# New Examples of Noncommutative $\Lambda(p)$ Sets

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**Abstract**

In this paper, we introduce a certain combinatorial property $Z^\star(k)$, which is defined for every integer $k \geq 2$, and show that every set $E \subset \mathbb{Z}$ with the property $Z^\star(k)$ is necessarily a noncommutative $\Lambda(2k)$ set. In particular, using number theoretic results about the number of solutions to so-called "$S$-unit equations," we show that for any finite set $Q$ of prime numbers, $E_Q$ is noncommutative $\Lambda(p)$ for every real number $2 < p < \infty$, where $E_Q$ is the set of natural numbers whose prime divisors all lie in the set $Q$.

# 1 Introduction

For any finite set $Q$ of prime numbers, let $E_Q \subset \mathbb{N}$ denote the set of all natural numbers $n$ such that every prime divisor of $n$ lies in $Q$. If $Q$ contains only a single prime $q$, then $E_Q = \{q^j \,|\, j \geq 0\}$ is a Hadamard set and therefore also a Sidon set; consequently, for every real number $2 < p < \infty$, the bound

$$\|f\|_{L^p} \leq C\|f\|_{L^2}$$

holds for every function $f \in L^p$ whose Fourier coefficients are supported on the set $E_Q$, where $C > 0$ is a constant depending only on $p$; in other words, the set $E_Q$ is of type $\Lambda(p)$. When $Q$ has cardinality $\#Q \geq 2$, the set $E_Q$ is neither Hadamard nor Sidon; however, number theoretic results about solutions to so-called "$S$-unit equations" imply that $E_Q$ is again a $\Lambda(p)$ set for $2 < p < \infty$.

In this paper, we show that for any finite set $Q$ of prime numbers and any real number $2 < p < \infty$, the set $E_Q$ satisfies a much stronger analytic property, namely the noncommutative $\Lambda(p)$ property; that is, $E_Q$ is of type $\Lambda(p)_{cb}$. More precisely, we show that the bound

$$\|f\|_{L^p(S^p)} \leq C \max\left\{ \left\| \Big( \sum_n \widehat{f}(n)^* \widehat{f}(n) \Big)^{1/2} \right\|_{S^p}, \left\| \Big( \sum_n \widehat{f}(n) \widehat{f}(n)^* \Big)^{1/2} \right\|_{S^p} \right\}$$

holds for every function $f \in L^p(S^p)$ whose Fourier coefficients are supported on the set $E_Q$, where the constant $C > 0$ depends only on $p$ and on the cardinality $\#Q$ of the set $Q$. Here $S^p$ denotes the Schatten $p$-class over the Hilbert space $\ell_2$; it is the Banach space of all compact operators $x : \ell_2 \to \ell_2$ with a finite norm given by

$$\|x\|_{S^p} = \Big( \mathrm{Tr}\,(x^* x)^{p/2} \Big)^{1/p},$$

where $\mathrm{Tr}(\,\cdot\,)$ denotes the usual trace. The Banach space $L^p(S^p)$ consists of all Bochner measurable $S^p$-valued functions defined on the unit circle, equipped with the norm

$$\|f\|_{L^p(S^p)} = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(t)\|_{S^p}^p \, dt \right)^{1/p},$$

where $dt$ is the Lebesgue measure.

To establish our results, we introduce a certain combinatorial property $Z^\star(k)$, defined for every integer $k \geq 2$, and show that every set $E$ with the property $Z^\star(k)$ is necessarily of type $\Lambda(2k)_{cb}$. In particular, we observe that for any finite set $Q$ of primes, the set $E_Q$ satisfies $Z^\star(k)$ for every $k \geq 2$; this follows from the number theoretic results mentioned earlier. Note that the sets $E_Q$

with $\#Q \geq 2$, along with their translations, dilations, etc., provide the only currently known examples of sets that are of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ but are not Sidon sets.

The paper is organized as follows. Sections 2–7 are entirely expository in nature; there we review the definitions and results that are needed in the sequel. In Section 8, we show that the $Z^\star(k)$ property implies the $\Lambda(2k)_{cb}$ property. In Section 9, we observe that every set $E_Q$ satisfies $Z^\star(k)$ for all $k \geq 2$, and that $E_Q$ is not a Sidon set if $\#Q \geq 2$. In Section 10, we give some concluding remarks.

## 2 Khintchine inequalities

For every $n \in \mathbb{N}$, let $\varepsilon_n : \{\pm 1\}^{\mathbb{N}} \longrightarrow \{\pm 1\}$ denote the $n$-th coordinate projection, let $\nu$ be the uniform probability measure on $\{\pm 1\}^{\mathbb{N}}$, and let $p$ be an arbitrary real number with $2 < p < \infty$.

The classical *Khintchine inequalities* show that there exists a constant $C > 0$, depending only on $p$, such that for all $m \geq 1$ and any sequence $x_1, x_2, \ldots, x_m$ in $\mathbb{C}$, one has

$$\Big\| \sum_{n=1}^{m} \varepsilon_n x_n \Big\|_{L^p(\{\pm 1\}^{\mathbb{N}},\, \nu,\, \mathbb{C})} \leq C \left( \sum_{n=1}^{m} |x_n|^2 \right)^{1/2}; \tag{2.1}$$

see [6], for example, for a proof of Khintchine inequalities in the general case $1 \leq p < \infty$. The inequalities (2.1) were later generalized to the noncommutative setting by Lust-Piquard [7], who showed that there exists a constant $C > 0$, depending only on $p$, such that for all $m \geq 1$ and any sequence of operators $x_1, x_2, \ldots, x_m$ in $S^p$, the following inequality holds:

$$\begin{aligned} &\Big\| \sum_{n=1}^{m} \varepsilon_n x_n \Big\|_{L^p(\{\pm 1\}^{\mathbb{N}},\, \nu,\, S^p)} \\ &\qquad \leq C \max \left\{ \left\| \left( \sum_{n=1}^{m} x_n^* x_n \right)^{1/2} \right\|_{S^p}, \left\| \left( \sum_{n=1}^{m} x_n x_n^* \right)^{1/2} \right\|_{S^p} \right\}; \end{aligned} \tag{2.2}$$

see [7] for a proof of the noncommutative Khintchine inequalities in the more general case where $1 < p < \infty$; see also [8] for the case $p = 1$.

# 3 $\Lambda(p)$ sets

The notion of a $\Lambda(p)$ set was first introduced in [16] and studied extensively by Rudin and many others. In this paper, we restrict ourselves to the case where $2 < p < \infty$, for simplicity. For any set $E \subset \mathbb{Z}$, let

$$L^p_E = \left\{ f \in L^p \,\middle|\, \widehat{f} \text{ is supported on } E \right\},$$

where $\widehat{f}$ denotes the Fourier transform of $f$. Then $E$ is said to be of type $\Lambda(p)$, or $E$ has the $\Lambda(p)$ property, if there exists a constant $C > 0$, depending only on $p$ and $E$, such that for every function in $L^p_E$, the following bound holds:

$$\|f\|_{L^p} \leq C \Big( \sum_{n \in E} |\widehat{f}(n)|^2 \Big)^{1/2}.$$

We denote by $\lambda_p(E)$ the smallest constant $C$ for which this inequality holds for all $f \in L^p_E$.

Using convexity, one sees that every $\Lambda(p)$ set is also a $\Lambda(q)$ set for any real number $2 < q < p$.

We also recall that, as shown in [16], there is a natural size limitation for the intersection of any $\Lambda(p)$ set with a fixed arithmetic progression. More precisely, if $2 < p < \infty$ is fixed, and $E$ is a $\Lambda(p)$ set, then

$$\#\big(E \cap \{a + b, a + 2b, \ldots, a + Nb\}\big) \leq 4 \left(\lambda_p(E)\right)^2 N^{2/p} \tag{3.3}$$

for all integers $a, b, N$ with $N \geq 1$. This result is optimal. Indeed, given $2 < p < \infty$, there is a subset $E_N$ of $\{1, \ldots, N\}$ for each integer $N$, satisfying $\#E_N \geq N^{2/p}$ and $\lambda_p(E_N) \leq C_p$ where the constant $C_p$ depends only on $p$. This result was first shown by Rudin for even integers (see [16]), then later by Bourgain for arbitrary real numbers (see [2], and also [19]). It follows that for every $2 < p < \infty$, there exists a $\Lambda(p)$ set that is not a $\Lambda(q)$ set for any $q > p$.

In [16], a certain combinatorial property has been considered which is not only stronger but often easier to deal with than the analytic property $\Lambda(2k)$. Let $k \geq 1$ be a fixed integer. A set $E \subset \mathbb{Z}$ is called a $Z^+(k)$ set if there exists a constant $C > 0$, depending only on $E$, such that for all $\gamma \in \mathbb{Z}$,

$$\#\left\{ (n_1, n_2, \ldots, n_k) \in E^k \,\middle|\, n_1 + n_2 + \ldots + n_k = \gamma \right\} \leq C.$$

It has been shown by Rudin [16] that every $Z^+(k)$ set is necessarily of type $\Lambda(2k)$. In particular, for any finite set $Q$ of primes, the set $E_Q$ satisfies $Z^+(k)$ for all $k \geq 1$ (see Section 9); hence it follows that $E_Q$ is of type $\Lambda(p)$ for every $2 < p < \infty$.

## 4 Noncommutative $\Lambda(p)$ sets

The notion of noncommutative $\Lambda(p)$ sets was first introduced and studied in [5]. For $2 < p < \infty$ and $E \subset \mathbb{Z}$, let

$$L^p_E(S^p) = \left\{ f \in L^p(S^p) \,\middle|\, \widehat{f} \text{ is supported on } E \right\}.$$

The set $E$ is called a noncommutative $\Lambda(p)$ set (or simply, a $\Lambda(p)_{cb}$ set) if there exists a constant $C > 0$, depending only on $p$ and $E$, such that for every function $f$ in $L^p_E(S^p)$, the bound

$$\left\| f \right\|_{L^p(S^p)} \leq C \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_p \tag{4.4}$$

holds, where the triple norm $\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \cdot \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_p$ is defined by

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_p = \max \left\{ \left\| \left( \sum_{n \in \mathbb{Z}} \widehat{f}(n)^* \widehat{f}(n) \right)^{1/2} \right\|_{S^p}, \left\| \left( \sum_{n \in \mathbb{Z}} \widehat{f}(n) \widehat{f}(n)^* \right)^{1/2} \right\|_{S^p} \right\}.$$

We denote by $\lambda^{cb}_p(E)$ the smallest constant $C$ for which the inequality (4.4) holds for all $f \in L^p_E(S^p)$. Note that, by convexity, the opposite inequality

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_p \leq \left\| f \right\|_{L^p(S^p)} \tag{4.5}$$

always holds for every $f \in L^p(S^p)$. We remark that the notation $cb$ is an abbreviation for the words "completely bounded." Harcharras [5] showed that a given set $E$ has the $\Lambda(p)_{cb}$ property if and only if every bounded sequence $(a_n)_{n \in E}$ can be extended to a completely bounded Fourier multiplier on the operator space $L^p$ when the latter is endowed with its canonical operator space structure as defined by Pisier [13].

It is clear from the definition that every $\Lambda(p)_{cb}$ set is necessarily a $\Lambda(p)$ set, therefore the size restriction (3.3) applies. On the other hand, it has been shown in [5] that there exist sets with the $\Lambda(p)$ property for every $p$ which do not have the $\Lambda(p)_{cb}$ property for any $p$; thus, the $\Lambda(p)_{cb}$ property is much stronger than the $\Lambda(p)$ property in general.

Note that, by convexity, a $\Lambda(p)_{cb}$ set is also a $\Lambda(q)_{cb}$ set if $2 < q < p < \infty$. Building on the work of Rudin [16], it has been shown in [5] that for every even integer $p = 2k > 2$, there exists a $\Lambda(p)_{cb}$ set that does not have the $\Lambda(q)$ property for any $q > p$; the general case is still open.

In [5], a combinatorial property has been considered which is stronger and easier to deal with than the analytic property $\Lambda(2k)_{cb}$. Let $k \geq 1$ be a fixed integer. A set $E \subset \mathbb{Z}$ is called a $Z(k)$ set if there exists a constant $C > 0$, depending only on $k$ and $E$, such that for all $\gamma \in \mathbb{Z}$,

$$\#\left\{(n_1, n_2, \ldots, n_k) \in E^k \,\middle|\, n_i \neq n_j \text{ if } i \neq j, \text{ and } \sum_{j=1}^{k} (-1)^{j+1} n_j = \gamma\right\} \leq C.$$

It has been shown in [5] that an arbitrary $Z^+(k)$ set need not possess the $\Lambda(2k)_{cb}$ property even though it is a $\Lambda(2k)$ set as mentioned earlier. However, any set with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$.

Our review of the combinatorial property $Z(k)$ has been intended primarily to motivate our consideration of the new property $Z^\star(k)$ introduced in Section 8. In many situations, it is useful to have combinatorial criteria like $Z(k)$ and $Z^\star(k)$ which imply the (albeit weaker) analytic property $\Lambda(2k)_{cb}$. For the purposes of this paper, the $Z(k)$ property alone is insufficient, since for an arbitrary finite set $Q$ of primes, the set $E_Q$ need not be of type $Z(k)$. For example, taking $Q = \{2, 3\}$, the relation

$$2^{i+3}3^j - 2^i 3^{j+2} + 2^i 3^j = 0, \qquad \forall\, i, j \geq 0,$$

implies that $E_Q$ is not of type $Z(3)$ even though it is of type $Z^\star(k)$ for all $k \geq 2$ and therefore of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ (see Section 9).

# 5 Sidon sets

A set $E \subset \mathbb{Z}$ is called a Sidon set if there exists a constant $C > 0$, depending only on $E$, such that for all functions $f \in L^\infty_E$, the following bound holds:

$$\sum_{n \in E} \left|\widehat{f}(n)\right| \leq C \left\|f\right\|_{L^\infty}. \tag{5.6}$$

We denote by $\lambda_\infty(E)$ the smallest constant $C$ for which this inequality holds for all $f \in L^\infty_E$.

It is well known that a Sidon set is a $\Lambda(p)$ set for every $2 < p < \infty$. In fact, it is a $\Lambda(p)_{cb}$ set for every $2 < p < \infty$ as shown in [5]. On the other hand, there is a natural size limitation for the intersections of any Sidon set with a fixed arithmetic progression. It has been shown in [16] that there exists an absolute constant $C > 0$ such that for every Sidon set $E$,

$$\#\big(E \cap \{a + b, a + 2b, \ldots, a + Nb\}\big) \leq C\big(\lambda_\infty(E)\big)^2 \log N$$

for all integers $a, b, N$ with $N \geq 1$.

## 6 Pisier's Rademacherization principle

In this section, we describe a result of [14] that can be used to determine nontrivial upper bounds for the norm of certain sums of products of operators in which various repetitions of the indices occur.

Given two partitions $\mathcal{P} = \{\mathcal{P}_j\}$ and $\mathcal{Q} = \{\mathcal{Q}_i\}$ of the set $\{1, 2, \ldots, k\}$, write $\mathcal{P} \le \mathcal{Q}$ if for every $j$, $\mathcal{P}_j \subset \mathcal{Q}_i$ for some $i$, and write $\mathcal{P} < \mathcal{Q}$ whenever $\mathcal{P} \le \mathcal{Q}$ and $\mathcal{P} \ne \mathcal{Q}$. It is easily verified that the relation $\le$ provides a partial order on the set of all partitions of $\{1, 2, \ldots, k\}$; the unique minimal element with respect to $\le$ is the partition $\mathcal{P}_{\min} = \big\{\{1\}, \{2\}, \ldots, \{k\}\big\}$, while $\mathcal{P}_{\max} = \big\{\{1, 2, \ldots, k\}\big\}$ is the unique maximal element.

Given a $k$-tuple $n = (n_1, n_2, \ldots, n_k) \in E^k$, where $E$ is an arbitrary set, let $\mathcal{P}_n = \{\mathcal{P}_{n,j}\}$ denote the canonical partition attached to $n$; that is, for all $1 \le i, \ell \le k$, both $i$ and $\ell$ belong to the same set $\mathcal{P}_{n,j}$ if and only if $n_i = n_\ell$.

**Proposition 1.** *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{\varepsilon_n\}_{n \in E}$ a family of independent random variables with*

$$\mathbb{P}\big(\{\varepsilon_n = 1\}\big) = \mathbb{P}\big(\{\varepsilon_n = -1\}\big) = 1/2, \qquad \forall\, n \in E.$$

*Let $k \ge 2$ be an arbitrary integer. For $1 \le j \le k$, let $X_j$ be a Banach space, and $f_j : E \longrightarrow X_j$ a finitely supported function. Let*

$$\varphi : X_1 \times X_2 \times \ldots \times X_k \longrightarrow X$$

*be a $k$-linear map of norm at most 1, where $X$ is a given Banach space. Finally, for any partition $\mathcal{P}$ of the set $\{1, 2, \ldots, k\}$, put*

$$A_{\mathcal{P}} = \big\{ j \in \{1, 2, \ldots, k\} \,\big|\, \{j\} \in \mathcal{P} \big\}.$$

*Then the following inequality holds:*

$$\Big\| \sum_{\substack{n=(n_1,\ldots,n_k)\in E^k \\ \mathcal{P}_n \ge \mathcal{P}}} \varphi\big(f_1(n_1), \ldots, f_k(n_k)\big) \Big\|_X \le \prod_{j \in A_{\mathcal{P}}} \Big\| \sum_{n \in E} f_j(n) \Big\|_{X_j} \prod_{\substack{1 \le j \le k \\ j \notin A_{\mathcal{P}}}} \bigg( \int_\Omega \Big\| \sum_{n \in E} \varepsilon_n f_j(n) \Big\|_{X_j}^k \, d\,\mathbb{P} \bigg)^{1/k}.$$

# 7 Some operator norm inequalities

**Proposition 2.** *Let $1 \leq p \leq \infty$, $a, b > 1$ with $a^{-1} + b^{-1} = 1$, $y$ a positive operator in $S^{ap}$, and $x_1, x_2, \ldots, x_m$ a sequence of operators each in $S^{2bp}$. Then the following inequality holds:*

$$\Big\| \sum_{n=1}^{m} x_n^* y x_n \Big\|_{S^p} \leq \|y\|_{S^{ap}} \max\Big\{ \Big\| \sum_{n=1}^{m} x_n^* x_n \Big\|_{S^{bp}}, \Big\| \sum_{n=1}^{m} x_n x_n^* \Big\|_{S^{bp}} \Big\}.$$

This proposition first appears in [7] when $x_1, x_2, \ldots x_n$ is a sequence of self-adjoint operators. The general case requires only the three line lemma and can be found in [15].

The following corollary follows from Proposition 2 by a simple inductive argument.

**Corollary 3.** *Let $1 \leq p \leq \infty$ and $k \geq 1$ be fixed. For each $1 \leq j \leq k$, let $E_j$ be a finite set of indices, let $a_j > 1$, and let $\{x_{j,n}\}_{n \in E_j}$ be a family of operators in $S^{2a_j p}$. Finally, suppose that $\sum_{j=1}^{k} a_j^{-1} = 1$. Then the following inequality holds:*

$$\Big\| \sum_{n_j \in E_j,\ 1 \leq j \leq k} x_{k,n_k}^* \ldots x_{2,n_2}^* x_{1,n_1}^* \, x_{1,n_1} x_{2,n_2} \ldots x_{k,n_k} \Big\|_{S^p} \leq \prod_{j=1}^{k} \max\Big\{ \Big\| \sum_{n \in E_j} x_{j,n}^* x_{j,n} \Big\|_{S^{a_j p}}, \Big\| \sum_{n \in E_j} x_{j,n} x_{j,n}^* \Big\|_{S^{a_j p}} \Big\}.$$

# 8 Main results

Throughout this section, let $k$ be a fixed integer with $k \geq 2$. Here we introduce a new combinatorial property for sets $E \subset \mathbb{Z}$, similar to the $Z(k)$ property described in Section 4.

We say that a set $E \subset \mathbb{Z}$ has the property $Z^\star(k)$ if there is a constant $C > 0$, depending only on $E$ and $k$, such that:

(*i*) For every nonzero $\gamma \in \mathbb{Z}$, the conditions

$$n_1 - n_2 + \ldots + (-1)^{k+1} n_k = \gamma \tag{8.7}$$

and

$$\sum_{j\in\mathcal{J}}(-1)^{j+1}n_j \neq 0 \qquad \text{for all } \emptyset \neq \mathcal{J} \subsetneq \{1,\ldots,k\} \tag{8.8}$$

are satisfied for at most $C$ elements $(n_1, n_2, \ldots, n_k) \in E^k$.

*(ii)* For every $\emptyset \neq \mathcal{J} \subset \{1,\ldots,k\}$, there are at most $C$ vectors $\mathrm{v}_\ell \in \mathbb{Q}^{\#\mathcal{J}}$ such that if the vector $n = (n_j)_{j\in\mathcal{J}} \in E^{\#\mathcal{J}}$ satisfies the conditions

$$\sum_{j\in\mathcal{J}}(-1)^{j+1}n_j = 0 \tag{8.9}$$

and

$$\sum_{j\in\mathcal{J}'}(-1)^{j+1}n_j \neq 0 \qquad \text{for all } \emptyset \neq \mathcal{J}' \subsetneq \mathcal{J}, \tag{8.10}$$

then $n = \eta\mathrm{v}_\ell$ for some $\eta \in E$ and some $1 \leq \ell \leq C$.

**Theorem 4.** *If a set $E \subset \mathbb{Z}$ has the property $Z^\star(k)$, then $E$ is a $\Lambda(2k)_{cb}$ set.*

*Proof.* Without loss of generality, one can assume that $E \subset \mathbb{N}$. Throughout the proof, the letter $C$ is used to denote any positive constant that occurs and depends only on $k$ and or $E$; its precise meaning might change from line to line.

For every $n = (n_1, n_2, \ldots, n_k) \in E^k$, let $R_n$ denote the collection of all subsets $\emptyset \neq \mathcal{J} \subsetneq \{1,\ldots,k\}$ such that (8.9) and (8.10) hold, and let $\mathcal{R}$ be the set of all collections obtained in this way; that is,

$$\mathcal{R} = \{R \,|\, R = R_n \text{ for some } n \in E^k\}.$$

For $R, R' \in \mathcal{R}$, write $R' < R$ or $R > R'$ whenever $\emptyset \neq R' \subsetneq R$. Then the relation $<$ defines a partial order on $\mathcal{R}$. We also put

$$d_0 = \max\{\#R \,|\, R \in \mathcal{R}\},$$

and for $0 \leq d \leq d_0$, let

$$\mathcal{R}(d) = \{R \in \mathcal{R} \,|\, \#R = d\}.$$

Then $\mathcal{R}$ is the disjoint union $\mathcal{R} = \bigcup_{d=0}^{d_0} \mathcal{R}(d)$.

Now let $f = \sum_{n\in E} x_n e^{int}$ be fixed; note that $x_n = \widehat{f}(n) \in S^{2k}$ for every $n \in E$. For simplicity, we assume that the Fourier transform $\widehat{f}$ is finitely supported.

For every $k$-tuple $n = (n_1, n_2, \ldots, n_k) \in E^k$, let

$$\widetilde{x}_n = x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \ldots x_{n_k}^{\mu_k} \in S^2,$$

where $\mu_j = 1$ if $j$ is odd, and $\mu_j = *$ if $j$ is even. Then

$$\|f\|_{L^{2k}(S^{2k})}^{2k} = \|f^{\mu_1} f^{\mu_2} \ldots f^{\mu_k}\|_{L^2(S^2)}^2 = \Big\| \sum_{\gamma \in \mathbb{Z}} e^{\mathrm{i}\gamma t} \sum_{n \in E^k(\gamma)} \widetilde{x}_n \Big\|_{L^2(S^2)}^2,$$

where for each $\gamma \in \mathbb{Z}$,

$$E^k(\gamma) = \{n = (n_1, n_2, \ldots, n_k) \in E^k \,|\, n_1 - n_2 + \ldots + (-1)^{k+1} n_k = \gamma\}.$$

It follows that

$$\|f\|_{L^{2k}(S^{2k})}^{2k} = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{n \in E^k(\gamma)} \widetilde{x}_n \Big\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \Big\|_{S^2}^2.$$

Thus,

$$\|f\|_{L^{2k}(S^{2k})}^{2k} \le C \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \Big\|_{S^2}^2 = C \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \mathcal{S}(R),$$

where we have set

$$\mathcal{S}(R) = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \Big\|_{S^2}^2.$$

For each collection $R$ with $0 < \#R < d_0$, one has

$$\begin{aligned} \mathcal{S}(R) &\le 2 \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \ge R}} \widetilde{x}_n \Big\|_{S^2}^2 + 2 \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n > R}} \widetilde{x}_n \Big\|_{S^2}^2 \\ &\le 2\, \widetilde{\mathcal{S}}(R) + C \sum_{\substack{d < d' \le d_0 \\ R' \in \mathcal{R}(d')}} \mathcal{S}(R'), \end{aligned}$$

where we have set

$$\widetilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \ge R}} \widetilde{x}_n \Big\|_{S^2}^2;$$

when $\#R = 0$ or $d_0$, it is clear that $\mathcal{S}(R) = \widetilde{\mathcal{S}}(R)$. Consequently,

$$\|f\|_{L^{2k}(S^{2k})}^{2k} \leq C \sum_{\substack{0\leq d\leq d_0\\ R\in\mathcal{R}(d)}} \widetilde{\mathcal{S}}(R). \tag{8.11}$$

*Step 1*. We start by showing that the inequality $\widetilde{\mathcal{S}}(\emptyset) \leq C|\!|\!| f |\!|\!|_{2k}^{2k}$ holds for some constant $C > 0$. Indeed,

$$\widetilde{\mathcal{S}}(\emptyset) = \mathcal{S}(\emptyset) = \sum_{\substack{\gamma\in\mathbb{Z}\\ \gamma\neq 0}} \Big\| \sum_{\substack{n\in E^k(\gamma)\\ R_n=\emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 + \Big\| \sum_{\substack{n\in E^k(0)\\ R_n=\emptyset}} \widetilde{x}_n \Big\|_{S^2}^2. \tag{8.12}$$

Since $E$ has the property $Z^\star(k)$, for every $\gamma \neq 0$ the equation (8.7) has at most $C$ solutions $n \in E^k$ such that (8.8) also holds. Thus,

$$\begin{aligned}
\sum_{\substack{\gamma\in\mathbb{Z}\\ \gamma\neq 0}} \Big\| \sum_{\substack{n\in E^k(\gamma)\\ R_n=\emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 &\leq C \sum_{\substack{\gamma\in\mathbb{Z}\\ \gamma\neq 0}} \sum_{\substack{n\in E^k(\gamma)\\ R_n=\emptyset}} \|\widetilde{x}_n\|_{S^2}^2 \\
&\leq C \sum_{n\in E^k} \|\widetilde{x}_n^*\widetilde{x}_n\|_{S^1} = C \Big\| \sum_{n\in E^k} \widetilde{x}_n^*\widetilde{x}_n \Big\|_{S^1} \\
&= C \Big\| \sum_{n_1,n_2,\ldots,n_k\in E} (x_{n_k}^{\mu_k})^* \ldots (x_{n_2}^{\mu_2})^* (x_{n_1}^{\mu_1})^* x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \ldots x_{n_k}^{\mu_k} \Big\|_{S^1} \\
&\leq C \prod_{j=1}^{k} \max\Big\{ \Big\| \sum_{n_j\in E} x_{n_j}^* x_{n_j} \Big\|_{S^k}, \Big\| \sum_{n_j\in E} x_{n_j} x_{n_j}^* \Big\|_{S^k} \Big\}
\end{aligned}$$

where for the last inequality, we have applied Corollary 3. It follows that

$$\sum_{\substack{\gamma\in\mathbb{Z}\\ \gamma\neq 0}} \Big\| \sum_{\substack{n\in E^k(\gamma)\\ R_n=\emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 \leq C\, |\!|\!| f |\!|\!|_{2k}^{2k}.$$

For every $n$ occurring in the second term of (8.12), since $\gamma = 0$, we see that the equation (8.9) holds with $\mathcal{J} = \{1, 2, \ldots, k\}$; since $R_n = \emptyset$, the condition (8.10) also applies. Hence, since $E$ has the property $Z^\star(k)$, there are at most $C$ vectors $\mathrm{v}_\ell \in \mathbb{Q}^k$ such that for each $n$ occurring in the second term of (8.12), $n = \eta\mathrm{v}_\ell$ for some $\eta \in E$ and $1 \leq \ell \leq C$. Using Cauchy-Schwarz's inequality, we see that

$$\Big\| \sum_{\substack{n\in E^k(0)\\ R_n=\emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 \leq C \sum_{1\leq\ell\leq C} \Big\| \sum_{\eta\in E} x_{\eta\mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta\mathrm{v}_{\ell,2}}^{\mu_2} \ldots x_{\eta\mathrm{v}_{\ell,k}}^{\mu_k} \Big\|_{S^2}^2.$$

Note that here and elsewhere in the proof, we write $x_z = 0$ if $z \in \mathbb{Q}$, $z \notin E$.

At this point, fix $1 \leq \ell \leq C$. We apply Proposition 1 with the following choices: $\Omega$ is $\{\pm 1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta \in E}$ is a family of coordinate projections, where $\varepsilon_\eta$ is the $m_\eta$-th projection on $\Omega$, for some enumeration $\{m_\eta \,|\, \eta \in E\}$ of the set $\mathbb{N}$; $\mathcal{P}$ is the maximal partition $\mathcal{P}_{\max}$; $\varphi$ is the $k$-linear contractive map that is simply the $k$-fold product from $S^{2k} \times S^{2k} \times \ldots \times S^{2k}$ into $S^2$; the functions $f_j : E \longrightarrow S^{2k}$ are defined by mapping $\eta \in E$ to $f_j(\eta) = x^{\mu_j}_{\eta \mathrm{v}_{\ell,j}}$ in $S^{2k}$, for each $1 \leq j \leq k$. By the proposition, it follows that

$$\Big\| \sum_{\eta \in E} x^{\mu_1}_{\eta \mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta \mathrm{v}_{\ell,2}} \ldots x^{\mu_k}_{\eta \mathrm{v}_{\ell,k}} \Big\|_{S^2} \leq \prod_{j=1}^{k} \left( \int_\Omega \Big\| \sum_{\eta \in E} \varepsilon_\eta \, x_{\eta \mathrm{v}_{\ell,j}} \Big\|^k_{S^{2k}} d\,\mathbb{P} \right)^{1/k}.$$

Now, apply Jensen's inequality followed by the noncommutative version of Khintchine inequalities (2.2) as follows:

$$\begin{aligned} \Big\| \sum_{\eta \in E} x^{\mu_1}_{\eta \mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta \mathrm{v}_{\ell,2}} \ldots x^{\mu_k}_{\eta \mathrm{v}_{\ell,k}} \Big\|_{S^2} &\leq \prod_{j=1}^{k} \left( \int_\Omega \Big\| \sum_{\eta \in E} \varepsilon_\eta \, x_{\eta \mathrm{v}_{\ell,j}} \Big\|^{2k}_{S^{2k}} d\,\mathbb{P} \right)^{1/(2k)} \\ &\leq C \prod_{j=1}^{k} \Big\| \sum_{\eta \in E} x_{\eta \mathrm{v}_{\ell,j}} \, e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \Big\|_{2k} \leq C \, \| f \|^k_{2k}, \end{aligned}$$

since for each $1 \leq j \leq C$,

$$\Big\| \sum_{\eta \in E} x_{\eta \mathrm{v}_{\ell,j}} \, e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \Big\|_{2k} \leq \| f \|_{2k}.$$

It follows that

$$\Big\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|^2_{S^2} \leq C \| f \|^{2k}_{2k},$$

which completes Step 1.

*Step 2.* Next, we show that the inequality $\widetilde{\mathcal{S}}(R) \leq C \big\| f \big\|^{2k-2}_{L^{2k}(S^{2k})} \| f \|^2_{2k}$ holds for every $1 \leq d \leq d_0$ and every $R \in \mathcal{R}(d)$.

For this aim, fix $1 \leq d \leq d_0$ and $R \in \mathcal{R}(d)$. There is a canonical equivalence relation $\mathcal{P}_R$ induced by the collection $R$ on the set $\{1, 2, \ldots, k\}$, defined as follows. Write $j \equiv \ell \pmod{\mathcal{P}_R}$ if and only if there exists a positive integer $t = t(j, \ell)$ and sets $\mathcal{J}_1, \ldots, \mathcal{J}_t$ in the collection $R$ such that:

(*i*) $j \in \mathcal{J}_1$ and $\ell \in \mathcal{J}_t$,

(*ii*) $\mathcal{J}_j \cap \mathcal{J}_{j+1} \neq \emptyset$ for $1 \leq j < t$.

Let $\mathcal{P}_R$ also denote the corresponding partition of $\{1, 2, \ldots, k\}$, and let $a_R$ denote the number of singleton sets in $\mathcal{P}_R$. Below we show the following more precise inequality:

$$\widetilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \Bigg\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \widetilde{x}_n \Bigg\|_{S^2}^2 \leq C \big\| f \big\|_{L^{2k}(S^{2k})}^{2a_R} \big\|\big| f \big\|\big|_{2k}^{2k-2a_R}.$$

Combining (*ii*) in property $Z^\star(k)$ with condition (*ii*) in our definition of the equivalence relation $\mathcal{P}_R$ above, it is not hard to see that there are at most $C$ vectors $\mathrm{v}_\ell = (\mathrm{v}_{\ell,j})_{j=1}^k \in \mathbb{Q}^k$ with the properties:

(*i*) $\mathrm{v}_{\ell,j} = 1$ if $\{j\} \in \mathcal{P}_R$,

(*ii*) For every $n = (n_1, n_2, \ldots, n_k) \in E^k$, the inequality $R_n \geq R$ holds if and only if for some $1 \leq \ell \leq C$ and some $\eta = (\eta_1, \eta_2, \ldots, \eta_k) \in E^k$ with $\eta_i = \eta_\ell$ whenever $i \equiv \ell \pmod{\mathcal{P}_R}$, $n_j = \eta_j \mathrm{v}_{\ell,j}$ for all $1 \leq j \leq k$.

Consequently,

$$\begin{aligned}
\widetilde{\mathcal{S}}(R) &= \sum_{\gamma \in \mathbb{Z}} \Bigg\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \widetilde{x}_n \Bigg\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \Bigg\| \sum_{\substack{n = (n_1, \ldots, n_k) \in E^k(\gamma) \\ R_n \geq R}} x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \cdots x_{n_k}^{\mu_k} \Bigg\|_{S^2}^2 \\
&= \sum_{\gamma \in \mathbb{Z}} \Bigg\| \sum_{1 \leq \ell \leq C} \sum_{\substack{\eta = (\eta_1, \ldots, \eta_k) \in E^k, \ \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1} - \eta_2 \mathrm{v}_{\ell,2} + \ldots + (-1)^{k+1} \eta_k \mathrm{v}_{\ell,k} = \gamma}} x_{\eta_1 \mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta_2 \mathrm{v}_{\ell,2}}^{\mu_2} \cdots x_{\eta_k \mathrm{v}_{\ell,k}}^{\mu_k} \Bigg\|_{S^2}^2 \\
&\leq C \sum_{1 \leq \ell \leq C} \sum_{\gamma \in \mathbb{Z}} \Bigg\| \sum_{\substack{\eta = (\eta_1, \ldots, \eta_k) \in E^k, \ \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1} - \eta_2 \mathrm{v}_{\ell,2} + \ldots + (-1)^{k+1} \eta_k \mathrm{v}_{\ell,k} = \gamma}} x_{\eta_1 \mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta_2 \mathrm{v}_{\ell,2}}^{\mu_2} \cdots x_{\eta_k \mathrm{v}_{\ell,k}}^{\mu_k} \Bigg\|_{S^2}^2 \\
&= C \sum_{1 \leq \ell \leq C} \widetilde{\mathcal{S}}_\ell(R),
\end{aligned}$$

where $\widetilde{\mathcal{S}}_\ell(R)$ denotes the inner summation for each $\ell$.

Let $1 \le \ell \le C$ be fixed; then we can estimate $\widetilde{\mathcal{S}}_\ell(R)$ as follows:

$$\begin{aligned}\widetilde{\mathcal{S}}_\ell(R) &= \Bigg\| \sum_{\gamma\in\mathbb{Z}} e^{\mathrm{i}\gamma t} \sum_{\substack{\eta=(\eta_1,\dots,\eta_k)\in E^k,\ \mathcal{P}_\eta\ge\mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1}-\eta_2\mathrm{v}_{\ell,2}+\dots+(-1)^{k+1}\eta_k\mathrm{v}_{\ell,k}=\gamma}} x^{\mu_1}_{\eta_1\mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta_2\mathrm{v}_{\ell,2}} \dots x^{\mu_k}_{\eta_k\mathrm{v}_{\ell,k}} \Bigg\|^2_{L^2(S^2)} \\ &= \Bigg\| \sum_{\substack{\eta=(\eta_1,\dots,\eta_k)\in E^k \\ \mathcal{P}_\eta\ge\mathcal{P}_R}} \Big( x_{\eta_1\mathrm{v}_{\ell,1}} e^{\mathrm{i}\eta_1\mathrm{v}_{\ell,1}}\Big)^{\mu_1} \dots \Big( x_{\eta_k\mathrm{v}_{\ell,k}} e^{\mathrm{i}\eta_k\mathrm{v}_{\ell,k}}\Big)^{\mu_k} \Bigg\|^2_{L^2(S^2)}.\end{aligned}$$

We apply Proposition 1 with the following choices: $\Omega$ is $\{\pm1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta\in E}$ is a family of coordinate projections, where $\varepsilon_\eta$ is the $m_\eta$-th projection on $\Omega$, for some enumeration $\{m_\eta \,|\, \eta\in E\}$ of the set $\mathbb{N}$; $\mathcal{P}$ is the partition $\mathcal{P}_R$; $\varphi$ is the $k$-linear contractive map that is simply the $k$-fold product from $L^{2k}(S^{2k}) \times L^{2k}(S^{2k}) \times \dots \times L^{2k}(S^{2k})$ into $L^2(S^2)$; the functions $f_j : E \longrightarrow L^{2k}(S^{2k})$ are defined by mapping $\eta\in E$ to

$$f_j(\eta) : t \mapsto \Big( x_{\eta\mathrm{v}_{\ell,j}} e^{\mathrm{i}\eta\mathrm{v}_{\ell,j} t}\Big)^{\mu_j}$$

for each $1 \le j \le k$. Note that each $f_j \in L^{2k}(S^{2k})$. By the proposition, it follows that

$$\begin{aligned}\widetilde{\mathcal{S}}_\ell(R)^{1/2} &\le \prod_{\substack{1\le j\le k \\ \{j\}\in\mathcal{P}_R}} \Big\| \sum_{\eta\in E} f_j(\eta)\Big\|_{L^{2k}(S^{2k})} \prod_{\substack{1\le j\le k \\ \{j\}\notin\mathcal{P}_R}} \Big( \int_\Omega \Big\| \sum_{\eta\in E} \varepsilon_\eta f_j(\eta)\Big\|^k_{L^{2k}(S^{2k})} d\,\mathbb{P}\Big)^{1/k} \\ &= \Big\| \sum_{\eta\in E} x_\eta e^{\mathrm{i}\eta t}\Big\|^{a_R}_{L^{2k}(S^{2k})} \prod_{\substack{1\le j\le k \\ \{j\}\notin\mathcal{P}_R}} \Big( \int_\Omega \Big\| \sum_{\eta\in E} \varepsilon_\eta\, x_{\eta\mathrm{v}_{\ell,j}} e^{\mathrm{i}\eta\mathrm{v}_{\ell,j} t}\Big\|^{2k}_{L^{2k}(S^{2k})} d\,\mathbb{P}\Big)^{1/2k} \\ &\le C \,\|f\|^{a_R}_{L^{2k}(S^{2k})} \prod_{\substack{1\le j\le k \\ \{j\}\notin\mathcal{P}_R}} \widetilde{\mathcal{S}}_j\end{aligned}$$

where for the second inequality, we have used Jensen's inequality, and for the third one, we have used Fubini's Theorem followed by (2.2), and where

$$\widetilde{\mathcal{S}}_j = \max\left\{ \Big\| \Big( \sum_{\eta\in E} x_{\eta\mathrm{v}_{\ell,j}} x^*_{\eta\mathrm{v}_{\ell,j}}\Big)^{1/2}\Big\|_{S^{2k}}, \Big\| \Big( \sum_{\eta\in E} x^*_{\eta\mathrm{v}_{\ell,j}} x_{\eta\mathrm{v}_{\ell,j}}\Big)^{1/2}\Big\|_{S^{2k}} \right\} \le |||f|||_{2k}$$

for every $1 \le j \le k$ with $\{j\} \notin \mathcal{P}_R$. Therefore, we have shown that for each $1 \le \ell \le C$,

$$\widetilde{\mathcal{S}}_\ell(R) \le C\,\|f\|^{2a_R}_{L^{2k}(S^{2k})} |||f|||^{2k-2a_R}_{2k}.$$

It follows that

$$\widetilde{\mathcal{S}}(R) \leq C \left\|f\right\|_{L^{2k}(S^{2k})}^{2a_R} \left\|\!\left|f\right|\!\right\|_{2k}^{2k-2a_R} \leq C \left\|f\right\|_{L^{2k}(S^{2k})}^{2k-2} \left\|\!\left|f\right|\!\right\|_{2k}^{2},$$

where for the second inequality, we have used (4.5). This completes Step 2.

*Step 3*. Combining our estimates from Steps 1 and 2, we have by (8.11):

$$\left\|f\right\|_{L^{2k}(S^{2k})}^{2k} \leq C \left( \left\|\!\left|f\right|\!\right\|_{2k}^{2k} + \left\|f\right\|_{L^{2k}(S^{2k})}^{2k-2} \left\|\!\left|f\right|\!\right\|_{2k}^{2} \right),$$

which clearly implies that

$$\left\|f\right\|_{L^{2k}(S^{2k})} \leq C \left\|\!\left|f\right|\!\right\|_{2k}.$$

This completes the proof. □

# 9 $S$-unit equations

In this section, we use some known number theoretic results to show that for an arbitrary finite set $Q$ of primes, the set $E_Q$ is of type $\Lambda(p)_{cb}$ for $2 < p < \infty$.

Let $K$ be an algebraic number field of degree $d$; that is, $K$ is a finite extension of the rationals $\mathbb{Q}$, with $d = [K : \mathbb{Q}]$. Let $S$ be a finite collection of places of $K$ containing all of the archimedean places, and let $\mathcal{U}_S$ be the group of $S$-units inside the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in $K$. Given nonzero elements $a_1, \ldots, a_k \in K$, one is interested in counting the number of nondegenerate solutions to the $S$-unit equation

$$a_1x_1 + a_2x_2 + \ldots + a_kx_k = 1, \qquad x_1, x_2, \ldots, x_k \in \mathcal{U}_S, \tag{9.13}$$

i.e., those where no proper subsum $a_{j_1}x_{j_1} + \ldots + a_{j_\ell}x_{j_\ell}$ vanishes.

Mahler [9] proved that for $k = 2$ and $K = \mathbb{Q}$, (9.13) has only finitely many solutions. Van der Poorten and Schlickewei [12] and Evertse [3] independently proved that for all $k \geq 2$ and every number field $K$, (9.13) has only finitely many solutions. This result was later extended by Evertse and Győry [4], who showed that the number of solutions is bounded by a constant which is independent of the coefficients $a_1, \ldots, a_k$. Later, Schlickewei showed that the constant depends only on $k$, on the cardinality $\#S$ of the set $S$, and on the degree $d$ (see [17] for the case $K = \mathbb{Q}$, and [18] for the general case).

In particular, when $K = \mathbb{Q}$, for any finite set $Q$ of primes, one can apply the results of [17] mentioned above with $S = Q \cup \{\infty\}$ to deduce that $E_Q$

satisfies both properties $Z^+(k)$ and $Z^\star(k)$ for all $k \geq 2$, where the constant $C > 0$ depends only on $k$ and on the cardinality $\#Q$ of the set $Q$. In fact, our definition of property $Z^\star(k)$ was chosen with precisely these sets in mind. Applying now Theorem 4 together with our remarks from Section 4, we obtain the following:

**Theorem 5.** *Let $Q$ be a nonempty finite set of prime numbers. Then the set $E_Q$ is of type $\Lambda(p)_{cb}$ for every real number $2 < p < \infty$.*

We conclude this section by observing that $E_Q$ is not a Sidon set whenever $\#Q \geq 2$. Indeed, let $s = \#Q$, and let $q_1 < q_2 < \ldots < q_s$ be the primes in $Q$. Then for all nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_s \leq (\log N)/(s \log q_s)$, the integer $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ lies in $E_Q$ and in $[1, N]$. Thus, if $N$ is sufficiently large,

$$\#\big(E_Q \cap [1, N]\big) \geq C\,(\log N)^s$$

where the constant $C > 0$ depends only on $Q$. This contradicts (5.6) (with $a = 0$ and $b = 1$) whenever $s = \#Q \geq 2$.

# 10 Remarks

The notions of $\Lambda(p)$ and $\Lambda(p)_{cb}$ sets and the properties $Z^+(k)$ and $Z(k)$ can be naturally defined for an arbitrary discrete group $G$. In this more general context, it has been shown that any subset of $G$ with the $Z^+(k)$ property is necessarily of type $\Lambda(2k)$. The argument is identical to that given by Rudin in the special case $G = \mathbb{Z}$; see [16]. It is also known that any subset of $G$ with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$ by the results of [5]. It would be interesting to find a suitable generalization of the property $Z^\star(k)$ for an arbitrary discrete group $G$ and to show that any subset of $G$ with the $Z^\star(k)$ property is necessarily of type $\Lambda(2k)_{cb}$. It would also be of interest to obtain explicit examples of $\Lambda(2k)_{cb}$ sets in $G$ that are similar to the sets $E_Q$ considered here.

Let $G$ be any discrete group and $k \geq 2$ a fixed integer. If a set $E \subset G$ has the $Z(k)$ property, then it is of type $\Lambda(2k)_{cb}$ as we have just mentioned. Consequently, the union of any finite number of sets with the $Z(k)$ property is also of type $\Lambda(2k)_{cb}$. It is natural to ask whether the converse statement is also true; this question was originally raised by Pisier when $G = \mathbb{Z}$ and is still open.

**Question 1.** *Let $G$ be a discrete group, and let $E \subset G$ be a set of type $\Lambda(2k)_{cb}$, where $k > 2$ is a fixed integer. Does there exist a finite collection*

$E_1, E_2, \ldots, E_c$ *of subsets of* $G$ *such that each* $E_j$ *has the* $Z(k)$ *property and such that* $E$ *is the union of the* $E_j$*?*

Using Mihăilescu's recent proof of the Catalan conjecture (see [10], and also [1]), one can show that every set $E_Q$ with $\#Q = 2$ can be decomposed into (at most) four sets, each with the $Z(3)$ property. In particular, this shows that $E_Q$ is of type $\Lambda(6)_{cb}$ without using our Theorem 4. However, we do not see how to generalize this to an arbitrary set $E_Q$ and an arbitrary integer $k \geq 2$, since the appropriate analogue to Mihăilescu's result is missing.

Finally, it has been shown in [5] that any noncommutative $\Lambda(p)$ set cannot contain the sum $A+A$ for any infinite set $A$. Neuwirth [11] later noticed that the arguments in [5] can be slightly modified to show that a noncommutative $\Lambda(p)$ set cannot contain the sum $A+B$ for any infinite sets $A$ and $B$. By Theorem 4, this can therefore be applied to any set $E$ with the property $Z^\star(k)$. For the special sets $E_Q$, stronger results are known: $E_Q$ cannot contain the sum $A+B$ for any infinite set $A$ and any set $B$ with at least two elements. This follows, for example, from a fairly deep result due to Mahler: for any finite set of primes $Q$, the gaps between consecutive integers free of primes outside of $Q$ tend to infinity. The authors wish to thank Carl Pomerance for bringing this to our attention.

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