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ON THE NORM OF AN IDEMPOTENT SCHUR MULTIPLIER ON THE SCHATTEN CLASS

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Abstract. We show that if the norm of an idempotent Schur multiplier on the Schatten class S^p lies sufficiently close to 1, then it is necessarily equal to 1. We also give a simple characterization of those idempotent Schur multipliers on S^p whose norm is 1.

1. Introduction

We study norms of idempotent Schur multipliers defined on the Schatten p-class with $1 < p < \infty$, $p \neq 2$. For any idempotent Schur multiplier ϕ, we show that if the norm of ϕ lies sufficiently close to 1, then it is necessarily equal to 1. More precisely, if ϕ is an idempotent Schur multiplier on the Schatten p-class, then $\phi = 0$, $\|\phi\| = 1$, or $\|\phi\| \geq 1 + \eta_p$, where η_p is a positive constant that depends only on p. We also obtain a simple characterization of those idempotent Schur multipliers whose norm is equal to 1. When $p = 1$ or ∞, these results have been obtained by Livshits [2], while for $p = 2$, every nonzero idempotent Schur multiplier has norm 1.

To state our results more explicitly, we need to fix some standard terminology. For every real number p in the range $1 \leq p < \infty$, denote by S^p the *Schatten p-class* over the Hilbert space ℓ_2; it is the Banach space of all compact operators $x : \ell_2 \to \ell_2$ with finite norm

$$\|x\|_{S^p} = \Big(\mathrm{Tr}\,(x^*x)^{p/2}\Big)^{1/p},$$

where $\mathrm{Tr}(\,\cdot\,)$ denotes the usual trace. For $p = \infty$, the space S^∞ is the Banach space of all compact operators $x : \ell_2 \to \ell_2$, equipped with the usual operator norm. The spaces S^p, $1 \leq p \leq \infty$, were considered in [4] as noncommutative analogues for the spaces ℓ_p, $1 \leq p \leq \infty$ (for a more modern reference, see [3] for example).

For $1 \leq p \leq \infty$ and a positive integer n, let S_n^p denote the Schatten p-class over the Hilbert space ℓ_2^n of dimension n.

In what follows, we make no distinction between an operator x on ℓ_2 and the corresponding matrix $(x_{ij})_{i,j\in\mathbb{N}}$ relative to the canonical basis $\{e_{ij}\}_{i,j\in\mathbb{N}}$ of S^p.

A set-theoretic map $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is said to be a *Schur multiplier* on S^p if the associated operator $T_\phi : S^p \to S^p$, defined by

$$T_\phi(x) = (\phi_{ij}\, x_{ij})_{i,j\in\mathbb{N}}, \qquad \forall\, x = (x_{ij})_{i,j\in\mathbb{N}} \in S^p,$$

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is well defined and bounded on S^p. In particular, this implies that ϕ itself is a bounded map. Let $\mathcal{M}(S^p)$ denote the space of all Schur multipliers on S^p. Then $\mathcal{M}(S^p)$ is a Banach algebra when it is equipped with the pointwise product and the norm

$$\|\phi\|_{\mathcal{M}(S^p)} = \|T_\phi : S^p \to S^p\|, \qquad \forall\, \phi \in \mathcal{M}(S^p).$$

It is well known that for pairs $1 \le p, q \le \infty$ with $p^{-1}+q^{-1} = 1$, the algebras $\mathcal{M}(S^p)$ and $\mathcal{M}(S^q)$ can be identified isometrically. These identifications can be done via the identity map by defining the duality between S^p and S^q with $\langle x, y\rangle = \mathrm{Tr}({}^t x\, y)$ for all $x \in S^p$ and $y \in S^q$.

In addition, the space $\mathcal{M}(S^2)$ can be identified isometrically with the Hilbert space $\ell_2(\mathbb{N} \times \mathbb{N})$. Consequently, when studying $\mathcal{M}(S^p)$ it suffices to reduce to the case where $2 < p \le \infty$.

Finally, a Schur multiplier $\phi \in \mathcal{M}(S^p)$ is said to be *idempotent* provided that $T_\phi \circ T_\phi = T_\phi$; clearly, this is equivalent to the condition that ϕ maps $\mathbb{N} \times \mathbb{N}$ into the set $\{0, 1\}$. For such multipliers, one has

$$\|\phi\|_{\mathcal{M}(S^p)} = \|\phi \cdot \phi\|_{\mathcal{M}(S^p)} \le \|\phi\|^2_{\mathcal{M}(S^p)}.$$

Hence, $\|\phi\|_{\mathcal{M}(S^p)} \ge 1$ whenever $\phi \ne 0$. Our main result is the following:

Theorem 1. *For every real number p with $1 < p < \infty$ and $p \ne 2$, there exists a constant $\eta_p > 0$ (depending only on p) such that for every nonzero idempotent Schur multiplier $\phi \in M(S^p)$ with $\|\phi\|_{\mathcal{M}(S^p)} \ne 1$, the following inequality holds:*

$$\|\phi\|_{\mathcal{M}(S^p)} \ge 1 + \eta_p.$$

By the remarks above, it suffices to consider the case where $2 < p < \infty$, which we assume throughout the sequel.

2. Proof of the Main Result

The proof of Theorem 1 can be split into three pieces, as follows.

Lemma 1. *Let $\Delta = (\Delta_{ij})_{1\le i,j\le 2}$ with $\Delta_{11} = \Delta_{12} = \Delta_{22} = 1$ and $\Delta_{21} = 0$. Then $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > \|\Delta\|_{\mathcal{M}(S_2^p)} > 1$ for $2 < p < p' \le \infty$.*

Proof. For every $c \in \mathbb{C}$, let $x^{(c)} = (x^{(c)}_{ij})_{1\le i,j\le 2}$, where $x^{(c)}_{11} = x^{(c)}_{12} = x^{(c)}_{22} = 1$ and $x^{(c)}_{21} = c$. One has

$$\|x^{(c)}\|_{S_2^p} = \Big(\mathrm{Tr}\left(x^{(c)*}x^{(c)}\right)^{p/2}\Big)^{1/p} = \Big(\lambda_{+,c}^{p/2} + \lambda_{-,c}^{p/2}\Big)^{1/p},$$

where

$$\lambda_{\pm,c} = \frac{1}{2}\left(3 + |c|^2 \pm \sqrt{5 + 8\,\Re(c) + 2|c|^2 + |c|^4}\right).$$

In particular, if we choose $c = (2-p)/2$, then

$$\|\Delta\|^p_{\mathcal{M}(S_2^p)} \ge \frac{\|\Delta(x^{(c)})\|^p_{S_2^p}}{\|x^{(c)}\|^p_{S_2^p}} = \frac{\|x^{(0)}\|^p_{S_2^p}}{\|x^{(c)}\|^p_{S_2^p}} = f(p),$$

where $f(p)$ is the function

$$\frac{2^p\big((3+\sqrt{5})^{p/2} + (3-\sqrt{5})^{p/2}\big)}{\big(p^2 - 4p + 16 + (p-4)\sqrt{p^2+16}\,\big)^{p/2} + \big(p^2 - 4p + 16 - (p-4)\sqrt{p^2+16}\,\big)^{p/2}}.$$

Since $f(2) = 1$, $f'(2) = 0$, and

$$f''(2) = \frac{\log(3+\sqrt{5})}{3\sqrt{5}} - \frac{\log(3-\sqrt{5})}{3\sqrt{5}} - \frac{1}{6} > 0,$$

the Taylor expansion for $f(p)$ near $p = 2$ shows that $f(p) > 1$ if $2 < p < 2 + \varepsilon$, for some $\varepsilon > 0$. Thus,

$$\|\Delta\|_{\mathcal{M}(S_2^p)} > 1, \qquad \forall\, 2 < p < 2 + \varepsilon.$$

Now let $p' > p > 2$ be arbitrary real numbers and let $0 < \theta < 1$ be chosen so that $1/p = (1-\theta)/2 + \theta/p'$. By the classical results of complex interpolation, we have $S_2^p = \left(S_2^2, S_2^{p'}\right)_\theta$ isometrically (for the definition and fundamental results on complex interpolation, the reader is referred to [1]); hence it follows that

$$\|\Delta\|_{\mathcal{M}(S_2^p)} \le \|\Delta\|_{\mathcal{M}(S_2^2)}^{1-\theta} \, \|\Delta\|_{\mathcal{M}(S_2^{p'})}^{\theta}.$$

Taking $2 < p < 2 + \varepsilon$ with ε sufficiently small, and using the obvious fact that $\|\Delta\|_{\mathcal{M}(S_2^2)} = 1$, the preceding relation and our results above imply that $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > 1$ for all $2 < p' \le \infty$. Since $0 < \theta < 1$, the above relation further implies that $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > \|\Delta\|_{\mathcal{M}(S_2^p)}$ for $2 < p < p' \le \infty$. This completes the proof. □

It has been shown in [2] that $\|\Delta\|_{\mathcal{M}(S_2^\infty)} = 2/\sqrt{3}$, which provides an upper bound for $\|\Delta\|_{\mathcal{M}(S_2^p)}$ for any $p > 2$. On the other hand, in the notation of Lemma 1 and taking $c = -1$, we have for $p > 2$,

$$\|\Delta\|_{\mathcal{M}(S_2^p)} \ge \frac{\|x^{(0)}\|_{S_2^p}}{\|x^{(-1)}\|_{S_2^p}} = \left(\frac{(3+\sqrt{5})^{p/2} + (3-\sqrt{5})^{p/2}}{2^{p+1}}\right)^{1/p} > \frac{\sqrt{3+\sqrt{5}}}{2^{1+1/p}}.$$

It remains an interesting question to determine the precise value of $\|\Delta\|_{\mathcal{M}(S_2^p)}$ for any p in the range $2 < p < \infty$; this will not be needed, however, in what follows.

We now define, for each p in the range $2 < p < \infty$,

$$\eta_p = -1 + \|\Delta\|_{\mathcal{M}(S_2^p)}.$$

In view of Lemma 1, η_p is strictly positive.

Definition. A map ϕ defined on $\mathbb{N} \times \mathbb{N}$ (or any of its subsets) is said to be *triangle-free* if there are no integers i, j, k, l such that $\phi_{ij} = \phi_{il} = \phi_{kj} = 1$ and $\phi_{kl} = 0$.

The following lemma is an easy consequence of Lemma 1; the proof is omitted.

Lemma 2. *Fix $p > 2$, and suppose that $\phi \in \mathcal{M}(S^p)$ is idempotent. If*

$$\|\phi\|_{\mathcal{M}(S^p)} < 1 + \eta_p,$$

then ϕ is triangle-free.

Finally, we have

Lemma 3. *If a map $\phi : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ is nonzero and triangle-free, then $\|\phi\|_{\mathcal{M}(S^p)} = 1$ for every real number $p > 2$.*

Proof. For any positive integer n, denote by $\phi^{(n)}$ the restriction of ϕ to the subset $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ of $\mathbb{N} \times \mathbb{N}$. Recalling the well-known fact

$$\|\phi\|_{\mathcal{M}(S^p)} = \sup_{n \ge 1} \|\phi^{(n)}\|_{\mathcal{M}(S_n^p)},$$

we see that it suffices to show that $\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = 1$ whenever $\phi^{(n)} \ne 0$.

To this end, let $n \geq 1$ be fixed with $\phi^{(n)} \neq 0$. For every integer $1 \leq i \leq n$, define the row sum

$$c_i = \#\{1 \leq j \leq n \,|\, \phi_{ij}^{(n)} = 1\}.$$

To show $\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = 1$, we may freely permute the rows and/or the columns of $\phi^{(n)}$ in any way that we want; in particular, without loss of generality, we may assume that

$$c_1 \geq c_2 \geq c_3 \geq \ldots \geq c_n,$$

and that

$$\phi_{11}^{(n)} = \phi_{12}^{(n)} = \phi_{13}^{(n)} = \ldots = \phi_{1c_1}^{(n)} = 1.$$

Since ϕ is triangle-free, for every $1 \leq i \leq n$ there are only two possibilities:

(α) $\phi_{ij} = 1$ for all $1 \leq j \leq c_1$, and $\phi_{ij} = 0$ for all $j > c_1$;

(β) $\phi_{ij} = 0$ for all $1 \leq j \leq c_1$.

After permuting the rows if necessary, we may assume that (α) occurs for $1 \leq i \leq r_1$, and that (β) occurs for $i > r_1$. Then

$$\phi^{(n)} = \phi_1 \oplus \phi_1',$$

where ϕ_1 is an $r_1 \times c_1$ rectangular matrix with every entry equal to 1, and ϕ_1' is an $(n - r_1) \times (n - c_1)$ rectangular matrix whose entries are equal to 0 or 1 and which is triangle-free. If $\phi_1' = 0$, we stop; otherwise, we repeat the same argument with $\phi^{(n)}$ replaced by ϕ_1', obtaining

$$\phi^{(n)} = \phi_1 \oplus \phi_2 \oplus \phi_2'.$$

We continue in this way until the process stops, at which point we have

$$\phi^{(n)} = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_s,$$

where every ϕ_k, $1 \leq k \leq s$, is an $r_k \times c_k$ rectangular matrix, all of the entries of $\phi_1, \ldots, \phi_{s-1}$ are equal to 1, and the entries of ϕ_s are all equal to 1 or all equal to 0. By adding some additional zero rows and/or zero columns to $\phi^{(n)}$ if necessary, we may also assume that $r_k = c_k$ for $1 \leq k \leq s$. Then

$$\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = \sup_{1 \leq k \leq s} \|\phi_k\|_{\mathcal{M}(S_{r_k}^p)} = 1,$$

and the result follows. $\square$

Theorem 1 is an immediate consequence of Lemmas 1–3, as the reader can easily verify.

Examining the proof of Theorem 1, we see that for a nonzero idempotent Schur multiplier ϕ, the following assertions are equivalent:

(a) for some $p > 2$, $\phi : S^p \to S^p$ has norm 1;
(b) ϕ is triangle-free;
(c) ϕ is equivalent to a multiplier of the form $\phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \ldots$, where each ϕ_j has all of its entries equal to 1 or all of its entries equal to 0;
(d) $\phi : S^\infty \to S^\infty$ has norm 1;
(e) for every p, $\phi : S^p \to S^p$ has norm 1.

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