Concatenations with Binary Recurrent Sequences

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Abstract
Given positive integers $A_1, \ldots, A_t$ and $b \geq 2$, we write $A_1 \cdots A_t(b)$ for the integer whose base-$b$ representation is the concatenation of the base-$b$ representations of $A_1, \ldots, A_t$. In this paper, we prove that if $(u_n)_{n \geq 0}$ is a binary recurrent sequence of integers satisfying some mild hypotheses, then for every fixed integer $t \geq 1$, there are at most finitely many nonnegative integers $n_1, \ldots, n_t$ such that $|u_{n_1}| \cdots |u_{n_t}|(b)$ is a member of the sequence $(|u_n|)_{n \geq 0}$. In particular, we compute all such instances in the special case that $b = 10$, $t = 2$, and $u_n = F_n$ is the sequence of Fibonacci numbers.

1 Introduction

A result of Senge and Straus [24, 25] asserts that if $b_1, b_2 \geq 2$ are multiplicatively independent integers, there are at most finitely many positive integers with the property that the sum of the digits in each of the two bases $b_1$ and $b_2$ lies below any prescribed bound. An effective
version of this statement is due to Stewart [28], who gave a lower bound on the overall sum of the digits of \( n \) in base \( b_1 \) and in base \( b_2 \). A somewhat more general version of Stewart’s result has been obtained by Luca [16].

A variety of arithmetical questions about integers whose base-\( b \) digits satisfy certain restrictions has been considered by many authors; see, for example, [1, 2, 3, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 16, 18, 19, 27] and the references contained therein. Here, we consider integers whose base-\( b \) digits are formed by concatenating (absolute values of) terms in a binary recurrent sequence.

Let \((u_n)_{n \geq 0}\) be a binary recurrent sequence of integers; i.e., a sequence of integers satisfying the recurrence relation

\[
  u_{n+2} = ru_{n+1} + su_n \quad (n \geq 0),
\]

where \( r \) and \( s \) are nonzero integers with \( r^2 + 4s \neq 0 \). It is well known that if \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - rx - s = 0 \), then \( u_n = c\alpha^n + d\beta^n \) holds for all \( n \geq 0 \), where \( c \) and \( d \) are constants given by

\[
  c = -\frac{\beta u_0 + u_1}{\alpha - \beta} \quad \text{and} \quad d = \frac{\alpha u_0 - u_1}{\alpha - \beta}.
\]

Throughout the paper, we assume that \((u_n)_{n \geq 0}\) is nondegenerate (i.e., \( \alpha/\beta \) is not a root of 1, and \( \alpha \beta \) \( cd \neq 0 \)). Reordering the roots if necessary, we can further assume that \( |\alpha| \geq |\beta| \) and \( |\alpha| > 1 \).

Let \( b \geq 2 \) be a fixed integer base. Given positive integers \( A_1, \ldots, A_t \), we denote by \( A_1 \cdots A_t(b) \) the integer whose base-\( b \) representation is equal to the concatenation (in order) of the base-\( b \) representations of \( A_1, \ldots, A_t \). Thus, if \( l_i \) is the smallest positive integer such that \( A_i < b^{l_i} \), we have

\[
  A_1 \cdots A_t(b) = b^{l_2 + \cdots + l_t} A_1 + b^{l_3 + \cdots + l_t} A_2 + \cdots + b^{l_t} A_{t-1} + A_t.
\]

We always assume that \( A_1 \neq 0 \), and in the special case that \( b = 10 \), we omit the subscript to simplify the notation.

In this paper, we study the set of positive integers \( |u_n| \), where \((u_n)_{n \geq 0}\) is a binary recurrent sequence, that are the base-\( b \) concatenations of other numbers of the form \( |u_{n_j}|, j = 1, \ldots, t \). We show that if \( t \geq 2 \) is fixed, then there are only finitely many instances of the equality

\[
  |u_n| = |u_{n_1}| \cdots |u_{n_t}|(b)
\]

provided that the sequence \((u_n)_{n \geq 0}\) satisfies certain mild hypotheses. Note that some assumptions are clearly needed in order to rule out certain obvious counterexamples; for instance, the result does not hold for the sequence \( u_n = b^n - 1, n \geq 0 \), since the concatenation of any two or more terms produces another term of the same sequence.

**Theorem 1.** Let \( u_n = c\alpha^n + d\beta^n \) be a nondegenerate binary recurrent sequence of integers, and let \( b \geq 2 \) be a fixed integer base. Assume that \( \dim_{\mathbb{Z}}(\log \alpha, \log \beta, \log b) \geq 2 \). Then for every fixed integer \( t \geq 2 \), there are at most finitely many positive integers \( n \) for which the equality

\[
  |u_n| = |u_{n_1}| \overbrace{0 \cdots 0}^{m_1} |u_{n_2}| \overbrace{0 \cdots 0}^{m_2} \cdots |u_{n_t}| \overbrace{0 \cdots 0}^{m_t}(b)
\]

holds for some nonnegative integers \( n_1, \ldots, n_t \) and \( m_1, \ldots, m_t \), with \( u_{n_1} \neq 0 \).
Here, log(·) stands for any fixed determination of the natural logarithm function, and \( \dim_{\mathbb{Z}}(\log \alpha, \log \beta, \log b) \) denotes the rank of (the free part of) the additive subgroup of \( \mathbb{C} \) generated by \{log \alpha, log \beta, log b\}.

Although our proof of Theorem 1 is ineffective, this result can be seen as an extension of the aforementioned results of Senge and Straus [24, 25].

In some special cases, one can employ effective methods to completely determine all the solutions to an equation such as (2). Perhaps the best known example of a binary recurrent sequence is the Fibonacci sequence \( (F_n)_{n \geq 0} \), where \( F_0 = 0 \) and \( F_1 = 1 \), and (1) holds with \( r = s = 1 \). In this case, one has \( \alpha = (1 + \sqrt{5})/2, \beta = \alpha^{-1}, c = 1/(\alpha - \beta) \), and \( d = -c \). For this special sequence, we obtain the following computational result:

**Theorem 2.** If \((m, n, k)\) is an ordered triple of nonnegative integers with \( m > 0 \) and such that \( F_m F_n = F_k \), then \( F_k \in \{13, 21, 55\} \).

Throughout the paper, we use the Vinogradov symbols \( \ll \) and \( \gg \), as well as the Landau symbol \( O \), with the understanding that the implied constants are computable and depend at most on the given data.

## 2 Preliminaries

Let \( \mathbb{L} \) be an algebraic number field of degree \( D \) over \( \mathbb{Q} \). Denote by \( \mathcal{O}_L \) the ring of algebraic integers and by \( \mathcal{M}_L \) the set of places. For a fractional ideal \( \mathcal{I} \) of \( \mathbb{L} \), let \( \text{Nm}_L(\mathcal{I}) \) be the usual norm; we recall that \( \text{Nm}_L(\mathcal{I}) = \#(\mathcal{O}_L/\mathcal{I}) \) if \( \mathcal{I} \) is an ideal of \( \mathcal{O}_L \), and the norm map is extended multiplicatively (using unique factorization) to all of the fractional ideals of \( \mathbb{L} \).

For a prime ideal \( \mathcal{P} \), we denote by \( \text{ord}_\mathcal{P}(x) \) the order at which \( \mathcal{P} \) appears in the ideal factorization of the principal ideal \([x]\) generated by \( x \) in \( \mathbb{L} \).

For a place \( \mu \in \mathcal{M}_L \) and a number \( x \in \mathbb{L} \), we define the absolute value \( |x|_\mu \) as follows:

(i) \( |x|_\mu = |\sigma(x)|^{1/D} \) if \( \mu \) corresponds to a real embedding \( \sigma : \mathbb{L} \rightarrow \mathbb{R} \);

(ii) \( |x|_\mu = |\sigma(x)|^{2/D} = |\sigma(x)|^{2/D} \) if \( \mu \) corresponds to some pair of complex conjugate embeddings \( \sigma, \overline{\sigma} : \mathbb{L} \rightarrow \mathbb{C} \);

(iii) \( |x|_\mu = \text{Nm}_{\mathbb{L}}(\mathcal{P})^{-\text{ord}_\mathcal{P}(x)/D} \) if \( \mu \) corresponds to a nonzero prime ideal \( \mathcal{P} \) of \( \mathcal{O}_L \).

In the case (i) or (ii), we say that \( \mu \) is real infinite or complex infinite, respectively; in the case (iii), we say that \( \mu \) is finite.

The set of absolute values are well known to satisfy the following product formula:

\[
\prod_{\mu \in \mathcal{M}_L} |x|_\mu = 1, \quad \text{for all } x \in \mathbb{L}^*. \tag{3}
\]

One of our principal tools is the following simplified version of a result of Schlickewei [22, 23], which is commonly known as the Subspace Theorem:
Theorem 3. Let $\mathbb{L}$ be an algebraic number field of degree $D$. Let $S$ be a finite set of places of $\mathbb{L}$ containing all the infinite ones. Let $\{L_{1,\mu}, \ldots, L_{N,\mu}\}$ for $\mu \in S$ be linearly independent sets of linear forms in $N$ variables with coefficients in $\mathbb{L}$. Then, for every fixed $0 < \varepsilon < 1$, the set of solutions $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{L}^N \setminus \{0\}$ to the inequality

$$\prod_{\mu \in S} \prod_{i=1}^N |L_{i,\mu}(\mathbf{x})|_{\mu} \leq (\max\{|x_i| : i = 1, \ldots, N\})^{-\varepsilon}$$

is contained in finitely many proper linear subspaces of $\mathbb{L}^N$.

Let $S$ be a finite subset of $\mathcal{M}_L$ containing all the infinite places. An element $x \in \mathbb{L}$ is called a $S$-unit if $|x|_{\mu} = 1$ for all $\mu \notin S$. An equation of the form

$$\sum_{i=1}^N a_i x_i = 0,$$

where each $a_i \in \mathbb{L}^*$, is called an $S$-unit equation if each $x_i$ is an $S$-unit; it is said to be nondegenerate if no proper subsum of the left hand side vanishes. It is clear that if $\mathbf{x} = (x_1, \ldots, x_N)$ is a solution of the $S$-unit equation (5), and $\rho$ is a $S$-unit in $\mathbb{L}^*$, then $\rho \mathbf{x} = (\rho x_1, \ldots, \rho x_N)$ is also a solution of (5); in this case, the solutions $\mathbf{x}$ and $\rho \mathbf{x}$ are said to be equivalent.

Theorem 4. Let $a_1, \ldots, a_N$ be fixed numbers in $\mathbb{L}^*$. Then the $S$-unit equation (5) has only finitely many equivalence classes of nondegenerate solutions $(x_1, \ldots, x_N)$. Moreover, the number of such equivalence classes is bounded by a constant that depends only on $N$ and the cardinality of $S$.

An immediate consequence of Theorem 4 is that if $\mathbf{x} = (x_1, \ldots, x_N)$ is a solution of the $S$-unit equation (5), then the ratios $x_i/x_j$ for $1 \leq i < j \leq N$ can assume only finitely many values.

We shall also need some estimates from the theory of lower bounds for linear forms in logarithms, both in the complex and the $p$-adic cases.

Let $\alpha_1$ and $\alpha_2$ be algebraic numbers. Put $\mathbb{L} = \mathbb{Q}[\alpha_1, \alpha_2]$, and let $D$ be the degree of $\mathbb{L}$ over $\mathbb{Q}$. Let $A_1$ and $A_2$ be two positive integers such that

$$\log A_i \geq \max \left\{ h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D} \right\} \quad (i = 1, 2).$$

Here, for an algebraic number $\alpha$ whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{i=1}^d (X - \alpha^{(i)})$, we write $h(\alpha)$ for the logarithmic height of $\alpha$, which is given by

$$h(\alpha) = \frac{1}{d} \left( \log |a| + \sum_{i=1}^d \log \left( \max\{1, |\alpha^{(i)}|\} \right) \right).$$

Let $b_1$ and $b_2$ be positive integers, and put $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$. Finally, let

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$
The following result is Corollaire 2 on page 288 of [20], which gives an effective lower bound on the size of \( \log |\Lambda| \):

**Theorem 5.** Assume that \( \alpha_1 \) and \( \alpha_2 \) are real, positive, and multiplicatively independent. Then

\[
\log |\Lambda| \geq -24.34 D^4 \left( \max \left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 \log A_1 \log A_2.
\]

We also need a \( p \)-adic lower bound on \( \Lambda \), that is, an upper bound on the order at which a prime ideal \( P \) can appear in the factorization of the principal ideal generated by \( \Lambda_1 = \alpha_1 b_1 - \alpha_2 b_2 \) inside \( \mathcal{O}_L \). For this, let \( p \) be the prime number such that \( P \mid p \) (i.e., \( p\mathbb{Z} = \mathcal{P} \cap \mathbb{Z} \)), and let \( f \) be such that the finite field \( \mathcal{O}_L/P \) has \( p^f \) elements. Let \( g \) be the smallest positive integer such that \( P \) divides both \( \alpha_1 g - 1 \) and \( \alpha_2 g - 1 \). Assume further that \( A \) satisfies the inequality (6) as well as the inequality \( \log A_i \geq f \log p / D \), for \( i = 1, 2 \). The following result is an easy consequence of Corollaire 2 on page 315 of [4]:

**Theorem 6.** Assume that \( \alpha_1 \) and \( \alpha_2 \) are multiplicatively independent. Then

\[
\ord_P (\Lambda_1) \leq \frac{24 pg D^5}{f^4 (p-1) \log p^4} \times \left( \max \left\{ \log b' + \log \log p + 0.4, \frac{10 f \log p}{D}, 10 \right\} \right)^2 \log A_1 \log A_2.
\]

### 3 Proof of Theorem 1

Our proof of Theorem 1 also treats the (slightly more general) case in which we allow \( t = 1 \), but in this case we add the additional hypothesis that \( m_1 \geq 1 \) (clearly, this condition is needed to insure that the number of solutions to (2) is finite).

Since \( \alpha / \beta \) is not a root of unity, at most one element of the sequence \( (u_n)_{n \geq 0} \) is equal to 0. Hence, if \( u_{n_i} = 0 \) for some \( i \) in (2), then \( n_i \) is uniquely determined. Note that \( i \neq 1 \). If this happens, then equation (2) can be viewed as an equation of the same form, but with \( t \) replaced by \( t - 1 \) (and with only \( 2t - 1 \) unknowns). Thus, to prove the theorem, it suffices to show that there at most finitely many solutions to (2) with \( u_{n_i} \neq 0, i = 1, \ldots, t \).

Let \( \mathbb{L} = \mathbb{Q}[\alpha, \beta] \), and let \( S \) be the set of all infinite places of \( \mathbb{L} \) and all finite places that divide \( s\mathbb{b} = -\alpha \beta b \). For a positive integer \( m \), let \( \ell_b(m) \) denote the number of the digits in the base-\( b \) representation of \( m \).

Equation (2) is equivalent to

\[
|u_n| = \sum_{i=1}^{t} |u_{n_i}| b^{s_i},
\]

where

\[
s_i = \sum_{j=i}^{t} m_j + \sum_{j=i+1}^{t} \ell_b(|u_{n_j}|) \quad (i = 1, \ldots, t).
\]
We remark that, if \( n = n_i \) for some \( i \), it follows that \( t = 1 \) (since each \( u_{n_i} \neq 0 \)) and \( s_1 = m_1 = 0 \) (since \( b \geq 2 \)), which contradicts our assumption that \( m_1 \geq 1 \) when \( t = 1 \). Hence, \( n \neq n_i \) for all \( i = 1, \ldots, t \). Now write (7) in the form

\[
\varepsilon_0(c\alpha^n + d\beta^n) = \sum_{i=1}^{t} \varepsilon_i (c\alpha^{n_i} + d\beta^{m_i}) b^{s_i},
\]

where \( \varepsilon_i \in \{\pm 1\} \) for \( i = 0, \ldots, t \).

Suppose first that \( n_i > n - \kappa \) for some \( i \in \{1, \ldots, t\} \), where \( \kappa \geq 0 \) is a constant to be specified later. From (7), we have that \( |u_n| \geq |u_{n_i}|b^{s_i} \). It is known that the estimate \( |u_m| = |\alpha|^{m+O(\log m)} \) holds for all positive integers \( m \geq 2 \) (see Theorem 3.1 on page 64 in [26]). Moreover, if \( \alpha \) is real, then \( |\alpha| > |\beta| \), and one has the estimate \( |u_m| = |\alpha|^{m+O(1)} \). Therefore, since \( |\alpha| > 1 \), the following bound holds if \( n_i > n - \kappa \):

\[
\max\{n_i - n, s_i\} \ll \begin{cases} 
1, & \text{if } \alpha \in \mathbb{R}; \\
\log n, & \text{if } \alpha \notin \mathbb{R} \quad \text{(i.e., } \alpha = \beta). \end{cases}
\]

Next, we show that if \( n_i \leq n - \kappa \) for every \( i = 1, \ldots, t \), then there exists an index \( i \in \{1, \ldots, t\} \) for which the following bound holds:

\[
\max\{n - n_i, s_i\} \ll 1.
\]

To do this, we first observe that (8) is an \( S \)-unit equation with \( N = 2t + 2 \) terms, coefficients \((a_1, \ldots, a_N) = (c, d, -c, -d, \ldots, -c, -d)\), and the \( S \)-unit unknowns \( x = (x_1, \ldots, x_N) = (\varepsilon_0\alpha^n, \varepsilon_0\beta^n, \varepsilon_1\alpha^{n_1}b^{s_1}, \ldots, \varepsilon_t\beta^{m_t}b^{s_t})\).

If the \( S \)-unit equation (8) is nondegenerate, then \( x_1/x_2 = (\alpha/\beta)^n \) can assume only finitely many values; since \( \alpha/\beta \) is not a root of unity, it follows that \( n \) can take at most finitely many values.

On the other hand, if the \( S \)-unit equation (8) is degenerate, let \( E_1 \) and \( E_2 \) be two (not necessarily distinct) nondegenerate subequations of (8) that contain the unknowns \( x_1 = \varepsilon_0\alpha^n \) and \( x_2 = \varepsilon_0\beta^n \), respectively. Clearly, \( E_1 \) and \( E_2 \) can be chosen in at most finitely many ways. The preceding argument shows that \( n \) can assume only finitely many values if the unknowns \( x_1 \) and \( x_2 \) both lie in \( E_1 \) or both lie in \( E_2 \). Therefore, we may assume that \( E_1 \) does not contain \( x_2 \), and \( E_2 \) does not contain \( x_1 \). We now distinguish the following cases:

(i) \( E_1 \) contains an unknown of the form \( x_{2i+1} = \varepsilon_i\alpha^{n_i}b^{s_i} \) for some \( i \geq 1 \) and \( E_2 \) contains an unknown of the form \( x_{2j} = \varepsilon_j\beta^{m_j}b^{s_j} \) for some \( j \geq 1 \).

In this case, both \( x_1/x_{2i+1} = \pm \alpha^{n-n_i}b^{-s_i} \) and \( x_2/x_{2j} = \pm \beta^{m-m_j}b^{-s_j} \) can assume only finitely many values. Since \( \text{dim}_{\mathbb{Z}}(\log \alpha, \log \beta, \log b) \geq 2 \), it follows that either the pair \((\alpha, b)\) or the pair \((\beta, b)\) is multiplicatively independent; thus, either \( \max\{n-n_i, s_i\} \ll 1 \) or \( \max\{n-n_j, s_j\} \ll 1 \).

(ii) \( E_1 \) contains only unknowns of the form \( x_{2i} = \varepsilon_i\beta^{m_i}b^{s_i} \) with \( i \geq 2 \) (except for \( x_1 \)) and \( E_2 \) contains only unknowns of the form \( x_{2j+1} = \varepsilon_j\alpha^{n_j}b^{s_j} \) with \( j \geq 1 \) (except for \( x_2 \)).

For each choice of the indices \( i \) and \( j \), both \( x_1/x_{2i} = \pm \alpha^{n-n_i}b^{-s_i} \) and \( x_2/x_{2j+1} = \pm \beta^{m-m_j}b^{-s_j} \) can have at most finitely many values. Since we may assume that \( n \)
takes infinitely many values (otherwise, there is nothing to prove), it follows that there
exist numbers $n^*, n_i^*, n_j^*, s_i^*$, and $s_j^*$ such that both relations
\begin{align}
\alpha^{n_i^*} b^{-s_i} &= \alpha^{n_j^*} b^{-s_j}, \\
\beta^{n_i^*} \alpha^{-n_j^*} b^{-s_j} &= \beta^{n_j^*} \alpha^{-n_j^*} b^{-s_j},
\end{align}
(11)
hold for arbitrarily large values of $n$. Among all possible choices for the quintuple
$(n^*, n_i^*, n_j^*, s_i^*, s_j^*)$ of such numbers, we fix one for which $n_i^*$ is as small as possible; thus,
$n_i \geq n_i^*$ whenever the relations (11) hold.

Since there are only finitely many possibilities for $E_1$ and $E_2$ and (once these are
fixed) for the indices $i$ and $j$, we obtain in this way a finite list of such quintuples
$(n^*, n_i^*, n_j^*, s_i^*, s_j^*)$. Hence, the constant $\kappa$ can be initially chosen such that the inequality
$\kappa > \max\{n^* - n_i^*, n^* - n_j^*\}$ holds in all cases.

Now let $E_1$, $E_2$, $i$, and $j$ be fixed, and suppose that the relations (11) hold with $n > n^*$. Taking
logarithms, we obtain that
\begin{align}
(n - n^*) \log \alpha &= (n_i - n_i^*) \log \beta + (s_i - s_i^*) \log b, \\
(n - n^*) \log \beta &= (n_j - n_j^*) \log \alpha + (s_j - s_j^*) \log b.
\end{align}

Let $v_1 = (n_i - n_i^*)/(n - n^*)$ and $v_2 = (n_j - n_j^*)/(n - n^*)$, and note that both numbers
are rational. Since we are assuming that $n_i \leq n - \kappa$ for $i = 1, \ldots, t$, it follows that
\begin{equation}
n^* - n_i^* < \kappa \leq n - n_i,
\end{equation}
which implies that $v_1 < 1$. Similarly, $v_2 < 1$. Since $n_i \geq n_i^*$ by our choice of the
quintuple $(n^*, n_i^*, n_j^*, s_i^*, s_j^*)$, we also see that $v_1 \geq 0$. These statements together imply
that $v_1 v_2 \neq 1$, which is all we need. From the preceding relations, we obtain that
\begin{equation}
\log \alpha = v_1 \log \beta + w_1 \log b = v_1 (v_2 \log \alpha + w_2 \log b) + w_1 \log b,
\end{equation}
where $w_1 = (s_i - s_i^*)/(n - n^*)$ and $w_2 = (s_j - s_j^*)/(n - n^*)$ are rational numbers. Since $v_1 v_2 \neq 1$, this implies that $\log \alpha / \log b$ is rational. Similarly, we see
that $\log \beta / \log b$ is rational. But these statements contradict our hypothesis that
$\dim_{\mathbb{Z}}(\log \alpha, \log \beta, \log b) \geq 2$; therefore, $n$ is bounded, and it follows that $n_i$, $n_j$, $s_i$, and
$s_j$ are bounded as well.

(iii) The remaining cases.

For the remaining cases, there are only two possibilities:
\begin{itemize}
  \item $E_1$ contains an unknown of the form $x_{2i+1} = \varepsilon_i \alpha^{n_i} b^{s_i}$ for some $i \geq 1$ and $E_2$
    contains only unknowns of the form $x_{2j+1} = \varepsilon_j \alpha^{n_j} b^{s_j}$ with $j \geq 1$ (except for $x_2$).
  \item $E_1$ contains only unknowns of the form $x_{2i} = \varepsilon_i \beta^{n_i} b^{s_i}$ with $i \geq 2$ (except for $x_1$)
    and $E_2$ contains an unknown of the form $x_{2j} = \varepsilon_j \beta^{n_j} b^{s_j}$ for some $j \geq 2$.
\end{itemize}
We treat only the first case, as the second case is similar.

We note that the ratio \( x_{2i}/x_{2i+1} = \pm \alpha^{n-n_i} b^{-s_i} \) assumes only finitely many values. If \( \alpha \) and \( b \) are multiplicatively independent, it follows that both \( n - n_i \) and \( s_i \) are bounded, and we are done. On the other hand, if \( n - n_i \) is not bounded, it follows that \( \log \alpha / \log b \) is rational. If \( j \) is such that \( x_{2j+1} \in E_2 \), then \( x_2/x_{2j+1} = \pm \beta^n \alpha^{-n_j} b^{-s_j} \) can take at most finitely many values. Since \( \alpha \) and \( b \) are multiplicatively dependent, \( \beta \) and \( b \) must be multiplicatively independent, and it follows that \( n \) can take only finitely many values. But this is impossible if \( n - n_i \) is unbounded.

The analysis above completes our proof that (10) holds for some \( i \) in the case that \( n_i \leq n - \kappa \) for all \( i = 1, \ldots, t \). Combining (9) and (10), we see that the bound

\[
\max\{|n-n_i|, s_i\} \ll \begin{cases} 1, & \text{if } \alpha \in \mathbb{R}; \\ \log n, & \text{if } \alpha \not\in \mathbb{R} \quad \text{(i.e., } \alpha = \overline{\beta}) \end{cases}
\]

(12)

holds for some \( i \in \{1, \ldots, t\} \) in every case.

We now select \( i \) such that (12) holds and rewrite (8) in the form

\[
ca^n + d\beta^n + Ab^{s_i-1} + c_1\alpha^{n_i} b^{s_i} + d_1\beta^{n_i} b^{s_i} + B = 0,
\]

(13)

where \( c_1 = -\varepsilon_i \varepsilon_0 c, d_1 = -\varepsilon_i \varepsilon_0 d, \)

\[
A = -\sum_{j=1}^{i-1} \varepsilon_j \varepsilon_0 u_{n_j} b^{s_j-s_{j-1}} \quad \text{and} \quad B = -\sum_{j=i+1}^{t} \varepsilon_i \varepsilon_0 u_{n_j} b^{s_j}.
\]

Since

\[
b^{s_{i-1}} \geq |u_{n_i}| \geq |\alpha|^{n_i+O(\log n_i)} = |\alpha|^{n+O(\log n)},
\]

we see that \( A = \exp(O(\log n)) \). Similarly, since \( b^{s_i} \geq B \), it follows that \( B = \exp(O(\log n)) \).

Assume first that both \( n - n_i \) and \( s_i \) are bounded (this is the case, for instance, if \( \mathbb{L} \) is real). In this case, \( A \) and \( B \) are bounded as well; hence, we can assume that they are fixed. Here, (13) becomes

\[
C_1\alpha^n + D_1\beta^n + Ab^{s_i-1} + B = 0,
\]

(14)

where \( C_1 = c + c_1\alpha^{n_i-n} \) and \( D_1 = d + d_1\beta^{n_i-n} \) can also be regarded as fixed numbers. The case \( A = B = C_1 = D_1 = 0 \) leads to \( i = t = 1, \alpha^{n-n_i} = -ce^{-1} = \pm 1 \) and \( \beta^{n-n_i} = -dd^{-1} = \pm 1 \); therefore, \( t = 1, n = n_1 \), and \( m_1 = 0 \), which contradicts our assumption that \( m_1 \geq 1 \) when \( t = 1 \). Consequently, the equation (14) is nontrivial. If any two of the coefficients \( A, B, C_1, D_1 \) are zero, then either \( n \) or \( s_{i-1} \) is bounded, and this leads to at most finitely many possibilities for \( n \). A similar argument based on Theorem 4 can be used if one of the coefficients \( A, B, C_1, D_1 \) is zero, or if \( ABC_1D_1 \neq 0 \), to show that there are at most finitely many possibilities for \( n \).

Thus, from now on, we can suppose that either \( n - n_i \) or \( s_i \) is unbounded over the set of solutions to (13). In this case, \( \alpha \) and \( \beta \) are complex conjugates.

Assume first that \( B \neq 0 \) in equation (13). Suppose also that \( A \neq 0 \). We apply Theorem 3 with \( N = 5 \), the linear forms \( L_{j,\mu}(x) = x_j \) for each \( j = 1, \ldots, 5 \), and \( \mu \in S \), except
when \(j = 1\) and \(\mu\) is infinite, in which case we take \(L_{1,\mu}(x) = cx_1 + dx_2 + x_3 + c_1 x_4 + d_1 x_5\) (note that, as \(L\) is complex quadratic, there is only one infinite place). We evaluate the double product appearing in Theorem 3 for our system of forms and the points \(x = (\alpha^n, \beta^n, Ab^{n_i-1}, \alpha^{n_i}b^{n_i}, \beta^{n_i}b^{n_i})\). Clearly,

\[
\prod_{\mu \in S} |L_{j,\mu}(x)| = 1 \quad (15)
\]

if \(j \in \{2, 4, 5\}\), since \(x_2, x_4\) and \(x_5\) are \(S\)-units. Moreover,

\[
\prod_{\mu \in S} |L_{3,\mu}(x)| \leq A = \exp(O(\log n)). \quad (16)
\]

Finally, since \(x_1\) is an \(S\)-unit, it follows from the product formula (3) that

\[
\prod_{\mu \in S, \mu \text{ finite}} |L_{1,\mu}(x)|_\mu = \frac{1}{|Nm_L(\alpha^n)|} \leq \frac{1}{|\alpha|^n}, \quad (17)
\]

while by equation (13), we have

\[
\prod_{\mu \in S, \mu \text{ infinite}} |L_{1,\mu}(x)|_\mu = B^2 \leq \exp(O(\log n)). \quad (18)
\]

Multiplying the estimates (15), (16), (17) and (18), we derive that

\[
\prod_{j=1}^N \prod_{\mu \in S} |L_{j,\mu}(x)| \leq \frac{AB^2}{\alpha^n} \leq \exp(-n \log \alpha + O(\log n)). \quad (19)
\]

Since \(\max\{|x_j| : j = 1, \ldots, N\} = |\alpha|^n\), the inequality (19) together with Theorem 3 (for example, with \(\varepsilon = 1/2\) and \(n > n_\varepsilon\)), imply that there exist finitely many proper subspaces of \(L^N\) containing all solutions \(x\). Thus, the relation

\[
C_2\alpha^{n_i} + D_2\beta^{n_i} + C_3\alpha^{n_i}b^{n_i} + D_3\beta^{n_i}b^{n_i} + EAb^{n_i-1} = 0 \quad (20)
\]

holds for some fixed coefficients \(C_2, D_2, C_3, D_3\) and \(E\) in \(L\), which are not all equal to zero. If \(A = 0\), then the same argument with \(N = 4\) also yields an identity of the shape (20). Finally, if \(B = 0\), then (13) is the same as (20) with \(C_2 = c, D_2 = d, C_3 = c_1, D_3 = d_1\), and \(E = 1\). Clearly, we may assume that \(C_2\) and \(D_2\) are conjugate (over \(L\)), that \(C_3\) and \(D_3\) are conjugate (over \(L\)), and that \(E \in \mathbb{Z}\) (if not, we can conjugate (20) and subtract the result from (20) to obtain a “shorter” nontrivial equation of the same type with the desired properties).

If \(E = 0\), then (20) is a \(S\)-unit equation. If it is nondegenerate, we see that \(\alpha^n\beta^{-n}\) can take only finitely many values; since \(\alpha/\beta\) is not a root of unity, there are at most finitely many possibilities for \(n\). If the \(S\)-unit equation is degenerate, then either \(C_2 = D_2 = 0\), in which case \(n_i\) can take only finitely many values (and since \(|n - n_i| \ll \log n\), it follows that
n is bounded as well), or $C_2D_2 \neq 0$ but $C_3 = D_3 = 0$, in which case $n$ can again take only finitely many values, or $C_2C_3D_2D_3 \neq 0$. In the last case, either $\alpha^{n-n_i}b^{-s_i}$ and $\beta^{n-n_i}b^{-s_i}$ can take only finitely many values, or $\alpha^n\beta^{-n_i}b^{-s_i}$ and $\beta^m\alpha^{-n_i}b^{-s_i}$ can take only finitely many values; but these are cases that have already been considered.

Finally, we are left with the possibility that $E \neq 0$, in which case we can assume that $E = 1$. We now rewrite (20) in the form

$$C_4\alpha^n + D_4\beta^n = -Ab^{s_i-1},$$

(21)

where $C_4 = C_2 + C_3\alpha^{n-n_i}b^{s_i}$ and $D_4 = D_2 + D_3\beta^{m-n_i}b^{s_i}$. Since $C_4$ and $D_4$ are conjugated in $\mathbb{L}$, it follows that they are simultaneously zero or nonzero.

Assume first that $C_4 = D_4 = 0$. Then both relations

$$C_2 = -C_3\alpha^{n-n_i}b^{s_i} \quad \text{and} \quad D_2 = -D_3\beta^{m-n_i}b^{s_i}$$

(22)

hold. If $C_2 = 0$ then $C_3 = 0$ (by (22)), $D_2 = 0$ (because $C_2$ and $D_2$ are conjugated), and therefore $D_3 = 0$ (by (22)); together with equation (20), these lead to $E = 0$, which is a contradiction. Thus, $C_2 \neq 0$, and the preceding argument implies that $C_2C_3D_2D_3 \neq 0$.

Now, equation (22) together with our hypothesis that $\dim_\mathbb{Q} (\log \alpha, \log \beta, \log b) \geq 2$ lead to the conclusion that both $n - n_i$ and $s_i$ are bounded, which is a case already treated.

We now assume that $C_4D_4 \neq 0$. Let $\ell = \gcd(r^2, s)$, where $r$ and $s$ are the coefficients of the recurrence (1). Set $\alpha_1 = \alpha^2/\ell$, $\beta_1 = \beta^2/\ell$. Applying Lemma A.10 on page 20 in [26], we see that $\alpha_1$ and $\beta_1$ are algebraic integers and that the principal ideals they generate in $\mathbb{L}$ are coprime. Clearly, $\alpha_1$ and $\beta_1$ are complex conjugates, and $|\alpha_1| > 1$. Write $n = 2m + \delta$, where $\delta \in \{0, 1\}$. Put $(C_5, D_5) = (C_4, D_4)$ if $\delta = 0$ and $(C_5, D_5) = (\alpha C_4, \beta D_4)$ if $\delta = 1$. Dividing both sides of equation (21) by $\ell^m$, we see that the expression

$$C_5\alpha_1^m + D_5\beta_1^m$$

is a rational number such that every prime factor of its numerator or denominator divides either $A\alpha \beta$ or one of the denominators of $C_2$, $D_2$, $C_3$, or $D_3$. Let $\mathcal{P} = \{p_1, \ldots, p_v\}$ be the set consisting of all of these primes, and write

$$C_5\alpha^m + D_5\beta^m = \prod_{i=1}^v p_i^{r_i}.$$

We now bound the order $r_i$ of $p_i$. Let $\pi_i$ be some prime ideal of $\mathbb{L}$ lying above $p_i$. If $\pi_i | \alpha_1$, then $\text{ord}_{\pi_i}(\alpha_1^m) \geq m \geq n/2 - 1$. On the other hand, it is clear that

$$\max\{|\text{ord}_{\pi_i}(C_5)|, |\text{ord}_{\pi_i}(D_5)|\} \ll \max\{|n - n_i|, s_i\} \ll \log n.$$

Thus, for large $n$, we get that

$$\text{ord}_{\pi_i}(C_5\alpha_1^m + D_5\beta_1^m) = \text{ord}_{\pi_i}(D_5\beta_1^m) = \text{ord}_{\pi_i}(D_5) \ll \log n,$$

(23)

since $\alpha_1$ and $\beta_1$ are coprime. A similar analysis can be used if $\pi_i | \beta_1$. Assume now that $\pi_i$ does not divide $\alpha_1\beta_1$. Then

$$r_i = \text{ord}_{\pi_i}(C_5\alpha_1^m + D_5\beta_1^m) = \text{ord}_{\pi_i}(C_5\beta_1^m) + \text{ord}_{\pi_i}((\alpha_1/\beta_1)^m - (D_5/C_5)).$$
Certainly,
\[ \text{ord}_{\pi_i}(C_5\beta_1^m) = \text{ord}_{\pi_i}(C_5) \ll \log n, \]
while from Theorem 6, we deduce that
\[ \text{ord}_{\pi_i}((\alpha_1/\beta_1)^m - (-D_5/C_5)) \ll (\log n)^2 |\log |C_5|| \ll (\log n)^3. \]
Thus,
\[ \text{ord}_{\pi_i}(C_5\alpha_1^m + D_5\beta_1^m) \ll (\log n)^3 \quad (24) \]
in this case. Comparing inequalities (23) and (24), we see that inequality (24) always holds. Since this is true for all \(i = 1, \ldots, v\), we conclude that
\[ \log |C_5\alpha_1^m + D_5\beta_1^m| \leq \sum_{i=1}^{v} r_i \log p_i \ll (\log n)^3. \quad (25) \]
On the other hand, we have
\[ \log |C_5\alpha_1^m + D_5\beta_1^m| = \log |C_5| + m \log |\alpha_1| + \log |1 + (D_5C_5^{-1}(\beta_1\alpha_1^{-1})^m)|. \]
Clearly,
\[ \log |C_5| \gg -\log n, \quad (26) \]
and using Theorem 5, we get that
\[ \log |1 + (D_5C_5^{-1}(\beta_1\alpha_1^{-1})^m)| \gg -(\log n)^2 |\log |C_5||. \quad (27) \]
Putting together inequalities (25), (26), (27), and using the fact that \(m \gg n\) and \(|\alpha_1| > 1\), we obtain that
\[ n \ll (\log n)^3, \]
which shows that \(n\) can take only finitely many values.
This completes the proof of Theorem 1.

4 Proof of Theorem 2
Before proceeding to the proof of Theorem 2, we gather a few useful facts about the Fibonacci sequence.
We first recall the following special case of the Primitive Divisor Theorem, which is due to Carmichael [5]:

**Lemma 7.** For all \(n \geq 13\), there exists a prime factor \(p\) of \(F_n\) such that \(p\) does not divide \(F_m\) for any positive integer \(m < n\). Furthermore, any such prime \(p\) satisfies \(p \equiv \pm 1 \mod n\).

Next, we record the following estimate for the function \(\ell(n) = \ell_{10}(F_n)\), which gives the number of digits in the decimal expansion of \(F_n\):
Lemma 8. For all $n \geq 1$, we have
\[
\frac{(n-2) \log \alpha}{\log 10} < \ell(n) \leq \frac{(n-1) \log \alpha}{\log 10} + 1.
\]

Proof. By induction on $k$, it is easy to see that $\alpha^{k-2} \leq F_k \leq \alpha^{k-1}$ holds for all $k \geq 1$. Since $\ell(k)$ is the unique integer for which $10^{\ell(k)-1} \leq F_k < 10^{\ell(k)}$, the result follows. 

We keep the notation used in the proof of Theorem 1. In particular, $\mathbb{L} = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_L = \mathbb{Z}[\alpha]$, and $D = 2$ is the degree of $\mathbb{L}$ over $\mathbb{Q}$. Notice that $\mathcal{O}_L$ is a UFD. We also put $\varpi = \sqrt{5}$ and $\mathcal{P} = [\varpi]$; then $[p] = [5] = \mathcal{P}^2$, and $f = 1$. We need the following elementary lemma:

Lemma 9. If $r \geq 2$, we have
\[
\text{ord}_{\mathcal{P}}(\alpha^r - 1) \leq \frac{2 \log(r/4)}{\log 5} + 1.
\]
The same inequality holds with $\alpha$ replaced by $\beta$.

Proof. The inequality for $\beta$ follows from the one for $\alpha$ by conjugation. Note that the right hand side of the stated inequality is positive for all $r \geq 2$. Since
\[
\alpha = \frac{1 + \sqrt{5}}{2} \equiv 2^{-1} \pmod{\varpi},
\]
it follows that $\text{ord}_{\mathcal{P}}(\alpha^r - 1) = 0$ if $4 \nmid r$; hence, it suffices to assume that $4 \mid r$ in what follows. Since
\[
\alpha^4 - 1 = \frac{5 + 3\sqrt{5}}{2},
\]
it follows that $\text{ord}_{\mathcal{P}}(\alpha^4 - 1) = 1$. Thus, we may write $\alpha^4 = 1 + \varpi u$, where $u$ is coprime to $\varpi$. If $s \geq 1$ is an integer and $5 \nmid s$, then
\[
\alpha^{4s} - 1 = (\alpha^4 - 1) \sum_{j=0}^{s-1} \alpha^{4j} \varpi u \sum_{j=0}^{s-1} (1 + \varpi u)^j \equiv \varpi us \pmod{\varpi},
\]
which shows that $\text{ord}_{\mathcal{P}}(\alpha^{4s} - 1) = 1$ as well. One checks similarly that if $s \geq 1$ and $5 \nmid s$, then $\text{ord}_{\mathcal{P}}(\alpha^{20s} - 1) = 3$.

We now claim that, for all $t \geq 0$ and $s \geq 1$ such that $5 \nmid s$, we have
\[
\text{ord}_{\mathcal{P}}(\alpha^{4s5^t} - 1) = 2t + 1.
\]
(28)

To prove this, we use induction on the parameter $t$. Since the claim is true for $t = 0$ or $1$, let us suppose that $t \geq 2$. Then,
\[
\alpha^{4s5^t} - 1 = (\alpha^{4s5^{t-1}} - 1) \sum_{j=0}^{4} \alpha^{4sj5^{t-1}}
\]
\[
= 5(\alpha^{4s5^{t-1}} - 1) + (\alpha^{4s5^{t-1}} - 1) \sum_{j=1}^{4} (\alpha^{4sj5^{t-1}} - 1).
\]
By the induction hypothesis, we have
\[
\text{ord}_P \left( 5(\alpha^{4s}5^{t-1} - 1) \right) = 2 + (2(t - 1) + 1) = 2t + 1,
\]
while
\[
\text{ord}_P \left( \left( \sum_{j=1}^{4} (\alpha^{4s}5^{t-1} - 1) \right) \right) \geq 2(2(t - 1) + 1) = 4t - 2 > 2t + 1,
\]
and (28) follows.

Finally, writing \( r = 4s \cdot 5^t \), where \( t \geq 0, s \geq 1, \) and \( 5 \nmid s \), we have
\[
\text{ord}_P(\alpha^r - 1) = 2t + 1 = \frac{2\log(r/4)}{\log 5} + 1 \leq \frac{2\log(r/4)}{\log 5} + 1,
\]
which finishes the proof. \( \square \)

**Lemma 10.** If \( (m, n, k) \) is an ordered triple of positive integers such that \( F_m F_n = F_k \), and \( (m, n, k) \neq (1, 4, 7) \) or \( (2, 4, 7) \), then \( m \geq 3 \) and \( k - n \geq 4 \).

**Proof.** Suppose that \( n \geq 13 \). First, suppose that \( m = 1 \) or \( m = 2 \). Then \( 10^{\ell(n)} + F_n = F_k \); hence, \( 2F_n \leq F_k \leq 11F_n \), which (by simple estimates) implies that \( n + 2 \leq k \leq n + 5 \). Since \( n \geq 13 \), we have that \( \ell(n) \geq 3 \), and thus,
\[
F_n \equiv F_k \pmod{8}.
\]

An analysis of the sequence of Fibonacci numbers modulo 8 shows that this congruence is not possible when \( k = n + 4 \) or \( k = n + 5 \); therefore, \( k = n + 2 \) or \( k = n + 3 \). If \( k = n + 2 \), then \( 10^{\ell(n)} = F_{n+1} \), while for \( k = n + 3 \), we have \( 10^{\ell(n)} = 2F_{n+1} \). However, by Lemma 7, there exists a prime \( p \geq n \) dividing \( F_{n+1} \), which is not possible in our cases. Consequently, if \( m \leq 2 \), we must have \( n \leq 12 \). Checking the remaining possibilities, the only solutions found are \( (1, 4, 7) \) and \( (2, 4, 7) \).

Assuming now that \( F_m F_n = F_k \), \( n \geq 15 \), and \( k \leq n + 3 \), we then have
\[
F_m \cdot 10^{\ell(n)} = F_k - F_n = \begin{cases} 
F_{n-1}, & \text{if } k = n + 1; \\
F_{n+1}, & \text{if } k = n + 2; \\
2F_{n+1}, & \text{if } k = n + 3.
\end{cases} \tag{29}
\]

Moreover, \( m < n - 1 \), for otherwise
\[
F_k = F_m \cdot 10^{\ell(n)} + F_n > 1000F_{n-1} > F_{n+3},
\]
contradicting our assumption that \( k \leq n + 3 \). Using Lemma 7 again, we see that there exist primes \( p \mid F_{n-1} \) and \( q \mid F_{n+1} \) with \( \gcd(pq, F_m) = 1 \) and \( \min\{p, q\} \geq 13 \), which is not possible in view of (29). Hence, if \( k \leq n + 3 \), we must have \( n \leq 14 \), and thus \( k \leq 17 \). Examining these possibilities reveals no solutions other than the two found in the previous case. \( \square \)
Lemma 11. If $r \geq 1$ is even, then

$$\frac{\alpha^r - 1}{\beta^r - 1} = -\alpha^r,$$

while if $r \geq 5$ is odd, then the numbers $(\alpha^r - 1)/(\beta^r - 1)$ and $\alpha$ are multiplicatively independent.

Proof. The first statement is trivial since $\alpha \beta = -1$. For the second statement, we note that if $r$ is odd then

$$\frac{\alpha^r - 1}{\beta^r - 1} = -\alpha^r \left(\frac{\alpha^r - 1}{\alpha^r + 1}\right).$$

We now observe that if $\mathcal{D}$ is the common divisor in $\mathcal{O}_L$ of $\alpha^r - 1$ and $\alpha^r + 1$, then $\mathcal{D} | 2$. Since 2 is inert in $\mathcal{O}_L$, it follows that $\mathcal{D} \in \{1, 2\}$. The above arguments show that if $(\alpha^r - 1)/(\beta^r - 1)$ and $\alpha$ are multiplicatively dependent, then so are $(\alpha^r - 1)/(\alpha^r + 1)$ and $\alpha$. Using the fact that $\mathcal{O}_L$ is a UFD and the computation of $\mathcal{D}$, it follows that $\alpha^r - 1$ is either a unit, or it is an associate of 2. Hence, we get an equation of the form

$$\alpha^r - 1 = \pm 2^\lambda \alpha^t$$

with integers $\lambda \in \{0, 1\}$ and $t$. Since $r > 3$, it follows that $\alpha^r - 1 > \alpha^3 - 1 > 2$; hence, the sign in this equation is positive, and $t \geq 1$. Clearly, $t < r$. Thus, $\alpha^r - 1 = 2^\lambda \alpha^t$. By conjugation, we also have $\beta^r - 1 = 2^\lambda \beta^t$. Subtracting these two equations and dividing the result by $\alpha - \beta$, we obtain that $F_r = 2^\lambda F_t$. If $r \geq 13$, this equation is impossible in view of Lemma 7. The fact that $F_r = 2^\lambda F_t$ is also impossible for $5 \leq r \leq 13$ can be checked by hand, and the result follows.

We are now ready to embark on the proof of Theorem 2. For this, let $(m, n, k)$ be a fixed triple of nonnegative integers for which $F_m F_n = F_k$ holds. We note that $n > 0$, since for $n = 0$ we have $10 F_m = F_k$, which has no positive integer solutions $(m, k)$ (by Lemma 7, for example). Put $r = k - n$, and assume that $k > 10^6$. By Lemma 10, we can further suppose that $m \geq 3$ and $r \geq 4$. Since $\beta = -1/\alpha$, we have

$$F_m \cdot 10^\ell(n) = F_k - F_n = \varpi^{-1}(\alpha^k - \beta^k - \alpha^n + \beta^n)$$

$$= \varpi^{-1}(\alpha^n(\alpha^r - 1) - \beta^n(\beta^r - 1))$$

$$= \varpi^{-1}\alpha^n(\beta^r - 1) \left(\frac{\alpha^r - 1}{\beta^r - 1} - (-\alpha^{-2})^n\right).$$

Consequently,

$$\text{ord}_P(F_m) + 2\ell(n) = -1 + \text{ord}_P(\beta^r - 1) + \text{ord}_P \left(\frac{\alpha^r - 1}{\beta^r - 1} - (-\alpha^{-2})^n\right).$$

Assume first that $r$ is odd. We apply Theorem 6 with the choices $\alpha_1 = (\alpha^r - 1)/(\beta^r - 1)$, $\alpha_2 = -\alpha^{-2}$, $b_1 = 1$, and $b_2 = n$. The condition that $\alpha_1$ and $\alpha_2$ are multiplicatively independent is satisfied by Lemma 11 because $r \geq 5$. Furthermore, note that

$$h(\alpha_1) \leq \frac{1}{2} (\log |(\alpha^r - 1)/(\beta^r - 1)| + \log |\alpha_1|) \leq \frac{1}{2} \log \alpha^{2r} = r \log \alpha,$$
and \( h(\alpha_2) = \log \alpha \). Since \( r \geq 5 \), we can choose \( A_1 = \alpha^r \) and \( A_2 = \varpi \); hence,

\[
b' = \frac{1}{\log 5} + \frac{n}{2r \log \alpha} \leq \frac{1}{2 \log \alpha} + \frac{n}{10 \log \alpha} \leq \frac{3n}{4 \log \alpha}.
\]

Finally, as \( \alpha \equiv \beta \pmod{\varpi} \), and \( \text{ord}_\mathcal{P}(\alpha^r - 1) = \text{ord}_\mathcal{P}(\beta^r - 1) = 0 \) (by Lemma 9), it follows that \( \mathcal{P} \) divides \( \alpha_1 - 1 \). Moreover, noting that \( -\alpha^{-2} \equiv 1 \pmod{\varpi} \), it follows that \( \mathcal{P} \) also divides \( \alpha_2 - 1 \). Thus, we can take \( g = 1 \). By Theorem 6, we obtain the bound

\[
\text{ord}_\mathcal{P} \left( \left( \frac{\alpha^r - 1}{\beta^r - 1} \right) - (-\alpha^{-2})^n \right) \leq \frac{480r \log \alpha}{(\log 5)^3} \left( \max \left\{ \log n + \log \left( \frac{3 \log 5}{4 \log \alpha} \right) + 0.4, 10 \right\} \right)^2 \leq 56r \left( \max \{ \log n + 2, 10 \} \right)^2.
\]

Next, consider the case that \( r \) is even; then

\[
\frac{\alpha^r - 1}{\beta^r - 1} - (-\alpha^{-2})^n = -\alpha^r - (-\alpha^{-2})^n = (-1)^{n+1} \alpha^{-2n}(\alpha^{k+n} \pm 1),
\]

and the last expression divides \( \alpha^{2k+2n} - 1 \) in \( \mathcal{O}_L \); hence, by Lemma 9, we obtain that

\[
\text{ord}_\mathcal{P} \left( \left( \frac{\alpha^r - 1}{\beta^r - 1} \right) - (-\alpha^{-2})^n \right) \leq \frac{2 \log((k + n)/2)}{\log 5} + 1.
\]

Substituting the estimates above into (30), and applying Lemmas 8 and 9, we derive that

\[
2 \frac{(n - 2) \log \alpha}{\log 10} < \ell(n) \leq \frac{2 \log(r/4)}{\log 5} + 56r \left( \max \{ \log n + 2, 10 \} \right)^2, \tag{31}
\]

if \( r \) is odd, and

\[
2 \frac{(n - 2) \log \alpha}{\log 10} < \ell(n) \leq \frac{2 \log(r/4)}{\log 5} + \frac{2 \log((k + n)/2)}{\log 5} + 1, \tag{32}
\]

if \( r \) is even.

From the equality \( F_mF_n = F_k \), we also see that

\[
\alpha^m \cdot 10^{\ell(n)} - \alpha^k = \beta^m \cdot 10^{\ell(n)} - \alpha^n + \beta^n - \beta^k, \tag{33}
\]

and, since \( 10^{\ell(n)} < 10F_n \) and \( m \geq 3 \), we have

\[
|\alpha^{m-k} \cdot 10^{\ell(n)} - 1| = \alpha^{-k} |\beta^m \cdot 10^{\ell(n)} - \alpha^n + \beta^n - \beta^k| \leq \alpha^{-k} (10|\beta|^3F_n + \alpha^n + 2) < 4\alpha^{-r}. \tag{34}
\]

Since \( m \geq 3 \), both sides of (33) are negative, and since \( r \geq 4 \), we have \( 4\alpha^{-r} < \frac{3}{5} \), thus,

\[
\frac{2}{5} < \alpha^{m-k} \cdot 10^{\ell(n)} < 1.
\]
It follows that
\[ |\alpha^{m-k} \cdot 10^{\ell(n)} - 1| > \frac{2}{5}(k - m) \log \alpha - \ell(n) \log 10. \]  (35)

We now apply Theorem 5 with the choices \( \Lambda = (k - m) \log \alpha - \ell(n) \log 10, \alpha_1 = 10, \alpha_2 = \alpha, \) 
\( b_1 = \ell(n), \) and \( b_2 = k - m. \) Here, \( h(\alpha_1) = \log 10 \) and \( h(\alpha_2) = \frac{1}{2} \log \alpha; \) hence, we can choose \( A_1 = 10, \) and \( A_2 = \alpha^2, \) and
\[
b' = \frac{\ell(n)}{4 \log \alpha} + \frac{k - m}{20} < b'' = \frac{\ell(n)}{4 \log \alpha} + \frac{k}{20}.
\]

Using Theorem 5, we get that
\[ |(k - m) \log \alpha - \ell(n) \log 10| \geq \exp \left( -864 \left( \max \{ \log b'' + 0.14, 10.5 \} \right)^2 \right). \]

Combining the above estimates, we derive the bound
\[ r < \frac{\log 10}{\log \alpha} + \frac{864}{\log \alpha} \left( \max \{ \log b'' + 0.14, 10.5 \} \right)^2. \]  (36)

Now, if \( k > 2n, \) then, by Lemma 8, we have
\[
b'' = \frac{\ell(n)}{4 \log \alpha} + \frac{k}{20} < \frac{n - 1}{4 \log 10} + \frac{1}{4 \log \alpha} + \frac{k}{20} < \frac{(k/2) - 1}{4 \log 10} + \frac{1}{4 \log \alpha} + \frac{k}{20},
\]
and \( r = k - n > k/2; \) hence, the inequality (36) is not possible for \( k > 500000. \) On the other hand, if \( k \leq 2n, \) then
\[
b'' = \frac{\ell(n)}{4 \log \alpha} + \frac{k}{20} < \frac{n - 1}{4 \log 10} + \frac{1}{4 \log \alpha} + \frac{n}{10}.
\]

When \( r \) is even, estimate (32) gives
\[
\frac{(n - 2) \log \alpha}{\log 10} < \frac{\log(n/4)}{\log 5} + \frac{\log(3n/2)}{\log 5} + \frac{1}{2},
\]
which implies that \( n < 20; \) hence, \( k < 40. \) When \( r \) is odd, by combining the inequalities (31), and (36), we obtain a contradiction unless \( n \leq 1.1 \times 10^{11} \) and \( k \leq 2n \leq 2.2 \times 10^{11}. \)

Although the preceding argument shows that there are only finitely many solutions \((m, n, k)\) to the equation \( F_m F_n = F_k, \) it would be computationally infeasible to search for solutions over the entire range \( k \leq 2.2 \times 10^{11}. \) In order to reduce the range further, we use a standard technique involving the continued fraction expansion of \((\log 10)/ (\log \alpha).\)

Suppose that \( n \leq 1.1 \times 10^{11} \) and \( r \geq 56. \) By (34) and (35), we have
\[
\left| \frac{\log 10}{\log \alpha} - \frac{(k - m)}{\ell(n)} \right| < \frac{10}{\alpha^r \ell(n)} < \frac{1}{2 \ell(n)^2}.
\]
Here, the last inequality is equivalent to \( 20 \ell(n) \leq \alpha^r, \) which holds (by Lemma 8) for this choice of parameters. By well known properties of continued fractions, it follows that the fraction \((k - m)/\ell(n)\) is a convergent of \((\log 10)/(\log \alpha).\) Writing \((k - m)/\ell(n) = p_j/q_j\) for
some \( j \geq 0 \), where \( p_j/q_j \) denotes the \( j \)th convergent to \((\log 10)/(\log \alpha)\), and using Lemma 8 again to bound \( \ell(n) \) for \( n \) in our range, we see that \( q_j \leq \ell(n) \leq 2.3 \times 10^{10} \), which implies that \( j \leq 23 \). Noting that
\[
10\alpha^{-r} > |\ell(n) \log 10 - (k - m) \log \alpha| \geq \min_{1 \leq j \leq 23} |q_j \log 10 - p_j \log \alpha| > 1.6 \times 10^{-11},
\]
we conclude that \( r \leq 57 \). Substituting this estimate into (31), we derive the more tractable upper bound \( n \leq 2.1 \times 10^6 \).

At this point, we turn to the computer. Note that if \( n \geq 74 \), one has \( \ell(n) \geq 15 \); therefore, if \( F_mF_n = F_k \), it follows that \( F_n \equiv F_k \pmod{10^{15}} \). However, a computer search quickly reveals that there is no solution to this congruence with \( 74 \leq n \leq 2.1 \times 10^6 \) and \( k \leq n+57 \). Thus, it remains only to search for solutions \((m, n, k)\) with \( n \leq 73 \) and \( k \leq n+57 \), and one obtains only solutions with \( k = 7, 8 \) or \( 10 \); that is \( F_k \in \{13, 21, 55\} \).

This completes the proof of Theorem 2.

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