases $k=0$ and $k=n-1$ are easy since $\mathbf{s}\big(\varpi^{\mathfrak{f}^{(n-k-1)}}\big)\in\widetilde{Z}$, hence we may assume that $n\geq 3$, and $1\leq k\leq n-2$. To simplify the notation, let $\mathbf{c}(\mathfrak{f}) := \mathbf{c}\big(\mathbf{s}(\varpi^{\mathfrak{f}})\big)$ for all $\mathfrak{f}\in\mathbb{Z}^{n-1}$. Our goal is to compute $\mathbf{c}(\mathfrak{f}^{(n-k-1)})$.

For every $m\in\mathbb{Z}$ and every non-negative integer i, let $(m)_i$ denote a string of i copies of m, and consider the set of elements in $\mathbb{Z}^{n-1}$ defined by:

$$\mathfrak{f}(i,j) := \big((1)_{n-k-i-2}, (-i-1)_j, j+1, (-i)_{k-j}, (k+1)_i\big),$$

for all $0\leq i\leq n-k-2$, $0\leq j\leq k$. Observe that:

$$\mathfrak{f}(0,0) = \big((1)_{n-k-1}, (0)_k\big) = \mathfrak{f}^{(n-k-1)}.$$

Also, the $\mathfrak{f}(i,j)$'s are related by the action of simple reflections:

$$s_{n-k-i+j-1}\big[\mathfrak{f}(i,j)\big] = \mathfrak{f}(i,j+1), \qquad \text{for all } 0\leq i\leq n-k-2,\ 0\leq j\leq k-1.$$

Applying Corollary 2.4 to this identity, it follows that:

$$\mathbf{c}\big(\mathfrak{f}(i,j)\big) = q^{i+j+2}\mathfrak{g}_{\overline{\psi}}^{(-i-j-2)}(\varpi,\varpi)_{\mathbb{F}}^{j(i+1)}\mathbf{c}\big(\mathfrak{f}(i,j+1)\big),$$

Consequently:

$$\mathbf{c}\big(\mathfrak{f}(i,0)\big) = \prod_{j=0}^{k-1} q^{i+j+2}\mathfrak{g}_{\overline{\psi}}^{(-i-j-2)}(\varpi,\varpi)_{\mathbb{F}}^{j(i+1)}\cdot\mathbf{c}\big(\mathfrak{f}(i,k)\big).$$

Now if $0 \le i \le n-k-3$, we have:

$$\mathfrak{f}(i,k) := \big((1)_{n-k-i-2}, (-i-1)_k, (k+1)_{i+1}\big) = \mathfrak{f}(i+1,0),$$

thus we obtain:

$$\mathbf{c}\big(\mathfrak{f}^{(n-k-1)}\big) = \prod_{i=0}^{n-k-2}\prod_{j=0}^{k-1} q^{i+j+2}\mathfrak{g}_{\overline{\psi}}^{(-i-j-2)}(\varpi,\varpi)_{\mathbb{F}}^{j(i+1)} \cdot \mathbf{c}\big(\mathfrak{f}(n-k-2,k)\big). \tag{2.15}$$

To evaluate the right side of equation (2.15), we first observe that:

$$\mathfrak{f}(n-k-2,k) = \big((-n+k+1)_k, (k+1)_{n-k-1}\big),$$

so we can apply Lemma 2.2. We find that:

$$\mathbf{c}\big(\mathfrak{f}(n-k-2,k)\big) = \chi(\varpi)^{k-n+1} q^{-k(n-k-1)(n+1)/2}(\varpi,\varpi)_{2,\mathbb{F}}^{\varepsilon_2 k(k+1)/2}\,\widehat{\mathfrak{g}}_{\psi}^{-\varepsilon_2(k+1)}. \tag{2.16}$$

Also:

$$\prod_{i=0}^{n-k-2}\prod_{j=0}^{k-1} q^{i+j+2} = q^{k(n-k-1)(n+1)/2}. \tag{2.17}$$

Hence, it remains only to evaluate:

$$F(k) := \prod_{i=0}^{n-k-2}\prod_{j=0}^{k-1} \mathfrak{g}_{\overline{\psi}}^{(-i-j-2)}(\varpi,\varpi)_{\mathbb{F}}^{j(i+1)} = \prod_{i=1}^{n-k-1}\prod_{j=1}^{k} \mathfrak{g}_{\overline{\psi}}^{(-i-j)}(\varpi,\varpi)_{\mathbb{F}}^{i(j+1)}. \tag{2.18}$$

For $k=1$, we have:

$$F(1) = \big(\mathfrak{g}_{\overline{\psi}}^{(-1)}\big)^{-1}\cdot \prod_{i=1}^{n-1}\mathfrak{g}_{\overline{\psi}}^{(i)} = q^{(n-1)/2}(\varpi,\varpi)_{\mathbb{F}}^{\varepsilon_2 n(n-2)/8}\,\widehat{\mathfrak{g}}_{\overline{\psi}}^{\varepsilon_2}\big(\mathfrak{g}_{\overline{\psi}}^{(-1)}\big)^{-1},$$

the second equality following from (1.9). Now for all $1 \le k \le n-3$, the relation:

$$F(k+1)/F(k) = \prod_{i=1}^{n-k-2}\mathfrak{g}_{\overline{\psi}}^{(-i-k-1)}(\varpi,\varpi)_{\mathbb{F}}^{ik}\prod_{j=1}^{k}\big(\mathfrak{g}_{\overline{\psi}}^{(-j+k+1)}\big)^{-1}(\varpi,\varpi)_{\mathbb{F}}^{(j+1)(k+1)}$$

follows easily from (2.18). Applying (2.8) and (2.9) again, this equation can be simplified to:

$$F(k+1)/F(k) = q^{-k+(n-1)/2}(\varpi,\varpi)_{\mathbb{F}}^{\varepsilon_2 n(n-2)/8}(\varpi,\varpi)_{2,\mathbb{F}}^{\varepsilon_2 k}\,\widehat{\mathfrak{g}}_{\overline{\psi}}^{\varepsilon_2}\left(\mathfrak{g}_{\overline{\psi}}^{(-k-1)}\right)^{-1}$$

By induction, it follows that:

$$F(k) = q^{k(n-k)/2}(\varpi,\varpi)_{\mathbb{F}}^{\varepsilon_2 kn(n-2)/8}(\varpi,\varpi)_{2,\mathbb{F}}^{\varepsilon_2 k(k-1)/2}\,\widehat{\mathfrak{g}}_{\overline{\psi}}^{\varepsilon_2 k}\prod_{i=1}^{k}\left(\mathfrak{g}_{\overline{\psi}}^{(-i)}\right)^{-1}. \tag{2.19}$$

Substituting (2.16), (2.17) and (2.19) into equation (2.15), we find that $\mathbf{c}\big(\mathfrak{f}^{(n-k-1)}\big)$ equals:

$$\chi(\varpi)^{k-n+1}q^{k(n-k)/2}(\varpi,\varpi)_{\mathbb{F}}^{\varepsilon_2 kn(n-2)/8}\,\widehat{\mathfrak{g}}_{\psi}^{-\varepsilon_2}\prod_{i=1}^{k}\left(\mathfrak{g}_{\overline{\psi}}^{(-i)}\right)^{-1}.$$

Here we have used the fact that $\widehat{\mathfrak{g}}_{\psi}\cdot\widehat{\mathfrak{g}}_{\overline{\psi}} = 1$, thus $\widehat{\mathfrak{g}}_{\psi}^{-1}\cdot\widehat{\mathfrak{g}}_{\overline{\psi}} = \widehat{\mathfrak{g}}_{\overline{\psi}}^{2} = (\varpi,\varpi)_{2,\mathbb{F}}$. Theorem 2.1 follows at once by substituting this expression into (2.13). $\square$

§3. The Whittaker function for the induced series

In this section, we will slightly modify the notation of §2 by appending the superscript prime (′) to the various symbols introduced there. Thus, we now write $G' := GL_{n-1}(\mathbb{F})$, $\widetilde{G}' := \widetilde{GL}_{n-1}^{(-1)}(\mathbb{F})$, $\sigma' := \sigma_{n-1}^{(-1)}$, $\mathbf{s}' : G' \to \widetilde{G}'$, $\psi' : N'^{*} \to \mathbb{C}^{\times}$, and so on. We continue to assume that $n \geq 2$, $|n|_{\mathbb{F}} = 1$, and q is odd.

Now let $G := GL_n(\mathbb{F})$, let $\widetilde{G} := \widetilde{GL}_n^{(0)}(\mathbb{F})$, and let $\sigma := \sigma_n$ (cf. §1). Let T be the subgroup of diagonal matrices in G. Then by (1.2):

$$\mathbf{s}(t)\,\mathbf{s}(t') = \mathbf{s}(tt') \cdot \prod_{i<j} (t_i, t'_j)_{\mathbb{F}}, \qquad \text{for all } t, t' \in T,$$

and therefore:

$$\mathbf{s}(t)\,\mathbf{s}(t')\,\mathbf{s}(t)^{-1}\mathbf{s}(t')^{-1} = \prod_i (t_i, t'_i)_{\mathbb{F}}^{-1} \cdot (\det t, \det t')_{\mathbb{F}}.$$

Let Z be the center of G, and $\widetilde{Z} := \mathbf{p}^{-1}(Z)$. Although $\widetilde{Z}$ is *not* the center of $\widetilde{G}$, this relation implies that $\widetilde{Z}$ is abelian.

Using the representation (θ, V_θ) introduced in §2, we will next construct a certain series of induced representations of the metaplectic group $\widetilde{G}$. Consider the embedding of G' into G given by:

$$\iota : G' \hookrightarrow G, \qquad g \mapsto \begin{pmatrix} g & \\ & \det\, g^{-1} \end{pmatrix}, \qquad \text{for all } g \in G'.$$

By Theorem 1.1, it follows that the map ι gives rise to an embedding of $\widetilde{G}'$ into $\widetilde{G}$:

$$\tilde{\iota} : \widetilde{G}' \hookrightarrow \widetilde{G}, \qquad (g, \xi) \mapsto \big(\iota(g), \xi\big), \qquad \text{for all } g \in G',\ \xi \in \mu_n.$$

In other words:

$$\tilde{\iota}\big(\mathbf{s}'(g)\xi\big) = \mathbf{s}\big(\iota(g)\big)\xi, \qquad \text{for all } g \in G',\ \xi \in \mu_n.$$

Now let P be the standard parabolic subgroup of type $(n-1,1)$ in G, M its Levi component, and U its unipotent radical. Then $M \cong G' \times \mathbb{F}^\times$, and U is isomorphic to $(n-1)$ copies of the additive group $\mathbb{F}$. Let $\widetilde{P} := \mathbf{p}^{-1}(P)$, $\widetilde{M} := \mathbf{p}^{-1}(M)$, and $U^* := \mathbf{s}(U)$. We define:

$$P_n := \left\{ p \in P \,\middle|\, \det p \in \mathbb{F}^{\times n} \right\}, \qquad \widetilde{P}_n := \mathbf{p}^{-1}(P_n).$$

Observe that $\widetilde{P}_n$ is the semidirect product of the groups $\tilde{\imath}(\widetilde{G}')$, $\tilde{\jmath}(\mathbb{F}^{\times n})$, and U^*, where:

$$\tilde{\jmath} : \mathbb{F}^\times \to \widetilde{G}, \qquad x \mapsto \mathbf{s}\begin{pmatrix} I' & \\ & x \end{pmatrix}, \qquad \text{for all } x \in \mathbb{F}^\times.$$

Here I' denotes the identity matrix in G'. Since the groups $\tilde{\imath}(\widetilde{G}')$ and $\tilde{\jmath}(\mathbb{F}^{\times n})$ commute, it follows that the representation (θ, V_θ) can be extended to a genuine representation $\theta_P : \widetilde{P}_n \to \mathrm{Aut}(V_\theta)$ by the formula:

$$\theta_P\big(\tilde{\imath}(g)\tilde{\jmath}(x)u\big)f := \theta(g)f, \qquad \text{for all } g \in G',\ x \in \mathbb{F}^{\times n},\ u \in U^*,\ f \in V_\theta.$$

Now let δ_P be the modular character of P. We will regard δ_P as a character of $\widetilde{P}$ that is trivial on μ_n. For every $s \in \mathbb{C}$, let (π_s, V_s) denote the (normalized) induced representation $\mathrm{Ind}_{\widetilde{P}_n}^{\widetilde{G}}(\delta_P^{s-\frac{1}{2n}}\,\theta_P)$. Here:

$$V_s := \left\{ f \in C^\infty(\widetilde{G}; V_\theta) \,\middle|\, f(pg) = \delta_P(p)^{s+\frac{n-1}{2n}}\theta_P(p)f(g) \text{ for all } p \in \widetilde{P}_n,\ g \in \widetilde{G} \right\},$$

where $C^\infty(\widetilde{G}; V_\theta)$ is the space of locally-constant functions $f : \widetilde{G} \to V_\theta$. The group $\widetilde{G}$ acts on V_s by right translation: $\pi_s(g)f(g') := f(g'g)$ for all $g, g' \in \widetilde{G}$, $f \in V_s$.

For the remainder of this section, we fix an element $a \in \mathcal{O}^\times$. We will next construct a certain Whittaker function $W_{s,a} : \widetilde{G} \to \mathbb{C}$ associated to the representation π_s, and the goal of this section is to calculate the special value $W_{s,a}(\widetilde{I})$. To define $W_{s,a}$, we first observe that the space V_s contains a unique normalized K^*-fixed vector. That is, there exists a unique vector $\phi_s \in V_s$ such that $\pi_s(k)\phi_s = \phi_s$ for all

$k \in K^*$, and $\phi_s(\widetilde{I}) = \phi_\theta$, where ϕ_θ is the normalized K'^*-fixed vector in the space of θ (cf. §2). More precisely:

$$\phi_s(g) = \begin{cases} \delta_P(p)^{s+\frac{n-1}{2n}}\theta_P(p)\phi_\theta & \text{if } g = pk \text{ for some } p \in \widetilde{P}_n,\ k \in K^*, \\ 0 & \text{otherwise,} \end{cases}$$

for all $g \in \widetilde{G}$. Next, let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be the nontrivial additive character chosen in §2, and let ψ' be the corresponding character of N'^*. Let ψ_a be the unique character of N^* that satisfies for all $x \in \mathbb{F}$:

$$\psi_a\big(\tilde{n}_i(x)\big) = \begin{cases} \psi(ax) & \text{if } i = 1, \\ \psi(x) & \text{if } 2 \le i \le n-1. \end{cases}$$

Finally, let $\lambda_{s,a} : V_s \to \mathbb{C}$ be the linear functional defined by:

$$\lambda_{s,a}(f) := \int\limits_{\mathbb{F}^{n-1}} \lambda_\theta f\left(\mathbf{s}\begin{pmatrix} & I' \\ 1 & x_1 \dots x_{n-1} \end{pmatrix}\right)\overline{\psi}(ax_1)\,dx, \qquad \text{for all } f \in V_s. \tag{3.1}$$

Here $dx := dx_1 \dots dx_{n-1}$, where each dx_i is the unique Haar measure for $\mathbb{F}$ such that $\mathrm{Vol}(\mathcal{O}; dx_i) = 1$. Note that if $\mathrm{Re}(s)$ is sufficiently large, the integrals defining $\lambda_{s,a}$ converge absolutely; otherwise, the integrals are understood to represent their regularized values. The functional $\lambda_{s,a}$ is clearly a ψ_a-Whittaker functional for π_s. Although the space of all such functionals has dimension n^2, the following theorem uniquely characterizes $\lambda_{s,a}$.

Theorem 3.1 *Up to multiplication by a scalar, $\lambda_{s,a}$ is the only linear functional $\lambda : V_s \to \mathbb{C}$ that satisfies the properties:*

$$\lambda\big(\pi_s(n)f\big) = \psi_a(n)\lambda(f), \qquad \text{for all } n \in N^*,\ f \in V_s, \tag{3.2}$$

and:

$$\lambda\big(\pi_s\big(\mathbf{s}(x\cdot I)\big)f\big) = \chi(x)^{n-1}\gamma_\psi(x)^{\varepsilon_2}\lambda(f), \qquad \text{for all } x \in \mathbb{F}^\times,\ f \in V_s. \tag{3.3}$$

Proof: The uniqueness assertion was proved by Bump and Lieman [BL]. The fact that $\lambda_{s,a}$ satisfies (3.2) and (3.3) is an immediate consequence of definition (3.1). To see that $\lambda_{s,a} \neq 0$, let $w := \begin{pmatrix} & I' \\ 1 & \end{pmatrix}$, and let $\tilde{w} := \mathbf{s}(w)$. Choosing $m \gg 0$, let ϕ_s' be the element of V_s defined by:

$$\phi_s'(g) = \begin{cases} \delta_P(p)^{s+\frac{n-1}{2n}}\theta_P(p)\phi_\theta & \text{if } g = p\tilde{w}k \text{ for some } p \in \widetilde{P}_n,\ k \in K_m^*, \\ 0 & \text{otherwise,} \end{cases}$$

for all $g \in \widetilde{G}$. Then it is easily seen that $\lambda_{s,a}(\phi_s') = q^{-m(n-1)} \neq 0$. $\square$

The Whittaker function $W_{s,a} : \widetilde{G} \to \mathbb{C}$ can now be defined as follows:

$$W_{s,a}(g) := \lambda_{s,a}\big(\pi_s(g)\phi_s\big), \qquad \text{for all } g \in \widetilde{G}.$$

Note that for all $\xi \in \mu_n$, $x \in \mathbb{F}^\times$, $n \in N^*$, $g \in \widetilde{G}$, $k \in K^*$:

$$W_{s,a}\big(\xi\mathbf{s}(x{\cdot}I)ngk\big) = \xi\chi(x)^{n-1}\gamma_\psi(x)^{\varepsilon_2}\psi_a(n)W_{s,a}(g).$$

The main result of this section is the following theorem.

Theorem 3.2 *Let $W_{s,a}$ be the Whittaker function defined above. If n is odd, or $n \equiv 2$ or $4 \pmod 8$, then:*

$$W_{s,a}(\widetilde{I}\,) = \frac{L\big(ns, \chi\,(\,\cdot\,,a)_{\mathbb{F}}\big)}{L(n^2s, \chi^n)}.$$

If n is odd, or $n \equiv 0$ or $6 \pmod 8$, then:

$$W_{s,a}(\widetilde{I}\,) = \frac{L\big(ns, \chi\,(\,\cdot\,,-a)_{\mathbb{F}}\big)}{L(n^2s, \chi^n)}.$$

Here $(\,\cdot\,,\pm a)_{\mathbb{F}}$ denotes the (unramified) quasicharacter given by $x \mapsto (x,\pm a)_{\mathbb{F}}$ for all $x \in \mathbb{F}^\times$, and for any unramified quasicharacter $\chi_\circ : \mathbb{F}^\times \to \mathbb{C}^\times$, $L(s,\chi_\circ)$ is the standard local L-function given by $L(s,\chi_\circ) := \big(1 - \chi_\circ(\varpi)q^{-s}\big)^{-1}$.

PROOF: By definition:

$$W_{s,a}(\widetilde{I}) = \lambda_{s,a}(\phi_s) = \int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\mathbf{s}\begin{pmatrix} & I' \\ 1 & x_1 \dots x_{n-1} \end{pmatrix} \right) \overline{\psi}(ax_1)\, dx.$$

Since:

$$\mathbf{s}\begin{pmatrix} & I' \\ 1 & x_1 \dots x_{n-1} \end{pmatrix} = \tilde{s}_{n-1}\tilde{n}_{n-1}(x_{n-1}) \dots \tilde{s}_1\tilde{n}_1(x_1) = \prod_{i=n-1}^{1} \tilde{s}_i\tilde{n}_i(x_i),$$

we have that:

$$W_{s,a}(\widetilde{I}) = \int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\prod_{i=n-1}^{2} \tilde{s}_i\tilde{n}_i(x_i) \cdot \tilde{s}_1\tilde{n}_1(x_1) \right) \overline{\psi}(ax_1)\, dx. \tag{3.4}$$

Now for all $x \in \mathbb{F}$, we introduce the notation:

$$\dot{x} = \begin{cases} 1 & \text{if } x \in \mathcal{O}, \\ x & \text{if } x \notin \mathcal{O}, \end{cases}$$

and:

$$\ddot{x} = \begin{cases} 0 & \text{if } x \in \mathcal{O}, \\ x^{-1} & \text{if } x \notin \mathcal{O}. \end{cases}$$

Then it is easily shown that:

$$\tilde{s}_i\tilde{n}_i(x) = \tilde{n}_i(\ddot{x})\tilde{h}_i(\dot{x}^{-1})\tilde{k}_i(x), \qquad \text{for all } x \in \mathbb{F},\ 1 \le i \le n-1, \tag{3.5}$$

where $\tilde{k}_i(x)$ is an element of K^*. Applying this relation with $i = 1$, and using the fact that ϕ_s is K^*-fixed, the integral in (3.4) becomes:

$$\int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\prod_{i=n-1}^{2} \tilde{s}_i\tilde{n}_i(x_i) \cdot \tilde{n}_1(\ddot{x}_1)\tilde{h}_1(\dot{x}_1^{-1}) \right) \overline{\psi}(ax_1)\, dx. \tag{3.6}$$

Next, we observe that:

$$\prod_{i=n-1}^{2} \tilde{s}_i\tilde{n}_i(x_i) \cdot \tilde{n}_1(\ddot{x}_1) = \mathbf{s}\begin{pmatrix} 1 & -\ddot{x}_1 x_2 \dots -\ddot{x}_1 x_{n-1} & \ddot{x}_1 \\ & I' & \end{pmatrix} \cdot \prod_{i=n-1}^{2} \tilde{s}_i\tilde{n}_i(x_i),$$

and:

$$\prod_{i=n-1}^{2} \tilde{s}_i \tilde{n}_i(x_i) \cdot \tilde{h}_1(\dot{x}_1^{-1}) = \tilde{h}_{1,n}(\dot{x}_1^{-1})(\dot{x}_1, \dot{x}_1)_{\mathbb{F}}^{n-2} \cdot \prod_{i=n-1}^{2} \tilde{s}_i \tilde{n}_i(x_i/\dot{x}_1).$$

Here we use the notation $\tilde{h}_{i,j}(x) := \mathbf{s}\big(h_{i,j}(x)\big)$, where $h_{i,j}(x)$ is the diagonal matrix with x in the i-th position, x^{-1} in the j-th position, and 1's elsewhere along the diagonal. After substituting the preceding identities into (3.6), we obtain:

$$\int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\mathbf{s}\begin{pmatrix} 1 & -\ddot{x}_1 x_2 \ldots - \ddot{x}_1 x_{n-1} & \ddot{x}_1 \\ & I' & \end{pmatrix} \tilde{h}_{1,n}(\dot{x}_1^{-1}) \cdot \prod_{i=n-1}^{2} \tilde{s}_i \tilde{n}_i(x_i/\dot{x}_1) \right)$$
$$\times \quad (\dot{x}_1, \dot{x}_1)_{\mathbb{F}}^{n-2}\, \overline{\psi}(a x_1)\, dx$$
$$= \int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\tilde{h}_{1,n}(\dot{x}_1^{-1}) \prod_{i=n-1}^{2} \tilde{s}_i \tilde{n}_i(x_i/\dot{x}_1) \right) (\dot{x}_1, \dot{x}_1)_{\mathbb{F}}^{n-2}\, \overline{\psi}(a x_1) \overline{\psi}(\ddot{x}_1 x_2)\, dx$$
$$= \int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\tilde{h}_{1,n}(\dot{x}_1^{-1}) \prod_{i=n-1}^{2} \tilde{s}_i \tilde{n}_i(x_i) \right) |\dot{x}_1|_{\mathbb{F}}^{n-2} (\dot{x}_1, \dot{x}_1)_{\mathbb{F}}^{n-2}\, \overline{\psi}(a x_1) \overline{\psi}(\dot{x}_1 \ddot{x}_1 x_2)\, dx.$$

Here we have made the change of variables $\left\{ x_i \mapsto \dot{x}_1 x_i \,\middle|\, 2 \le i \le n-1 \right\}$. Similarly, using relation (3.5) with $i = 2$, it follows that $W_{s,a}(\widetilde{I}\,)$ equals:

$$\int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\tilde{h}_{1,n}(\dot{x}_1^{-1}) \tilde{h}_{2,n}(\dot{x}_2^{-1}) \cdot \prod_{i=n-1}^{3} \tilde{s}_i \tilde{n}_i(x_i) \right)$$
$$\times \quad |\dot{x}_1|_{\mathbb{F}}^{n-2} |\dot{x}_2|_{\mathbb{F}}^{n-3} (\dot{x}_1, \dot{x}_1)_{\mathbb{F}}^{n-2} (\dot{x}_2, \dot{x}_2)_{\mathbb{F}}^{n-3}\, \overline{\psi}(a x_1) \overline{\psi}(\dot{x}_1 \ddot{x}_1 x_2) \overline{\psi}(\dot{x}_2 \ddot{x}_2 x_3)\, dx.$$

Continuing inductively in this manner, we find that $W_{s,a}(\widetilde{I}\,)$ is equal to:

(3.7)
$$\int\limits_{\mathbb{F}^{n-1}} \lambda_\theta \phi_s \left(\prod_{i=1}^{n-1} \tilde{h}_{i,n}(\dot{x}_i^{-1}) \right) \prod_{i=1}^{n-1} |\dot{x}_i|_{\mathbb{F}}^{n-i-1} (\dot{x}_i, \dot{x}_i)_{\mathbb{F}}^{n-i-1} \cdot \overline{\psi}(a x_1) \prod_{j=2}^{n-1} \overline{\psi}(\dot{x}_{j-1} \ddot{x}_{j-1} x_j)\, dx.$$

To evaluate the integral (3.7), note that we can restrict the domain of integration to $(\mathbb{F}-\{0\})^{n-1}$ without affecting the result. We regard this new domain as a disjoint union:

$$(\mathbb{F} - \{0\})^{n-1} = \bigcup_{\mathfrak{f} \in \mathbb{Z}^{n-1}} R(\mathfrak{f}),$$

where for all $\mathfrak{f} = (\mathfrak{f}_1, \dots, \mathfrak{f}_{n-1}) \in \mathbb{Z}^{n-1}$:

$$R(\mathfrak{f}) := \left\{(x_1, \dots, x_{n-1}) \in \mathbb{F}^{n-1} \,\middle|\, v(x_i) = \mathfrak{f}_i \text{ for all } i\right\}.$$

For fixed $\mathfrak{f} \in \mathbb{Z}^{n-1}$, the contribution of region $R(\mathfrak{f})$ to the integral (3.7) can be evaluated as follows. For $1 \leq i \leq n-1$, let:

$$\delta_i := \begin{cases} 0 & \text{if } \mathfrak{f}_i \geq 0, \\ 1 & \text{if } \mathfrak{f}_i < 0. \end{cases}$$

Then for all $(x_1, \dots, x_{n-1}) \in R(\mathfrak{f})$, we have $\dot{x}_i = x_i^{\delta_i}$, $\ddot{x}_i = \delta_i x_i^{-1}$, and $\dot{x}_i \ddot{x}_i = \delta_i$, and our goal is therefore to compute:

(3.8)
$$\int\limits_{R(\mathfrak{f})} \lambda_\theta \phi_s \left(\prod_{i=1}^{n-1} \tilde{h}_{i,n}(x_i^{-\delta_i})\right) \prod_{i=1}^{n-1} |x_i|_{\mathbb{F}}^{\delta_i(n-i-1)} (x_i, x_i)_{\mathbb{F}}^{\delta_i(n-i-1)} \cdot \overline{\psi}(a x_1) \prod_{j=2}^{n-1} \overline{\psi}(\delta_{j-1} x_j)\, dx.$$

After the change of variables $\left\{x_i \mapsto \varpi^{\mathfrak{f}_i} x_i \,\middle|\, 1 \leq i \leq n-1\right\}$, we obtain:

$$\prod_{i=1}^{n-1} q^{-\delta_i \mathfrak{f}_i (n-i-1) - \mathfrak{f}_i} (\varpi, \varpi)_{\mathbb{F}}^{\delta_i \mathfrak{f}_i (n-i-1)}$$
$$\times \int\limits_{x_1, \dots, x_{n-1} \in \mathcal{O}^\times} \lambda_\theta \phi_s \left(\prod_{i=1}^{n-1} \tilde{h}_{i,n}(\varpi^{-\delta_i \mathfrak{f}_i} x_i^{-\delta_i})\right) \overline{\psi}(a \varpi^{\mathfrak{f}_1} x_1) \prod_{j=2}^{n-1} \overline{\psi}(\varpi^{\delta_{j-1} \mathfrak{f}_j} x_j)\, dx.$$

By a straightforward cocycle calculation:

$$\prod_{i=1}^{n-1} \tilde{h}_{i,n}(\varpi^{-\delta_i \mathfrak{f}_i} x_i^{-\delta_i}) = \tilde{\iota}\big(\mathbf{s}'(\varpi^{-\delta \mathfrak{f}})\big) \prod_{i=1}^{n-1} \tilde{h}_{i,n}(x_i^{-\delta_i}) \cdot \prod_{j=1}^{n-1} \prod_{i=j}^{n-1} (\varpi^{\delta_i \mathfrak{f}_i}, x_j^{\delta_j})_{\mathbb{F}},$$

where $\delta \mathfrak{f} \in \mathbb{Z}^{n-1}$ is defined by $(\delta \mathfrak{f})_i := \delta_i \mathfrak{f}_i = \min(\mathfrak{f}_i, 0)$ for all i. Since $\tilde{h}_{i,n}(x_i^{-\delta_i})$ lies in K^* for all $x_i \in \mathcal{O}^\times$, it follows from the definition of ϕ_s that (3.8) is equal to:

(3.9)
$$W_\theta\big(\mathbf{s}'(\varpi^{-\delta \mathfrak{f}})\big) \prod_{i=1}^{n-1} q^{-\delta_i \mathfrak{f}_i (-ns + \frac{n-1}{2} - i) - \mathfrak{f}_i} (\varpi, \varpi)_{\mathbb{F}}^{\delta_i \mathfrak{f}_i (n-i-1)}$$
$$\times \int\limits_{x_1, \dots, x_{n-1} \in \mathcal{O}^\times} \prod_{j=1}^{n-1} \prod_{i=j}^{n-1} (\varpi^{\delta_i \mathfrak{f}_i}, x_j^{\delta_j})_{\mathbb{F}} \cdot \overline{\psi}(a \varpi^{\mathfrak{f}_1} x_1) \prod_{j=2}^{n-1} \overline{\psi}(\varpi^{\delta_{j-1} \mathfrak{f}_j} x_j)\, dx.$$

Now we define:

$$G(i\,;\,j) := \int_{x\in\mathcal{O}^\times} (\varpi, x)_{\mathbb{F}}^{i}\,\overline{\psi}(\varpi^j x)\,dx, \qquad \text{for all } i, j \in \mathbb{Z}.$$

It is easy to verify that:

$$G(i\,;\,j) = \begin{cases} 1 - q^{-1} & \text{if } i \equiv 0 \,(\mathrm{mod}\ n) \text{ and } j \geq 0, \\ q^{-1}\mathfrak{g}_{\overline{\psi}}^{(i)} & \text{if } j = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

By Fubini's theorem, the integral in (3.9) is the product of:

(3.11)

$$\int_{x_1\in\mathcal{O}^\times} \prod_{i=1}^{n-1} (\varpi^{\delta_i\mathfrak{f}_i}, x_1^{\delta_1})_{\mathbb{F}}\,\overline{\psi}(a\varpi^{\mathfrak{f}_1}x_1)\,dx_1 = \prod_{i=1}^{n-1} (\varpi^{\delta_i\mathfrak{f}_i}, a^{-\delta_1})_{\mathbb{F}} \cdot G\big(\delta_1 \sum_{i=1}^{n-1} \delta_i\mathfrak{f}_i\,;\,\mathfrak{f}_1\big),$$

and:

$$\int_{x_j\in\mathcal{O}^\times} \prod_{i=j}^{n-1} (\varpi^{\delta_i\mathfrak{f}_i}, x_j^{\delta_j})_{\mathbb{F}}\,\overline{\psi}(\varpi^{\delta_{j-1}\mathfrak{f}_i}x_j)\,dx_j = G\big(\delta_j \sum_{i=j}^{n-1} \delta_i\mathfrak{f}_i\,;\,\delta_{j-1}\mathfrak{f}_j\big) \tag{3.12}$$

for all $2 \leq j \leq n-1$.

Now according to [KP] Theorem I.4.2, $W_\theta\big(\mathbf{s}'(\varpi^{-\delta\mathfrak{f}})\big) = 0$ unless:

$$\delta_1\mathfrak{f}_1 \leq \delta_2\mathfrak{f}_2 \leq \ldots \leq \delta_{n-1}\mathfrak{f}_{n-1}.$$

This implies that (3.9) vanishes unless $\mathfrak{f}_1 \leq \ldots \leq \mathfrak{f}_k < 0$ and $\mathfrak{f}_{k+1}, \ldots, \mathfrak{f}_{n-1} \geq 0$ for some k with $0 \leq k \leq n-1$. On the other hand, it follows from (3.10) that right side of (3.11) vanishes unless $\mathfrak{f}_1 \geq -1$. Hence, we may assume that $\mathfrak{f}$ has the form $\big((-1)_k, \mathfrak{f}_{k+1}, \ldots, \mathfrak{f}_{n-1}\big)$ with $\mathfrak{f}_{k+1}, \ldots, \mathfrak{f}_{n-1} \geq 0$. In this case, $\delta_i = 1$ if $1 \leq i \leq k$, and $\delta_i = 0$ otherwise. Consequently:

$$W_\theta\big(\mathbf{s}'(\varpi^{-\delta\mathfrak{f}})\big) = W_\theta\big(\mathbf{s}'(\varpi^{\mathfrak{f}^{(k)}})\big),$$

where $\mathfrak{f}^{(k)}$ is the special element $\big((1)_k,(0)_{n-k-1}\big)\in\mathbb{Z}^{n-1}$ considered in §2. When $\mathfrak{f}$ has the special form above, the first product in (3.9) simplifies to:

$$q^{-k(2ns-n+k)/2}(\varpi,\varpi)_{\mathbb{F}}^{k(k-1)/2}\prod_{i=k+1}^{n-1}q^{-\mathfrak{f}_i}.$$

Using (3.10), we also find that (3.11) equals:

$$\prod_{i=1}^{n-1}(\varpi^{\delta_i\mathfrak{f}_i},a^{-\delta_1})_{\mathbb{F}}\cdot G\big(\delta_1\sum_{i=1}^{n-1}\delta_i\mathfrak{f}_i\,;\,\mathfrak{f}_1\big)=\begin{cases}1-q^{-1} & \text{if } k=0,\\ q^{-1}(\varpi,a)_{\mathbb{F}}^k\,\mathfrak{g}_{\psi}^{(-k)} & \text{if } k\ge 1,\end{cases}$$

and for all $2\le j\le n-1$, (3.12) is equal to:

$$G\big(\delta_j\sum_{i=j}^{n-1}\delta_i\mathfrak{f}_i\,;\,\delta_{j-1}\mathfrak{f}_j\big)=\begin{cases}q^{-1}\mathfrak{g}_{\psi}^{(j-k-1)} & \text{if } 2\le j\le k,\\ 1-q^{-1} & \text{if } k+1\le j\le n-1.\end{cases}$$

Combining all of these results, it follows that (3.9) is equal to:

$$W_\theta\big(\mathbf{s}'(\varpi^{\mathfrak{f}^{(k)}})\big)q^{-k(2ns-n+k+2)/2}(\varpi,a)_{\mathbb{F}}^k(\varpi,\varpi)_{\mathbb{F}}^{k(k-1)/2}\prod_{i=1}^{k}\mathfrak{g}_{\psi}^{(-i)}\prod_{i=k+1}^{n-1}q^{-\mathfrak{f}_i}(1-q^{-1})$$

Now to compute the integral (3.7), we apply the preceding result, summing the contributions from all regions $R(\mathfrak{f})$ such that $\mathfrak{f}$ has the form $\big((-1)_k,\mathfrak{f}_{k+1},\dots,\mathfrak{f}_{n-1}\big)$, $\mathfrak{f}_{k+1},\dots,\mathfrak{f}_{n-1}\ge 0$, with $0\le k\le n-1$. If we collect together the contributions for each fixed value of k and use the fact that:

$$\sum_{\mathfrak{f}_{k+1},\dots,\mathfrak{f}_{n-1}\ge 0}\ \prod_{i=k+1}^{n-1}q^{-\mathfrak{f}_i}(1-q^{-1})=1,$$

it follows that:

$$W_{s,a}(\widetilde{I}\,)=\sum_{k=0}^{n-1}W_\theta\big(\mathbf{s}'(\varpi^{\mathfrak{f}^{(k)}})\big)q^{-k(2ns-n+k+2)/2}(\varpi,a)_{\mathbb{F}}^k(\varpi,\varpi)_{\mathbb{F}}^{k(k-1)/2}\prod_{i=1}^{k}\mathfrak{g}_{\psi}^{(-i)}.$$

Finally, we substitute the explicit value of $W_\theta\big(\mathbf{s}'(\varpi^{\mathfrak{f}^{(k)}})\big)$ given by Theorem 2.1, and we obtain:

$$\begin{aligned}W_{s,a}(\widetilde{I}\,)&=\sum_{k=0}^{n-1}\chi(\varpi)^k(\varpi,a)_{\mathbb{F}}^k(\varpi,\varpi)_{\mathbb{F}}^{e_2k(n^2+2n+8)/8}q^{-kns}\\&=\frac{1-\chi(\varpi)^nq^{-n^2s}}{1-\chi(\varpi)(\varpi,a)_{\mathbb{F}}(\varpi,-1)_{\mathbb{F}}^{e_2(n^2+2n+8)/8}q^{-ns}}\\&=\frac{L\big(ns,\chi\,(\,\cdot\,,a)_{\mathbb{F}}\,(\,\cdot\,,-1)_{\mathbb{F}}^{e_2(n^2+2n+8)/8}\big)}{L(n^2s,\chi^n)}.\end{aligned}$$

This completes the proof. $\square$

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