QUASI-METRIC GEOMETRY: SMOOTHNESS AND CONVERGENCE RESULTS

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

QUASI-METRIC GEOMETRY: SMOOTHNESS AND CONVERGENCE RESULTS

presented by Dan Brigham,

a candidate for the degree of Master of Arts and hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

This thesis has two distinct yet related parts, the first pertaining to Geometry on quasi-metric spaces with emphasis on the Hausdorff outer-measure, the natural extension of the Gromov-Pompeiu-Hausdorff distance to quasi-metric spaces, and smoothness of quasi-metric spaces, and the second dealing with Euclidean Geometry, namely the role Geometry plays in analysis, particularly in the characterization of Lipschitz domains via cones, domains of class $C^{1,\omega}$, where ω is a modulus of continuity, via pseudo-balls, which includes Lyapunov domains, a sharp version of the Hopf-Oleinik Boundary Point Principle, and subsequently the Strong Maximum Principle. Large portions of the material from the initial part will appear in [62] while the second part is joint work to appear in [9].

Chapter 1

Introduction

Pertaining to the first part of this thesis, while much of the development of Geometrical Analysis has so far taken place in the context of metric spaces, dictated by recent advancements in PDE's, Numerical Analysis, Harmonic Analysis, Function Space Theory, et cetera, it has become desirable to reconsider the role of Geometry in settings where only a quasi-metric is available. This thesis deals with topics in Quasi-Metric Geometry with a special emphasis on smoothness and convergence. In particular, a generalization of Gromov-Pompeiu-Hausdorff distance between metric spaces to the quasi-metric space context is presented. The majority of this work is seen in [62].

The aim of the second chapter is to develop a metrization theorem which will be called upon several times throughout the first part of the thesis. In general, quasi-metrics may be very poorly behaved; a quasi-metric may not be continuous with respect to the topology it induces and open balls may not be open in the the topology, among other deficiencies. The metrization theorem corrects these issues. Its fundamental component is the act of taking an arbitrary quasi-metric and defining a new quasi-metric, which remains related to the original, by examining the most efficient path from one point to another:

$$\rho_\alpha(x,y) := \inf\Big\{\Big(\sum_{i=1}^{N}\rho(\xi_i,\xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N\in\mathbb{N} \text{ and } \xi_1,\dots,\xi_{N+1}\in X \tag{1.1}$$
$$\text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\},$$

where X is a nonempty set, ρ is a quasi-metric on X and the exponent $\alpha \in (0,+\infty)$ plays an important role in optimal constants. The second chapter is divided into two subsections, the prior containing the subadditive-regularization of a non-negative function defined on a nonempty set cross itself, the symmetrization of such functions, pointwise equivalence of functions and how these three notions interact in the appropriate settings. The latter consists of constructing a genuine metric from a quasi-metric which, when raised to the proper proper, is pointwise equivalent to the original quasi-metric, and properties of the constructed metric.

The third chapter explores in more depth the notion of quasi-metric spaces, beginning with several examples of quasi-metric spaces, then leading to the notion of a class of quasi-metric spaces and certain optimal constants. The next section of the third chapter is devoted to studying topologies induced by quasi-metrics, touching on familiar topics such as balls, the distance from one set to another, the diameter of a set, completeness, totally boundedness, bi-Lipschitz maps, separability, compactness and Vitali's covering lemma, to name a few. The chapter concludes with a discussion of extensions of the Kuratowski and Fréchet embedding theorems.

Certain measure theoretic concepts, such as the notions of a measure, outer-measure, Borel measure, regular with respect to a sigma-algebra, equivalence of measures, Radon measure, etc., which play a crucial role are developed in the first two sections of the fourth chapter. These culminate to the definition of the Hausdorff outer-measure defined on quasi-metric spaces,

$$\mathcal{H}^d_{X,\rho}(E) := \lim_{\varepsilon\to 0^+}\Big(\inf\Big\{\sum_{j=1}^{\infty}(\mathrm{diam}_\rho(A_j))^d : \\ E\subseteq\bigcup_{j=1}^{\infty}A_j \text{ and } \mathrm{diam}_\rho(A_j)\leq\varepsilon \text{ for every } j\Big\}\Big), \tag{1.2}$$

where X is a nonempty set, $E \subseteq X$ is also nonempty and $d > 0$. The chapter finishes with properties of the Hausdorff outer-measure on quasi-metric spaces and the definition of the Hausdorff dimension of a set.

The fifth chapter, which is also the largest, ultimately leads to convergence of a family of quasi-metric spaces in the appropriate sense. First, a neighborhood of a set is defined, which leads to the Pompeiu-Hausdorff distance between two subsets of the same quasi-metric space. From there, the Gromov-Pompeiu-Hausdorff distance between metrics spaces is defined, followed by a version of this suitable to the setting of quasi-metric spaces, and the chapter is concluded with a discussion of convergence with respect to these distances.

The sixth and final chapter deals with smoothness of quasi-metric spaces, including Assouad dimension and Assouad's convexity index, and how these relate to richness (or lack thereof) of Hölder functions defined on quasi-metric spaces. This is also related to snowflaked version of metric spaces, including the famous fractal the Koch curve.

The second part of this thesis, seen in Chapters 7-16, is joint work completed with R. Alvarado, V. Maz'ya, M. Mitrea and E. Ziadé in [9] and has two parts which intertwine closely. One is of a predominantly geometric flavor and is aimed at describing the smoothness of domains (as classically formulated in analytical terms) in a purely geometric language. The other, having a more pronounced analytical nature, studies how the ability of expressing regularity in a geometric fashion is helpful in establishing sharp results in partial differential equations. We begin by motivating the material belonging to the first part just described.

Over the past few decades, analysis on classes of domains defined in terms of specific geometrical and measure theoretical properties has been a driving force behind many notable advances in partial differential equations and harmonic analysis. Examples of categories of domains with analytic and geometric measure theoretic characteristics are specifically designed to meet the demands and needs of work in the aforementioned fields include the class of nontangentially accessible (NTA) domains introduced in [40] by D. Jerison and C. Kenig (NTA domains form the most general class of regions where the pointwise nontangential behavior of harmonic functions at boundary points is meaningful), the class of (ε, δ)-domains considered in [41] by P. Jones (these are the most general type of domains known to date for which linear extension operators which preserve regularity measured on Sobolev scales may be constructed), uniformly rectifiable (UR) domains introduced in [17] by G. David and S. Semmes (making up the largest class of domains with the property that singular integral operators of Calderón-Zygmund type defined on their boundaries are continuous on L^p, $1 < p < \infty$), and the class of Semmes-Kenig-Toro (SKT) domains defined in [35] (SKT domains make up the most general class of domains for which Fredholm theory for boundary layer potentials, as originally envisioned by I. Fredholm, can be carried out).

In the process, more progress has been registered in our understanding of more familiar (and widely used) classes of domains such as the family of Lipschitz domains, as well as domains exhibiting low regularity assumptions. For example, the following theorem, which characterizes the smoothness of a domain of locally finite perimeter in terms of the regularity properties of the geometric measure theoretic outward unit normal, has been recently proved in [36]:

Theorem 1.1. *Assume that Ω is an open, nonempty, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter and which lies on only one side of its topological boundary, i.e.,*

$$\partial\Omega = \partial(\overline{\Omega}). \tag{1.3}$$

Denote by ν the outward unit normal to Ω, defined in the geometric measure theoretic sense at each point belonging to $\partial^*\Omega$, the reduced boundary of Ω. Finally, fix $\alpha \in (0,1]$. Then Ω is locally of class $\mathscr{C}^{1,\alpha}$ if and only if ν extends to an S^{n-1}-valued function on the boundary $\partial\Omega$ which is locally Hölder of order α. In particular,

$$\begin{gathered} \Omega \textit{ is a locally } \mathscr{C}^{1,1}\textit{-domain} \iff \\ \textit{the Gauss map } \nu : \partial^*\Omega \to S^{n-1} \textit{ is locally Lipschitz.} \end{gathered} \tag{1.4}$$

Finally, corresponding to the limiting case $\alpha = 0$, one has that Ω is a locally $\mathscr{C}^1$ domain if and only if the Gauss map $\nu : \partial^*\Omega \to S^{n-1}$ has a continuous extension to $\partial\Omega$.

Open subsets of $\mathbb{R}^n$ (of locally finite perimeter) whose outward unit normal is Hölder are typically called Lyapunov domains (cf., e.g., [30], [31, Chapter I]). Theorem 1.1 shows that, with this definition, Lyapunov domains are precisely those open sets whose boundaries may be locally described by graphs of functions with Hölder first-order derivatives (in a suitable system of coordinates). All these considerations are of an analytical, or measure theoretical flavor.

By way of contrast, in this part of the thesis we are concerned with finding an intrinsic description of a purely geometrical nature for the class of Lyapunov domains in $\mathbb{R}^n$. In order to be able to elaborate, let us define what we term here to be an hour-glass shape. Concretely, given $a, b > 0$ and $\alpha \in [0, +\infty)$, introduce

$$\mathscr{HG}^{\alpha}_{a,b} := \big\{x \in \mathbb{R}^n : a|x|^{1+\alpha} < |x_n| < b\big\}. \tag{1.5}$$

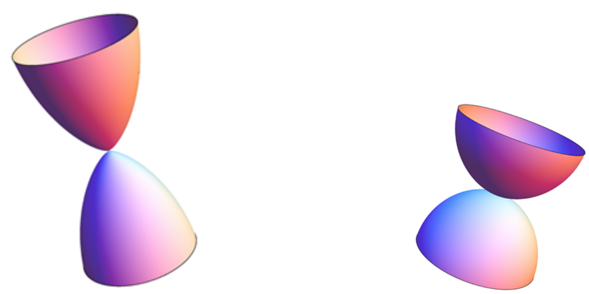

Figure 1. *The figure on the left is an hour-glass shape with α near 0, while the figure depicted on the right is an hour-glass shape with α near 1.*

With this piece of terminology, one of our geometric regularity results may be formulated as follows.

Theorem 1.2. *A nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is Lyapunov if and only if there exist $a, b > 0$ and $\alpha \in (0, 1]$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mathcal{R}(0) = x_0 \quad \textit{and} \quad \partial\Omega \cap \mathcal{R}\big(\mathscr{HG}^{\alpha}_{a,b}\big) = \varnothing. \tag{1.6}$$

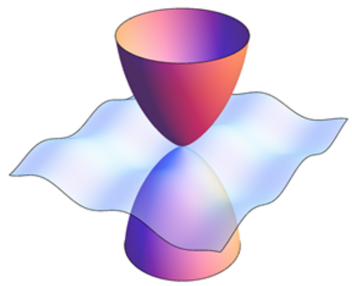

Figure 2. *Threading the boundary of a domain Ω in between the two rounded components of an hour-glass shape with direction vector along the vertical axis.*

The reader is referred to Theorem 11.3 in the body of the thesis for a more precise statement, which is stronger than Theorem 1.2 on two accounts: it is local in nature, and it allows for more general regions than those considered in (1.5). See (1.11) in this regard. Equally important, Theorem 11.3 makes it clear that the Hölder order of the normal is precisely the exponent $\alpha \in (0,1]$ used in the definition of the hour-glass region (1.5). As a corollary of Theorem 1.2, we note the following purely geometric characterization of domains of class $\mathscr{C}^{1,1}$: *a nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.* The latter condition amounts to requesting that there exists $r > 0$ along with an arbitrary function $h : \partial\Omega \to S^{n-1}$ with the property that

$$B(x + rh(x), r) \subseteq \Omega \quad \text{and} \quad B(x - rh(x), r) \subseteq \mathbb{R}^n \setminus \Omega \quad \text{for all } \ x \in \partial\Omega. \tag{1.7}$$

The idea is that the configuration consisting of two open, disjoint, congruent balls in $\mathbb{R}^n$ sharing a common boundary point may be rigidly transported so that it contains an hour-glass region $\mathscr{H}\mathscr{G}^{\alpha}_{a,b}$ with $\alpha = 1$ and some suitable choice of the parameters a, b (depending only on the radius r appearing in (1.7)).

The limiting case $\alpha = 0$ of Theorem 1.2 is also true, although the nature of the result changes in a natural fashion. Specifically, if $a \in (0,1)$ then, corresponding to $\alpha = 0$, the hour-glass region $\mathscr{HG}^{\alpha}_{a,b}$ from (1.5) becomes the two-component, open, circular, upright, truncated cone with vertex at the origin

$$\Gamma_{\theta,b} := \big\{x \in \mathbb{R}^n : \cos(\theta/2)|x| < |x_n| < b\big\}, \tag{1.8}$$

where $\theta := 2\arccos(a) \in (0,\pi)$ is the (total) aperture of the cone. This yields the following characterization of Lipschitzianity: a nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is a Lipschitz domain if and only if there exist $\theta \in (0,\pi)$ and $b > 0$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{R}(x_0) = 0 \quad \text{and} \quad \mathcal{R}(\partial\Omega) \cap \Gamma_{\theta,b} = \varnothing. \tag{1.9}$$

Our characterizations of Lipschitz domains in terms of uniform cone conditions are of independent interest and, in fact, the result just mentioned is the starting point in the proof of Theorem 1.2. Concretely, the strategy for proving the aforementioned geometric characterization of Lyapunov domains in terms of a uniform hour-glass condition with exponent $\alpha \in (0,1]$ consists of three steps: (1) show that the domain in question is Lipschitz, (2) show that the unit normal satisfies a Hölder condition of order $\alpha/(\alpha+1)$, and (3) show that the boundary of the original domain may be locally described as a piece of the graph of a function whose first-order derivatives are Hölder of order α.

In fact, we shall prove a more general result than Theorem 1.2 (cf. Theorem 1.3 below), where the (components of the) hour-glass shape (1.5) are replaced by a more general family of subsets of $\mathbb{R}^n$, which we call pseudo-balls (for the justification of this piece of terminology see item (*iii*) in Lemma 7.1). To formally introduce this class of sets, consider

$$\begin{gathered} R\in(0,+\infty) \text{ and } \omega:[0,R]\to[0,+\infty) \text{ a continuous function} \\ \text{with the properties that } \omega(0)=0 \text{ and } \omega(t)>0\ \ \forall\, t\in(0,R]. \end{gathered} \tag{1.10}$$

Then the pseudo-ball with apex at $x_0\in\mathbb{R}^n$, axis of symmetry along $h\in S^{n-1}$, height $b>0$, aperture $a>0$ and shape function ω as in (1.10), is defined as

$$\mathscr{G}^{\omega}_{a,b}(x_0,h):=\{x\in B(x_0,R):\ a|x-x_0|\,\omega(|x-x_0|)<h\cdot(x-x_0)<b\}\,. \tag{1.11}$$

For certain geometric considerations, it will be convenient to impose the following two additional conditions on the shape function ω:

$$\lim_{\lambda\to 0^+}\Big(\sup_{t\in(0,\min\{R,R/\lambda\}]}\tfrac{\omega(\lambda\,t)}{\omega(t)}\Big)=0, \quad \text{and } \omega \text{ strictly increasing.} \tag{1.12}$$

Also, in the second half of part II of this thesis, in relation to problems in partial differential equations, we shall work with functions $\widetilde{\omega}:[0,R]\to[0,+\infty)$ satisfying Dini's integrability condition

$$\int_0^R\frac{\widetilde{\omega}(t)}{t}\,dt<+\infty. \tag{1.13}$$

Of significant interest for us in this thesis is the class of functions $\omega_{\alpha,\beta}$, indexed by pairs of numbers $\alpha\in[0,1]$, $\beta\in\mathbb{R}$, such that $\beta<0$ if $\alpha=0$, defined as follows (convening that $\frac{\beta}{0}:=+\infty$ for any $\beta\in\mathbb{R}$):

$$\begin{gathered} \omega_{\alpha,\beta}:\big[0,\min\{e^{\frac{\beta}{\alpha}},e^{\frac{\beta}{\alpha-1}}\}\big]\to[0,+\infty), \\ \omega_{\alpha,\beta}(t):=t^{\alpha}(-\ln t)^{\beta} \text{ if } t>0 \text{ and } \omega_{\alpha,\beta}(0):=0. \end{gathered} \tag{1.14}$$

Corresponding to $\beta=0$, abbreviate $\omega_\alpha:=\omega_{\alpha,0}$. Note $\omega_{\alpha,\beta}$ satisfies all conditions listed in (1.10), (1.12) and (1.13) given $\alpha\in(0,1]$ and $\beta\in\mathbb{R}$. In addition, we also have that $t \mapsto \omega_{\alpha,\beta}(t)/t$ is decreasing. However, if $\alpha = 0$ then $\omega_{\alpha,\beta}$ satisfies Dini's integrability condition if and only if $\beta < -1$.

If $\alpha \in (0,1]$ and $a, b > 0$ then, corresponding to ω_α as in (1.14), the pseudo-ball

$$\begin{aligned} \mathscr{G}^{\alpha}_{a,b}(x_0,h) &:= \mathscr{G}^{\omega_\alpha}_{a,b}(x_0,h) \\ &= \big\{x \in B(x_0,1) \subseteq \mathbb{R}^n : a|x-x_0|^{1+\alpha} < h\cdot(x-x_0) < b\big\} \end{aligned} \tag{1.15}$$

is designed so that the hour-glass region (1.5) consists precisely of the union between the sets $\mathscr{G}^{\alpha}_{a,b}(0,\mathbf{e}_n)$ and $\mathscr{G}^{\alpha}_{a,b}(0,-\mathbf{e}_n)$, where $\mathbf{e}_n$ is the canonical unit vector along the vertical direction in $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$. The pseudo-balls (1.15) naturally make the transition between cones and genuine balls in $\mathbb{R}^n$ in the sense that, corresponding to $\alpha = 1$, the pseudo-ball $\mathscr{G}^{1}_{a,b}(x_0,h)$ is a solid spherical cap of an ordinary Euclidean ball, whereas corresponding to the limiting case when one formally takes $\alpha = 0$ in (1.15), the pseudo-ball $\mathscr{G}^{0}_{a,b}(x_0,h)$ is a one-component, circular, truncated, open cone (cf. Lemma 7.1 in the body of the thesis for more details).

In order to state the more general version of Theorem 1.2 alluded to above, we need one more definition. Call an open set $\Omega \subseteq \mathbb{R}^n$ a domain of class $\mathscr{C}^{1,\omega}$ if, near boundary points, its interior may be described (up to an isometric change of variables) in terms of upper-graphs of $\mathscr{C}^1$ functions whose first-order partial derivatives are continuous with modulus ω. Then a version of Theorem 1.2 capable of dealing with the more general type of pseudo-balls introduced in (1.11) reads as follows.

Theorem 1.3. *Let ω be a function as in* (1.10) *and* (1.12). *Then a nonempty, open, proper subset Ω of $\mathbb{R}^n$, with compact boundary is of class $\mathscr{C}^{1,\omega}$ if and only if there* *exist $a > 0$, $b > 0$ and two functions $h_{\pm} : \partial\Omega \to S^{n-1}$ with the property that*

$$\mathscr{G}^{\omega}_{a,b}(x, h_+(x)) \subseteq \Omega \quad \text{and} \quad \mathscr{G}^{\omega}_{a,b}(x, h_-(x)) \subseteq \mathbb{R}^n \setminus \Omega \quad \text{for each } \; x \in \partial\Omega. \tag{1.16}$$

Moreover, in the case when $\Omega \subseteq \mathbb{R}^n$ is known to be of class $\mathscr{C}^{1,\omega}$, one necessarily has $h_- = -h_+$.

This more general version of Theorem 1.2 is justified by the applications to partial differential equations we have in mind. Indeed, as we shall see momentarily, this more general hour-glass shape is important since it permits a desirable degree of flexibility (which happens to be optimal) in constructing certain types of barrier functions, adapted to the operator in question.

More specifically, in the second half of part II of this thesis we deal with maximum principles for second-order, non-divergence form differential operators. Traditionally, the three most basic maximum principles are labeled as weak, boundary point, and strong (cf. the discussion in [26], [76]). Among these, it is the Boundary Point Principle which has the most obvious geometrical character, both in its formulation and proof. For example, M.S. Zaremba [89], E. Hopf [37] and O.A. Oleinik [70] have proved such Boundary Point Principles in domains satisfying an interior ball condition. Our goal here is to prove a sharper version of their results with the interior ball condition replaced by an interior pseudo-ball condition. In fact, it is this goal that has largely motivated the portion of the research in this thesis described earlier.

Being able to use pseudo-balls as a replacement of standard Euclidean balls allows us to relax both the assumptions on the underlying domain, as well as those on the coefficients of the differential operator by considering semi-elliptic operators with singular lower-order terms (drift). Besides its own intrinsic merit, relaxing the regularity assumptions on the coefficients is particularly significant in view of applications to nonlinear partial differential equations.

To state a version of our main result in this regard (cf. Theorem 13.3), we make one definition. Given a real-valued function u of class $\mathscr{C}^2$ in an open subset of $\mathbb{R}^n$, denote by $\nabla^2 u$ the Hessian matrix of u, i.e., $\nabla^2 u := \big(\partial_i \partial_j u\big)_{1 \le i,j \le n}$. We then have the following Boundary Point Principle, relating the type of degeneracy in the ellipticity, as well as the nature of the singularities in the coefficients of the differential operator, to geometry of the underlying domain.

Theorem 1.4. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$ and assume that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) :\, a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\} \subseteq \Omega, \tag{1.17}$$

for some parameters $a,b,R \in (0,+\infty)$, a direction vector $h \in S^{n-1}$, and a real-valued shape function $\omega \in \mathscr{C}^0([0,R])$, which is positive and non-decreasing on $(0,R]$, and with the property that the mapping $(0,R] \ni t \mapsto \omega(t)/t \in (0,+\infty)$ is non-increasing. Also, consider a non-divergence form, second-order, differential operator L in Ω acting on functions $u \in \mathscr{C}^2(\Omega)$ according to

$$Lu := -\mathrm{Tr}\big(A\,\nabla^2 u\big) + \vec{b}\cdot\nabla u = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b^i \partial_i u \quad \text{in } \Omega, \tag{1.18}$$

whose coefficients $A = \big(a^{ij}\big)_{1\le i,j\le n} : \Omega \to \mathbb{R}^{n\times n}$ and $\vec{b} = (b^i)_{1\le i\le n} : \Omega \to \mathbb{R}^n$ satisfy

$$\inf_{x\in\mathscr{G}^{\omega}_{a,b}(x_0,h)}\ \inf_{\xi\in S^{n-1}} (A(x)\xi)\cdot\xi \ge 0, \quad (A(x)h)\cdot h > 0 \quad \text{for each } x \in \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.19}$$

In addition, suppose that there exists a real-valued function $\widetilde{\omega} \in \mathscr{C}^0\big([0,R]\big)$*, which is positive on* $(0,R]$ *and satisfying Dini's integrability condition* $\int_0^R t^{-1}\widetilde{\omega}(t)\,dt < +\infty$*, with the property that*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\mathrm{Tr}\,A(x)\Big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big((A(x)h)\cdot h\Big)} < +\infty, \tag{1.20}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\,\vec{b}(x)\cdot h\Big\} + \Big(\sum\limits_{i=1}^{n}\max\{0,-b^i(x)\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big((A(x)h)\cdot h\Big)} < +\infty. \tag{1.21}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ *for which* $\vec{\ell}\cdot h > 0$*, and suppose that*

$u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ *is a function satisfying*

$$(Lu)(x) \geq 0 \quad \textit{and} \quad u(x_0) < u(x) \quad \textit{for each} \quad x \in \Omega. \tag{1.22}$$

Then

$$\liminf_{t\to 0^+} \frac{u(x_0+t\vec{\ell}) - u(x_0)}{t} > 0. \tag{1.23}$$

For example, if $\partial\Omega \in \mathscr{C}^{1,\alpha}$ for some $\alpha \in (0,1)$ and if ν denotes the outward unit normal to $\partial\Omega$, then (1.23) holds provided $\vec{\ell}\cdot\nu(x_0) < 0$ and the coefficients of the semi-elliptic operator L, as in (13.20), satisfy for some $\varepsilon \in (0,\alpha)$

$$(A(x)\nu(x_0))\cdot\nu(x_0) > 0 \;\; \text{for each } x \in \Omega \text{ near } x_0, \text{ and} \tag{1.24}$$

$$\limsup_{\Omega\ni x\to x_0} \frac{|x-x_0|^{\alpha-\varepsilon}\Big(\mathrm{Tr}\,A(x)\Big) + |x-x_0|^{1-\varepsilon}|\vec{b}(x)|}{(A(x)\nu(x_0))\cdot\nu(x_0)} < +\infty. \tag{1.25}$$

Also, it can be readily verified that if the coefficients of the operator L are bounded near x_0, then a sufficient condition guaranteeing the validity of (1.20)-(1.21) is the existence of some $c > 0$ such that

$$(A(x)h)\cdot h \geq c\frac{\big((x-x_0)\cdot h\big)\,\omega(|x-x_0|)}{|x-x_0|\widetilde{\omega}((x-x_0)\cdot h)}, \qquad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.26}$$

This should be thought of as an admissible degree of degeneracy in the ellipticity's uniformity of the operator L (a phenomenon concretely illustrated by considering the case when $\omega(t) = t^\alpha$ and $\widetilde{\omega}(t) = t^\beta$ for some $0 < \beta < \alpha < 1$).

It is illuminating to note that the geometry of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ affects (through its direction vector h and shape function ω) the conditions (1.19)-(1.21) imposed on the coefficients of the differential operator L. This is also the case for the proof of Theorem 1.4 in which we employ a barrier function which is suitably adapted both to the nature of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$, as well as to the degree of degeneracy of the ellipticity of the operator L (manifested through $\widetilde{\omega}$ and ω). Concretely, this barrier function is defined at each $x \in \mathscr{G}^{\omega}_{a,b}(x_0, h)$ as

$$\begin{aligned} v(x) := (x - x_0) \cdot h \quad &+ \quad C_0 \int_0^{(x-x_0)\cdot h} \int_0^{\xi} \frac{\widetilde{\omega}(t)}{t} dt d\xi \\ &- \quad C_1 \int_0^{|x-x_0|} \int_0^{\xi} \frac{\omega(t)}{t} \Big(\frac{t}{\xi}\Big)^{\gamma-1} dt d\xi, \end{aligned} \tag{1.27}$$

where $\gamma > 1$ is a fine-tuning parameter, and $C_0, C_1 > 0$ are suitably chosen constants (depending on Ω and L), whose role is to ensure that v satisfies the properties described below. The linear part in the right-hand side of (1.27) is included in order to guarantee that

$$\vec{\ell} \cdot (\nabla v)(x_0) > 0, \tag{1.28}$$

while the constants C_0, C_1 are chosen such that

$$Lv \leq 0 \ \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0, h), \ \text{ and } \exists\, \varepsilon > 0 \text{ so that } \varepsilon v \leq u - u(x_0) \ \text{ on } \partial\mathscr{G}^{\omega}_{a,b}(x_0, h). \tag{1.29}$$

Then (1.23) follows from (1.28)-(1.29) and the Weak Maximum Principle.

Note that the class of second-order, nondivergence form, differential operators considered in Theorem 1.4 is invariant under multiplication by arbitrary positive functions, and that no measurability assumptions are made on the coefficients.

Although a more refined version of Theorem 1.4 is proved later in the thesis (cf. Theorem 13.3), we wish to note here that this result is already quantitatively optimal. To see this, consider the case when $\Omega := \{x \in \mathbb{R}^n_+ : x_n < 1\}$, the point x_0 is the origin in $\mathbb{R}^n$, and

$$L := -\Delta + \frac{\psi(x_n)}{x_n}\frac{\partial}{\partial x_n} \qquad \text{in } \ \Omega, \tag{1.30}$$

where $\psi : (0, 1] \to (0, +\infty)$ is a continuous function with the property that

$$\int_0^1 \frac{\psi(t)}{t}\, dt = +\infty. \tag{1.31}$$

Then, if $\vec{\ell} := \mathbf{e}_n := (0, ..., 0, 1) \in \mathbb{R}^n$ and

$$u(x_1, ..., x_n) := \int_0^{x_n} \exp\Big\{-\int_\xi^1 \frac{\psi(t)}{t}\, dt\Big\}\, d\xi, \qquad \forall\, (x_1, ..., x_n) \in \Omega, \tag{1.32}$$

it follows that $u \in \mathscr{C}^2(\Omega)$, u may be continuously extended at $0 \in \mathbb{R}^n$ by setting $u(0) := 0$, and $u > 0$ in Ω. Furthermore,

$$\frac{\partial u}{\partial x_n} = \exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\, dt\Big\}, \quad \text{and} \tag{1.33}$$

$$\frac{\partial^2 u}{\partial x_n^2} = \frac{\psi(x_n)}{x_n} \exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\, dt\Big\} = \frac{\psi(x_n)}{x_n}\frac{\partial u}{\partial x_n} \ \text{ in } \ \Omega, \tag{1.34}$$

from which we deduce that $Lu = 0$ in $\mathbb{R}^n_+$, and $(\nabla u)(0) = 0$, thanks to (1.31). Hence (1.23), the conclusion of the Boundary Point Principle formulated in Theorem 1.4, fails in this case. The sole cause of this breakdown is the inability to find a shape function $\widetilde{\omega}$ satisfying Dini's integrability condition and such that (1.21) holds. Indeed, the latter condition reduces, in the current setting, to

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)\ni x\to 0}\Big(\frac{\max\{0,\vec{b}(x)\cdot\mathbf{e}_n\}}{x_n^{-1}\widetilde{\omega}(x_n)}\Big)<+\infty, \qquad \text{where}$$
$$\vec{b}(x):=\big(0,...,0,\psi(x_n)/x_n\big) \ \text{ for } \ x=(x_1,...,x_n)\in\Omega, \tag{1.35}$$

which, if true, would force $\widetilde{\omega}(t)\geq c\,\psi(t)$ for all $t>0$ small (for some fixed $c>0$). However, in light of (1.31), this would prevent $\widetilde{\omega}$ from satisfying Dini's integrability condition. This proves the optimality of condition (1.21) in Theorem 1.4. A variant of this counterexample also shows the optimality of condition (1.20). Specifically, let Ω, $\vec{\ell}$, x_0, u be as before and, this time, consider

$$L:=-\Big(\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}+\frac{x_n}{\psi(x_n)}\frac{\partial^2}{\partial x_n^2}\Big)+\frac{\partial}{\partial x_n} \qquad \text{in } \ \Omega. \tag{1.36}$$

Obviously, $Lu=0$ in Ω and Ω satisfies an interior pseudo-ball condition at the origin with shape function $\omega(t):=t$. As such, condition (1.20) would entail (for this choice of ω, after some simple algebra), $\widetilde{\omega}(t)\geq c\psi(t)$ for all $t>0$ small. In concert with (1.31) this would, of course, prevents $\widetilde{\omega}$ from satisfying Dini's integrability condition. Other aspects of the sharpness of Theorem 1.4 are discussed later, in Chapter 14.

As a consequence of our Boundary Point Principle, we obtain a Strong Maximum Principle for a class of non-uniformly elliptic operators with singular (and possibly non-measurable) drift terms. More specifically, we have the following theorem.

Theorem 1.5. *Let Ω be a nonempty, connected, open subset of $\mathbb{R}^n$, and suppose that L, written as in* (1.18)*, is a (possibly non-uniformly) elliptic second-order differential operator in non-divergence form (without a zeroth-order term) in Ω. Also, assume that for each $x_0\in\Omega$ and each $\xi\in S^{n-1}$ there exists a real-valued function $\widetilde{\omega}=\widetilde{\omega}_{x_0,\xi}$ which is continuous on* $[0,1]$*, positive on* $(0,1]$*, satisfies $\int_0^1\frac{\widetilde{\omega}(t)}{t}\,dt<+\infty$, and with* *the property that*

$$\limsup_{\substack{(x-x_0)\cdot\xi>0\\ x\to x_0}} \frac{\Big(\mathrm{Tr}\,A(x)\Big)+\big|\vec{b}(x)\cdot\xi\big|+\big|\vec{b}(x)\big||x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}\Big((A(x)\xi)\cdot\xi\Big)}<+\infty. \tag{1.37}$$

Then if $u\in\mathscr{C}^2(\Omega)$ satisfies $(Lu)(x)\geq 0$ for all $x\in\Omega$ and assumes a global minimum value at some point in Ω, it follows that u is constant in Ω.

See Theorem 15.1 for a slightly more refined version, though such a result is already quantitatively sharp. The following example sheds light in this regard. Concretely, in the n-dimensional Euclidean unit ball centered at the origin, consider

$$L:=-\tfrac{1}{n+2}\Delta+\vec{b}(x)\cdot\nabla,\quad \text{where}\quad \vec{b}(x):=\begin{cases}|x|^{-2}x & \text{if } x\in B(0,1)\setminus\{0\},\\ 0 & \text{if } x=0.\end{cases} \tag{1.38}$$
and $u:\overline{B(0,1)}\longrightarrow\mathbb{R}$ given by $u(x):=|x|^4$ for each $x\in\overline{B(0,1)}$.

It follows that

$$u\in\mathscr{C}^2\big(\overline{B(0,1)}\big),\quad (\nabla u)(x)=4|x|^2x \quad\text{and} \tag{1.39}$$

$$(\Delta u)(x)=4(n+2)|x|^2,\quad \forall\, x\in\overline{B(0,1)}. \tag{1.40}$$

Consequently,

$$(Lu)(x)=0\ \text{ for each }\ x\in B(0,1),\qquad u\geq 0\ \text{ in }\ B(0,1), \tag{1.41}$$

$$u(0)=0\ \text{ and }\ u\big|_{\partial B(0,1)}=1, \tag{1.42}$$

which shows that the Strong Maximum Principle fails in this case. To understand the nature of this failure, observe that given a function $\widetilde{\omega}:(0,1)\to(0,+\infty)$ and a vector $\xi\in S^{n-1}$, condition (1.37) entails

$$\limsup_{\substack{x\to 0\\ x\cdot\xi>0}} \frac{|x|^{-2}\,x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}}<+\infty \tag{1.43}$$

which, when specialized to the case when x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0, +\infty)$ such that $\widetilde{\omega}(t) \geq c$ for all small $t > 0$. Of course, this would prevent $\widetilde{\omega}$ from satisfying Dini's integrability condition.

In the last part of of this chapter we briefly review some of the most common notational conventions used in the sequel. Throughout the thesis, we shall assume that $n \geq 2$ is a fixed integer, $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$, and '$\cdot$' denotes the canonical dot product of vectors in $\mathbb{R}^n$. Also, as usual, S^{n-1} is the unit sphere centered at the origin in $\mathbb{R}^n$ and by $B(x, r)$ we denote the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, i.e., $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. Whenever necessary to stress the dependence of a ball on the dimension of the ambient Euclidean space we shall write $B_n(x, r)$ in place of $\{y \in \mathbb{R}^n : |x - y| < r\}$. We let $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ denote the canonical orthonormal basis in $\mathbb{R}^n$. In particular, $\mathbf{e}_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$, and we shall use the abbreviation (x', x_n) in place of $(x_1, \ldots, x_n) \in \mathbb{R}^n$. By $0'$ we typically denote the origin in $\mathbb{R}^{n-1}$, often regarded as a subspace of $\mathbb{R}^n$ under the canonical identification $\mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1} \times \{0\}$. Next, given $E \subseteq \mathbb{R}^n$, we use E^c, E°, $\overline{E}$ and ∂E to denote, respectively, the complement of E (relative to $\mathbb{R}^n$, i.e., $E^c := \mathbb{R}^n \setminus E$), the interior, the closure and the boundary of E. One other useful piece of terminology is as follows. Let $E \subseteq \mathbb{R}^n$ be a set of cardinality ≥ 2 and assume that $(X, \|\cdot\|)$ is a normed vector space. Then $\mathscr{C}^\alpha(E, X)$ will denote the vector space of functions $f : E \to X$ which are Hölder of order $\alpha > 0$, i.e., for which

$$\|f\|_{\mathscr{C}^\alpha(E,X)} := \sup_{x,y \in E,\, x \neq y} \frac{\|f(x) - f(y)\|}{|x - y|^\alpha} < +\infty. \tag{1.44}$$

As is customary, functions which are Hölder of order $\alpha = 1$ will be referred to as

Lipschitz functions. Also, corresponding to the limiting case $\alpha = 0$, we agree that $\mathscr{C}^0$ stands for the class of continuous functions (in the given context).

More generally, given a modulus of continuity ω, a real-valued function f is said to be of class $\mathscr{C}^\omega$ provided there exists $C \in (0, +\infty)$ such that $|f(x)-f(y)| \leq C\,\omega(|x-y|)$ for $|x-y|$ small. Functions of class $\mathscr{C}^{1,\omega}$ are then defined by requiring that their first-order partial derivatives exist and are in $\mathscr{C}^\omega$.

Finally, by $\mathrm{Tr}\,A$ and $A^\top$ we shall denote, respectively, the trace and transpose of the matrix A.

Chapter 2

Metrization Theorems

This begins the main body of the first half of the work, much of which will appear in [62].

In this preliminary chapter we develop the most essential tool in the study of the geometry of quasi-metric spaces, a certain regularization process. Given an arbitrary, nonempty set and a nonnegative function defined on the set cross itself, we can construct a strongly related, yet much nicer function. From there, after adding additional assumptions on the first function, our produced function ultimately is a quasi-metric which, when raised to a suitable power, is a metric.

Usually we'll be interested in taking a given quasi-metric and improving it while not straying too far from the original quasi-distance. To give the reader an idea of why such a process is useful, consider an arbitrary quasi-metric space. The quasi-distance induces a topology on its ambient space. However, the quasi-metric may fail to be continuous with respect to the topology it induces. In fact, the open balls may not even be open in its topology! This is not the case in metric spaces. Fortunately, the regularized version of our function fixes both the continuity and open ball problems.

We will not see any applications in this chapter but we'll call upon it in crucial moments throughout the text.

Another interesting fact about our regularized quasi-distance is that, when raised to suitable powers, it is a genuine distance function. This gives us a bridge between a subject which is already well-understood, the geometry of metric spaces, and a much broader setting, the geometry of quasi-metric spaces.

Lastly, throughout this entire text, we'll usually think of X as a set of cardinality at least 2, though we'll be explicit when it is necessary that X not be a set of cardinality 1 (we always assume nonempty). Quasi-metrics defined on a singleton cross itself are not very interesting.

2.1 Regularization and Symmetrization of Functions

We begin with the most basic definition:

Definition 2.1. *Let X be a nonempty set. Then a* `quasi-metric` *on X (or $X \times X$) is any function $\rho : X \times X \to [0, +\infty]$ (this function is allowed to take the value infinity) satisfying the following three conditions for all $x, y, z \in X$:*

$$\textit{(nondegeneracy)} \qquad \rho(x,y) = 0 \Longleftrightarrow x = y \tag{2.1}$$

$$\textit{(quasi-symmetry)} \qquad \rho(x,y) \leq C_0 \rho(y,x) \tag{2.2}$$

$$\textit{(quasi-triangle inequality)} \qquad \rho(x,y) \leq C_1(\rho(x,z) + \rho(z,y)) \tag{2.3}$$

for some fixed constants $C_0, C \geq 1$.

If instead of quasi-symmetry we have `symmetry` (that is, $\rho(x,y) = \rho(y,x)$ for any given $x, y \in X$) and in the third condition above we have $C = 1$ then ρ is a regular metric (satisfying the standard triangle inequality).

Also, why would we want to allow our function to take the value $+\infty$? If X consists of two disconnected regions, say U and V, then we would want $\rho(x, y) = +\infty$ for any $x \in U$ and any $y \in V$ as there is no way to pass from one disconnected region to another.

Remark 2.1. *Given X a nonempty set and ρ a quasi-metric,* (2.3) *is equivalent to the following condition:*

$$\textit{(quasi-subadditivity)}\ \rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\} \tag{2.4}$$

for some fixed constant $C_1 \geq 1$ (usually different from C) and for all $x, y, z \in X$.

For the remainder of the text, when referring to the quasi-triangle inequality or quasi-subadditivity, we shall always use C for (2.3) and C_1 for (2.4).

Also, if the constant in (2.4) is 1 then ρ is called an `ultrametric` (and hence sometimes (2.4) is called the `quasi-ultrametric condition`). Furthermore, in addition to calling ρ quasi-subadditive, we will say ρ satisfies the `quasi-subadditive condition`.

Though property (2.4) may seem oddly named (usually subadditivity makes one think of something involving addition), its origins become more apparent after reading Definition (2.2).

Keep these definitions in mind. For now we shall temporarily step away from functions which are already quasi-metrics, instead focusing on arbitrary, nonnegative functions defined on a set cross itself. The goal, which will be fully realized at the end of the chapter, is to get as much as possible out of the least assumptions (for example, we are not assuming the following functions are symmetric or non-degenerate).

Next we prove a lemma which will be of great use later.

Lemma 2.1. *Let X be a nonempty set and $\rho : X \times X \to [0, +\infty]$ be such that*

$$\rho(x, y) \leq 2 \max\{\rho(x, z), \rho(z, y)\} \text{ for any } x, y, z \in X. \tag{2.5}$$

Then, for every $N \in \mathbb{N}$ and every $x = \xi_1, \dots, \xi_{N+1} = y \in X$ there holds

$$\rho(x, y) \leq 2\rho(x, \xi_2) + 4 \sum_{i=2}^{N-1} \rho(\xi_i, \xi_{i+1}) + 2\rho(\xi_N, y), \tag{2.6}$$

with the convention that the sum in the right-hand side of (2.6) *is disregarded if* $N \leq 2$.

Proof. We shall prove the lemma by induction on the parameter $N \in \mathbb{N}$. Given that ρ is nonnegative, the statement corresponding to $N = 1$ is obvious, while the statement corresponding to $N = 2$ is a direct consequence of (2.5). Assume that $N \in \mathbb{N}$, $N \geq 2$, is such that

$$\text{(2.6) holds with } N \text{ replaced by } K \in \{1, \dots, N\} \text{ and } x = \xi_1, \dots, \xi_{N+1} = y \in X \tag{2.7}$$

Pick next $x = \xi_1, \dots, \xi_{N+2} = y \in X$. We want to prove that

$$\rho(x, y) \leq 2\rho(x, \xi_2) + 4 \sum_{i=2}^{N} \rho(\xi_i, \xi_{i+1}) + 2\rho(\xi_{N+1}, y), \tag{2.8}$$

For $M \in \{1, \dots, N + 1\}$ consider the inequality

$$\rho(x, y) \leq 2\rho(\xi_M, y). \tag{2.9}$$

Denote by M_0 the largest number $M \in \{1, \dots, N + 1\}$ for which (2.9) holds. Since (2.9) is verified for $M = 1$, it follows that $M_0 \in \{1, \dots, N + 1\}$ is well-defined. If $M_0 = N + 1$, then (2.8) holds as well (since ρ is nonnegative). Hence, it suffices to analyze the case when $1 \leq M_0 \leq N$. By design,

$$\rho(x,y) \leq 2\rho(\xi_{M_0}, y). \tag{2.10}$$

and

$$\rho(x,y) > 2\rho(\xi_{M_0+1}, y). \tag{2.11}$$

On the other hand, by (2.5), we have

$$\rho(x,y) \leq 2\max\Big\{\rho(x,\xi_{M_0+1}), \rho(\xi_{M_0+1}, y)\Big\}. \tag{2.12}$$

In light of (2.11), we can infer from (2.12) that

$$\rho(x,y) \leq 2\rho(x, \xi_{M_0+1}). \tag{2.13}$$

If $M_0 = 1$, given that ρ is nonnegative, it follows from (2.13) that (2.8) holds and we are done. Thus, there remains to study the case when $2 \leq M_0 \leq N$. Under this assumption, summing (2.10) and (2.13) gives us

$$\rho(x,y) \leq \rho(x,\xi_{M_0+1}) + \rho(\xi_{M_0}, y) \tag{2.14}$$

By the induction hypothesis (2.7) twice, once with $K = M_0$ and once more with $K = N - M_0 + 2$, we have

$$\begin{aligned} \rho(x,\xi_{M_0+1}) &\leq 2\rho(x,\xi_2) + 4\sum_{i=2}^{M_0-1}\rho(\xi_i,\xi_{i+1}) + 2\rho(\xi_{M_0},\xi_{M_0+1}), && (2.15) \\ \rho(\xi_{M_0}, y) &\leq 2\rho(\xi_{M_0},\xi_{M_0+1}) + 4\sum_{i=M_0+1}^{N}\rho(\xi_i,\xi_{i+1}) + 2\rho(\xi_{N+1}, y). && (2.16) \end{aligned}$$

Summing up inequalities (2.15) and (2.16) while recalling (2.14), it follows that (2.8) also holds when $2 \leq M_0 \leq N$. This completes the proof of the lemma. □

Going further, we now describe a regularization procedure for a given, arbitrary (real-valued) nonnegative function defined on a set cross itself.

Definition 2.2. *Assume that X is a nonempty set. A function $\rho : X \times X \to [0, +\infty]$ is said to be α-`quasi-subadditive` for some $\alpha \in (0, +\infty]$ provided there exists a finite constant $C \geq 2^{-1/\alpha}$ such that*

$$\rho(x, y) \leq C\Big([\rho(x, z)]^{\alpha} + [\rho(z, y)]^{\alpha}\Big)^{\frac{1}{\alpha}}, \qquad \forall\, x, y, z \in X \tag{2.17}$$

if $\alpha < +\infty$ and, corresponding to $\alpha = +\infty$ (in which scenario it is assumed that $C \geq 1$),

$$\rho(x, y) \leq C \max\big\{\rho(x, z)\, ,\, \rho(z, y)\big\}, \qquad \forall\, x, y, z \in X. \tag{2.18}$$

Furthermore, "∞-quasi-subadditive" may be abbreviated simply as `quasi-subadditive` *(introduced above in (2.3) but with a less relaxed condition on the constant involved).*

As a side note, 1-quasi-subadditivity is the same as the quasi-triangle inequality, though again given a quasi-metric on a set of cardinality at least two it necessarily happens that $C \geq 1$.

Finally, say that $\rho : X \times X \to [0, +\infty]$ is α-`subadditive` for some $\alpha \in (0, +\infty]$ if one may take $C = 1$ in (2.17) when $\alpha < +\infty$ and in (2.18) when $\alpha = +\infty$.

Lemma 2.2. *For the following, fix a nonempty set X, a function $\rho : X \times X \to [0, +\infty]$ and an exponent $\alpha \in (0, +\infty]$.*

(1) ρ is α-quasi-subadditive $\Longleftrightarrow$ ρ is quasi-subadditive. However, the constants in (2.17) and (2.18) are generally different.

(2) ρ is α-quasi-subadditive $\Longrightarrow$ ρ is β-quasi-subadditive for any $\beta \in (0, \alpha]$ (with the same constant).

(3) If ρ is a quasi-subadditive then so is $X \times X \ni (x, y) \mapsto \min\{\rho(x, y), 1\} \in [0, 1]$.

(4) If ρ and $\widetilde{\rho} : X \times X \to [0, +\infty]$ are quasi-subadditive functions then so are $\max\{\rho, \widetilde{\rho}\}$ and $\rho + \widetilde{\rho}$.

Proof. All claims follow immediately from definitions. $\square$

The following definition, describing the α-subadditive regularization of a real-valued, nonnegative function defined on a a nonempty set cross itself, plays a key role in much of our subsequent work.

Definition 2.3. *Given a nonempty set X, a function $\rho : X \times X \to [0, +\infty]$ and an exponent $\alpha \in (0, +\infty)$, define $\rho_\alpha : X \times X \to [0, +\infty]$ by setting for each $x, y \in X$*

$$\rho_\alpha(x, y) := \inf\Big\{ \Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha \Big)^{\frac{1}{\alpha}} : \qquad (2.19)$$
$$N \in \mathbb{N},\, \xi_1, \dots, \xi_{N+1} \in X,\ x = \xi_1, y = \xi_{N+1} \Big\}.$$

Also, corresponding to $\alpha = +\infty$, define $\rho_\infty : X \times X \to [0, +\infty]$ by setting for each $x, y \in X$

$$\rho_\infty(x, y) := \inf\Big\{ \max_{1 \leq i \leq N} \rho(\xi_i, \xi_{i+1}) : \qquad (2.20)$$
$$N \in \mathbb{N},\, \xi_1, \dots, \xi_{N+1} \in X,\ x = \xi_1, y = \xi_{N+1} \Big\}. \qquad (2.21)$$

Finally, call the function ρ_α the α-`subadditive regularization` of ρ.

In other words, the α-subadditive regularization process is an operator acting on the collection of functions defined on a nonempty set cross itself, outputting another function defined on the same set cross itself.

The lemma below substantiates the heuristic principle that while ρ_α continues to be closely related to ρ, in general it enjoys better functional analytic properties. In fact, the reason for using the terminology "α-subadditive regularization" for the function ρ_α, defined according to the regularization scheme in Definition 2.3, becomes clear from part (3) of the lemma.

Lemma 2.3. *Given a nonempty set X, an exponent $\alpha \in (0, +\infty]$ and a function $\rho : X \times X \to [0, +\infty]$, the following properties hold.*

(1) $\rho_\alpha : X \times X \to [0, +\infty]$ is well-defined and has the property that for every $C \in [0, +\infty)$ and $\beta \in (0, \alpha]$,

$$\rho_\alpha \leq \rho_\beta, \qquad (C\,\rho)_\alpha = C\,\rho_\alpha \quad \textit{and} \quad \rho_\alpha \leq \rho \qquad \textit{on} \;\; X \times X. \tag{2.22}$$

(2) Given another function $\widetilde{\rho} : X \times X \to [0, +\infty]$, one has

$$\rho \leq \widetilde{\rho} \;\; \textit{on} \;\; X \times X \Longrightarrow (\rho)_\alpha \leq (\widetilde{\rho})_\alpha \;\; \textit{on} \;\; X \times X. \tag{2.23}$$

(3) For each $\beta \in (0, \alpha]$, the function ρ_α is β-subadditive. That is, if β is finite there holds

$$\rho_\alpha(x, y) \leq \Big([\rho_\alpha(x, z)]^\beta + [\rho_\alpha(z, y)]^\beta \Big)^{\frac{1}{\beta}}, \qquad \forall\, x, y, z \in X, \tag{2.24}$$

and, corresponding to the case when $\beta = \alpha = +\infty$,

$$\rho_\infty(x, y) \leq \max\big\{\rho_\infty(x, z)\,,\, \rho_\infty(z, y)\big\}, \qquad \forall\, x, y, z \in X. \tag{2.25}$$

In particular, ρ_α is α-subadditive.

(4) *ρ_α is quasi-subadditive, in the precise sense that*

$$\rho_\alpha(x,y) \le 2^{1/\alpha} \max\big\{\rho_\alpha(x,z)\,,\,\rho_\alpha(z,y)\big\}, \qquad \forall\, x,y,z \in X. \tag{2.26}$$

(5) *One has the following equivalence:*

$$\rho = \rho_\alpha \iff \rho \text{ is } \alpha\text{-subadditive.} \tag{2.27}$$

As a consequence of this, as well as part (3)*, one has* $(\rho_\alpha)_\beta = \rho_\alpha$*. In particular,*

$$(\rho_\alpha)_\alpha = \rho_\alpha. \tag{2.28}$$

In other words, taking the α-subadditive regularization of a function is an idempotent operation.

(6) *ρ_α can be characterized as the largest nonnegative function defined on $X \times X$ with the property that it is both α-subadditive and less than or equal to ρ on $X \times X$.*

(7) *If α is finite and $\rho_i : X \times X \to [0,+\infty]$, $i \in \mathbb{N}$, are α-subadditive functions, then for each $\beta \in [\alpha,+\infty)$ the function*

$$\rho : X \times X \longrightarrow [0,+\infty], \qquad \rho := \Big(\sum_{i=1}^{\infty} \rho_i^\beta\Big)^{1/\beta}, \tag{2.29}$$

is also α-subadditive.

(8) *The regularization process described in Definition 2.3 is invariant under surjections, in the following precise sense. Suppose Y is another nonempty set and*

that ϕ is a function mapping from the set Y to the set X. Then we have that $\rho_\alpha \circ (\phi(y_1), \phi(y_2)) \leq \big(\rho \circ (\phi, \phi)\big)_\alpha (y_1, y_2)$ for every $y_1, y_2 \in Y$.

The invariance truly comes from the fact that if ϕ is onto then it follows that $\rho_\alpha \circ (\phi, \phi) = \big(\rho \circ (\phi, \phi)\big)_\alpha$ on $Y \times Y$.

(9) *For each exponent $\beta \in (0, +\infty)$, $(\rho^\beta)_\alpha = (\rho_{\alpha\beta})^\beta$.*

Proof. The claims in (1) and (2) are easily seen through definitions.

To prove (3), fix $\alpha \in (0, +\infty)$ and $x, y, z \in X$. Also, assume that $M, N \in \mathbb{N}$ and the points $\xi_1, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M+1} \in X$, are such that $x = \xi_1$, $z = \xi_{N+1}$ and $y = \xi_{N+M+1}$. Consequently, by (2.19),

$$\big[\rho_\alpha(x, y)\big]^\alpha \leq \sum_{i=1}^{N+M} [\rho(\xi_i, \xi_{i+1})]^\alpha = \sum_{i=1}^{N} [\rho(\xi_i, \xi_{i+1})]^\alpha + \sum_{i=1}^{M} [\rho(\xi_{N+i}, \xi_{N+i+1})]^\alpha. \tag{2.30}$$

Taking the infimum over all $N, M \in \mathbb{N}$ and $\xi_1, \ldots, \xi_{N+M+1}$ as above, we obtain that

$$\rho_\alpha(x, y) \leq \Big([\rho_\alpha(x, z)]^\alpha + [\rho_\alpha(z, y)]^\alpha\Big)^{\frac{1}{\alpha}}. \tag{2.31}$$

Let now $\beta \in (0, \alpha]$ be arbitrary, fixed. Examining the right-hand side of (2.31) and since $\beta/\alpha \leq 1$, we have

$$\begin{aligned} \Big([\rho_\alpha(x, z)]^\alpha + [\rho_\alpha(z, y)]^\alpha\Big)^{\frac{\beta}{\alpha}} &= \Big(\big([\rho_\alpha(x, z)]^\beta\big)^{\frac{\alpha}{\beta}} + \big([\rho_\alpha(z, y)]^\beta\big)^{\frac{\alpha}{\beta}}\Big)^{\frac{\beta}{\alpha}} \\ &\leq [\rho_\alpha(x, z)]^\beta + [\rho_\alpha(z, y)]^\beta, \end{aligned} \tag{2.32}$$

so that (2.24) is proved in the case when $\alpha \in (0, +\infty)$ by combining (2.31) and (2.32).

The case when α is infinite and β is finite and the case when both α and β are infinite follow similarly.

Consider next the claim made in (4). The case when $\alpha = +\infty$ is contained in (2.25), so it suffices to treat the situation when $\alpha \in (0, +\infty)$. In this scenario, the inequality proved in (3) written for $\beta := \alpha$ gives that for each $x, y, z \in X$ we have

$$\begin{aligned} \rho_\alpha(x,y) &\leq \Big([\rho_\alpha(x,z)]^\alpha + [\rho_\alpha(z,y)]^\alpha\Big)^{\frac{1}{\alpha}} \\ &\leq \Big(2\,\max\big\{[\rho_\alpha(x,z)]^\alpha\,,\,[\rho_\alpha(z,y)]^\alpha\big\}\Big)^{\frac{1}{\alpha}} \\ &= 2^{\frac{1}{\alpha}}\,\max\big\{\rho_\alpha(x,z)\,,\,\rho_\alpha(z,y)\big\}, \end{aligned} \tag{2.33}$$

hence (2.26) is proved.

As far as (5) is concerned, the right-pointing implication in (2.27) is a direct consequence of (2.24) and (2.25). To prove the opposite implication, consider first the case $\alpha < +\infty$ and note that if $x, y \in X$, $N \in \mathbb{N}$ and $\xi_1, \cdots \xi_{N+1}$ are such that $x = \xi_1, y = \xi_{N+1}$ then an inductive argument (on the parameter N) shows that

$$\rho(x,y)^\alpha \leq \sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha. \tag{2.34}$$

Fixing $x, y \in X$ and taking the infimum of both sides of (2.34) over all $\xi_1, \cdots \xi_{N+1} \in X$ satisfying $x = \xi_1, y = \xi_{N+1}$ with $N \in \mathbb{N}$ then yields $\rho(x,y)^\alpha \leq \rho_\alpha(x,y)^\alpha$ for all $x, y \in X$, i.e. $\rho \leq \rho_\alpha$. Since the opposite inequality is contained in (2.22), we eventually conclude that $\rho = \rho_\alpha$, as desired. Finally, the reasoning in the case when $\alpha = +\infty$ is similar except that, this time, one has

$$\rho(x,y) \leq \max_{1 \leq i \leq N} \rho(\xi_i, \xi_{i+1}) \tag{2.35}$$

in place of (2.34). The proof of (5) is therefore complete.

Concerning the maximality claim made in (6), thanks to (3) and the last inequality in (1), it suffices to show that if the functions $\rho, \phi : X \times X \to [0, +\infty]$ and the exponent $\alpha \in (0, +\infty]$ are such that $\phi \leq \rho$ on $X \times X$ and ϕ is α-subadditive then necessarily

$\phi \le \rho_\alpha$ on $X \times X$. This, however, follows from (2.27) and (2) which allow us to write that $\phi = \phi_\alpha \le \rho_\alpha$ on $X \times X$.

The claim in (7) is readily implied by the special case when all but finitely many ρ_i's are identically zero, via a limiting argument. Thus pick $x, y, z \in X$, $N \in \mathbb{N}$ and $\rho_1, \cdots, \rho_N$, N α-subadditive functions defined on $X \times X$ and taking values in $[0, +\infty]$. Then

$$\begin{aligned}
&\big(\rho_1(x,y)^\beta + \cdots + \rho_N(x,y)^\beta\big)^{1/\beta} \\
&\quad \le \Big[\big(\rho_1(x,z)^\alpha + \rho_1(z,y)^\alpha\big)^{\beta/\alpha} + \cdots + \big(\rho_N(x,z)^\alpha + \rho_N(z,y)^\alpha\big)^{\beta/\alpha}\Big]^{1/\beta} \\
&\quad =: \big[(a_1 + b_1)^p + \cdots + (a_N + b_N)^p\big]^{1/\beta}, \qquad (2.36)
\end{aligned}$$

where we have set $a_i := \rho_i(x,z)^\alpha$, $y_i := \rho_i(z,y)^\alpha$, where $1 \le i \le N$, and $p := \beta/\alpha$. Since $p \ge 1$ and $a_i, b_i \ge 0$, the discrete version of Minkowski's Inequality gives

$$\big[(a_1 + b_1)^p + \cdots + (a_N + b_N)^p\big]^{1/p} \le (a_1^p + \cdots a_N^p)^{1/p} + (b_1^p + \cdots b_N^p)^{1/p}. \qquad (2.37)$$

Using this back in (2.36) and unraveling notation then yields

$$\begin{aligned}
&\big(\rho_1(x,y)^\beta + \cdots + \rho_N(x,y)^\beta\big)^{\frac{1}{\beta}} \\
&\quad \le \Big[\big(\rho_1(x,z)^\beta + \cdots + \rho_N(x,z)^\beta\big)^{\frac{\alpha}{\beta}} + \big(\rho_1(z,y)^\beta + \cdots + \rho_N(z,y)^\beta\big)^{\frac{\alpha}{\beta}}\Big]^{\frac{1}{\alpha}}, (2.38)
\end{aligned}$$

which shows that $\left(\sum_{i=1}^N \rho_i^\beta\right)^{1/\beta}$ is an α-subadditive function. Taking the limit as N tends to infinity then completes the proof of (7).

Regarding the claim made in (8), the difference in $\rho_\alpha \circ (\phi, \phi)$ and $(\rho \circ (\phi, \phi))_\alpha$ is the set we are taking the infimum over.

Or, thinking of the α-subadditive regularization process as an operator, the α in $\rho_\alpha \circ (\phi, \phi)$ sends a function defined on $X \times X$ to a function defined on $X \times X$ while the

α in $(\rho \circ (\phi, \phi))_\alpha$ sends a function defined on $Y \times Y$ to a function defined on $Y \times Y$.

By definition, we have for $y_1, y_2 \in Y$

$$\rho_\alpha \circ (\phi(y_1), \phi(y_2)) = \\ \inf\Big\{ \Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha \Big)^{\frac{1}{\alpha}} : N \in \mathbb{N},\, \phi(y_1) = \xi_1, \ldots, \xi_{N+1} = \phi(y_2) \in X \Big\} \tag{2.39}$$

while

$$(\rho \circ (\phi, \phi))_\alpha(y_1, y_2) = \\ \inf\Big\{ \Big(\sum_{i=1}^{N} \rho(\phi(\eta_i), \phi(\eta_{i+1}))^\alpha \Big)^{\frac{1}{\alpha}} : N \in \mathbb{N},\, y_1 = \eta_1, \ldots, \eta_{N+1} = y_2 \in Y \Big\} \tag{2.40}$$

In the infimum in (2.39), ρ is allowed inputs from all of X, while in the infimum in (2.40), ρ's inputs only come from $\phi(Y)$. Since $\phi(Y) \subseteq X$, we have the following inequality: $\rho_\alpha \circ (\phi(y_1), \phi(y_2)) \leq \big(\rho \circ \phi\big)_\alpha(y_1, y_2)$.

If we know that ϕ is onto, or $\phi(Y) = X$, then every $\rho(\xi_i, \xi_{i+1})$ corresponds to (at least) one $\rho(\phi(\eta_i), \phi(\eta_{i+1}))$. This implies the infimums are equal. This finishes the proof of (8).

The claim made in part (9) of Lemma 2.3 follows by unraveling the definition of ρ_α.

And this completes the proof of Lemma 2.3. $\square$

For the purpose of comparing various functions we now introduce the following equivalence relation.

Definition 2.4. *Let X be a nonempty set. Call $\rho_1, \rho_2 : X \times X \to [0, +\infty]$* `equivalent`*, and write $\rho_1 \approx \rho_2$, if there exist $C', C'' \in (0, +\infty)$ with the property that*

$$C'\rho_1(x, y) \leq \rho_2(x, y) \leq C''\rho_1(x, y), \qquad \forall\, x, y \in X. \tag{2.41}$$

Remark 2.2. *For any two functions* $\rho_1, \rho_2 : X \times X \to [0, +\infty]$, *we have that* $\rho_1 \approx \rho_2$ *if and only if there exists* $\eta : X \times X \to (C', C'')$, *where* $C', C'' \in (0, +\infty)$, *with the property that*

$$\rho_2(x, y) = \eta(x, y)\rho_1(x, y), \qquad \forall\, x, y \in X. \tag{2.42}$$

Our next theorem, which should be contrasted to (2.27), shows that if ρ is a real-valued, nonnegative function defined on a nonempty set X cross itself which, for some $\alpha \in (0, +\infty]$, is equivalent (in the sense of Definition 2.4) with its α-subadditive regularization ρ_α (cf. Definition 2.3) then ρ is also α-quasi-subadditive or, equivalently, quasi-subadditive (cf. Lemma 2.2, part (1)). Also, in the converse direction, if ρ is quasi-subadditive then ρ is equivalent with its α-subadditive regularization ρ_α for a judicious choice of the index α (depending on the constant involved in the quasi-subadditivity condition satisfied by the function ρ). The latter result is of paramount importance for our work.

Theorem 2.4. *Let* X *be a nonempty set.*

(I) Assume that $\rho : X \times X \to [0, +\infty]$ *is a function which is quasi-subadditive on* $X \times X$ *in the sense of* (2.4). *That is, that there exists a finite constant* $C_1 \geq 1$ *with the property that*

$$\rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\}, \quad \textit{for all} \ \ x, y, z \in X. \tag{2.43}$$

Introduce

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty] \tag{2.44}$$

and define the function $\rho_\alpha : X \times X \to [0,+\infty]$ as in Definition 2.3.

Then $\rho_\alpha \approx \rho$. More specifically, with C_1 the same constant as in (2.4),

$$C_1^{-2}\rho \leq \rho_\alpha \leq \rho \qquad \text{on } \ X \times X. \tag{2.45}$$

In particular,

$$(\rho_\alpha)^{-1}(\{0\}) = \rho^{-1}(\{0\}). \tag{2.46}$$

More generally, if $\beta \in (0,\alpha]$ then

$$2^{-2/\beta}\rho \leq \rho_\beta \leq \rho \qquad \text{on } \ X \times X. \tag{2.47}$$

(II) Conversely, if $\rho : X \times X \to [0,+\infty]$ is a function for which there exist some $\alpha \in (0,+\infty]$ and some finite constant $C \geq 1$ with the property that

$$\rho \leq C\,\rho_\alpha \quad \text{on } \ X \times X, \tag{2.48}$$

(hence $\rho \approx \rho_\alpha$ since the estimate $\rho_\alpha \leq \rho$ is always true) then ρ satisfies the estimate (2.4) *for the choice $C_1 := C2^{1/\alpha}$ (hence, ρ is quasi-subadditive).*

Proof. Consider the claims made in part (I) of the theorem. The case $C_1 = 1$, corresponding to $\alpha = +\infty$, is immediate from Lemma 2.3, therefore we shall assume in what follows that $\alpha < +\infty$. From (2.22) we know that $\rho_\alpha(x,y) \leq \rho(x,y)$ for all $x,y \in X$. In the opposite direction, the idea is to apply Lemma 2.1 to the function $(\rho)^\alpha$. Note that the choice (2.44) ensures that $C_1^\alpha = 2$ and, hence, $(\rho)^\alpha$ satisfies an inequality as described in (2.5). This allows to conclude that

$$\big[\rho(x,y)\big]^\alpha \leq 4\sum_{i=1}^{N}\big[\rho(\xi_i,\xi_{i+1})\big]^\alpha \tag{2.49}$$

each time $N \in \mathbb{N}$ and $x = \xi_1, \dots, \xi_{N+1} = y \in X$. In particular, this implies that

$$\rho(x,y) \leq 4^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{N} \big[\rho(\xi_i, \xi_{i+1})\big]^{\alpha} \Big)^{\frac{1}{\alpha}}. \tag{2.50}$$

Starting with arbitrary $x, y \in X$ and then taking the infimum in (2.50) over all $N \in \mathbb{N}$ and $\xi_1, \dots, \xi_{N+1} \in X$ satisfying $x = \xi_1, \dots, \xi_{N+1} = y$, allows us to conclude that

$$\rho(x,y) \leq 4^{\frac{1}{\alpha}} \rho_\alpha(x,y) = C_1^2 \rho_\alpha(x,y), \qquad \forall\, x, y \in X. \tag{2.51}$$

This completes the proof of (2.45). Note that (2.46) readily follows from (2.45). To see (2.47), observe that if $\beta \in (0, \alpha]$ then $C_1 \leq 2^{1/\beta}$. Hence, we have the inequality that $\rho(x,y) \leq 2^{1/\beta} \max\{\rho(x,z), \rho(z,y)\}$ for all $x, y, z \in X$. Consequently, (2.45) may be used with α replaced by β and C_1 replaced by $2^{1/\beta}$. This yields (2.47) and finishes the proof of (I).

Finally, regarding (II), note that based on (2.48), (2.26) and (2.22), for each elements $x, y, z \in X$ we may write

$$\begin{aligned} \rho(x,y) &\leq C\rho_\alpha(x,y) \\ &\leq C2^{1/\alpha} \max\big\{\rho_\alpha(x,z), \rho_\alpha(z,y)\big\} \\ &\leq C2^{1/\alpha} \max\big\{\rho(x,z), \rho(z,y)\big\}, \end{aligned} \tag{2.52}$$

hence ρ satisfies a quasi-subadditivity condition with $C_1 := C2^{1/\alpha}$. $\square$

Remark 2.3. *In the context of the first part of Theorem 2.4, if in place of the quasi-subadditivity condition* (2.4) *the function $\rho : X \times X \to [0, +\infty]$ is assumed to have the property that there exist $p \in (0, +\infty)$ and $C \geq 2^{-1/p}$ such that*

$$\rho(x,y) \leq C(\rho(x,z)^p + \rho(z,y)^p)^{1/p}, \quad \textit{for all} \;\; x, y, z \in X, \tag{2.53}$$

i.e., ρ *satisfies a* p*-quasi-subadditivity condition, then for any* $x, y, z \in X$,

$$\rho(x,y) \leq C\Big([\rho(x,z)]^p + [\rho(z,y)]^p\Big)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} C \max\{\rho(x,z), \rho(z,y)\} \tag{2.54}$$

which means that estimate (2.4) *holds for the choice* $C_1 := 2^{\frac{1}{p}}C$. *Note that we have the equality* $\big(\log_2[2^{\frac{1}{p}}C]\big)^{-1} = p/(1 + p\log_2 C)$. *Hence, part* ($I$) *in Theorem 2.4 gives that* $\rho \approx \rho_\alpha$ *where, instead of* (2.44), *this time we take* α *to be*

$$\frac{p}{1 + p\,\log_2 C} \in (0, +\infty]. \tag{2.55}$$

Recall the notions of quasi-symmetry and symmetry defined in and immediately after Definition (2.1). It is now our goal to see how these two properties interact with the different regularization procedures (in the sense of Definition 2.3). We begin by making the following definition.

Definition 2.5. *Let* X *be a nonempty set and* $\rho : X \times X \to [0, +\infty]$. *Then define the function* $\rho_\iota : X \times X \to [0, +\infty]$ *by* $\rho_\iota(x,y) = \rho(y,x)\ \forall\, x, y \in X$.

The subscript iota in this definition comes from the word `involution`.

Lemma 2.5. *Fix* X *a nonempty set,* $\rho : X \times X \to [0, +\infty]$ *and let* ρ_ι *be as in Definition* (2.5). *Prove the following:*

(1) ρ *is symmetric if and only if* $\rho = \rho_\iota$.

(2) $(\rho_\iota)_\iota = \rho$.

(3) $\rho_\iota \approx \rho$ *if and only if* ρ *is quasi-symmetric.*

(4) ρ_ι *is a quasi-metric if and only if* ρ *is a quasi-metric.*

Proof. All claims are straightforward consequences of definitions. $\square$

In the next lemma we introduce a certain symmetrization procedure of an arbitrary function and study some of its most basic properties. An alternative symmetrization process will be introduced later, in Definition (3.4).

Lemma 2.6. *Suppose that X is a nonempty set and $\rho : X \times X \to [0, +\infty]$. Consider the function $\rho_{sym} : X \times X \to [0, +\infty]$, ρ's* `max-symmetrization`*, defined by*

$$\rho_{sym}(x, y) := \max\{\rho(x, y), \rho_\iota(x, y)\}, \quad \forall\, x, y \in X. \tag{2.56}$$

Then the following properties hold.

(1) The function ρ_{sym} is symmetric.

(2) The function ρ is quasi-symmetric if and only if there exists another function $\rho' : X \times X \to [0, +\infty]$ which is symmetric and such that $\rho \approx \rho'$.

(3) The function ρ_{sym} may be characterized as the smallest $[0, +\infty]$-valued function defined on $X \times X$ that is symmetric and which is $\geq \rho$ pointwise on $X \times X$.

(4) If ρ is quasi-subadditive on $X \times X$ it follows that so is ρ_{sym}, and with the same constant. More precisely, if ρ satisfies (2.4) for some finite constant $C_1 \geq 1$ then also

$$\rho_{sym}(x, y) \leq C_1 \max\{\rho_{sym}(x, z), \rho_{sym}(z, y)\}, \quad \textit{for all } \ x, y, z \in X. \tag{2.57}$$

Proof. For every $x, y \in X$ we may write

$$\rho_{sym}(x, y) = \max\big\{\rho(x, y), \rho(y, x)\big\} = \max\big\{\rho(y, x), \rho(x, y)\big\} = \rho_{sym}(y, x). \tag{2.58}$$

Hence, ρ_{sym} is symmetric, proving (1).

Both directions in the claim made in part (2) follow by definitions.

To prove the claim in part (3), observe that, on the one hand, ρ_{sym} is symmetric and satisfies $\rho_{sym} \geq \rho$ on $X \times X$.

On the other hand, given a function $\rho' : X \times X \to [0, +\infty]$ which is symmetric and which satisfies $\rho'(x, y) \geq \rho(x, y)$ for all $x, y \in X$, we have $\rho'(x, y) = \rho'(y, x) \geq \rho(y, x)$ for all $x, y \in X$. Thus, ultimately, $\rho'(x, y) \geq \max\{\rho(x, y), \rho(y, x)\} = \rho_{sym}(x, y)$ for all $x, y \in X$. Thus, $\rho' \geq \rho_{sym}$ on $X \times X$, as wanted.

Finally, concerning (4), we know from part (4) of Lemma 2.5 if ρ is quasi-subadditive then so is ρ_ι. Lemma 2.2, part (4) then tells us ρ_{sym} must be quasi-subadditive, but this does not tell us anything about the constants involved. Taking a different approach, if ρ satisfies the quasi-subadditivity condition (2.4) for some finite constant $C_1 \geq 1$ then for all $x, y, z \in X$ we may write

$$
\begin{aligned}
\rho_{sym}(x, y) &= \max\big\{\rho(x, y), \rho(y, x)\big\} \\
&\leq C_1 \max\Big\{\max\{\rho(x, z), \rho(z, y)\}, \max\{\rho(y, z), \rho(z, x)\}\Big\} \\
&= C_1 \max\Big\{\max\{\rho(x, z), \rho(z, x)\}, \max\{\rho(z, y), \rho(y, z)\}\Big\} \\
&= C_1 \max\big\{\rho_{sym}(x, z), \rho_{sym}(z, y)\big\}. \qquad (2.59)
\end{aligned}
$$

As a result, ρ_{sym} satisfies the same estimate as ρ in (2.4) and, as claimed, with the same constant C_1 as in (2.4). This finishes the proof of the lemma. $\square$

We next discuss how the regularization procedure from Definition 2.3 interacts with the above notions of symmetry and quasi-symmetry.

Lemma 2.7. *Assume that X is a nonempty set and fix an index $\alpha \in (0, +\infty]$. Then for any function $\rho : X \times X \to [0, +\infty]$ the following claims are valid:*

(1) There holds $(\rho_\alpha)_\iota = (\rho_\iota)_\alpha$ on $X \times X$. In particular, if ρ is symmetric then so is ρ_α.

(2) If ρ is quasi-symmetric then so is ρ_α. More precisely, if $C_0 \geq 0$ is such that $\rho_\iota \leq C_0\, \rho$ on $X \times X$ then $(\rho_\alpha)_\iota \leq C_0\, \rho_\alpha$ on $X \times X$.

Proof. First we will consider the case when $\alpha < +\infty$ since the argument for $\alpha = +\infty$ is similar. Thus, assuming that $\alpha < +\infty$, for all $x, y \in X$ we have

$$\begin{aligned}
(\rho_\alpha)_\iota(x,y) &= \inf\Big\{ \Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha \Big)^{\frac{1}{\alpha}} : N \in \mathbb{N},\, \xi_1, \dots, \xi_N \in X,\ y = \xi_1, x = \xi_{N+1} \Big\} \\
&= \inf\Big\{ \Big(\sum_{i=1}^{N} (\rho_\iota)(\xi_{i+1}, \xi_i)^\alpha \Big)^{\frac{1}{\alpha}} : N \in \mathbb{N},\, \xi_1, \dots, \xi_{N+1} \in X,\ x = \xi_{N+1}, \cdots y = \xi_1 \Big\} \\
&= (\rho_\iota)_\alpha(x,y), \qquad (2.60)
\end{aligned}$$

proving (1). Next, if ρ is quasi-symmetric then there exists $C_0 \geq 0$ such that $\rho_\iota \leq C_0\, \rho$ on $X \times X$. In light of (2.22)-(2.23), and given what we have proved so far, this entails

$$(\rho_\alpha)_\iota = (\rho_\iota)_\alpha \leq (C_0\, \rho)_\alpha = C_0\, \rho_\alpha, \qquad (2.61)$$

proving (2). The case $\alpha = +\infty$ follows similarly, thus the proof of the lemma is therefore complete. □

Our next result is a version of Theorem 2.4 in which, given a quasi-subadditive function ρ defined on a non-empty set X cross itself, we seek to construct an equivalent, symmetric, regularized version of it. As part (2) of Lemma 2.6 shows, for this type of conclusion it is actually necessary to assume that ρ is quasi-symmetric.

Theorem 2.8. *Let X be a non-empty set and $\rho : X \times X \to [0, +\infty]$ be a function which is quasi-symmetric and has the property that there exists $C_1 \geq 1$ such that* (2.4) *holds. Finally, define the exponent $\alpha \in (0, +\infty]$ as in* (2.44).

Then the function $(\rho_{sym})_\alpha$, defined according with the recipe from Definition 2.3 (but using ρ_{sym} in place of ρ), satisfies the following properties.

(i) $(\rho_{sym})_\alpha$ *is symmetric.*

(ii) $(\rho_{sym})_\alpha \approx \rho$. *More specifically, with C_1 the same constant as in* (2.4) *and with C_0 the constant appearing in* (2.2), *one has*

$$C_1^{-2}\rho \leq (\rho_{sym})_\alpha \leq \max\{1, C_0\}\,\rho \quad \text{on} \quad X \times X. \tag{2.62}$$

Consequently,

$$(\rho_{sym})_\alpha^{-1}(\{0\}) = \rho^{-1}(\{0\}). \tag{2.63}$$

(iii) For each $\beta \in (0, \alpha]$ the function $(\rho_{sym})_\alpha$ is β-subadditive. That is, one has (with a natural interpretation when $\beta = \alpha = +\infty$),

$$(\rho_{sym})_\alpha(x, y) \leq \Big([(\rho_{sym})_\alpha(x, z)]^\beta + [(\rho_{sym})_\alpha(z, y)]^\beta\Big)^{\frac{1}{\beta}}, \ \forall\, x, y, z \in X. \tag{2.64}$$

(iv) The function $(\rho_{sym})_\alpha$ is quasi-subadditive. More precisely, for the same constant C_1 as in (2.4), *one has, for all $x, y, z \in X$,*

$$(\rho_{sym})_\alpha(x, y) \leq C_1 \max\big\{(\rho_{sym})_\alpha(x, z)\,,\,(\rho_{sym})_\alpha(z, y)\big\}. \tag{2.65}$$

Proof. From (1) and (4) in Lemma 2.6 we know that ρ_{sym} is symmetric and that it satisfies the same estimate as ρ in (2.4), with the same constant C_1. Based on this, part (1) in Lemma 2.7 then gives that $(\rho_{sym})_\alpha$ is also symmetric, as claimed in (i). Also,

$$C_1^{-2}\rho \leq C_1^{-2}\rho_{sym} \leq (\rho_{sym})_\alpha \leq \rho_{sym} \leq \max\{1, C_0\}\,\rho \quad \text{on} \quad X \times X, \tag{2.66}$$

where the second inequality comes from (2.45), the first and are by the definition of ρ_{sym} and $\rho_\alpha \leq \rho$ always. This proves (2.62) in (ii). Finally, (iii) is a direct consequence of part (3) in Lemma 2.3 while (iv) is clear from part (4) in Lemma 2.3.

$\square$

2.2 Construction of a Metric from a Quasi-metric

Before moving onto the next result, we must discuss the notion of a topology induced by a quasi-metric. Given a non-empty set X and a quasi-distance ρ, ρ naturally induces a topology τ_ρ on X. Specifically, the collection $\mathscr{O}_\rho$ of open sets in this topology can be described as

$$\begin{aligned} O \in \mathscr{O}_\rho \overset{def}{\Longleftrightarrow} O \subseteq X \text{ and } \forall\, x \in O\ \exists\, r > 0 \text{ such that} \\ B_\rho(x,r) := \{y \in X:\ \rho(x,y) < r\} \subseteq O. \end{aligned} \tag{2.67}$$

Now that we have developed sufficient background we are in a position to construct a new function out of our ρ_α which is more akin to a metric. We start by imposing as few conditions as possible as our starting function ρ, then gradually demand more of ρ and ultimately create a function which is strongly related (not quite equivalent) to a distance (the upcoming d in $d_{\rho,\beta}$ is for `distance`).

Theorem 2.9. *Let X be a non-empty set and suppose that $\rho : X \times X \to [0, +\infty]$ is a function for which there exists $C_1 \geq 1$ such that (2.4) holds. Define α as in (2.44) and construct ρ_α as in Definition 2.3. Then, for any finite $\beta \in (0, \alpha]$, the function*

$$d_{\rho,\beta} : X \times X \to [0, +\infty], \qquad d_{\rho,\beta}(x, y) := \big[\rho_\alpha(x, y)\big]^\beta, \qquad \forall\, x, y \in X, \tag{2.68}$$

satisfies the following properties.

(i) $d_{\rho,\beta}$ *satisfies the triangle inequality (in the sense of 2.3 with $C = 1$).*

(ii) If ρ is quasi-symmetric, i.e., there exists $C_0 \geq 0$ for which

$$\rho(x, y) \leq C_0\, \rho(y, x), \qquad \forall\, x, y \in X, \tag{2.69}$$

then

$$d_{\rho,\beta}(x, y) \leq C_0^\beta\, d_{\rho,\beta}(y, x) \quad \text{for all } x, y \in X. \tag{2.70}$$

In fact,

$$\text{if } \rho \text{ is symmetric then } d_{\rho,\beta} \text{ is symmetric}, \tag{2.71}$$

(iii) One has

$$C_1^{-2}\rho(x, y) \leq \big[d_{\rho,\beta}(x, y)\big]^{1/\beta} \leq \rho(x, y), \quad \forall\, x, y \in X. \tag{2.72}$$

(iv) Given $diag(X) := \{(x, x) : x \in X\}$,

$$\begin{aligned} diag(X) = \rho^{-1}(\{0\}) &\iff d_{\rho,\beta} \text{ is nondegenerate} \\ \text{and} \quad diag(X) \subseteq \rho^{-1}(\{0\}) &\iff d_{\rho,\beta} \text{ is pseudo-nondegenerate}, \end{aligned} \tag{2.73}$$

with nondegeneracy understood in the sense of Definition 2.1. A function which is pseudo-nondegenerate is allowed to vanish off of the diagonal.

(v) If ρ is symmetric and $\operatorname{diag}(X) \subseteq \rho^{-1}(\{0\})$*, then $d_{\rho,\beta}$ is a pseudo-distance on X and the $\tau_{d_{\rho,\beta}}$, the topology induced by $d_{\rho,\beta}$ on X coincides with τ_ρ, the topology induced by ρ. Furthermore, if in fact* $\operatorname{diag}(X) = \rho^{-1}(\{0\})$ *then $d_{\rho,\beta}$ is actually a distance on X.*

Proof. Assume that $x, y, z \in X$. Then, using the definition of ρ_α,

$$\begin{aligned} d_{\rho,\beta}(x,y) &= \big[\rho_\alpha(x,y)\big]^\beta \\ &\leq \big[\rho_\alpha(x,z)\big]^\beta + \big[\rho_\alpha(z,y)\big]^\beta \\ &= d_{\rho,\beta}(x,z) + d_{\rho,\beta}(z,y). \end{aligned} \tag{2.74}$$

This proves (i).

Suppose next that ρ satisfies (2.69). If $x, y \in X$ then

$$d_{\rho,\beta}(x,y) = \big[\rho_\alpha(x,y)\big]^\beta \leq \; C_0^\beta \big[\rho_\alpha(y,x)\big]^\beta = C_0^\beta \, d_{\rho,\beta}(y,x) \tag{2.75}$$

where the inequality above uses part (2) in Lemma 2.7. This proves (2.70). If ρ is actually symmetric then, from part (1) in Lemma 2.7 we deduce that ρ_α is also symmetric. Granted this and repeating the argument in (2.75) (with $C_0 = 1$) twice, once to show $d_{\rho,\beta}(x,y) \leq d_{\rho,\beta}(y,x)$ and once for $d_{\rho,\beta}(y,x) \leq d_{\rho,\beta}(x,y)$ completes the proof of (ii). Going further, (iii) follows directly from (2.45).

Regarding (iv), note that (iii) gives us $\big(d_{\rho,\beta}\big)^{\frac{1}{\beta}} \approx \rho$. As equivalent functions vanish on the same set, supposing $\operatorname{diag}(X) = \rho^{-1}(\{0\})$ immediately implies $d_{\rho,\beta}(x,y) = 0$ if and only if $x = y$, hence $d_{\rho,\beta}$ is nondegenerate by definition.

Conversely, assuming $d_{\rho,\beta}$ is nondegenerate is tantamount to $d_{\rho,\beta}^{-1}(\{0\}) = \operatorname{diag}(X)$, and as again equivalent functions share the same kernel, we conclude that the preimage of 0 $\rho^{-1}(\{0\}) = \operatorname{diag}(X)$, proving the first part of (iv).

The second claim in (2.73) follows in a straightforward manner by definitions.

Part (v) of the theorem also follows by simply double inclusion. Recall given $O \subseteq X$, to have $O \in \tau_\rho$, we must have $\forall\, x \in O\ \exists\, r > 0$ such that $B_\rho(x, r) \subset O$, where $B_\rho(x, r)$, the ρ-ball centered at x of radius r, is defined as $\{y \in X : \rho(x, y) < r\}$ for any $x \in X$ and $r > 0$. $\square$

The next result in this section pertains to the (Hölder-type) regularity of the regularization (in the sense of Definition 2.3) of a nonnegative, symmetric, quasi-subadditive, nondegenerate function defined on a nonempty set cross itself.

Theorem 2.10. *Let X be a nonempty set and assume that $\rho : X \times X \to [0, +\infty)$ is a function satisfying the following two properties:*

$$\text{there exists } C_1 \in [1, +\infty) \text{ such that (2.4) holds, and} \tag{2.76}$$

$$\text{the function } \rho \text{ is symmetric.} \tag{2.77}$$

Lastly, define α as in (2.44). Then for each exponent $\beta \in (0, \min\{1, \alpha\}]$, the function ρ_α, constructed as in Definition 2.3, satisfies the following local Hölder regularity condition of order β: for each $r > 0$ and $x, y, z \in X$ such that

$$\max\big\{\rho_\alpha(x, z), \rho_\alpha(x, y)\big\} \leq r, \tag{2.78}$$

one has

$$\big|\rho_\alpha(x, z) - \rho_\alpha(x, y)\big| \leq C\big[\rho_\alpha(z, y)\big]^\beta. \tag{2.79}$$

If, on the other hand, $1 < \beta \leq \alpha$, then (2.79) holds whenever $x, y, z \in X$ and

$r > 0$ *are such that*

$$\min\big\{\rho_\alpha(x,z), \rho_\alpha(x,y)\big\} \geq r. \tag{2.80}$$

Proof. Fix α as in (2.44) and $\beta \in (0, \alpha]$. Let $x, y, z \in X$ be arbitrary. The triangle inequality and the symmetry condition for $d_{\rho,\beta}$ then yield

$$\begin{aligned} &d_{\rho,\beta}(x,y) \leq d_{\rho,\beta}(x,z) + d_{\rho,\beta}(z,y), \qquad \text{and} \\ &d_{\rho,\beta}(x,z) \leq d_{\rho,\beta}(x,y) + d_{\rho,\beta}(y,z) = d_{\rho,\beta}(x,y) + d_{\rho,\beta}(z,y). \end{aligned} \tag{2.81}$$

In concert, the inequalities in (2.81) further imply

$$\big|d_{\rho,\beta}(x,y) - d_{\rho,\beta}(x,z)\big| \leq d_{\rho,\beta}(z,y). \tag{2.82}$$

An elementary fact which is useful in this context is the estimate

$$|a^\gamma - b^\gamma| \leq \gamma\,|a-b|\,\big[\max\{a,b\}\big]^{\gamma-1} \quad \text{ if } a, b \in [0,+\infty) \text{ and } \gamma \geq 1. \tag{2.83}$$

Writing (2.83) for $a := \big[\rho_\alpha(x,y)\big]^\beta$, $b := \big[\rho_\alpha(x,z)\big]^\beta$ and $\gamma := 1/\beta \geq 1$ yields

$$\begin{aligned} &\big|\rho_\alpha(x,y) - \rho_\alpha(x,z)\big| \\ &\qquad \leq \tfrac{1}{\beta}\,\big|d_{\rho,\beta}(x,y) - d_{\rho,\beta}(x,z)\big|\Big[\max\big\{d_{\rho,\beta}(x,y), d_{\rho,\beta}(x,z)\big\}\Big]^{\frac{1}{\beta}-1} \\ &\qquad \leq \tfrac{1}{\beta}\,d_{\rho,\beta}(z,y)\Big[\max\big\{\rho_\alpha(x,y), \rho_\alpha(x,z)\big\}\Big]^{1-\beta} \\ &\qquad = \tfrac{1}{\beta}\big[\rho_\alpha(z,y)\big]^\beta\Big[\max\big\{\rho_\alpha(x,y), \rho_\alpha(x,z)\big\}\Big]^{1-\beta}, \end{aligned} \tag{2.84}$$

where for the second inequality in (2.84) we have used (2.82). Based on this, (2.79) follows whenever (2.78) holds. Finally, the last claim in the statement of the theorem is proved in a similar fashion, this time making use of the fact that

$$|a^\gamma - b^\gamma| \leq \gamma\,|a-b|\,\big[\min\{a,b\}\big]^{\gamma-1} \quad \text{ if } a, b \in [0,+\infty) \text{ and } \gamma \in (0,1). \tag{2.85}$$

This completes the proof of the theorem. $\square$

Lemma 2.11. *The Hölder-type regularity in Theorem 2.10 is sharp in the following precise sense. Given $C_1 \in [1,+\infty)$, there exists a set X of cardinality at least 2, a symmetric quasi-distance $\rho : X \times X \to [0,+\infty)$ satisfying the quasi-subadditive condition with constant C_1, and if ρ' is such that $\rho' \approx \rho$ and there exists a $\beta \in (0,+\infty)$ and a constant $C \in [0,+\infty)$ for which*

$$\big|\rho'(x,y) - \rho'(x,z)\big| \leq C \max\big\{\rho'(x,y)^{1-\beta}, \rho'(x,z)^{1-\beta}\big\}\big[\rho'(y,z)\big]^{\beta} \tag{2.86}$$

whenever $x,y,z \in X$ (and also $x \notin \{y,z\}$ if $\beta > 1$) then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \tag{2.87}$$

Proof. Fix $C_1 \in (1,+\infty)$, let $X := \mathbb{R}$ and set $s := \log_2 C_1 \in (0,+\infty)$. Finally, define

$$\rho : \mathbb{R} \times \mathbb{R} \to [0,+\infty), \qquad \rho(x,y) := |x-y|^s, \quad \forall\, x,y \in \mathbb{R}. \tag{2.88}$$

The choice of s is designed so that this function satisfies (2.1),(2.2) and (2.4) for the given C_1 and with $C_0 = 1$. Assume now that $\rho' : \mathbb{R} \times \mathbb{R} \to [0+\infty)$ is a function such that $\rho' \approx \rho$ and there exist $\beta \in (0,+\infty)$ and $C \in [0,+\infty)$ for which (2.86) holds.

Fix $x,y \in \mathbb{R}$ arbitrary and $z \in \mathbb{R}$ such that $y \neq z$. Using the equivalence of ρ and ρ' in (2.86) (with the understanding that we also assume $x \notin \{y,z\}$ if $\beta > 1$) yields

$$\begin{aligned} \big|\rho'(x,y) - \rho'(x,z)\big| &\leq C \max\big\{\rho'(x,y)^{1-\beta}, \rho'(x,z)^{1-\beta}\big\}\big[\rho'(y,z)\big]^{\beta} \\ &\leq \widetilde{C} \max\big\{|x-y|^{s(1-\beta)}, |x-z|^{s(1-\beta)}\big\}|y-z|^{s\beta}, \end{aligned} \tag{2.89}$$

where the constant in the second line of (2.89) stems from the constant in the first line and the equivalence of ρ and ρ'. As $y \neq z$ we may divide through by $|y-z|$, resulting in

$$\frac{\big|\rho'(x,y) - \rho'(x,z)\big|}{|y-z|} \leq \widetilde{C} \max\big\{|x-y|^{s(1-\beta)}, |x-z|^{s(1-\beta)}\big\}|y-z|^{s\beta-1}. \tag{2.90}$$

Suppose $s\beta > 1$, hence $s\beta - 1 > 0$. For every fixed x and y, as z tends to y we have that $\max\{|x-y|^{s(1-\beta)}, |x-z|^{s(1-\beta)}\}$ approaches $|x-y|^{s(1-\beta)}$, which is finite. Since $s\beta - 1 > 0$, it follows that $|y-z|^{s\beta-1} \to 0$ as $z \to y$. Thus

$$\lim_{z\to y} \frac{\big|\rho'(x,y) - \rho'(x,z)\big|}{|y-z|} = 0, \quad \text{hence} \quad \lim_{z\to y} \frac{\rho'(x,y) - \rho'(x,z)}{y-z} = 0, \tag{2.91}$$

for every x and y in $\mathbb{R}$. In turn, this implies that $\rho'(x,\cdot)$ is differentiable and its derivate is 0. In particular $\rho'(x,\cdot)$ is constant for any $x \in \mathbb{R}$. This contradicts the fact that $\rho'(x,y) \approx |x-y|^s \to +\infty$ as $y \to +\infty$, so our supposition that $s\beta > 1$ is not valid. Hence, necessarily, $\beta \leq 1/s$, i.e., (2.87) holds.

When $C_1 = 1$, in which case we take $s := 0$, there is nothing to prove as we are demanding $\beta \leq +\infty$. $\square$

After so much developmental work, we are finally able to present the heart of this chapter, an absolutely crucial tool in the development of quasi-metric geometry.

Theorem 2.12. *Assume that X is a nonempty set. Given $\rho : X \times X \to [0, +\infty]$ and an arbitrary exponent $\alpha \in (0, +\infty]$ define the function*

$$\rho_\alpha : X \times X \longrightarrow [0, +\infty] \tag{2.92}$$

by setting for each $x, y \in X$

$$\rho_\alpha(x,y) := \inf\Big\{\Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X \tag{2.93}$$
$$\text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\},$$

whenever $\alpha < +\infty$ and its natural, endpoint counterpart corresponding to the case *when* $\alpha = +\infty$, *i.e.*,

$$\rho_\infty(x,y) := \inf\Big\{\max_{1\le i\le N}\rho(\xi_i,\xi_{i+1}) :\ N\in\mathbb{N} \text{ and } \xi_1,\dots,\,\xi_{N+1}\in X \tag{2.94}$$
$$\text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\},$$

Then the following conclusions hold for any function $\rho : X\times X \to [0,+\infty]$ *and any exponent* $\alpha\in(0,+\infty]$*:*

(1) For every $\alpha\in(0,+\infty]$ *one has:*

$$\rho_\alpha \text{ is well-defined and } \rho_\alpha \le \rho \text{ on } X\times X, \tag{2.95}$$

$$\beta\in(0,\alpha] \text{ finite} \Rightarrow \rho_\alpha(x,y) \le \Big([\rho_\alpha(x,z)]^\beta + [\rho_\alpha(z,y)]^\beta\Big)^{\frac{1}{\beta}}, \quad \forall\, x,y,z\in X, \tag{2.96}$$

$$\beta=\alpha=+\infty \ \Rightarrow\ \rho_\infty(x,y) \le \max\big\{\rho_\infty(x,z)\,,\,\rho_\infty(z,y)\big\}, \quad \forall\, x,y,z\in X, \tag{2.97}$$

$$\rho_\alpha(x,y) \le 2^{1/\alpha}\max\big\{\rho_\alpha(x,z)\,,\,\rho_\alpha(z,y)\big\}, \quad \forall\, x,y,z\in X, \tag{2.98}$$

$$\rho=\rho_\alpha \Leftrightarrow \rho\ \alpha\text{-subadditive: } \rho(x,y) \le \big(\rho(x,z)^\alpha + \rho(z,y)^\alpha\big)^{\frac{1}{\alpha}},\ \forall\, x,y,z\in X, \tag{2.99}$$

$$(\rho_\alpha)_\beta = \rho_\alpha \quad \text{ whenever } \ \beta\in(0,\alpha], \tag{2.100}$$

$$(\rho^\beta)_\alpha = (\rho_{\alpha\beta})^\beta \quad \text{ whenever } \ \beta\in(0,+\infty). \tag{2.101}$$

(2) If ρ_ι *is defined by* $\rho_\iota(x,y) := \rho(y,x)$ *for every* $x,y\in X$*, then*

$$(\rho_\alpha)_\iota = (\rho_\iota)_\alpha \quad \text{ on } \ X\times X, \tag{2.102}$$

$$\rho \text{ symmetric} \Longrightarrow \rho_\alpha \text{ symmetric}, \tag{2.103}$$

$$\rho \text{ quasi-symmetric} \Longrightarrow \rho_\alpha \text{ quasi-symmetric}. \tag{2.104}$$

(3) *Consider the function* $\rho_{sym} : X \times X \to [0, +\infty]$ *defined by*

$$\rho_{sym}(x, y) := \max\{\rho(x, y), \rho_{\iota}(x, y)\}, \quad \forall\, x, y \in X. \tag{2.105}$$

Then

$$\rho_{sym} \text{ is symmetric, i.e., } \rho_{sym}(x, y) = \rho_{sym}(y, x) \text{ for every } x, y \in X, \tag{2.106}$$

$$\rho \text{ is symmetric} \iff \rho = \rho_{sym}, \tag{2.107}$$

$$(\rho_{sym})_{sym} = \rho_{sym}, \tag{2.108}$$

$$(\rho_{sym})^{\beta} = (\rho^{\beta})_{sym}, \quad \forall\, \beta \in (0, +\infty). \tag{2.109}$$

Moreover, ρ_{sym} *may be characterized as the smallest* $[0, +\infty]$*-valued function defined on* $X \times X$ *which is symmetric and pointwise* $\geq \rho$.

(4) *One has* $\rho \approx \rho_{sym}$ *if and only if* ρ *is quasi-symmetric, i.e., there exists a finite constant* $C_0 \geq 0$ *such that*

$$\rho(y, x) \leq C_0\, \rho(x, y), \qquad \forall\, x, y \in X. \tag{2.110}$$

Furthermore, in the case when (2.110) *holds one actually has*

$$\rho \leq \rho_{sym} \leq \max\{1, C_0\}\, \rho. \tag{2.111}$$

(5) *The function* ρ *is quasi-symmetric if and only if there exists another function* $\rho' : X \times X \to [0, +\infty]$ *which is symmetric and such that* $\rho \approx \rho'$*. More precisely, if* ρ *satisfies* (2.110) *then so does* ρ_{α} *and with the same constant* C_0*.*

Say that $\rho : X \times X \to [0, +\infty]$ *is quasi-subadditive if there exists a finite constant* $C_1 \geq 1$ *with the property that*

$$\rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\}, \qquad \forall\, x, y, z \in X. \tag{2.112}$$

Moreover, whenever this is the case, define

$$\alpha := \frac{1}{\log_2 C_1} \in (0, +\infty]. \tag{2.113}$$

Then the following properties hold.

(6) If $\rho : X \times X \to [0, +\infty]$ *satisfies the quasi-subadditive condition* (2.112) *then its symmetrized version,* ρ_{sym} *defined as in* (2.105)*, also satisfies the quasi-subadditive condition and with the same constant as in the case of* ρ*.*

(7) If $\rho : X \times X \to [0, +\infty]$ *is quasi-subadditive and* α *is as in* (2.113) *then* $\rho_\alpha \approx \rho$*. More specifically, with* C_1 *the same constant as in* (2.112)*, one has*

$$C_1^{-2}\rho \leq \rho_\alpha \leq \rho \qquad \textit{on} \;\; X \times X. \tag{2.114}$$

In particular, ρ_α *is finite precisely when* ρ *is finite and they vanish simultaneously, i.e.,*

$$(\rho_\alpha)^{-1}(\{0\}) = \rho^{-1}(\{0\}). \tag{2.115}$$

(8) Conversely, if $\rho : X \times X \to [0, +\infty]$ *is a function for which there is* $\alpha' \in (0, +\infty]$ *and some finite constant* $C \geq 1$ *with the property that*

$$\rho \leq C\, \rho_{\alpha'} \qquad \textit{on} \;\; X \times X \tag{2.116}$$

(hence $\rho \approx \rho_{\alpha'}$ *since the estimate* $\rho_{\alpha'} \leq \rho$ *is always true), then* ρ *satisfies the quasi-subadditive condition* (2.112) *for the choice* $C_1 := C2^{1/\alpha'}$.

(9) *Assume that* $\rho : X \times X \to [0, +\infty]$ *is a function satisfying the quasi-subadditive condition* (2.112) *for some finite constant* $C_1 \geq 1$. *Define* α *as in* (2.113) *and construct* ρ_α *as described in* (2.93)-(2.94). *Finally, fix a finite number* $\beta \in (0, \alpha]$. *Then the function*

$$d_{\rho,\beta} : X \times X \to [0, +\infty], \; d_{\rho,\beta}(x, y) := \big[\rho_\alpha(x, y)\big]^\beta, \qquad \forall\, x, y \in X, \tag{2.117}$$

satisfies the following properties:

$$d_{\rho,\beta}(x, y) \leq d_{\rho,\beta}(x, z) + d_{\rho,\beta}(z, y), \qquad \forall\, x, y, z \in X, \tag{2.118}$$

$$\rho \text{ satisfies } (2.110) \Rightarrow d_{\rho,\beta}(y, x) \leq C_0^\beta\, d_{\rho,\beta}(x, y) \quad \forall\, x, y \in X, \tag{2.119}$$

$$\rho \text{ symmetric} \iff d_{\rho,\beta} \text{ symmetric}, \tag{2.120}$$

$$C_1^{-2}\rho(x, y) \leq \big[d_{\rho,\beta}(x, y)\big]^{1/\beta} \leq \rho(x, y), \quad \forall\, x, y \in X, \tag{2.121}$$

$$\rho^{-1}(\{0\}) = d_{\rho,\beta}^{-1}(\{0\}). \tag{2.122}$$

(10) *Assume that* $\rho : X \times X \to [0, +\infty)$ *is a symmetric function satisfying the quasi-subadditive condition* (2.112) *for some finite constant* $C_1 \geq 1$. *Define* α *as in* (2.113), *construct* ρ_α *as in* (2.93)-(2.94) *and, for a fixed, arbitrary, finite number* $\beta \in (0, \alpha]$, *define the function* $d_{\rho,\beta}$ *as in* (2.117). *Then*

$$\begin{gathered} \rho^{-1}(\{0\}) = \operatorname{diag}(X) \iff d_{\rho,\beta} \text{ is a distance on } X, \\ \operatorname{diag}(X) \subseteq \rho^{-1}(\{0\}) \iff d_{\rho,\beta} \text{ is a pseudo-distance on } X. \end{gathered} \tag{2.123}$$

Furthermore, if $\mathrm{diag}\,(X) \subseteq \rho^{-1}(\{0\})$ *then the pseudo-distance* $d_{\rho,\beta}$ *induces the same topology on* X *as* ρ.

(11) Assume that $\rho : X \times X \to [0, +\infty]$ *satisfies both the quasi-subadditive condition* (2.112) *and the quasi-symmetry condition* (2.110). *Introduce* ρ_{sym} *as in* (2.105) *and, with* α *as in* (2.113), *construct the canonical regularization of* ρ, *namely*

$$\rho_{\#} := (\rho_{sym})_\alpha, \tag{2.124}$$

(pronounced "rho sharp") as described in (2.93)-(2.94) *but using* ρ_{sym} *in place of* ρ. *Finally, let* $\beta \in (0, \alpha]$ *be arbitrary. Then, with* C_0 *and* C_1 *denoting the same constants as in* (2.110) *and* (2.112), *respectively, the following properties hold:*

$$\rho_{\#} \text{ is symmetric,} \tag{2.125}$$

$$\rho_{\#} \approx \rho \text{ in the precise sense that } C_1^{-2}\rho \leq \rho_{\#} \leq \max\{1, C_0\}\,\rho, \tag{2.126}$$

$$\rho_{\#}^{-1}(\{0\}) = \rho^{-1}(\{0\}), \tag{2.127}$$

$$\rho_{\#}(x,y) \leq \Big(\rho_{\#}(x,z)^\beta + \rho_{\#}(z,y)^\beta\Big)^{\frac{1}{\beta}}, \quad \forall\, x,y,z \in X, \tag{2.128}$$

$$\rho_{\#}(x,y) \leq C_1 \max\big\{\rho_{\#}(x,z)\,,\ \rho_{\#}(z,y)\big\}, \quad \forall\, x,y,z \in X, \tag{2.129}$$

$$C_1 = 1 \Longrightarrow \rho_{\#} = (\rho_{sym})_\infty = \rho_{sym}. \tag{2.130}$$

In addition, for any set Y,

$$\begin{aligned} &\text{if } \phi : Y \to X \text{ is any function mapping } Y \text{ to } X, \text{ then } \forall\, y_1, y_2, y_3 \in Y,\\ &\quad \rho(\phi(y_1), \phi(y_2)) \leq C_0 \rho(\phi(y_2), \phi(y_1)),\\ &\quad \rho(\phi(y_1), \phi(y_2)) \leq C_1 \max\big\{\rho(\phi(y_1), \phi(y_3)), \rho(\phi(y_3), \phi(y_2))\big\}\\ &\quad \text{and } \rho_{\#}(\phi,\phi) \leq \big(\rho \circ (\phi,\phi)\big)_{\#} \text{ on } Y \times Y. \end{aligned} \tag{2.131}$$

Furthermore, if ϕ is onto then $\rho_{\#}(\phi,\phi) = \big(\rho\circ(\phi,\phi)\big)_{\#}$ on $Y \times Y$.

(12) *Let $\rho : X \times X \to [0,+\infty)$ satisfy the quasi-subadditive condition* (2.112)*, non-degeneracy condition $\rho^{-1}(\{0\}) = \operatorname{diag}(X)$, and the quasi-symmetry condition* (2.110)*. Define α as in* (2.113) *and introduce $\rho_{\#}$, the canonical regularization of ρ, as in* (2.124)*.*

Then the function $\rho_{\#}$ is a quasi-distance on X (which is equivalent to ρ). In addition, the quasi-distance $\rho_{\#}$ is actually a genuine distance on X if α is in the closed interval $[1,+\infty]$ (i.e., when $C_1 \in [1,2]$) and, in fact,

$$\rho \text{ distance on } X \text{ and } \alpha = 1 \text{ (i.e., } C_1 = 2\text{)} \Longrightarrow \rho_{\#} = \rho. \tag{2.132}$$

Furthermore, if for each finite number $\beta \in (0,\alpha]$ one considers the function

$$d_{\rho_{\#},\beta} : X \times X \to [0,+\infty),\ d_{\rho_{\#},\beta}(x,y) := \big[\rho_{\#}(x,y)\big]^{\beta},\ \forall\, x,y \in X, \tag{2.133}$$

then $d_{\rho_{\#},\beta}$ is a distance on X which induces the same topology on X as ρ and which, in fact, has the property that

$$C_1^{-2}\rho(x,y) \leq \big[d_{\rho_{\#},\beta}(x,y)\big]^{1/\beta} \leq \max\{1, C_0\}\,\rho(x,y), \quad \forall\, x,y \in X. \tag{2.134}$$

Moreover, for each finite number $\beta \in \big(0,\alpha]$, the canonical regularization $\rho_{\#}$ satisfies the following Hölder-type regularity condition of order β:

$$\big|\rho_{\#}(x,y) - \rho_{\#}(x,z)\big| \leq \tfrac{1}{\beta}\max\big\{\rho_{\#}(x,y)^{1-\beta}, \rho_{\#}(x,z)^{1-\beta}\big\}\big[\rho_{\#}(y,z)\big]^{\beta} \tag{2.135}$$

whenever $x,y,z \in X$ (with the understanding that when $\beta > 1$ one also imposes the condition that $x \notin \{y,z\}$).

(13) The Hölder-type regularity result described in part (12) *above is sharp, in the following precise sense. Given a finite number $C_1 > 1$, there exist a nonempty set X and a symmetric quasi-distance $\rho : X \times X \to [0 + \infty)$ satisfying the quasi-subadditive condition for the given C_1 and which has the property that if $\rho' : X \times X \to [0 + \infty)$ is such that $\rho' \approx \rho$ and there exist $\beta \in (0, +\infty)$ and $C \in [0, +\infty)$ for which*

$$\big|\rho'(x,y) - \rho'(x,z)\big| \leq C \max\big\{\rho'(x,y)^{1-\beta}, \rho'(x,z)^{1-\beta}\big\}\big[\rho'(y,z)\big]^{\beta} \tag{2.136}$$

whenever $x, y, z \in X$ (and also $x \notin \{y, z\}$ if $\beta > 1$) then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \tag{2.137}$$

(14) If throughout the claims above the quasi-subadditive condition (2.112) *is replaced by the condition stating that*

$$\rho(x,y) \leq C_2\Big([\rho(x,z)]^p + [\rho(z,y)]^p\Big)^{1/p}, \quad \textit{for all} \ \ x, y, z \in X, \tag{2.138}$$

for some exponent $p \in (0, +\infty)$ and some finite constant $C_2 \geq 2^{-1/p}$, then the same conclusions as before hold if in place of (2.113) *one defines $\alpha = \alpha_p$ to be*

$$\alpha_p := \frac{p}{1 + p\log_2 C_2} \in (0, +\infty]. \tag{2.139}$$

Proof. This is a compilation of the results developed throughout the chapter. □

Chapter 3

Analysis on Quasi-metric Spaces

While the goal of the last chapter was to develop a tool which we will be calling upon in crucial moments, we did so without much explorations of quasi-metrics. In fact, for most of the chapter our function ρ was not required to satisfy any special properties. In this chapter we take a closer look at quasi-metrics, specifically at quasi-metric considerations and topological considerations.

We start by seeing some examples of quasi-metrics, followed by new definitions, then we revisit notions previously seen (regularization, symmetrization, equivalence of functions) and see how they interact.

The goal of this chapter is to substantiate the claim that there is already a good deal of analysis which can be carried out in the context of quasi-metric spaces. Concerning quasi-metric spaces, not only is this category more inclusive than the category of metric spaces but, at the same time, for many practical endeavors the former constitutes a more natural and flexible setting than the latter. For example, the category of quasi-metric spaces contains the family all quasi-Banach spaces which, in turn, encompasses a multitude of function spaces (measuring smoothness, on various scales) which are of fundamental importance in analysis. Indeed, this is the case for significant portions of the following familiar scales of spaces: Lebesgue spaces, weak-Lebesgue spaces, Lorentz spaces, Hardy spaces, weak-Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel-Lizorkin spaces, as well as weighted versions of these spaces.

3.1 Quasi-metric Considerations

We are still not quite in a position to write out the definition of a quasi-metric space. Let us first see some examples after establishing a convenience.

Convention 3.1. *Henceforth we will make the following standing assumption: given a quasi-metric ρ on X, we shall assume that the set X is nonempty and ρ is finite (i.e., ρ takes values in $[0, +\infty)$).*

Example 3.2. *Trivially, on any given set a quasi-distance always exists. For example, one may take $\rho : X \times X \to [0, +\infty)$ defined by $\rho(x, y) := 1 - \delta_{xy}$, where δ_{xy} is the Kronecker symbol, $x, y \in X$. Hence, $\rho(x, y) = 1$ if $x \neq y$ and $\rho(x, x) = 0$, for all $x, y \in X$. ρ is called the* `discrete metric` *on X.*

The discrete metric is symmetric and satisfies the triangle inequality, as the name suggests.

Example 3.3. *Imagine measuring distance as the amount of time it takes to get from location to another. With this notion, points uphill would be farther from points downhill than vice versa. To make this example concrete, let $X \subseteq \mathbb{R}$ and consider (X, ρ), where $\rho(x, y) := sin(\pi/4)|x - y|$ if $x, y \in X$ are such that $x \geq y$, and define $\rho(x, y) := (\sin(\pi/4))^{-1}|x - y|$ if $x, y \in X$ are such that $x < y$. One way to envision this is as an upward hill at a $45°$ incline.*

Then ρ is a quasi-metric with constant C_0 (the quasi-symmetry constant as in (2.2)*) equal to* 2 *and C_1 (the quasi-subadditive constant as in* (2.3)*) equal to* 3*. Actually, in the definition we may take any constants at least* 2 *and* 3*, respectively, but these are the optimal constants.*

Example 3.4. *Let X be the collection of English words (or any combinations of letters using any alphabet) and ρ of the same word twice be* 0 *and otherwise be 2^{-j} where j is the slot of the first character that differs between them. So $\rho(cat, cog) = 1/4$, $\rho(a, apple) = 1/4$ and $\rho(quasi, semi) = 1/2$.*

Then ρ is symmetric and satisfies the triangle inequality and so ρ is a metric. In fact, ρ satisfies the quasi-ultrametric condition (2.4) *with constant $C_1 = 1$, thus ρ is an ultrametric.*

Example 3.5. *One can trivially construct examples of quasi-metrics which do not satisfy the triangle inequality but are symmetric. Let $X = \{x, y, z\}$ and define ρ so that $\rho(x, y) := 1 =: \rho(y, z)$ but $\rho(x, z) := 3$ (with ρ symmetric and nondegenerate). Then ρ satisfies* (2.3) *with $C = 3/2$ and* (2.4) *with $C_1 = 3$. It is expected that such a metric would not satisfy the triangle inequality as there is no such thing as a standard triangle whose sides have length $1, 1$ and* 3*.*

Similarly, by taking the same X but instead defining $\rho(x, y) = \rho(y, z) = \rho(z, x) = 1$ and $\rho(y, x) = \rho(z, y) = \rho(x, z) = 2$, one acquires a quasi-metric which satisfies the usual triangle inequality, the quasi-symmetry condition (2.2) *with $C_0 = 2$ and the quasi-ultrametric condition* (2.4) *with $C_1 = 2$.*

Example 3.6. *Expanding upon the above example, instead of having our ambient* *space be 3 points, take X to be a circle, say $X := \{re^{i\theta} \in \mathbb{R}^2 : \theta \in [0, 2\pi)\}$ for some $r > 0$, and fix $c > 1$. The idea is traversing the circle clockwise takes longer than traversing the circle counterclockwise (or vice versa). For example, flying east generally takes longer than flying west thanks to the rotation of the earth (though the earth is not a circle, nor exactly a sphere for that matter). Define*

$$\rho\big(re^{i\theta_1}, re^{i\theta_2})\big) := \begin{cases} 0 & \text{if } \theta_1 = \theta_2 \\ cr(\theta_2 - \theta_1) & \text{if } 0 < \big(\theta_2 - \theta_1\big) < \pi \\ -r(\theta_2 - \theta_1) & \text{if } 0 \geq (\theta_2 - \theta_1) \geq -\pi \end{cases} \tag{3.1}$$

for all $re^{i\theta_1}, re^{i\theta_2} \in X$.

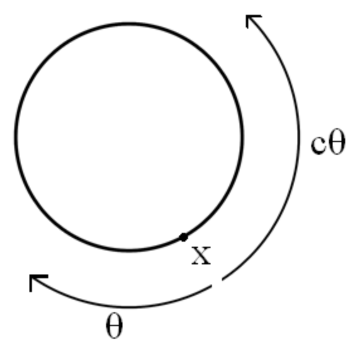

Figure 3. *Traveling counterclockwise takes longer than clockwise.*

This can be envisioned as traveling along the circle with counterclockwise travel being longer than traveling along the circle in a clockwise manner. Note that this is not the optimal way to travel for all pairs of points in X. As $c > 1$, it is better to go clockwise past the antipodal point from where one started to reach one's destination. The more efficient quasi-distance would be,

$$\rho'\big(re^{i\theta_1}, re^{i\theta_2}\big) := \begin{cases} 0 & \text{if } \theta_1 = \theta_2 \\ cr(\theta_2 - \theta_1) & \text{if } 0 < \big(\theta_2 - \theta_1\big) < (\pi/2)(c+1) \\ -r(\theta_2 - \theta_1) & \text{if } 0 \geq (\theta_2 - \theta_1) \geq -(\pi/2)(c+1) \end{cases} \tag{3.2}$$

for all $re^{i\theta_1}, re^{i\theta_2} \in X$. Note both C_ρ and $C_{\rho'}$ are $c + 1$ and, while C_0 is c.

To consider distance on the sphere, given $x, y \in rS^{n-1}, x \neq \pm y$, some fixed $r > 0$, simply take the great circle passing through x and y to be X and refer to (3.2). *In other words, X is the intersection of rS^{n-1} and the two-dimensional plane spanned by x and y. In the case that x and y are antipodal, that is $x = -y$, any great circle will yield $\rho'(x, y) = r\pi$. Lastly, if $x = y$ then the distance between them is* 0.

Example 3.7. *In $\mathbb{R}$, define a* `dyadic interval` *to be any interval with endpoints $j/2^n$ and $(j+1)/2^n$ for $j, n \in \mathbb{Z}$ (of course $j/2^n$ can be thought of as $j \cdot 2^n$ when n varies through $\mathbb{Z}$, but the latter is less suggestive). Depending on what we want, we may include or exclude either endpoint.*

First, we'll take our dyadic intervals to contain, say, the left endpoints. Let $X \subseteq \mathbb{R}$ and $\rho(x, y)$ be the length of the smallest dyadic interval containing both x and y for all $x, y \in X$ (and $\rho(x, x) = 0$ for any $x \in X$). Then ρ satisfies the quasi-ultrametric condition (2.4)*with constant $C_1 = 1$, so ρ is an ultrametric (symmetry and nondegeneracy are clear). However, if we allow the dyadic intervals to contain both endpoints, ρ is no longer an ultrametric (one can easily find an X and $x, y, z \in X$ such that $\rho(x, y) > \max\{\rho(x, z), \rho(z, y)\}$). Under this setting ρ satisfies the quasi-triangle inequality with $C_1 = 1$, hence ρ is a metric.*

This can easily be extended to subsets of $\mathbb{R}^n$ by crossing dyadic cubes (of the same length) with themselves n times; alternatively, one can define the cubes by their corners, typically denoted by $2^{-k}\mathbb{Z} := \{2^{-k}(x_1, \cdots, x_n) : k, x_i \in \mathbb{Z}, i = 1, \cdots, n\}$. Also, there is nothing particularly special (at least in this example) about working with powers of 2. *That is, we could have considered intervals such as $[j/3^n, (j+1)/3^n)$ or $[j/\pi^n, (j+1)/\pi^n)$.*

Example 3.8. *Let $X \subseteq \mathbb{R}^n$ and define*

$$\rho_1(x,y) := \begin{cases} 0 \text{ if } x = y \\ j \text{ if } j-1 < |\,x-y| \leq j, j \in \mathbb{N} \end{cases} \qquad \forall\, x,y \in \mathbb{R}^n, \tag{3.3}$$

and

$$\rho_2(x,y) := \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } 0 < |\,x-y| < 1 \\ j+1 \text{ if } j \leq |\,x-y| < j+1, j \in \mathbb{N} \end{cases} \qquad \forall\, x,y \in \mathbb{R}^n. \tag{3.4}$$

Then ρ_1 and ρ_2 are both nondegenerate, symmetric, and satisfy (2.4) *with $C_1 = 2$. In fact, both ρ's are genuine metrics.*

Example 3.9. *Let $X := \mathbb{R}$ and $\rho(x,y) := |\,x-y|^\alpha$ for all x and $y \in \mathbb{R}$, where $\alpha \in (0,+\infty]$. Then ρ is clearly symmetric and nondegenerate. To see ρ satisfies is quasi-subadditive with constant $C_{1,\alpha}$ dependent upon α, fix $x,y \in \mathbb{R}$ and observe*

$$\begin{aligned} |\,x-y|^\alpha &= |\,x-z+z-y|^\alpha \text{ for all } z \in \mathbb{R} &(3.5)\\ &\leq \big|\,|x-z|+|z-y|\big|^\alpha = \big|\,(|x-z|^\alpha)^{1/\alpha} + (|z-y|^\alpha)^{1/\alpha}\big|^\alpha &(3.6)\\ &\leq C_{1,\alpha}\big(|x-z|^\alpha + |z-y|^\alpha\big), &(3.7) \end{aligned}$$

where in (3.7) *we have used the elementary inequality*

$$(a^\gamma + b^\gamma)^{1/\gamma} \leq \begin{cases} a+b & \text{if } 1 \leq \gamma < +\infty, \\ 2^{\frac{1}{\gamma}-1}(a+b) & \text{if } 0 < \gamma \leq 1, \end{cases} \qquad \forall\, a,b \geq 0. \tag{3.8}$$

Thus $C_{1,\alpha} = 2$ for $\alpha \leq 1$ and $C_{1,\alpha} = 2^\alpha$ for $\alpha \geq 1$.

This setup can be extended to $(\mathbb{R}^n, \sum\limits_{i=1}^{n} |\,x_i - y_i|^{\alpha_i})$, where $\alpha_i \in (0,+\infty)$ and for $i \in \{1,\cdots,n\}$. In which case $C_1 = max\{C_{1,\alpha_1},\cdots,C_{1,\alpha_n}\}$. Note the $1/\alpha$ which usually appears on the outside of the metric for homogeneity has been dropped due to the α_i's possibly differing.

Looking at the above examples, one sees symmetry and the quasi-triangle inequality, as well as the constants involved, are independent, as expected.

Next we see how to construct numerous quasi-metrics from given quasi-metrics.

Lemma 3.10. *Let $\lambda > 0$ be fixed, ρ a quasi-metric on X and $\widetilde{\rho}$ a quasi-metric on Y. Then*

(1) $(\lambda\rho)$ is a quasi-metric on X.

(2) $\rho + \widetilde{\rho}$ is a quasi-metric on X when $X = Y$.

(3) $\min\{\rho\,,\,1\}$ is a quasi-metric on X.

(4) $\max\{\rho\,,\,\widetilde{\rho}\}$ is a quasi-metric on X when $X = Y$.

(5) $(\rho \times \widetilde{\rho})\big((x_1, y_1), (x_2, y_2)\big) := \rho(x_1, x_2) + \widetilde{\rho}(y_1, y_2), \ \forall\, x_1, x_2 \in X, y_1, y_2 \in Y$ is a quasi-metric on $X \times Y$.

Furthermore, if our above starting functions are symmetric, then the produced functions are also symmetric.

Proof. The above claims are direct consequences of definitions. □

Another, more significant method of attaining a new quasi-metric from an old is raising a quasi-metric to any positive power. Specifically, given $\alpha \in (0, +\infty)$, we shall refer to the transformation $\rho \mapsto \rho^\alpha$ as a `power-rescaling` (of order α).

Remark 3.1. *Given a quasi-metric ρ on X, ρ^α is also a quasi-metric for any exponent $\alpha \in (0, +\infty)$ (and is symmetric when ρ is symmetric). Furthermore, if ρ is an actual metric, ρ^α remains a metric for any $\alpha \in (0, 1]$)*

The second statement in Remark 3.1 cannot be extended to any positive α. That is, unlike metrics, the category of quasi-metrics is stable under any positive power-rescaling. Also, we have already run across these ideas in Chapter (2), most notably in Theorem 2.9 (and Theorem 2.12, parts (9) and (12)).

Remark 3.2. *The quality of being a quasi-distance is hereditary in the sense that if ρ is a quasi-metric defined on X and $Y \subseteq X$ is nonempty, then the restriction $\rho\lfloor Y$ of ρ to Y, naturally defined by setting*

$$(\rho\lfloor Y)(x, y) := \rho(x, y), \qquad \forall\, x, y \in Y, \tag{3.9}$$

becomes a quasi-distance on Y. Moreover, $\rho\lfloor Y$ is symmetric if ρ is symmetric and $(\rho\lfloor Y)^\alpha = \rho^\alpha\lfloor Y$ for any $\alpha \in (0, +\infty)$.

Up until now, we have only been working with given constants for the quasi-triangle inequality and quasi-ultrametric condition. However, in practice we are interested in the optimal such constant. Let ρ be a quasi-metric defined on X and recall the constant C_1 appearing in the quasi-triangle inequality in (2.3) and quasi-subadditivity condition (2.4).

For every $\alpha \in (0, +\infty]$, denote by $C_{\rho,\alpha}$ the optimal constant in the estimate

$$\rho(x, y) \leq C_{\rho,\alpha}\big(\rho(x, z)^\alpha + \rho(z, y)^\alpha\big)^{\frac{1}{\alpha}}, \quad \forall\, x, y, z \in X, \tag{3.10}$$

with the natural convention that the format of (3.10) when $\alpha = +\infty$ is

$$\rho(x, y) \leq C_{\rho,\infty} \max\{\rho(x, z), \rho(z, y)\}, \quad \forall\, x, y, z \in X. \tag{3.11}$$

Thus with this notation $C_{\rho,1}$ is the best constant in (2.3) and $C_{\rho,\infty}$ is the best constant in (2.4).

In other words,

$$C_{\rho,\alpha} = \begin{cases} \Big(\sup\limits_{\substack{x,y,z\in X \\ \text{not all equal}}} \dfrac{\rho(x,y)^\alpha}{\rho(x,z)^\alpha + \rho(z,y)^\alpha} \Big)^{\frac{1}{\alpha}} & \text{if } 0 < \alpha < +\infty, \\ \sup\limits_{\substack{x,y,z\in X \\ \text{not all equal}}} \dfrac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}} & \text{if } \alpha = +\infty, \end{cases} \tag{3.12}$$

and the fact that ρ is a quasi-distance ensures that $C_{\rho,\alpha}$ is a well-defined number in $[1, +\infty)$. In fact, (3.8) shows that $C_{\rho,\alpha}$ satisfies

$$1 \leq C_{\rho,\alpha} \leq \begin{cases} C_1 & \text{if } 0 < \alpha \leq 1, \\ 2^{1-\frac{1}{\alpha}} C_1 & \text{if } 1 \leq \alpha \leq +\infty. \end{cases} \tag{3.13}$$

Conversely, estimates such as (3.10)-(3.11) guarantee that ρ satisfies the quasi-triangle inequality (2.3) with

$$C_1 := \begin{cases} C_{\rho,\alpha} & \text{if } 1 \leq \alpha \leq +\infty, \\ 2^{\frac{1}{p}-1} C_{\rho,\alpha} & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{3.14}$$

Recalling terminology from Definition (2.2), $C_{\rho,\alpha} = 1$ corresponds to the quasi-metric ρ being α-subadditive and $C_{\rho,\infty} = 1$ corresponds to subadditivity of ρ. In these cases we have

$$\rho(x,y) \leq \begin{cases} \big(\rho(x,z)^\alpha + \rho(z,y)^\alpha\big)^{\frac{1}{\alpha}}, & \forall\, x,y,z \in X, \quad \text{if } 0 < \alpha < +\infty, \\ \max\{\rho(x,z), \rho(z,y)\}, & \forall\, x,y,z \in X, \quad \text{if } \alpha = +\infty. \end{cases} \tag{3.15}$$

In this vein, observe that

$$\rho \text{ is symmetric and } \alpha\text{-subadditive with } \alpha \in [1, +\infty] \Longrightarrow \rho \text{ is a distance,} \tag{3.16}$$

$$\rho \text{ is symmetric and } \alpha\text{-subadditive with } \alpha \in (0, 1] \Longrightarrow \rho^\alpha \text{ is a distance.} \tag{3.17}$$

Recall from Definition 2.4 that, given a set X, two functions ρ_1, ρ_2 mapping $X \times X$ to $[0, +\infty)$ are called equivalent, a condition denoted by $\rho_1 \approx \rho_2$, provided there exist finite constants $C', C'' > 0$ with the property that

$$C' \rho_1(x,y) \leq \rho_2(x,y) \leq C'' \rho_1(x,y), \qquad \forall\, x,y \in X. \tag{3.18}$$

As already pointed out in Remark 2.2, $\approx$ is an equivalence relation on the set of extended real-valued, nonnegative functions defined on $X \times X$.

Let us also note here that if $\rho_1, \rho_2 : X \times X \to [0, +\infty)$ are two functions with the property that $\rho_1 \approx \rho_2$, then ρ_1 is a quasi-distance on X if and only if ρ_2 is a quasi-distance on X. With this in mind, in the future we'll be interested in working with a class of quasi-metrics on a given set X rather than specifying some ρ.

Definition 3.1. *Given a set X (of cardinality ≥ 2), denote by $\mathfrak{Q}(X)$ the collection of all quasi-distances on X, and by $\mathfrak{Q}_{sym}(X)$ the family of all symmetric quasi-distances on X. That is,*

$$\mathfrak{Q}_{sym}(X) := \{\rho \in \mathfrak{Q}(X) :\ \rho(y,x) = \rho(x,y),\ \forall\, x, y \in X\}. \tag{3.19}$$

Also, for each $\rho \in \mathfrak{Q}(X)$ abbreviate $C_{\rho,\infty}$ from (3.12) simply as C_ρ, i.e.,

$$C_\rho := \sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}}. \tag{3.20}$$

Hence, in the context of the above definition, for every $\rho \in \mathfrak{Q}(X)$ one has

$$\rho(x,y) \leq C_\rho \max\{\rho(x,z), \rho(z,y)\}, \quad \text{for all } x, y, z \in X. \tag{3.21}$$

One may then readily check that

Lemma 3.11. *Given a quasi-metric ρ defined on $X \times X$, the following hold:*

(1) $\forall\, \rho \in \mathfrak{Q}(X)$ *(with* $\#\, X \geq 2), C_\rho \in [1, +\infty)$,

(2) ρ *ultrametric on* $X \iff \rho \in \mathfrak{Q}_{sym}(X)$ *and* $C_\rho = 1$,

(3) $\forall\, \rho \in \mathfrak{Q}(X)$ *and* $Y \subseteq X$*, nonempty,* $C_{\rho\lfloor Y} \leq C_\rho$.

Proof. These follow from definitions. □

As noted earlier, any power dilation preserves both $\mathfrak{Q}(X)$ and $\mathfrak{Q}_{sym}(X)$ and, in addition, given any $\rho \in \mathfrak{Q}(X)$ and $\beta \in (0, +\infty)$, $C_{\rho^\beta} = (C_\rho)^\beta$. Recall $\rho_\# := (\rho_{sym})_\alpha$ with $\alpha := \frac{1}{\log_2 C_1}$, C_1 the quasi-subadditivity constant of ρ. As we have seen, the constant C_1 is, in general, not optimal. Thus we make the following convention.

Convention 3.12. *Given a set X and $\rho \in \mathfrak{Q}(X)$, it is agreed that for the remainder of this work the α appearing in the definition of $\rho_\#$ is taken as $\alpha := [\log_2 C_\rho]^{-1}$ with C_ρ as in* (3.20).

Remark 3.3. *For any set X and any quasi-distance $\rho \in \mathfrak{Q}(X)$,*

$$(\rho^\beta)_\# = (\rho_\#)^\beta, \qquad \forall\, \beta \in (0, +\infty). \tag{3.22}$$

Note property (2.126) also gives

$$\rho_\# \approx \rho \ \text{ in the precise sense that } \ C_\rho^{-2}\rho \le \rho_\# \le C_0\, \rho. \tag{3.23}$$

In addition, by (2.125) and (2.129),

$$\rho_\# \in \mathfrak{Q}_{sym}(X) \quad \text{ and } \quad C_{\rho_\#} \le C_\rho. \tag{3.24}$$

Recall that we have a natural equivalence relation on $\mathfrak{Q}(X)$, as described in Definition 2.4, which leads to the following definition.

Definition 3.2. *Given a nonempty set X, call each equivalence class $\mathbf{q} \in \mathfrak{Q}(X)/_\approx$ a* `quasi-metric space structure` *on X.*

Moreover, for each $\rho \in \mathfrak{Q}(X)$, denote by $[\rho] \in \mathfrak{Q}(X)/_\approx$ the equivalence class of ρ. Hence,

$$[\rho] = \big\{\eta\rho : \ \eta : X \times X \to (0, +\infty),\, \eta \text{ as in Remark 2.2}\big\}. \tag{3.25}$$

Definition 3.3. *By a* `quasi-metric space` *we shall understand a pair* $(X, \mathbf{q})$ *where* X *is a set of cardinality* ≥ 2*, and* $\mathbf{q} \in \mathfrak{Q}(X)/_{\approx}$.

Thus, a quasi-metric space structure on a set X is simply a choice of an equivalence class in the set of all quasi-distances on X with respect to the equivalence relation described in (3.18). If X is a set of cardinality ≥ 2 and $\rho \in \mathfrak{Q}(X)$, we shall frequently use the simpler notation (X, ρ) in place of $(X, [\rho])$, and still refer to (X, ρ) as a quasi-metric space.

Corresponding to Definition (3.3), call (X, ρ) a `metric space` if ρ is actually a distance on X. Clearly, the family of metric spaces makes up a subclass of the family of all quasi-metric spaces.

Recall the max-symmetrization introduced in Lemma 2.6.

Definition 3.4. *Let* X *be an arbitrary, nonempty fixed set and suppose that* ρ *is a given quasi-metric on* X*. For each number* $p \in (0, +\infty)$ *define the* p`-th power symmetric part of` ρ *to be the function* $\rho_{sym,p} : X \times X \to [0, +\infty)$ *defined by*

$$\rho_{sym,p}(x, y) := 2^{-1/p}\big(\rho(x, y)^p + \rho(y, x)^p\big)^{1/p} \quad \forall\, x, y \in X. \tag{3.26}$$

Thus our old ρ_{sym} corresponds to $\rho_{sym,\infty}$, and we now extend the above definition to include $+\infty$, that is $\rho_{sym,\infty} := \rho_{sym}$. For each $p \in (0, +\infty]$, the function $\rho_{sym,p}$ is a symmetric quasi-distance on X. Moreover, if $p \in (0, +\infty)$ and $x, y, z \in X$ are arbitrary, then from

$$\rho(x, y)^p \leq (C_{\rho,p})^p(\rho(x, z)^p + \rho(z, y)^p) \tag{3.27}$$

and $\rho(y, x)^p \leq (C_{\rho,p})^p(\rho(y, z)^p + \rho(z, x)^p)$, we deduce that $\rho_{sym,p}(x,y) \leq C_{\rho,p}\big(\rho_{sym,p}(x,z) + \rho_{sym,p}(z,y)\big)$ for every $x, y, z \in X$ hence, ultimately,

$$C_{(\rho_{sym,p}),p} \leq C_{\rho,p}, \tag{3.28}$$

using notation introduced in (3.12). In addition, if C_0 is the constant appearing in the quasi-symmetry condition (2.2), then for every $p \in (0, +\infty)$

$$2^{-\frac{1}{p}}\big(1 + (C_0)^{-p}\big)^{\frac{1}{p}}\rho(x,y) \leq \rho_{sym,p}(x,y) \leq 2^{-\frac{1}{p}}\big(1 + (C_0)^{p}\big)^{\frac{1}{p}}\rho(x,y), \quad \forall\, x, y \in X, \tag{3.29}$$

This shows that $\rho \approx \rho_{sym,p}$ for every $p \in (0, +\infty)$, while a simple extension using part (1) from Lemma 2.5 includes the case $p = +\infty$. Moreover,

$$\rho \text{ is symmetric} \iff \rho = \rho_{sym,p} \text{ for some (hence, every) } p \in (0, +\infty]. \tag{3.30}$$

In summary, given a quasi-distance ρ on an arbitrary, nonempty set X and some number $p \in (0, +\infty]$, there exists a symmetric quasi-distance on X which is equivalent with ρ and such that its corresponding optimal constant in an inequality of the form described in (3.10)-(3.11) does not exceed the corresponding optimal constant associated with ρ (see (3.28)).

3.2 Topological Considerations

Before discussing topologies, it is natural to define balls. The following notion appeared briefly in (2.67), which we now explore in more depth.

Definition 3.5. *Let X be a nonempty set and assume that ρ is a quasi-distance on X. Then for each $x \in X$ and $r \in (0, +\infty)$ define*

$$B_\rho(x,r) := \{y \in X : \rho(x,y) < r\} \tag{3.31}$$

and refer to it as being a ρ-`ball`$BdXr@B_\rho(x,r)$ *(or, simply, a* `ball`*, when there is no confusion as to the nature of the quasi-distance). Call* x *and* r *the* `center` *and the* `radius`*, respectively, of the* ρ*-ball* $B_\rho(x,r)$*. Whenever the dependence on the set* X *needs to be emphasized we shall write* $B_{(X,\rho)}(x,r)$ *in place of* $B_\rho(x,r)$*.*

Finally, given an arbitrary ρ*-ball* $B := B_\rho(x,r)$ *in* X*, define* $\mathrm{rad}\,(B) := r$ *and, for any* $\lambda \in (0,+\infty)$ *set* $\lambda B := B_\rho(x,\lambda r)$*.*

Remark 3.4. *Let* (X,ρ) *be a quasi-metric space and define the closed ball as the set* $B_\rho[x_0,r] := \{x \in X : \rho(x_0,x) \leq r\}$ *for* $r > 0$*, it is not necessarily the case that* $B_\rho[x_0,r] = \overline{B_\rho(x_0,r)}$*. The right to left containment always holds, however.*

It is useful to observe that if (X,ρ) is a quasi-metric space then the ρ-balls in X transform naturally under power-rescalings of the quasi-distance. Specifically, we have

$$B_{\rho^\gamma}(x,r) = B_\rho(x,r^{1/\gamma}), \quad \forall\,\gamma > 0, \quad \forall\,x \in X, \quad \forall\,r > 0, \tag{3.32}$$

We continue our discussion by introducing a notion of “distance" between subsets of an ambient set equipped with a quasi-distance, as well as the concept of diameter in such a setting.

Definition 3.6. *Given a nonempty set* X *and* $\rho \in \mathfrak{Q}(X)$*, define*

$$\mathrm{dist}_\rho(A,B) := \inf\{\rho(x,y) :\, x \in A,\, y \in B\}, \qquad A, B \subseteq X,\ \textit{nonempty}, \tag{3.33}$$

and abbreviate $\mathrm{dist}_\rho(x,E) := \mathrm{dist}_\rho(\{x\},E)$ *for any* $x \in X$ *and* $E \subseteq X$*, nonempty. Also, define*

$$\mathrm{diam}_\rho(E) := \sup\{\rho(x,y) :\, x,y \in E\}, \qquad E \subseteq X, \tag{3.34}$$

with the convention that $\mathrm{diam}_\rho(\emptyset) := 0$. *Finally, call a set* $E \subseteq X$ `bounded` *provided one has* $\mathrm{diam}_\rho(E) < +\infty$.

Along with any definition comes new properties. Here is a brief collection of facts relating to the above definition.

Lemma 3.13. *If* (X, ρ) *is a quasi-metric space we then have:*

$$E \text{ is bounded} \iff E \text{ is contained in a } \rho\text{-ball}, \qquad \forall\, E \subseteq X, \tag{3.35}$$

$$\mathrm{dist}_{\rho^\gamma}(A, B) = \big[\mathrm{dist}_\rho(A, B)\big]^\gamma, \quad \forall\, \gamma > 0, \ \ \forall\, A, B \subseteq X, \ \text{nonempty}, \tag{3.36}$$

$$\mathrm{dist}_\rho(A, B) \leq C_0\, \mathrm{dist}_\rho(B, A), \quad \forall\, A, B \subseteq X, \ \text{nonempty, with } C_0 \text{ as in (2.2)}, \tag{3.37}$$

$$\mathrm{dist}_\rho(A, B) \geq \mathrm{dist}_\rho(\widetilde{A}, \widetilde{B}) \ \ \text{if } A \subseteq \widetilde{A} \subseteq X, \ B \subseteq \widetilde{B} \subseteq X, \ A, B \text{ nonempty}, \tag{3.38}$$

$$\mathrm{dist}_\rho\big(B_\rho(x, r), X \setminus B_\rho(x, Cr)\big) \geq C_\rho^{-1} C r \ \ \text{if } x \in X, \ r > 0, \ C > C_\rho, \tag{3.39}$$

$$\rho' \approx \rho \implies \mathrm{dist}_\rho(A, B) \approx \mathrm{dist}_{\rho'}(A, B), \quad \text{uniformly for } A, B \subseteq X, \ \text{nonempty}, \tag{3.40}$$

$$\mathrm{diam}_\rho(B_\rho(x, r)) \leq C_\rho C_0 r, \quad \forall\, x \in X, \ \forall\, r > 0, \tag{3.41}$$

$$\mathrm{diam}_\rho(A) \leq \mathrm{diam}_\rho(B), \quad \forall\, A, B \subseteq X, \ \text{nonempty, with } A \subseteq B, \tag{3.42}$$

$$\rho' \approx \rho \implies \mathrm{diam}_\rho(E) \approx \mathrm{diam}_{\rho'}(E), \quad \text{uniformly for } E \subseteq X. \tag{3.43}$$

Proof. The items presented here follow in a straightforward manner from definitions.

□

In particular, given a quasi-metric space $(X, \mathbf{q})$, it is unequivocal to say that $E \subseteq X$ is bounded if there exists $\rho \in \mathbf{q}$ such that $\mathrm{diam}_\rho(E) < +\infty$.

Recall the topology τ_ρ induced by ρ on X as described in (2.67). A natural observation is that the topology just described depends only on the quasi-metric space structure induced by ρ on X (i.e., only on $[\rho]$). More specifically, for any two quasi-distances ρ_1, ρ_2 on X one has

$$\rho_1 \approx \rho_2 \implies \tau_{\rho_1} = \tau_{\rho_2}, \tag{3.44}$$

i.e., equivalent quasi-distances induce the same topology on X. As a consequence, any quasi-metric space $(X, \mathbf{q})$ has a canonical topology, denoted $\tau_{\mathbf{q}}$, which is unequivocally defined as the topology τ_ρ naturally induced by a choice of a quasi-distance ρ in $\mathbf{q}$. Let us also note here that (3.32) entails

$$\tau_\rho = \tau_{\rho^\alpha}, \qquad \forall\, \alpha > 0. \tag{3.45}$$

In keeping with earlier conventions, we make the following definition.

Definition 3.7. *Given a quasi-metric space $(X, \mathbf{q})$, denote by $\mathscr{C}^0(X) = \mathscr{C}^0(X, \tau_{\mathbf{q}})$ the space of real-valued continuous functions defined on the topological space $(X, \tau_{\mathbf{q}})$ and by $\mathscr{C}^0_c(X) = \mathscr{C}^0_c(X, \tau_{\mathbf{q}})$ the subspace of $\mathscr{C}^0(X, \tau_{\mathbf{q}})$ consisting of functions which vanish identically outside of a bounded subset of X.*

As earlier states, given a quasi-metric space (X, ρ), the ρ-balls may not necessarily be open in the topology τ_ρ when $C_1 > 1$. Indeed, the following result holds.

Proposition 3.14. *For each real number $C > 1$ there exists a symmetric quasi-distance ρ on $\mathbb{R}^n$ such that τ_ρ is the ordinary topology in $\mathbb{R}^n$, ρ is equivalent with the Euclidean distance, $C_1 \leq C$ and not all ρ-balls are open.*

Proof. Fix $C > 1$ and consider the mapping $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ given by

$$\rho(x, y) := \phi(|x - y|) \ \text{ where } \ \phi(t) := \begin{cases} t & \text{if } \ t \in [0, 1], \\ Ct & \text{if } \ t \in (1, +\infty). \end{cases} \tag{3.46}$$

Then, since

$$\phi \text{ is increasing, } t \le \phi(t) \le Ct \text{ and} \tag{3.47}$$
$$\phi(t+s) \le C(\phi(t)+\phi(s)), \quad \forall\, t,s \in [0,+\infty),$$

it is straightforward to check based on definitions that ρ is a symmetric quasi-distance on $\mathbb{R}^n$ equivalent to the Euclidean distance, and such that $C_1 \le C$. In particular, τ_ρ is the canonical topology in $\mathbb{R}^n$. However, for each $r \in (1, C)$ we have

$$B_\rho(0,r) = \{x \in \mathbb{R}^n : \rho(x,0) = \phi(|x|) < r\} = \{x \in \mathbb{R}^n : |x| \le 1\}. \tag{3.48}$$

Thus, $B_\rho(0,r)$ is not open in the canonical Euclidean topology, hence, not open in τ_ρ. $\square$

On the positive side, we make the following readily verified observation.

Remark 3.5. *If (X,ρ) is a quasi-metric space then*

$$\begin{array}{c}\rho(x,\cdot) \textit{ upper semi-continuous on} \\ (X,\tau_\rho) \textit{ for every fixed point } x \in X\end{array} \Longleftrightarrow \textit{all } \rho\textit{-balls in } X \textit{ are open in } \tau_\rho. \tag{3.49}$$

In particular,

$$\rho(x,\cdot) \in \mathscr{C}^0(X,\tau_\rho) \textit{ for every } x \in X \Longrightarrow \textit{all } \rho\textit{-balls in } X \textit{ are open in } \tau_\rho. \tag{3.50}$$

Given a quasi-metric space $(X, \mathbf{q})$ and $E \subseteq X$, we let $\overline{E}$ and E° denote, respectively, the closure and interior of E in the topology $\tau_{\mathbf{q}}$.

Lemma 3.15. *If (X, ρ) is a quasi-metric space, the following properties hold:*

$$\operatorname{dist}_\rho(x, E) = 0 \iff x \in \overline{E}, \quad \forall\, x \in X,\ \forall\, E \subseteq X,\ \textit{nonempty}, \tag{3.51}$$

$$\operatorname{dist}_\rho(x, E) > 0 \iff x \in (X \setminus E)^\circ, \quad \forall\, x \in X,\ \forall\, E \subseteq X,\ \textit{nonempty}, \tag{3.52}$$

$$\operatorname{dist}_\rho(x, \overline{E}) \leq \operatorname{dist}_\rho(x, E) \leq C_\rho \operatorname{dist}_\rho(x, \overline{E}), \forall\, x \in X,\ \forall\, E \subseteq X,\ \textit{nonempty}, \tag{3.53}$$

$$\operatorname{diam}_\rho(E) \leq \operatorname{diam}_\rho(\overline{E}) \leq C_\rho^2 \operatorname{diam}_\rho(E), \quad \forall\, E \subseteq X. \tag{3.54}$$

$$\theta \in (0, C_\rho^{-1}) \implies \overline{B_\rho(x, \theta r)} \subseteq B_\rho(x, r) \subseteq \big(B_\rho(x, \theta^{-1} r)\big)^\circ, \quad \forall\, x \in X,\, r > 0. \tag{3.55}$$

In particular, (3.55) gives us any ρ-ball has nonempty interior and, in fact,

$$\forall\, x \in X,\ \forall\, r > 0 \implies \big(B_\rho(x, r)\big)^\circ \text{ is an open neighborhood of } x. \tag{3.56}$$

Consequently, based on (2.67) and (3.56), we conclude that

$$\Big\{\big(B_\rho(x, r)\big)^\circ\Big\}_{x \in X,\, r > 0} \quad \text{is a base for the topology } \tau_\rho. \tag{3.57}$$

Recall that a topological space is said to be `first-countable` if each of its points has a sequence of open neighborhoods with the property that any open set containing the point in question also contains one of these neighborhoods. Restricting ourselves to interiors of ρ-balls with rational radii in (3.57) then proves that

$$(X, \rho) \text{ quasi-metric space} \implies \left\{ \begin{array}{l} (X, \tau_\rho) \text{ is a first-countable,} \\ \text{Hausdorff topological space.} \end{array} \right. \tag{3.58}$$

In fact, a result much stronger than (3.58) holds, since the Alexandroff-Urysohn metrization theorem, originally appearing in [8], gives that the topology induced by the quasi-metric space structure on any given quasi-metric space is metrizable. Actually, more can be said if Theorem 2.12 is used in place of the Alexandroff-Urysohn metrization theorem. We shall do so momentarily, after making the following two definitions.

Definition 3.8. *Let* $(X, \mathbf{q})$ *be a quasi-metric space. A sequence* $\{x_j\}_{j\in\mathbb{N}} \subseteq X$ *is said to be* `Cauchy` *(with respect to the quasi-metric space structure* $\mathbf{q}$*) if there exists* $\rho \in \mathbf{q}$ *with the property that*

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that}\\ &\rho(x_j, x_k) < \varepsilon \text{ whenever } j, k \in \mathbb{N} \text{ satisfy } j, k \geq N_\varepsilon.\end{aligned} \tag{3.59}$$

It is clear that if (3.59) holds for some $\rho \in \mathbf{q}$ then it actually holds for any choice of ρ in $\mathbf{q}$. Thus, ultimately, the quality of being a Cauchy sequence is intrinsically determined by the given quasi-metric space structure $\mathbf{q}$.

Definition 3.9. *Call a quasi-metric space* $(X, \mathbf{q})$ `complete` *if for any sequence of points* $\{x_j\}_{j\in\mathbb{N}}$ *in* X *which is Cauchy (with respect to the quasi-metric space structure* $\mathbf{q}$*) there exists* $x_* \in X$ *with the property that for some (hence any)* $\rho \in \mathbf{q}$ *there holds*

$$\lim_{j\to\infty} \rho(x_*, x_j) = 0. \tag{3.60}$$

In addition, call a subset $Y \subseteq X$ `complete` *(with respect to* $\mathbf{q}$*) provided there exists* $\rho \in \mathbf{q}$ *such that* $(Y, \rho\lfloor Y)$ *is a complete quasi-metric space.*

Here is the result alluded to above.

Proposition 3.16. *If* $(X, \mathbf{q})$ *is a quasi-metric space then the topology* $\tau_{\mathbf{q}}$ *is metrizable. In fact, there exists a distance* d *on* X *with the property that* $\tau_d = \tau_{\mathbf{q}}$ *and such that a sequence of points in* X *is Cauchy with respect to* $\mathbf{q}$ *if and only if it is Cauchy with respect to the quasi-metric space structure induced by* d *on* X*. In particular, if* $(X, \mathbf{q})$ *is complete, then so is* (X, d)*.*

Proof. This is a direct consequence of Theorem 2.12 and (3.44)-(3.45)). $\square$

Proposition 3.16 shows that there is no distinction between the topology of metric and quasi-metric spaces. As a consequence, any statement about the nature of the topology of generic metric spaces continues to hold for the larger class of quasi-metric spaces. Let us briefly digress in order to make the following definition.

Definition 3.10. *Let (X, ρ) be a quasi-metric space. Then $E \subseteq X$ is called* ρ-`totally bounded` *provided for any $\varepsilon > 0$ the set E may be covered by finitely many ρ-balls in X of radii ε. Equivalently, $E \subseteq X$ is ρ-totally bounded if for every $\varepsilon > 0$ there exist finitely many points $\{x_j\}_{1 \leq j \leq N} \subseteq E$ (where $N \in \mathbb{N}$) such that*

$$E \subseteq \bigcup_{j=1}^{N} B_\rho(x_j, \varepsilon). \tag{3.61}$$

Recall that a subset E of a topological space (X, τ) is called compact if from every open cover of E one may extract a finite subcover. Also, a subset E of a quasi-metric space (X, ρ) is called compact if E is compact in topological space (X, τ_ρ). With these pieces of terminology we then have the following result.

Theorem 3.17. *Let (X, ρ) be a quasi-metric space, and assume that E is an arbitrary subset of X. Then the following conditions are equivalent:*

(i) E is a compact subset of the topological space (X, τ_ρ);

(ii) any sequence of points in E has a subsequence which converges (in the topology τ_ρ) to a limit in E;

(iii) any infinite subset of E has (in the topology τ_ρ) an accumulation point in E;

(iv) E is complete (with respect to $[\rho]$) and ρ-totally bounded.

Proof. This is a direct consequence of Proposition 3.16 and known results about the nature of the topology on metric spaces (cf., e.g., [14, Theorem 1.6.5, p. 14]). $\square$

Definition 3.11. *Given two arbitrary quasi-metric spaces $(X_0, \mathbf{q}_0)$ and $(X_1, \mathbf{q}_1)$, a mapping Φ which sends $(X_0, \mathbf{q}_0)$ to $(X_1, \mathbf{q}_1)$ is called* `bi-Lipschitz` *provided for some (hence, any) $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, one has $\rho_1(\Phi(x), \Phi(y)) \approx \rho_0(x, y)$, uniformly for $x, y \in X_0$.*

Of course, in the context of the above definition, the specific choice of the quasi-distances $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, used to define the notion of bi-Lipschitzianity is immaterial. In particular, being bi-Lipschitz is a quality which intrinsically depends only on the quasi-metric space structures of the spaces involved.

Remark 3.6. *Any bi-Lipschitz mapping is a homeomorphism onto its image. In particular, a bi-Lipschitz mapping $\Phi : (X_0, \mathbf{q}_0) \to (X_1, \mathbf{q}_1)$ between two quasi-metric spaces is surjective if and only if $\Phi : (X_0, \tau_{\mathbf{q}_0}) \to (X_1, \tau_{\mathbf{q}_1})$ is a homeomorphism.*

Definition 3.12. *Let (X, ρ) be a quasi-metric space. A quasi-metric space $(\widetilde{X}, \widetilde{\rho})$ is said to be a* `completion` *of (X, ρ) provided the following conditions are satisfies:*

(1) $X \subseteq \widetilde{X}$ and $\widetilde{\rho}\lfloor X \approx \rho$;

(2) X is dense in $(\widetilde{X}, \tau_{\widetilde{\rho}})$;

(3) $(\widetilde{X}, \widetilde{\rho})$ is complete.

Theorem 3.18. *Any quasi-metric space (X, ρ) has a completion $(\widetilde{X}, \widetilde{\rho})$ which is unique up to bi-Lipschitz homeomorphisms. More precisely, for any two completions*

$(\widetilde{X}_1, \widetilde{\rho}_1)$ and $(\widetilde{X}_2, \widetilde{\rho}_2)$ of (X, ρ) there exists a bi-Lipschitz homeomorphism, Φ which maps $(\widetilde{X}_1, \widetilde{\rho}_1)$ to $(\widetilde{X}_2, \widetilde{\rho}_2)$, with the property that $\Phi|_X$ is the identity mapping of X.

Proof. A result of this nature is well-known in the context of metric spaces (in which scenario, the uniqueness portion of the results is phrased in terms of isometries). Consult, for example, the source [14, Theorem 1.5.10, p. 12]. Our strategy for dealing with the case of quasi-metric spaces is to invoke Theorem 2.12 in order to reduce matters to the former setting. Specifically, given a quasi-metric space (X, ρ) and a real number β such that $0 < \beta \leq [\log_2 C_\rho]^{-1}$, it follows from Theorem 2.12 that $(X, (\rho_\#)^\beta)$ is a metric space. Denote by $(\widetilde{X}, d)$ the completion of this metric space in the sense described in [14, Theorem 1.5.10, p. 12]. Then, if we define $\widetilde{\rho} := d^{1/\beta} \in \mathfrak{Q}_{sym}(\widetilde{X})$, it follows that $(\widetilde{X}, \widetilde{\rho})$ is a complete quasi-metric space and $\widetilde{\rho}\lfloor X = \rho_\# \approx \rho$. Furthermore, X is dense in $\widetilde{X}$ in the topology $\tau_d = \tau_{\widetilde{\rho}}$. Hence, $(\widetilde{X}, \widetilde{\rho})$ is a completion of (X, ρ).

Suppose now that $(\widetilde{X}_1, \widetilde{\rho}_1)$ and $(\widetilde{X}_2, \widetilde{\rho}_2)$ are two completions of (X, ρ). Consider the mapping $\Phi : X \to X$ given by $\Phi(x) := x$ for every $x \in X$. We claim that this extends to a bi-Lipschitz homeomorphism $\Phi : (\widetilde{X}_1, \widetilde{\rho}_1) \to (\widetilde{X}_2, \widetilde{\rho}_2)$. Since, by design, this extension satisfies $\Phi|_X = \mathrm{id}_X$, the identity on X, this finishes the proof of the uniqueness part. To justify this claim, consider an arbitrary point $x \in \widetilde{X}_1$ and pick a sequence $\{x_k\}_{k \in \mathbb{N}} \subseteq X$ with the property that $\widetilde{\rho}_1(x, x_k) \to 0$ as $k \to \infty$. Then for each $k, l \in \mathbb{N}$ we may write

$$\begin{aligned}
\widetilde{\rho}_2(\Phi(x_k), \Phi(x_l)) &= \widetilde{\rho}_2(x_k, x_l) = \rho(x_k, x_l) = \widetilde{\rho}_1(x_k, x_l) \\
&\leq C_{\widetilde{\rho}_1} \max\{\widetilde{\rho}_1(x_k, x), \widetilde{\rho}_1(x, x_l)\} \\
&\leq C_{\widetilde{\rho}_1} K_{\widetilde{\rho}_1} \max\{\widetilde{\rho}_1(x, x_k), \widetilde{\rho}_1(x, x_l)\}. \qquad (3.62)
\end{aligned}$$

Thus, $\{\Phi(x_k)\}_k$ is Cauchy in $(\widetilde{X}_2, \widetilde{\rho}_2)$, hence convergent in $(\widetilde{X}_2, \widetilde{\rho}_2)$ to some point $y \in \widetilde{X}_2$. We then define $\Phi(x) := y$. The function Φ thus extended from $\widetilde{X}_1$ into $\widetilde{X}_2$ is unambiguously defined (which is readily seen by mixing sequences) and satisfies $\Phi|_X = \mathrm{id}_X$. In addition, given $x, y \in \widetilde{X}_1$ and sequences $\{x_k\}_{k\in\mathbb{N}} \subseteq X$, $\{y_k\}_{k\in\mathbb{N}} \subseteq X$ such that $\widetilde{\rho}_1(x, x_k) \to 0$ and $\widetilde{\rho}_1(y, y_k) \to 0$ as $k \to \infty$, passing to limit, $k \to \infty$, in the equivalence (seen much as in the first line in (3.62), with the help of (2.126))

$$(\widetilde{\rho}_2)_\#(\Phi(x_k), \Phi(y_k)) \approx (\widetilde{\rho}_1)_\#(x_k, y_k), \quad \text{uniformly in } k \in \mathbb{N}, \tag{3.63}$$

yields, on account of the continuity of the regularized quasi-distances $(\widetilde{\rho}_i)_\#$ in the topological space $(\widetilde{X}_i \times \widetilde{X}_j, \tau_{\widetilde{\rho}_i} \times \tau_{\widetilde{\rho}_i})$, for $i = 1, 2$, (itself, a consequence of Theorem 2.12) that

$$(\widetilde{\rho}_2)_\#(\Phi(x), \Phi(y)) \approx (\widetilde{\rho}_1)_\#(x, y), \tag{3.64}$$

with proportionality constants independent of x, y. Another reference to (2.126) then allows us to conclude from (3.64) that $\Phi : (\widetilde{X}_1, \widetilde{\rho}_1) \to (\widetilde{X}_2, \widetilde{\rho}_2)$ is bi-Lipschitz. Finally, the same type of arguments as above show that this mapping is surjective. On account of Remark 3.6, this justifies the claim and finishes the proof of the theorem. $\square$

Definition 3.13. *Let X be an arbitrary set. A family $(O_i)_{i\in I}$ of subsets of X is said to be a* `cover` *of X if $X = \cup_{i\in I} O_i$. Also, a collection $(W_j)_{j\in J}$ of subsets of X is said to be a* `refinement` *of the the cover $(O_i)_{i\in I}$ provided it is a cover of X and there exists a function $\sigma : J \to I$ with the property that $W_j \subseteq O_{\sigma(j)}$ for every $j \in J$.*

Proposition 3.19. *Let $(X, \mathbf{q})$ be a quasi-metric space. Then the topological space $(X, \tau_{\mathbf{q}})$ is paracompact, i.e., any open cover of X has a locally finite open refinement.*

Proof. This is a consequence of Proposition 3.16 and [71, Corollary 1.8, p. 60]. $\square$

Recall that a topological space (X, τ) is said to be `normal` if for each pair of disjoint closed subsets E_0, E_1 of (X, τ) there exist two disjoint open subsets U_0, U_1 of (X, τ) with the property that $E_0 \subseteq U_0$ and $E_1 \subseteq U_1$. A classical extension result is the Tietze-Urysohn theorem which states that a topological space (X, τ) is normal if and only if every real-valued continuous function defined on a closed subset of X may be extended to the entire space with preservation of continuity. See, for example, the souces [71, Proposition 3.7, p. 19], [18, Theorem 2.1.8].

Assume that (X, τ) is a metrizable topological space, i.e., there exists a distance d on X such that $\tau = \tau_d$, the topology canonically induced on X by the distance d. In this setting, if E_0, E_1 are two disjoint closed subsets of (X, τ), then the function $f : (X, \tau) \to \mathbb{R}$ given by

$$f(x) := \frac{\mathrm{dist}_d(x, E_1)}{\mathrm{dist}_d(x, E_0) + \mathrm{dist}_d(x, E_1)}, \quad \forall\, x \in X, \tag{3.65}$$

is continuous and has the property that $f \equiv 0$ on E_0 and $f \equiv 1$ on E_1. Hence, if we define $U_0 := f^{-1}((-\infty, \frac{1}{2}))$ and $U_1 := f^{-1}((\frac{1}{2}, +\infty))$, then U_0, U_1 are disjoint open subsets of X satisfying $E_0 \subseteq U_0$ and $E_1 \subseteq U_1$. In concert with Proposition 3.16, this shows that

$$\text{if } (X, \mathbf{q}) \text{ is a quasi-metric space,} \tag{3.66}$$

then $(X, \tau_{\mathbf{q}})$ is a normal topological space.

In turn, the above property opens the door for constructing partitions of unity, consisting of continuous functions, in quasi-metric spaces. Concretely, the following result holds.

Proposition 3.20. *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that $(O_i)_{i\in I}$ is a cover of X consisting of open sets in $(X, \tau_{\mathbf{q}})$. Then for each index $i \in I$ there exists a function $\phi_i \in \mathscr{C}^0(X, \tau_{\mathbf{q}})$ such that*

$$\begin{aligned} &0 \leq \phi_i \leq 1, \quad S_i := \{x \in X : \phi_i(x) > 0\} \subseteq O_i \text{ for every } i \in I, \\ &(S_i)_{i\in I} \text{ is locally finite, and } \textstyle\sum_{i\in I} \phi_i(x) = 1 \text{ for all } x \in X. \end{aligned} \tag{3.67}$$

Proof. This follows from (3.66) and [71, Proposition 3.12, p. 22]. □

Regarding the distribution of points in a quasi-metric space we make the following definition.

Definition 3.14. *A quasi-metric space (X, ρ) is said to be* `uniformly distributed` *provided any ρ-ball in X may contain only an at most countable family of mutually disjoint ρ-balls of a given, fixed positive radius.*

Here is an equivalent reformulation and a sufficient condition for the above uniform distribution condition.

Lemma 3.21. *(1) A quasi-metric space (X, ρ) is uniformly distributed if and only if, for every ρ-ball $B \subseteq X$ and every $\varepsilon > 0$, it follows that any set $A \subseteq B$ with the property that*

$$\rho(x, y) > \varepsilon \text{ for every } x, y \in A \text{ with } x \neq y \tag{3.68}$$

is at most countable.

(2) Assume that the quasi-metric space (X, ρ) has the property that for every ρ-ball

$B \subseteq X$ and every $\varepsilon > 0$ there exists a number $N = N(B, \varepsilon) \in \mathbb{N}$ such that

$$\forall\, A \subseteq B \text{ such that } \rho(x,y) > \varepsilon \text{ for every } x, y \in A \qquad (3.69)$$
$$\text{with } x \neq y \Longrightarrow \#\,A \leq N,$$

where $\#\,A$ denotes the cardinality of the set A. Then (X, ρ) is uniformly distributed.

Proof. Both items follow with ease from definitions. □

Lemma 3.22. *If (X, ρ) is a uniformly distributed quasi-metric space then the topological space (X, τ_ρ) is separable (recall a topological space is separable if it contains a countable, dense subset).*

Proof. For each fixed $j \in \mathbb{N}$ consider the set

$$\mathcal{X}_j := \{A \subseteq X :\ \rho(x,y) > 1/j \text{ for every } x, y \in A \text{ with } x \neq y\}. \qquad (3.70)$$

When equipped with the inclusion, this set becomes partially ordered and any of its totally ordered subsets has an upper bound. Therefore, by Zorn's lemma, the set $\mathcal{X}_j$ contains a maximal element which we will denote by A_j. We claim that for each $j \in \mathbb{N}$ the set A_j is at most countable. Indeed, if $x_o \in X$ is a fixed point then for each $k \in \mathbb{N}$ the set $A_j \cap B_\rho(x_o, k)$ is, thanks to the fact that the quasi-metric space (X, ρ) is uniformly distributed, an at most countable set, and since $A_j = \cup_{k\in\mathbb{N}}(A_j \cap B_\rho(x_o, k))$ the desired conclusion follows.

Having established that A_j is at most countable for each $j \in \mathbb{N}$, it follows that

$$A_* := \bigcup_{j\in\mathbb{N}} A_j \qquad (3.71)$$

is an at most countable subset of X and we claim that A_* is also dense in (X, τ_ρ). To this end, fix an arbitrary number $\varepsilon > 0$ along with some arbitrary point $x \in X$ and pick $j \in \mathbb{N}$ such that $j > C_0/\varepsilon$. By the maximality of A_j it follows that there exists $x_j \in A_j$ such that either $\rho(x, x_j) \leq 1/j$, or $\rho(x_j, x) \leq 1/j$. Thus, in any event, for any $j \in \mathbb{N}$ there exists $x_j \in A_*$ with the property that $\rho(x, x_j) < \varepsilon$. This proves the claim and finishes the proof of the lemma. $\square$

We wish to further elaborate on the nature of the topological structure induced by a quasi-metric. First we establish the following basic covering result (in the less general setting of spaces of homogeneous type cf. [15, Lemma 3, p. 299]).

Lemma 3.23 (Vitali's covering lemma)**.** *Let (X, ρ) be a quasi-metric space and fix a finite constant $C > C_\rho^2 C_0$. Consider a family of ρ-balls*

$$\mathcal{F} = \big\{B_\rho(x_\alpha, r_\alpha)\big\}_{\alpha \in I}, \quad x_\alpha \in X,\ r_\alpha > 0 \ \textit{ for every } \alpha \in I, \tag{3.72}$$

such that

$$\sup_{\alpha \in I} r_\alpha < +\infty. \tag{3.73}$$

In addition, suppose that either

$$(X, \tau_\rho) \textit{ is separable,} \tag{3.74}$$

(recall from Lemma 3.22 that this is condition is always satisfied if the quasi-metric space (X, ρ) is uniformly distributed), or

$$\begin{aligned} &\textit{for every sequence } \{B_\rho(x_j, r_j)\}_{j \in \mathbb{N}} \subseteq \mathcal{F} \textit{ consisting} \\ &\textit{of mutually disjoint of } \rho\textit{-balls one has } \lim_{j \to \infty} r_j = 0. \end{aligned} \tag{3.75}$$

Then there exists an at most countable set $J \subseteq I$ with the property that

$$B_\rho(x_j, r_j) \cap B_\rho(x_{j'}, r_{j'}) = \emptyset \quad \forall\, j, j' \in J \quad \text{with} \quad j \neq j', \tag{3.76}$$

and each ρ-ball from $\mathcal{F}$ is contained in a dilated ρ-ball of the form $B_\rho(x_j, Cr_j)$ for some $j \in J$. In particular,

$$\bigcup_{\alpha \in I} B_\rho(x_\alpha, r_\alpha) \subseteq \bigcup_{j \in J} B_\rho(x_j, Cr_j). \tag{3.77}$$

Proof. This is a slight variant of a result proved in [34]. □

In turn, Vitali's covering lemma is the main ingredient in establishing the following result pertaining to the nature of the open sets in the topology induced by a quasi-metric.

Lemma 3.24. *Let (X, ρ) be a separable quasi-metric space (recall from Lemma 3.22 the separability condition holds whenever (X, ρ) is uniformly distributed), let $\mathcal{O} \subseteq X$ be open (in the topology τ_ρ) and fix $\varepsilon > 0$. Then there exist a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ in X and a sequence of positive numbers $\{r_j\}_{j \in \mathbb{N}}$ such that the following properties hold:*

(i) $0 < r_j < \varepsilon$ *for all* $j \in \mathbb{N}$*;*

(ii) $\mathcal{O} = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j) = \bigcup_{j \in \mathbb{N}} \overline{B_\rho(x_j, r_j)} = \bigcup_{j \in \mathbb{N}} \big(B_\rho(x_j, r_j)\big)^\circ$*;*

(iii) there exists $\theta \in (0, 1)$ with the property that the ρ-balls $B_\rho(x_j, \theta r_j)$, $j \in \mathbb{N}$, are mutually disjoint.

Proof. Assume that $\varepsilon > 0$ is given and fix a number $M > 4C_\rho^4 C_0$. Since $\mathcal{O}$ is open, it follows that for every $x \in \mathcal{O}$ there exists $r(x) > 0$ such that $B_\rho(x, r(x)) \subseteq \mathcal{O}$. Set

$\overline{r}(x) := \min\{\varepsilon, r(x)\}$ and apply Vitali's lemma to the family of ρ-balls with bounded radii

$$\Big\{B_\rho\big(x, \tfrac{\overline{r}(x)}{M}\big)\Big\}_{x\in\mathcal{O}}. \tag{3.78}$$

Hence, since the quasi-metric space (X, ρ) is uniformly distributed, Lemma 3.23 applies and gives the existence of a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in $\mathcal{O}$ with the property that the ρ-balls

$$B_\rho\big(x_j, \tfrac{\overline{r}(x_j)}{M}\big), \quad j \in \mathbb{N}, \quad \text{are mutually disjoint,} \tag{3.79}$$

and

$$\forall\, x \in \mathcal{O}\;\; \exists\, j = j(x) \in \mathbb{N} \text{ such that } \; B_\rho\big(x, \tfrac{\overline{r}(x)}{M}\big) \subseteq B_\rho\big(x_j, \tfrac{2C_\rho^2 C_0\, \overline{r}(x_j)}{M}\big). \tag{3.80}$$

Define $r_j := \frac{4C_\rho^3 C_0\, \overline{r}(x_j)}{M}$ for each $j \in \mathbb{N}$. We claim that the x_j's and r_j's just constructed are such that properties (i)-(ii) are satisfied. To see this, note that since $M > L$ and $\overline{r}(x_j) < \varepsilon$, it is immediate that $r_j < \varepsilon$ for every $j \in \mathbb{N}$. Moreover, the above choices ensure that

$$B_\rho\big(x_j, \tfrac{2C_\rho^2 C_0\, \overline{r}(x_j)}{M}\big) = B_\rho\big(x_j, \tfrac{r_j}{2C_\rho}\big) \subseteq \big(B_\rho(x_j, r_j)\big)^\circ, \quad \text{for every } j \in \mathbb{N}, \tag{3.81}$$

thanks to (3.55). Based on (3.81) and (3.80), we may therefore conclude that

$$\mathcal{O} \subseteq \bigcup_{j\in\mathbb{N}} \big(B_\rho(x_j, r_j)\big)^\circ. \tag{3.82}$$

Moving on, if $\lambda \in \big(C_\rho, \frac{M}{4C_\rho^3 C_0}\big)$, which is a nondegenerate interval given the fact that $M > 4C_\rho^4 C_0$, then $\lambda r_j \leq \overline{r}(x_j) \leq r(x_j)$ for every $j \in \mathbb{N}$ so that, by (3.55),

$$\overline{B_\rho(x_j, r_j)} \subseteq B_\rho(x_j, \lambda r_j) \subseteq B_\rho(x_j, r(x_j)) \subseteq \mathcal{O}, \quad \forall\, j \in \mathbb{N}. \tag{3.83}$$

Hence,

$$\bigcup_{j\in\mathbb{N}} \overline{B_\rho(x_j, r_j)} \subseteq \mathcal{O}. \tag{3.84}$$

By combining (3.82) and (3.84), we may therefore conclude that (ii) holds. Finally, choose $\theta \in (0,1)$ so that $0 < \theta < \frac{1}{4C_\rho^3 C_0}$. Then $\theta r_j \leq \frac{\overline{r}(x_j)}{M}$ which, in view of (3.79), shows that (iii) holds for this choice of θ, completing the proof of the lemma. $\square$

3.3 An Extension of the Kuratowski and Fréchet Embedding Theorems

Here we shall establish the quasi-metric version of Kuratowski's and Fréchet's embedding theorems, originally formulated in the context of metric spaces. Given an arbitrary, nonempty set X, denote by $L^\infty(X)$ the space of all bounded functions $f : X \to \mathbb{R}$. When equipped with the norm

$$\|f\|_{L^\infty(X)} := \sup_{x\in X} |f(x)|, \tag{3.85}$$

this may be canonically identified with the Lebesgue space $L^\infty(X, \mu_c)$, where μ_c is the counting measure on X and, as such, $(L^\infty(X), \|\cdot\|_{L^\infty(X)})$ becomes a Banach space. As it is customary, corresponding to the case when $X = \mathbb{N}$, we shall write $\ell^\infty(\mathbb{N})$ in place of $L^\infty(\mathbb{N})$.

The classical Kuratowski's embedding theorem (cf. [55]) asserts that any metric space (X, d) embeds isometrically into $L^\infty(X)$, and our theorem below extends this result to the setting of quasi-metric space.

Theorem 3.25 (Quasi-metric space v. of Kuratowski's embedding theorem)**.** *Assume that* $(X, \mathbf{q})$ *is a quasi-metric space and that* β *is such that there exists a* *quasi-distance* $\rho \in \mathbf{q}$ *with the property that* $0 < \beta \leq [\log_2 C_\rho]^{-1}$. *Then* $(X, \mathbf{q})$ *admits a bi-Lipschitz embedding into the quasi-Banach space* $\big(L^\infty(X), \|\cdot\|_{L^\infty(X)}^{1/\beta}\big)$.

Proof. Let $(X, \mathbf{q})$ be a quasi-metric space and suppose that the quasi-distance $\rho \in \mathbf{q}$ and the real number β are such that $0 < \beta \leq [\log_2 C_\rho]^{-1}$. Also, denote by $\rho_\#$ the regularized version of ρ (as in Theorem 2.12), and fix a point $x_o \in X$. Then, for each $x \in X$, consider the function

$$\phi^x : X \to \mathbb{R}, \quad \phi^x(z) := \big[\rho_\#(x,z)\big]^\beta - \big[\rho_\#(z,x_o)\big]^\beta, \quad \forall\, z \in X. \tag{3.86}$$

Note that thanks to (2.117)-(2.134)

$$|\phi^x(y) - \phi^y(y)| = \big[\rho_\#(x,y)\big]^\beta \geq (C_\rho)^{-2\beta} \rho(x,y)^\beta, \quad \forall\, x,y \in X, \tag{3.87}$$

and

$$\begin{aligned} |\phi^x(z) - \phi^y(z)| &= \big|\big[\rho_\#(x,z)\big]^\beta - \big[\rho_\#(y,z)\big]^\beta\big| \\ &\leq \big[\rho_\#(x,y)\big]^\beta \leq (C_0)^\beta \rho(x,y)^\beta, \quad \forall\, x,y,z \in X. \end{aligned} \tag{3.88}$$

Hence, if we define

$$\Phi : X \to L^\infty(X), \quad \Phi(x) := \phi^x, \quad \forall\, x \in X, \tag{3.89}$$

from (3.87)-(3.88) we may deduce that

$$C_\rho^{-2} \rho(x,y) \leq \|\Phi(x) - \Phi(y)\|_{L^\infty(X)}^{1/\beta} \leq C_0\, \rho(x,y), \quad \forall\, x,y \in X. \tag{3.90}$$

Then the claim in the first part of the statement of the theorem follows from this. □

A drawback of Theorem 3.25 is the fact that the embedding discussed there takes place into a space which depends on the original quasi-metric space. The classical

Fréchet's embedding theorem (cf. [24]), on the other hand, corrects the aforementioned deficiency, stating any separable metric space (X, d) embeds isometrically into $\ell^\infty(\mathbb{N})$. A version of this theorem suitable for quasi-metric spaces is presented next.

Theorem 3.26 (The quasi-metric space version of Fréchet's embedding theorem). *Let $(X, \mathbf{q})$ be a quasi-metric space with the property that the topological space $(X, \tau_{\mathbf{q}})$ is separable. Also, assume that the real number β is such that there exists a quasi-distance $\rho \in \mathbf{q}$ for which $0 < \beta \leq [\log_2 C_\rho]^{-1}$.*

Then the quasi-metric space $(X, \mathbf{q})$ admits a bi-Lipschitz embedding into the quasi-Banach space $\big(\ell^\infty(\mathbb{N}), \|\cdot\|^{1/\beta}_{\ell^\infty(\mathbb{N})}\big)$. More precisely, if $\rho_\#$ denotes the regularized version of ρ (as in Theorem 2.12) then there exists a mapping $\Psi : X \to \ell^\infty(\mathbb{N})$ which satisfies

$$\|\Psi(x) - \Psi(y)\|^{1/\beta}_{\ell^\infty(\mathbb{N})} = \rho_\#(x, y), \quad \forall\, x, y \in X. \tag{3.91}$$

Proof. Assume that the quasi-distance $\rho \in \mathbf{q}$ and the real number β are as in the first part of the theorem, and select a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ which is dense in $(X, \tau_{\mathbf{q}})$. As before, we let $\rho_\#$ be the regularized version of ρ (defined in Theorem 2.12). To proceed, consider the mapping

$$\begin{aligned} &\Phi : \{x_j : j \in \mathbb{N}\} \longrightarrow \ell^\infty(\mathbb{N}), \ \text{ defined for each } j \in \mathbb{N} \text{ by} \\ &\Phi(x_j) := \Big\{ \big[\rho_\#(x_j, x_k)\big]^\beta - \big[\rho_\#(x_k, x_1)\big]^\beta \Big\}_{k \in \mathbb{N}}. \end{aligned} \tag{3.92}$$

Based on the fact that the function defined in (2.117) is a distance, we may then estimate

$$\begin{aligned} \big[\rho_\#(x_i, x_j)\big]^\beta &= \big(\Phi(x_i) - \Phi(x_j)\big)_j \\ &\leq \|\Phi(x_i) - \Phi(x_j)\|_{\ell^\infty(\mathbb{N})} \leq \big[\rho_\#(x_i, x_j)\big]^\beta, \quad \forall\, i, j \in \mathbb{N} \end{aligned} \tag{3.93}$$

hence, ultimately,

$$\big[\rho_{\#}(x_i, x_j)\big]^{\beta} = \|\Phi(x_i) - \Phi(x_j)\|_{\ell^\infty(\mathbb{N})}, \quad \forall\, i, j \in \mathbb{N}. \tag{3.94}$$

From the density of $\{x_j\}_{j\in\mathbb{N}}$ in $(X, \tau_{\mathbf{q}})$, the continuity of the function $\rho_{\#}$ which maps $(X \times X, \tau_{\mathbf{q}} \times \tau_{\mathbf{q}})$ to $[0, +\infty)$ (which is a consequence of results established in Theorem 2.12), and (3.94) it follows that the function Φ extends to a mapping $\Psi : X \to \ell^\infty(\mathbb{N})$ which satisfies (3.91). In concert with (2.134), this estimate then gives

$$C_\rho^{-2}\rho(x, y) \le \|\Psi(x) - \Psi(y)\|^{1/\beta}_{\ell^\infty(\mathbb{N})} \le C_0\, \rho(x, y), \quad \forall\, x, y \in X. \tag{3.95}$$

Hence, Ψ is a bi-Lipschitz embedding of $(X, \mathbf{q})$ into $\big(\ell^\infty(\mathbb{N}), \|\cdot\|^{1/\beta}_{\ell^\infty(\mathbb{N})}\big)$ which also satisfies (3.91). $\square$

Chapter 4

Outer Measures

For the majority of this chapter we step away from quasi-metric spaces, though they'll arise at the end. Between now and then we'll develop some measure theory which will be of great use when discussing the Hausdorff outer measure on quasi-metric spaces, the topic of section (4.3). To begin, we lay down a number of definitions.

Definition 4.1. *Given a set X, a* `sigma-algebra` $\mathfrak{M}$ `of subsets of` X*, or, more simply, a* `sigma-algebra` $\mathfrak{M}$ `on` X *is any* $\mathfrak{M} \subseteq 2^X$ *(with 2^X denoting the collection of all subsets of X) satisfying:*

$$X \in \mathfrak{M} \tag{4.1}$$

$$\text{if } A \in \mathfrak{M} \text{ then } X \setminus A \in \mathfrak{M} \tag{4.2}$$

$$\text{if } A_j \in \mathfrak{M}, j \in \mathbb{N}, \text{then } \bigcup_{j\in\mathbb{N}} A_j \in \mathfrak{M}. \tag{4.3}$$

The pair $(X, \mathfrak{M})$ is called a `measurable space`*.*

For example, the simplest sigma-algebra of a set X is just X itself and the empty set. The power set of X is also a sigma-algebra. As is the collection of subsets of X which are either countable or whose complements are countable.

Definition 4.2. *Let X be a set and $\mathfrak{M}$ a sigma-algebra on X. A* `measure` *μ on X is an extended real-valued, nonnegative function defined on $\mathfrak{M}$ $\mu^* : \mathfrak{M} \to [0, +\infty]$ satisfying*

$$\mu(\emptyset) = 0 \tag{4.4}$$

$$\begin{aligned} &\mu(A) = \sum_{j \in \mathbb{N}} \mu(A_j) \text{ if } A, A_j \in \mathfrak{M}, \\ &\qquad j \in \mathbb{N}, \text{ with } A = \bigcup_{j \in \mathbb{N}} A_j,\ A_j \text{ mutually disjoint.} \end{aligned} \tag{4.5}$$

A set $A \subseteq X$ is called μ-`measurable` *provided $A \in \mathfrak{M}$. Call the triplet $(X, \mathfrak{M}, \mu)$ a* `measure space`.

Lastly, a measure space is said to be `complete` *if every subset of any set of measure 0 is also measurable.*

Note to have a measure one must have a sigma-algebra.

Lemma 4.1. *Given a measure space $(X, \mathfrak{M}, \mu)$, the following hold:*

(1) (4.5)$\Longrightarrow$(4.4) *provided there is at least one element in $\mathfrak{M}$ (other than the empty set) with finite measure.*

(2) If $A_1, A_2 \in \mathfrak{M}$ such that $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.

(3) $\mu(\bigcup\limits_{j \in \mathbb{N}} A_j) \leq \sum_{j \in \mathbb{N}} \mu(A_j)$ *for* $A_j \in \mathfrak{M}, j \in \mathbb{N}$.

(4) $\mu(\bigcup\limits_{j \in \mathbb{N}} A_j) = \lim\limits_{j \to \infty} \mu(A_j)$ *for* $A_j \in \mathfrak{M}, A_j \subseteq A_{j+1}, j \in \mathbb{N}$.

(5) $\mu(\bigcap\limits_{j \in \mathbb{N}} A_j) = \lim\limits_{j \to \infty} \mu(A_j)$ *for* $A_j \in \mathfrak{M}, A_{j+1} \subseteq A_j, j \in \mathbb{N}$ *provided at least one A_j has finite measure (think of a counterexample when A_j has infinite measure for all j).*

Proof. The above properties are straightforward consequences of definitions. □

Definition 4.3. *Let X be a fixed, arbitrary set. An* `outer measure` *μ^* on X is an extended real-valued, nonnegative function $\mu^* : 2^X \to [0, +\infty]$ satisfying*

$$\mu^*(\emptyset) = 0 \text{ and } \mu^*(A) \leq \sum_{j \in \mathbb{N}} \mu^*(A_j) \quad \text{if } A, A_j \subseteq X,\ j \in \mathbb{N}, \text{ with } A \subseteq \bigcup_{j \in \mathbb{N}} A_j. \tag{4.6}$$

A set $A \subseteq X$ is called μ^*-`measurable` *provided*

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A), \quad \text{for each } \ Y \subseteq X. \tag{4.7}$$

On the other hand, an outer measure is defined on every subset of the ambient space. We set

$$\mathfrak{M}_{\mu^*} := \{A \subseteq X : \ A \text{ is } \mu^*\text{-measurable}\}. \tag{4.8}$$

There is a simple procedure which associates to μ a complete measure $\overline{\mu}$ on X. Specifically,

$$\overline{\mathfrak{M}} := \{A \cup E : \ A \in \mathfrak{M},\ E \subseteq B \in \mathfrak{M},\ \mu(B) = 0\}, \tag{4.9}$$

is a sigma-algebra on X which contains $\mathfrak{M}$, and $\overline{\mu} : \overline{\mathfrak{M}} \to [0, +\infty]$ defined by

$$\overline{\mu}(A \cup E) := \mu(A) \text{ whenever } A \in \mathfrak{M} \tag{4.10}$$

$$\text{and } E \subseteq B \in \mathfrak{M} \text{ are such that } \mu(B) = 0, \tag{4.11}$$

is a well-defined measure which is complete and extends μ. It is then easy to check that the null sets for $\overline{\mu}$ are precisely all subsets of null-sets of μ, i.e.,

$$\{A \in \overline{\mathfrak{M}} : \ \overline{\mu}(A) = 0\} = \{E \subseteq X : \ \exists\, B \in \mathfrak{M} \tag{4.12}$$

$$\text{such that } E \subseteq B \text{ and } \mu(B) = 0\}.$$

Given an arbitrary set $E \subseteq X$, the restriction of the outer measure μ^* to E is the outer measure $\mu^*\lfloor E$ on E defined by

$$\mu^*\lfloor E := \mu^*\Big|_{2^E}. \tag{4.13}$$

Remark 4.1. *For any $E \subseteq X$,*

$$\mu^*\lfloor E \textit{ is an outer measure on } E, \textit{ and } \{A \cap E : A \in \mathfrak{M}_{\mu^*}\} \subseteq \mathfrak{M}_{\mu^*\lfloor E}. \tag{4.14}$$

Remark 4.2. *A closely related version of the restriction of the outer measure to an arbitrary set as defined in* (4.13) *is as follows. Assume that X is a given set, $E \subseteq X$ and μ^* is an outer measure on E. Then it can be easily verified that μ^*_E defined by*

$$\mu^*_E(A) := \mu^*(A \cap E), \quad \forall\, A \subseteq X, \tag{4.15}$$

is an outer measure on X. We stress that while μ^*_E is an outer measure defined on X, the outer measure $\mu^*\lfloor E$ is defined on E. Depending on circumstances there may be some advantages in considering one version of the restriction of outer measures over the other. For now, we wish to point out that, as is seen from (4.8),

$$\mu^* \textit{ outer measure on } X \textit{ and } E \subseteq X \Longrightarrow \mathfrak{M}_{\mu^*} \subseteq \mathfrak{M}_{\mu^*_E}. \tag{4.16}$$

Definition 4.4. *Assume that X is a given set, $E \subseteq X$ and μ^* is an outer measure on E. Define $(\mu^*)^X$, the* `lifting` *of μ^* from E to X by setting*

$$(\mu^*)^X(A) := \mu^*(A \cap E), \quad \forall\, A \subseteq X. \tag{4.17}$$

Then the following properties can be verified based on definitions.

Remark 4.3. *Let X be a set and μ^* an outer measure on X, then*

(1) $(\mu^*)^X$ *is an outer measure on* X *and*

(2) $\{A \subseteq X : A \cap E \in \mathfrak{M}_{\mu^*}\} \subseteq \mathfrak{M}_{(\mu^*)^X}$.

Passing from an given outer measure to a genuine measure is typically done using Carathéodory's classical theorem, according to which, if μ^* is an outer measure on X then

$$\mathfrak{M}_{\mu^*} \text{ is a sigma-algebra, and } \widetilde{\mu} := \mu^*\big|_{\mathfrak{M}_{\mu^*}} \text{ is a complete measure.} \tag{4.18}$$

Conversely, to any given measure space $(X, \mathfrak{M}, \mu)$ one can associate an outer measure μ^* by setting, for each $A \subseteq X$,

$$\begin{aligned} \mu^*(A) &:= \inf\Big\{\sum_{j\in\mathbb{N}} \mu(A_j) : A_j \in \mathfrak{M},\ j \in \mathbb{N},\ A \subseteq \bigcup_{j\in\mathbb{N}} A_j\Big\} \\ &= \inf\Big\{\mu(E) : A \subseteq E \in \mathfrak{M}\Big\}. \end{aligned} \tag{4.19}$$

Then (see, e.g., [85, # 9, p. 68]),

$$\left.\begin{array}{l}\text{whenever } (X, \mathfrak{M}, \mu) \text{ is a measure space and the} \\ \text{outer measure } \mu^* \text{ is associated with } \mu \text{ as in (4.19)}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \mathfrak{M}_{\mu^*} = \overline{\mathfrak{M}} \text{ and} \\ \mu^*\big|_{\mathfrak{M}_{\mu^*}} = \overline{\mu}, \end{array}\right. \tag{4.20}$$

i.e., the measure $\widetilde{\mu}$ (defined in (4.18)) becomes the completion $\overline{\mu}$ of μ (given in (4.10)). Furthermore, under the same assumptions as in the left-hand side of (4.20), it follows that

$$\text{for every } A \subseteq X \text{ there exists } B \in \mathfrak{M} \text{ such that } A \subseteq B \text{ and } \mu^*(A) = \mu(B). \tag{4.21}$$

See, e.g., the last statement of [79, Proposition 6, p. 293].

Definition 4.5. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and assume that* $\mathfrak{M}_0$ *is a sigma-algebra of subsets of* X *contained in* $\mathfrak{M}$*. Then the measure* μ *is said to be* $\mathfrak{M}_0$-`regular` *provided*

$$\forall\, A \in \mathfrak{M}\ \ \exists\, B \in \mathfrak{M}_0 \text{ such that } A \subseteq B \text{ and } \mu(A) = \mu(B). \tag{4.22}$$

In analogy with Definition 4.5, we also introduce the following piece of terminology.

Definition 4.6. *Let X be an arbitrary set and assume that μ^* is an outer measure on X. Furthermore, suppose that $\mathfrak{M}$ is a sigma-algebra of subsets of X. Then the outer measure μ^* is called* $\mathfrak{M}$-`regular` *provided $\mathfrak{M} \subseteq \mathfrak{M}_{\mu^*}$ and*

$$\forall\, A \subseteq X\ \ \exists\, B \in \mathfrak{M} \text{ such that } A \subseteq B \text{ and } \mu^*(A) = \mu^*(B). \tag{4.23}$$

There is a close relationship between the concept of regularity for genuine measures, as discussed in Definition 4.5, and the notion of regularity for outer measures, from Definition 4.6. More specifically, the following result holds.

Lemma 4.2. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and assume that $\mathfrak{M}_0$ is a sigma-algebra of subsets of X contained in $\mathfrak{M}$. Denote by μ^* the outer measure associated with μ as in* (4.19)*. Then*

the measure μ is $\mathfrak{M}_0$-regular (in the sense of Definition 4.5) $\Longleftrightarrow$

the outer measure μ^* is $\mathfrak{M}_0$-regular (in the sense of Definition 4.6). (4.24)

As a corollary,

$$\text{the outer measure } \left(\mu\big|_{\mathfrak{M}_0}\right)^* \text{ is } \mathfrak{M}_0\text{-regular.} \tag{4.25}$$

Proof. The left-pointing implication in (4.24) is a simple consequence of (4.21) and definitions, so we shall focus on the right-pointing implication in (4.24). To this end, fix an arbitrary set $E \subseteq X$. If $\mu^*(E) = +\infty$ then $\mu^*(X) = +\infty$ and $E \subseteq X \in \mathfrak{M}_0$.

There remains to treat the case when $\mu^*(E) < +\infty$. In this scenario, from (4.19) we know that for each $j \in \mathbb{N}$ there exists $A_j \in \mathfrak{M}_0$ with the property that $E \subseteq A_j$ and $\mu(A_j) < \mu^*(E) + 1/j$. Next, using the fact that the measure μ is $\mathfrak{M}_0$-regular, for each $j \in \mathbb{N}$ it is possible to find $B_j \in \mathfrak{M}_0$ such that $A_j \subseteq B_j$ and $\mu(A_j) = \mu(B_j)$. Consequently,

$$E \subseteq B_j \in \mathfrak{M}_0 \text{ and } \mu^*(E) \leq \mu(B_j) < \mu^*(E) + 1/j \text{ for very } j \in \mathbb{N}. \tag{4.26}$$

Hence, if we set $B := \bigcap_{j \in \mathbb{N}} B_j$, it follows that

$$E \subseteq B \in \mathfrak{M}_0 \text{ and } \mu^*(E) \leq \mu(B) < \mu^*(E) + 1/j \text{ for very } j \in \mathbb{N}. \tag{4.27}$$

Note that the above double inequality forces the equality $\mu^*(E) = \mu(B)$ hence, ultimately, $\mu^*(E) = \mu^*(B)$ since $B \in \mathfrak{M}$. All in all, the above reasoning shows that μ^* is a $\mathfrak{M}_0$-regular outer measure.

Finally, (4.25) is a direct consequence of (4.24) and the obvious fact that $\mu\big|_{\mathfrak{M}_0}$ is $\mathfrak{M}_0$-regular. $\square$

Lemma 4.3. *Assume that X is an arbitrary set and that μ^* is an outer measure on X. Also, suppose that $\mathfrak{M}$ is a sigma-algebra of subsets of X with the property that $\mathfrak{M} \subseteq \mathfrak{M}_{\mu^*}$. Then*

$$\mu^* \textit{ is } \mathfrak{M}\textit{-regular} \iff \mu^* = \Big(\mu^*\big|_{\mathfrak{M}}\Big)^*. \tag{4.28}$$

Proof. To prove the right-pointing implication in (4.28), observe that if $A \subseteq X$ is arbitrary then, on the one hand,

$$\Big(\mu^*\big|_{\mathfrak{M}}\Big)^*(A) = \inf\Big\{\mu^*(E) : A \subseteq E \in \mathfrak{M}\Big\} \geq \mu^*(A), \tag{4.29}$$

by the monotonicity of μ^*. On the other hand, given that the outer measure μ^* is $\mathfrak{M}$-regular, it follows that there exists a set $B \in \mathfrak{M}$ such that $A \subseteq B$ and

$$\mu^*(A) = \mu^*(B).$$

In turn, this shows that the opposite of the inequality in (4.29) also holds and, hence, $\left(\mu^*\Big|_{\mathfrak{M}}\right)^*(A) = \mu^*(A)$. Since the left-pointing implication in (4.28) is a direct consequence of (4.25) in Lemma 4.2, the desired conclusion follows. □

Definition 4.7. *Assume that X is a given set and that $\mathfrak{M}$ is a sigma-algebra of subsets of X. Call two measures $(\mathfrak{M}, \mu_1)$ and $(\mathfrak{M}, \mu_2)$ on X* `equivalent`*, and write $\mu_1 \approx \mu_2$, provided there exist $c, C \in (0, +\infty)$ such that*

$$c\mu_1(E) \leq \mu_2(E) \leq C\mu_1(E), \quad \forall\, E \in \mathfrak{M}. \tag{4.30}$$

Moreover, call two outer measures μ_1^* and μ_2^* on X `equivalent`*, and write $\mu_1^* \approx \mu_2^*$, if there exist $c, C \in (0, +\infty)$ such that*

$$c\mu_1^*(E) \leq \mu_2^*(E) \leq C\mu_1^*(E), \quad \forall\, E \subseteq X. \tag{4.31}$$

It is clear that $\approx$ is an equivalent relation and that if μ_1, μ_2 are defined on a common sigma-algebra then

$$\mu_1 \approx \mu_2 \text{ as measures} \Longrightarrow (\mu_1)^* \approx (\mu_2)^* \text{ as outer measures.} \tag{4.32}$$

Furthermore, if μ_1^* and μ_2^* are two outer measures on X then

$$\mu_1^* \approx \mu_2^* \text{ and } \mathfrak{M} \subseteq \mathfrak{M}_{\mu_1^*} \cap \mathfrak{M}_{\mu_2^*} \text{ sigma-algebra} \Longrightarrow \mu_1^*\Big|_{\mathfrak{M}} \approx \mu_2^*\Big|_{\mathfrak{M}}. \tag{4.33}$$

The last result in this section gives a complete characterization of the equivalence of measures in terms of the nature of Radon-Nikodym derivatives.

Proposition 4.4. *Suppose that* $(X, \mathfrak{M}, \mu)$ *is a sigma-finite measure space. The following are equivalent:*

(a) $\mu' : \mathfrak{M} \to [0, +\infty]$ *is a measure with the property that* $\mu' \approx \mu$ *(in the sense of Definition 4.7).*

(b) There exists a unique function f *satisfying the following properties:*

$$\begin{aligned} &(i)\ f \text{ is } \mathfrak{M}-\text{measurable}, \\ &(ii)\ \exists\, A \in \mathfrak{M} \text{ with } \mu(A) = 0 \text{ such that } c \leq f \leq C \text{ on } X \setminus A, \\ &(iii)\ \mu' = f\mu, \end{aligned} \tag{4.34}$$

where the constants $c, C \in (0, +\infty)$ *are as in* (4.30).

Proof. Consider the implication $(a) \Rightarrow (b)$. Since

$$\mu' \approx \mu \Longrightarrow \mu' \ll \mu, \tag{4.35}$$

the Radon-Nikodym Theorem gives the existence of a nonnegative function f satisfying (i) and (iii) in (4.34). Moreover, (see [80, Theorem 1.40, p. 30]) there exists $A \in \mathfrak{M}$ with the property that $\mu(A) = 0$ and

$$f(x) \in \overline{\left\{ \tfrac{1}{\mu(E)} \int_E f \, d\mu : \, E \in \mathfrak{M}, \, \mu(E) > 0 \right\}} \quad \text{for } x \in X \setminus A, \tag{4.36}$$

with the closure taken in the canonical topology of $\mathbb{R}$. On the other hand, if $E \in \mathfrak{M}$ is such that $\mu(E) > 0$, the equivalence of μ and μ' (written as in (4.30)) gives

$$\frac{1}{\mu(E)} \int_E f \, d\mu = \frac{\mu'(E)}{\mu(E)} \in [c, C]. \tag{4.37}$$

From this and (4.36) it then follows that f also satisfies $c \leq f(x) \leq C$ for each $x \in X \setminus A$, for some $A \in \mathfrak{M}$ with $\mu(A) = 0$. This completes the proof of $(a) \Rightarrow (b)$. Finally, the implication $(b) \Rightarrow (a)$ is a direct consequence of definitions. $\square$

4.1 Borel and Radon Measures

While we have already encountered the notion of a topology generated by a quasi-metric, we have not seen the general definition of a topology.

Definition 4.8. *A* `topological space` *is a set X equipped with a topology τ, i.e., a family $\mathscr{O} \subseteq 2^X$ satisfying the following properties:*

(1) for every family $\{\mathcal{O}_j\}_{j\in J}$ with $\mathcal{O}_j \in \mathscr{O}$ for all $j \in J$ there holds

$\bigcup_{j\in J} \mathcal{O}_j \in \mathscr{O}$;

(2) for every $\{\mathcal{O}_j\}_{j\in J}$ with $\#J$ finite and $\mathcal{O}_j \in \mathscr{O}$ for all $j \in J$ there holds

$\bigcap_{j\in J} \mathcal{O}_j \in \mathscr{O}$;

(3) $\emptyset$, $X \in \mathscr{O}$.

Then $Borel(X) = Borel_\tau(X)$ is the smallest sigma-algebra containing $\mathscr{O}$. We shall refer to $\mathscr{O}$ as the collection of `open sets` *in X (in the topology τ) and to*

$$\mathscr{O}^c := \{X \setminus \mathcal{O}\}_{\mathcal{O}\in\mathscr{O}} \tag{4.38}$$

as the collection of `closed sets` *in X (in the topology τ).*

Finally, if (X, τ) is a topological space and $A \subseteq X$ then $\tau|_A$, the topology induced by τ on A, is defined by taking $\{A \cap O : O \in \mathscr{O}\}$ to be the collection of open sets in A.

Of course, given a topological space (X, τ) and $A \subseteq X$, then $(A, \tau|_A)$ becomes a topological space itself. In this connection, it is useful to remark that for any $A \subseteq X$,

$$\{A \cap B : B \in Borel_\tau(X)\} = Borel_{\tau|_A}(A). \tag{4.39}$$

Indeed, if we consider

$$\mathcal{F} := \{A \cap B : B \in Borel_\tau(X)\}, \quad \mathcal{G} := \{B \subseteq X : B \cap A \in Borel_{\tau|_A}(A)\}, \quad (4.40)$$

then it is easily checked that $\mathcal{F}$ is a sigma-algebra of subsets of A which contains the open subsets of $(A, \tau|_A)$, whereas $\mathcal{G}$ is a sigma-algebra of subsets of X which contains the open subsets of (X, τ). Consequently, $Borel_{\tau|_A}(A) \subseteq \mathcal{F}$ and $Borel_\tau(X) \subseteq \mathcal{G}$. Now, the first of these two inclusions yields the right-to-left inclusion in (4.39), while the second one gives the left-to-right inclusion in (4.39). Hence, (4.39) follows.

Definition 4.9. *Let (X, τ) be a topological space. A measure $(\mathfrak{M}, \mu)$ on X is called a* `Borel measure` *(or a* `Borelian measure`*) if $Borel_\tau(X) \subseteq \mathfrak{M}$. Also, call μ a* `Borel regular measure` *if μ is $Borel_\tau(X)$-regular.*

Definition 4.10. *If (X, τ) is a topological space and μ^* is an outer measure on X, call μ^* a* `Borel outer measure` *on X if $Borel_\tau(X) \subseteq \mathfrak{M}_{\mu^*}$. Furthermore, call μ^* a* `Borel regular outer measure` *if μ^* is $Borel_\tau(X)$-regular.*

Remark 4.4. *Given a topological space (X, τ) and a Borel outer measure μ^* on X, it follows from Lemma 4.3 that*

$$\mu^* \text{ Borel regular outer measure} \Longleftrightarrow \mu^* = \left(\mu^*\big|_{Borel_\tau(X)}\right)^*. \quad (4.41)$$

Lemma 4.5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space such that μ is either complete or Borelian. If μ^* denotes the outer measure associated with μ as in (4.19) then*

$$\mu \text{ is a Borel regular measure} \Longleftrightarrow \mu^* \text{ is a Borel regular outer measure.} \quad (4.42)$$

Proof. This follows from Lemma 4.2, (4.20) and definitions. □

Lemma 4.6. *Let (X, τ) be a topological space and assume that μ is a Borel measure on X. Then*

$$\left(\mu\big|_{Borel_\tau(X)}\right)^* \text{ is a Borel regular outer measure on } X. \tag{4.43}$$

Moreover,

$$\mu^* \text{ Borel regular outer measure} \Longleftrightarrow \mu^* = \left(\mu\big|_{Borel_\tau(X)}\right)^*. \tag{4.44}$$

Proof. The claim in (4.43) follows from definitions and the second part in Lemma 4.2, whereas (4.44) is easily seen from (4.43) and Lemma 4.3. □

Observe that, by (4.16) and definitions, if (X, τ) is a topological space and μ^* is a Borel outer measure on X then for every $A \subseteq X$ it follows that μ^*_A is also a Borel outer measure on X. Similar issues for the "floor" restriction of outer measures are explored in the lemma below.

Lemma 4.7. *Let (X, τ) be a topological space and assume that μ^* is a Borel outer measure on X. Then for every $A \subseteq X$ it follows that $\mu^* \lfloor A$ is a Borel outer measure on $(A, \tau|_A)$.*

If, in addition, μ^* is Borel regular, then for every $A \subseteq X$ the Borel outer measure $\mu^* \lfloor A$ on $(A, \tau|_A)$ is regular.

Proof. To deal with the claim made in the first part of the statement of the lemma, fix an arbitrary set $A \subseteq X$. From (4.14) we know that $\mu^* \lfloor A$ is an outer measure on A and, by (4.39) and the fact that $Borel_\tau(X) \subseteq \mathfrak{M}_{\mu^*}$,

$$\begin{aligned} Borel_{\tau|_A}(A) &= \{A \cap B : B \in Borel_\tau(X)\} \\ &\subseteq \{A \cap B : B \in \mathfrak{M}_{\mu^*}\} \subseteq \mathfrak{M}_{\mu^* \lfloor A}, \end{aligned} \tag{4.45}$$

where we have also used the second part in (4.14). This allows us to conclude that $\mu^*\lfloor A$ is a Borel outer measure on A.

There remains to settle the regularity issue, in the second part of the statement of the lemma. To this end, assume that μ^* is a Borel regular outer measure on X and let $E \subseteq A$ be arbitrary. Then there exists $B \in Borel_\tau(X)$ such that $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$. Then if we set $B_o := B \cap A$, we have $E \subseteq B_o \subseteq A$ and $B_o \in Borel_{\tau|_A}(A)$ by (4.39). By the monotonicity of the outer measures μ^* and $\mu^*\lfloor A$, and keeping in mind these inclusions, we may write

$$(\mu^*\lfloor A)(B_o) = \mu^*(B_o) \leq \mu^*(B) = \mu^*(E) = (\mu^*\lfloor A)(E) \leq (\mu^*\lfloor A)(B_o). \tag{4.46}$$

Hence, $(\mu^*\lfloor A)(E) = (\mu^*\lfloor A)(B_o)$ which, given what we have shown already, proves that $\mu^*\lfloor A$ is a Borel regular outer measure on A. $\square$

Remark 4.5. *Let (X, τ) be a topological space, $E \subseteq X$ arbitrary, and assume that μ^* is a Borel outer measure on E. Lift μ^* to an outer measure $(\mu^*)^X$ on X as in Remark 4.4. It follows then from property (ii) in Remark 4.4 and* (4.39) *that if μ^* is a Borel outer measure on E then $(\mu^*)^X$ is a Borel outer measure on X.*

For the next lemma see also [20, Lemma 1, p. 6] for the case $X = \mathbb{R}^n$ and also [21, Theorem 2.2.2, p. 60] for the case when X is a metric space.

Lemma 4.8. *Assume that (X, τ) is a topological space and that μ^* is a Borel outer measure on X. Denote by $\mathscr{O}$ the collection of open sets in the topology τ and suppose that (X, τ) has the property that*

$$\begin{array}{c}\textit{any open set (in the topology } \tau\textit{) can be written as}\\ \textit{a countable union of closed sets (in the topology } \tau\textit{).}\end{array} \tag{4.47}$$

Then one has

$$B \in Borel_\tau(X) \textit{ and } \mu^*(B) < +\infty \Longrightarrow \mu^*(B) = \sup_{C \in \mathscr{O}^c,\, C \subseteq B} \mu^*(C). \tag{4.48}$$

Proof. Fix $B \in Borel_\tau(X)$ for which $\mu^*(B) < +\infty$ and let $\nu := \mu^*\lfloor_B$. Then ν is a finite Borel outer measure on B. Define the set

$$\begin{aligned} \mathcal{F} \;:=\; \{A \subseteq X :\ & A \text{ is } \mu\text{-measurable and } \forall\, \varepsilon > 0 \text{ there exists} \\ & C \subseteq A \text{ closed set such that } \nu(A \setminus C) < \varepsilon\}. \end{aligned} \tag{4.49}$$

Then clearly all closed sets in X belong to $\mathcal{F}$. We next claim that

$$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F} \Longrightarrow \bigcap_{i\in\mathbb{N}} A_i \in \mathcal{F} \quad \text{and} \quad \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}. \tag{4.50}$$

To prove (4.50), assume that $A_i \in \mathcal{F}$ for each $i \in \mathbb{N}$ and fix an arbitrary $\varepsilon > 0$. Then, for each $i \in \mathbb{N}$, there exists a closed set $C_i \subseteq A_i$ such that $\nu(A_i \setminus C_i) < \varepsilon/2^i$. Consequently, $\bigcap\limits_{i\in\mathbb{N}} C_i$ is a closed set contained in $\bigcap\limits_{i\in\mathbb{N}} A_i$, and we have

$$\nu\Big(\bigcap_{i\in\mathbb{N}} A_i \setminus \bigcap_{i\in\mathbb{N}} C_i\Big) \leq \nu\Big(\bigcup_{i\in\mathbb{N}} (A_i \setminus C_i)\Big) \leq \sum_{i\in\mathbb{N}} \nu(A_i \setminus C_i) < \sum_{i\in\mathbb{N}} 2^{-i}\varepsilon = \varepsilon, \tag{4.51}$$

proving that $\bigcap\limits_{i\in\mathbb{N}} A_i \in \mathcal{F}$. Also, since ν is finite, we can apply part (4) from Lemma 4.1 (note that, by (4.18), $\mu^*\Big|_{\mathfrak{M}_{\mu^*}}$ is a measure on X) to write

$$\begin{aligned} \lim_{N\to\infty} \nu\Big(\bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{i=1}^{N} C_i\Big) &= \nu\Big(\bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{i\in\mathbb{N}} C_i\Big) \leq \nu\Big(\bigcup_{i\in\mathbb{N}} (A_i \setminus C_i)\Big) \\ &\leq \sum_{i\in\mathbb{N}} \nu(A_i \setminus C_i) < \sum_{i\in\mathbb{N}} 2^{-i}\varepsilon = \varepsilon. \end{aligned} \tag{4.52}$$

Hence, there exists $N_o \in \mathbb{N}$ such that $\nu\Big(\bigcup\limits_{i\in\mathbb{N}} A_i \setminus \bigcup\limits_{i=1}^{N_o} C_i\Big) < \varepsilon$. The latter, together with the fact that $\bigcup\limits_{i=1}^{N_o} C_i$ is closed proves that $\bigcup\limits_{i\in\mathbb{N}} A_i \in \mathcal{F}$. This completes the proof of (4.50).

In light of (4.47), what we proved so far also implies that all open sets in X are contained in $\mathcal{F}$.

Consider next the set

$$\mathcal{G} := \{A \in \mathcal{F} : X \setminus A \in \mathcal{F}\}. \tag{4.53}$$

It is trivial that if $A \in \mathcal{G}$ then $X \setminus A \in \mathcal{G}$, so $\mathcal{G}$ is closed under taking complements. Since we proved that $\mathcal{F}$ contains all open and closed sets of X, it follows that $\mathcal{G}$ also contains all open and closed sets of X. Moreover, $\mathcal{G}$ is closed under taking countable unions. Indeed, if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{G}$, then by definition $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{X \setminus A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$, so that by (4.50) we have $\bigcup\limits_{i\in\mathbb{N}} A_i \in \mathcal{F}$ and $X \setminus \bigcup\limits_{i\in\mathbb{N}} A_i = \bigcap_{i\in\mathbb{N}}(X \setminus A_i) \in \mathcal{F}$. This proves that $\bigcup\limits_{i\in\mathbb{N}} A_i \in \mathcal{G}$ as desired. Summing up, we have proved that $\mathcal{G}$ is a sigma-algebra containing all open sets of X. This implies that $\mathcal{G}$ contains also $Borel_\tau(X)$ and, in particular, $B \in \mathcal{G}$. The latter implies that $B \in \mathcal{F}$ and satisfies (4.48). $\square$

Lemma 4.9. *Assume that (X, τ) is a topological space which satisfies (4.47) and let $\mathscr{O}$ denote the collection of open sets in the topology τ. Suppose that μ^* is a Borel outer measure on X with the property that*

$$\begin{gathered} \textit{there exist } \{O_j\}_{j\in\mathbb{N}} \subseteq \mathscr{O} \textit{ so that } X = \bigcup_{j\in\mathbb{N}} O_j \\ \textit{and } \mu^*(O_j) < +\infty \quad \forall\, j \in \mathbb{N}. \end{gathered} \tag{4.54}$$

Then

$$\forall\, B \in Borel_\tau(X),\ \forall\, \varepsilon > 0,\ \exists\, O \in \mathscr{O} \textit{ with } B \subseteq O \textit{ and } \mu^*(O \setminus B) < \varepsilon. \tag{4.55}$$

Moreover,

$$\begin{gathered} \mu^* \textit{ Borel regular outer measure on } X \textit{ satisfying (4.54)} \\ \Longrightarrow \mu^*(A) = \inf_{O\in\mathscr{O},\, A\subseteq O} \mu^*(O), \quad \forall\, A \subseteq X. \end{gathered} \tag{4.56}$$

Proof. Introduce

$$U_i := \bigcup_{1 \leq j \leq i} O_j, \quad \forall\, i \in \mathbb{N}, \tag{4.57}$$

so that

$$X = \bigcup_{i \in \mathbb{N}} U_i \quad \text{and} \quad U_i \in \mathscr{O}, \quad \mu^*(U_i) < +\infty, \quad U_i \subseteq U_{i+1} \quad \forall\, i \in \mathbb{N}. \tag{4.58}$$

Also, fix some set $B \in Borel_\tau(X)$ along with an arbitrary number $\varepsilon > 0$. Then for each $i \in \mathbb{N}$ we have that $U_i \setminus B \in Borel_\tau(X)$ and $\mu(U_i \setminus B) < +\infty$. Consequently, we may invoke Lemma 4.8 and deduce that there exists a set $C_i \in \mathscr{O}^c$ with the property that $C_i \subseteq U_i \setminus B$ and

$$\mu^*\big((U_i \setminus B) \setminus C_i\big) < 2^{-i}\varepsilon. \tag{4.59}$$

Note that

$$O := \bigcup_{i \in \mathbb{N}} (U_i \setminus C_i) \Longrightarrow O \in \mathscr{O}. \tag{4.60}$$

Since for each $i \in \mathbb{N}$ we have $C_i \subseteq X \setminus B$ it follows that $U_i \cap B \subseteq U_i \setminus C_i$, so that

$$B = \bigcup_{i \in \mathbb{N}} (U_i \cap B) \subseteq \bigcup_{i \in \mathbb{N}} (U_i \setminus C_i) = O. \tag{4.61}$$

Furthermore, by (4.60), (4.59) and the subadditivity of the outer measure μ^*,

$$\begin{aligned} \mu^*(O \setminus B) &= \mu^*\Big(\bigcup_{i \in \mathbb{N}} \big((U_i \setminus C_i) \setminus B\big)\Big) \leq \sum_{i \in \mathbb{N}} \mu^*\big((U_i \setminus C_i) \setminus B\big) \\ &= \sum_{i \in \mathbb{N}} \mu^*\big((U_i \setminus B) \setminus C_i\big) \leq \sum_{i \in \mathbb{N}} 2^{-i}\varepsilon = \varepsilon. \end{aligned} \tag{4.62}$$

Now, (4.55) follows from (4.60), (4.61) and (4.62).

As far as (4.56) is concerned, assume that μ^* is a Borel regular outer measure on X and let $A \subseteq X$ be arbitrary. If $\mu^*(A) = +\infty$ there is nothing to prove, so assume in what follows that $\mu^*(A) < +\infty$. Also, fix an arbitrary $\varepsilon > 0$. Given that μ^* is a Borel regular outer measure, there exists $B \in Borel_\tau(X)$ with the property that $A \subseteq B$ and $\mu^*(B) = \mu^*(A)$. Going further, by (4.55), one may find $O \in \mathscr{O}$ such that $B \subseteq O$ and $\mu^*(O \setminus B) < \varepsilon$. This entails $A \subseteq O$ and since $O = (O \setminus B) \cup B$, the subadditivity of μ^* gives

$$\mu^*(O) \leq \mu^*(O \setminus B) + \mu^*(B) < \varepsilon + \mu^*(A). \tag{4.63}$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\mu^*(A) \geq \inf_{O \in \mathscr{O},\, A \subseteq O} \mu^*(O). \tag{4.64}$$

The opposite inequality in (4.64) is clear from the monotonicity of μ^* and this finishes the proof of (4.56). $\square$

Proposition 4.10. *Let (X, τ) be a topological space and denote by $\mathscr{O}$ the collection of open sets in the topology τ. Also, assume that $(\mathfrak{M}, \mu)$ is a Borel measure on X. Then the following statements are true.*

(1) If (X, τ) satisfies (4.47) then

$$\forall\, B \in Borel_\tau(X) \text{ with } \mu(B) < +\infty \Longrightarrow \mu(B) = \sup_{C \in \mathscr{O}^c,\, C \subseteq B} \mu(C). \tag{4.65}$$

(2) If (X, τ) satisfies (4.47) and is such that

$$\text{there exist } \{O_j\}_{j\in\mathbb{N}} \subseteq \mathscr{O} \text{ so that } X = \bigcup_{j\in\mathbb{N}} O_j \text{ and } \mu(O_j) < +\infty \quad \forall\, j \in \mathbb{N}, \tag{4.66}$$

then

$$\forall\, B \in Borel_\tau(X),\ \forall\, \varepsilon > 0,\ \exists\, O \in \mathscr{O} \text{ with } B \subseteq O \text{ and } \mu(O \setminus B) < \varepsilon. \tag{4.67}$$

(3) If (X,τ) satisfies (4.47) and μ is a Borel regular measure satisfying (4.66), then μ satisfies the outer regularity condition

$$\mu(E) = \inf_{O\in\mathscr{O},\, E\subseteq O} \mu(O), \quad \forall\, E \in \mathfrak{M}, \tag{4.68}$$

as well as the inner-regularity condition

$$\mu(E) = \sup_{C\in\mathscr{O}^c,\, C\subseteq E} \mu(C), \quad \forall\, E \in \mathfrak{M}. \tag{4.69}$$

Proof. Let μ^* be the outer measure associated with μ as in (4.19). Hence, by (4.20), μ^* is a Borel outer measure. Then parts (1), (2) as well as (4.68) in part (3) in the statement of the proposition are direct consequence of Lemma 4.8, (4.55) and (4.56) in Lemma 4.9, respectively, given that μ and μ^* agree on $\mathfrak{M}$ which, in turn, contains $Borel_\tau(X)$.

There remains to prove (4.69) under the assumption that (X,τ) satisfies (4.47) and μ is a Borel regular measure which satisfies (4.66). To this end, fix $E \in \mathfrak{M}$ and note, obviously,

$$\mu(E) \geq \sup_{C\in\mathscr{O}^c,\, C\subseteq E} \mu(C). \tag{4.70}$$

To prove the opposite inequality, assume first that

$$\mu(E) < +\infty \tag{4.71}$$

and fix an arbitrary $\varepsilon > 0$. Since μ is a Borel regular measure, there exists

$$B \in Borel_\tau(X) \text{ with the property that } E \subseteq B \text{ and } \mu(E) = \mu(B). \tag{4.72}$$

In particular, thanks to (4.71), $\mu(B) < +\infty$ so (4.65) applies and yields

$$C \in \mathscr{O}^c \text{ with the property that } C \subseteq B \text{ and } \mu(B) < \mu(C) + \varepsilon/2. \tag{4.73}$$

On the other hand, from (4.68) applied to the set $B \setminus E \in \mathfrak{M}$, we know that there exists

$$O \in \mathscr{O} \quad \text{with the property that } B \setminus E \subseteq O \text{ and } \mu(O) < \mu(B \setminus E) + \varepsilon/2. \tag{4.74}$$

At this stage, define $C_\varepsilon := C \setminus O \subseteq X$ and observe that, since $B \setminus E \subseteq O$ and $C \subseteq B$, we have $C_\varepsilon \subseteq C \setminus (B \setminus E) = C \cap E$. Hence,

$$C_\varepsilon \in \mathscr{O}^c \quad \text{and} \quad C_\varepsilon \subseteq E. \tag{4.75}$$

Furthermore, since $E \setminus C \subseteq B \setminus C$ and $E \cap O = O \setminus (B \setminus E)$, we have

$$E \setminus C_\varepsilon = (E \setminus C) \cup (E \cap O) \subseteq (B \setminus C) \cup [O \setminus (B \setminus E)]. \tag{4.76}$$

Consequently, from (4.76), (4.73) and (4.74), we obtain

$$\mu(E \setminus C_\varepsilon) \leq \mu(B \setminus C) + \mu(O \setminus (B \setminus E)) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{4.77}$$

Thus, $\mu(E) < \mu(C_\varepsilon) + \varepsilon$ which, when used in concert with (4.75), justifies the opposite inequality in (4.70), completing the proof of (4.69) under the additional hypothesis that (4.71) holds.

Finally, there remains to prove (4.69) as stated. To this end, assume that $E \in \mathfrak{M}$ is such that $\mu(E) = +\infty$ and recall the sequence $\{O_j\}_{j \in \mathbb{N}}$ from (4.66). Furthermore, let the U_i's retain the same significance as in (4.57), so that so that

$$X = \bigcup_{i \in \mathbb{N}} U_i \quad \text{and} \quad U_i \in \mathscr{O}, \quad \mu(U_i) < +\infty, \quad U_i \subseteq U_{i+1} \quad \forall\, i \in \mathbb{N}. \tag{4.78}$$

Then $E \cap U_i \in \mathfrak{M}$ and $\mu(E \cap U_i) < +\infty$ for each $i \in \mathbb{N}$, so what we have proved up to this point in relation to (4.69) applies and gives that for each $i \in \mathbb{N}$ there exists

$C_i \in \mathscr{O}^c$ with $C_i \subseteq E \cap U_i$ and $\mu(C_i) + 1/i > \mu(E \cap U_i)$. Hence,

$$\lim_{i\to\infty} \mu(C_i) \geq \lim_{i\to\infty} \big(\mu(E \cap U_i) - 1/i\big) = \mu(E) = +\infty, \tag{4.79}$$

which proves that there are closed subsets of E of arbitrarily large measure. As a result, (4.69) also holds in the case when $E \in \mathfrak{M}$ satisfies $\mu(E) = +\infty$. This finishes the proof of the proposition. $\square$

Remark 4.6. *Let (X, τ) be a locally compact, Hausdorff topological space with the property that every open set in X is sigma-compact, i.e., if $\mathscr{O}$ denotes the collection of open sets in X, then*

$$\forall\, O \in \mathscr{O},\ \exists\, K_j \subseteq X \ \textit{compact},\ j \in \mathbb{N},\ \textit{such that}\ \ O = \bigcup_{j\in\mathbb{N}} K_j \tag{4.80}$$

(parenthetically, note that any separable, locally compact, topological space whose topology is induced by a metric satisfies (4.80)*).*

Furthermore, if $(\mathfrak{M}, \mu)$ is a Borel measure on X such that $\mu(K) < +\infty$ for every compact set $K \subseteq X$, it follows that conditions (4.47) *and* (4.66) *hold. Likewise, if μ^* is a Borel outer measure on X which is finite on compact subsets of X, then conditions* (4.47) *and* (4.54) *hold as well.*

Remark 4.7. *Assume that (X, τ) is a topological space and $(\mathfrak{M}, \mu)$ is a Borel measure on X which satisfies* (4.68)*. Then μ is a Borel regular measure.*

Proposition 4.11. *Let (X, τ) be a locally compact Hausdorff topological space in which every open set can be written as a countable union of compact sets. Assume that $(\mathfrak{M}, \mu)$ is a Borel measure on X such that $\mu(K) < +\infty$ for every compact set*

$K \subseteq X$. *Then*

$$\mu(E) = \inf_{O \in \mathscr{O},\, E \subseteq O} \mu(O), \quad \forall\, E \in Borel_\tau(X), \tag{4.81}$$

and

$$\mu(E) = \sup_{K \text{ compact},\, K \subseteq E} \mu(K), \quad \forall\, E \in Borel_\tau(X). \tag{4.82}$$

In particular, (4.81)-(4.82) *are valid for any locally finite Borel measure μ on a locally compact, separable metric space X (equipped with the topology τ canonically induced by the metric).*

Proof. The outer regularity formula (4.81) is a consequence of part (2) in Proposition 4.10 and Remark 4.6. As far as the inner-regularity formula (4.82) is concerned, let us first treat the case when $\mu(E) < +\infty$. In this scenario, thanks to part (1) in Proposition 4.10 and Remark 4.6 it suffices to observe that if

$$\bigcup_{j \in \mathbb{N}} K_j = X \text{ with } K_j \subseteq X \text{ compact and } K_j \subseteq K_{j+1} \text{ for every } j \in \mathbb{N}, \tag{4.83}$$

then for every $C \in \mathscr{O}^c$ we have $\mu(C \cap K_j) \to \mu(C)$ as $j \to \infty$, and each $C \cap K_j$ is a compact set (since X is Hausdorff). In the case when $\mu(E) = +\infty$ consider the pairwise disjoint Borel sets $D_j := K_{j+1} \setminus K_j$, $j \in \mathbb{N}$, and note that since

$$E = \cup_{j \in \mathbb{N}} (D_j \cap E),$$

it follows that $+\infty = \mu(E) = \sum_{j \in \mathbb{N}} \mu(D_j \cap E)$. On the other hand, since $D_j \cap E$ is a Borel set of finite measure, what we have proved already gives that for each $j \in \mathbb{N}$ one can find a compact set $C_j \subseteq D_j \cap E$ with the property that $\mu(C_j) \geq \mu(D_j \cap E) - 2^{-j}$. Then, since the C_j's are disjoint, we obtain

$$\lim_{k \to \infty} \mu\Big(\bigcup_{j=1}^{k} C_j\Big) = \sum_{j=1}^{\infty} \mu(C_j) \geq \sum_{j=1}^{\infty} \big(\mu(D_j \cap E) - 2^{-j}\big) = +\infty. \tag{4.84}$$

Since for each fixed k the set $\bigcup_{j=1}^{k} C_j$ is compact and contained in E, it follows that (4.82) is valid in the case when $\mu(E) = +\infty$ as well.

Finally, the very last claim in the statement of the proposition is a corollary of what has just been proved, given that any separable metric space is Lindelöf. $\square$

Definition 4.11. *Let (X, τ) be a topological space. A measure μ on X is called a* `Radon measure` *if μ is a Borel regular measure and $\mu(K) < +\infty$ for each compact set $K \subseteq X$. Likewise, an outer measure μ^* on X is called a* `Radon outer measure` *if μ^* is a Borel regular outer measure and $\mu^*(K) < +\infty$ for each compact set $K \subseteq X$.*

Remark 4.8. *Let (X, τ) be a topological space. Then Lemma 4.5 shows that a measure μ on X is a Radon measure if and only if its associated outer measure (as in* (4.19)*) is a Radon outer measure.*

Proposition 4.12. *Suppose that (X, τ) is a topological space, μ^* is a Borel regular outer measure on X, and that $A \subseteq X$ is an arbitrary set. Make the hypothesis that μ^* assumes finite values on compact subsets of X contained in A. Then $\mu^* \lfloor A$ is a Radon outer measure on the topological space $(A, \tau|_A)$.*

Proof. Set $\nu^* := \mu^* \lfloor A$. Thanks to Lemma 4.7, it follows that ν^* is a Borel regular outer measure on A. Furthermore, $\nu^*(K) < +\infty$ for every set $K \subseteq A$ which is compact in the topology induced by τ on A since any such K is also compact when viewed as a subset of the topological space X. Hence, $\mu^* \lfloor A$ is a Radon outer measure on the topological space $(A, \tau|_A)$. $\square$

Proposition 4.13. *Let (X, τ) be a locally compact Hausdorff topological space in which every open set is sigma-compact (i.e., can be written as a countable union of* *compact sets). Assume that μ^* is a Radon outer measure on X. Then*

$$\mu^*(E) = \inf_{O \in \mathscr{O},\, E \subseteq O} \mu^*(O), \quad \forall\, E \subseteq X, \tag{4.85}$$

and

$$\mu^*(E) = \sup_{K \text{ compact},\, K \subseteq E} \mu^*(K), \quad \forall\, E \in \mathfrak{M}_{\mu^*}. \tag{4.86}$$

Proof. Formula (4.85) is a consequence of (4.56) and Remark 4.6. Finally, formula (4.86) is proved along similar lines to (4.82) since all properties of Borel measures used to justify the latter have natural counterparts for Borel outer measures. □

Corollary 4.14. *Assume that (X, τ) is a locally compact Hausdorff topological space in which every open set is sigma-compact and suppose that $(\mathfrak{M}, \mu)$ is a Radon measure on X. Then μ is both outer-regular and inner-regular, i.e.,*

$$\mu(E) = \inf_{O \in \mathscr{O},\, E \subseteq O} \mu(O), \quad \forall\, E \in \mathfrak{M}, \tag{4.87}$$

and

$$\mu(E) = \sup_{K \text{ compact},\, K \subseteq E} \mu(K), \quad \forall\, E \in \mathfrak{M}. \tag{4.88}$$

Proof. This is a direct consequence of Proposition 4.13 applied to the outer measure μ^* associated with μ as in (4.19) (plus the observation made in Remark 4.8). □

Given topological space (X, τ), denote by $\mathscr{C}^0(X) = \mathscr{C}^0(X, \tau)$ the space of real-valued continuous functions on X and by $\mathscr{C}^0_{comp}(X) = \mathscr{C}^0_{comp}(X, \tau)$ the subspace of $\mathscr{C}^0(X, \tau)$ consisting of functions which vanish identically outside of a compact set in X.

Proposition 4.15. *Assume that (X, τ) is a locally compact Hausdorff topological space in which every open set is sigma-compact (recall that the latter condition automatically holds if (X, τ) is metrizable and separable), and suppose that $(\mathfrak{M}, \mu)$ is a Radon measure on X. Then*

$$\mathscr{C}^0_{comp}(X, \tau) \hookrightarrow L^p(X, \mu) \;\; \textit{densely, whenever} \;\; p \in (0, \infty). \tag{4.89}$$

Proof. Thanks to the fact that the simple functions on X embed densely (by an `embedding` we understand a mapping between two topological spaces which is a homeomorphism onto its image) into the space of L^p functions on X for $p \in (0, \infty)$, it suffices to show that for any $E \in \mathfrak{M}$ with $\mu(E) < +\infty$ it is possible to approximate $\mathbf{1}_E$ arbitrarily well in $L^p(X, \mu)$ with functions from $\mathscr{C}^0_{comp}(X, \tau)$.

With this goal in mind, given an arbitrary $\varepsilon > 0$, we make use of Corollary 4.14 in order to obtain an open set O containing E along with a compact set K contained in E with the property that $\mu(O \setminus K) < \varepsilon$. Urysohn's lemma then furnishes some $f \in \mathscr{C}^0_{comp}(X, \tau)$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_O$ which, in turn, ensures that $|\mathbf{1}_E - f| \leq \mathbf{1}_{O \setminus K}$ so that, ultimately, $\|\mathbf{1}_E - f\|_{L^p(X,\mu)} \leq \mu(O \setminus K)^{1/p} < \varepsilon^{1/p}$. From this, the desired conclusion follows. $\square$

4.2 Measures Defined on the Sigma-algebra Generated by Balls

Remark 4.9. *It follows from Lemma 3.24 that any separable quasi-metric space satisfies condition* (4.47).

Definition 4.12. *Given a quasi-metric space (X, ρ), denote by $\mathfrak{M}_\rho(X)$ the sigma-algebra generated by the collection of all ρ-balls in X.*

Lemma 4.16. *For any separable quasi-metric space* (X, ρ) *there holds*

$$Borel_{\tau_\rho}(X) \subseteq \mathfrak{M}_\rho(X). \tag{4.90}$$

As a corollary, any measure μ *on* X *with the property that the* ρ*-balls are* μ*-measurable is a Borel measure. Likewise, any outer measure* μ^* *on* X *with the property that the* ρ*-balls are* μ^**-measurable is a Borel outer measure.*

Proof. This is an immediate consequence of Lemma 3.24. □

Another basic tool for establishing the Borelianity of an outer measure is the following version of Caratheodory's criterion adapted to the setting of quasi-metric spaces.

Lemma 4.17. *Let* (X, ρ) *be a quasi-metric space and assume that* μ^* *is an outer measure on* X *with the property that*

$$\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B) \quad \textit{for every } A, B \subseteq X \textit{ with } \mathrm{dist}_\rho(A, B) > 0. \tag{4.91}$$

Then μ^* *is a Borel outer measure.*

Proof. Let C be an arbitrary closed set in (X, τ_ρ). The desired conclusion follows as soon as we show that C is μ^*-measurable, i.e.,

$$\mu^*(A) \geq \mu^*(A \cap C) + \mu^*(A \setminus C), \quad \forall\, A \subseteq X, \tag{4.92}$$

since the opposite inequality is a trivial consequence of the subadditivity of μ^*. To proceed, fix an arbitrary set $A \subseteq X$ and note that (4.92) is trivially true if $\mu^*(A) = +\infty$. Hence, we assume in what follows that $\mu^*(A) < +\infty$. Also, recall from Theorem 2.12

that there exists a distance d on X and $\alpha > 0$ with the property that $\rho \approx d^{1/\alpha}$. Next, define

$$C_j := \big\{x \in X : \operatorname{dist}_d(x, C) \leq 1/j\big\}, \quad j \in \mathbb{N}, \tag{4.93}$$

so that $C \subseteq C_j$ for every $j \in \mathbb{N}$. Given that for every $j \in \mathbb{N}$ we have, thanks to (3.36)-(3.40),

$$\operatorname{dist}_\rho(A \setminus C_j, A \cap C) \approx \big[\operatorname{dist}_d(A \setminus C_j, A \cap C)\big]^{1/\alpha} \geq (1/j)^{1/\alpha} > 0, \tag{4.94}$$

the hypothesis (4.91) applies and yields (on account of the subadditivity of μ^*)

$$\mu^*(A \setminus C_j) + \mu^*(A \cap C) = \mu^*\big((A \setminus C_j) \cup (A \cap C)\big) \leq \mu^*(A), \quad \forall\, j \in \mathbb{N}. \tag{4.95}$$

At this stage we make the claim that

$$\lim_{j \to \infty} \mu^*(A \setminus C_j) = \mu^*(A \setminus C). \tag{4.96}$$

In order to justify this claim, recall from Theorem 2.12 that there exists a distance d and the number $\alpha > 0$ with the property that $\rho \approx d^{1/\alpha}$. Based on this, we then introduce

$$D_k := \big\{x \in A : (k+1)^{-1} < \operatorname{dist}_d(x, C) \leq k^{-1}\big\} \subseteq A, \quad k \in \mathbb{N}. \tag{4.97}$$

A simple argument gives that $\operatorname{dist}_d(D_k, D_{k'}) > 0$ if $k, k' \in \mathbb{N}$ are such that $|k - k'| \geq 2$. Based on this and (3.36)-(3.40) we may therefore conclude that

$$\operatorname{dist}_\rho(D_k, D_{k'}) \approx \big[\operatorname{dist}_d(D_k, D_{k'})\big]^{1/\alpha} > 0 \quad \text{for } k, k' \in \mathbb{N} \text{ with } |k - k'| \geq 2. \tag{4.98}$$

Consequently, for any $m \in \mathbb{N}$,

$$\operatorname{dist}_\rho\Big(\bigcup_{k=1}^{m} D_{2k-1}\,,\; D_{2m+1}\Big) \geq \min\Big\{\operatorname{dist}_\rho(D_{2k-1}, D_{2m+1}) : 1 \leq k \leq m\Big\} > 0 \tag{4.99}$$

and

$$\mathrm{dist}_\rho\Big(\bigcup_{k=1}^{m} D_{2k}\,,\,D_{2m+2}\Big) \geq \min\Big\{\mathrm{dist}_\rho(D_{2k},D_{2m+2}) :\, 1 \leq k \leq m\Big\} > 0. \tag{4.100}$$

Keeping in mind (4.99)-(4.100), for each fixed number $N \in \mathbb{N}$ we then proceed to estimate

$$\begin{aligned} \sum_{k=1}^{2N} \mu^*(D_k) &= \sum_{k=1}^{N} \mu^*(D_{2k-1}) + \sum_{k=1}^{N} \mu^*(D_{2k}) \\ &= \mu^*\Big(\bigcup_{k=1}^{N} D_{2k-1}\Big) + \mu^*\Big(\bigcup_{k=1}^{N} D_{2k}\Big) \leq 2\mu^*(A), \end{aligned} \tag{4.101}$$

where we have made repeated use of our hypothesis (4.91) (written with equality). By passing to the limit $N \to +\infty$ we therefore arrive at

$$\sum_{k=1}^{\infty} \mu^*(D_k) \leq 2\mu^*(A) < +\infty. \tag{4.102}$$

Going further, the fact that C is closed entails $A \setminus C = \{x \in A :\, \mathrm{dist}_d(x, C) > 0\}$ hence, further,

$$A \setminus C = (A \setminus C_j) \cup \Big(\bigcup_{k=j}^{\infty} D_k\Big), \quad \forall\, j \in \mathbb{N}. \tag{4.103}$$

As a result, for every $j \in \mathbb{N}$ we have

$$\mu^*(A \setminus C_j) \leq \mu^*(A \setminus C) \leq \mu^*(A \setminus C_j) + \sum_{k=j}^{\infty} \mu^*(D_k) \tag{4.104}$$

which, in turn, allows us to estimate

$$\begin{aligned} \lim_{j\to\infty} \mu^*(A \setminus C_j) &\leq \mu^*(A \setminus C) \\ &\leq \lim_{j\to\infty} \mu^*(A \setminus C_j) + \lim_{j\to\infty} \sum_{k=j}^{\infty} \mu^*(D_k) \\ &= \lim_{j\to\infty} \mu^*(A \setminus C_j), \end{aligned} \tag{4.105}$$

by (4.102). This finishes the proof of the claim made in (4.96). With this in hand and recalling (4.95), we may write

$$\begin{aligned}\mu^*(A \setminus C) + \mu^*(A \cap C) &= \lim_{j\to\infty} \mu^*(A \setminus C_j) + \mu^*(A \cap C) \\ &= \lim_{j\to\infty} \Big[\mu^*(A \setminus C_j) + \mu^*(A \cap C)\Big] \leq \mu^*(A), \qquad (4.106)\end{aligned}$$

completing the proof of (4.92). □

The next definition gives an expected correlation between sets in a quasi-metric space and measures defined on the quasi-metric space equipped with its natural topology.

Definition 4.13. *Assume that (X, ρ) is a quasi-metric space and suppose that $(\mathfrak{M}, \mu)$ is a Borel measure on X. Call the measure μ* `locally finite` *provided it satisfies*

$$\mu(B) < +\infty \text{ for any bounded set } B \in Borel_{\tau_\rho}(X). \qquad (4.107)$$

We next discuss the particular form which Proposition 4.10 takes when the topology on X is induced by a quasi-metric space structure.

Proposition 4.18. *Let (X, ρ) be a quasi-metric space and denote by $\mathscr{O}$ the collection of open sets in the topology τ_ρ. Also, assume that $(\mathfrak{M}, \mu)$ is a Borel measure on X. Then the following statements are true.*

(i) If (X, τ_ρ) is separable, then one has

$$\forall\, B \in Borel_{\tau_\rho}(X) \text{ with } \mu(B) < +\infty \Longrightarrow \mu(B) = \sup_{C \in \mathscr{O}^c,\, C \subseteq B} \mu(C). \qquad (4.108)$$

(ii) *If* (X, τ_ρ) *is separable and the given Borel measure is locally finite (i.e. it satisfies* (4.107)*) then*

$$\forall\, B \in Borel_{\tau_\rho}(X),\ \ \forall\, \varepsilon > 0 \Longrightarrow \exists\, O \in \mathscr{O} \text{ with } B \subseteq O \text{ and } \mu(O \setminus B) < \varepsilon, \tag{4.109}$$

and

$$\mu(B) = \sup_{\substack{C \in \mathscr{O}^c,\, C \subseteq B \\ C \text{ bounded}}} \mu(C), \quad \forall\, B \in Borel_{\tau_\rho}(X). \tag{4.110}$$

(iii) *If* μ *is a Borel regular measure which is locally finite then* μ *satisfies the outer-regularity condition*

$$\mu(E) = \inf_{O \in \mathscr{O},\, E \subseteq O} \mu(O), \quad \forall\, E \in \mathfrak{M}, \tag{4.111}$$

as well as the inner-regularity condition

$$\mu(E) = \sup_{\substack{C \in \mathscr{O}^c,\, C \subseteq E \\ C \text{ bounded}}} \mu(C), \quad \forall\, E \in \mathfrak{M}. \tag{4.112}$$

Proof. As already stated in Remark (4.9), (X, τ_ρ) satisfies (4.47) so (i) follows from (1) in Proposition 4.10. Next, granted (4.107), (X, τ_ρ) also satisfies (4.66) so the first part of (ii) and the first part of (iii) are, respectively, consequences of (2) and the first part of (3) in Proposition 4.10. With an eye on (4.110), observe first that if $B \in Borel_{\tau_\rho}(X)$ is arbitrary and $x \in X$ is a fixed point then $B_j := B \cap \overline{B_\rho(x, j)}$ is a bounded Borel subset of B for every $j \in \mathbb{N}$ and $B_j \nearrow B$ as $j \to \infty$. Consequently, (4.110) follows by noting that $\mu(B_j) \nearrow \mu(B)$ as $j \to \infty$ and invoking (4.108) for each j.

Finally, as regards the second part of (iii), given the second part of (3) in Proposition 4.10 it suffices to observe that if $C \subseteq X$ is a closed set and $x \in X$ is a fixed point then $C_j := C \cap \overline{B_\rho(x,j)}$ is a closed subset of C for every $j \in \mathbb{N}$ and $\mu(C_j) \nearrow \mu(C)$ as $j \to \infty$, since $C_j \nearrow C$ as $j \to \infty$. $\square$

4.3 The Hausdorff Outer Measure on Quasi-metric Spaces

The goal of this section is to show that the Hausdorff outer measure defined on quasi-metric spaces continues to enjoy some of the most basic properties of its counterpart from the setting of Euclidean spaces. A good reference for the latter context is [20].

A basic result in geometric measure theory is that in the classical setting of $\mathbb{R}^n$ (viewed as a metric space when equipped with the canonical Euclidean distance), the Hausdorff outer measure is a Borel regular outer measure. In the general setting of a set X on which a quasi-metric space structure has been specified, a Hausdorff outer measure of given dimensionality can be associated for any choice of a quasi-metric compatible with that structure. For any such choice, the corresponding Hausdorff outer measure turns out to be a Borel outer measure though in general it is not expected to be regular. It is remarkable however that one can always choose a quasi-metric compatible with the original structure for which the associated Hausdorff outer measure turns out to be a Borel regular outer measure.

Definition 4.14. *Let (X, ρ) be a quasi-metric space, and fix $d \geq 0$. Given a set*

$E \subseteq X$, *for every* $\varepsilon > 0$ *define*

$$\mathcal{H}^d_{X,\rho,\varepsilon}(E) := \inf \Big\{ \sum_{j=1}^{\infty} (\mathrm{diam}_\rho(A_j))^d : \, E \subseteq \bigcup_{j=1}^{\infty} A_j \quad \text{and } \mathrm{diam}_\rho(A_j) \leq \varepsilon \text{ for every } j \Big\} \tag{4.113}$$

then take

$$\mathcal{H}^d_{X,\rho}(E) := \lim_{\varepsilon \to 0^+} \mathcal{H}^d_{X,\rho,\varepsilon}(E) = \sup_{\varepsilon > 0} \mathcal{H}^d_{X,\rho,\varepsilon}(E) \in [0, +\infty]. \tag{4.114}$$

The quantity $\mathcal{H}^d_{X,\rho}(E)$ *is called the* d-`dimensional Hausdorff outer measure` *in* (X, ρ) *of the set* E. *Whenever the choice of the quasi-distance* ρ *is irrelevant or clear from the context,* $\mathcal{H}^d_{X,\rho}(E)$ *is abbreviated as* $\mathcal{H}^d_X(E)$.

Similarly, given $E \subseteq X$, *for every* $\varepsilon > 0$ *define*

$$\widetilde{\mathcal{H}}^d_{X,\rho,\varepsilon}(E) := \inf \Big\{ \sum_{j=1}^{\infty} r_j^d : \, E \subseteq \bigcup_{j=1}^{\infty} B_\rho(x_j, r_j) \text{ and } r_j \leq \varepsilon \text{ for every } j \Big\}, \tag{4.115}$$

set

$$\widetilde{\mathcal{H}}^d_{X,\rho}(E) := \lim_{\varepsilon \to 0^+} \widetilde{\mathcal{H}}^d_{X,\varepsilon}(E) = \sup_{\varepsilon > 0} \widetilde{\mathcal{H}}^d_{X,\varepsilon}(E) \in [0, +\infty], \tag{4.116}$$

and abbreviate $\widetilde{\mathcal{H}}^d_{X,\rho}(E)$ *as* $\widetilde{\mathcal{H}}^d_X(E)$ *whenever unambiguous.*

Remark 4.10. *(i) If for some* $\varepsilon > 0$ *the set over which the infimum is taken in* (4.113) *is empty, we adopt the convention that*

$$\inf \emptyset := +\infty. \tag{4.117}$$

A similar convention applies in the case of (4.115).

(ii) In the case when the quasi-metric space (X, ρ) *is separable (recall we have that from Lemma 3.22 that this is condition is always satisfied if the quasi-metric space*

(X,ρ) is uniformly distributed) it follows that for each $\varepsilon > 0$ the set over which the infimum is taken in (4.115) *is nonempty (for example, this can be seen using Vitali's lemma–see, for example, Lemma 3.23– applied to the family $\{B_\rho(x,\varepsilon/(3C_\rho^2))\}_{x\in E}$). This then show that a similar conclusion holds in the context of* (4.113).

Note that $\mathcal{H}^0_{X,\rho}$ is equivalent to (in the sense of Definition (4.7)) the counting measure. More basic properties of the Hausdorff measure are collected in the proposition below.

Proposition 4.19. *Let (X,ρ) be a quasi-metric space and fix $d \geq 0$. Then the following properties hold:*

(1) $\mathcal{H}^d_{X,\rho}$ is a Borel outer measure on X and there exists a quasi-distance ρ' on X such that $\rho' \approx \rho$ and for which $\mathcal{H}^d_{X,\rho'}$ is a Borel regular outer measure on X.

(2) One has $\widetilde{\mathcal{H}}^d_{X,\rho} \approx \mathcal{H}^d_{X,\rho}$. More precisely, there exist two finite constants $C_1, C_2 > 0$ which depend only on ρ such that

$$C_1\,\widetilde{\mathcal{H}}^d_{X,\rho}(E) \leq \mathcal{H}^d_{X,\rho}(E) \leq C_2\,\widetilde{\mathcal{H}}^d_{X,\rho}(E) \quad \text{for all } E \subseteq X. \tag{4.118}$$

In particular, $\widetilde{\mathcal{H}}^d_{X,\rho}$ and $\mathcal{H}^d_{X,\rho}$ are related to one another as in Proposition 4.4.

(3) Assume that $\Sigma \subseteq X$, and consider the quasi-metric space $(\Sigma, \rho|_\Sigma)$, where $\rho|_\Sigma$ denotes the restriction of ρ to $\Sigma \times \Sigma$. Then the d-dimensional Hausdorff outer measure in $(\Sigma,\rho|_\Sigma)$ is equivalent to the restriction to Σ of the d-dimensional Hausdorff outer measure in X. That is,

$$\mathcal{H}^d_{\Sigma,\rho|_\Sigma} \approx \mathcal{H}^d_{X,\rho}\lfloor\Sigma. \tag{4.119}$$

As such, $\mathcal{H}^d_{\Sigma,\rho|_\Sigma}$ and $\mathcal{H}^d_{X,\rho}\lfloor\Sigma$ are related to each other as in Proposition 4.4.

(4) Assume that $0 \leq d_1 < d_2 < +\infty$. Then for each $E \subseteq X$ one has

$$\mathcal{H}^{d_1}_{X,\rho}(E) < +\infty \Longrightarrow \mathcal{H}^{d_2}_{X,\rho}(E) = 0, \tag{4.120}$$

$$\mathcal{H}^{d_2}_{X,\rho}(E) > 0 \Longrightarrow \mathcal{H}^{d_1}_{X,\rho}(E) = +\infty. \tag{4.121}$$

(5) One has $\mathcal{H}^d_{X,\rho'} \approx \mathcal{H}^d_{X,\rho}$ whenever ρ' is a quasi-distance on X with the property that $\rho' \approx \rho$. More precisely, there exist two finite constants $C_1, C_2 > 0$, which depend only on ρ and ρ' such that

$$C_1\,\mathcal{H}^d_{X,\rho}(E) \leq \mathcal{H}^d_{X,\rho'}(E) \leq C_2\,\mathcal{H}^d_{X,\rho}(E) \quad \textit{for all } E \subseteq X. \tag{4.122}$$

In fact there exists a unique function, henceforth denoted by

$$f := \Big(\frac{\mathrm{d}\rho'}{\mathrm{d}\rho}\Big)_{\mathcal{H}^d} \tag{4.123}$$

and referred to as the d-`Hausdorff derivative` of the quasi-distance ρ' with respect to the quasi-distance ρ, satisfying the following properties (with C_1, C_2 as in (4.122)*):*

$$\begin{aligned} &(i)\ f \text{ is } Borel_{\tau_\rho}(X) - \text{measurable},\\ &(ii)\ \exists\, A \in Borel_{\tau_\rho}(X) \text{ with } \mathcal{H}^d_{X,\rho}(A) = 0\\ &\qquad \text{such that } C_1 \leq f \leq C_2 \text{ on } X \setminus A,\\ &(iii)\ \mathcal{H}^d_{X,\rho'}\big|_{Borel_{\tau_\rho}(X)} = f\,\mathcal{H}^d_{X,\rho}\big|_{Borel_{\tau_\rho}(X)}. \end{aligned} \tag{4.124}$$

Proof. We start by establishing that $\mathcal{H}^d_{X,\rho,\varepsilon}$ is an outer measure for each fixed $\varepsilon > 0$. Let us first focus on the subadditivity property. To see this, assume that $E_j \subseteq X$,

$j \in \mathbb{N}$, are given, arbitrary sets. The goal is to show that

$$\mathcal{H}^d_{X,\rho,\varepsilon}\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) \leq \sum_{j=1}^{\infty} \mathcal{H}^d_{X,\rho,\varepsilon}(E_j). \tag{4.125}$$

If there exists $j \in \mathbb{N}$ with the property that $\mathcal{H}^d_{X,\rho,\varepsilon}(E_j) = +\infty$ then (4.125) trivially holds. To treat the remaining case, select an arbitrary $\theta > 0$ and, for every $j \in \mathbb{N}$, choose a family of sets $A_{j,k} \subseteq X$, $k \in \mathbb{N}$, with $\operatorname{diam}_\rho(A_{j,k}) \leq \varepsilon$ for every $k \in \mathbb{N}$ and for which

$$E_j \subseteq \bigcup_{k\in\mathbb{N}} A_{j,k}, \quad \mathcal{H}^d_{X,\rho,\varepsilon}(E_j) + \theta 2^{-j} \geq \sum_{k=1}^{\infty} \big(\operatorname{diam}_\rho(A_{j,k})\big)^d. \tag{4.126}$$

Note that since we are currently assuming that $\mathcal{H}^d_{X,\rho,\varepsilon}(E_j) < +\infty$ for every $j \in \mathbb{N}$, (4.117) ensures that such a family of sets always exists. Then the family $\{A_{j,k}\}_{j,k\in\mathbb{N}}$ covers $\bigcup_{j\in\mathbb{N}} E_j$ and consists of sets with ρ-diameters $\leq \varepsilon$. Hence, by definition,

$$\begin{aligned} \mathcal{H}^d_{X,\rho,\varepsilon}\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) &\leq \sum_{j,k=1}^{\infty} \big(\operatorname{diam}_\rho(A_{j,k})\big)^d = \sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty} \big(\operatorname{diam}_\rho(A_{j,k})\big)^d\Big) \\ &\leq \sum_{j=1}^{\infty}\Big(\mathcal{H}^d_{X,\rho,\varepsilon}(E_j) + \theta 2^{-j}\Big) \leq \theta + \sum_{j=1}^{\infty} \mathcal{H}^d_{X,\rho,\varepsilon}(E_j). \end{aligned} \tag{4.127}$$

Since $\theta > 0$ is arbitrary, this shows that (4.125) holds, as desired. Trivially, $\mathcal{H}^d_{X,\rho,\varepsilon}$ is monotone and $\mathcal{H}^d_{X,\rho,\varepsilon}(\emptyset) = 0$, so $\mathcal{H}^d_{X,\rho,\varepsilon}$ is indeed an outer measure for each fixed $\varepsilon > 0$.

We next proceed to show that $\mathcal{H}^d_{X,\rho}$ is an outer measure. This follows easily by observing that if $E_j \subseteq X$, $j \in \mathbb{N}$ are given then by (4.125) and (4.114)

$$\mathcal{H}^d_{X,\rho,\varepsilon}\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) \leq \sum_{j=1}^{\infty} \mathcal{H}^d_{X,\rho,\varepsilon}(E_j) \leq \sum_{j=1}^{\infty} \mathcal{H}^d_{X,\rho}(E_j), \quad \forall\, \varepsilon > 0, \tag{4.128}$$

then letting $\varepsilon \to 0^+$.

The next step is devoted to proving that $\mathcal{H}^d_{X,\rho}$ is a Borel outer measure. We shall do so with the help of Caratheodory's criterion from Lemma 4.17. With this goal in mind, choose $A, B \subseteq X$ such that $\mathrm{dist}_\rho(A,B) > 0$ and our goal is to show that

$$\mathcal{H}^d_{X,\rho}(A \cup B) \geq \mathcal{H}^d_{X,\rho}(A) + \mathcal{H}^d_{X,\rho}(B). \tag{4.129}$$

Without loss of generality it may be assumed that $\mathcal{H}^d_{X,\rho}(A \cup B) < +\infty$ (since otherwise there is nothing to prove), in which scenario we fix $\varepsilon \in \big(0, (1/4C_\rho)\,\mathrm{dist}_\rho(A,B)\big)$ and select sets $C_j \subseteq X$, $j \in \mathbb{N}$ such that

$$A \cup B \subseteq \bigcup_{j \in \mathbb{N}} C_j \quad \text{and} \quad \mathrm{diam}_\rho(C_j) \leq \varepsilon \quad \text{for every} \quad j \in \mathbb{N}. \tag{4.130}$$

Introduce $J_A := \{j \in \mathbb{N} : \, C_j \cap A \neq \emptyset\}$ and $J_B := \{j \in \mathbb{N} : \, C_j \cap B \neq \emptyset\}$ so that

$$A \subseteq \bigcup_{j \in J_A} C_j, \quad B \subseteq \bigcup_{j \in J_B} C_j, \quad \text{and} \quad C_j \cap C_{j'} = \emptyset \quad \text{for} \quad j \in J_A, \; j' \in J_B. \tag{4.131}$$

Consequently,

$$\begin{aligned} \sum_{j=1}^{\infty} \big(\mathrm{diam}_\rho(C_j)\big)^d \;&\geq\; \sum_{j \in J_A} \big(\mathrm{diam}_\rho(C_j)\big)^d + \sum_{j \in J_B} \big(\mathrm{diam}_\rho(C_j)\big)^d \\ &\geq\; \mathcal{H}^d_{X,\rho,\varepsilon}(A) + \mathcal{H}^d_{X,\rho,\varepsilon}(B). \end{aligned} \tag{4.132}$$

Taking the infimum over all possible choices of the family $\{C_j\}_{j \in \mathbb{N}}$ with the aforementioned properties leads to the conclusion that $\mathcal{H}^d_{X,\rho,\varepsilon}(A \cup B) \geq \mathcal{H}^d_{X,\rho,\varepsilon}(A) + \mathcal{H}^d_{X,\rho,\varepsilon}(B)$. Finally, passing $\varepsilon \to 0^+$ yields (4.129). Thus, Lemma 4.17 applies and finishes the proof of the claim. This concludes the proof of the first claim made in part (1) in the statement of the proposition.

Our next task is to prove that there exists a quasi-distance ρ' which is equivalent to ρ and such that $\mathcal{H}^d_{X,\rho'}$ is a Borel regular outer measure. Of course, for any quasi-distance ρ' with $\rho \approx \rho'$ we know that $\mathcal{H}^d_{X,\rho'}$ is a Borel outer measure so we focus on the regularity aspect. To this end, recall from Theorem 2.12 that there exists a distance δ on X and some $\alpha > 0$ with the property that $\rho \approx \delta^{1/\alpha}$. If we now define $\rho' := \delta^{1/\alpha}$ then ρ, ρ' and δ induce the same topology on X (i.e., $\tau_\rho = \tau_{\rho'} = \tau_\delta$). Hence, for every $E \subseteq X$ we have

$$\begin{aligned}\mathcal{H}^d_{X,\rho',\varepsilon}(E) &= \inf\Big\{\sum_{j=1}^{\infty}(\mathrm{diam}_\delta(A_j))^{d/\alpha} : E \subseteq \bigcup_{j=1}^{\infty} A_j \text{ and } \mathrm{diam}_\delta(A_j) \leq \varepsilon^\alpha \text{ for every } j\Big\} \\ &= \inf\Big\{\sum_{j=1}^{\infty}(\mathrm{diam}_\delta(A_j))^{d/\alpha} : E \subseteq \bigcup_{j=1}^{\infty} A_j, \;\; A_j \subseteq X \;\text{ such that} \\ &\qquad\qquad A_j = \overline{A_j} \;\text{ and }\; \mathrm{diam}_\delta(A_j) \leq \varepsilon^\alpha \text{ for every } j \in \mathbb{N}\Big\}, \end{aligned} \tag{4.133}$$

where the first equality follows from (4.113) and (3.36), while the second equality is a consequence of (3.54) (and the fact that $C_\delta = 1$). Consider next an arbitrary $E \subseteq X$ such that $\mathcal{H}^d_{X,\rho'}(E) < +\infty$. In particular, $\mathcal{H}^d_{X,\rho',\varepsilon}(E) < +\infty$ for every $\varepsilon > 0$. For each $k \in \mathbb{N}$ select a family of closed sets $A_{j,k} \subseteq X$, $j \in \mathbb{N}$, with the following properties

$$\mathrm{diam}_\delta(A_{j,k}) \leq k^{-\alpha}, \quad E \subseteq \bigcup_{j\in\mathbb{N}} A_{j,k} \quad \text{and} \tag{4.134}$$

$$\sum_{j=1}^{\infty}(\mathrm{diam}_\delta(A_{j,k}))^{d/\alpha} \leq \mathcal{H}^d_{X,\rho',1/k}(E) + \frac{1}{k}. \tag{4.135}$$

Thus, if we now introduce

$$B := \bigcap_{k\in\mathbb{N}}\Big(\bigcup_{j\in\mathbb{N}} A_{j,k}\Big), \tag{4.136}$$

it follows that $B \in Borel_{\tau_\delta}(X) = Borel_{\tau_\rho}(X)$ and $E \subseteq B$ (cf. (4.134)). Moreover, since we have $B \subseteq \bigcup_{j\in\mathbb{N}} A_{j,k}$ for each $k \in \mathbb{N}$, as seen from (4.136), we obtain from (4.133) and (4.134)-(4.135) that

$$\mathcal{H}^d_{X,\rho',1/k}(B) \leq \sum_{j=1}^{\infty}(\mathrm{diam}_\delta(A_{j,k}))^{d/\alpha} \leq \mathcal{H}^d_{X,\rho',1/k}(E) + \frac{1}{k}, \quad \forall\, k \in \mathbb{N}. \tag{4.137}$$

Passing to the limit $k \to +\infty$ then yields $\mathcal{H}^d_{X,\rho'}(B) \le \mathcal{H}^d_{X,\rho'}(E)$ hence, ultimately, it holds that $\mathcal{H}^d_{X,\rho'}(B) = \mathcal{H}^d_{X,\rho'}(E)$ given that $E \subseteq B$. There remains to observe that in the case when $E \subseteq X$ is such that $\mathcal{H}^d_{X,\rho'}(E) = +\infty$, by monotonicity we necessarily have $\mathcal{H}^d_{X,\rho'}(X) = +\infty$, so we may simply take $B := X$ and the same conclusion holds. This finishes the proof of the fact that $\mathcal{H}^d_{X,\rho'}$ is a Borel regular outer measure, and completes the treatment of part (1) in the statement of the proposition.

To prove part (2), recall (an equation dealing with diameter of rho-balls) which in the present context implies

$$\mathcal{H}^d_{X,\rho,\varepsilon}(E) \le (2C_\rho)^d\, \widetilde{\mathcal{H}}^d_{X,\rho,2C_\rho\varepsilon}(E) \quad \text{for all } E \subseteq X, \tag{4.138}$$

and therefore the second inequality in (4.118) holds with $C_2 = (2C_\rho)^d$. In order to prove the first inequality in (4.118), fix an arbitrary set $E \subseteq X$. If $\mathcal{H}^d_{X,\rho,\varepsilon}(E) = +\infty$ there is nothing to prove so suppose next that $\mathcal{H}^d_{X,\rho,\varepsilon}(E) < +\infty$. In this scenario, assume that $E \subseteq \bigcup_{j=1}^\infty A_j$ for some $A_j \subseteq X$ with $\operatorname{diam}_\rho(A_j) < \varepsilon$, for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ pick a point $x_j \in A_j$ and observe that $A_j \subseteq B_\rho(x_j, \operatorname{diam}_\rho A_j)$, hence $E \subseteq \bigcup_{j=1}^\infty B_\rho(x_j, \operatorname{diam}_\rho(A_j))$. In turn, this implies

$$\widetilde{\mathcal{H}}^d_{X,\rho,\varepsilon}(E) \le \mathcal{H}^d_{X,\rho,\varepsilon}(E) \quad \text{for all } E \subseteq X, \tag{4.139}$$

and therefore the first inequality in (4.118) holds with $C_1 = 1$.

Next we deal with the claim made in item (3) in the statement of the proposition. To get started, select a set $E \subseteq \Sigma$ with the goal of proving that

$$\mathcal{H}^d_{X,\rho}(E) \le \mathcal{H}^d_{\Sigma,\rho|_\Sigma}(E). \tag{4.140}$$

Obviously, we may assume that $\mathcal{H}^d_{\Sigma,\rho|_\Sigma}(E) < +\infty$. In this case, fix an arbitrary number $\varepsilon > 0$ and assume that $E \subseteq \bigcup_{j=1}^{\infty} B_{(\Sigma,\rho|_\Sigma)}(x_j, r_j)$ with $0 < r_j < \varepsilon$ and $x_j \in \Sigma$, $j \in \mathbb{N}$. Then

$$E \subseteq \bigcup_{j=1}^{\infty} B_{(\Sigma,\rho|_\Sigma)}(x_j, r_j) = \bigcup_{j=1}^{\infty} B_{(X,\rho)}(x_j, r_j) \cap \Sigma \subseteq \bigcup_{j=1}^{\infty} B_{(X,\rho)}(x_j, r_j), \tag{4.141}$$

which readily yields (4.140). Next, we shall show that for any $E \subseteq \Sigma$ there holds

$$\mathcal{H}^d_{\Sigma,\rho|_\Sigma}(E) \leq (2C_\rho)^d \mathcal{H}^d_{X,\rho}(E). \tag{4.142}$$

To this end, there is no loss of generality in assuming that $\mathcal{H}^d_{X,\rho}(E) < +\infty$. Next, fix $\varepsilon > 0$ and assume that $E \subseteq \bigcup_{j=1}^{\infty} B_{(X,\rho)}(x_j, r_j)$ with $0 < r_j < \varepsilon$ and $x_j \in X$, $j \in \mathbb{N}$. Hence, we have $E \subseteq \bigcup_{j=1}^{\infty} \big(B_{(X,\rho)}(x_j, r_j) \cap \Sigma\big)$.

Introduce $J := \{j \in \mathbb{N} : B_{(X,\rho)}(x_j, r_j) \cap \Sigma \neq \emptyset\}$ and note that for each $j \in J$ one may select $x_j^* \in B_{(X,\rho)}(x_j, r_j) \cap \Sigma$ and note that

$$B_{(X,\rho)}(x_j, r_j) \cap \Sigma \subseteq B_{(\Sigma,\rho|_\Sigma)}(x_j^*, 2C_\rho r_j). \tag{4.143}$$

Consequently,

$$E \subseteq \bigcup_{j\in J} B_{(\Sigma,\rho|_\Sigma)}(x_j^*, 2C_\rho r_j), \tag{4.144}$$

and, hence,

$$\mathcal{H}^d_{\Sigma,\rho|_\Sigma,2C_\rho\varepsilon}(E) \leq (2C_\rho)^d \mathcal{H}^d_{X,\rho,\varepsilon}(E), \tag{4.145}$$

from which (4.142) readily follows. Together with (4.140), this shows that $\mathcal{H}^d_{\Sigma,\rho|_\Sigma}$ is equivalent to the restriction to Σ of the d-dimensional Hausdorff outer measure in X.

As far as (4.120)-(4.121) are concerned, if $0 \leq d_1 < d_2 < +\infty$ and $E \subseteq X$, it follows from definitions that for each $\varepsilon > 0$ one has

$$\mathcal{H}^{d_2}_{X,\rho,\varepsilon}(E) \leq \varepsilon^{d_2-d_1} \mathcal{H}^{d_1}_{X,\rho,\varepsilon}(E) \leq \varepsilon^{d_2-d_1} \mathcal{H}^{d_1}_{X,\rho}(E). \tag{4.146}$$

Thus, (4.120) follows by letting $\varepsilon \to 0^+$ and using $\mathcal{H}^{d_1}_{X,\rho}(E) < +\infty$. Concerning (4.121), if we have $\mathcal{H}^{d_2}_{X,\rho}(E) > 0$ it follows that there exist $c > 0$ and $\varepsilon_* > 0$ with the property that $\mathcal{H}^{d_2}_{X,\rho,\varepsilon}(E) \geq c$ for every $\varepsilon \in (0, \varepsilon_*)$. Based on this and (4.146) we then deduce that $\mathcal{H}^{d_1}_{X,\rho}(E) \geq c\,\varepsilon^{d_1 - d_2}$ for every $\varepsilon \in (0, \varepsilon_*)$. Hence, $\mathcal{H}^{d_1}_{X,\rho}(E) = +\infty$, as wanted.

Finally, the first part of claim in (5) is a direct consequence of definitions and the fact that equivalent quasi-distances produce comparable balls, whereas the existence of a function f satisfying (4.124) follows from Proposition 4.4. $\square$

Remark 4.11. *(i) An inspection of the above proof shows that if (X, δ) is a metric space and $\gamma > 0$ then $\mathcal{H}^{d}_{X,\delta^\gamma}$ is a Borel regular outer measure on (X, τ_δ) for every $d \geq 0$.*

(ii) Given a quasi-metric space (X, ρ), the `Hausdorff dimension` *of a set $E \subseteq X$ is defined as*

$$\dim_{\mathcal{H}_X}(E) := \inf \{ d \geq 0 : \mathcal{H}^{d}_{X,\rho}(E) = 0 \}. \tag{4.147}$$

It follows from Proposition 4.19 that the above definition depends only on the quasi-metric space structure induced by ρ on X.

Chapter 5

Distances Between Quasi-metric Spaces

The primary goal is this section is to develop a notion of how close two different quasi-metric spaces are. We begin by exploring the distance between two subsets of the same metric space, then this opens the door to measuring the distance between two separate quasi-metric spaces by embedding them into a common quasi-metric space. Lastly, this will lead to notions of convergence of quasi-metric spaces.

An excellent discussion on the development and uses of the Gromov-Pompeiu-Hausdorff on metric spaces is the seventh chapter of [14]. We will refer to their work often.

5.1 The Pompeiu-Hausdorff Quasi-distance on Quasi-metric Spaces

In this subsection we introduce and study a version of the Pompeiu-Hausdorff distance[1] in the context of quasi-metric spaces. To set the stage, given a set X, a

[1] What we here call the Pompeiu-Hausdorff distance has typically been referred to in the literature as the Hausdorff distance. For historical accuracy, however, it is significant to note that D. Pompeiu was the first to introduce (a slight version of) this concept in his thesis (written under the supervision of H. Poincaré). Pompeiu's thesis has appeared in print in [72], published in 1905, where Pompeiu calls this notion *écart (mutuel)* between two sets. Subsequently, F. Hausdorff has revisited this topic

quasi-distance $\rho \in \mathfrak{Q}(X)$, a subset A of X and a number $r \in (0, +\infty)$, define

$$N_{\rho,r}(A) := \{x \in X : \operatorname{dist}_\rho(x, A) \leq r\}. \tag{5.1}$$

Here the N is for `neighborhood`.

Lemma 5.1. *If $\rho, \rho' \in \mathfrak{Q}(X)$, $r, r_0, r_1, \alpha, \lambda > 0$, $p \in (0, +\infty)$ and $U, V \subseteq X$, then the following properties hold:*

$$N_{\rho^\alpha,r}(U) = N_{\rho,r^{1/\alpha}}(U), \quad N_{\rho,r}(U) \subseteq N_{\rho,r}(V) \ \ \textit{if} \ \ U \subseteq V, \tag{5.2}$$

$$N_{\lambda\rho,r}(U) = N_{\rho,r/\lambda}(U), \quad N_{\rho,r}(U) \subseteq N_{\rho',r}(U) \ \ \textit{if} \ \ \rho' \leq \rho, \tag{5.3}$$

$$\overline{N_{\rho,r}(U)} \subseteq N_{\rho,C_\rho r}(U) \ \ \textit{and} \ \ N_{\rho,r}(\overline{U}) \subseteq N_{\rho,C_\rho r}(U), \tag{5.4}$$

$$N_{\rho,r_0}\big(N_{\rho,r_1}(U)\big) \subseteq N_{\rho,r}(U) \ \ \textit{if} \ \ r := C_{\rho,p}(r_0^p + r_1^p)^{1/p}, \tag{5.5}$$

$$\textit{or} \ \ r := C_\rho \max\{r_0, r_1\},$$

$$\bigcap_{r>0} N_{\rho_\#,r}(U) = \overline{U} \quad \textit{and} \quad N_{\rho_\#,r}(U) = N_{\rho_\#,r}(\overline{U}), \quad \forall\, U \subseteq X, \tag{5.6}$$

with $\rho_\#$ as in Theorem 2.12, as usual.

Lastly, if ρ is symmetric and $x_0 \in X$, defining $B_\rho[x_0, r] := \{x \in X : \rho(x_0, x) \leq r\}$,

$$N_{\rho,r}(\{x_0\}) = B_\rho[x_0, r]. \tag{5.7}$$

Proof. Every item in this list of properties may easily be shown based on definitions.

□

Proposition 5.2. *Given (X, d) a compact metric space, the following characterization of $N_{d,r}$ holds:*

$$N_{d,r}(A) = \bigcup_{x \in \bar{A}, 0 < r' \leq r} B_d[x, r'] \quad \forall\ A \subseteq X, r > 0, \tag{5.8}$$

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in 1914, and on page 463 of his book [33] he correctly attributes the introduction of this notion to Pompeiu.

where $B_d[x_0, r] := \{x \in X : d(x_0, x) \leq r\}$ *and the closure of* A *is taken in the topology which* d *induces.*

Proof. Fix $A \subseteq X$ and $r > 0$. Let $x \in N_{d,r}(A)$. We wish to find an $x_0 \in \bar{A}$ such that $x \in B_d[x_0, r]$. As $x \in N_{d,r}(A)$, by definition $\inf\{d(x, y) : y \in A\} = r' \leq r$. Thus given $n \in \mathbb{N}$, there is some $x_n \in A$ satisfying

$$d(x, x_n) < r' + 1/n. \tag{5.9}$$

In particular, $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in a compact metric space, and so by Theorem 3.17 (with E taken to be $\overline{A}$, where closed subsets of compact spaces are known to be compact), $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence. Take $x_0 \in X$ to be the limit of the convergent subsequence. As x_0 is a limit point of A, we have $x_0 \in \overline{A}$. Since d is a metric it is continuous in both components, hence $d(x_0, x) \leq r$ by passing to the limit as n tends to infinity in (5.9). Lastly, it follows that $x \in B_d[x_0, r]$, as desired, proving

$$N_{d,r}(A) \subseteq \bigcup_{x \in \bar{A}, 0 < r' \leq r} B_d[x, r']. \tag{5.10}$$

Next fix x in the union over all $x \in \overline{A}$ and $0 < r' \leq r$ of $B_d[x, r']$, so there is some $x_0 \in \overline{A}$ and some $r' \in (0, r]$ satisfying $x \in B_d[x_0, r']$. Clearly

$$\inf\{d(x, y) : y \in A\} \leq d(x_0, y) \leq r$$

, thus

$$N_{d,r}(A) \supseteq \bigcup_{x \in \bar{A}, 0 < r' \leq r} B_d[x, r'], \tag{5.11}$$

which completes the proof. $\square$

Remark 5.1. *In general, the hypotheses of Theorem 5.2 cannot be relaxed. Let* (X, ρ) *be a quasi-metric space and* $A \subseteq X$*. While* $N_{\rho,r}$ *does satisfy several expected, desirable properties, other properties which seem reasonable do not hold. Let*

$$B_\rho[x_0, r] := \{x \in X : \rho(x_0, x) \leq r\}$$

. Then the following do not hold:

$$N_{\rho,r}(A) = \bigcup_{x \in A, 0<r'<r} B_\rho[x, r'] \quad , \quad N_{\rho,r}(A) = \bigcup_{x \in A, 0<r' \leq r} B_\rho[x, r'] \tag{5.12}$$

$$N_{\rho,r}(A) = \bigcup_{x \in \overline{A}, 0<r'<r} B_\rho[x, r'] \quad , \quad N_{\rho,r}(A) = \bigcup_{x \in \overline{A}, 0<r' \leq r} B_\rho[x, r']. \tag{5.13}$$

However, as we have seen, if we further assume that ρ *is a genuine metric and* (X, ρ) *is compact then the second equation in* (5.13) *does hold.*

Definition 5.1. *Given a metric space* (X, d)*, for each* $U, V \subseteq X$*, introduce the* `Pompeiu-Hausdorff quasi-distance` *between* U *and* V *as*

$$D_\rho(U, V) := D_{(X,\rho)}(U, V) := \inf\big\{r > 0 : U \subseteq N_{\rho,r}(V) \text{ and } V \subseteq N_{\rho,r}(U)\big\}. \tag{5.14}$$

Proposition 5.3. *Given a quasi-metric space* (X, ρ)*, where* ρ *is symmetric,* $x_0 \in X$ *and* $A \subseteq X$*, one has* $D_\rho(\{x_0\}, A) = \inf\{r > 0 : A \subseteq B_\rho[x_0, r]\}$.

Proof. Let (X, ρ) be a quasi-metric space. Fix $x_0 \in X$ and $A \subseteq X$. It suffices to show

$$\big(\{x_0\} \subseteq N_{\rho,r}(A) \text{ and } A \subseteq N_{\rho,r}(\{x_0\})\big) \iff A \subseteq B_\rho[x_0, r] \quad \forall\, r > 0. \tag{5.15}$$

First observe that, by unraveling the definitions of $N_{\rho,r}(\{x_0\})$ and $\mathrm{dist}_\rho(x, \{x_0\})$, it is apparent that $N_{\rho,r}(\{x_0\}) = \big\{x \in X : \rho(x, x_0) \leq r\big\} = B_\rho[x_0, r]$ (this step may fail if

ρ is not symmetric), which was presented earlier as (5.7) in Lemma 5.1. Putting this into (5.15) reduces matters to showing

$$\big(x_0 \in N_{\rho,r}(A) \text{ and } A \subseteq B_\rho[x_0, r]\big) \iff A \subseteq B_\rho[x_0, r] \quad \forall\, r > 0. \tag{5.16}$$

So fix $r > 0$. Then we must prove

$$A \subseteq B_\rho[x_0, r] \implies x_0 \in N_{\rho,r}(A). \tag{5.17}$$

The fact that $A \subseteq B_\rho[x_0, r]$ tells us given $y \in A$, $\rho(x_0, y) \leq r$, while $x_0 \in N_{\rho,r}(A)$ occurs if $\operatorname{dist}(x_0, A) \leq r$, which is equivalent to $\inf\{\rho(x_0, y) : y \in A\} \leq r$. As $\rho(x_0, y) \leq r$ for any $y \in A$, of course the infimum is less than or equal to r. $\square$

Remark 5.2. *A simple counterexample showing ρ must be symmetric in Proposition 5.3 is as follows: let (X, ρ) be as the second part of Example 3.5. Then it holds that $D_\rho(\{x\}, \{y\}) = 2$ but $\inf\{r > 0 : \{y\} \subseteq B_\rho[x, r]\} = 1$.*

Also, to see $N_{\rho,r}(\{x_0\}) = B_\rho[x_0, r]$ may fail if ρ is not symmetric, using the same quasi-metric space, we have $N_{\rho,1}(\{x\}) = \{x, z\}$ but $B_\rho[x, 1] = \{x, y\}$.

Given a topological space (X, τ), define

$$\mathrm{Cl}\,(X, \tau) := \{E \subseteq X : E \text{ is closed in } \tau\}, \tag{5.18}$$

$$\mathrm{Cp}\,(X, \tau) := \{E \subseteq X : E \text{ is compact in } \tau\}. \tag{5.19}$$

Previously $\mathscr{O}^c$ was used to denote the closed sets in (X, τ). Here we adopt $\mathrm{Cl}\,(X, \tau)$ for two reasons: first, it makes it clear which topological space we are considering; second, the notation makes a stronger connection with (5.19).

Theorem 5.4. *For each fixed set X the following properties are valid.*

(1) *Recall that* 2^X *denotes the set of all subsets of a given set* X *and a. As a real-valued, nonnegative function defined on* $2^X \times 2^X$, *the Pompeiu-Hausdorff quasi-distance satisfies*

$$D_{\lambda\rho} = \lambda D_\rho, \qquad \forall\,\rho \in \mathfrak{Q}(X), \quad \forall\,\lambda \in (0, +\infty), \tag{5.20}$$

$$D_{\rho^\alpha} = \big[D_\rho\big]^\alpha, \quad \forall\,\rho \in \mathfrak{Q}(X), \quad \forall\,\alpha \in (0, +\infty), \tag{5.21}$$

$$D_{\rho'} \le D_\rho, \qquad \forall\,\rho, \rho' \in \mathfrak{Q}(X) \ \ \textit{with} \ \ \rho' \le \rho \ \ \textit{on} \ \ X. \tag{5.22}$$

Also, for each $\rho \in \mathfrak{Q}(X)$, *the function* D_ρ *is symmetric in the sense that*

$$D_\rho(U, V) = D_\rho(V, U), \quad \forall\,U, V \subseteq X. \tag{5.23}$$

In addition, $D_\rho \approx D_{\rho'}$ *whenever* $\rho, \rho' \in \mathfrak{Q}(X)$ *are such that* $\rho \approx \rho'$. *More precisely,*

$$C'\,D_\rho \le D_{\rho'} \le C''\,D_\rho \ \ \textit{if} \ \ \rho, \rho' \in \mathfrak{Q}(X) \tag{5.24}$$

$$\textit{satisfy} \ \ C'\rho \le \rho' \le C''\rho \ \ \textit{on} \ \ X.$$

(2) *If* $\rho \in \mathfrak{Q}(X)$, $p \in (0, +\infty)$, *and*

$$C_{\rho,p} := \sup_{\substack{x,y,z\in X\\ \textit{not all equal}}} \frac{\rho(x,y)}{([\rho(x,z)]^p + [\rho(z,y)]^p)^{1/p}}, \tag{5.25}$$

then

$$D_\rho(U, V) \le C_{\rho,p}\Big(\big[D_\rho(U, W)\big]^p + \big[D_\rho(W, V)\big]^p\Big)^{1/p}, \quad \forall\,U, V, W \subseteq X. \tag{5.26}$$

Furthermore, formally corresponding to the case $p = +\infty$, *there holds*

$$D_\rho(U, V) \le C_\rho \max\Big\{D_\rho(U, W), D_\rho(W, V)\Big\}, \quad \forall\,U, V, W \subseteq X, \tag{5.27}$$

where, as usual, C_ρ is as in (3.20)*.*

(4) Assume that $\rho \in \mathfrak{Q}(X)$. Then for every $U, V \subseteq X$ one has

$$D_\rho(U,V) = 0 \iff \overline{U} = \overline{V}, \tag{5.28}$$

where the closures are taken with respect to τ_ρ, the topology canonically induced by ρ on X. Moreover, one has

$$C_\rho^{-1} D_\rho(U,V) \leq D_\rho(\overline{U}, \overline{V}) \leq C_\rho\, D_\rho(U,V), \qquad \textit{for every } \ U, V \subseteq X. \tag{5.29}$$

(5) Let $\rho \in \mathfrak{Q}(X)$ be an arbitrary, fixed quasi-distance and recall (5.18)*. Then*

$$D_\rho \in \mathfrak{Q}\big(\mathrm{Cl}\,(X, \tau_\rho)\big), \tag{5.30}$$

i.e., the Pompeiu-Hausdorff quasi-distance is a genuine quasi-distance when restricted to the collection of all closed subsets of (X, τ_ρ). Moreover,

$$\begin{array}{c}\textit{if the quasi-metric space } (X,\rho) \textit{ is complete then} \\ \big(\mathrm{Cl}\,(X,\tau_\rho), D_\rho\big) \textit{ is a complete quasi-metric space,}\end{array} \tag{5.31}$$

and

$$\begin{array}{c}\textit{if the topological space } (X,\tau_\rho) \textit{ is compact then} \\ \textit{the topological space } \big(\mathrm{Cp}\,(X,\tau_\rho), \tau_{D_\rho}\big) \textit{ is compact.}\end{array} \tag{5.32}$$

(6) Assume that $\rho \in \mathfrak{Q}(X)$ and $\beta \in \mathbb{R}$ are such that $0 < \beta \leq [\log_2 C_\rho]^{-1}$. For every $U, V \subseteq X$, define

$$\begin{aligned} D_{\rho,\beta}(U,V) := \inf\Big\{ \Big(\sum_{i=1}^{N} \big[D_\rho(A_i, A_{i+1})\big]^\beta\Big)^{\frac{1}{\beta}} : N \in \mathbb{N},\ (A_i)_{1\leq i\leq N+1} \subseteq 2^X \\ \textit{such that } \ A_1 = U,\ A_{N+1} = V \Big\}. \end{aligned} \tag{5.33}$$

Then the function $D_{\rho,\beta} : 2^X \times 2^X \to [0,+\infty)$ *has the following properties:*

$$D_{\rho,\beta} \text{ is symmetric,} \tag{5.34}$$

$$D_{\rho,\beta} \text{ restricted to } \mathrm{Cl}\,(X,\tau_\rho) \times \mathrm{Cl}\,(X,\tau_\rho) \text{ belongs to } \mathfrak{Q}\big(\mathrm{Cl}\,(X,\tau_\rho)\big), \tag{5.35}$$

$$D_{\rho,\beta} \approx D_\rho \text{ in the precise sense that } 2^{-2/\beta} D_\rho \leq D_{\rho,\beta} \leq D_\rho, \tag{5.36}$$

$$D_{\rho,\beta}(U,V) \leq \Big(\big[D_{\rho,\beta}(U,W)\big]^\beta + \big[D_{\rho,\beta}(W,V)\big]^\beta\Big)^{\frac{1}{\beta}}, \quad \forall\, U,V,W \subseteq X, \tag{5.37}$$

$$D_{\rho,\beta}(U,V) \leq 2^{1/\beta} \max\big\{D_{\rho,\beta}(U,W)\,,\, D_{\rho,\beta}(W,V)\big\}, \quad \forall\, U,V,W \subseteq X. \tag{5.38}$$

Moreover, for each $\gamma \in (0,\beta]$, *the following Hölder-type regularity condition of order* γ *holds:*

$$\begin{aligned} &\big|D_{\rho,\beta}(U,V) - D_{\rho,\beta}(U,W)\big| \\ &\qquad \leq \tfrac{1}{\gamma} \max\Big\{\big[D_{\rho,\beta}(U,V)\big]^{1-\gamma}, \big[D_{\rho,\beta}(U,W)\big]^{1-\gamma}\Big\}\big[D_{\rho,\beta}(V,W)\big]^\gamma \end{aligned} \tag{5.39}$$

whenever $U,V,W \subseteq X$, *with the understanding that if* $\gamma > 1$ *then one also imposes the condition that* $\overline{U} \neq \overline{V}$ *and* $\overline{U} \neq \overline{W}$, *where the closures are taken in* (X,τ_ρ).

Proof. The properties of $N_{\rho,r}$ in Lemma 5.1 readily yield the claims made in parts (1)-(4) in the statement of the theorem. In turn, from (5.27), (5.30) and Theorem 2.12 we deduce that

$$\begin{array}{l} \big(X,(\rho_\#)^\beta\big) \text{ is a metric space, } \tau_{(\rho_\#)^\beta} = \tau_\rho \text{ and} \\ \big(\mathrm{Cl}\,(X,\tau_\rho), D_{(\rho_\#)^\beta}\big) \text{ is a metric space, whenever} \\ \rho \in \mathfrak{Q}(X) \text{ and } \beta \in \mathbb{R} \text{ satisfy } 0 < \beta \leq [\log_2 C_\rho]^{-1}. \end{array} \tag{5.40}$$

Then (5.31)-(5.32) follow from (5.40) and similar properties known for metric spaces (cf., e.g, [14, Proposition 7.3.7 and Theorem 7.3.8]). Finally, the claims made in part (6) are consequences of Theorem 2.12 and properties established earlier in the proof. □

5.2 The Gromov-Pompeiu-Hausdorff Distance from One Quasi-metric Spaces to Another

The aim of this subsection is to adapt the notion of Gromov-Pompeiu-Hausdorff distance between metric spaces to the more general context of quasi-metric spaces. We begin by making a couple of definitions.

Definition 5.2. *Given two quasi-metric spaces* $(X_0, \rho_0), (X_1, \rho_1)$ *along with a function* $f : X_0 \to X_1$ *and a number* $\gamma \in (0, +\infty)$*, define the* `distortion of order` γ *of* f *with respect to* ρ_0, ρ_1 *as*

$$\mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) := \sup_{x,y\in X_0} \big| \big[\rho_1(f(x), f(y))\big]^{\gamma} - \big[\rho_0(x,y)\big]^{\gamma} \big|. \tag{5.41}$$

If $\gamma = 1$*, it is agreed that the abbreviation* $\mathrm{Dis}_{\rho_0,\rho_1}(f)$ *may be used in place of* $\mathrm{Dis}^{1}_{\rho_0,\rho_1}(f)$.

Example 5.5. *(1) If* $X_0 = X_1$ *then* $\mathrm{Dis}^{\gamma}_{\rho_0,\rho_0}(\mathrm{id}_{X_0}) = 0$ *for any* $\gamma \in (0,\infty)$*, where* id_{X_0} *is used to denote the identity function from* X_0 *to itself.*

(2) If $X_0 \subseteq \mathbb{R}^n, X_1 := \mathbb{R}^n$*,* $f(x) := cx$ *for some nonzero constant* c*,* $\gamma \in (0,\infty)$ *and* ρ_0, ρ_1 *are the standard Euclidean norm, then* $\mathrm{Dis}^{\gamma}_{\rho_0,\rho_0}(f) = (\mathrm{diam}(X_0))^{\gamma} \big| |c|^{\gamma} - 1 \big|$.

(3) If $X_0 := (0, +\infty), X_1 := \mathbb{R}$*,* $f(x) := x^{\alpha}$ *for some positive* α *not* 1*,* $\gamma \in (0,\infty)$ *and taking* $\rho_0(x,y) := |x-y| =: \rho_1(x,y)$*, then* $\mathrm{Dis}^{\gamma}_{\rho_0,\rho_0}(f) = \infty$.

(4) If, on the other hand, if $X_0 := (a,b)$*, where* a *is a nonnegative real number and* b *is greater than* a*,* $X_1 := \mathbb{R}, f(x) = x^{\alpha}$ *for some positive* α*,* $\gamma \in (0,\infty)$ *and*

both $\rho_0(x,y)$ and $\rho_1(x,y)$ are taken to be the standard 1-dimensional Euclidean norm, then $\mathrm{Dis}^{\gamma}_{\rho_0,\rho_0}(f) = |b-a|^{\alpha\gamma} - |b-a|^{\gamma}$.

Definition 5.3. *Given a non-empty set Y, a quasi-metric spaces (X,ρ) and two arbitrary functions $f, g : Y \to X$, define the* `deviation` *of f from g, with respect to the quasi-distance ρ, as*

$$\mathrm{Dev}_{\rho}(f,g) := \sup_{x\in Y} \rho(f(x), g(x)). \tag{5.42}$$

Note Y need not have a quasi-distance defined on it (and, if it does, the quasi-distance is irrelevant).

Example 5.6. *(1) If $f = g$ then* $\mathrm{Dev}_{\rho}(f,g) = 0$ *regardless of X, Y and ρ.*

(2) If $Y := \mathbb{R}^n =: X$, $f(x) := c_0 x, g(x) := c_1 x$ for some nonzero constants c_0, c_1 not equal and ρ is the standard Euclidean norm, then $\mathrm{Dev}_{\rho}(f,g) = \infty$. *If, however, Y is bounded, then the deviation is finite.*

(3) If $Y := \mathbb{R}^n =: X$, $f(x) := e^{1/|x|^2-1}, g(x) := -e^{1/|x|^2-1}$ and ρ is again the Euclidean norm, then $\mathrm{Dev}_{\rho}(f,g) = 2e$.

(4) If $Y := (0,\infty) =: X$, $f(x) := 1/x, g(x) := x$ for every $x \in (0,\infty)$ and $\rho(x,y)$ is the discrete metric, then $\mathrm{Dev}_{\rho}(f,g) = 1$.

Thus the boundedness of the domain, the boundedness of the functions in question and the quasi-metric all contribute to the finiteness of the deviation.

Lemma 5.7. *(1) Let (X_0,ρ_0), (X_1,ρ_1) and (X_2,ρ_2) be three quasi-metric spaces and assume that $f : X_0 \to X_1$ and $g : X_1 \to X_2$ are given functions. Then for every* *number* $\gamma \in (0, +\infty)$ *there holds*

$$\mathrm{Dis}^{\gamma}_{\rho_0,\rho_2}(g \circ f) \leq \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f) + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(g). \tag{5.43}$$

(2) Suppose that (X_0, ρ_0) *and* (X_1, ρ_1) *are two quasi-metric spaces and that the functions* $f : X_0 \to X_1$ *and* $g : X_1 \to X_0$ *are given. Then for every number* $\gamma \in (0, +\infty)$ *one has*

$$\big[\mathrm{Dev}_{\rho_0}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma} \leq \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f), \tag{5.44}$$

and

$$\big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma} \leq \big[\mathrm{Dev}_{\rho_0}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_0}(g). \tag{5.45}$$

(3) Let X_0 *be a non-empty set,* (X_1, ρ_1) *and* (X_2, ρ_2) *be two quasi-metric spaces and assume that* $h : X_0 \to x_1$, $f : X_1 \to X_2$, $g : X_1 \to X_1$, *and* $k : X_0 \to X_2$ *are given functions. Then for every* $\gamma \in (0, +\infty)$ *one has*

$$\begin{aligned} &\big[\mathrm{Dev}_{\rho_2}(k, f \circ g \circ h)\big]^{\gamma} \\ &\qquad \leq (C_{\rho_2})^{\gamma} \max\big\{\big[\mathrm{Dev}_{\rho_2}(k, f \circ h)\big]^{\gamma}, \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, g)\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(f)\big\}. \end{aligned} \tag{5.46}$$

(4) Assume that (X_0, d_0) *and* (X_1, d_1) *are two metric spaces and let* $f : X_0 \to X_1$ *and* $g : X_1 \to X_0$ *be two given functions. Then*

$$\mathrm{Dis}_{d_1,d_0}(g) \leq \mathrm{Dis}_{d_0,d_1}(f) + 2\,\mathrm{Dev}_{d_1}(\mathrm{id}_{X_1}, f \circ g), \tag{5.47}$$

$$\mathrm{Dis}_{d_0,d_1}(f) \leq \mathrm{Dis}_{d_1,d_0}(g) + 2\,\mathrm{Dev}_{d_0}(\mathrm{id}_{X_0}, g \circ f). \tag{5.48}$$

Proof. To deal with the claim made in part (1), fix $\gamma \in (0, +\infty)$. Using Definition 5.41

we may then write

$$\begin{aligned}
\mathrm{Dis}^{\gamma}_{\rho_0,\rho_2}(g\circ f) &= \sup_{x,y\in X_0}\big|\big[\rho_2(g(f(x)),g(f(y)))\big]^{\gamma}-\big[\rho_0(x,y)\big]^{\gamma}\big| \\
&\leq \sup_{x,y\in X_0}\Big\{\big|\big[\rho_2(g(f(x)),g(f(y)))\big]^{\gamma}-\big[\rho_1(f(x),f(y))\big]^{\gamma}\big| \\
&\qquad +\big|\big[\rho_1(f(x),f(y))\big]^{\gamma}-\big[\rho_0(x,y)\big]^{\gamma}\big|\Big\} \\
&\leq \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f)+\mathrm{Dis}^{\gamma}_{\rho_1,\rho_2}(g), \qquad (5.49)
\end{aligned}$$

as wanted. Consider next the claim made in part (2). For each $x\in X_0$ we have

$$\begin{aligned}
\big[\rho_0(x,g(f(x)))\big]^{\gamma} &\leq \big|\big[\rho_0(x,g(f(x)))\big]^{\gamma}-\big[\rho_1(f(x),f(g(f(x))))\big]^{\gamma}\big| \\
&\qquad +\big[\rho_1(f(x),f(g(f(x))))\big]^{\gamma} \\
&\leq \mathrm{Dis}^{\gamma}_{\rho_0,\rho_1}(f)+\big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1},f\circ g)\big]^{\gamma}. \qquad (5.50)
\end{aligned}$$

Taking the supremum over all $x\in X_0$ then yields (5.44) Finally, (5.45) is proved in a similar manner.

To deal with the claim made in part (3), making use of Definition 5.3 and the quasi-ultrametric property of ρ_2, we have

$$\begin{aligned}
\big[\mathrm{Dev}_{\rho_2}(k,f\circ g\circ h)\big]^{\gamma} &= \sup_{x\in X_0}\big[\rho_2(k(x),f(g(h(x))))\big]^{\gamma} \qquad (5.51)\\
&\leq \sup_{x\in X_0}\Big\{(C_{\rho_2})^{\gamma}\max\big\{\big[\rho_2(k(x),f(h(x)))\big]^{\gamma},\big[\rho_2(f(h(x)),f(g(h(x))))\big]^{\gamma}\big\}\Big\} \\
&\leq \sup_{x\in X_0}\Big\{(C_{\rho_2})^{\gamma}\max\big\{\big[\mathrm{Dev}_{\rho_2}(k,f\circ h)\big]^{\gamma},\big[\rho_2(f(h(x)),f(g(h(x))))\big]^{\gamma}\big\}\Big\}.
\end{aligned}$$

In addition, for each $x \in X_0$, we have

$$\begin{aligned}
\big[\rho_2(f(h(x)), f(g(h(x))))\big]^\gamma &\leq \big[\rho_1(h(x), g(h(x)))\big]^\gamma \\
&\quad + \Big| \big[\rho_2(f(h(x)), f(g(h(x))))\big]^\gamma - \big[\rho_1(h(x), g(h(x)))\big]^\gamma \Big| \\
&\leq \big[\mathrm{Dev}_{\rho_1}(\mathrm{id}_{X_1}, g)\big]^\gamma + \mathrm{Dis}^\gamma_{\rho_1,\rho_2}(f). \qquad (5.52)
\end{aligned}$$

Now (3) follows by combining (5.51)-(5.52).

As regards the claims made in part (4), for each $x, y \in X_1$ we may, on the one hand, write

$$\begin{aligned}
\big| d_0(g(x), g(y)) - d_1(x, y) \big| &\leq \big| d_1(f(g(x)), f(g(y))) - d_0(g(x), g(y)) \big| \\
&\quad + \big| d_1(f(g(x)), f(g(y))) - d_1(x, y) \big| \qquad (5.53) \\
&\leq \mathrm{Dis}_{d_0,d_1}(f) + \big| d_1(f(g(x)), f(g(y))) - d_1(x, y) \big|.
\end{aligned}$$

On the other hand, since d_1 is a genuine distance, based on the triangle inequality we may estimate

$$\begin{aligned}
&\big| d_1(f(g(x)), f(g(y))) - d_1(x, y) \big| \\
&\qquad \leq \big| d_1(f(g(x)), f(g(y))) - d_1(f(g(x)), y) \big| + \big| d_1(f(g(x)), y) - d_1(x, y) \big| \\
&\qquad \leq d_1(f(g(y)), y) + d_1(f(g(x)), x) \\
&\qquad \leq 2\, \mathrm{Dev}_{d_1}(\mathrm{id}_{X_1}, f \circ g). \qquad (5.54)
\end{aligned}$$

At this stage, a combination of (5.53), (5.54) yields (5.47), after taking the supremum over all $x, y \in X_1$. Finally, (5.48) is proved in a similar manner, and this completes the proof of the lemma. $\square$

Definition 5.4. *Given two quasi-metric spaces* (X_0, ρ_0) *and* (X_1, ρ_1) *along with two arbitrary functions* $f : X_0 \to X_1$, $g : X_1 \to X_0$ *and two numbers* $\alpha \in (0, +\infty]$ *and* $\gamma \in (0, \infty)$, *set*

$$[f, g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} := \max \Big\{ \mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_\alpha,((\rho_1)_{sym})_\alpha}(f), \mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(g),$$
$$\big[\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0}, g \circ f)\big]^{\gamma}, \big[\mathrm{Dev}_{((\rho_1)_{sym})_\alpha}(\mathrm{id}_{X_1}, f \circ g)\big]^{\gamma} \Big\}^{1/\gamma} \tag{5.55}$$

where $((\rho_0)_{sym})_\alpha$ *and* $((\rho_1)_{sym})_\alpha$ *are the* α*-subadditive regularizations of the symmetric versions of* ρ_0, ρ_1, *constructed as in Theorem 2.12. Then define*

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) := \inf \big\{ [f, g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} : \ f : X_0 \to X_1, \ g : X_1 \to X_0 \big\}. \tag{5.56}$$

Note we are not using the $\rho_\#$ notation as the α above is not necessarily the same α as in Convention 3.12.

On the collection of all quasi-metric spaces, the function $d_{\alpha,\gamma}$ satisfies the symmetry and the finiteness condition described in the next proposition.

Proposition 5.8. *Fix two numbers* $\alpha, \gamma \in (0, +\infty)$ *and assume that* (X_0, ρ_0) *and* (X_1, ρ_1) *are quasi-metric spaces. Then* $d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)$ *is a well-defined number belonging to* $[0, +\infty]$ *and the following symmetry property holds:*

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) = d_{\alpha,\gamma}\big((X_1, \rho_1), (X_0, \rho_0)\big). \tag{5.57}$$

Furthermore,

$$d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big) \leq \max \big\{ \mathrm{diam}_{\rho_0}(X_0), \mathrm{diam}_{\rho_1}(X_1) \big\}. \tag{5.58}$$

Proof. The claims in the first part of the statement are clear from Definition 5.4. As for (5.58), if $x_j \in X_j$, $j = 0, 1$, are two fixed points, consider the functions

$f : X_0 \to X_1$, $f(x) := x_1$ for every $x \in X_0$, and $g : X_1 \to X_0$, $g(x) := x_0$ for every $x \in X_1$. It follows then from (5.41), (5.42) and (2.126) that

$$\begin{aligned}
\mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_\alpha,((\rho_1)_{sym})_\alpha}(f) &= \Big[\sup_{x,y\in X_0} ((\rho_0)_{sym})_\alpha(x,y)\Big]^{\gamma} \\
&= \big[\mathrm{diam}_{((\rho_0)_{sym})_\alpha}(X_0)\big]^{\gamma} \leq \big[\mathrm{diam}_{\rho_0}(X_0)\big]^{\gamma}, \qquad (5.59)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(g) &= \Big[\sup_{x,y\in X_1} ((\rho_1)_{sym})_\alpha(x,y)\Big]^{\gamma} \\
&= \big[\mathrm{diam}_{((\rho_1)_{sym})_\alpha}(X_1)\big]^{\gamma} \leq \big[\mathrm{diam}_{\rho_1}(X_1)\big]^{\gamma}, \qquad (5.60)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0}, g\circ f) &= \sup_{x\in X_0} ((\rho_0)_{sym})_\alpha(x,x_0) \\
&\leq \sup_{x\in X_0} (\rho_0(x,x_0)_{sym}) \leq \mathrm{diam}_{\rho_0}(X_0), \qquad (5.61)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Dev}_{((\rho_1)_{sym})_\alpha}(\mathrm{id}_{X_1}, f\circ g) &= \sup_{x\in X_1} ((\rho_1)_{sym})_\alpha(x,x_1) \\
&\leq \sup_{x\in X_1} (\rho_1(x,x_1)_{sym}) \leq \mathrm{diam}_{\rho_1}(X_1). \qquad (5.62)
\end{aligned}$$

Now, (5.58) follows from (5.59)-(5.62) and (5.55)-(5.56). □

Proposition 5.9. *Let (X_0, ρ_0), (X_1, ρ_1) and (X_2, ρ_2) be three quasi-metric spaces and fix two numbers $\alpha, \gamma \in (0, +\infty)$. Then*

$$\begin{aligned}
&d_{\alpha,\gamma}\big((X_0,\rho_0),(X_2,\rho_2)\big) \qquad (5.63)\\
&\quad \leq 2^{1/\alpha}\Big[\big[d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)\big]^{\gamma} + \big[d_{\alpha,\gamma}\big((X_1,\rho_1),(X_2,\rho_2)\big)\big]^{\gamma}\Big]^{1/\gamma}.
\end{aligned}$$

In particular, there holds

$$\begin{aligned}
&d_{\alpha,\gamma}\big((X_0,\rho_0),(X_2,\rho_2)\big) \qquad (5.64)\\
&\quad \leq 2^{1/\gamma+1/\alpha}\max\Big\{d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big), d_{\alpha,\gamma}\big((X_1,\rho_1),(X_2,\rho_2)\big)\Big\}.
\end{aligned}$$

Proof. Fix $r_{01} > d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)$ and $r_{12} > d_{\alpha,\gamma}\big((X_1,\rho_1),(X_2,\rho_2)\big)$. Hence, there exist functions $f_{01}: X_0 \to X_1$, $f_{10}: X_1 \to X_0$, $f_{12}: X_1 \to X_2$ and $f_{21}: X_2 \to X_1$ such that

$$[f_{01}, f_{10}]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)} < r_{01} \quad \text{and} \quad [f_{12}, f_{21}]^{\alpha,\gamma}_{(X_1,\rho_1),(X_2,\rho_2)} < r_{12}. \tag{5.65}$$

Consider the functions $f := f_{12} \circ f_{01} : X_0 \to X_2$ and $g := f_{10} \circ f_{21} : X_2 \to X_0$. Then, applying (5.43) in concert with (5.65) we obtain

$$\begin{aligned}
&\mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_\alpha,((\rho_2)_{sym})_\alpha}(f) \qquad (5.66)\\
&\qquad \le \mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_2)_{sym})_\alpha}(f_{12}) + \mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_\alpha,((\rho_1)_{sym})_\alpha}(f_{01}) < r_{01}^{\gamma} + r_{12}^{\gamma},\\
&\mathrm{Dis}^{\gamma}_{((\rho_2)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(g) \qquad (5.67)\\
&\qquad \le \mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(f_{10}) + \mathrm{Dis}^{\gamma}_{((\rho_2)_{sym})_\alpha,((\rho_1)_{sym})_\alpha}(f_{21}) < r_{01}^{\gamma} + r_{12}^{\gamma}.
\end{aligned}$$

Next, we use part (3) of Lemma 5.7 and the fact that $C_{((\rho_0)_{sym})_\alpha} \le 2^{1/\alpha}$ (itself, a consequence of (2.26) and part (6) in Theorem 2.12) in order to be able to write

$$\begin{aligned}
\big[\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0}, g\circ f)\big]^{\gamma} &\le (C_{((\rho_0)_{sym})_\alpha})^{\gamma} \max\Big\{ \big[\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0}, f_{10}\circ f_{01})\big]^{\gamma},\\
&\qquad \big[\mathrm{Dev}_{((\rho_1)_{sym})_\alpha}(\mathrm{id}_{X_1}, f_{21}\circ f_{12})\big]^{\gamma} + \mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(f_{10})\Big\}\\
&\le 2^{\gamma/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma}).
\end{aligned} \tag{5.68}$$

Similarly, we obtain

$$\big[\mathrm{Dev}_{((\rho_2)_{sym})_\alpha}(\mathrm{id}_{X_2}, f\circ g)\big]^{\gamma} \le 2^{\gamma/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma}). \tag{5.69}$$

In concert, (5.66)-(5.69) and (5.55)-(5.56) imply that

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_2,\rho_2)\big) \le 2^{1/\alpha}(r_{01}^{\gamma} + r_{12}^{\gamma})^{1/\gamma}. \tag{5.70}$$

Now letting $r_{01} \searrow d_{\alpha,\gamma}\big((X_0, \rho_0), (X_1, \rho_1)\big)$ and $r_{12} \searrow d_{\alpha,\gamma}\big((X_1, \rho_1), (X_2, \rho_2)\big)$ in (5.70) we arrive at (5.63), from which (5.64) also follows. $\square$

Proposition 5.10. *If the quasi-metric spaces (X_0, ρ_0) and (X_1, ρ_1) are bi-Lipschitzly homeomorphic with one another then there exists $\widetilde{\rho}_0 \in \mathfrak{Q}(X_0)$ such that*

$$\widetilde{\rho}_0 \approx \rho_0 \quad \textit{and} \quad d_{\alpha,\gamma}\big((X_0, \widetilde{\rho}_0), (X_1, \rho_1)\big) = 0 \ \ \textit{for every} \ \ \alpha, \gamma \in (0, +\infty). \tag{5.71}$$

More specifically, suppose that $\phi : X_0 \to X_1$ is a bijection with the property that there exist constants $c', c'' \in (0, +\infty)$ such that

$$c'\rho_0(x, y) \leq \rho_1(\phi(x), \phi(y)) \leq c''\rho_0(x, y), \quad \forall\, x, y \in X_0. \tag{5.72}$$

If one then defines

$$\widetilde{\rho}_0(x, y) := \rho_1(\phi(x), \phi(y)), \quad \forall\, x, y \in X_0, \tag{5.73}$$

it follows that

$$\begin{aligned} &\widetilde{\rho}_0 \in \mathfrak{Q}(X_0), \quad c'\rho_0 \leq \widetilde{\rho}_0 \leq c''\rho_0, \qquad \textit{and} \\ &d_{\alpha,\gamma}\big((X_0, \widetilde{\rho}_0), (X_1, \rho_1)\big) = 0 \quad \forall\ \alpha, \gamma \in (0, +\infty). \end{aligned} \tag{5.74}$$

Proof. Assume that the function $\phi : X_0 \to X_1$ is bijective and that there exist two constants $c', c'' \in (0, +\infty)$ with the property that (5.72) holds. Define $\widetilde{\rho}_0$ as in (5.73). Then, by design, $\widetilde{\rho}_0 \in \mathfrak{Q}(X_0)$ and $c'\rho_0 \leq \widetilde{\rho}_0 \leq c''\rho_0$ on $X_0 \times X_0$. Furthermore, from part (8) in Lemma 2.3 it follows that

$$((\widetilde{\rho}_0)_{sym})_\alpha(x, y) = ((\rho_1)_{sym})_\alpha(\phi(x), \phi(y)), \quad \forall\, x, y \in X_0, \tag{5.75}$$

and, hence, for every $\alpha, \gamma \in (0, +\infty)$,

$$\mathrm{Dis}^{\gamma}_{((\widetilde{\rho}_0)_{sym})_\alpha, ((\rho_1)_{sym})_\alpha}(\phi) = 0 \quad \text{and} \quad \mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha, ((\widetilde{\rho}_0)_{sym})_\alpha}(\phi^{-1}) = 0. \tag{5.76}$$

Since, clearly, for each $\alpha, \gamma \in (0, +\infty)$ we also have

$$\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0}, \phi^{-1} \circ \phi) = \mathrm{Dev}_{((\rho_1)_{sym})_\alpha}(\mathrm{id}_{X_1}, \phi \circ \phi^{-1}) = 0, \tag{5.77}$$

we deduce that $[\phi, \phi^{-1}]^{\alpha,\gamma}_{(X_0,\widetilde{\rho}_0),(X_1,\rho_1)} = 0$, for every $\alpha, \gamma \in (0, +\infty)$. Thus, ultimately, it follows that $d_{\alpha,\gamma}\big((X_0, \widetilde{\rho}_0), (X_1, \rho_1)\big) = 0$ for every $\alpha, \gamma \in (0, +\infty)$, finishing the proof of (5.74). □

Note even though $d_{\alpha,\gamma}$ has properties very similar to a quasi-metric (symmetry by Proposition 5.8, quasi-triangle inequality by Proposition 5.9, vanishes if a certain power of the regularized version of its inputs are isometric (this is made precise in (5.89) and (5.90)), it is not a quasi-metric as it vanishes off the diagonal ($d_{\alpha,\gamma}$ can be nonzero for bi-Lipschitz homeomorphic quasi-metric spaces, though this discussion is saved until Remark (5.6)).

Definition 5.5. *Define the* `Gromov-Pompeiu-Hausdorff distance`

$$d_{GPH}\big((X_0, d_0), (X_1, d_1)\big) \tag{5.78}$$

between any two given metric spaces (X_0, d_0) *and* (X_1, d_1) *as the infimum of all* $r > 0$ *with the property that there exist a metric space* (X, d) *along with isometric embeddings* $\phi_0 : (X_0, d_0) \to (X, d)$ *and* $\phi_1 : (X_1, d_1) \to (X, d)$ *for which*

$$D_{(X,d)}\big(\phi_0(X_0), \phi_1(X_1)\big) \leq r, \tag{5.79}$$

where $D_{(X,d)}$ *denotes the Pompeiu-Hausdorff distance in* (X, d) *(cf.* (5.14)*).*

In the case that $D_{(X,d)}\big(\phi_0(X_0), \phi_1(X_1)\big) > r$ *for any* $r \in \mathbb{R}$ *regardless of* (X, d), ϕ_0 *and* ϕ_1, $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big)$ *is taken to be* $+\infty$. *That is, the infimum is taken over the empty set, which we earlier convened to be* $+\infty$ *in Remark* (4.10)*.*

Before moving on, we wish to note that $d_{\alpha,\gamma}$, defined in Definition 5.4, when restricted to acting on metric spaces is equivalent to the Gromov-Pompeiu-Hausdorff distance. In particular, when $d_{1,1}$ acts on metric spaces (in which case $\alpha = \gamma = 1$ is admissible as $C_d \leq 2$ for any metric d), we have $d_{1,1} \approx d_{GPH}$. For justification, see inequalities (5.89) and (5.90), noting $(d_{sym})_1^1 = d$ whenever d is a metric.

Remark 5.3. *Given a metric space (X, d), one has $D_d(U, V)$ is geate than or equal to $d_{GPH}\big((U, d\lfloor U), (V, d\lfloor V)\big)$ for any $U, V \subseteq X$. That is, the Gromov-Pompeiu-Hausdorff distance between two subsets of the same metric space always dominates their Pompeiu-Hausdorff distance.*

The following is presented in [14, 7.3.14, p. 255].

Proposition 5.11. *Given two metric spaces $(X_j, d_j), j = 0, 1$ with $\operatorname{diam}_{d_0}(X_0)$ finite, it follows that $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big) \geq 1/2|\operatorname{diam}_{d_1}(X_1) - \operatorname{diam}_{d_0}(X_0)|$.*

Proof. Fix (X_0, d_0) and (X_1, d_1), two metric spaces. If $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big) = +\infty$ then there is nothing to prove and we are done. So suppose we have the inequality that $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big) < +\infty$. Let r positive be such that there is a metric space (X, d) and two isometric embeddings $\phi_j : (X_j, d_j) \to (X, d)$, $j = 0, 1$ with $D_{(X,d)}\big(\phi_0(X_0), \phi_1(X_1)\big) < r$. That this is possible is guaranteed by the finiteness of $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big)$. At this stage, if we can show

$$D_{(X,d)}\big(\phi_0(X_0), \phi_1(X_1)\big) \geq 1/2|\operatorname{diam}_{d_1}(X_1) - \operatorname{diam}_{d_0}(X_0)| \tag{5.80}$$

then we will be done as $(X, d), \phi_0$ and ϕ_1 were chosen for an arbitrary r in the infimum of the definition of Gromov-Pompeiu-Hausdorff distance.

To this end, fix $r' > 0$ such that $\phi_0(X_0) \subseteq N_{d,r'}(\phi_1(X_1))$ and $\phi_1(X_1) \subseteq N_{d,r'}(\phi_0(X_0))$ as in the definition of $D_{(X,d)}$. Using the fact

$$\begin{gathered}\text{for } X_0, X_1 \subseteq X, (X,d) \text{ a metric space}, r > 0 \\ X_0 \subseteq N_{d,r}(X_1) \implies \operatorname{diam}_d(X_0) \leq \operatorname{diam}_d(X_1) + 2r,\end{gathered} \tag{5.81}$$

$\operatorname{diam}_d(\phi_0(X_0)) \leq \operatorname{diam}_d(\phi_1(X_1)) + 2r'$ and $\operatorname{diam}_d(\phi_1(X_1)) \leq \operatorname{diam}_d(\phi_0(X_0)) + 2r'$. Invoking the assumption that $\operatorname{diam}_{d_0}(X_0)$ is finite and shuffling around inequalities in the appropriate manner yields $2r' \geq |\operatorname{diam}_d(\phi_1(X_1)) - \operatorname{diam}_d(\phi_0(X_0))|$. Upon observing that, as ϕ_0 and ϕ_1 are isometries, we have $\operatorname{diam}_d(\phi_0(X_0)) = \operatorname{diam}_{d_0}(X_0)$ and $\operatorname{diam}_d(\phi_1(X_1)) = \operatorname{diam}_{d_1}(X_1)$, hence

$$r' \geq 1/2|\operatorname{diam}_{d_1}(X_1) - \operatorname{diam}_{d_0}(X_0)|. \tag{5.82}$$

As the choice of r' in the definition of $D_{(X,d)}$ was arbitrary, (5.80) follows. $\square$

Remark 5.4. *Note* (5.81) *does not, in general, hold for quasi-metric spaces.*

A simple consequence of Proposition 5.11 is as follows: if (X_0, d_0) has finite d_0-diameter and $d_{GPH}\big((X_0, d_0), (X_1, d_1)\big)$ is finite for some metric space (X_1, d_1), then necessarily (X_1, d_1) has finite d_1-diameter. However, for any given metric spaces $(X_i, d_i), i = 1, 2$, the Gromov-Pompeiu-Hausdorff distance between them being finite does not necessarily entail both metric spaces have finite diameters. For example, $d_{GPH}\big((\mathbb{R}, |\cdot - \cdot|), (\mathbb{R}, |\cdot - \cdot|)\big) = 0$.

Remark 5.5. *Applying Proposition 5.11 with (X_0, d_0) an arbitrary metric space and defining $(X_1, d_1) := (\{x_1\}, d_1)$, where d_1 is irrelevant (any metric or quasi-metric defined on a singleton cross itself only takes the value zero), we conclude*

$$d_{GPH}\big((X_0, d_0), (\{x_1\}, d_1)\big) \geq \operatorname{diam}_{d_0}(X_0)/2, \tag{5.83}$$

as the diameter of any singleton metric space is zero. However, the opposite inequality need not hold; that is, it is not necessarily true that

$$d_{GPH}\big((X_0, d_0), (\{x_1\}, d_1)\big) = \mathrm{diam}_{d_0}(X_0)/2.$$

Take, for example, X_0 an isosceles triangle in the plane whose sides have length 2 and d_0 the Euclidean distance. Then the d_0-diameter of X_0 is 2 but

$$d_{GPH}\big((X_0, d_0), (\{x_1\}, d_1)\big) = 2/\sqrt{3}.$$

This corrects an inaccuracy found in [14, Exercise 7.3.15].

However, the following inequality does hold.

$$\mathrm{diam}_{d_0}(X_0)/2 \leq d_{GPH}\big((X_0, d_0), (\{x_1\}, d_1)\big) \leq \mathrm{diam}_{d_0}(X_0), \tag{5.84}$$

as one can isometrically embed (X_0, d_0) via $\phi : X_0 \to X$ into any metric space (X, d) one wishes, then map $\{x_1\}$ anywhere into the closure of $\phi(X)$ relative to the topology τ_d. Thus we have the equivalence $\mathrm{diam} \approx d_{GPH}(\,\cdot\,, \{x_0\})$ *for any singleton $\{x_0\}$.*

Definition 5.6. *Consider two metric spaces (X_0, d_0), (X_1, d_1). Given an arbitrary number $\varepsilon > 0$, a function $f : X_0 \to X_1$ is said to be an ε-*`isometry` *between (X_0, d_0) and (X_1, d_1) provided*

$$\mathrm{Dis}_{d_0,d_1}(f) < \varepsilon \quad \textit{and} \quad X_1 = \bigcup_{x \in X_0} B_{d_1}(f(x), \varepsilon). \tag{5.85}$$

Denote by ε-Iso$\big((X_0, d_0), (X_1, d_1)\big)$ the collection of all ε-isometries between (X_0, d_0) and (X_1, d_1).

With this piece of terminology, Corollary 7.3.28 in [14] then shows that the following implications are valid:

$$d_{GPH}\big((X_0,d_0),(X_1,d_1)\big) < \varepsilon \implies (2\varepsilon)\text{-Iso}\,\big((X_0,d_0),(X_1,d_1)\big) \neq \emptyset, \tag{5.86}$$

$$\varepsilon\text{-Iso}\,\big((X_0,d_0),(X_1,d_1)\big) \neq \emptyset \implies d_{GPH}\big((X_0,d_0),(X_1,d_1)\big) < 2\varepsilon. \tag{5.87}$$

Proposition 5.12. *Assume that (X_j,ρ_j), $j=0,1$, are two quasi-metric spaces and suppose that $\alpha \in (0,+\infty]$ and the real number γ satisfy*

$$\gamma \in (0,\alpha] \quad \text{ and } \quad 0<\alpha \le [\log_2 C_{\rho_j}]^{-1}, \quad j=0,1. \tag{5.88}$$

Then $(X_j,((\rho_j)_{sym})^{\gamma}_{\alpha})$, $j=0,1$, are metric spaces (where, for $j=0,1$, $((\rho_j)_{sym})^{\gamma}_{\alpha}$ abbreviates $(((\rho_j)_{sym})_{\alpha})^{\gamma})$, and

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big) \le 6^{1/\gamma}\Big[d_{GPH}\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha}\big)\Big]^{1/\gamma}, \tag{5.89}$$

$$\Big[d_{GPH}\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha}\big)\Big]^{1/\gamma} \le 2^{1/\gamma} d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big). \tag{5.90}$$

Proof. That $(X_j,((\rho_j)_{sym})^{\gamma}_{\alpha})$, $j=0,1$, are metric spaces whenever (5.88) holds is a consequence of Theorem 2.12. Next, let $\varepsilon > d_{GPH}\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha}\big)$. Make use of (5.86) and select $f \in (2\varepsilon)\text{-Iso}\,\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha})\big)$. In particular, $X_1 = \bigcup\limits_{x\in X_0} B_{((\rho_1)_{sym})^{\gamma}_{\alpha}}(f(x),2\varepsilon)$ and we may invoke the axiom of choice in order to construct $g: X_1 \to X_0$ with the property that $[((\rho_1)_{sym})_{\alpha}(f(g(x)),x)]^{\gamma} < 2\varepsilon$ for every $x \in X_1$. Thus, we have

$$\big[\mathrm{Dev}_{((\rho_1)_{sym})_{\alpha}}(\mathrm{id}_{X_1}, f\circ g)\big]^{\gamma} < 2\varepsilon, \tag{5.91}$$

$$\mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_{\alpha},(\rho_1)_{\alpha}}(f) = \mathrm{Dis}_{((\rho_0)_{sym})^{\gamma}_{\alpha},((\rho_1)_{sym})^{\gamma}_{\alpha}}(f) < 2\varepsilon, \tag{5.92}$$

where the fact that $f \in (2\varepsilon)\text{-Iso}\,\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha})\big)$ has been used in the last inequality above. Based on (5.91)-(5.92) and parts (2) and (4) in Lemma 5.7, we then deduce that

$$\begin{aligned}&\mathrm{Dis}^{\gamma}_{((\rho_1)_{sym})_\alpha,((\rho_0)_{sym})_\alpha}(g)=\mathrm{Dis}_{((\rho_1)_{sym})^\gamma_\alpha,((\rho_0)_{sym})^\gamma_\alpha}(g)<6\,\varepsilon,\\&\big[\mathrm{Dev}_{((\rho_0)_{sym})_\alpha}(\mathrm{id}_{X_0},g\circ f)\big]^{\gamma}<4\varepsilon.\end{aligned}\tag{5.93}$$

Collectively, (5.91)-(5.93) prove that

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)\leq[f,g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)}<(6\,\varepsilon)^{1/\gamma}.\tag{5.94}$$

After letting $\varepsilon\searrow d_{GPH}\big((X_0,((\rho_0)_{sym})^\gamma_\alpha),(X_1,((\rho_1)_{sym})^\gamma_\alpha\big)$, this yields (5.89).

To justify (5.90), consider $\varepsilon>d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)$. Then there exist two functions, $f:X_0\to X_1$ and $g:X_1\to X_0$, with the property that $[f,g]^{\alpha,\gamma}_{(X_0,\rho_0),(X_1,\rho_1)}<\varepsilon$. In particular,

$$\mathrm{Dis}_{((\rho_0)_{sym})^\gamma_\alpha,((\rho_1)_{sym})^\gamma_\alpha}(f)=\mathrm{Dis}^{\gamma}_{((\rho_0)_{sym})_\alpha,((\rho_1)_{sym})_\alpha}(f)<\varepsilon^\gamma,\tag{5.95}$$

$$\mathrm{Dev}_{((\rho_1)_{sym})^\gamma_\alpha}(\mathrm{id}_{X_1},f\circ g)=\big[\mathrm{Dev}_{((\rho_1)_{sym})_\alpha}(\mathrm{id}_{X_1},f\circ g)\big]^{\gamma}<\varepsilon^\gamma.\tag{5.96}$$

Upon observing that (5.96) entails

$$X_1=\bigcup_{x\in X_0}B_{((\rho_1)_{sym})^\gamma_\alpha}(f(x),\varepsilon^\gamma),\tag{5.97}$$

it follows from (5.95) and (5.97) that ε^γ-Iso $\big((X_0,((\rho_0)_{sym})^\gamma_\alpha),(X_1,((\rho_1)_{sym})^\gamma_\alpha)\big)\neq\emptyset$. With this in hand, (5.87) then gives $d_{GPH}\big((X_0,((\rho_0)_{sym})^\gamma_\alpha),(X_1,((\rho_1)_{sym})^\gamma_\alpha\big)<2\varepsilon^\gamma$. By letting ε approach $d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)$ from above, estimate (5.90) follows. This finishes the proof of the proposition. $\square$

Remark 5.6. *Let (X_j,ρ_j), $j=0,1$ be two quasi-metric spaces with* $\mathrm{diam}_{\rho_0}(X_0)$ *finite and suppose $\alpha,\gamma\in(0,+\infty]$ satisfy* (5.88)*. Then, by combining Proposition 5.11 and*

5.90 of Proposition 5.12, it follows that

$$\begin{aligned} 2d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big)^{\gamma} &\geq d_{GPH}\big((X_0,((\rho_0)_{sym})^{\gamma}_{\alpha}),(X_1,((\rho_1)_{sym})^{\gamma}_{\alpha}\big) \\ &\geq 1/2|\operatorname{diam}_{((\rho_0)_{sym})^{\gamma}_{\alpha}}(X_0)-\operatorname{diam}_{((\rho_1)_{sym})^{\gamma}_{\alpha}}(X_1)\,|, \quad (5.98) \end{aligned}$$

and thus

$$d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big) \geq \big(1/4|\operatorname{diam}_{((\rho_0)_{sym})^{\gamma}_{\alpha}}(X_0)-\operatorname{diam}_{((\rho_1)_{sym})^{\gamma}_{\alpha}}(X_1)\,|\big)^{1/\gamma}. \quad (5.99)$$

Taking $X_0 := [0,1], X_1 := [0,2]$ and ρ_0, ρ_1 to be the Euclidean distance on $\mathbb{R}$, we have (X_0,ρ_0) and (X_1,ρ_1) are bi-Lipschitz homeomorphic but $d_{\alpha,\gamma}\big((X_0,\rho_0),(X_1,\rho_1)\big) > 0$ for $\alpha := 1 =: \gamma$.

Although $d_{1,1} \approx d_{GPH}$ when acting on metric spaces, in which case $d_{\alpha,\gamma}$ could be seen as an extension of the Gromov-Pompeiu-Hausdorff distance, it does not satisfy the triangle inequality, nor does any nice power of $d_{\alpha,\gamma}$. The best we can get is (5.63) and (5.64). For this matter, we wish to consider a regularized version of it.

We are now ready to introduce a version of the Gromov-Pompeiu-Hausdorff distance which is suitably adapted to the context of quasi-metric spaces.

Definition 5.7. *Fix $\alpha \in (0,+\infty]$, a number $\gamma \in (0,\alpha]$ finite and assume that the quasi-metric spaces (X,ρ_X), (Y,ρ_Y) have the property that*

$$0 < \alpha \leq \min\Big\{[\log_2 C_{\rho_X}]^{-1}, [\log_2 C_{\rho_Y}]^{-1}\Big\}. \quad (5.100)$$

Then for every finite $\beta \in (0,(1/\alpha+1/\gamma)^{-1}]$ define

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) := \inf\Big\{\Big(\sum_{i=1}^{N}\big[d_{\alpha,\gamma}\big((Z_i,\rho_i),(Z_{i+1},\rho_{i+1})\big)\big]^{\beta}\Big)^{\frac{1}{\beta}}\Big\} \quad (5.101)$$

where $d_{\alpha,\gamma}$ is as in Definition 5.4 and the above infimum is taken over all numbers $N \in \mathbb{N}$ and all families $(Z_i, \rho_i)_{1\leq i\leq N+1}$ of quasi-metric spaces with the property that

$$\begin{aligned} &(Z_0, \rho_0) = (X, \rho_X), \qquad (Z_{N+1}, \rho_{N+1}) = (Y, \rho_Y), \\ &\textit{and} \quad 0 < \alpha \leq [\log_2 C_{\rho_i}]^{-1}, \quad \forall\, i \in \{1, ..., N+1\}. \end{aligned} \tag{5.102}$$

Before delving into the properties of this new metric, we first wish to comment that $\delta_{\alpha,\gamma,\beta}$ is, indeed, a natural adaptation of d_{GPH} (see item (i) of Theorem 5.13 below). That is, when $\delta_{\alpha,\gamma,\beta}$ is restricted to acting on metric spaces it becomes equivalent to the Gromov-Pompeiu-Hausdorff distance.

The theorem below, summarizing some of the most basic properties of the function $\delta_{\alpha,\gamma,\beta}$ introduced in Definition 5.7, is the main result in this subsection.

Theorem 5.13. *Fix $\alpha \in (0, +\infty]$, $\gamma \in (0, \alpha]$ finite and assume that the paramete $\beta \in (0, (1/\alpha + 1/\gamma)^{-1}]$. Then the following properties are valid:*

(i) Whenever (X, ρ_X) and (Y, ρ_Y) are two quasi-metric spaces satisfying the inequality $[\log_2 C_{\rho_X}]^{-1} \geq \alpha$ and $[\log_2 C_{\rho_Y}]^{-1} \geq \alpha$, the following symmetry condition holds:

$$\delta_{\alpha,\gamma,\beta}\big((X, \rho_X), (Y, \rho_Y)\big) = \delta_{\alpha,\gamma,\beta}\big((Y, \rho_Y), (X, \rho_X)\big). \tag{5.103}$$

Moreover,

$$2^{-\frac{2}{\beta}} d_{\alpha,\gamma} \leq \delta_{\alpha,\gamma,\beta} \leq d_{\alpha,\gamma} \tag{5.104}$$

hence, in particular,

$$\delta_{\alpha,\gamma,\beta}\big((X, \rho_X), (Y, \rho_Y)\big) \leq \max\big\{\mathrm{diam}_{\rho_X}(X), \mathrm{diam}_{\rho_Y}(Y)\big\}. \tag{5.105}$$

In addition,

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) \tag{5.106}$$

$$\leq 6^{1/\gamma}\Big[d_{GPH}\big((X,((\rho_X)_{sym})^{\gamma}_{\alpha}),(Y,((\rho_Y)_{sym})^{\gamma}_{\alpha})\big)\Big]^{1/\gamma}, \tag{5.107}$$

$$\Big[d_{GPH}\big((X,((\rho_X)_{sym})^{\gamma}_{\alpha}),(Y,((\rho_Y)_{sym})^{\gamma}_{\alpha})\big)\Big]^{1/\gamma}$$

$$\leq 2^{1/\alpha+1/\beta+1/\gamma}\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big). \tag{5.108}$$

In particular, if all quasi-metric spaces involved are metric spaces it follows from (5.106) *and* (5.108) *that* $\delta_{1,1,\beta} \approx d_{GPH}$ *for any fixed* $\beta \in (0,1/2]$.

(ii) The function $(\delta_{\alpha,\gamma,\beta})^{\beta}$ *satisfies the triangle inequality in the sense that if* (X,ρ_X), (Y,ρ_Y), (Z,ρ_Z) *are three quasi-metric spaces with the property that*

$$0 < \alpha \leq \min\Big\{[\log_2 C_{\rho_X}]^{-1},[\log_2 C_{\rho_Y}]^{-1},[\log_2 C_{\rho_Z}]^{-1}\Big\}, \tag{5.109}$$

then

$$\big[\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big)\big]^{\beta} \tag{5.110}$$

$$\leq \big[\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Z,\rho_Z)\big)\big]^{\beta} + \big[\delta_{\alpha,\gamma,\beta}\big((Z,\rho_Z),(Y,\rho_Y)\big)\big]^{\beta}.$$

(iii) Suppose that (X,ρ_X) *and* (Y,ρ_Y) *are quasi-metric spaces for which there holds* $[\log_2 C_{\rho_X}]^{-1} \geq \alpha$ *and* $[\log_2 C_{\rho_Y}]^{-1} \geq \alpha$, *and such that the topological spaces* (X,τ_{ρ_X}) *and* (Y,τ_{ρ_Y}) *are compact. Then*

$$\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big) = 0 \tag{5.111}$$

$$\Longleftrightarrow (X,((\rho_X)_{sym})_{\alpha}) \text{ and } (Y,((\rho_Y)_{sym})_{\alpha}) \text{ are isometric.}$$

In particular, $\delta_{\alpha,\gamma,\beta}\big((X,\rho_X),(Y,\rho_Y)\big)=0$ implies that (X,ρ_X) and (Y,ρ_Y) are bi-Lipschitzly homeomorphic.

Conversely, given two bi-Lipschitzly homeomorphic quasi-metric spaces (X,ρ_X) and (Y,ρ_Y) which satisfy $[\log_2 C_{\rho_X}]^{-1}\geq\alpha$ and $[\log_2 C_{\rho_Y}]^{-1}\geq\alpha$, there exists a quasi-distance $\widetilde{\rho}_X\in\mathfrak{Q}(X)$ such that $\widetilde{\rho}_X\approx\rho_X$ and $\delta_{\alpha,\gamma,\beta}\big((X,\widetilde{\rho}_X),(Y,\rho_Y)\big)=0$.

Proof. Recall that estimate (5.64) holds and that, by hypothesis, the exponent β satisfies the inequality $0<\beta\leq[\log_2 2^{1/\gamma+1/\alpha}]^{-1}$. Keeping these observations in mind, then the symmetry property (5.103) is a consequence of (5.57) and (2.125) in part (11) of Theorem 2.12, whereas the double inequality in (5.104) follows from (2.47). Furthermore, (5.105) follows from (5.104) and (5.58). A combination of (5.104) and Proposition 5.12 proves (5.106)-(5.108). This concludes the proof of (*i*). Next, based on our earlier observations and (2.128) in part (11) of Theorem 2.12, we deduce that the claim in (*ii*) holds.

Moving on, the first claim in (*iii*) is a corollary of (5.108) and [14, Theorem 7.3.30]. Together, (5.104) and Proposition 5.10 then imply the second claim in (*iii*), finishing the proof of the theorem. □

5.3 Convergence of Quasi-metric Spaces

With the $\delta_{\alpha,\gamma,\beta}$ introduced in Definition 5.7 we may now talk about convergence of quasi-metric spaces. We will primarily be interested in convergence of compact quasi-metric spaces as convergence of non-compact quasi-metric spaces is usually very poor. The notion of convergence of quasi-metric spaces has already been encountered in the final claim of Theorem 5.13.

However, before stating what it means for a sequence of quasi-metric spaces to converge, we'll start with convergence of metric spaces. This notion will be useful thanks to how strongly related $\delta_{\alpha,\gamma,\beta}$ and d_{GPH} are (and a substantial amount of information is known regarding convergence with respect to d_{GPH}).

Definition 5.8. *An infinite family of compact metric spaces* $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ *is said to converge to a compact metric space* (X, d) *if* $\lim\limits_{n\to\infty} d_{GPH}((X_n, d_n), (X, d)) = 0$*. In this case* (X, d) *is called the* `Gromov-Pompeiu-Hausdorff limit` *of* $\{(X_n, d_n)\}_{n\in\mathbb{N}}$.

Note by (5.111) of Theorem 5.13, it follows that the limit is unique up to isometries.

Proposition 5.14. *Every compact metric space is the Gromov-Pompeiu-Hausdorff limit of metric spaces of finite cardinality.*

Proof. Fix (X, d) a metric space of finite cardinality. As X is compact, we can find finitely many points $X_n := \{x_i\}_{i=1}^N$, such that $X = \bigcup\limits_{x\in X_n}^{N} B_d(x, 1/n)$. Next, specializing Definition 5.6 to when $(X_0, d_0) := (X, d)$, $(X_1, d_1) := (X_n, d\lfloor X_n)$ and $f := f_n$ the identity, we see f_n is a $1/n$-isometry between $(X_n, d\lfloor X_n)$ and (X, d).

Then (5.87) kicks in and tell us $d_{GPH}((X_n, d\lfloor X_n), (X, d)) \leq 2/n$ for every natural number n, and so the result follows. □

One interpretation of Proposition 5.14 is that metric spaces of finite cardinality are dense in compact metric spaces. That is, Proposition 5.14 will prove to be incredibly useful as results that hold in metric spaces of finite cardinality will hold in the more general setting of compact metric spaces. Conversely, when given something we wish to prove in compact metric spaces, in many cases matters may be reduced to simply studying finite metric spaces.

In this vein, let us establish a few more properties regarding convergence involving finite metric spaces.

Proposition 5.15. *A sequence of metric spaces $\big\{(X_n, d_n)\big\}_{n\in\mathbb{N}}$ converges to a singleton (metric space) if and only if $\lim\limits_{n\to\infty} \operatorname{diam}_{d_n}(X_n) = 0$.*

Proof. By definition, a sequence of metric spaces can only converge to another metric space, but any metric defined on a singleton cross itself only takes the value 0. Both implications follow from the discussion in Remark 5.5. $\square$

The next proposition encompasses the previous as we classify when a family of metric spaces converges to a metric space with finite cardinality (rather than simply cardinality 1), as seen in [14, 7.4.7, p. 261].

Proposition 5.16. *Let $\big\{(X_n, d_n)\big\}_{n\in\mathbb{N}}$ be a family of metric spaces and (X, d) be a metric space of cardinality N. Then (X, d) is the Gromov-Pompeiu-Hausdorff limit of $\big\{(X_n, d_n)\big\}_{n\in\mathbb{N}}$ if and only if, for every natural number $n \in \mathbb{N}$ from a rank on, (X_n, d_n) can be split into a finite family of sets $\big\{(X_{n,1}, d\lfloor X_{n,1}), \cdots, (X_{n,N}, d\lfloor X_{n,N})\big\}$ satisfying*

$$\begin{gathered} X_n \text{ is the disjoint union of the } X_{n,i}\text{'s}, \\ \operatorname{diam}_{d_{n,i}}(X_{n,i}) \to 0 \text{ and } \operatorname{dist}_{d_n}(X_{n,i}, X_{n,j}) \to d(x_i, x_j). \end{gathered} \tag{5.112}$$

Definition 5.9. *Fix $\alpha \in (0, +\infty]$, $\gamma \in (0, \alpha]$ finite, a family of compact quasi-metric spaces $\big\{(X_n, \rho_n)\big\}_{n\in\mathbb{N}}$ and another compact quasi-metric space (X, ρ) which all satisfy the following inequality: $\alpha \leq \min\big\{\inf\limits_{n\in\mathbb{N}} [\log_2 C_{\rho_n}]^{-1}, [\log_2 C_\rho]^{-1}\big\}$. Then $\big\{(X_n, \rho_n)\big\}_{n\in\mathbb{N}}$ is said to converge to (X, ρ) if $\lim\limits_{n\to\infty} \delta_{\alpha,\gamma,\beta}((X_n, \rho_n), (X, \rho)) = 0$ for some (hence, any) $\beta \in (0, (1/\alpha + 1/\gamma)^{-1}]$. In this case (X, ρ) is called the $\delta_{\alpha,\gamma}$* `limit` *of $\big\{(X_n, d_n)\big\}_{n\in\mathbb{N}}$.*

First of all, notice that if we tailor the items in Definition 5.9 in the appropriate manner then $\delta_{\alpha,\gamma}$ convergence reduces to precisely Gromov-Pompeiu-Hausdorff convergence, as desired. Specifically, if (X_n, ρ_n) are actually metric spaces for every $n \in \mathbb{N}$ then we may take α and γ to be 1 and, furthermore, $((\rho_n)_{sym})_{\alpha}^{\gamma} = \rho_n$ for all $n \in \mathbb{N}$. Inequalities (5.106) and (5.108) then tell us $\delta_{\alpha,\gamma,\beta}$ is pointwise equivalent to d_{GPH} (in the above setting they are defined on the same objects), thus convergence is the same.

In summary, $\delta_{\alpha,\gamma}$ convergence is a generalization of Gromov-Pompeiu-Hausdorff convergence. Now the goal is to see what properties of Gromov-Pompeiu-Hausdorff convergence hold in the more general setting.

Theorem 5.17. *Every compact quasi-metric space (X, ρ) is the $\delta_{\alpha,\gamma}$ limit of quasi-metric spaces of finite cardinality for any α, γ satisfying $\alpha \in (0, [\log_2 C_\rho]^{-1}]$ and γ finite is in the interval $(0, \alpha]$.*

Proof. The idea of the proof is not to invoke Proposition 5.14 but rather to mimic its proof. That being said, fix (X, ρ) a compact quasi-metric space and find α positive such that $\alpha \leq [\log_2 C_\rho]^{-1}$. As X is compact, for every natural number n there exist finitely many nonempty, open (in the topology induced by ρ) subsets A_i^n of X such that

$$\operatorname{diam}_\rho(A_i^n) \leq 1/n \text{ for every } i \in \{1, \cdots, N_n\} \text{ and } X \subseteq \bigcup_{i=1}^{N_n} A_i^n, \tag{5.113}$$

where N_n a natural number depending on n. For $n \in \mathbb{N}$ (and thus N_n), pick $x_i^n \in A_i^n$ for all i between 1 and N_n and define $X_n := \{x_i\}_{i=1}^{N_n}$ and $\rho_n := \rho\lfloor X_n$.

Then $C_{\rho_n} \leq C_\rho$ for every natural number n by (3) of Lemma 3.11 and so the inequality $\alpha \leq [\log_2 C_{\rho_n}]^{-1}$ follows. Now fix $\gamma \in (0, \alpha]$ finite and, for every $n \in \mathbb{N}$, define $f_n : (X_n, ((\rho_n)_{sym})^\gamma_\alpha) \to (X, (\rho_{sym})^\gamma_\alpha)$ as the identity. In order to get an ε-isometry we need $X = \bigcup_{x \in X_n} B_{(\rho_{sym})^\gamma_\alpha}(f_n(x), \varepsilon)$ as the distortion of the identity is 0. It would suffice to show, for every $n \in \mathbb{N}$,

$$\exists\, r_n > 0 \text{ such that } A_i^n \subseteq B_{(\rho_{sym})^\gamma_\alpha}(x_i^n, r). \tag{5.114}$$

By construction $x_i^n \in A_i^n$, thus $\rho(x_i^n, x) \leq C_0/n$ for any $x \in A_i^n$, where C_0 is the constant appearing in the quasi-symmetry condition of ρ. With the aid of (2.126), it follows that $(\rho_{sym})^\gamma_\alpha(x_i^n, x) \leq C_0^{2\gamma}/n^\gamma$. By definition, $x \in B_{(\rho_{sym})^\gamma_\alpha}(x_i^n, C_0^{2\gamma}/n^\gamma)$ for every $x \in A_i^n$, so (5.114) is proven with $r_n := C_0^{2\gamma}/n^\gamma$. Thus f_n is a $2C_0^{2\gamma}/n^\gamma$-isometry between $(X, ((\rho)_{sym})^\gamma_\alpha)$ and $(X_n, ((\rho_n)_{sym})^\gamma_\alpha)$. Then (5.87) implies

$$d_{GPH}((X, (\rho_{sym})^\gamma_\alpha), (X_n, ((\rho_n)_{sym})^\gamma_\alpha)) \leq 2C_0^{2\gamma}/n^\gamma$$

for every natural number n, and so by (5.106) we have, for any $\beta \in (0, (1/\alpha + 1/\gamma)^{-1}]$,

$$\begin{aligned} \delta_{\alpha,\gamma,\beta}\big((X,\rho),(X_n,\rho_n)\big) &\leq 6^{1/\gamma}\Big[d_{GPH}\big((X,(\rho_{sym})^\gamma_\alpha),(X_n,((\rho_n)_{sym})^\gamma_\alpha)\big)\Big]^{1/\gamma} && (5.115)\\ &\leq 12^{1/\gamma} C_0/n. && (5.116)\end{aligned}$$

Taking the limit as n tends to infinity finishes the proof. □

Note instead of using balls in the proof of Theorem 5.17 arbitrary open sets A_i were used. The reason is ρ-balls are not necessarily open in the topology they induce (and compactness has to do with open covers).

Recall the definition of a metric space (X, ρ) being ρ-totally bounded, Definition 3.10. We now wish to consider a family of compact metric spaces which are all totally bounded.

Definition 5.10. *Let $\{(X_i, \rho_i)\}_{i\in I}$, I some indexing set, be a collection of (quasi-)metric spaces. Then $\{(X_i, \rho_i)\}_{i\in I}$ is called* `uniformly totally bounded` *provided there is a uniform, nonnegative constant K such that $\operatorname{diam}_{\rho_i}(X_i) \le K$ for all $i \in I$ and for any $\varepsilon > 0$ there is a natural number n depending on ε such that $X_i = \bigcup_{j=1}^{n} B_\rho(x_j^i, \varepsilon)$ for all $i \in I$, where $\{x_j^i\}_{j=1}^n \subseteq X_i$.*

The following is a slight variant of [14, Theorem 7.4.15], Gromov's compactness theorem. In particular, here we consider an arbitrary infinite indexing set I, though the proof follows directly by fixing $J \subseteq I$ countable.

Theorem 5.18. *Let $\{(X_i, d_i)\}_{i\in I}$, I some infinite indexing set, be a uniformly totally bounded collection of compact metric spaces. Then every countable $J \subseteq I$ contains a d_{GPH}-convergent subsequence to a compact metric space.*

Heuristically, a convenient way to think of Theorem 5.18 is any uniformly totally bounded collection of compact metric spaces is precompact in the Gromov-Pompeiu-Hausdorff topology.

Theorem 5.19. *Assume that $\big\{(X_i, \rho_i)\big\}_{i\in I}$ is an infinite family of quasi-metric spaces satisfying the following conditions:*

$$\text{the topological space } \ (X_i, \tau_{\rho_i}) \ \text{ is compact for every } \ i \in I, \tag{5.117}$$

$$\inf_{i\in I} [\log_2 C_{\rho_i}]^{-1} \ge \alpha \quad \text{and} \quad \sup_{i\in I} \operatorname{diam}_{\rho_i}(X_i) < +\infty, \tag{5.118}$$

and

$$\begin{aligned} &\forall\, \varepsilon > 0 \;\exists\, N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall\, i \in I \;\exists\, A_i \subseteq X_i \ \text{ with} \\ &\text{the property that } \ \#\,A_i \le N \ \text{ and } \ \bigcup_{x\in A_i} B_{\rho_i}(x, \varepsilon) = X_i. \end{aligned} \tag{5.119}$$

In addition, assume that α is finite. Then for every countable set $J \subseteq I$ there exist a countable set $\{j_k\}_{k\in\mathbb{N}} \subseteq J$ and a quasi-metric space (X, ρ) with the property that (X, τ_ρ) is compact, $[\log_2 C_\rho]^{-1}$ is greater than or equal to α, and

$$\lim_{k\to\infty} \delta_{\alpha,\alpha,\beta}\big((X_{j_k}, \rho_{j_k}), (X, \rho)\big) = 0. \tag{5.120}$$

Proof. The key observation is that if α is finite then the current hypotheses imply that $\big\{(X_i, (\rho_i)^\alpha_\alpha)\big\}_{i\in I}$ is a uniformly totally bounded family of compact metric spaces. As such, Gromov's compactness theorem, Theorem 5.18, gives that for every countable set $J \subseteq I$ there exist a countable set $\{j_k\}_{k\in\mathbb{N}} \subseteq J$ and a metric space (X, d) with the property that (X, τ_ρ) is compact and

$$\lim_{k\to\infty} d_{GPH}\big((X_{j_k}, (((\rho_{j_k})_{sym})_\alpha)^\alpha), (X, d)\big) = 0. \tag{5.121}$$

Hence, if we now define $\rho := d^{1/\alpha}$ on $X \times X$, then (X, ρ) is a symmetric quasi-metric space and, since, $\tau_{d^{1/\alpha}} = \tau_d$, the topological space (X, τ_ρ) is compact. Furthermore, we have the inequality $C_\rho = C_{d^{1/\alpha}} = (C_d)^{1/\alpha} \leq 2^{1/\alpha}$, since $C_d \leq 2$ given that d is a distance. Consequently, $[\log_2 C_\rho]^{-1} \geq \alpha$. Going further, observe that

$$((\rho_{sym})_\alpha)^\alpha = \big((d^{1/\alpha})_\alpha\big)^\alpha = d \tag{5.122}$$

by parts (5) and (9) in Lemma 2.3 since, as a distance, d is 1-subadditive. This shows that we have $(X, d) = (X, ((\rho_{sym})_\alpha)^\alpha)$. With this in hand, (5.120) follows by virtue of (5.122) and (5.106). This concludes the treatment of the proof of the theorem. $\square$

Chapter 6

Smoothness of Quasi-metric Spaces

In this section we explore different methods of measuring how smooth a given quasi-metric space is. In particular, we'll investigate lower smoothness index, upper smoothness index and Assouad dimension.

This will open the door to the study of fractals and quasi-metric spaces.

6.1 Lower and Upper Smoothness Indices and Assouad Dimension

The goal here is to introduce some new, natural, concepts of lower and upper smoothness indices for a quasi-metric space, and to highlight the basic role they play in describing the structural richness of Hölder spaces. We indicate how these indices compare to one another and study the relationship with Assouad's convexity index introduced in [12]. We have already run across the precursor to the lower smoothness index of a quasi-metric space (and have used it on several occasions). Before introducing smoothness, however, we need the counterpart from which the upper smoothness index will arise.

Assume that X is a fixed set of cardinality ≥ 2. For each $\rho \in \mathfrak{Q}(X)$ we then define

$$c_\rho := \inf_{\substack{x,y\in X\\ x\neq y}} \Big(\sup_{z\in X} \frac{\rho(x,y)}{\max\{\rho(x,z),\rho(z,y)\}} \Big). \tag{6.1}$$

Recall the constant C_ρ introduced earlier in (3.20). It is then readily seen from definitions that

$$\forall\,\rho \in \mathfrak{Q}(X) \implies c_\rho \in [1, C_\rho], \tag{6.2}$$

$$\rho \text{ ultrametric on } X \implies c_\rho = 1, \tag{6.3}$$

$$\forall\,\rho \in \mathfrak{Q}(X),\ \ \forall\,\beta \in (0,+\infty) \implies c_{\rho^\beta} = (c_\rho)^\beta. \tag{6.4}$$

Let us point out that, in general, it may happen that $\rho \in \mathfrak{Q}(X)$ is such that $c_\rho < C_\rho$; see the discussion in Comment 6.1. In order to make the relationship between c_ρ and C_ρ more transparent, we make the following definition.

Definition 6.1. *Let X be a given set of cardinality ≥ 2. For each $\rho \in \mathfrak{Q}(X)$ define*

$$\rho_M(x,y) := \inf_{z\in X} \big(\max\{\rho(x,z),\rho(z,y)\}\big), \quad \forall\,x,y \in X, \tag{6.5}$$

and refer to the function $\rho_M : X \times X \to [0,+\infty)$ defined as in (6.5) *as the* `mid-point version` *of the quasi-distance ρ.*

The terminology introduced in the above definition is suggested by formula (6.6) below.

Proposition 6.1. *Let X be a given set of cardinality ≥ 2 and assume that $\rho \in \mathfrak{Q}(X)$. Then the mid-point version of the quasi-distance ρ satisfies*

$$\rho_M(x,y) = \inf\big\{r > 0 :\ B_\rho(x,r) \cap B_\rho(y,r) \neq \emptyset\big\}, \quad \forall\,x,y \in X, \tag{6.6}$$

and

$$(\rho^\beta)_M = (\rho_M)^\beta, \quad \forall\, \beta \in (0, +\infty). \tag{6.7}$$

Also, $\rho_M \in \mathfrak{Q}(X)$ *and* $\rho_M \approx \rho$. *More precisely, there holds*

$$c_\rho \leq \frac{\rho(x,y)}{\rho_M(x,y)} \leq C_\rho, \quad \forall\, x,y \in X \text{ with } x \neq y, \tag{6.8}$$

and the above inequalities are optimal in the sense that

$$c_\rho = \inf_{\substack{x,y\in X\\ x\neq y}} \Big(\frac{\rho(x,y)}{\rho_M(x,y)}\Big) \quad \text{ and } \quad C_\rho = \sup_{\substack{x,y\in X\\ x\neq y}} \Big(\frac{\rho(x,y)}{\rho_M(x,y)}\Big). \tag{6.9}$$

We are now properly set up to formally define smoothness indices.

Proof. All claims are straightforward consequences of definitions. □

Definition 6.2. *The* `upper smoothness index` *of a given a quasi-metric space* $(X, \mathbf{q})$ *is defined as*

$$\mathrm{Ind}\,(X, \mathbf{q}) := \inf \big\{[\log_2 c_\rho]^{-1} :\ \rho \in \mathbf{q}\big\} \tag{6.10}$$

where, for every $\rho \in \mathfrak{Q}(X)$, *the constant* c_ρ *has been introduced in* (6.1). *In addition, define the* `lower smoothness index` *of* $(X, \mathbf{q})$ *by setting*

$$\mathrm{ind}\,(X, \mathbf{q}) := \sup \big\{[\log_2 C_\rho]^{-1} :\ \rho \in \mathbf{q}\big\} \tag{6.11}$$

where, for every $\rho \in \mathfrak{Q}(X)$, *the constant* C_ρ *has been introduced in* (3.20).

Finally, if X *is an arbitrary set of cardinality* ≥ 2 *and* $\rho \in \mathfrak{Q}(X)$, *it is agreed to abbreviate* $\mathrm{Ind}\,(X, \rho) := \mathrm{Ind}\,(X, [\rho])$ *and* $\mathrm{ind}\,(X, \rho) := \mathrm{ind}\,(X, [\rho])$.

Some elementary properties of the upper and lower smoothness indices are recorded next.

Lemma 6.2. *For any set* X*,*

$$\rho \text{ ultrametric on } X \Longrightarrow \operatorname{ind}(X,\rho) = +\infty, \tag{6.12}$$

$$\rho \text{ distance on } X \Longrightarrow \operatorname{ind}(X,\rho) \geq 1. \tag{6.13}$$

Furthermore, for any quasi-distance $q \in \mathfrak{Q}(X)$ *one has*

$$\operatorname{ind}(X,\rho^\alpha) = \tfrac{1}{\alpha}\operatorname{ind}(X,\rho) \quad \text{and} \quad \operatorname{Ind}(X,\rho^\alpha) = \tfrac{1}{\alpha}\operatorname{Ind}(X,\rho), \quad \forall\,\alpha \in (0,+\infty), \tag{6.14}$$

and the following bounds hold:

$$[\log_2 C_\rho]^{-1} \leq \operatorname{ind}(X,\rho), \quad \operatorname{Ind}(X,\rho) \leq [\log_2 c_\rho]^{-1}. \tag{6.15}$$

Proof. All claims are straightforward consequences of definitions. □

Remark 6.1. Recall from (6.15) that, by design, for any $\rho \in \mathfrak{Q}(X)$ we have the lower bound $\operatorname{ind}(X,\rho) \geq [\log_2 C_\rho]^{-1}$. However, as a simple example shows, it may be the case that $\rho \in \mathfrak{Q}(X)$ is such that $\operatorname{ind}(X,\rho)$ is substantially larger than $[\log_2 C_\rho]^{-1}$. To see this, for each fixed parameter $\lambda > 0$, consider $X := \{x_1, x_2, x_3\}$ equipped with the quasi-distance $\rho_\lambda : X \times X \to [0,+\infty)$ given by

$$\begin{aligned} \rho_\lambda(x_1,x_2) &:= \rho_\lambda(x_2,x_1) := \lambda, \\ \rho_\lambda(x_1,x_3) &:= \rho_\lambda(x_3,x_1) := 1, \\ \rho_\lambda(x_2,x_3) &:= \rho_\lambda(x_3,x_2) := 1, \\ \rho_\lambda(x_1,x_1) := \rho_\lambda(x_2,x_2) &:= \rho_\lambda(x_3,x_3) := 0. \end{aligned} \tag{6.16}$$

Hence, corresponding to $\lambda = 1$, ρ_1 is the discrete metric on X, i.e., $\rho_1(x_j,x_k) = 1-\delta_{jk}$ for every $j,k \in \{1,2,3\}$ (where δ_{jk} denotes the usual Kronecker symbol). Then, on the one hand, $\rho_\lambda \approx \rho_1$ for every $\lambda > 0$ which, in light of (6.12), gives that for each $\lambda > 0$ one has the equality $\operatorname{ind}(X,\rho_\lambda) = \operatorname{ind}(X,\rho_1) = +\infty$. On the other hand, if $\lambda > 1$ then $C_{\rho_\lambda} = \lambda$ and, hence, $[\log_2 C_{\rho_\lambda}]^{-1} \to 0$ as $\lambda \to +\infty$. In the same setting, it may be checked that $c_\rho = 1$ whenever the quasi-distance $\rho \in \mathfrak{Q}(X)$ is such that $\rho \approx \rho_\lambda$. Note that this entails $c_{\rho_\lambda} < C_{\rho_\lambda}$ when $\lambda > 1$.

Our next result shows (among other things) that the lower smoothness index is invariant under bi-Lipschitz homeomorphisms.

Proposition 6.3. *If $(X_j, \mathbf{q}_j)$, $j = 0, 1$, are two given quasi-metric spaces for which there is a bi-Lipschitz mapping $\Phi : (X_0, \mathbf{q}_0) \to (X_1, \mathbf{q}_1)$ then* $\mathrm{ind}\,(X_0, \mathbf{q}_0) \geq \mathrm{ind}\,(X_1, \mathbf{q}_1)$. *Consequently, if two quasi-metric spaces are bi-Lipschitz homeomorphic then they have the same lower smoothness index.*

As a corollary, if (X, ρ) is a quasi-metric space and $Y \subseteq X$ has cardinality ≥ 2 then it holds that $\mathrm{ind}\,(Y, \rho) \geq \mathrm{ind}\,(X, \rho)$.

Proof. For every $\rho \in \mathbf{q}_1$ define the function $\widetilde{\rho} : X_0 \times X_0 \to [0, +\infty)$ by setting $\widetilde{\rho}(x, y) := \rho(\Phi(x), \Phi(x))$ for each $x, y \in X_0$. The fact that Φ is bi-Lipschitz implies that $\widetilde{\rho} \in \mathbf{q}_0$. Also, by design, $C_{\widetilde{\rho}} \leq C_\rho$. Hence,

$$\mathrm{ind}\,(X_0, \mathbf{q}_0) = \mathrm{ind}\,(X_0, \widetilde{\rho}) \geq [\log_2 C_{\widetilde{\rho}}]^{-1} \geq [\log_2 C_\rho]^{-1}. \tag{6.17}$$

Taking the supremum over all $\rho \in \mathbf{q}_1$ then yields $\mathrm{ind}\,(X_0, \mathbf{q}_0) \geq \mathrm{ind}\,(X_1, \mathbf{q}_1)$.

To prove the last claim in the statement of the proposition, observe that the canonical inclusion map $\iota : (Y, \rho) \hookrightarrow (X, \rho)$ is bi-Lipschitz, so $\mathrm{ind}\,(Y, \rho) \geq \mathrm{ind}\,(X, \rho)$ by what we have proved in the first part. $\square$

We continue by discussing how the upper and lower smoothness indices are related. As a preamble, we first establish the following characterization of the lower smoothness index.

Proposition 6.4. *If X is an arbitrary set of cardinality ≥ 2 then for every quasi-distance $\rho \in \mathfrak{Q}(X)$ one has*

$$\begin{aligned} \operatorname{ind}(X,\rho) &= \sup\big\{[\log_2 C_{\rho'}]^{-1} : \rho' \in [\rho]\big\} \qquad (6.18)\\ &= \sup_{\substack{\theta: X\times X\to\mathbb{R}\\ 0<\inf\theta\leq\sup\theta<+\infty}} \left[\log_2\Big(\sup_{\substack{x,y,z\in X\\ not\ all\ equal}} \frac{(\theta\rho)(x,y)}{\max\{(\theta\rho)(x,z),(\theta\rho)(y,z)\}}\Big)\right]^{-1}. \end{aligned}$$

Proof. The first equality in (6.18) is just a re-writing of the definition (6.11). Also, the fact the first supremum in (6.18) is dominated by the second supremum in (6.18) follows from Remark 2.2. Consider next a function $\theta : X \times X \to \mathbb{R}$ with the property that $0 < \inf \theta \leq \sup \theta < +\infty$ and set $\rho' := \theta\rho$. Then, by part (6) in Theorem 2.12, $(\rho')_{sym}$ is a quasi-distance on X with $C_{(\rho')_{sym}} = C_{\rho'}$. Also, by part (3) in Theorem 2.12, we have $(\rho')_{sym} \approx \rho' \approx \rho$. Consequently, $\operatorname{ind}(X,\rho) \geq [\log_2 C_{\rho'}]^{-1}$ which proves that the second supremum in (6.18) is majorized by $\operatorname{ind}(X,\rho)$. The desired conclusion follows. $\square$

Here is the theorem alluded to above. It asserts that, for any quasi-metric space, the lower smoothness index is less than or equal to the upper smoothness index.

Theorem 6.5. *For any quasi-metric space* $(X, \mathbf{q})$ *one has* $\operatorname{ind}(X,\mathbf{q}) \leq \operatorname{Ind}(X,\mathbf{q})$.

Proof. Fix some $\rho \in \mathbf{q}$ and consider $\theta : X \times X \to \mathbb{R}$ such that

$$0 < \inf \theta \leq \sup \theta < +\infty.$$

Thus, if we set $M := \sup\{\theta(x,y) : x, y \in X,\ x \neq y\}$, it follows that $0 < M < +\infty$. Fix an arbitrary $\varepsilon \in (0,1)$ and select $x_\varepsilon, y_\varepsilon \in X$ such that $x_\varepsilon \neq y_\varepsilon$ and

$$\theta(x_\varepsilon, y_\varepsilon) > M - \varepsilon.$$

Use the definition of c_ρ from (6.1) in order to conclude that there exists some $z_\varepsilon \in X$ for which

$$\frac{\rho(x_\varepsilon, y_\varepsilon)}{\max\{\rho(x_\varepsilon, z_\varepsilon), \rho(z_\varepsilon, y_\varepsilon)\}} > c_\rho - \varepsilon. \tag{6.19}$$

In the case when $z_\varepsilon \neq x_\varepsilon$, $z_\varepsilon \neq y_\varepsilon$ we have $0 < \theta(x_\varepsilon, z_\varepsilon) \leq M$ and $0 < \theta(z_\varepsilon, y_\varepsilon) \leq M$ which, together with the previous observations, allow us to estimate

$$\begin{aligned}\frac{(\theta\rho)(x_\varepsilon, y_\varepsilon)}{\max\{(\theta\rho)(x_\varepsilon, z_\varepsilon), (\theta\rho)(z_\varepsilon, y_\varepsilon)\}} &> \frac{(M-\varepsilon)\rho(x_\varepsilon, y_\varepsilon)}{M\ \max\{\rho(x_\varepsilon, z_\varepsilon), \rho(z_\varepsilon, y_\varepsilon)\}} \\ &> \Big(1 - \frac{\varepsilon}{M}\Big)(c_\rho - \varepsilon).\end{aligned} \tag{6.20}$$

This implies that

$$\frac{(\theta\rho)(x_\varepsilon, y_\varepsilon)}{\max\{(\theta\rho)(x_\varepsilon, z_\varepsilon), (\theta\rho)(z_\varepsilon, y_\varepsilon)\}} > \Big(1 - \frac{\varepsilon}{M}\Big)(c_\rho - \varepsilon) \tag{6.21}$$

whenever $z_\varepsilon \notin \{x_\varepsilon, y_\varepsilon\}$. However, in the case when $z_\varepsilon = x_\varepsilon$, or $z_\varepsilon = y_\varepsilon$, estimate (6.21) follows directly from (6.19). Thus, (6.21) holds in all cases and we may conclude, based on it, that

$$\sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{(\theta\rho)(x, y)}{\max\{(\theta\rho)(x, z), (\theta\rho)(y, z)\}} \geq c_\rho. \tag{6.22}$$

Hence, since θ was arbitrarily chosen amongst the class of functions with the specified properties, it follows from (6.22) that

$$\sup_{\substack{\theta: X\times X\to\mathbb{R}\\ 0<\inf\theta\leq\sup\theta<+\infty}} \left[\log_2\Big(\sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{(\theta\rho)(x, y)}{\max\{(\theta\rho)(x, z), (\theta\rho)(y, z)\}}\Big)\right]^{-1} \leq [\log_2 c_\rho]^{-1}. \tag{6.23}$$

Having established this, Proposition 6.4 then gives $\operatorname{ind}(X, \rho) \leq [\log_2 c_\rho]^{-1}$ which further yields $\operatorname{ind}(X, \rho) \leq \operatorname{Ind}(X, \rho)$ after taking the infimum over all $\rho \in \mathbf{q}$. $\square$

An equivalent way of expressing the conclusion in Theorem 6.5 is to say that

$$c_{\rho_1} \leq C_{\rho_2} \quad \text{for all} \quad \rho_1, \rho_2 \in \mathbf{q}. \tag{6.24}$$

Also, it follows from (6.12)-(6.13) and Theorem 6.5 that for any set X we have

$$\rho \text{ ultrametric on } X \implies \mathrm{Ind}\,(X,\rho) = +\infty, \tag{6.25}$$

$$\rho \text{ distance on } X \implies \mathrm{Ind}\,(X,\rho) \geq 1. \tag{6.26}$$

Other significant consequences are discussed in the corollary below.

Corollary 6.6. *If $(X,\mathbf{q})$ is a quasi-metric space with the property that there exists $\rho \in \mathbf{q}$ such that*

$$\forall\, x,y \in X \;\; \exists\, z \in X \quad \textit{with} \;\; \rho(x,z) \leq \tfrac{1}{2}\rho(x,y) \;\; \textit{and} \;\; \rho(z,y) \leq \tfrac{1}{2}\rho(x,y). \tag{6.27}$$

then $\mathrm{Ind}\,(X,\mathbf{q}) \leq 1$. *In particular, if $(X,\|\cdot\|)$ is a normed vector space and $\mathbf{q}$ stands for the quasi-metric space structure induced by the norm $\|\cdot\|$, then*

$$\mathrm{ind}\,(Y,\mathbf{q}) = \mathrm{Ind}\,(Y,\mathbf{q}) = 1, \quad \textit{for any convex subset } Y \textit{ of } X. \tag{6.28}$$

As a consequence, for any $n \in \mathbb{N}$ one has

$$\begin{aligned} &\mathrm{ind}\,(\mathbb{R}^n, |\cdot - \cdot|) = \mathrm{Ind}\,(\mathbb{R}^n, |\cdot - \cdot|) = 1,\\ &\mathrm{ind}\,([0,1]^n, |\cdot - \cdot|) = \mathrm{Ind}\,([0,1]^n, |\cdot - \cdot|) = 1, \end{aligned} \tag{6.29}$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

Proof. If $\rho \in \mathbf{q}$ satisfy (6.27) then $\rho_M \leq \frac{1}{2}\rho$. In concert with (6.9), this forces $c_\rho \geq 2$ hence, further, $\mathrm{Ind}\,(X,\mathbf{q}) \leq 1$. This proves the first claim in the statement of the corollary. Note that, when used together with Theorem 6.5, this gives that $\mathrm{ind}\,(X,\mathbf{q}) \leq \mathrm{Ind}\,(X,\mathbf{q}) \leq 1$. Next, if Y is a convex subset of a normed vector space $(X,\|\cdot\|)$, then for any $x,y \in Y$ we have $\frac{x+y}{2} \in Y$ and

$$\|x - (\frac{x+y}{2})\| = \|y - (\frac{x+y}{2})\| = \frac{1}{2}\|x-y\|$$

. This shows that if $\mathbf{q}$ denotes the quasi-metric space structure induced by the norm then $\operatorname{Ind}(X, \mathbf{q}) \leq 1$ by what we have proved so far. When combined with Theorem 6.5 and (6.13), this shows that $\operatorname{ind}(X, \mathbf{q}) = \operatorname{Ind}(X, \mathbf{q}) = 1$, as claimed. Finally, (6.29) is an obvious consequence of the more general situation considered earlier in the proof. □

Corollary 6.7. *If the interval* $[0, 1]$ *may be bi-Lipschitzly embedded into the quasi-metric space* $(X, \mathbf{q})$ *then* $\operatorname{ind}(X, \mathbf{q}) \leq 1$.

Also, if $(X, \mathbf{q})$ *is a quasi-metric space with the property that* $\operatorname{ind}(X, \mathbf{q}) < 1$ *then* $(X, \mathbf{q})$ *cannot be bi-Lipschitzly embedded into some* $\mathbb{R}^n$.

Proof. Both claims in the statement of the corollary are consequences of the last part in Proposition 6.3 and (6.29). □

Definition 6.3. *Let* X *be a nonempty set and assume that* $\rho \in \mathfrak{Q}(X)$, $\beta \in (0, +\infty)$, *and* $E \subseteq X$ *has cardinality* ≥ 2. *Given a real-valued function* f *on* E, *define its* `Hölder semi-norm` *(of order* β, *relative to the quasi-distance* ρ*) by setting*

$$\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} := \sup_{x,y \in E,\, x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\beta}. \tag{6.30}$$

Next, if $\mathbf{q}$ *is a quasi-metric space structure on* X, *we define the homogeneous Hölder space* $\dot{\mathscr{C}}^\beta(E, \mathbf{q})$ *as*

$$\begin{aligned} \dot{\mathscr{C}}^\beta(E, \mathbf{q}) &:= \big\{f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < +\infty \textit{ for some } \rho \in \mathbf{q}\big\} \\ &= \big\{f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < +\infty \textit{ for every } \rho \in \mathbf{q}\big\}. \end{aligned} \tag{6.31}$$

Given any $\beta > 0$, it follows that $\{\|\cdot\|_{\dot{\mathscr{C}}^\beta(E,\rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent semi-norms on $\dot{\mathscr{C}}^\beta(E, \mathbf{q})$.

Theorem 6.8. *Let $(X, \mathbf{q})$ be quasi-metric space and assume that $\beta \in \mathbb{R}$ is such that $\beta > \mathrm{Ind}\,(X, \mathbf{q})$. Then $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$ contains only constant functions.*

Proof. Let $\beta \in \mathbb{R}$ and $\rho \in \mathbf{q}$ be such that $\beta > [\log_2 c_\rho]^{-1} \in (0, +\infty]$. This entails $2^{1/\beta} < c_\rho$ and we select $\gamma \in (0, \beta)$ with the property that $2^{1/\gamma} < c_\rho$. Based on this and the definition of c_ρ in (6.1) we then conclude that if $\kappa := 2^{-1/\gamma}$ then

$$\begin{aligned} &\forall\,(x,y) \in X \times X \setminus \mathrm{diag}\,(X) \quad \exists\, z \in X \text{ such that} \\ &\qquad \rho(x,z) \leq \kappa\rho(x,y) \ \text{ and } \ \rho(z,y) \leq \kappa\rho(x,y). \end{aligned} \tag{6.32}$$

Pick now $f \in \dot{\mathscr{C}}^\beta(X, \rho)$, with the goal of showing that, necessarily, f is constant. To this end, let $x, y \in X$, $x \neq y$ and set $L := \rho(x,y) > 0$. Based on (6.32), for each $N \in \mathbb{N}$ we may inductively construct a family of points $\{z_i^N \in X : 0 \leq i \leq 2^N\}$ satisfying

$$z_0^N = x,\ z_{2^N}^N = y \ \text{ and } \ \rho(z_i^N, z_{i+1}^N) \leq \kappa^N L \ \text{ whenever } \ 0 \leq i \leq 2^N. \tag{6.33}$$

Hence, using (6.33) and the fact that $f \in \dot{\mathscr{C}}^\beta(X, \rho)$ we may write

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{i=0}^{2^N} |f(z_i^N, z_{i+1}^N)| \leq C \sum_{i=0}^{2^N} \rho(z_i^N, z_{i+1}^N)^\beta \\ &= C(L\kappa^N)^\beta 2^N = CL^\beta (2\kappa^\beta)^N. \end{aligned} \tag{6.34}$$

On the other hand, the fact that $0 < \gamma < \beta$ forces $2\kappa^\beta = 2^{1-\beta/\gamma} \in (0,1)$ and, hence, $(2\kappa^\beta)^N \to 0$ as $N \to +\infty$. Together with (6.34), this implies that $f(x) = f(y)$. This shows that f must be constant, completing the proof of the proposition. $\square$

Moving on, we recall the notion of convexity index (*l'indice de convexité*) introduced by Assouad in [12, Définition 3, p. 732].

Definition 6.4. *Assoud's* `convexity index` *of a quasi-metric space* $(X, \mathbf{q})$ *is defined as*

$$\begin{aligned}\mathrm{Cv}\,(X, \mathbf{q}) := \inf \big\{ & p \in (0, +\infty) : \exists\, \rho \in \mathbf{q} \text{ such that} \\ & \rho^{1/p} \text{ is equivalent to a distance on } X \big\}. \end{aligned} \tag{6.35}$$

Our next result is the theorem below asserting that, for any quasi-metric space, our lower smoothness index coincides with the reciprocal of Assouad's convexity index.

Theorem 6.9. *For any quasi-metric space* $(X, \mathbf{q})$ *one has*

$$\mathrm{ind}\,(X, \mathbf{q}) = \frac{1}{\mathrm{Cv}\,(X, \mathbf{q})}. \tag{6.36}$$

Proof. Let $p \in (0, +\infty)$ be such that there exists $\rho \in \mathbf{q}$ and a distance d on X with the property that $\rho^{1/p} \approx d$. It follows that $d^p \in \mathbf{q}$ and $C_{d^p} = (C_d)^p \leq 2^p$ since $C_d \leq 2$ given that d is a distance. This proves that $\mathrm{ind}\,(X, \mathbf{q}) \geq [\log_2 C_{d^p}]^{-1} \geq 1/p$ hence, ultimately, $\mathrm{ind}\,(X, \mathbf{q}) \geq [\mathrm{Cv}\,(X, \mathbf{q})]^{-1}$. In the opposite direction, pick $\rho \in \mathbf{q}$ and assume that $0 < \beta < [\log_2 C_\rho]^{-1}$. Then part (12) in Theorem 2.12 shows that there exists a distance d on X with the property that $d^{1/\beta} \in \mathbf{q}$. In turn, after unraveling definitions, this condition readily implies $1/\beta \geq \mathrm{Cv}\,(X, \mathbf{q})$. This forces $[\log_2 C_\rho]^{-1} \leq [\mathrm{Cv}\,(X, \mathbf{q})]^{-1}$, hence

$$\mathrm{ind}\,(X, \mathbf{q}) \leq [\mathrm{Cv}\,(X, \mathbf{q})]^{-1}.$$

All in all, (6.36) follows. $\square$

6.2 Fractals and Quasi-metric Spaces

Before delving into the relation between fractals and quasi-metric spaces, we present a well-known fractal: the Koch curve[1]. Its construction is quite simple. In the plane, start with any closed line segment (we will work with $[0,1] \times \{0\}$). Divide the initial line segment into thirds and, using the middle third as a base, construct an equilateral triangle and remove the segment used as the base. Apply the same procedure to the four line segments attained after the first step is complete, constructing the triangles in the same direction. Repeat this process indefinitely.

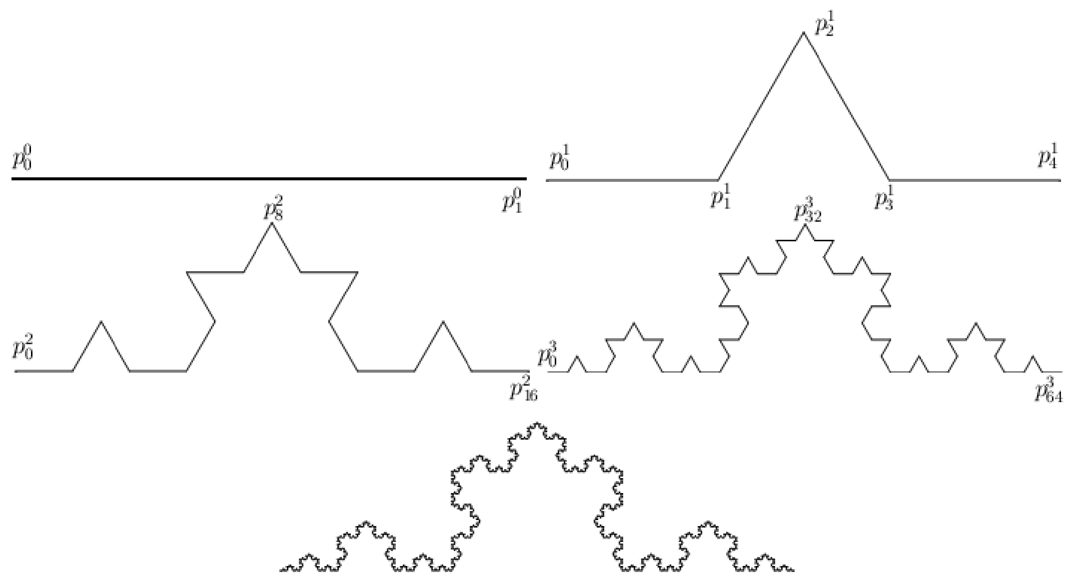

Figure 4. *Construction of the Kock curve.*

To be more precise, define p_j^i as the j^{th} endpoint of the i^{th} step of construction for all numbers $i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, j \in \{0, \cdots, 4^i\}$. Then, if K^i denotes the i^{th} figure in the construction of the Koch curve, $K^i = \bigcup_{j=0}^{4^i} [p_j^i, p_{j+1}^i]$.

[1]The Koch curve, sometimes called the Koch snowflake (though Koch snowflake, Koch star and Koch island refer to a different, yet strongly related, object), is one of the earliest fractals, first described by H. Koch in [52].

For K^0 taken to be $[0,1]\times\{0\}$, $p_0^i=(0,0)$ and $p_{4^i}^i=(1,0)$ for every nonnegative integer i. Also, $p_{2\cdot 4^{i-1}}^i=(1/2,\sqrt{3}/6)$ for every natural number i and $p_j^i=p_{4j}^{i+1}$ for $i\in\mathbb{N}_0$ and $j\in\{0,\cdots,4^i\}$. Another simple observation is the p_j^i's have $3^{i-1}+1$ distinct $y-$values for every $i\in\mathbb{N}_0$. The length of K^i (which is also its 1-dimensional Hausdorff outer measure, $\mathcal{H}^1_{\mathbb{R}^2,|\cdot-\cdot|}(K^i)$) is $(4/3)^i$ for every $i\in\mathbb{N}_0$. Thus the Koch curve itself has infinite length, or the 1-dimensional Hausdorff outer measure is unsuitable. In fact, as is well known, the Hausdorff dimension of K is $\dim_{\mathcal{H}_{\mathbb{R}^2}}(K)=\ln 4/\ln 3$.

With this preliminary discussion we are in a suitable position to present

Proposition 6.10. *With K denoting the Koch curve, $([0,1],|\cdot-\cdot|^{\ln 3/\ln 4})$ is bi-Lipschitz homeomorphic to $(K,|\cdot-\cdot|\lfloor K)$.*

Proof. First note the ln3/ln4 appearing in the exponent of the 1-dimensional Euclidean distance is the reciprocal of the Koch curve's Hausdorff dimension.

By [73, Theorem 1, p. 1048], there exists a homeomorphism $\phi:[0,1]\to K$ satisfying the following inequality for all x and y in $[0,1]$:

$$\frac{\sin(\pi/2)}{8\cos^3(\pi/6)}|x-y|^{\log_2 2\cos(\pi/6)}\leq|\phi(x)-\phi(y)|\leq 4|x-y|^{\log_2 2\cos(\pi/6)}. \tag{6.37}$$

As $0<\dfrac{\sin(\pi/2)}{8\cos^3(\pi/6)}<4<\infty$, it follows that $([0,1],|\cdot-\cdot|^{\log_2\cos(\pi/6)})$ is bi-Lipschitz homeomorphic to $(K,|\cdot-\cdot|\lfloor K)$. All that remains is to observe

$$\log_2(2\cos\pi/6)=\log_2\sqrt{3}=(\ln\sqrt{3})/(\ln 2)=(\ln 3)/(2\ln 2)=(\ln 3)/(\ln 4), \tag{6.38}$$

which finishes the proof of Proposition 6.10. $\square$

Before proceeding, we wish to note that in [73] S. Ponomarev constructed a a family of fractals like the Koch curve (the Koch curve itself is included in this family). Specifically, instead of starting with a line and constructing triangles, Ponomarev starts with a solid triangle and deletes inner triangles. In particular, using complex numbers, Ponomarev considers an initial triangle with vertices $0, 1$ and $(1 + i/\tan\theta)/2$, then removes the open triangle with vertices $(2\cos\theta)^{-2}, 1-(2\cos\theta)^{-2}$ and $(1+i/\tan\theta)/2$. The process of removing open triangles is continued. When $\theta = \pi/6$ the final result is the Koch curve, which is why the $\pi/6$ appears in the proof of Proposition 6.10.

Given a metric space (X, d) and $\varepsilon \in (0, 1)$, call (X, d^{ε}), which is also a metric space, the ε-`snowflaked version` of (X, d). This terminology is suggested by the fact that for each $\varepsilon \in (\frac{1}{2}, 1)$ the quasi-metric space $(\mathbb{R}, |\cdot - \cdot|^{\varepsilon})$ admits a bi-Lipschitz embedding into $\mathbb{R}^2$ whose image is reminiscent of the boundary of a domain depicting a snowflake (recall that an `embedding` is a mapping between two topological spaces which is a homeomorphism onto its image).

Corollary 6.11. *A quasi-metric space is bi-Lipschitz homeomorphic to a snowflaked version of a metric space if and only if its lower smoothness index is* > 1.

Proof. Assume the quasi-metric space $(X, \mathbf{q})$ and the metric space (Y, d) are such that there exists a number $\varepsilon \in (0, 1)$ with the property that $(X, \mathbf{q})$ and (Y, d^{ε}) are bi-Lipschitz homeomorphic. Then Proposition 6.3 gives that

$$\operatorname{ind}(X, \mathbf{q}) = \operatorname{ind}(Y, d^{\varepsilon}) > \varepsilon^{-1},$$

where the last inequality follows from (6.11), after observing that

$$C_{d^{\varepsilon}} = (C_d)^{\varepsilon} \leq 2^{\varepsilon}.$$

This proves the right-pointing implication in the statement of the corollary. As regards the opposite implication, note that if $(X, \mathbf{q})$ is a quasi-metric space satisfying

$\operatorname{ind}(X, \mathbf{q}) > 1$ then, by Theorem 6.9, we have $\operatorname{Cv}(X, \mathbf{q}) < 1$. Hence, upon recalling (6.35), it follows that there exist $\rho \in \mathbf{q}$, $p \in (0, 1)$, and a distance d on X with the property that $\rho^{1/p} \approx d$. Thus, the identity mapping is a bi-Lipschitz homeomorphism of $(X, \mathbf{q})$ onto the snowflaked version (X, d^p) of the metric space (X, d). $\square$

Before presenting an application dealing with bi-Lipschitz Euclidean embeddings of quasi-metric spaces, we make a few definitions.

Definition 6.5. *A quasi-metric space* $(X, \mathbf{q})$ *is called* `geometric doubling` *if there exists* $\rho \in \mathbf{q}$ *for which one can find a number* $N \in \mathbb{N}$*, called the* `geometric doubling constant of` $(X, \mathbf{q})$*, with the property that any* ρ*-ball of radius* r *in* X *may be covered by at most* N ρ*-balls in* X *of radii* $r/2$*. Finally, if* X *is an arbitrary, nonempty set and* $\rho \in \mathfrak{Q}(X)$*, call* (X, ρ) *geometric doubling if* $(X, [\rho])$ *is geometric doubling.*

Note that if $(X, \mathbf{q})$ is a geometric doubling quasi-metric space then

$$\begin{array}{l}\forall\, \rho \in \mathbf{q}\ \ \forall\, \theta \in (0,1)\ \ \exists\, N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r \\ \text{in } X \text{ may be covered by at most } N\ \rho\text{-balls in } X \text{ of radii } \theta r.\end{array} \tag{6.39}$$

In particular, this ensures that the last part in Definition 6.5 is meaningful. Another useful consequence of the geometric doubling property for a quasi-metric space $(X, \mathbf{q})$ is as follows.

Definition 6.6. *Given* $\varepsilon > 0$ *and* $\rho \in \mathbf{q}$*, a set* $E \subseteq X$ *is said to be* (ε, ρ)-`disperse` *provided*

$$\rho(x, y) \geq \varepsilon, \quad \forall\, x, y \in E \ \ \textit{with} \ \ x \neq y. \tag{6.40}$$

Then, if $(X, \mathbf{q})$ is a geometric doubling quasi-metric space, $\rho \in \mathbf{q}$ and $\varepsilon > 0$,

$$\begin{array}{c}\text{any family of } \rho\text{-balls in } X \text{ of bounded radii, and whose centers} \\ \text{make up a } (\varepsilon, \rho)\text{-disperse subset of } X \text{, have bounded overlap,}\end{array} \tag{6.41}$$

for some bound depending only on ε, the bound on the radii of the ρ-balls in question, and the geometric doubling constant of the quasi-metric space $(X, \mathbf{q})$.

Lastly, let us point out here that

$$\begin{array}{l}\text{if } (X, \mathbf{q}) \text{ is a geometric doubling quasi-metric space then} \\ \text{the topological space } (X, \tau_{\mathbf{q}}) \text{ is separable.}\end{array} \tag{6.42}$$

Definition 6.7. *Assume that (X, ρ) is a quasi-metric space.*

(1) Given $s \in [0, +\infty]$ call (X, ρ) s-`homogeneous` *if there exists a finite constant $c \geq 0$ with the property that whenever $Y \subseteq X$ is such that there exist $b \geq a > 0$ satisfying $a \leq \rho(x, y) \leq b$ for all $x, y \in Y$ with $x \neq y$, then the cardinality of Y is $\leq c\big(\frac{b}{a}\big)^s$.*

(2) The `Assouad dimension` *of (X, ρ) is*

$$\dim_A(X, \rho) := \inf \big\{ s \in [0, +\infty] : \ (X, \rho) \textit{ is } s\textit{-homogeneous} \big\}. \tag{6.43}$$

Recall that given a quasi-metric space (X, ρ), denote by

$$\operatorname{diam}_\rho(X) := \sup \{ \rho(x, y) : \ x, y \in X \}$$

its diameter (relative to the quasi-distance ρ). Then $\dim_A(X, \rho) < +\infty$ if and only if there exist $C \geq 1$ and $s \in [0, +\infty)$ such that, for every $x \in X$ and for any real numbers R, r satisfying the inequality $\operatorname{diam}_\rho(X) \geq R \geq r > 0$, the ρ-ball $B_\rho(x, R)$ can be covered by at most $C\big(\frac{R}{r}\big)^s$ ρ-balls of radii $\leq r$. The latter property is equivalent to the geometric doubling condition (6.39), hence a quasi-metric space is geometric doubling if and only if it has finite Assouad dimension. In fact, sometimes this is taken to be the definition of Assouad dimension, as in Definition 10.15 of [34].

The following properties can be verified directly from definitions (refer also to [12, Proposition 2, p. 733], as well as [13, (a)-(c), p. 435] in the case of metric spaces).

Lemma 6.12. *Let (X, ρ) be a quasi-metric space. Then the following are true.*

(i) For every $\alpha > 0$ there holds $\dim_A(X, \rho^\alpha) = \frac{1}{\alpha} \dim_A(X, \rho)$.

(ii) If ρ' is a quasi-distance on X such that $\rho' \approx \rho$, then $\dim_A(X, \rho) = \dim_A(X, \rho')$.

(iii) For every set $\widetilde{X} \subseteq X$ there holds $\dim_A(\widetilde{X}, \rho) \leq \dim_A(X, \rho)$. Moreover, if $\widetilde{X}$ is a dense subset of X then actually $\dim_A(\widetilde{X}, \rho) = \dim_A(X, \rho)$.

(iv) The Assouad dimension is invariant under bi-Lipschitz surjections, in the sense that if (X_j, ρ_j), $j = 0, 1$, are two quasi-metric spaces and if

$$\Phi : (X_0, \rho_0) \to (X_1, \rho_1)$$

is a bi-Lipschitz surjection then $\dim_A(X_0, \rho_0) = \dim_A(X_1, \rho_1)$.

(v) For every $n \in \mathbb{N}$, one has $\dim_A(\mathbb{R}^n, |\cdot - \cdot|) = n$, where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

Proof. The above items may easily be seen by referring to definitions. $\square$

Thanks to property *(ii)* in Lemma 6.12, the following definition is meaningful.

Definition 6.8. *Given an arbitrary, fixed quasi-metric space $(X, \mathbf{q})$ define its Assouad dimension as $\dim_A(X, \mathbf{q}) := \dim_A(X, \rho)$ for some (hence, any) $\rho \in \mathbf{q}$.*

P. Assouad's embedding theorem (cf. [11, Proposition 1.30, p. I.29], [12, Remarque 2, p. 732], and [13, Proposition 2.6, p. 436]) asserts that each snowflaked version of a metric space of finite Assouad dimension admits a bi-Lipschitz embedding in some (finite dimensional) Euclidean space. We may now state and prove an extension of this result to the setting of quasi-metric spaces of finite Assouad dimension. Our theorem shows that the example of a subset of some $\mathbb{R}^n$ endowed with a power of the standard Euclidean distance is prototypical (i.e., always realizable, up to a bi-Lipschitz homeomorphism) in the class of all quasi-metric spaces of finite Assouad dimension.

Theorem 6.13. *Let $(X, \mathbf{q})$ be a quasi-metric space. Then the following three conditions are equivalent:*

(1) for any $\beta \in \mathbb{R}$ satisfying the inequality $0 < \beta < \operatorname{ind}(X, \mathbf{q})$ there exist a number $n \in \mathbb{N}$, a set $\widetilde{X} \subseteq \mathbb{R}^n$, a quasi-distance $\rho \in \mathbf{q}$, and a bi-Lipschitz homeomorphism

$$\Phi : (X, \rho) \longrightarrow (\widetilde{X}, |\cdot - \cdot|^{1/\beta}), \tag{6.44}$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$;

(2) the same embedding property as in (1) *but for just some number $\beta \in \mathbb{R}$ satisfying the inequality $0 < \beta < \operatorname{ind}(X, \mathbf{q})$;*

(3) $(X, \mathbf{q})$ is of finite Assouad dimension, i.e., $\dim_A(X, \mathbf{q}) < +\infty$.

Proof. That (1) $\Rightarrow$ (2) is obvious. As regards the implication (2) $\Rightarrow$ (3), note that for the given β, ρ and $\widetilde{X}$ we may write

$$\begin{aligned} \dim_A(X, \rho) &= \dim_A(\widetilde{X}, |\cdot - \cdot|^{1/\beta}) = \beta \dim_A(\widetilde{X}, |\cdot - \cdot|) \\ &\leq \beta \dim_A(\mathbb{R}^n, |\cdot - \cdot|) = \beta n < +\infty, \end{aligned} \tag{6.45}$$

by (iv), (i), (iii), (v) in Lemma 6.12. The desired conclusion follows.

To justify the implication (3) $\Rightarrow$ (1), given $\beta \in \big(0, \mathrm{ind}\,(X, \mathbf{q})\big)$ consider a quasi-distance $\rho \in \mathbf{q}$ with the property that $0 < \beta < [\log_2 C_\rho]^{-1}$ and select $\varepsilon \in (0,1)$ such that $\beta/\varepsilon < [\log_2 C_\rho]^{-1}$. Then, if $\rho_\#$ is the regularized version of ρ as in (12) of Theorem 2.12 we may conclude that $(X, (\rho_\#)^{\beta/\varepsilon})$ is a metric space. Furthermore, thanks to the properties (i)-(ii) listed in Lemma 6.12, we have

$$\dim_A(X, (\rho_\#)^{\beta/\varepsilon}) < +\infty.$$

Hence, Assouad's embedding theorem cited above ensures the existence of some $n \in \mathbb{N}$ and of a bi-Lipschitz function from the ε-snowflaked version of

$$(X, (\rho_\#)^{\beta/\varepsilon})$$

into $(\mathbb{R}^n, |\cdot - \cdot|)$. Thus, Φ satisfies

$$|\Phi(x) - \Phi(y)| \approx [\rho(x,y)]^\beta, \quad \text{uniformly for } x, y \in X, \tag{6.46}$$

so that, in particular, H maps X homeomorphically onto its image. Consequently, by taking the set $\widetilde{X} := \Phi(X) \subseteq \mathbb{R}^n$, the implication (3) $\Rightarrow$ (1) follows. $\square$

The above theorem extends [86, Theorem 1.192, p. 114], where a smaller range of β's has been treated (more specifically, $0 < \beta < \alpha$ where α is taken to be

$$\alpha := [\log_2 [c(2c+1)]]^{-1} \in (0,1)$$

). As is apparent from the work in [86] (and the references therein), the significance of such an embedding result stems from the fact that this allows one to transport certain notions of smoothness from Euclidean spaces onto the quasi-metric space much as local coordinate charts are used to define smoothness spaces on a manifold.

Theorem 6.13 may also be used to construct examples of subsets K of Euclidean spaces which are homeomorphic to a unit cube and such that the upper smoothness index of K (viewed as metric space, when equipped with the Euclidean distance) is larger than any a priori given bound. The relevance of such examples is substantiated in [32]. Here is the precise statement of the result sketched above.

Corollary 6.14. *Let $(X, \mathbf{q})$ be a quasi-metric space of finite Assouad dimension and assume that $\gamma \in (0, +\infty)$ has the property that there exists $\rho \in \mathbf{q}$ such that $\log_2 C_\rho < \gamma$. Then there exist $n \in \mathbb{N}$, $Y \subseteq \mathbb{R}^n$, a finite constant $C \geq 0$, and a function $f : Y \to X$ which is a homeomorphism and satisfies*

$$\rho(f(x), f(y)) \leq C|x - y|^\gamma, \quad \forall\, x, y \in Y. \tag{6.47}$$

In particular, for every $m \in \mathbb{N}$ and $\gamma \in (0, +\infty)$ there exist $n \in \mathbb{N}$ and a compact set $K \subseteq \mathbb{R}^n$ with the property that K is homeomorphic to $[0, 1]^m$ and there are nonconstant, real-valued, Hölder functions of order γ defined on K. Consequently, $\mathrm{Ind}\,(K, |\cdot - \cdot|) \geq \gamma$.

Proof. The claim in the first part of the statement of the corollary follows from Theorem 6.13, by taking $f := \Phi^{-1}$, $\beta := 1/\gamma$ and $Y := \widetilde{X}$. The second claim is a consequence of this result, specialized to the case when $(X, \mathbf{q}) := ([0, 1]^m, |\cdot - \cdot|)$ and $K := \Phi([0, 1]^m)$. Finally, the fact that $\mathrm{Ind}\,(K, |\cdot - \cdot|) \geq \gamma$ is seen with the help of Theorem 6.8. $\square$

Chapter 7

The Geometry of Pseudo-Balls

This begins the main body of the second part of the work of this thesis. It will appear in [9].

In this chapter we introduce a category of sets which contains both the cones and balls in $\mathbb{R}^n$, and which we shall call pseudo-balls. This concept is going to play a basic role for the entire subsequent discussion. As a preamble, we describe the class of cones in the Euclidean space. Concretely, by an open, truncated, one-component circular cone in $\mathbb{R}^n$ we understand any set of the form

$$\Gamma_{\theta,b}(x_0,h) := \{x \in \mathbb{R}^n : \cos(\theta/2)\,|x - x_0| < (x - x_0)\cdot h < b\}, \tag{7.1}$$

where $x_0 \in \mathbb{R}^n$ is the vertex of the cone, $h \in S^{n-1}$ is the direction of the axis, $\theta \in (0,\pi)$ is the (full) aperture of the cone, and $b \in (0,+\infty)$ is the height of the cone.

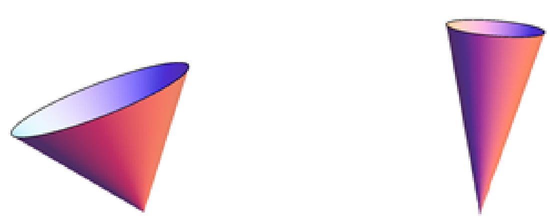

Figure 5. *One-component circular cones. The aperture of the cone*

on the left is larger than that of the cone on the right.

Definition 7.1. *Assume* (1.10) *and suppose that the point* $x_0 \in \mathbb{R}^n$*, vector* $h \in S^{n-1}$ *and numbers* $a, b \in (0, +\infty)$ *are given. Then the* $\texttt{pseudo} - \texttt{ball}$ *with apex at* x_0*, axis of symmetry along* h*, height* b*, amplitude* a*, and shape function* ω *is defined by*

$$\mathscr{G}^{\omega}_{a,b}(x_0, h) := \{x \in B(x_0, R) \subseteq \mathbb{R}^n : a|x - x_0|\, \omega(|x - x_0|) < h \cdot (x - x_0) < b\}\,. \tag{7.2}$$

Collectively, a, b *and* ω *constitute the geometrical characteristics of the named pseudo-ball.*

In the sequel, given a, b *and* α *positive numbers, abbreviate* $\mathscr{G}^{\alpha}_{a,b}(x_0, h) := \mathscr{G}^{\omega_\alpha}_{a,b}(x_0, h)$ *with* ω_α *as in* (1.14)*, i.e., define*

$$\mathscr{G}^{\alpha}_{a,b}(x_0, h) := \big\{x \in B(x_0, 1) \subseteq \mathbb{R}^n : a|x - x_0|^{1+\alpha} < h \cdot (x - x_0) < b\big\}\,. \tag{7.3}$$

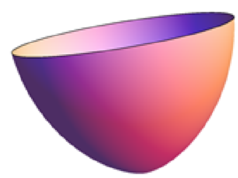

Figure 6. *A pseudo-ball with shape function* $\omega(t) = t^{1/2}$.

Some basic, elementary properties of pseudo-balls are collected in the lemma below. In particular, item *(iii)* justifies the terminology employed in Definition 7.1.

Lemma 7.1. *Assume* (1.10) *and, in addition, suppose that* ω *is strictly increasing. Also, fix two parameters* $a, b \in (0, +\infty)$*, a point* $x_0 \in \mathbb{R}^n$ *and a vector* $h \in S^{n-1}$*. Then the following hold.*

(i) The pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ *is a nonempty open subset of* $\mathbb{R}^n$ *(in fact, it contains a line segment of the form* $\{x_0 + th : 0 < t < \varepsilon\}$ *for some small* $\varepsilon > 0$*), which*

is included in the ball $B(x_0, R)$, *and with the property that* $x_0 \in \partial\mathscr{G}^{\omega}_{a,b}(x_0, h)$. *Corresponding to the choice* $x_0 := 0 \in \mathbb{R}^n$ *and* $h := \mathbf{e}_n \in S^{n-1}$, *one has*

$$\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n : |x| < R,\ a|x|\,\omega(|x|) < x_n < b\}. \tag{7.4}$$

Furthermore,

$$\begin{gathered} \text{if } b \in (0, R\,\omega(R)) \text{ and if } t_b \in (0, R) \text{ satisfies} \\ t_b\,\omega(t_b) = b \text{ then } \mathscr{G}^{\omega}_{a,b}(x_0, h) \subseteq B(x_0, t_b). \end{gathered} \tag{7.5}$$

(ii) Assume that $a \in (0, 1)$. *Then, corresponding to the limiting case when* $\alpha = 0$, *the pseudo-ball introduced in* (7.3) *coincides with the one-component, circular, open cone with vertex at* x_0, *unit axis* h, *aperture* $\theta := 2 \arccos a \in (0, \pi)$, *and which is truncated at height* b. *That is,*

$$\mathscr{G}^{0}_{a,b}(x_0, h) = \Gamma_{\theta,b}(x_0, h), \quad \text{for } \ \theta := 2 \arccos a \in (0, \pi). \tag{7.6}$$

(iii) In the case when $\alpha = 1$, *then for each* $a > 0$ *the pseudo-ball defined in* (7.3) *coincides with the solid spherical cap obtained by intersecting the open ball in* $\mathbb{R}^n$ *with center at* $x_0 + h/(2a)$ *and radius* $r := 1/(2a)$ *with the half-space* $H(x_0, h, b) := \{x \in \mathbb{R}^n : (x - x_0) \cdot h < b\}$. *In other words*[1],

$$\mathscr{G}^{1}_{a,b}(x_0, h) = B\big(x_0 + h/(2a), 1/(2a)\big) \cap H(x_0, h, b). \tag{7.7}$$

Furthermore, when and $b \geq 1/a$, *one actually has*

$$\mathscr{G}^{1}_{a,b}(x_0, h) = B\big(x_0 + h/(2a), 1/(2a)\big).$$

[1]The term “pseudo-ball" has been chosen, *faute de mieux*, primarily because of this observation.

(iv) Let $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry; hence, $\mathcal{R} = T \circ \mathscr{R}$, where $\mathscr{R}$ is a rotation about the origin in $\mathbb{R}^n$ and T is a translation in $\mathbb{R}^n$. Then

$$\mathcal{R}\big(\mathscr{G}^{\omega}_{a,b}(x_0, h)\big) = \mathscr{G}^{\omega}_{a,b}(\mathcal{R}(x_0), \mathscr{R}h). \tag{7.8}$$

In particular, $x_1 + \mathscr{G}^{\omega}_{a,b}(x_0, h) = \mathscr{G}^{\omega}_{a,b}(x_0 + x_1, h)$ for every $x_1 \in \mathbb{R}^n$.

(v) Pick $t_* \in (0, R)$ with the property that $\omega(t_*) < 1$. Then whenever the number b_0 and the angle θ satisfy

$$0 < b_0 < \min\{b, t_*\}, \quad 2\max\Big\{\arccos\big(\omega(t_*)\big)\,,\,\arccos\big(b_0/t_*\big)\Big\} \leq \theta < \pi, \tag{7.9}$$

one has

$$\Gamma_{\theta,b_0}(x_0, h) \subseteq \mathscr{G}^{\omega}_{a,b}(x_0, h). \tag{7.10}$$

As a consequence, the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ contains truncated circular cones (with vertex at x_0 and axis h) of apertures arbitrarily close to π.

Proof. These are all straightforward consequences of definitions. □

For the remainder of this chapter we shall assume that the shape function ω as in (1.10) also satisfies the conditions listed in (1.12). That is (after a slight rephrasing of the first condition in (1.12)),

$$\begin{gathered} R \in (0, +\infty) \text{ and } \omega : [0, R] \to [0, +\infty) \text{ is a continuous, (strictly)} \\ \text{increasing function, with the property that } \omega(0) = 0 \text{ and there exists} \\ \text{a function } \eta : (0, +\infty) \to (0, +\infty) \text{which satisfies } \lim_{\lambda \to 0^+} \eta(\lambda) = 0 \\ \text{and } \omega(\lambda t) \leq \eta(\lambda)\,\omega(t) \text{ for all } \lambda > 0 \text{ and } t \in \big[0, \min\{R, R/\lambda\}\big]. \end{gathered} \tag{7.11}$$

For future reference, let us note here that conditions (7.11) entail that

$$\begin{aligned}&\omega : [0, R] \to [0, \omega(R)] \ \text{ is invertible and its inverse}\\ &\omega^{-1} : [0, \omega(R)] \to [0, R] \text{ is a continuous function which} \\ &\text{is (strictly) increasing and satisfies } \ \omega^{-1}(0) = 0.\end{aligned} \tag{7.12}$$

In order to facilitate the presentation of the proof of the main result in this paper we shall now present a series of technical, preliminary lemmas pertaining to the geometry of pseudo-balls. The key ingredient is the fact that a pseudo-ball has positive, finite curvature near the apex. A concrete manifestation of this property is the fact that two pseudo-balls with apex at the origin and whose direction vectors do not point in opposite ways necessarily have a substantial overlap.

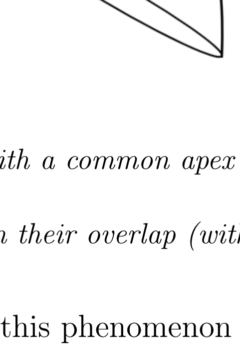

Figure 7. *Any two pseudo-balls with a common apex and whose direction vectors are not opposite contain a ball in their overlap (with quantitative control of its size).*

A precise, quantitative aspect of this phenomenon is discussed in Lemma 7.2 below.

Lemma 7.2. *Assume* (7.11) *and let* $a, b \in (0, +\infty)$ *be given. Then there exists* $\varepsilon > 0$*, which depends only on* $\eta, \omega, R,$ a *and* b*, such that for any* $h_0, h_1 \in S^{n-1}$ *the following* *implication holds:*

$$
\begin{gathered}
x \in \mathbb{R}^n \quad \text{and} \quad \Big| x - \varepsilon\, \omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\,\tfrac{h_0+h_1}{|h_0+h_1|}\Big| < \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big| \Longrightarrow \\
|x| < R \quad \text{and} \quad a|x|\omega(|x|) < \min\big\{x\cdot h_0, x\cdot h_1\big\} \quad \text{and} \quad \max\big\{x\cdot h_0, x\cdot h_1\big\} < b,
\end{gathered} \tag{7.13}
$$

with the convention that $\frac{h_0+h_1}{|h_0+h_1|} := 0$ *if* $h_0 + h_1 = 0$. *In other words, for each vectors* $h_0, h_1 \in S^{n-1}$, *the first line in* (7.13) *implies that* $x \in \mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(0,h_1)$.

Proof. Fix a real number ε such that

$$
0 < \varepsilon < \min\Big\{\tfrac{2b}{3R}\,,\,\tfrac{2}{3}\Big\} \quad \text{and} \quad \eta\big(\tfrac{3\varepsilon}{2}\big) < \big[3a\,\omega(R)\big]^{-1}. \tag{7.14}
$$

That this is possible is ensured by the last line in (7.11). Next, pick two arbitrary vectors $h_0, h_1 \in S^{n-1}$ and introduce $v := \frac{h_0+h_1}{2}$. Then, if x is as in the first line in (7.13), we may estimate (keeping in mind that $|v| \leq 1$):

$$
\begin{aligned}
|x| \;&\leq\; \Big| x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big| + \Big|\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big| \\
&<\; \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| + \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big) \leq \tfrac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big).
\end{aligned} \tag{7.15}
$$

Granted the first condition in (7.14), this further implies (recall that ω^{-1} is increasing and $|v| \leq 1$) that

$$
|x| \leq \tfrac{3\varepsilon}{2} R < \min\{R, b\} \tag{7.16}
$$

so the first estimate in the second line of (7.13) is taken care of. In order to prove the remaining estimates in the second line of (7.13), it is enough to show that

$$
a|x|\omega(|x|) < x\cdot h_0 < b \;\; \text{if } x \in \mathbb{R}^n \text{ is such that} \tag{7.17}
$$

$$
\Big| x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big| < \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|,
$$

since the roles of h_0 and h_1 in (7.13) are interchangeable. To this end, assume that x is as in the last part of (7.17) and write

$$x \cdot h_0 = \Big(x - \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big) \cdot h_0 + \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|} \cdot h_0 \tag{7.18}$$

and observe that $v \cdot h_0 = \frac{1}{2}(1 + h_0 \cdot h_1) = |v|^2$. Thus, on the one hand, we have

$$\varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|} \cdot h_0 = \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{7.19}$$

On the other hand, based on the Cauchy-Schwarz inequality and the assumption on x we obtain

$$\begin{aligned} \Big|\Big(x - \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big) \cdot h_0\Big| &\leq \Big|x - \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big| \\ &< \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)|v|. \end{aligned} \tag{7.20}$$

From this it follows that

$$x \cdot h_0 > \varepsilon\, \omega^{-1}\big(\omega(R)|v|\big)|v| - \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)|v| = \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{7.21}$$

At this stage, we make the claim that $\frac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)|v| > a|x|\omega(|x|)$ which, when used in concert with the estimate just derived, yields $x \cdot h_0 > a|x|\omega(|x|)$. To justify this claim, based on (7.15) and (7.11) we may then write

$$\begin{aligned} a|x|\omega(|x|) &\leq a\tfrac{3\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)\, \omega\Big(\tfrac{3\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)\Big) \\ &\leq a\tfrac{3\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)\, \eta\big(\tfrac{3\varepsilon}{2}\big)\, \omega(R)|v| \\ &< \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)|v|\big)|v|, \end{aligned} \tag{7.22}$$

where the third inequality is a consequence of (7.14). This finishes the proof of the claim. There remains to observe that, thanks to (7.16), $x \cdot h_0 \leq |x| < b$, completing the proof of Lemma 7.2. $\square$

The main application of Lemma 7.2 is the following result asserting, in a quantitative manner, that two pseudo-balls necessarily overlap if their apexes are sufficiently close to one another relative to the degree of proximity of their axes.

Lemma 7.3. *Assume* (7.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Also, suppose that the parameter* $\varepsilon = \varepsilon(\omega, \eta, R, a, b) > 0$ *is as in Lemma 7.2. Then for every* $x_0, x_1 \in \mathbb{R}^n$ *and every* $h_0, h_1 \in S^{n-1}$ *one has*

$$|x_0 - x_1| < \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big| \Longrightarrow \mathscr{G}^{\omega}_{a,b}(x_0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1, h_1) \neq \varnothing. \tag{7.23}$$

Proof. To set the stage, let $\varepsilon > 0$ be as in Lemma 7.2 and assume that $x_0, x_1 \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that the estimate in the left-hand side of (7.23) holds. In particular, $|h_0 + h_1| > 0$ and

$$\begin{aligned} &B\Big(\varepsilon\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\tfrac{h_0+h_1}{|h_0+h_1|}, \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big|\Big) \\ &\qquad\qquad \subseteq \mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(0, h_1). \end{aligned} \tag{7.24}$$

Indeed, this is simply a rephrasing of the conclusion in Lemma 7.2. Henceforth, we denote the ball in the left-hand side of (7.24) by $\mathcal{B}_{h_0,h_1}$ and denote its center and its radius by c_{h_0,h_1} and r_{h_0,h_1}, respectively. To proceed, consider $y := c_{h_0,h_1} + x_0 - x_1 \in \mathbb{R}^n$ and note that $|y - c_{h_0,h_1}| = |x_0 - x_1| < r_{h_0,h_1}$. This implies that $y \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0, h_1)$, thus, ultimately,

$$c_{h_0,h_1} = (x_1 - x_0) + y \in x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0, h_1).$$

Since we also have $c_{h_0,h_1} \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0, h_0)$, this analysis shows that

$$c_{h_0,h_1} \in \mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \Big(x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0, h_1)\Big).$$

Upon recalling from item (iv) in Lemma 7.1 that $x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0, h_1) = \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1)$, we deduce that $\mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1) \neq \varnothing$. Finally, translating by x_0 yields $\mathscr{G}^{\omega}_{a,b}(x_0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1, h_1) \neq \varnothing$. This completes the proof of Lemma 7.3. $\square$

We conclude this chapter by presenting a consequence of Lemma 7.3 to the effect that two pseudo-balls sharing a common apex are disjoint if and only if their axes point in opposite directions.

Corollary 7.4. *Assume* (7.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Then for each point* $x \in \mathbb{R}^n$ *and any pair of vectors* $h_0, h_1 \in S^{n-1}$ *one has*

$$\mathscr{G}^{\omega}_{a,b}(x, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x, h_1) = \varnothing \Longleftrightarrow h_0 + h_1 = 0. \tag{7.25}$$

Proof. If $x \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that $|h_0 + h_1| > 0$ and yet

$$\mathscr{G}^{\omega}_{a,b}(x, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x, h_1) = \varnothing,$$

then Lemma 7.3 (used with $x_0 := x =: x_1$) yields a contradiction. This proves the left-to-right implication in (7.25). For the converse implication, observe that if

$$y \in \mathscr{G}^{\omega}_{a,b}(x, h) \cap \mathscr{G}^{\omega}_{a,b}(x, -h)$$

for some $x \in \mathbb{R}^n$ and $h \in S^{n-1}$ then $a|y - x|\,\omega(|y - x|) < h \cdot (y - x)$ and

$$a|y - x|\,\omega(|y - x|) < (-h) \cdot (y - x).$$

Hence $h \cdot (y - x) > 0$ and $(-h) \cdot (y - x) > 0$, a contradiction which concludes the proof of the corollary. $\square$

Chapter 8

Sets of Locally Finite Perimeter

Given $E \subseteq \mathbb{R}^n$, denote by $\mathbf{1}_E$ the characteristic function of E. A Lebesgue measurable set $E \subseteq \mathbb{R}^n$ is said to be of locally finite perimeter provided

$$\mu := \nabla \mathbf{1}_E \tag{8.1}$$

is a locally finite $\mathbb{R}^n$-valued measure. For a set of locally finite perimeter which has a compact boundary we agree to drop the adverb 'locally'. Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter we denote by σ the total variation measure of μ; σ is then a locally finite positive measure supported on ∂E, the topological boundary of E. Also, clearly, each component of μ is absolutely continuous with respect to σ. It follows from the Radon-Nikodym theorem that

$$\mu = \nabla \mathbf{1}_E = -\nu\sigma, \tag{8.2}$$

where

$$\begin{array}{l} \nu \in L^\infty(\partial E, d\sigma) \text{ is an } \mathbb{R}^n\text{-valued function} \\ \text{satisfying } |\nu(x)| = 1, \text{ for } \sigma\text{-a.e. } x \in \partial E. \end{array} \tag{8.3}$$

It is customary to identify σ with its restriction to ∂E with no special mention. We shall refer to ν and σ, respectively, as the (geometric measure theoretic) outward unit normal and the surface measure on ∂E. Note that ν defined by (8.2) can only be specified up to a set of σ-measure zero. To eliminate this ambiguity, we redefine $\nu(x)$, for every x, as being

$$\lim_{r\to 0} \fint_{B(x,r)} \nu \, d\sigma \tag{8.4}$$

whenever the above limit exists, and zero otherwise. In doing so, we shall make the convention that $\fint_{B(x,r)} \nu \, d\sigma := (\sigma(B(x,r)))^{-1} \int_{B(x,r)} \nu \, d\sigma$ if $\sigma(B(x,r)) > 0$, and zero otherwise. The Besicovitch Differentiation Theorem (cf., e.g., [20]) ensures that ν in (8.2) agrees with (8.4) for σ-a.e. x.

The reduced boundary of E is then defined as

$$\partial^* E := \big\{x \in \partial E : \, |\nu(x)| = 1\big\}. \tag{8.5}$$

This is essentially the point of view adopted in [90] (cf. Definition 5.5.1 on p. 233). Let us remark that this definition is slightly different from that given on p. 194 of [20]. The reduced boundary introduced there depends on the choice of the unit normal in the class of functions agreeing with it σ-a.e. and, consequently, can be pointwise specified only up to a certain set of zero surface measure. Nonetheless, any such representative is a subset of $\partial^* E$ defined above and differs from it by a set of σ-measure zero.

Moving on, it follows from (8.5) and the Besicovitch Differentiation Theorem that σ is supported on $\partial^* E$, in the sense that $\sigma(\mathbb{R}^n \setminus \partial^* E) = 0$. From the work of Federer and of De Giorgi it is also known that, if $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^* E. \tag{8.6}$$

Recall that, generally speaking, given a Radon measure μ in $\mathbb{R}^n$ and a set $A \subseteq \mathbb{R}^n$, the restriction of μ to A is defined as $\mu \lfloor A := \mathbf{1}_A \, \mu$. In particular, $\mu \lfloor A \ll \mu$ and $d(\mu \lfloor A)/d\mu = \mathbf{1}_A$. Thus,

$$\sigma \ll \mathcal{H}^{n-1} \quad \text{and} \quad \frac{d\sigma}{d\mathcal{H}^{n-1}} = \mathbf{1}_{\partial^* E}. \tag{8.7}$$

Furthermore (cf. Lemma 5.9.5 on p. 252 in [90], and p. 208 in [20]) one has

$$\partial^* E \subseteq \partial_* E \subseteq \partial E, \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0, \tag{8.8}$$

where $\partial_* E$, the *measure-theoretic boundary* of E, is defined by

$$\partial_* E := \Big\{ x \in \partial E : \limsup_{r \to 0^+} r^{-n} \mathcal{H}^n \big(B(x,r) \cap E^{\pm} \big) > 0 \Big\}. \tag{8.9}$$

Above, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (i.e., up to normalization, the Lebesgue measure) in $\mathbb{R}^n$, and we have set $E^+ := E$, $E^- := \mathbb{R}^n \setminus E$.

Let us also record here a useful criterion for deciding whether a Lebesgue measurable subset E of $\mathbb{R}^n$ is of locally finite perimeter in $\mathbb{R}^n$ (cf. [20], p. 222):

$$E \;\text{ has locally finite perimeter } \Longleftrightarrow \tag{8.10}$$

$$\mathcal{H}^{n-1}(\partial_* E \cap K) < +\infty, \quad \forall\, K \subseteq \mathbb{R}^n \;\text{ compact}.$$

We conclude this chapter by proving the following result of geometric measure theoretic flavor (which is a slight extension of Proposition 2.9 in [36]), establishing a link between the cone property and the direction of the geometric measure theoretic unit normal.

Lemma 8.1. *Let E be a subset of $\mathbb{R}^n$ of locally finite perimeter. Fix a point x_0 belonging to $\partial^* E$ (the reduced boundary of E) with the property that there exist $b > 0$,*

$\theta \in (0, \pi)$ and $h \in S^{n-1}$ such that

$$\Gamma_{\theta,b}(x_0, h) \subseteq E. \tag{8.11}$$

Then, if $\nu(x_0)$ denotes the geometric measure theoretic outward unit normal to E at x_0, there holds

$$\nu(x_0) \in \Gamma_{\pi-\theta,1}(0, -h). \tag{8.12}$$

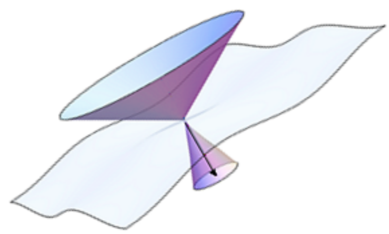

Figure 8. *Where the outward unit normal must be.*

Proof. The idea is to use a blow-up argument. Specifically, consider the half-space

$$H(x_0) := \{x \in \mathbb{R}^n : \nu(x_0) \cdot (x - x_0) < 0\} \subseteq \mathbb{R}^n \tag{8.13}$$

and, for each $r > 0$ and $A \subseteq \mathbb{R}^n$, set

$$A_r := \{x \in \mathbb{R}^n : r(x - x_0) + x_0 \in A\}. \tag{8.14}$$

Also, abbreviate $\Gamma := \Gamma_{\theta,b}(x_0, h)$ and denote by $\widetilde{\Gamma}$ the circular, open, infinite cone which coincides with $\Gamma_{\theta,b}(x_0, h)$ near its vertex. The theorem concerning the blow-up of the reduced boundary of a set of locally finite perimeter (cf., e.g., p. 199 in [20]) gives that

$$\mathbf{1}_{E_r} \longrightarrow \mathbf{1}_{H(x_0)} \quad \text{in} \quad L^1_{loc}(\mathbb{R}^n), \quad \text{as} \quad r \to 0^+. \tag{8.15}$$

On the other hand, it is clear that $\Gamma_r \subseteq E_r$ and $\mathbf{1}_{\Gamma_r} \longrightarrow \mathbf{1}_{\widetilde{\Gamma}}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0^+$. This and (8.14) then allow us to write

$$\begin{aligned} \mathbf{1}_{\widetilde{\Gamma}} &= \lim_{r\to 0^+} \mathbf{1}_{\Gamma_r} = \lim_{r\to 0^+} \Big(\mathbf{1}_{\Gamma_r} \cdot \mathbf{1}_{E_r}\Big) \\ &= \Big(\lim_{r\to 0^+} \mathbf{1}_{\Gamma_r}\Big) \cdot \Big(\lim_{r\to 0^+} \mathbf{1}_{E_r}\Big) = \mathbf{1}_{\widetilde{\Gamma}} \cdot \mathbf{1}_{H(x_0)} \\ &= \mathbf{1}_{\widetilde{\Gamma}\cap H(x_0)}, \end{aligned} \tag{8.16}$$

in a pointwise $\mathcal{H}^n$-a.e. sense in $\mathbb{R}^n$. In turn, this implies

$$\widetilde{\Gamma} \subseteq \overline{H(x_0)}. \tag{8.17}$$

Now, (8.12) readily follows from this, (8.13), and simple geometrical considerations.

□

Chapter 9

Measuring the Smoothness of Euclidean Domains in Analytical Terms

We begin by giving the formal definition of the category of Lipschitz domains and domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1]$. The reader is reminded that the superscript c is the operation of taking the complement of a set, relative to $\mathbb{R}^n$.

Definition 9.1. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$. Also, fix a point $x_0 \in \partial\Omega$. Call Ω a* `Lipschitz domain near` *x_0 if there exist $r, c > 0$ with the following significance. There exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, a choice N of the unit normal to H, and an open cylinder*

$$\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, \ |t| < c\}$$

(called coordinate cylinder near x_0) such that

$$\mathcal{C}_{r,c} \cap \Omega = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H, \ t > \varphi(x')\}, \tag{9.1}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$, called the defining function for $\partial\Omega$ near x_0, satisfying

$$\varphi(x_0) = 0 \quad \textit{and} \quad |\varphi(x')| < c \quad \textit{if} \ \ |x' - x_0| \leq r. \tag{9.2}$$

Collectively, the pair $(\mathcal{C}_{r,c}, \varphi)$ will be referred to as a local chart near x_0, whose geometrical characteristics consist of r, c and the Lipschitz constant of φ.

Moreover, call Ω a `locally Lipschitz domain` *if it is a Lipschitz domain near every point $x \in \partial\Omega$. Finally, Ω is simply called a* `Lipschitz domain` *if it is locally Lipschitz and such that the geometrical characteristics of the local charts associated with each boundary point are independent of the point in question.*

The categories of $\mathscr{C}^{1,\alpha}$ domains with $\alpha \in (0, 1]$, as well as their local versions, are defined analogously, requiring that the defining functions φ have first-order directional derivatives (along vectors parallel to the hyperplane H) which are of class $\mathscr{C}^{\alpha}$ (the Hölder space of order α).

A few useful observations related to the property of an open set $\Omega \subseteq \mathbb{R}^n$ of being a Lipschitz domain near a point $x_0 \in \partial\Omega$ are collected below.

Proposition 9.1. *Assume that Ω is a nonempty, proper, open subset of $\mathbb{R}^n$, and fix $x_0 \in \partial\Omega$.*

(i) If Ω is a Lipschitz domain near x_0 and if $(\mathcal{C}_{r,c}, \varphi)$ is a local chart near x_0 (in the sense of Definition 9.1) then, in addition to (9.1), one also has

$$\mathcal{C}_{r,c} \cap \partial\Omega = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t = \varphi(x')\}, \tag{9.3}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^c = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t < \varphi(x')\}. \tag{9.4}$$

Furthermore,

$$\mathcal{C}_{r,c} \cap \overline{\Omega} = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t \geq \varphi(x')\}, \tag{9.5}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^{\circ} = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t > \varphi(x')\}, \tag{9.6}$$

and, consequently,

$$E \cap \partial\Omega = E \cap \partial(\overline{\Omega}), \quad \forall\, E \subseteq \mathcal{C}_{r,c}. \tag{9.7}$$

(ii) Assume that there exist an $(n-1)$*-dimensional plane* $H \subseteq \mathbb{R}^n$ *passing through* x_0*, a choice* N *of the unit normal to* H*, an open cylinder* $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$ *and a Lipschitz function* $\varphi : H \to \mathbb{R}$ *satisfying* (9.2) *such that* (9.3) *holds. Then, if* $x_0 \notin (\overline{\Omega})^\circ$*, it follows that* Ω *is a Lipschitz domain near* x_0*.*

Proof. The fact that (9.1) implies (9.3) is a consequence of the general fact

$$\begin{aligned} &\mathcal{O}, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n \ \text{ open sets such that } \ \mathcal{O} \cap \Omega_1 = \mathcal{O} \cap \Omega_2 \\ &\qquad\qquad\qquad \Longrightarrow \mathcal{O} \cap \partial\Omega_1 = \mathcal{O} \cap \partial\Omega_2, \end{aligned} \tag{9.8}$$

used with $\mathcal{O} := \mathcal{C}_{r,c}$, $\Omega_1 := \Omega$ and Ω_2 the upper-graph of φ. In order to justify (9.8), we make the elementary observation that

$$E \subseteq \mathbb{R}^n \text{ arbitrary set and } \mathcal{O} \subseteq \mathbb{R}^n \text{ open set } \Longrightarrow \overline{E} \cap \mathcal{O} \subseteq \overline{E \cap \Omega}. \tag{9.9}$$

Then, in the context of (9.8), based on assumptions and (9.9) we may write

$$\begin{aligned} \mathcal{O} \cap \partial\Omega_1 \subseteq (\mathcal{O} \cap \overline{\Omega_1}) \setminus (\mathcal{O} \cap \Omega_1) \ &\subseteq \ \overline{\mathcal{O} \cap \Omega_1} \setminus (\mathcal{O} \cap \Omega_1) \\ &= \ \overline{\mathcal{O} \cap \Omega_2} \setminus (\mathcal{O} \cap \Omega_2) \subseteq \overline{\Omega_2} \setminus \Omega_2 = \partial\Omega_2. \end{aligned} \tag{9.10}$$

This further entails $\mathcal{O} \cap \partial\Omega_1 \subseteq \mathcal{O} \cap \partial\Omega_2$ from which (9.8) follows by interchanging the roles of Ω_1 and Ω_2. As mentioned earlier, this establishes (9.3). Thus, in order to prove (9.4), it suffices to show that

$$(9.1) \ \text{ and } \ (9.3) \Longrightarrow (9.4), \tag{9.11}$$

In turn, (9.11) follows by writing

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^c = \mathcal{C}_{r,c} \setminus (\mathcal{C}_{r,c} \cap \overline{\Omega}) = \mathcal{C}_{r,c} \setminus \Big((\mathcal{C}_{r,c} \cap \Omega) \cup (\mathcal{C}_{r,c} \cap \partial\Omega) \Big)$$

$$= \mathcal{C}_{r,c} \setminus \Big((\mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t > \varphi(x')\})$$

$$\cup (\mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t = \varphi(x')\}) \Big)$$

$$= \mathcal{C}_{r,c} \setminus \Big(\mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t \geq \varphi(x')\} \Big)$$

$$= \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t < \varphi(x')\}, \tag{9.12}$$

as desired. Next, (9.5) is a consequence of (9.1) and (9.3), while (9.6) follows from (9.5) by passing to interiors. In concert, (9.5)-(9.6) and (9.3) give that

$$\mathcal{C}_{r,c} \cap \partial(\overline{\Omega}) = \mathcal{C}_{r,c} \cap \partial\Omega, \tag{9.13}$$

which further implies (9.7) by taking the intersection of both sides with a given set $E \subseteq \mathcal{C}_{r,c}$. This completes the proof of part (i). As far as (ii) is concerned, it suffices to show that, up to reversing the sense on the vertical axis in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$x_0 \notin (\overline{\Omega})^\circ \implies (9.1), \; (9.4). \tag{9.14}$$

In turn, (9.14) follows from Lemma 9.2, stated and proved below. $\square$

Here is the topological result which has been invoked earlier, in the proof of the implication (9.14).

Lemma 9.2. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, proper, open set, and fix $x_0 \in \partial\Omega$. Also, assume that $B' \subseteq \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional open ball, $I \subseteq \mathbb{R}$ is an open interval, and that $\varphi : B' \to I$ is a continuous function. Denote the graph of φ by*

$\Sigma := \{(x', \varphi(x')) : x' \in B'\}$. *Assume that the open cylinder*

$$\mathcal{C} := B' \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$$

contains x_0 *and satisfies* $\Sigma = \mathcal{C} \cap \partial\Omega$. *Finally, set*

$$D^+ := \{(x', x_n) \in \mathcal{C} : \varphi(x') < x_n\}, \quad D^- := \{(x', x_n) \in \mathcal{C} : \varphi(x') > x_n\}. \tag{9.15}$$

Then one of the following three alternatives holds:

$$\Omega \cap \mathcal{C} = D^+ \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^-, \tag{9.16}$$

$$\Omega \cap \mathcal{C} = D^- \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \tag{9.17}$$

$$x_0 \in (\overline{\Omega})^\circ. \tag{9.18}$$

Proof. We begin by noting that $D^\pm$ are connected sets. To see this, consider D^+, as the argument for D^- is similar. It suffices to show that the set in question is pathwise connected, and a continuous curve γ contained in D^+ joining any two given points $x, y \in D^+$ may be taken to consist of three line segments, L_1, L_2, L_3, defined as follows. Take L_1 and L_2 to be the vertical line segments contained in D^+ which emerge from x and y, respectively, and then choose L_3 to be a horizontal line segment making the transition between L_1 and L_2 near the very top of $\mathcal{C}$. Moving on, we claim that one of the following situations necessarily happens:

$$\begin{aligned} &(i)\ D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq (\overline{\Omega})^c, \qquad \text{or} \\ &(ii)\ D^- \subseteq \Omega \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c, \qquad \text{or} \\ &(iii)\ D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq \Omega, \qquad \text{or} \\ &(iv)\ D^- \subseteq (\overline{\Omega})^c \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c. \end{aligned} \tag{9.19}$$

To prove this, note that $D^\pm$ are disjoint from Σ and, hence, from $\partial\Omega \cap \mathcal{C}$. In turn, this further entails that $D^\pm$ are disjoint from $\partial\Omega$. Based on this and the fact that

$\mathbb{R}^n = \Omega \cup (\overline{\Omega})^c \cup \partial\Omega$, we conclude that

$$D^{\pm} \subseteq \Omega \cup (\overline{\Omega})^c. \tag{9.20}$$

Now, recall that $D^{\pm}$ are connected sets, and observe that Ω are $(\overline{\Omega})^c$ open, disjoint sets. In concert with the definition of connectivity, (9.20) then implies that each of the two sets D^+, D^- is contained in either Ω, or $(\overline{\Omega})^c$. Unraveling the various possibilities now proves that one of the four scenarios in (9.19) must hold. This concludes the proof of the claim made about (9.19). The next step is to show that if conditions (i) in (9.19) happen, then conditions (9.16) happen as well. To see this, assume (i) holds, i.e., $D^+ \subseteq \Omega$, $D^- \subseteq (\overline{\Omega})^c$ and recall that $\Sigma = \partial\Omega \cap \mathcal{C}$. Then $D^+ = \Omega \cap \mathcal{C}$. Indeed, the left-to-right inclusion is clear from what we assume. For the opposite inclusion, we reason by contradiction and assume that there exists $x \in \Omega \cap \mathcal{C}$ such that $x \notin D^+$, thus $x \in \Omega$, $x \in \mathcal{C}$, $x \notin D^+$. Since

$$\mathcal{C} = D^+ \cup D^- \cup \Sigma, \quad \text{disjoint unions,} \tag{9.21}$$

we obtain

$$\mathcal{C} = D^+ \cup D^- \cup (\mathcal{C} \cap \partial\Omega), \quad \text{disjoint unions.} \tag{9.22}$$

From the assumptions on x we have that $x \notin D^+$, $x \notin \mathcal{C} \cap \partial\Omega$ (since $x \in \Omega$ and $\Omega \cap \partial\Omega = \varnothing$) and consequently, using also (9.22), $x \in D^- \subseteq (\overline{\Omega})^c$. This yields that $x \notin \Omega$, contradicting the assumption that $x \in \Omega$. This completes the proof of the fact that $D^+ = \Omega \cap \mathcal{C}$. In a similar fashion, we also obtain that $D^- = (\overline{\Omega})^c \cap \mathcal{C}$.

We next propose to show that if condition (ii) in (9.19) happens, then condition

(9.17) happens as well. In particular, if (ii) happens, then

$$\Omega \cap \mathcal{C} = D^-, \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \quad \partial\Omega \cap \mathcal{C} = \Sigma. \tag{9.23}$$

To see this, assume (ii) happens. The first observation is that $D^+ = (\overline{\Omega})^c \cap \mathcal{C}$. The left-to-right inclusion is clear. Assume next that there exists $x \in \mathcal{C}$ such that $x \notin \overline{\Omega}$ (thus $x \notin \partial\Omega$) and $x \notin D^+$. Together with (9.22), these imply $x \in D^-$ so $x \in D^- \subseteq \Omega \subseteq \overline{\Omega}$. which is in contradiction with our assumptions. Moving on, the second observation is that $D^- = \Omega \cap \mathcal{C}$. The left-to-right inclusion is obvious. In the opposite direction, assume that there exists $x \in \mathcal{C}$ such that $x \in \Omega$ (hence $x \notin \partial\Omega$) yet $x \notin D^-$. Invoking (9.22) it follows that $x \in D^+ \subseteq (\overline{\Omega})^c$, hence $x \notin \overline{\Omega}$ contradicting the assumption on x.

Next, we shall show that if condition (iii) in (9.19) happens, then $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, i.e. (9.18) happens. To this end, let $x_* \in \partial\Omega \cap \mathcal{C}$. Then there exists $r > 0$ such that $B(x_*, r) \subseteq \mathcal{C}$. We claim that

$$B(x_*, r) \subseteq \overline{\Omega} \tag{9.24}$$

Indeed, by (9.21), we have

$$B(x_*, r) = \Big(B(x_*, r) \cap D^+\Big) \cup \Big(B(x_*, r) \cap D^-\Big) \cup \Big(B(x_*, r) \cap \Sigma\Big). \tag{9.25}$$

Making use of the inclusion $B(x_*, r) \subseteq \mathcal{C}$, we then obtain

$$\begin{aligned} &B(x_*, r) \cap D^+ \subseteq D^+ \subseteq \Omega, \\ &B(x_*, r) \cap D^- \subseteq D^- \subseteq \Omega, \\ &B(x_*, r) \cap \Sigma \subseteq \Sigma = \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega. \end{aligned} \tag{9.26}$$

Combining all these with (9.25), it follows that $B(x_*, r) \subseteq \Omega \cap \partial\Omega = \overline{\Omega}$, proving (9.24). In turn, (9.24) implies that $x_* \in (\overline{\Omega})^\circ \cap \partial\Omega$ so that, ultimately,

$$\mathcal{C} \cap \partial\Omega \subseteq (\overline{\Omega})^\circ \cap \partial\Omega. \tag{9.27}$$

Since $x_0 \in \mathcal{C} \cap \partial\Omega$, this forces $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, as claimed.

At this stage in the proof, there remains to show that condition (vi) in (9.19) never happens. Reasoning by assume (iv) actually does happen, i.e.,

$$D^- \subseteq (\overline{\Omega})^c, \quad D^+ \subseteq (\overline{\Omega})^c \quad \text{and} \quad \Sigma = \partial\Omega \cap \mathcal{C}. \tag{9.28}$$

Taking the union of the first two inclusions above yields

$$D^+ \cup D^- \subseteq (\overline{\Omega})^c \Longrightarrow \mathcal{C} \setminus \Sigma \subseteq (\overline{\Omega})^c \Longrightarrow \mathcal{C} \cap (\Sigma^c) \subseteq (\overline{\Omega})^c \Longrightarrow \overline{\Omega} \subseteq \Sigma \cup \mathcal{C}^c, \tag{9.29}$$

where the last implication follows by taking complements. Taking the intersection with $\mathcal{C}$, this yields $\mathcal{C} \cap \overline{\Omega} \subseteq \Sigma = \mathcal{C} \cap \partial\Omega$, thanks to the fact that $\Sigma = \mathcal{C} \cap \partial\Omega$. Thus, $\mathcal{C} \cap \Omega \subseteq \mathcal{C} \cap \overline{\Omega} \subseteq \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega$, i.e.,

$$\mathcal{C} \cap \Omega \subseteq \partial\Omega. \tag{9.30}$$

Since, by assumption, $\mathcal{C}$ is an open neighborhood of the point $x_0 \in \partial\Omega$, the definition of the boundary implies that $\mathcal{C} \cap \Omega \neq \varnothing$. Therefore, there exists $x_* \in \mathcal{C} \cap \Omega$. From (9.30) it follows that $x_* \in \partial\Omega$, which forces us to conclude that the open set Ω contains some of its own boundary points. This is a contradiction which shows that (iv) in (9.19) never happens. The proof of the lemma is therefore complete. $\square$

The proposition below formalizes the idea that a connected, proper, open subset of $\mathbb{R}^n$ whose boundary is a compact Lipschitz surface is a Lipschitz domain. Before stating this, we wish to note that the connectivity assumption is necessary since, otherwise, $\Omega := \{x \in \mathbb{R}^n : \ |x| < 2 \ \text{ and } \ |x| \neq 1\}$ would serve as a counterexample.

Theorem 9.3. *Let Ω be a nonempty, connected, proper, open subset of $\mathbb{R}^n$, with $\partial\Omega$ bounded and suppose that for each $x_0 \in \partial\Omega$ there exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : \ x' \in H, \ |x' - x_0| < r, \ |t| < c\}$ and a Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying* (9.2) *such that* (9.3) *holds. Then Ω is a Lipschitz domain.*

In the proof of the above result, the following generalization of the Jordan-Brouwer separation theorem for arbitrary compact topological hypersurfaces in $\mathbb{R}^n$, established in [7, Theorem 1, p. 284], plays a key role.

Proposition 9.4. *Let Σ be a compact, connected, topological $(n-1)$-dimensional submanifold (without boundary) of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus \Sigma$ consists of two connected components, each with boundary Σ.*

Proof of Theorem 9.3. Let Σ be a connected component of $\partial\Omega$ (in the relative topology induced by $\mathbb{R}^n$ on $\partial\Omega$). We claim that

$$\Sigma \subseteq \partial(\overline{\Omega}). \tag{9.31}$$

To justify this claim, we first observe that, granted the current assumptions, it follows that $\partial\Omega$ is a compact, $(n-1)$-dimensional Lipschitz sub-manifold (without boundary) of $\mathbb{R}^n$. Hence, in particular, Σ is a compact, connected, $(n-1)$-dimensional topological manifold (without boundary in $\mathbb{R}^n$). Invoking Proposition 9.4 we may then conclude that there exist $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} &\mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2 \quad \text{and} \quad \mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing, \\ &\mathcal{O}_j \ \text{ open, connected, } \ \partial\mathcal{O}_j = \Sigma \ \text{ for } \ j = 1, 2. \end{aligned} \tag{9.32}$$

Let us also observe that these conditions further entail

$$\partial(\overline{\mathcal{O}_j}) = \Sigma \ \text{ for } \ j = 1, 2. \tag{9.33}$$

Indeed, $\overline{\mathcal{O}_1} = \mathcal{O}_1 \cup \partial\mathcal{O}_1 = \mathcal{O}_1 \cup \Sigma = (\mathcal{O}_2)^c$ which forces $\partial(\overline{\mathcal{O}_1}) = \partial[(\mathcal{O}_2)^c] = \partial\mathcal{O}_2 = \Sigma$, from which (9.33) follows. Moreover, since Ω is a connected set contained in $\mathbb{R}^n \setminus \partial\Omega \subseteq \mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2$, it follows that Ω is contained in one of the sets $\mathcal{O}_1, \mathcal{O}_2$. To fix ideas, assume that $\Omega \subseteq \mathcal{O}_1$. Then $\overline{\Omega} \subseteq \overline{\mathcal{O}_1}$ and, hence,

$$(\overline{\Omega})^\circ \subseteq (\overline{\mathcal{O}_1})^\circ = \overline{\mathcal{O}_1} \setminus \partial(\overline{\mathcal{O}_1}) = (\mathcal{O}_1 \cup \partial\mathcal{O}_1) \setminus \partial\mathcal{O}_1 = \mathcal{O}_1, \tag{9.34}$$

where the next-to-last equality is a consequence of (9.33) (and (9.32)). The relevant observation for us here is that, in concert with the second line in (9.32), the inclusion in (9.34) forces

$$\Sigma \cap (\overline{\Omega})^\circ = \varnothing. \tag{9.35}$$

To proceed, note that since $\Sigma \subseteq \partial\Omega \subseteq \overline{\Omega}$, we also have

$$\Sigma \cap (\overline{\Omega})^c = \varnothing. \tag{9.36}$$

Thus, since

$$\Sigma \subseteq \mathbb{R}^n = (\overline{\Omega})^\circ \cup \partial(\overline{\Omega}) \cup (\overline{\Omega})^c, \tag{9.37}$$

we may ultimately deduce from (9.35)-(9.37) that (9.31) holds. The end-game in the proof of the proposition is then as follows. Taking the union of all connected components of $\partial\Omega$, we see from (9.31) that $\partial\Omega \subseteq \partial(\overline{\Omega})$. Consequently, since the opposite inclusion is always true, we arrive at the conclusion that

$$\partial\Omega = \partial(\overline{\Omega}). \tag{9.38}$$

Therefore, $\partial\Omega \cap (\overline{\Omega})^\circ = \partial(\overline{\Omega}) \cap (\overline{\Omega})^\circ = \varnothing$ and, as such, given any $x_0 \in \partial\Omega$ it follows that necessarily $x_0 \notin (\overline{\Omega})^\circ$. With this in hand, the fact that Ω is a Lipschitz domain now follows from part (ii) of Proposition 9.1. $\square$

Definition 9.1 and (i) in Proposition 9.1 show that if $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain near a boundary point x_0 then, in a neighborhood of x_0, $\partial\Omega$ agrees with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, considered in a suitably chosen system of coordinates (which is isometric with the original one). Then the outward unit normal has an explicit formula in terms of $\nabla\varphi$, namely, in the new system of coordinates,

$$\nu(x', \varphi(x')) = \frac{(\nabla'\varphi(x'), -1)}{\sqrt{1 + |\nabla'\varphi(x')|^2}}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \text{ near } x_0', \tag{9.39}$$

where the gradient $\nabla\varphi(x')$ of φ exists by the classical Rademacher theorem for $\mathcal{H}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. This readily implies that if $\Omega \subseteq \mathbb{R}^n$ is a $\mathscr{C}^{1,\alpha}$ domain for some $\alpha \in (0, 1]$ then the outward unit normal $\nu : \partial\Omega \to S^{n-1}$ is Hölder of order α.

We next discuss a cone property enjoyed by Lipschitz domains whose significance will become more apparent later.

Lemma 9.5. *Assume that $\Omega \subseteq \mathbb{R}^n$ is Lipschitz near $x_0 \in \partial\Omega$. More specifically, suppose that the $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, the unit normal N to H, the Lipschitz function $\varphi : H \to \mathbb{R}$ and the cylinder $\mathcal{C}_{r,c}$*

are such that (9.1)-(9.2) *hold. Denote by* M *the Lipschitz constant of* φ *and fix* $\theta \in (0, 2\arctan(\frac{1}{M})]$. *Finally, select* $\lambda \in (0,1)$. *Then there exists* $b > 0$ *such that*

$$\Gamma_{\theta,b}(x,N) \subseteq \Omega \quad \textit{and} \quad \Gamma_{\theta,b}(x,-N) \subseteq \mathbb{R}^n \setminus \Omega \quad \textit{for each} \;\; x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega. \tag{9.40}$$

Proof. Let $\theta \in \big(0, 2\arctan(\frac{1}{M})\big]$, where $M > 0$ is the Lipschitz constant of φ, and pick $b > 0$ such that

$$b < \min\Big\{c, \, \tfrac{(1-\lambda)r}{\tan(\theta/2)}\Big\}. \tag{9.41}$$

These conditions guarantee that $\Gamma_{\theta,b}(x, \pm N) \subseteq \mathcal{C}_{r,c}$ for each $x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega$ so, as far as the first inclusion in (9.40) is concerned, it suffices to show that

$$x', y' \in H, \;\; s \in \mathbb{R} \;\text{ so that }\; y' + sN \in \Gamma_{\theta,b}\big(x' + \varphi(x')N, N\big) \tag{9.42}$$
$$\Longrightarrow s > \varphi(y').$$

Fix x', y', s as in the left-hand side of (9.42). Then

$$\cos(\theta/2)\sqrt{|y'-x'|^2 + (s-\varphi(x'))^2} < s - \varphi(x'). \tag{9.43}$$

Consequently,

$$s = s - \varphi(x') + \varphi(x') > \cos(\tfrac{\theta}{2})(|y'-x'|^2 + (s-\varphi(x'))^2)^{\frac{1}{2}} + (\varphi(x') - \varphi(y')) + \varphi(y').$$

So to prove that $s > \varphi(y')$ it is enough to show that

$\cos(\frac{\theta}{2})(|y'-x'|^2 + (s-\varphi(x'))^2)^{\frac{1}{2}} + (\varphi(x') - \varphi(y')) \geq 0$. This is trivially true if $y' = x'$, so there remains to consider the situation when $x' \neq y'$. Assuming that this is the case, define $A := \dfrac{|s-\varphi(x')|^2}{|y'-x'|^2}$ and $B := \dfrac{\varphi(x') - \varphi(y')}{|x'-y'|}$, in which scenario we must show that $\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}} + B \geq 0$. By construction, $A \geq 0$ and $B \in [-M, M]$, so it suffices to prove that

$$\cos(\tfrac{\theta}{2})(1+A)^{\frac{1}{2}} \geq M. \tag{9.44}$$

As a preamble, observe that $\cos(\frac{\theta}{2})(|y'-x'|^2+(s-\varphi(x'))^2)^{\frac{1}{2}} < s-\varphi(x')$ entails $\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}} < A^{\frac{1}{2}}$, or $\cos^2(\frac{\theta}{2})(1+A) < A$. Thus, $\dfrac{\cos^2(\frac{\theta}{2})}{1-\cos^2(\frac{\theta}{2})} < A$ and, further, $A > \cot^2(\frac{\theta}{2})$. Using this this lower bound on A in (9.44) yields

$$\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}} > \cos(\frac{\theta}{2})(1+\cot^2(\frac{\theta}{2}))^{\frac{1}{2}} = \cot^2(\frac{\theta}{2}).$$

Now $\cot^2(\frac{\theta}{2}) \geq M$ if and only if $\tan^2(\frac{\theta}{2}) \leq \frac{1}{M}$, which is true by our original choice of θ. This completes the proof of (9.42) and finishes the proof of the first inclusion in (9.40). The second inclusion in (9.40) is established in a similar fashion, completing the proof of the lemma. $\square$

Our next result shows that suitable rotations of graphs of differentiable functions continue to be graphs of functions (enjoying the same degree of regularity as the original ones). This is going to be useful later, in the Proof of Theorem 11.3.

Lemma 9.6. *Assume that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open neighborhood of the origin and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function satisfying $\varphi(0') = 0$, which is differentiable and whose derivative is continuous at $0' \in \mathbb{R}^{n-1}$. Let $\mathcal{R}$ be a rotation about the origin in $\mathbb{R}^n$ with the property that*

$$\mathcal{R} \ \textit{maps the vector} \ \frac{(\nabla\varphi(0'),-1)}{\sqrt{1+|\nabla\varphi(0')|^2}} \ \textit{into} \ -\mathbf{e}_n \in \mathbb{R}^n. \tag{9.45}$$

Then there exists a continuous, real-valued function ψ defined in a small neighborhood of $0' \in \mathbb{R}^{n-1}$ with the property that $\psi(0') = 0$ and whose graph coincides, in a small neighborhood of $0 \in \mathbb{R}^n$, with the graph of φ rotated by $\mathcal{R}$.

Furthermore, φ is of class $\mathscr{C}^{1,\alpha}$, for some $\alpha \in (0,1]$, if and only if so is ψ.

Proof. Matching the graph of φ, after being rotated by $\mathcal{R}$, by that of a function ψ comes down to ensuring that ψ is such that $\mathcal{R}(x', \varphi(x')) = (y', \psi(y'))$ can be solved both for x' in terms y', as well as for y' in terms x', near the origin in $\mathbb{R}^{n-1}$ in each instance. Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection map of $\mathbb{R}^n$ onto the first $n-1$ coordinates, and denote by $\pi_n : \mathbb{R}^n \to \mathbb{R}$ the coordinate projection map of $\mathbb{R}^n$ onto the last coordinate. Then,

$$\begin{aligned}
(y', y_n) = \mathcal{R}(x', \varphi(x')) \quad &\Leftrightarrow \quad \mathcal{R}^{-1}(y', y_n) = (x', \varphi(x')) \\
&\Leftrightarrow \quad \pi'\mathcal{R}^{-1}(y', y_n) = x' \ \text{ and } \ \pi_n\mathcal{R}^{-1}(y', y_n) = \varphi(x') \\
&\Leftrightarrow \quad F(y', y_n) = 0 \ \text{ and } \ x' = \pi'\mathcal{R}^{-1}(y', y_n), \qquad (9.46)
\end{aligned}$$

where F is the real-valued function defined in a neighborhood of the origin in $\mathbb{R}^n$ by

$$F(y', y_n) := \varphi(\pi'\mathcal{R}^{-1}(y', y_n)) - \pi_n\mathcal{R}^{-1}(y', y_n). \qquad (9.47)$$

Then a direct calculation shows that $F(0', 0) = 0$ and

$$\begin{aligned}
\partial_n F(y', y_n) &= \sum_{j=1}^{n-1} (\partial_j\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))(\mathcal{R}^{-1}\mathbf{e}_n) \cdot \mathbf{e}_j - (\mathcal{R}^{-1}\mathbf{e}_n) \cdot \mathbf{e}_n \\
&= (\mathcal{R}^{-1}\mathbf{e}_n) \cdot ((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\, -1) \\
&= \mathbf{e}_n \cdot \mathcal{R}((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\, -1). \qquad (9.48)
\end{aligned}$$

In particular, by (9.45),

$$\partial_n F(0', 0) = -\sqrt{1 + |\nabla\varphi(0')|^2} \neq 0. \qquad (9.49)$$

Thus, by the Implicit Function Theorem, there exists a continuous real-valued function ψ defined in a small neighborhood of $0' \in \mathbb{R}^{n-1}$ such that $\psi(0') = 0$ and for which

$$F(y', y_n) = 0 \Longleftrightarrow y_n = \psi(y') \quad \text{whenever } (y', y_n) \text{ is near } 0. \tag{9.50}$$

From this and (9.46), all desired conclusions follow. $\square$

Chapter 10

Characterization of Lipschitz Domains in Terms of Cones

The main goal in this chapter is to discuss several types of cones conditions which fully characterize the class of Lipschitz domains in $\mathbb{R}^n$. The results presented here build on and generalize those from § 2 in [36]. To help put matters in the proper perspective, it is worth recalling that an open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary and the property that there exists an open, circular, truncated, one-component cone Γ with vertex at $0 \in \mathbb{R}^n$ such that for every $x_0 \in \partial\Omega$ there exist $r > 0$ and a rotation $\mathcal{R}$ about the origin such that

$$x + \mathcal{R}(\Gamma) \subseteq \Omega, \quad \forall\, x \in B(x_0, r) \cap \overline{\Omega} \tag{10.1}$$

is necessarily Lipschitz (the converse is also true). See Theorem 1.2.2.2 on p. 12 in [29] for a proof.

A different type of condition which characterizes Lipschitzianity has been recently discovered in [36]. This involves the notion of a transversal vector field to the boundary of a domain $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter which we now record. As a preamble, we remind the reader that $\partial^*\Omega$ denotes the reduced boundary of Ω and that $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff (outer-)measure in $\mathbb{R}^n$.

Definition 10.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of locally finite perimeter, with outward unit normal ν, and fix a point $x_0 \in \partial\Omega$. Then, it is said that Ω has a* `continuous transversal vector field near` x_0 *provided there exists a continuous vector field h which is uniformly (outwardly) transverse to $\partial\Omega$ near x_0, in the sense that there exist $r > 0$, $\kappa > 0$ so that $h : \partial\Omega \cap B(x_0, r) \to \mathbb{R}^n$ is continuous and*

$$\nu \cdot h \geq \kappa \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad B(x_0, r) \cap \partial^*\Omega. \tag{10.2}$$

Here is the statement of the result proved in [36] alluded to above.

Theorem 10.1. *Assume that Ω is a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter, and fix $x_0 \in \partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if it has a continuous transversal vector field near x_0 and there exists $r > 0$ such that*

$$\partial(\Omega \cap B(x_0, r)) = \partial(\overline{\Omega \cap B(x_0, r)}). \tag{10.3}$$

We momentarily digress for the purpose of discussing an elementary result of topological nature which is going to be used shortly.

Lemma 10.2. *Let E_1, E_2 be two subsets of $\mathbb{R}^n$ with the property that*

$$\big(\partial E_1 \setminus \partial(\overline{E_1})\big) \cap \overline{E_2} = \varnothing \quad \text{and} \quad \big(\partial E_2 \setminus \partial(\overline{E_2})\big) \cap \overline{E_1} = \varnothing. \tag{10.4}$$

Then

$$\partial(E_1 \cap E_2) = \partial(\overline{E_1 \cap E_2}). \tag{10.5}$$

Proof. Since $\partial(\overline{E}) \subseteq \partial E$ for any set $E \subseteq \mathbb{R}^n$, the right-to-left inclusion in (10.5) always holds, so there remains to show that, granted (10.4), one has

$$\partial(E_1 \cap E_2) \subseteq \partial(\overline{E_1 \cap E_2}). \tag{10.6}$$

To this end, recall that

$$\partial(A \cap B) \subseteq (\overline{A} \cap \partial B) \cup (\partial A \cap \overline{B}), \quad \forall\, A, B \subseteq \mathbb{R}^n, \tag{10.7}$$

which further implies

$$\partial(A \cap B) = \Big(\partial(A \cap B) \cap \overline{A} \cap \partial B\Big) \cup \Big(\partial(A \cap B) \cap \overline{B} \cap \partial A\Big). \tag{10.8}$$

From this and simple symmetry considerations we see that (10.6) will follow as soon as we check the validity of the inclusion

$$\partial(E_1 \cap E_2) \cap (\overline{E_1} \cap \partial E_2) \subseteq \partial(\overline{E_1 \cap E_2}). \tag{10.9}$$

To this end, we reason by contradiction and assume that there exist a point x and a number $r > 0$ satisfying

$$\begin{gathered} x \in \partial(E_1 \cap E_2), \quad x \in \partial E_2, \quad \text{and} \\ \text{either} \;\; B(x, r) \cap (\overline{E_1 \cap E_2}) = \varnothing, \;\; \text{or} \;\; B(x, r) \subseteq \overline{E_1 \cap E_2}. \end{gathered} \tag{10.10}$$

Note that if $B(x, r) \cap (\overline{E_1 \cap E_2}) = \varnothing$ then also $B(x, r) \cap (E_1 \cap E_2) = \varnothing$, contradicting the fact that $x \in \partial(E_1 \cap E_2)$. Thus, necessarily, $B(x, r) \subseteq \overline{E_1 \cap E_2}$. However, this entails

$$x \in (\overline{E_1 \cap E_2})^\circ \cap \partial E_2 \subseteq \overline{E_1} \cap (\overline{E_2})^\circ \cap \partial E_2 = \overline{E_1} \cap (\partial E_2 \setminus \partial(\overline{E_2})) = \varnothing, \tag{10.11}$$

by (10.4). This shows that the conditions listed in (10.10) are contradictory and, hence, proves (10.9). □

Definition 10.2. *A proper, nonempty open Ω subset of $\mathbb{R}^n$ is said to satisfy an* `exterior, uniform, continuously varying cone condition near a point` *$x_0 \in \partial\Omega$ provided there exist two numbers $r, b > 0$, an angle $\theta \in (0, \pi)$, and a function*

h defined on $B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is continuous at x_0 and such that

$$\Gamma_{\theta,b}(x, h(x)) \subseteq \mathbb{R}^n \setminus \Omega, \quad \forall\, x \in B(x_0, r) \cap \partial\Omega. \tag{10.12}$$

Also, a nonempty, open set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy a `global, exterior, uniform, continuously varying cone condition` *if Ω satisfies an interior uniform continuously varying cone condition near each point on $\partial\Omega$.*

Finally, define an `interior uniform continuously varying cone condition` *(near a boundary point, or globally) in an analogous manner, replacing $\mathbb{R}^n \setminus \Omega$ by Ω in* (10.12)*.*

The global, interior, uniform, continuously varying cone condition has earlier appeared in [64] where N.S. Nadirashvili has used it as the main background geometrical hypothesis for the class of domains in which he proves a uniqueness theorem for the oblique derivative boundary value problem (cf. [64, Theorem 1, p. 327]). We shall revisit the latter topic in § 4. For now, our goal is to establish the following proposition, refining a result of similar flavor proved in [36][1].

Proposition 10.3. *Assume that Ω is a proper, nonempty open subset of $\mathbb{R}^n$ and suppose that $x_0 \in \partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if Ω satisfies an exterior, uniform, continuously varying cone condition near x_0.*

Proof. In one direction, if Ω is a Lipschitz domain near x_0 then the existence of $r, b > 0$, $\theta \in (0, \pi)$ and a function $h : B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is actually constant and such that (10.12) holds, follows from Lemma 9.5. The crux of the matter is, of course, dealing with the converse implication. In doing so, we shall employ the

[1] In the process, we also use the opportunity to correct a minor gap in the treatment in [36].

notation introduced in Definition 10.2. We begin by observing that condition (10.12) forces $B(x_0,r)\cap\partial\Omega\subseteq\overline{[(\Omega^c)^\circ]}$. In concert with the readily verified formula $(\Omega^c)^\circ=(\overline{\Omega})^c$, this yields $B(x_0,r)\cap\partial\Omega\subseteq\overline{[(\overline{\Omega})^c]}$. Hence,

$$B(x_0,r)\cap\partial\Omega\subseteq\overline{\Omega}\cap\overline{[(\overline{\Omega})^c]}=\partial(\overline{\Omega})$$

and, further, $B(x_0,r)\cap\partial\Omega\subseteq B(x_0,r)\cap\partial(\overline{\Omega})$. Since the opposite inclusion is always true, we may ultimately deduce that

$$B(x_0,r)\cap\partial\Omega=B(x_0,r)\cap\partial(\overline{\Omega}). \tag{10.13}$$

As a consequence of (10.13), we obtain

$$\begin{aligned} B(x_0,r)\cap(\overline{\Omega})^\circ &= B(x_0,r)\cap\big(\overline{\Omega}\setminus\partial(\overline{\Omega})\big)=\big(B(x_0,r)\cap\overline{\Omega}\big)\setminus\big(B(x_0,r)\cap\partial(\overline{\Omega})\big)\\ &= \big(B(x_0,r)\cap\overline{\Omega}\big)\setminus\big(B(x_0,r)\cap\partial\Omega\big)=B(x_0,r)\cap\big(\overline{\Omega}\setminus\partial\Omega\big)\\ &= B(x_0,r)\cap\Omega, \end{aligned} \tag{10.14}$$

hence

$$\Omega\cap B(x_0,r)=(\overline{\Omega})^\circ\cap B(x_0,r). \tag{10.15}$$

Next, fix $b_0\in(0,b)$ along with $\varepsilon\in(0,1-\cos(\theta/2))$. Then there exists $\theta_0\in(0,\theta)$ with the property that

$$\cos(\theta_0/2)-\varepsilon>\cos(\theta/2)\quad\text{ and }\quad\frac{b_0}{\cos(\theta_0/2)}<b. \tag{10.16}$$

Next, with $\varepsilon>0$ as above, select $r_0\in(0,r)$ such that

$$|h(x)-h(x_0)|<\varepsilon\quad\text{ whenever }\; x\in B(x_0,r_0)\cap\partial\Omega. \tag{10.17}$$

That this is possible is ensured by the continuity of the function h at x_0. We then claim that

$$\Gamma_{\theta_0,b_0}(x,h(x_0)) \subseteq \Gamma_{\theta,b}(x,h(x)), \quad \forall\, x \in B(x_0,r_0) \cap \partial\Omega. \tag{10.18}$$

Indeed, if $x \in B(x_0,r_0) \cap \partial\Omega$ and $y \in \Gamma_{\theta_0,b_0}(x,h(x_0))$ then

$$\begin{aligned}
(y-x)\cdot h(x) &= (y-x)\cdot h(x_0) + (y-x)\cdot(h(x)-h(x_0)) \\
&> \cos(\theta_0/2)|y-x| - \varepsilon|y-x| = (\cos(\theta_0/2)-\varepsilon)|y-x| \\
&> \cos(\theta/2)|y-x|,
\end{aligned} \tag{10.19}$$

by the Cauchy-Schwarz inequality, the first inequality in (10.16) and condition (10.17). In addition, since $y \in \Gamma_{\theta_0,b_0}(x,h(x_0))$ forces $|y-x| < (\cos(\theta_0/2))^{-1}b_0$, it follows that

$$(y-x)\cdot h(x) \leq |y-x| < \frac{b_0}{\cos(\theta_0/2)} < b, \tag{10.20}$$

by the Cauchy-Schwarz inequality and the second inequality in (10.16). All together, this analysis proves (10.18). With this in hand, we deduce from (10.12) that

$$\Gamma_{\theta_0,b_0}(x,h(x_0)) \subseteq \mathbb{R}^n \setminus \Omega, \quad \forall\, x \in B(x_0,r_0) \cap \partial\Omega. \tag{10.21}$$

Moving on, consider the open, proper, subset of $\mathbb{R}^n$ given by

$$D := (\Omega^c)^\circ \cap B(x_0,r_0). \tag{10.22}$$

Since, by (10.12), $\Gamma_{\theta,b}(x_0,h(x_0)) \subseteq (\Omega^c)^\circ$, it follows that D is also nonempty. The first claim we make about the set D is that

$$\partial D = \partial(\overline{D}). \tag{10.23}$$

To justify this, observe that $D = (\overline{\Omega})^c \cap B(x_0,r_0)$ and note that since

$$\partial E \setminus \partial(\overline{E}) = \partial E \cap (\overline{E})^\circ, \quad \forall\, E \subseteq \mathbb{R}^n, \tag{10.24}$$

we have

$$\partial((\overline{\Omega})^c) \setminus \partial(\overline{(\overline{\Omega})^c}) = \partial(\overline{\Omega}) \cap \big(\overline{(\overline{\Omega})^c}\big)^\circ \;=\; \partial(\overline{\Omega}) \cap \big(\overline{((\overline{\Omega})^\circ)}\big)^c \qquad (10.25)$$

$$\subseteq\; \partial(\overline{\Omega}) \cap \big(\overline{\Omega}\big)^c = \varnothing.$$

Having established this, (10.23) follows from Lemma 10.2.

Going further, the second claim we make about the set D introduced in (10.22) is that

$$\partial D \subseteq \big(\partial\Omega \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0). \qquad (10.26)$$

To see this, with the help of (10.7) we write

$$\begin{aligned}\partial D \;&\subseteq\; \big(\partial\big((\Omega^c)^\circ\big) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0) = \big(\partial\big((\overline{\Omega})^c\big) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0)\\ &=\; \big(\partial(\overline{\Omega}) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0) = \big(\partial\Omega \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0), \qquad (10.27)\end{aligned}$$

where the last equality is a consequence of (10.13). This proves (10.26). Let us note here that, as a consequence of this, (10.21) and elementary geometrical considerations, we have

$$\eta := \min\big\{b_0,\, \cos(\theta_0/2)\, r_0/2\big\} \Longrightarrow \Gamma_{\theta_0,\eta}(x, h(x_0)) \subseteq D \qquad (10.28)$$

$$\forall\, x \in B(x_0, r_0/2) \cap \partial D.$$

The third claim we make about the set D from (10.22) is that

$$\mathcal{H}^{n-1}(\partial D) < +\infty. \qquad (10.29)$$

Of course, given (10.26), it suffices to show that there exists a finite constant $C = C(\theta, b) > 0$ with the property that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B(x_0, r_0)) \leq C r_0^{n-1}. \qquad (10.30)$$

With this goal in mind, recall first that, in general, $\mathcal{H}^{n-1}(E) \leq C_n \lim_{\delta \to 0^+} \mathcal{H}^{n-1}_\delta(E)$, where $\mathcal{H}^{n-1}_\delta(E)$ denotes the infimum of all sums $\sum_{B\in\mathcal{B}}(\text{radius}\, B)^{n-1}$, associated with all covers $\mathcal{B}$ of E with open balls B of radii $\leq \delta$. Next, abbreviate $\Gamma := \Gamma_{\theta_0,b_0}(0, h(x_0))$ so that (10.21) reads $x + \Gamma \subseteq \Omega^c$ for every $x \in B(x_0, r_0) \cap \partial\Omega$. Denote by L the one-dimensional space spanned by the vector $h(x_0)$ in $\mathbb{R}^n$. For some fixed $\lambda \in (0, 1)$, to be specified later, consider $\Gamma_\lambda \subseteq \Gamma$ to be the open, truncated, circular, one-component cone of aperture $\lambda\,\theta_0$ with vertex at $0 \in \mathbb{R}^n$ and having the same height b_0 and symmetry axis L as Γ. Elementary geometry gives

$$|x - y| < h,\ x \notin y + \Gamma,\ y \notin x + \Gamma \Longrightarrow |x - y| \leq \frac{\text{dist}\,(x + L\,,\, y + L)}{\sin(\theta_0/2)}. \tag{10.31}$$

In subsequent considerations, it can be assumed that r_0 is smaller than a fixed fraction of b_0. To fix ideas, suppose henceforth that $0 < r_0 \leq b_0/10$.

In order to continue, select a small number $\delta \in (0, r_0)$ and cover $\partial\Omega \cap B(x_0, r_0)$ by a family of balls $\{B(x_j, r_j)\}_{j\in J}$ with $x_j \in \partial\Omega$, $0 < r_j \leq \delta$, for each $j \in J$. By Vitali's lemma, there is no loss of generality in assuming that $\{B(x_j, r_j/5)\}_{j\in J}$ are mutually disjoint. Then it holds that $\mathcal{H}^{n-1}_\delta(\partial\Omega \cap B(x_0, r_0)) \leq C_n \sum_{j\in J} r_j^{n-1}$. Let π be a fixed $(n-1)$-plane perpendicular to the axis of Γ and denote by A_j the projection of $(x_j + \Gamma_\lambda) \cap B(x_j, r_j/5)$ onto π. Clearly, we have $\mathcal{H}^{n-1}(A_j) \approx r_j^{n-1}$, for every $j \in J$, and there exists an $(n-1)$-dimensional ball of radius $3r$ in π containing all A_j's.

We now claim that $\lambda > 0$ can be chosen sufficiently small as to ensure that the A_j's are mutually disjoint. Indeed, if $A_{j_1} \cap A_{j_2} \neq \varnothing$, for some $j_1, j_2 \in J$, then we have the inequality that $\text{dist}\,(x_{j_1} + L\,,\, x_{j_2} + L) \leq (r_{j_1} + r_{j_2})\,\sin(\lambda\,\theta_0/2)$. Also,

$|x_{j_1} - x_{j_2}| \geq (r_{j_1} + r_{j_2})/5$, as $B(x_{j_1}, r_{j_1}/5) \cap B(x_{j_2}, r_{j_2}/5) = \varnothing$. Note that

$$|x_{j_1} - x_{j_2}| \leq 4r < b_0.$$

Since also we have $\partial\Omega \ni x_{j_1} \notin x_{j_2} + \Gamma \subseteq (\Omega^c)^\circ$ plus a similar condition with the roles of j_1 and j_2 reversed, it follows from (10.31) that

$$(r_{j_1} + r_{j_2})/5 \leq (r_{j_1} + r_{j_2}) \sin(\lambda\,\theta_0/2)/\sin(\theta_0/2)$$

, or $\sin(\theta_0/2) < 5\,\sin(\lambda\,\theta_0/2)$. Taking $\lambda \in (0,1)$ sufficiently small, this leads to a contradiction. This finishes the proof of the claim that the A_j's are mutually disjoint if λ is small enough. Assuming that this is the case, we obtain

$$\sum_{j\in J} r_j^{n-1} \leq C \sum_{j\in J} \mathcal{H}^{n-1}(A_j) \leq C\mathcal{H}^{n-1}(\cup_{j\in J} A_j) \leq C r_0^{n-1}$$

, given the containment condition on the A_j's. As a consequence,

$$\mathcal{H}_\delta^{n-1}(\partial\Omega \cap B(x_0, r_0)) \leq C r_0^{n-1},$$

so by taking the supremum over $\delta > 0$ we arrive at $\mathcal{H}^{n-1}(\partial\Omega \cap B(x_0, r_0)) \leq C r_0^{n-1}$. This finishes the proof of (10.30) and, hence, (10.29) holds.

In summary, the above analysis shows that D is a proper, nonempty open subset of $\mathbb{R}^n$, of finite perimeter and such that (10.28) holds. Granted this, if follows from Lemma 8.1 that if ν_D is the geometric measure theoretic outer unit normal to D then

$$\nu_D(x) \in \Gamma_{\pi-\theta_0,\eta}(0, h(x_0)) \quad \text{for each } x \in B(x_0, r_0/2) \cap \partial^* D. \tag{10.32}$$

Hence, the vector $h(x_0) \in S^{n-1}$ is transversal to ∂D near x_0 in the precise sense that

$$\nu_D(x) \cdot h(x_0) \geq \cos((\pi - \theta_0)/2) > 0 \quad \text{for each } x \in B(x_0, r_0/2) \cap \partial^* D. \tag{10.33}$$

From (10.23) (cf. also Lemma 10.2) and (10.33) we deduce that D is a Lipschitz domain near x_0.

The end-game in the proof of the proposition is as follows. Since D is a Lipschitz domain near x_0, it follows that $(\bar{D})^c$ is also a Lipschitz domain near x_0. In turn, this and the fact that, thanks to (10.15), we have $\Omega \cap B(x_0, r_0/2) = (\bar{D})^c \cap B(x_0, r_0/2)$, we may finally conclude that Ω is a Lipschitz domain near the point x_0. $\square$

Proposition 10.4. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty open set which is not dense in $\mathbb{R}^n$, and suppose that $x_0 \in \partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if there exist two numbers $r, b > 0$, an angle $\theta \in (0, \pi)$, and a function h defined on $B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is continuous at x_0 and such that*

$$B(x_0, r) \cap \partial\Omega = B(x_0, r) \cap \partial(\overline{\Omega}) \quad \textit{and} \tag{10.34}$$

$$\Gamma_{\theta,b}(x, h(x)) \subseteq \Omega, \quad \forall\, x \in B(x_0, r) \cap \partial\Omega. \tag{10.35}$$

Proof. This follows from applying Proposition 10.3 to the open, nonempty, proper subset $(\overline{\Omega})^c$ of $\mathbb{R}^n$ and keeping in mind (9.7). $\square$

It is instructive to observe that there is a weaker version of Propositions 10.3-10.4 (same conclusion, yet stronger hypotheses) but whose proof makes no use of results or tools from geometric measure theory. This is presented next.

Proposition 10.5. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, proper, open set and that $x_* \in \partial\Omega$. Then Ω is Lipschitz near x_* if and only if there exist $b, r > 0$, $\theta \in (0, \pi)$ and a function h defined on $B(x_*, r) \cap \partial\Omega$ taking values in S^{n-1} which is continuous at x_* and with the property that*

$$\Gamma_{\theta,b}(x, h(x)) \subseteq \Omega \quad \textit{and} \quad \Gamma_{\theta,b}(x, -h(x)) \subseteq \mathbb{R}^n \setminus \Omega, \quad \forall\, x \in B(x_*, r) \cap \partial\Omega. \tag{10.36}$$

Proof. Assume first that the nonempty, proper, open set $\Omega \subsetneq \mathbb{R}^n$ and the point $x_* \in \partial\Omega$ are such that (10.36) holds. Thanks to the analysis in (10.16)-(10.18), there is no loss of generality in assuming that the function $h : B(x_*, r) \cap \partial\Omega \to S^{n-1}$ is constant, say $h(x) \equiv v \in S^{n-1}$ for each $x \in B(x_*, r) \cap \partial\Omega$. Furthermore, since for any rotation $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\mathcal{R}(\Gamma_{\theta,b}(x, \pm v)) = \Gamma_{\theta,b}(\mathcal{R}(x), \pm\mathcal{R}(v)), \tag{10.37}$$

there is no loss of generality in assuming that $v = \mathbf{e}_n$. Finally, performing a suitable translation, we can assume that $x_* = 0 \in \mathbb{R}^n$. Granted these, fix some small positive number c, say,

$$0 < c < \min\left\{b \cos(\theta/2)\,,\, \frac{r}{\sqrt{1 + (\cos(\theta/2))^2}}\right\}, \tag{10.38}$$

and consider the cylinder

$$\mathcal{C} := B_{n-1}\big(0', c\cos(\theta/2)\big) \times (-c, c) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n. \tag{10.39}$$

Then the top lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0, v) \subseteq \Omega$, whereas the bottom lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0, -v) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ = (\Omega^c)^\circ = (\overline{\Omega})^c$. We now make the claim that for every point x' in the ball $B_{n-1}(0', c\cos(\theta/2))$,

the interior of the line segment $L(x') := [(x', c), (x', -c)]$ intersects $\partial\Omega$. (10.40)

Indeed, if $x' \in B_{n-1}(0', c\cos(\theta/2))$ is such that the (relative) interior of $L(x')$ is disjoint from $\partial\Omega$, the fact that $\mathbb{R}^n = \Omega \cup \partial\Omega \cup (\overline{\Omega})^c$ with the three sets appearing in the right-hand side mutually disjoint, implies that Ω and $(\overline{\Omega})^c$ form an open cover of $L(x')$. Since $L(x') \cap \Omega$ is nonempty (as it contains (x', c)), $L(x') \cap (\overline{\Omega})^c$ is nonempty (as it contains $(x', -c)$), and $\Omega \cap (\overline{\Omega})^c = \varnothing$, this contradicts the fact that $L(x')$ is connected. This proves that there exists $x_0 \in L(x')$ with the property that $x_0 \in \partial\Omega$. There remains to observe that, necessarily, x_0 is different from the endpoints of $L(x')$ in order to conclude that this point actually belongs to the (relative) interior of $L(x')$. This finishes the proof of (10.40).

Our next claim is that, in fact (with $\#E$ denoting the cardinality of the set E)

$$\#(L(x') \cap \partial\Omega) = 1, \quad \forall\, x' \in B_{n-1}(0', c\,\cos(\theta/2)). \tag{10.41}$$

To justify this, let $x = (x', x_n) \in L(x') \cap \partial\Omega$. Then

$$|x| = \sqrt{|x'|^2 + x_n^2} \leq \sqrt{c^2(\cos(\theta/2))^2 + c^2} = c\sqrt{1 + (\cos(\theta/2))^2} < r, \tag{10.42}$$

so $x \in B(0, r) \cap \partial\Omega$. Consequently, from (10.36) and conventions,

$$\Gamma_{\theta,b}(x, \mathbf{e}_n) \subseteq \Omega \quad \text{ and } \quad \Gamma_{\theta,b}(x, -\mathbf{e}_n) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ. \tag{10.43}$$

In turn, this forces (with $\mathcal{I}(y, z)$ denoting the relatively open line segment with endpoints $y, z \in \mathbb{R}^n$)

$$\mathcal{I}(x, x + b\,\mathbf{e}_n) \subseteq \Omega \quad \text{ and } \quad \mathcal{I}(x, x - b\,\mathbf{e}_n) \subseteq (\overline{\Omega})^c \tag{10.44}$$

and, hence, $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n) \cap \partial\Omega = \{x\}$. With this in hand, (10.41) follows after noticing that the (relative) interior of $L(x')$ is contained in $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n)$ since, by design, $c < b\,\cos(\theta/2) < b$.

Having established (10.41), it is then possible to define a function

$$\varphi : B_{n-1}(0', c\,\cos(\theta/2)) \longrightarrow (-c, c) \tag{10.45}$$

in an unambiguous fashion by setting, for every $x' \in B_{n-1}(0', c\,\cos(\theta/2))$,

$$\varphi(x') := x_n \quad \text{if} \quad (x', x_n) \in L(x') \cap \partial\Omega. \tag{10.46}$$

Then, by design (recall (10.39)), we have

$$\mathcal{C} \cap \partial\Omega = \{x = (x', x_n) \in \mathcal{C} : x_n = \varphi(x')\}, \tag{10.47}$$

and we now proceed to show that φ defined in (10.45)-(10.46) is a Lipschitz function. Concretely, if we now select two arbitrary points $x', y' \in B_{n-1}(0', c\cos(\theta/2))$, then $(y', \varphi(y'))$ belongs to $\partial\Omega$, therefore $(y', \varphi(y')) \notin \Gamma_{\theta,b}((x', \varphi(x')), \pm\mathbf{e}_n)$. This implies

$$\begin{aligned} \pm((y', \varphi(y')) - (x', \varphi(x'))) \cdot \mathbf{e}_n &\leq \cos(\theta/2)\,|(y', \varphi(y')) - (x', \varphi(x'))| \\ &\leq \cos(\theta/2)\,|y' - x'|. \end{aligned} \tag{10.48}$$

Thus, ultimately, $|\varphi(y') - \varphi(x')| \leq \cos(\theta/2)\,|y' - x'|$, which shows that φ is a Lipschitz function, with Lipschitz constant $\leq \cos(\theta/2)$. Based on the classical result of E.J. McShane [60] and H. Whitney [88], the function (10.45) may be extended to the entire Euclidean space $\mathbb{R}^{n-1}$ to a Lipschitz function, with Lipschitz constant $\leq \cos(\theta/2)$.

Going further, since the cone condition (10.36) also entails that the point $x_0 \in \partial\Omega$ is the limit of points from $\Gamma_{\theta,b}(x_0, h(x_0)) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ$, we may conclude that $x_0 \in \overline{(\overline{\Omega})^c}$, i.e., $x_0 \notin (\overline{\Omega})^\circ$. With this and (10.47) in hand, we may then invoke Proposition 9.1 in order to conclude that Ω is a Lipschitz domain near 0.

Finally, the converse implication in the statement of the proposition is a direct consequence of Lemma 9.5. $\square$

Definition 10.3. *Call a set* $\Omega \subseteq \mathbb{R}^n$ `starlike with respect to` $x_0 \in \Omega$ *if* $\mathcal{I}(x, x_0) \subseteq \Omega$ *for all* $x \in \Omega$, *where* $\mathcal{I}(x, x_0)$ *denotes the open line segment in* $\mathbb{R}^n$ *with endpoints* x *and* x_0.

Also, call a set $\Omega \subseteq \mathbb{R}^n$ `starlike with respect to a ball` $B \subseteq \Omega$ *if*

$\mathcal{I}(x,y) \subseteq \Omega$ for all $x \in \Omega$ and $y \in B$ (that is, Ω is starlike with respect to any point in B).

Theorem 10.6. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Then Ω is a locally Lipschitz domain if and only if every $x_* \in \partial\Omega$ has an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ with the property that $\Omega \cap \mathcal{O}$ is starlike with respect to some ball.*

In particular, any bounded convex domain is Lipschitz.

Proof. Pick an arbitrary point $x_* \in \partial\Omega$ and let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open neighborhood of x_* with the property that $\Omega \cap \mathcal{O}$ is starlike with respect to a ball $B(x_0, r) \subseteq \Omega \cap \mathcal{O}$. For each $x \in \mathbb{R}^n \setminus \overline{B(x_0, r/2)}$, consider the circular cone with vertex at x and axis along $x_0 - x$ described as

$$C(x) := \Big\{ y \in \mathbb{R}^n : \sqrt{1 - \tfrac{r}{2|x-x_0|}}\,|y-x| < (y-x) \cdot \tfrac{x_0 - x}{|x_0 - x|} < \tfrac{|x_0-x|^2 - (r/2)^2}{|x_0 - x|} \Big\}. \tag{10.49}$$

Elementary geometry then shows that

$$C(x) \subseteq \bigcup_{y \in B(x_0, r/2)} \mathcal{I}(x,y) \subseteq \mathcal{O} \cap \Omega, \quad \forall\, x \in (\mathcal{O} \cap \Omega) \setminus \overline{B(x_0, r/2)}, \tag{10.50}$$

where the second inclusion is a consequence of the fact that $\mathcal{O} \cap \Omega$ is starlike with respect to $B(x_0, r/2)$. Then, for each $x \in \mathcal{O} \cap \partial\Omega$, there exists a sequence $\{x_j\}_{j \in \mathbb{N}}$ of points in $\mathcal{O} \cap \Omega$ such that $x_j \to x$ as $j \to +\infty$, hence

$$C(x) \subseteq \bigcup_{j \in \mathbb{N}} C(x_j). \tag{10.51}$$

In concert with (10.50), this implies that

$$C(x) \subseteq \mathcal{O} \cap \Omega, \quad \forall\, x \in \mathcal{O} \cap \partial\Omega. \tag{10.52}$$

Next, for each $b > 0$ and $x \in \mathbb{R}^n \setminus \overline{B(x_0, r/2)}$ denote by $\widetilde{C}_b(x)$ the cone with vertex at x, same aperture as $C(x)$, axis pointing in the opposite direction to that of $C(x)$, and height b. We then claim that there exist $b > 0$ and $\rho > 0$ with the property that

$$\widetilde{C}_b(x) \subseteq \mathbb{R}^n \setminus \Omega, \quad \forall\, x \in B(x_*, \rho) \cap \partial\Omega. \tag{10.53}$$

To justify this claim, note that since $\mathcal{O}$ is an open neighborhood of x_* it is possible to select $b, \rho > 0$ sufficiently small so that

$$\widetilde{C}_b(x) \subseteq \mathcal{O} \quad \forall\, x \in B(x_*, \rho). \tag{10.54}$$

Assuming that this is the case, the existence of a point $x \in B(x_*, \rho) \cap \partial\Omega$ for which there exists $\widehat{x} \in \widetilde{C}_b(x) \cap \Omega$ would entail, thanks to (10.54) and (10.50),

$$x \in C(\widehat{x}) \subseteq \mathcal{O} \cap \Omega, \tag{10.55}$$

which contradicts the fact that $x \in \partial\Omega$. This finishes the proof of the claim made in (10.53).

Having established (10.52) and (10.53), Proposition 10.5 applies and yields that Ω is Lipschitz near x_*. Since $x_* \in \partial\Omega$ has been arbitrarily chosen we may therefore conclude that Ω is locally Lipschitz. This establishes one of the implications in the equivalence formulated in the statement of the theorem.

In the opposite direction, observe that if $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $M > 0$ and if $x_0' \in \mathbb{R}^{n-1}$ and $t > 0$ are given, then

$$\begin{array}{l}\text{the open segment with endpoints } (x_0', t + \varphi(x_0')) \text{ and } (x', \varphi(x')) \text{ belongs} \\ \text{to the (open) upper-graph of } \varphi \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ satisfies } |x'| < t/M.\end{array} \tag{10.56}$$

Then the desired conclusion (i.e., that Ω is locally starlike in the sense explained in the statement of the theorem) follows from this and (9.1). $\square$

Chapter 11

Characterizing Lyapunov Domains in Terms of Pseudo-Balls

This chapter contains the main result in this paper of geometrical flavor, namely the geometric characterization of Lyapunov domains in terms of a uniform, two-sided pseudo-ball condition. To set the stage, we first make the following definition.

Definition 11.1. *Let E be an arbitrary, proper, nonempty, subset of $\mathbb{R}^n$.*

(i) The set E is said to satisfy an `interior pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.10) *provided there exist a, $b > 0$ and $h \in S^{n-1}$ such that $\mathscr{G}^{\omega}_{a,b}(x_0, h) \subseteq E$.*

(ii) The set E is said to satisfy an `exterior pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.10) *provided $E^c := \mathbb{R}^n \setminus E$ satisfies an interior pseudo-ball condition at the point x_0 with shape function ω.*

(iii) The set E is said to satisfy a `two-sided pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.10) *provided E satisfies both an interior and an exterior pseudo-ball condition at $x_0 \in \partial E$ with shape function ω.*

(iv) The set E is said to satisfy a `uniform hour-glass condition near` *$x_0 \in \partial E$*

with shape function ω as in (1.10) *provided there exists $r > 0$ such that E satisfies a two-sided pseudo-ball condition at each point $x \in B(x_0, r) \cap \partial E$ with shape function ω and truncation height independent of x.*

(v) Finally, the set E is said to satisfy a `uniform hour-glass condition` *with shape function ω as in* (1.10) *provided both E and E^c satisfy a pseudo-ball condition at each point $x \in \partial E$ with shape function ω and height independent of x.*

While Definition 11.1 only requires that ω is as in (1.10), for the rest of this chapter we will also assume that ω satisfies (1.12), i.e., ω is as in (7.11).

That the terminology "hour-glass condition" employed above is justified is made transparent in the lemma below.

Lemma 11.1. *Let E be a subset of $\mathbb{R}^n$ which satisfies a two-sided pseudo-ball condition at a point $x_0 \in \partial E$ with shape function ω as in* (7.11)*. That is, there exist a, $b > 0$ and $h_\pm \in S^{n-1}$ such that $\mathscr{G}^\omega_{a,b}(x_0, h_+) \subseteq E$ and $\mathscr{G}^\omega_{a,b}(x_0, h_-) \subseteq E^c := \mathbb{R}^n \setminus E$. Then necessarily $h_+ = -h_-$.*

Proof. This is an immediate consequence of Corollary 7.4. $\square$

Remarkably, if $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition then the function $h : \partial E \to S^{n-1}$, assigning to each boundary point $x \in \partial E$ the direction $h(x) \in S^{n-1}$ of the pseudo-ball with apex at x contained in E, turns out to be continuous. A precise, local version of this result is recorded next.

Lemma 11.2. *Assume that the set $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition near $x_* \in \partial E$ with shape function ω as in* (7.11)*, height $b > 0$ and aperture $a > 0$.*

Let $\varepsilon = \varepsilon(\omega, \eta, R, a, b) > 0$ be as in Lemma 7.3 and define

$$\widehat{\omega} : [0,1] \to \big[0, \tfrac{\varepsilon R}{2}\big], \quad \widehat{\omega}(t) := \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)t\big)t, \quad \forall\, t \in [0,1]. \tag{11.1}$$

Since $\widehat{\omega}$ is continuous, increasing and bijective, it is meaningful to consider its inverse, i.e., the function

$$\widetilde{\omega} : \big[0, \tfrac{\varepsilon R}{2}\big] \to [0,1], \quad \widetilde{\omega}(t) := \widehat{\omega}^{\,-1}(t), \quad \forall\, t \in \big[0, \tfrac{\varepsilon R}{2}\big], \tag{11.2}$$

which is also continuous and increasing.

Then there exists a number $r > 0$ such that the function $h : B(x_*, r) \cap \partial E \to S^{n-1}$, defined at each point $x \in B(x_*, r) \cap \partial E$ by the demand that $h(x)$ is the unique vector in S^{n-1} with the property that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, is well-defined and continuous. In fact, with $\widetilde{\omega}$ as in (11.2), one has

$$h \in \mathscr{C}^{\,\widetilde{\omega}}\big(B(x_*, r) \cap \partial E\,,\, S^{n-1}\big). \tag{11.3}$$

Proof. Let $r > 0$, ω as in (7.11), and $a, b > 0$ be such that E satisfies a two-sided pseudo-ball condition at each point $x \in B(x_*, r) \cap \partial E$ with shape function ω, height b and aperture a. The fact that for each $x \in B(x_*, r) \cap \partial E$ there exists a unique vector $h(x) \in S^{n-1}$, which is unequivocally determined by the demand that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, follows from our assumption on E and Lemma 11.1. Consequently, we also have $\mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \mathbb{R}^n \setminus E$.

We are left with proving that the mapping $B(x_*, r) \cap \partial E \ni x \mapsto h(x) \in S^{n-1}$ is continuous and, in the process, estimate its modulus of continuity. With this goal in mind, pick two arbitrary points $x_0, x_1 \in B(x_*, r) \cap \partial E$. We then have $\mathscr{G}^{\omega}_{a,b}(x_0, h(x_0)) \cap \mathscr{G}^{\omega}_{a,b}(x_1, -h(x_1)) = \varnothing$ since the former set is contained in E and the latter set is contained in $\mathbb{R}^n \setminus E$. In turn, from this, Lemma 7.3, and (11.1) we infer that

$$|x_0 - x_1| \geq \tfrac{\varepsilon}{2}\, \omega^{-1}\Big(\omega(R)\tfrac{|h(x_0)-h(x_1)|}{2}\Big)\Big|\tfrac{h(x_0)-h(x_1)}{2}\Big| = \widehat{\omega}\Big(\tfrac{|h(x_0)-h(x_1)|}{2}\Big). \tag{11.4}$$

As a consequence, if $0 < r < \frac{\varepsilon R}{4}$ to begin with, we obtain from (11.4) and (11.2) that

$$|h(x_0) - h(x_1)| \leq 2\,\widetilde{\omega}(|x_0 - x_1|), \quad \forall\, x_0, x_1 \in B(x_*, r) \cap \partial E. \tag{11.5}$$

This shows that $h \in \mathscr{C}^{\widetilde{\omega}}(B(x_*, r) \cap \partial E, S^{n-1})$, as desired. $\square$

We are now in a position to formulate the main result in this chapter.

Theorem 11.3. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$ and assume that ω is as in (7.11) and $x_0 \in \partial\Omega$. Then Ω satisfies a uniform hour-glass condition with shape function ω near x_0 if and only if Ω is of class $\mathscr{C}^{1,\omega}$ near x_0.*

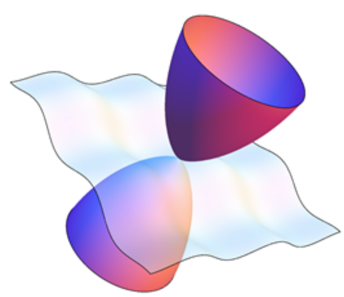

Figure 9. *A tilted hour-glass shape.*

Let us momentarily pause to record an immediate consequence of Theorem 11.3 which is particularly useful in applications.

Corollary 11.4. *Given ω as in (7.11), an open proper nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,\omega}$ if and only if Ω satisfies a uniform hour-glass condition with shape function ω.*

As a corollary, an open proper nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.

Proof. The first claim in the statement is a direct consequence of Theorem 11.3, while the last claim follows from the first with the help of part (iii) in Lemma 7.1. $\square$

One useful ingredient in the proof of Theorem 11.3, of independent interest, is the differentiability criterion of geometrical nature presented in the proposition below.

Proposition 11.5. *Assume that $U \subseteq \mathbb{R}^{n-1}$ is an arbitrary set, and that $x_* \in U^\circ$. Given a function $f : U \to \mathbb{R}$, denote by G_f the graph of f, i.e.,*

$$G_f := \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n.$$

Then f is differentiable at the point x_* if and only if f is continuous at x_* and there exists a non-horizontal vector $N \in \mathbb{R}^n$ (i.e., satisfying $N \cdot \mathbf{e}_n \neq 0$) with the following significance. For every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ with the property that $G_f \cap B((x_*, f(x_*)), \delta)$ lies in between the cones $\Gamma_{\theta,\delta}((x_*, f(x_*)), N)$ and $\Gamma_{\theta,\delta}((x_*, f(x_*)), -N)$, i.e.,

$$\begin{aligned} G_f \cap B((x_*, f(x_*)), \delta) \\ \subseteq \mathbb{R}^n \setminus \Big[\Gamma_{\theta,\delta}((x_*, f(x_*)), N) \cup \Gamma_{\theta,\delta}((x_*, f(x_*)), -N)\Big]. \end{aligned} \tag{11.6}$$

If this happens, then necessarily N is a scalar multiple of $(\nabla f(x_*), -1) \in \mathbb{R}^n$.

Proof. Assume that f is differentiable at x_*. Then f is continuous at x_*. To proceed, take

$$N := \frac{(\nabla f(x_*), -1)}{\sqrt{1 + |\nabla f(x_*)|^2}} \in \mathbb{R}^n. \tag{11.7}$$

Clearly, $|N| = 1$ and $N \cdot \mathbf{e}_n = -(1 + |\nabla f(x_*)|^2)^{-1/2} \neq 0$, so N is non-horizontal. Then, given $\theta \in (0, \pi)$, the fact that f is differentiable at x_* implies that there exists

$\delta > 0$ for which

$$|f(x) - f(x_*) - (\nabla f(x_*)) \cdot (x - x_*)| < \cos(\theta/2)|x - x_*| \quad \forall\, x \in B(x_*, \delta) \cap U. \quad (11.8)$$

For any $x \in B(x_*, \delta) \cap U$ we may then estimate

$$\begin{aligned}
\Big|\big((x, f(x)) - (x_*, f(x_*))\big) \cdot N\Big| &= \frac{|(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)|}{\sqrt{1 + |\nabla f(x_*)|^2}} \\
&\leq |(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)| < \cos(\theta/2)|x - x_*| \\
&< \cos(\theta/2)|(x, f(x)) - (x_*, f(x_*))|, \qquad (11.9)
\end{aligned}$$

which (recall that $|N| = 1$) shows that

$$x \in B(x_*, \delta) \cap U \Longrightarrow (x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N). \qquad (11.10)$$

Upon observing that any point in $G_f \cap B((x_*, f(x_*)), \delta)$ is of the form $(x, f(x))$ for some point $x \in B(x_*, \delta) \cap U$, based on (11.10) we may conclude that (11.6) holds.

For the converse implication, suppose that f is continuous at x_* and assume that there exists a non-horizontal vector $N \in \mathbb{R}^n$ with the property that for every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ such that (11.6) holds. By dividing N by the nonzero number $-N \cdot \mathbf{e}_n$, we may assume that the n-th component of N is -1 to begin with. That is, $N = (N', -1)$ for some $N' \in \mathbb{R}^{n-1}$.

Fix an arbitrary number $\varepsilon \in (0, 1/2)$ and pick an angle $\theta \in (0, \pi)$ sufficiently close to π so that $0 < \cos(\theta/2) < \varepsilon/\sqrt{1 + |N'|^2}$. Then, by assumption, there exists $\delta_0 > 0$ with the property that if $x \in U$ is such that $|(x, f(x)) - (x_*, f(x_*))| < \delta_0$ then $(x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N)$, i.e.,

$$\begin{aligned}
\Big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\Big| &\leq \cos(\theta/2)|(N', -1)||(x, f(x)) - (x_*, f(x_*))| \\
&\leq \varepsilon\sqrt{|x - x_*|^2 + (f(x) - f(x_*))^2} \leq \varepsilon\big[|x - x_*| + |f(x) - f(x_*)|\big]. \qquad (11.11)
\end{aligned}$$

In turn, this forces (recall that $0 < \varepsilon < \frac{1}{2}$)

$$\begin{aligned}
|f(x) - f(x_*)| &\leq \Big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\Big| + |(x - x_*) \cdot N'| \\
&\leq \varepsilon\Big[|x - x_*| + |f(x) - f(x_*)|\Big] + |x - x_*||N'| \\
&\leq (\tfrac{1}{2} + |N'|)|x - x_*| + \tfrac{1}{2}|f(x) - f(x_*)|. \qquad (11.12)
\end{aligned}$$

Absorbing the last term above in the left-most side of (11.12) yields

$$\tfrac{1}{2}|f(x) - f(x_*)| \leq (\tfrac{1}{2} + |N'|)|x - x_*|. \qquad (11.13)$$

We have therefore proved that there exists $\delta_0 > 0$ for which

$$\begin{aligned}
&x \in U \ \text{ and } \ |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \\
&\quad \Longrightarrow |f(x) - f(x_*)| \leq (1 + 2|N'|)|x - x_*|.
\end{aligned} \qquad (11.14)$$

Returning with this back in (11.11) then yields

$$\begin{aligned}
&x \in U \ \text{ and } \ |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \Longrightarrow \\
&\quad \Big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\Big| \leq 2\varepsilon(1 + |N'|)|x - x_*|.
\end{aligned} \qquad (11.15)$$

Since we are assuming that f is continuous at the point x_*, it follows that there exists $\delta_1 > 0$ with the property that

$$x \in U \ \text{ and } \ |x - x_*| < \delta_1 \Longrightarrow |f(x) - f(x_*)| < \delta_0/\sqrt{2}. \qquad (11.16)$$

Introducing $\delta := \min\{\delta_1, \delta_0/\sqrt{2}\}$, implication (11.16) therefore guarantees that

$$x \in U \ \text{ and } \ |x - x_*| < \delta \Longrightarrow |(x, f(x)) - (x_*, f(x_*))| < \delta_0. \qquad (11.17)$$

Consequently, from this and (11.15) we deduce that

$$x \in B(x_*, \delta) \cap U \Longrightarrow \left|\frac{((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)}{|x - x_*|}\right| \leq 2\varepsilon(1 + |N'|). \qquad (11.18)$$

Since $\varepsilon \in (0, 1/2)$ was arbitrary, this translates into saying that

$$\lim_{x \to x_*,\, x \in U} \frac{f(x) - f(x_*) - N' \cdot (x - x_*)}{|x - x_*|} = 0. \qquad (11.19)$$

This proves that f is differentiable at x_* and, in fact, $\nabla f(x_*) = N'$. Hence, in particular, N is a scalar multiple of $(N', -1) = (\nabla f(x_*), -1)$. The proof of the proposition is therefore finished. $\square$

We are now ready to discuss the

Proof of Theorem 11.3. In a first stage, assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which satisfies a uniform hour-glass condition with shape function ω (as in (7.11)) near $x_* \in \partial\Omega$. In other words, there exist $b > 0$ and $r_* > 0$, along with a function $h : B(x_*, r_*) \cap \partial\Omega \to S^{n-1}$ such that

$$\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq \Omega \ \text{ and } \ \mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \Omega^c \ \text{ for every } \ x \in B(x_*, r_*) \cap \partial\Omega. \tag{11.20}$$

Note that the uniform hour-glass condition, originally introduced in part (iv) of Definition 11.1, may be written as above thanks to Corollary 7.4. Going further, Lemma 11.2 then guarantees (by eventually decreasing $r_* > 0$ if necessary) that the function $h : B(x_*, r_*) \cap \partial E \to S^{n-1}$ belongs to $\mathscr{C}^{\widetilde{\omega}}$, where $\widetilde{\omega}$ is as in (11.2). Hence, in particular, h is continuous at x_*. Having established this, from part (v) of Lemma 7.1 and Proposition 10.3 we then deduce that Ω is a Lipschitz domain near x_*. Hence, there exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_*, a choice of the unit normal N to H, a Lipschitz function $\varphi : H \to \mathbb{R}$ and a cylinder $\mathcal{C}_{r,c}$ such that (9.1)-(9.2) hold. Without loss of generality we may assume that x_* is the origin in $\mathbb{R}^n$, that H is the canonical horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ and that $N = \mathbf{e}_n$. In this setting, our goal is to show that

$$\text{the Lipschitz function } \ \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \ \text{ is of class } \mathscr{C}^{1,\omega} \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{11.21}$$

As a preamble, we shall show that

$$h(x) \cdot \mathbf{e}_n \neq 0 \quad \text{for every} \quad x \in B(0, r_*) \cap \partial\Omega. \tag{11.22}$$

To prove (11.22), assume that there exists $x_0 \in B(0, r_*) \cap \partial\Omega$ with the property that $h(x_0) \cdot \mathbf{e}_n = 0$, with the goal of deriving a contradiction. Then, on the one hand, (11.20) gives that $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \subseteq \Omega^c$, whereas Lemma 9.5 guarantees that $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \subseteq \Omega$ if $\theta_0 := 2 \arctan\left(\frac{1}{M}\right)$ and $b_0 > 0$ is sufficiently small, where M is the Lipschitz constant of the function φ. Given the locations of the aforementioned pseudo-ball and cone, the desired contradiction will follow as soon as we show that

$$\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \cap \Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \neq \varnothing. \tag{11.23}$$

To this end, it suffices to look at the cross-section of $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0))$ and $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n)$ with the two-dimensional plane π spanned by the orthogonal unit vectors $h(x_0)$ and $\mathbf{e}_n$. To fix ideas, choose a system of coordinates in π so that $\mathbf{e}_n$ is vertical and $-h(x_0)$ is horizontal, both pointing in the positive directions of these respective axes. In such a setting, it follows that there exists $m \in (0, +\infty)$ with the property that the cross-section of the truncated cone contains all points (x, y) with coordinates satisfying $y > mx$ for $x > 0$ sufficiently small. On the other hand, the portion of the boundary of the cross-section of the pseudo-ball lying in the first quadrant near the origin is described by the equation $\sqrt{x^2 + y^2}\, \omega(\sqrt{x^2 + y^2}) = x$. Hence, $\omega\big(x\sqrt{1 + (y/x)^2}\big) = 1/\sqrt{1 + (y/x)^2}$ and, given that $\omega(t) \searrow 0$ as $t \searrow 0$, this forces $y/x \to +\infty$ as $x \searrow 0$. From this, the desired conclusion follows, completing the proof of (11.22).

Moving on, based on (11.22), the fact that φ is continuous, part (v) of Lemma 7.1, and the geometric differentiability criterion presented in Proposition 11.5, we deduce that φ is differentiable at each point near $0' \in \mathbb{R}^{n-1}$ and, in addition,

$$h(x', \varphi(x')) \text{ is parallel to } (\nabla\varphi(x'), -1) \text{ for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{11.24}$$

We now make the claim that for each x' near $0' \in \mathbb{R}^{n-1}$ the vector $(\nabla\varphi(x'), -1)$ points away from Ω, in the sense that

$$(x', \varphi(x')) - t(\nabla\varphi(x'), -1) \in \Omega \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1} \text{ if } t > 0 \text{ is small.} \tag{11.25}$$

This amounts to checking that if x' is near $0' \in \mathbb{R}^{n-1}$ and if $t > 0$ is small then we have that $\varphi\big(x' - t\nabla\varphi(x')\big) < \varphi(x') + t$ which, in turn, follows by observing that (recall that φ is differentiable at points near $0'$)

$$\lim_{t \to 0^+} \frac{\varphi\big(x' - t\nabla\varphi(x')\big) - \varphi(x')}{t} = \frac{d}{dt}\Big[\varphi\big(x' - t\nabla\varphi(x')\big)\Big]\Big|_{t=0} = -|\nabla\varphi(x')|^2 < 1. \tag{11.26}$$

Thus (11.25) holds and, when considered together with the fact that $-h(x', \varphi(x'))$ is a unit vector which also points away from Ω (recall that this is the axis of the pseudo-ball with apex at $(x', \varphi(x'))$ which is contained in Ω^c) ultimately gives that

$$h(x', \varphi(x')) = \frac{(-\nabla\varphi(x'), 1)}{\sqrt{1 + |\nabla\varphi(x')|^2}} \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{11.27}$$

Note that since $\mathbb{R}^{n-1} \ni x' \mapsto (x', \varphi(x')) \in \partial\Omega$ is Lipschitz, and since $h \in \mathscr{C}^{\widetilde{\omega}}$ it follows that the mapping $x' \mapsto h(x', \varphi(x'))$ defined for x' near $0' \in \mathbb{R}^{n-1}$ belongs to $\mathscr{C}^{\widetilde{\omega}}$ as well. Moreover, (11.27) also shows that $h_n(x', \varphi(x')) \geq (1 + M^2)^{-\frac{1}{2}}$, where $M > 0$ is the Lipschitz constant of φ, and

$$\partial_j\varphi(x') = -\frac{h_j(x', \varphi(x'))}{h_n(x', \varphi(x'))}, \quad j = 1, ..., n-1, \tag{11.28}$$

granted that x' is near $0' \in \mathbb{R}^{n-1}$. Based on this it follows that $\nabla\varphi$ is of class $\mathscr{C}^{\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$, where $\widetilde{\omega}$ is as in (11.2). Thus, φ is of class $\mathscr{C}^{1,\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$. While this is a step in the right direction, more work is required in order to justify the stronger claim made in (11.21).

We wish to show that there exists $C > 0$ such that

$$|\nabla\varphi(x_0') - \nabla\varphi(x_1')| \leq C\omega(|x_0' - x_1'|)$$
$$\text{whenever } x_0' \text{ and } x_1' \text{ are near } 0' \in \mathbb{R}^{n-1}. \tag{11.29}$$

Thanks to Lemma 9.6 we may, without loss of generality, assume that $(x_0', \varphi(x_0')) = (0', 0)$ and that $\nabla\varphi(x_0') = 0'$. As such, matters are reduced to proving that

$$|\nabla\varphi(x_1')| \leq C\,\omega(|x_1'|) \quad \text{for } x_1' \text{ near } 0'. \tag{11.30}$$

Since this is trivially true when $|\nabla\varphi(x_1')| = 0$, it suffices to focus on the case when $|\nabla\varphi(x_1')| \neq 0$. In this scenario, define

$$x_2' := x_1' + |x_1'|\tfrac{\nabla\varphi(x_1')}{|\nabla\varphi(x_1')|} \tag{11.31}$$

and note that, by the triangle inequality, $|x_2'| \leq 2|x_1'|$. As the point $(x_2', \varphi(x_2'))$ lies on $\partial\Omega$, it does not belong to $\mathscr{G}^{\omega}_{a,b}((x_1', \varphi(x_1')), \pm h((x_1', \varphi(x_1'))))$. As a consequence, we either have

$$\begin{aligned} |(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\,\omega\big(|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\big) \\ \geq |h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|, \end{aligned} \tag{11.32}$$

or

$$|h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))| \geq b. \tag{11.33}$$

However, given that φ is Lipschitz, the latter eventuality never materializes if we choose x_1' sufficiently close to $0'$. Note that (11.31) forces

$$|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))| \leq \sqrt{1+M^2}|x_1'|$$

where $M > 0$ is the Lipschitz constant of φ. Since ω is increasing and satisfies the condition recorded in the last line of (7.11), we may write

$$\omega\big(|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\big) \leq C\,\omega\big(\sqrt{1+M^2}|x_1'|\big) \leq C\,\eta(\sqrt{1+M^2})\omega(|x_1'|), \quad (11.34)$$

for x_1' near $0'$ and x_2' as in (11.31). The bottom line of this portion of our analysis is that for some finite constant $C > 0$

$$\big|h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))\big| \leq C\,\omega(|x_1'|)$$
$$\text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (11.31).} \quad (11.35)$$

In view of (11.27) and the fact that φ is Lipschitz, we obtain from (11.35) that

$$\big|-\nabla\varphi(x_1') \cdot (x_2' - x_1') + \varphi(x_2') - \varphi(x_1')\big| \leq C|x_1'|\,\omega(|x_1'|)$$
$$\text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (11.31).} \quad (11.36)$$

This estimate further entails $|x_1'||\nabla\varphi(x_1')| \leq C|x_1'|\,\omega(|x_1'|) + C|\varphi(x_1')| + C|\varphi(x_2')|$ for some $C > 0$ independent of x_1' near $0'$ (again, x_2' as in (11.31)). Let us now examine $|\varphi(x_1')|$. Given that the point $(x_1', \varphi(x_1'))$ lies on the boundary of Ω, it does not belong to $\mathscr{G}^{\omega}_{a,b}(0, \pm\mathbf{e}_n)$. Much as before, this necessarily implies $|(x_1', \varphi(x_1'))|\,\omega\big(|(x_1', \varphi(x_1'))|\big) \geq |\varphi(x_1')|$. Since it is assumed that $\varphi(0') = 0$ we further deduce that

$$\big|(x_1', \varphi(x_1'))\big| = (|x_1'|^2 + (\varphi(x_1') - \varphi(0'))^2)^{\frac{1}{2}} \leq C|x_1'|, \quad (11.37)$$

by the Lipschitzianity of φ. Hence, ultimately, $|\varphi(x_1')| \leq C|x_1'|\,\omega(|x_1'|)$, by arguing as before. Likewise, $|\varphi(x_2')| \leq C|x_2'|\,\omega(|x_2'|)$ and since $|x_2'| \leq 2|x_1'|$, we see that

$$|\varphi(x_2')| \leq C|x_1'|\,\omega(|x_1'|).$$

All in all, the above reasoning gives

$$|x_1'||\nabla\varphi(x_1')| \leq C|x_1'|\,\omega(|x_1'|) + C|\varphi(x_1')| + C|\varphi(x_2')| \leq C|x_1'|\,\omega(|x_1'|).$$

Dividing the most extreme sides of this inequality by $|x_1'|$ then yields

$$|\nabla\varphi(x_1')| \leq C\,\omega(|x_1'|),$$

as desired. This concludes the proof of (11.21) and, hence, Ω is of class $\mathscr{C}^{1,\omega}$ near x_*.

Consider now the scenario when the proper, open nonempty set $\Omega \subseteq \mathbb{R}^n$ is of class $\mathscr{C}^{1,\omega}$ near some boundary point $x_* \in \partial\Omega$, where ω is as in (7.11). In particular, $\omega : [0,R] \to [0,+\infty)$ is continuous, strictly increasing and such that $\omega(0) = 0$. The goal is to show that Ω satisfies a uniform hour-glass condition near x_* with shape function ω. To this end, based on Definition 9.1 and Lemma 9.6, there is no loss of generality in assuming that x_* is the origin in $\mathbb{R}^n$ and that if $(\mathcal{C}_{r,c}, \varphi)$ is the local chart near $0 \in \mathbb{R}^n$ then

$$\begin{array}{c}\text{the symmetry axis of the cylinder } \mathcal{C}_{r,c} \text{ is in the vertical direction } \mathbf{e}_n, \\ \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \ \text{ is of class } \ \mathscr{C}^{1,\omega}, \quad \varphi(0') = 0, \quad \text{and} \quad \nabla\varphi(0') = 0'.\end{array} \tag{11.38}$$

Fix a constant $C \in (0,+\infty)$ with the property that

$$C \geq \sup_{\substack{x',y' \in \mathbb{R}^{n-1} \\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{\omega(|x'-y'|)}. \tag{11.39}$$

The job at hand is to determine $b > 0$, depending only on r, c and φ, with the property that

$$\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n) \subseteq \mathcal{C}_{r,c} \cap (\text{upper-graph of } \varphi), \tag{11.40}$$

$$\mathscr{G}^{\omega}_{a,b}(0, -\mathbf{e}_n) \subseteq \mathcal{C}_{r,c} \cap (\text{lower-graph of } \varphi). \tag{11.41}$$

Recall (7.5). Given that the mapping $t \mapsto t\,\omega(t)$ is increasing, it follows that $t_b \searrow 0$ as $b \searrow 0$. Consequently, we may select

$$b \in (0, R\,\omega(R)) \;\text{ small enough so that }\; t_{b/C} < \min\{r, c\}. \tag{11.42}$$

By part (i) in Lemma 7.1 such a choice ensures that $\mathscr{G}^{\omega}_{a,b}(0, \pm\mathbf{e}_n) \subseteq B(0, t_b) \subseteq \mathcal{C}_{r,c}$. Pick now an arbitrary point $x = (x', x_n) \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$. Then, on the one hand, we have $C|x|\,\omega(|x|) < x_n < b$. On the other hand, (11.38) and the Mean Value Theorem ensure the existence of some $\theta = \theta(x') \in (0, 1)$ with the property that $\varphi(x') = x' \cdot (\nabla\varphi(\theta x') - \nabla\varphi(0'))$. This and the fact that φ is of class $\mathscr{C}^{1,\omega}$ then allow us to estimate $\varphi(x') \leq |x'||\nabla\varphi(\theta x') - \nabla\varphi(0')| \leq C|x'|\,\omega(|x'|) \leq C|x|\,\omega(|x|) < x_n$. This estimate shows that the point x belongs to the upper graph of the function φ. In summary, this discussion proves that (11.40) holds in the current setting. The same type of analysis as above (this time, writing

$$\varphi(x') \geq -C|x'|\,\omega(|x'|) \geq -C|x|\,\omega(|x|) > x_n),$$

shows that (11.41) also holds under these conditions. All in all, Ω satisfies a two-sided pseudo-ball condition at 0 with shape function ω, aperture C and height depending only on the $\mathscr{C}^{1,\omega}$ nature of Ω. This, of course, suffices to complete the proof of the theorem. $\square$

Chapter 12

A Historical Perspective on the Hopf-Oleinik Boundary Point Principle

The question of how the geometric properties of the boundary of a domain influence the behavior of a solution to a second-order elliptic equation is of fundamental importance and has attracted an enormous amount of attention. A significant topic, with distinguished pedigree, belonging to this line of research is the understanding of the sign of oblique directional derivatives of such a solution at boundary points. A celebrated result in this regard, known as the "Boundary Point Principle", states that an oblique directional derivative of a nonconstant $\mathscr{C}^2$ solution to a second-order, uniformly elliptic operator L in non-divergence form[1] with bounded coefficients, at an extremal point located on the boundary of the underlying domain $\Omega \subseteq \mathbb{R}^n$ is necessarily nonzero provided the domain is sufficiently regular at that point. Part of the importance of this result stems from its role in the development of the Strong

[1]As is well-known, the Boundary Point Principle fails in the class of divergence form second-order uniformly elliptic operators with bounded coefficients, even when these coefficients are continuous at the boundary point (cf. [25, p. 169], [76, p. 39], [26, Problem 3.9, pp. 49-50], [65]), though does hold if the coefficients are Hölder continuous at the boundary point – cf. [22].

Maximum Principle[2], as well as its applications to the issue of regularity near the boundary and uniqueness for a number of basic boundary value problems (such as Neumann, Robin, and mixed).

In the (by now) familiar version in which the regularity demand on the domain in question amounts to an interior ball condition, and when the second-order, non-divergence form, differential operator is uniformly elliptic and has bounded coefficients, this principle is due to E. Hopf and O.A. Oleinik who have done basic work on this topic in the early 1950's. However, the history of this problem is surprisingly rich, stretching back for more than a century and involving many contributors. Since the narrative of this endeavor does not appear to be well-known[3], below we attempt a brief survey of some of the main stages in the development of this topic.

Special cases of the Boundary Point Principle have been known for a long time since this contains, in particular, the fact that Green's function associated with a uniformly elliptic operator L in a domain Ω has a positive conormal derivative at boundary points provided $\partial\Omega$ and the coefficients of L are sufficiently regular. Some of the early references on this theme are the works of C. Neumann in [68] and A. Korn in [54] in the case of the Laplacian, and L. Lichtenstein [57] for more general operators.

In his pioneering 1910 paper [89][4], M.S. Zaremba dealt with the case of the Lapla-

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[2]This is referred to in [76, p. 1] as a "bedrock result of the theory of second-order elliptic partial differential equations."

[3]For example, Zaremba's pioneering work at the beginning of the 20-th century is occasionally misrepresented as having been carried out in $\mathscr{C}^2$ domains when, in fact, in 1910 Zaremba has proved a Boundary Point Principle (for the Laplacian) in domains satisfying an interior ball condition at the point in question (a geometrical hypothesis which will remain the norm for the next 50 years).

[4]Zaremba's original motivation in this paper is the treatment of Dirichlet-Neumann mixed boundary value problems for the Laplacian. The nowadays familiar name "Zaremba's problem" has been eventually adopted in recognition of his early work in [89] (interestingly, in the preamble of this

cian in a three-dimensional domain Ω satisfying an interior ball condition at $x_0 \in \partial\Omega$ (cf. [89, Lemme, pp. 316-317]). His proof makes use of a barrier function, constructed with the help of Poisson's formula for harmonic functions in a ball. Concretely if, say, $B(0,r) \subseteq \Omega \subseteq \mathbb{R}^3$ and $x_0 \in \partial\Omega \cap \partial B(0,r)$, then Zaremba takes (cf. [89, p. 317])

$$v(x) := \frac{r^2 - |x|^2}{r} \int_{\partial B(0,r)} \frac{\psi(y)}{|x-y|^3}\, d\mathcal{H}^2(y), \quad x \in \overline{B(0,r)}, \tag{12.1}$$

where ψ is a continuous, nonnegative function defined on $\partial B(0,r)$, which is zero near x_0 but otherwise does not vanish identically. As such, the function in (12.1) is harmonic, nonnegative and vanishes at points on $\partial B(0,r)$ near x_0, and satisfies[5]

$$\big(-\frac{x_0}{r}\big) \cdot (\nabla v)(x_0) = 2 \int_{\partial B(0,r)} \frac{\psi(y)}{|x_0-y|^3}\, d\mathcal{H}^2(y) > 0. \tag{12.2}$$

These are the key features which virtually all subsequent generalizations based on barrier arguments will emulate in one form or another[6]. This being said, proofs based on other methods have been proposed over the years.

In 1932 G. Giraud managed to extend the Boundary Point Principle to a larger class of elliptic operators (containing the Laplacian), though this was done at the expense of imposing more restrictive conditions on the domain Ω. Specifically, in the souce [27, Théorème 5, p. 343][7] he requires that Ω is of class $\mathscr{C}^{1,1}$ (cf. Definition 9.1) which, as indicated in the second part of Corollary 11.4, is equivalent to the requirement that Ω satisfies a uniform two-sided ball condition. The strategy adopted by

paper, Zaremba attributes the question of considering such a mixed boundary value problem to Wilhelm Wirtinger).

[5]In essence, this itself is a manifestation of the boundary point principle but in the very special case of a harmonic function in a ball.

[6]It is worth noting that Zaremba's approach works virtually verbatim for oblique derivative problems for the Laplacian.

[7]In the footnote on page 343 of his 1932 paper, Giraud's acknowledges on this occasion the earlier work done in 1931 by Marcel Brelot in his Thèse, pp. 27-28.

Giraud in the proof of this result (cf. [27, pp. 343-346]) is essentially to reduce matters to the case of the Laplacian by freezing the coefficients and changing variables in a manner in which the Green function associated with the original differential operator may now be regarded as a perturbation of that for the Laplacian. Since the latter has an explicit formula, much as in the work by Zaremba, the desired conclusion follows. Shortly thereafter, in his 1933 paper [28], Giraud was able to sharpen the results he obtained earlier in [27] as to allow second-order elliptic operators whose top-order coefficients are Hölder while the coefficients of the lower-order terms are continuous[8], on domains of class $\mathscr{C}^{1,\alpha}$ where $\alpha \in (0,1)$; cf. [28, p. 50][9]. Giraud's proof of this more general result is a fairly laborious argument based on a change of variables (locally flattening the boundary).

Giraud's progress seems to have created a conundrum at this stage in the early development of the subject, namely there appeared to be two sets of conditions of geometric/analytic nature (which overlap but are otherwise unrelated) ensuring the validity of the Boundary Point Principle: on the one hand this holds for the Laplacian in domains satisfying an interior ball condition, while on the other hand this also holds for more general elliptic operators in domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$[10].

A few years later, in 1937, motivated by the question of uniqueness for the Neumann problem for the Laplacian[11], M. Keldysch and M. Lavrentiev have proved in

[8]The regularity conditions on the coefficients are not natural since, as is trivially verified, the class of differential operators for which the Boundary Point Principle holds is stable under multiplication by arbitrary (hence, possibly discontinuous) functions.

[9]Giraud's result is restated in [61, Theorem 3, IV, p. 7] for $\mathscr{C}^{1,\alpha}$ domains, though the proof given there is in the spirit of [37] and actually requires smoother boundaries.

[10]Typically, this is indicative of the fact that a more general phenomenon is at play. Alas, it will take about another 40 years for this issue to be resolved.

[11]This issue of uniqueness for the Neumann problem for the Laplacian has been raised by N. Gunther in his influential 1934 monograph on potential theory; cf. [30, Remarque, p. 99]. In

[51] a version of the Boundary Point Principle for the Laplacian in three-dimensional domains satisfying a more flexible property than the interior ball condition. Specifically, if $a, b \in (0, +\infty)$ and $\alpha \in (0, 1]$, consider the three-dimensional, open, truncated paraboloid of revolution (about the z-axis) with apex at $0 \in \mathbb{R}^3$,

$$\mathscr{P}^{\alpha}_{a,b} := \big\{(x, y, z) \in \mathbb{R}^3 : \, a(x^2 + y^2)^{\frac{1+\alpha}{2}} < z < b\big\}, \tag{12.3}$$

and say that $\Omega \subseteq \mathbb{R}$ satisfies an *interior paraboloid condition* at a boundary point $x_0 \in \partial\Omega$ provided one can place a congruent version of $\mathscr{P}^{\alpha}_{a,b}$ (for some choice of the exponent $\alpha \in (0, 1]$ and the geometrical parameters $a, b > 0$) inside Ω in such a manner that the apex is repositioned at x_0. With this piece of terminology, M. Keldysch and M. Lavrentiev's 1937 result then states that the Boundary Point Principle holds for the Laplacian in any domain satisfying an interior paraboloid condition at the point in question. This extends Zaremba's 1910 work in [89] by allowing considerably more general domains and, at the same time, is more in line with the geometrical context in Giraud's 1933 paper [28] since any domain of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0, 1)$ satisfies a paraboloid condition (for the same α in (12.3); e.g., this is implicit in the proof of Theorem 11.3). However, the conundrum described in the previous paragraph continued to persist.

As in Zaremba's approach, M. Keldysch and M. Lavrentiev's proof also relies upon the construction of a barrier function, albeit this is now adapted to the nature of the paraboloid (12.3). Specifically, in [51, p. 142] these authors consider following the

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this connection, we wish to note that in the 1967 English translation [31] of the original 1934 version of N. Gunther's book, this particular question has been omitted, and replaced by its solution given by M. Keldysch and M. Lavrentiev in [51].

barrier in $\mathscr{P}^{\alpha}_{a,b}$:

$$v(x,y,z) := z + \lambda\, r^{1+\beta} P_{1+\beta}(z/r), \quad \forall\,(x,y,z) \in \mathscr{P}^{\alpha}_{a,b}, \tag{12.4}$$

where $\beta \in (0,\alpha)$, $\lambda > 0$ is a normalization constant, $r := \sqrt{x^2+y^2+z^2}$, and $P_{1+\beta}$ is the (regular, normalized) solution to Legendre's differential equation[12] of order $1+\beta$:

$$(1-t^2)\frac{d^2}{dt^2}P_{1+\beta}(t) - 2t\frac{d}{dt}P_{1+\beta}(t) + (1+\beta)(2+\beta)P_{1+\beta}(t) = 0. \tag{12.5}$$

Then β and λ may be chosen so that v in (12.5) has the same key features as in the earlier work of Zaremba. Of course, the case $\alpha = 1$ corresponds to Zaremba's interior ball condition.

As a corollary of their Boundary Point Principle, M. Keldysch and M. Lavrentiev then establish the uniqueness for the Neumann problem (classically formulated[13]) for a family of domains which contains all bounded domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$. The issue whether this uniqueness result also holds for bounded domains of class $\mathscr{C}^1$ has subsequently become known as the Lavrentiev-Keldysch problem (cf. [53, p. 96]), and it will only be settled later. Momentarily fast-forwarding in time to 1981, it was N.S. Nadirashvili who in [64] proved a weaker version[14] of the Boundary Point Principle in bounded domains satisfying a global interior uniform cone condition (as discussed in Definition 10.2) which nonetheless suffices to deduce uniqueness in the Neumann and oblique boundary value problems in such a setting[15] (refer also to the souce [43, p. 307] for further refinements of Nadirashvili's theorem).

[12] A higher dimensional analogue of the Keldysch-Lavrentiev barrier requires considering Gegenbauer functions in place of solutions of (12.5).

[13] That is, the solution is assumed to be twice continuously differentiable inside the domain and continuous on the closure of the domain, with the normal derivative understood in as a one-sided directional derivative along the unit normal.

[14] Indeed, the Boundary Point Principle fails in the general class of Lipschitz domains; see [81, p. 4] for a simple counterexample in a two dimensional sector.

[15] The crux of Nadirashvili's paper [64] is that, for domains satisfying a uniform cone condition,

The coming of age of the work initiated by Zaremba in the 1910 is marked by the publication in 1952 of the papers [37], [70], in which E. Hopf[16] and O.A. Oleinik[17] have simultaneously and independently established a version of the Boundary Point Principle for domains satisfying an interior ball condition and for general, non-divergence form, uniformly elliptic operators with bounded coefficients[18]. In fact, Hopf and Oleinik's proofs differ only by their choice of barrier functions. In [37, p. 792], Hopf considered a barrier function in an annulus[19] given by

$$v(x) := e^{a|x|^2} - e^{ar^2}, \quad \forall\, x \in B(0,r) \setminus \overline{B(0,r/2)}, \quad r > 0, \tag{12.6}$$

where $a > 0$ is a sufficiently large constant (chosen in terms of the coefficients of L)[20]. Oleinik took a different approach to the construction of a barrier and in [70, p. 696] considered the following function[21] defined in a ball:

$$v(x) := C_1 x_n + x_n^2 - C_2 \sum_{i=1}^{n-1} x_i^2 \quad \forall\, x = (x_1, ..., x_n) \in B(r\mathbf{e}_n, r), \quad r > 0, \tag{12.7}$$

where $C_1, C_2 > 0$ are suitably chosen constants (depending on the size of the differ-

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while the directional derivative of a supersolution of a uniformly elliptic differential operator in non-divergence form may vanish at an extremal point located on the boundary, it does not, however, vanish identically in any neighborhood of that point.

[16]The crucial observation Hopf makes in 1952 is that the comparison method he employed in his 1927 paper [39, Section I] may be used to establish, similarly yet independently of the Strong Maximum Principle itself, a remarkably versatile version of the Boundary Point Principle.

[17]Oleinik's paper was published two years before she defended her doctoral dissertation, entitled "Boundary-value problems for PDE's with small parameter in the highest derivative and the Cauchy problem in the large for non-linear equations" in 1954.

[18]Strictly speaking, both Hopf and Oleinik ask in [37], [70] that the coefficients of the differential operator in question are continuous, but their proofs go trough verbatim under the weaker assumption of boundedness.

[19]The idea of considering this type of region apparently originated with D. Gilbarg who used it in [38, pp. 312-313].

[20]An elegant alternative to Hopf's barrier function (12.6) in the same annulus is $\widetilde{v}(x) := |x|^{-\lambda} - r^{-\lambda}$ for a sufficiently large constant $\lambda > 0$; see the discussion in [56, § 1.3].

[21]Interestingly, in the limiting case $\alpha = \beta = 1$, the Keldysch-Lavrentiev barrier (12.4) becomes (given the known formula $P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$ for the second-order Lagrange polynomial) precisely $v(x, y, x) = z + z^2 - \frac{1}{2}(x^2 + y^2)$, which strongly resembles Oleinik's barrier (12.7) in the three-dimensional setting.

ential operator L).

In this format, the Hopf-Oleinik Boundary Point Principle has become very popular and, even more than half a century later, is still routinely reproduced in basic text-books on partial differential equations (cf., e.g., [19], [76], [23], as well as the older monographs [26], [56], [61], [77]). However, the interior ball condition is unnecessarily restrictive and, as such, attempts were made to generalize Hopf and Oleinik's result (in a conciliatory manner with Giraud's 1933 result valid for domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1)$). Motivated by A.D. Aleksandrov's basic work in [1]-[6], in a series of papers beginning in the early 1970's (cf. [45], [44], [47], [48]) L.I. Kamynin and B.N. Khimchenko[22] succeeded[23] in extending the validity range of the Boundary Point Principle for general elliptic operators in non-divergence form with bounded coefficients to the class of domains satisfying an interior paraboloid condition, more general yet reminiscent of that considered by M. Keldysch and M. Lavrentiev in their publication [51, p. 141]. More specifically, Kamynin and Khimchenko define in place of (12.3)

$$\mathscr{P}^{\omega}_{a,b} := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} :\, a|x'|\,\omega(|x'|) < x_n < b\big\}\,, \tag{12.8}$$

where $a, b > 0$ and the (modulus of continuity, or) shape function $\omega \in \mathscr{C}^0\big([0,R]\big)$ is nonnegative, vanishes at the origin, and is required to satisfy certain differential/integral properties. For example, in [47], under the assumptions that

$$\omega \in \mathscr{C}^2\big((0,R)\big), \quad \omega'(t) \geq 0 \;\text{ and }\; \omega''(t) \leq 0 \;\text{ for every }\; t \in (0,R), \tag{12.9}$$

[22]Occasionally also spelled "Himčenko."
[23]Earlier, related results are due to R. Výborný in [87].

and granted that ω also satisfies a Dini integrability condition

$$\int_0^R \frac{\omega(t)}{t}\,dt < +\infty, \tag{12.10}$$

Kamynin and Khimchenko propose (cf. p. 84 in the English translation of [47]) the following exponential-type barrier which involves the above modulus of continuity

$$v(x) := x_n \exp\Big\{C_1 \int_0^{x_n} \frac{\widehat{\omega}(t)}{t}\,dt\Big\} - C_2\,|x|\,\omega(|x|), \quad \forall\, x = (x_1, ..., x_n) \in \mathscr{P}^{\omega}_{a,b}, \tag{12.11}$$

where $C_1, C_2 > 0$ are two suitably chosen constants. Here, $\widehat{\omega}$ is yet another modulus of continuity, satisfying the same type of conditions as in (12.9), and which is related to (in the terminology used in [47]) the nature of the degeneracy of the characteristic part of the differential operator L. A further refinement of this result, which applies to certain classes of differential operators with unbounded coefficients, has subsequently been worked out in [49] (cf. also [43]). Results of similar nature, but for domains satisfying an interior ball condition have been proved earlier by C. Pucci in [74], [75].

While the Dini condition (12.10) may not be omitted (cf. the discussion on pp. 85-88 in the English translation of [47]), the necessity of the differentiability conditions in (12.9) may be called into question. In this regard, see the discussion on p. 6 of [81], a paper in which M. Safonov proposes another approach to the Boundary Point Principle. His proof of [81, Theorem 1.8, p. 5] does not involve the use of a barrier function and, instead, is based on estimates for quotients u_2/u_1 of positive solutions of $Lu = 0$ in a Lipschitz domain Ω, which vanish on a portion of $\partial\Omega$. The main geometrical hypothesis in [81] is what the author terms interior Q-condition (replacing the earlier interior ball and paraboloid conditions), which essentially states that a region congruent to

$$Q := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < R, \ 0 < x_n - |x'|\omega(|x'|) < R\} \tag{12.12}$$

may be placed inside Ω so as to make contact with the boundary at a desired point. In this scenario[24], Safonov retains (12.10) and, in place of (12.9), only assumes a monotonicity condition, to the effect that

$$\begin{aligned} &\omega : [0, R] \to [0, 1] \text{ is such that the mapping} \\ &[0, R] \ni t \mapsto t\omega(t) \in [0, R] \text{ is non-decreasing.} \end{aligned} \tag{12.13}$$

This being said, the method employed by Safonov requires that $u(x) = x_n$ is a solution of the operator L and, as such, he imposes the restriction that L is a differential operator without lower-order terms, i.e., $L = \sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j$, which is uniformly elliptic and has bounded coefficients. However, from the perspective of the Boundary Point Principle, a uniformly ellipticity condition is unnecessarily strong (as already noted in [47]) and, in fact, so is the boundedness assumption on the coefficients. Indeed, as is trivially verified, if the Boundary Point Principle is valid for a certain differential operator L, then it remains valid for the operator ψL where ψ is an arbitrary (thus, possibly unbounded) positive function.

The topic of Boundary Point Principles for partial differential equations remains an active area of research, with significant work completed in the recent past. See, for example, [81], [82], [65], [66], [67], [58], among others, and we have already commented on the contents of some of these papers. Here, we only wish to note that in [58,

[24]Our notation is slightly different than that employed in [81], where the author works with $\psi(t) := t\omega(t)$ in place of ω.

Theorem 4.1, p. 346] G.M. Libermann establishes a version of the Boundary Point Principle which, though weaker than that due to L.I. Kamynin and B.N. Khimchenko, has a conceptually simpler proof, which works in any $\mathscr{C}^1$ domain whose unit normal has a modulus of continuity satisfying a Dini integrability condition[25].

Finally, it should be mentioned that adaptations of this body of results to parabolic differential operators have been worked out by L. Nirenberg [69], L.I. Kamynin [42], L.I. Kamynin and B.N. Khimchenko [46], [50], to cite a few, and that a significant portion of the theory continues to hold for nonlinear partial differential equations (cf., e.g., [76] and the references therein).

[25]The class of domains considered in [58] is, however, not optimal.

Chapter 13

Boundary Point Principle for Semi-elliptic Operators with Singular Drift

Our main result in this chapter, formulated in Theorem 13.3 below, is a sharp version of the Hopf-Oleinik Boundary Point Principle. The proof presented here, which is a refinement of work recently completed in [16], is based on a barrier construction in a pseudo-ball (cf. (7.2)). This is done under less demanding assumptions on the shape function ω than those stipulated by Kamynin and Khimchenko in (12.9) and, at the same time, our pseudo-ball $\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$ (cf. (7.4)) is a smaller set than the paraboloid $\mathscr{P}^{\omega}_{a,b}$ considered by Kamynin and Khimchenko in (12.8). Significantly, the coefficients of the differential operators for which our theorem holds are not necessarily bounded or measurable (in contrast to [37], [70], [44], [47], [48], and others), the matrix of top coefficients is only degenerately elliptic, and the coefficients of the lower-order terms are allowed to blow up at a rate related to the geometry of the domain[1]. Furthermore, by means of concrete counterexamples we show that that our result is sharp.

To set the stage, we first dispense of a number of preliminary matters.

[1]This addresses an issue raised in [82, p. 226].

Definition 13.1. *Let Ω be a nonempty, open, proper subset of $\mathbb{R}^n$, and fix a point $x_0 \in \partial\Omega$. We say that a vector $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$* `points inside` Ω `at` x_0 *provided there exists $\varepsilon > 0$ with the property that $x_0 + t\vec{\ell} \in \Omega$ whenever $t \in (0, \varepsilon)$. Given a function $u \in \mathscr{C}^0(\Omega \cup \{x_0\})$ and a vector $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$ pointing inside Ω at x_0, define the* `lower and upper directional derivatives of` u `at` x_0 `along` $\vec{\ell}$ *as*

$$
\begin{aligned}
&D_{\vec{\ell}}^{(inf)} u(x_0) := \liminf_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}, \quad \textit{and} \\
&D_{\vec{\ell}}^{(sup)} u(x_0) := \limsup_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}.
\end{aligned}
\tag{13.1}
$$

Of course, in the same geometric setting as above, $D_{\vec{\ell}}^{(\mathrm{inf})} u(x_0)$, $D_{\vec{\ell}}^{(\mathrm{sup})} u(x_0)$ are meaningfully defined in $\overline{\mathbb{R}} := [-\infty, +\infty]$, there holds $D_{\vec{\ell}}^{(\mathrm{inf})} u(x_0) \leq D_{\vec{\ell}}^{(\mathrm{sup})} u(x_0)$ and, as a simple application of the Mean Value Theorem shows,

$$
\left.
\begin{array}{l}
u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^1(\Omega) \text{ and the limit} \\
\nabla u(x_0) := \lim\limits_{t \to 0^+} (\nabla u)(x_0 + t\vec{\ell}) \ \text{ exists in } \mathbb{R}^n
\end{array}
\right\} \Rightarrow
\begin{aligned}
D_{\vec{\ell}}^{(\mathrm{inf})} u(x_0) &= D_{\vec{\ell}}^{(\mathrm{sup})} u(x_0) \\
&= \vec{\ell} \cdot \nabla u(x_0).
\end{aligned}
\tag{13.2}
$$

Shortly, we shall need a suitable version of the Weak Minimum Principle. In order to facilitate the subsequent discussion, we first make a few definitions. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and consider a second-order differential operator L in Ω:

$$
L := -\sum_{i,j=1}^{n} a^{ij} \partial_i \partial_j + \sum_{i=1}^{n} b^i \partial_i, \quad \text{where} \quad a^{ij}, b^i : \Omega \to \mathbb{R}, \quad i, j \in \{1, ..., n\}. \tag{13.3}
$$

Hence, L is in non-divergence form, without a zeroth-order term, and the reader is alerted to the presence of the minus sign in front of second-order part of L. In this context, recall that L is called *semi-elliptic* in Ω provided the coefficient matrix $A = (a^{ij})_{1 \leq i,j \leq n}$ is semi-positive definite at each point in Ω, i.e.,

$$
\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq 0 \ \text{ for every } x \in \Omega \text{ and every } \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n. \tag{13.4}
$$

Clearly, the semi-ellipticity condition for L in Ω is equivalent to the requirement that, at each point in Ω, the symmetric part of the coefficient matrix $A := (a^{ij})_{1\leq i,j\leq n}$, i.e., $\frac{1}{2}(A + A^\top)$ where $A^\top$ denotes the transpose of A, has only nonnegative eigenvalues. Also, we shall say that L (as above) is *non-degenerate along* $\xi^* = (\xi_1^*, ..., \xi_n^*) \in S^{n-1}$ in Ω provided

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^* > 0 \quad \text{for every } x \in \Omega. \tag{13.5}$$

For further use, let us also agree to call L *uniformly elliptic near* $x_0 \in \overline{\Omega}$ if there exists $r > 0$ such that

$$\inf_{x\in B(x_0,r)\cap\Omega}\ \inf_{\xi\in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > 0, \tag{13.6}$$

and simply *uniformly elliptic* provided

$$\inf_{x\in\Omega}\ \inf_{\xi\in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > 0. \tag{13.7}$$

Here is the variant of the Weak Minimum Principle alluded to above.

Proposition 13.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, nonempty, open set and assume that L is a second-order differential operator in non-divergence form (without a zeroth-order term) as in* (13.3) *which is semi-elliptic and non-degenerate along a vector $\xi^* = (\xi_1^*, ..., \xi_n^*) \in S^{n-1}$. In addition, suppose that the function*

$$\Omega \ni x \mapsto \frac{\sum_{i=1}^{n} b^i(x)\xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^*} \in \mathbb{R} \qquad \textit{is locally bounded from above in } \Omega. \tag{13.8}$$

Then for every real-valued function $u \in \mathscr{C}^2(\Omega)$ with the property that

$$(Lu)(x) \geq 0 \quad \textit{for every} \quad x \in \Omega, \tag{13.9}$$

it follows that

$$\inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega}\Big(\liminf_{\Omega\ni y\to x} u(y)\Big). \tag{13.10}$$

In particular, if u is also continuous on $\overline{\Omega}$, then the minimum of u in $\overline{\Omega}$ is achieved on the topological boundary $\partial\Omega$, i.e.,

$$\min_{x\in\overline{\Omega}} u(x) = \inf_{x\in\Omega} u(x) = \min_{x\in\partial\Omega} u(x). \tag{13.11}$$

Proof. Though the proof of this result follows a well-established pattern, we include it for the sake of completeness. For starters, since $u \in \mathscr{C}^2(\Omega)$, by replacing a^{ij} with $\widetilde{a}^{ij} := \frac{1}{2}(a^{ij} + a^{ji})$, $1 \leq i,j \leq n$ (a transformation which preserves (13.4) and (13.8)), there is no loss of generality in assuming that the coefficient matrix $A = (a^{ij})_{1\leq i,j\leq n}$ is symmetric at every point in Ω. Furthermore, observe that (13.10) is implied by the version of (13.11) in which Ω is replaced by any relatively compact subset of Ω, say, of the form $\Omega_k := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/k\}$ where $k \in \mathbb{N}$, by passing to the limit $k \to +\infty$. Hence, there is no loss of generality in assuming that the function defined in (13.8) is actually globally bounded in Ω. With these adjustments in mind, the fact that

$$Lu > 0 \quad \text{in} \quad \Omega \Longrightarrow \min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x) \tag{13.12}$$

is then a simple consequence of the semi-positive definiteness of the (symmetric) matrix-coefficient (cf. (13.4)), and the Second Derivative Test for functions of class $\mathscr{C}^2$ (cf., e.g., [26, Theorem 3.1, p. 32]). Finally, in the case when the weaker condition (13.9) holds, one makes use of (13.12) with u replaced by $u + \varepsilon v$, where $\varepsilon > 0$ is arbitrary, the function $v : \Omega \to \mathbb{R}$ is given by (recall that $\xi^* \in S^{n-1}$ is as in (13.8))

$$v(x) := -e^{\lambda\, x\cdot\xi^*}, \quad x \in \Omega, \tag{13.13}$$

and $\lambda \in (0, +\infty)$ is a fixed, sufficiently large constant. Concretely, since for every point $x \in \Omega$ we have

$$
\begin{aligned}
(Lv)(x) &= \lambda^2 \Big(\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^* \Big) e^{\lambda\, x \cdot \xi^*} - \lambda \Big(\sum_{i=1}^{n} b^i(x) \xi_i^* \Big) e^{\lambda\, x \cdot \xi^*} \\
&= \lambda\, e^{\lambda\, x \cdot \xi^*} \Big(\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^* \Big) \Big(\lambda - \frac{\sum_{i=1}^{n} b^i(x) \xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^*} \Big),
\end{aligned}
\tag{13.14}
$$

it follows (cf. also (13.5)) that

$$
\lambda > \sup_{x \in \Omega} \left(\frac{\sum_{i=1}^{n} b^i(x) \xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^*} \right) \Longrightarrow Lv > 0 \;\text{ in }\; \Omega. \tag{13.15}
$$

Hence, $\min\{(u + \varepsilon v)(x) : x \in \overline{\Omega}\} = \min\{(u + \varepsilon v)(x) : x \in \partial\Omega\}$ for each $\varepsilon > 0$, so (13.11) follows by letting $\varepsilon \to 0^+$. □

Shortly, we shall also require the following simple algebraic lemma.

Lemma 13.2. *Let A be an $n \times n$ matrix, with real entries, which is semi-positive definite, i.e., it satisfies $(A\xi) \cdot \xi \geq 0$ for every $\xi \in \mathbb{R}^n$. Then, with $\mathrm{Tr}\,(A)$ denoting the trace of A, there holds*

$$
\sup_{\xi \in S^{n-1}} [(A\xi) \cdot \xi] \leq \mathrm{Tr}\,(A). \tag{13.16}
$$

Proof. Working with $\frac{1}{2}(A + A^\top)$ in place of A, there is no loss of generality in assuming that A is symmetric. Then there exists a unitary $n \times n$ matrix, U, and a diagonal $n \times n$ matrix, D, such that $A = U^{-1} D U$. If $\lambda_1, ..., \lambda_n$ are the entries on the diagonal of D, then $\lambda_i \geq 0$ for each $i \in \{1, ..., n\}$, and $\mathrm{Tr}\,(A) = \lambda_1 + \cdots + \lambda_n$. On the other hand, $\sup_{\xi \in S^{n-1}} [(A\xi) \cdot \xi] = \max\{\lambda_i : 1 \leq i \leq n\}$, so the desired conclusion follows. □

As a final preliminary matter to discussing the theorem below, we make a couple of more definitions. Concretely, call a real-valued function f defined on an interval

$I \subseteq \mathbb{R}$ *quasi-decreasing* provided there exists $C \in (0,+\infty)$ with the property that $f(t_1) \leq Cf(t_0)$ whenever $t_0, t_1 \in I$ are such that $t_0 \leq t_1$. Moreover, call f *quasi-increasing* if $-f$ is quasi-decreasing. Of course, the class of quasi-increasing (respectively, quasi-decreasing) functions contains the class of non-decreasing (respectively, non-increasing) functions, but the inclusion is strict[2]. In fact, if ϕ is non-decreasing and $C \geq 1$, then any function f with the property that $\phi \leq f \leq C\phi$ is quasi-increasing. Conversely, given a quasi-increasing function f, defining
$\phi(t) := \inf_{s \geq t} f(s)$ yields a non-decreasing function for which $\phi \leq f \leq C\phi$ for some $C \geq 1$.

We are now prepared to state and prove the main result in this chapter.

Theorem 13.3. *Suppose that Ω is a nonempty, proper, open subset of $\mathbb{R}^n$ and that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) : \, a|x-x_0|\,\omega(|x-x_0|) < h \cdot (x-x_0) < b\} \subseteq \Omega, \quad (13.17)$$

for some parameters $a, b, R \in (0,+\infty)$, direction vector $h = (h_1, ..., h_n) \in S^{n-1}$, and a shape function $\omega : [0,R] \to [0,+\infty)$ exhibiting the following features:

$$\omega \text{ is continuous on } [0,R],\ \omega(t) > 0 \text{ for } t \in (0,R], \quad \sup_{0<t\leq R}\Big(\frac{\omega(t/2)}{\omega(t)}\Big) < \infty, \quad (13.18)$$

$$\text{and the mapping } \ (0,R] \ni t \mapsto \frac{\omega(t)}{t} \in (0,+\infty) \ \text{ is quasi-decreasing.} \quad (13.19)$$

Also, consider a non-divergence form, second-order, differential operator (without a

[2]For example, if $\alpha > 0$ then $\omega(t) := (2+\sin(t^{-1}))t^{\alpha}$, $t > 0$, is a quasi-increasing function which is not monotone in any interval of the form $(0,\varepsilon)$.

zeroth-order term)

$$L := -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i, \quad a^{ij}, b^i : \Omega \longrightarrow \mathbb{R}, \quad 1 \le i, j \le n, \tag{13.20}$$

$$L \text{ semi-elliptic in } \Omega \text{ and non-degenerate along } h \in S^{n-1} \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0, h). \tag{13.21}$$

In addition, suppose that there exists a real-valued function

$$\widetilde{\omega} \in \mathscr{C}^0\big([0, R]\big), \quad \widetilde{\omega}(t) > 0 \ \text{ for each } \ t \in (0, R], \quad \text{and} \quad \int_0^R \frac{\widetilde{\omega}(t)}{t}\, dt < +\infty, \tag{13.22}$$

with the property that

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < \infty, \tag{13.23}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\ \sum\limits_{i=1}^{n} b^i(x)h_i\Big\} + \Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\ -b^i(x)\big\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < \infty. \tag{13.24}$$

Finally, suppose that $u : \Omega \cup \{x_0\} \to \mathbb{R}$ *is a function satisfying*

$$u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega), \tag{13.25}$$

$$(Lu)(x) \ge 0 \ \text{ for each } \ x \in \Omega, \tag{13.26}$$

$$u(x_0) < u(x) \ \text{ for each } \ x \in \Omega, \tag{13.27}$$

and fix a vector $\vec{\ell} \in S^{n-1}$ *satisfying the transversality condition*

$$\vec{\ell} \cdot h > 0. \tag{13.28}$$

Then $\vec{\ell}$ *points inside* Ω *at* x_0*, and there exist a compact subset* K *of* Ω *which depends only on the geometrical characteristics of* $\mathscr{G}^{\omega}_{a,b}(x_0, h)$*, and a constant* $\kappa > 0$

which depends only on

$$\begin{aligned} &\textit{the quantities in } (13.23)\text{-}(13.24), \quad \big(\inf_K u\big) - u(x_0), \\ &\vec{\ell} \cdot h, \quad \textit{and the pseudo-ball character of } \Omega \textit{ at } x_0, \end{aligned} \tag{13.29}$$

with the property that

$$(D_{\vec{\ell}}^{(inf)} u)(x_0) \geq \kappa. \tag{13.30}$$

Proof. We debut with a few comments pertaining to the nature of the functions ω, $\widetilde{\omega}$, and also make a suitable (isometric) change of variables in order to facilitate the subsequent discussion. First, the fact that $\widetilde{\omega}$ is continuous on $[0, R]$, positive on $(0, R]$ and satisfies Dini's integrability condition forces $\widetilde{\omega}(0) = 0$. Second, for further reference, let us fix a constant $\eta \in (0, +\infty)$ with the property that (cf. (13.19))

$$\frac{\omega(t_1)}{t_1} \leq \eta \frac{\omega(t_0)}{t_0} \qquad \text{whenever } 0 < t_0 \leq t_1 \leq R. \tag{13.31}$$

Third, from (13.23) and Lemma 13.2 it follows that there exists $C > 0$ with the property that

$$\omega(t) \leq C\,\widetilde{\omega}(t), \qquad \text{for all } \ t \in [0, R]. \tag{13.32}$$

As a consequence of this and (13.22), we deduce that ω also satisfies Dini's integrability condition, i.e.,

$$\int_0^R \frac{\omega(t)}{t}\, dt < +\infty. \tag{13.33}$$

Moreover, it is also apparent from (13.18) and the Dini condition satisfied by ω that

$$\omega(0) = 0. \tag{13.34}$$

Fourth, we claim that there exist $M \in (0, +\infty)$ and $\gamma \in (1, +\infty)$ such that

$$(\eta\gamma)^{-1} \xi^{\gamma-1} \omega(\xi) \leq \int_0^\xi \omega(t) t^{\gamma-2}\, dt \leq M \xi^{\gamma-1} \omega(\xi), \quad \forall\, \xi \in (0, R]. \tag{13.35}$$

To justify this claim, observe that if N stands for the supremum in the last condition in (13.18) then $N \in (0, +\infty)$ and

$$\omega(2^{-k}t) \leq N^k \omega(t), \quad \forall\, t \in (0, R], \quad \forall\, k \in \mathbb{N}. \tag{13.36}$$

Next, fix a number $\gamma \in \mathbb{R}$ such that

$$\gamma > 1 + \max\{0\,,\,\log_2 N\}, \tag{13.37}$$

and recall that the function $(0, R] \ni t \mapsto \omega(t)/t \in (0, +\infty)$ is quasi-increasing. Then, if $\eta \in (0, +\infty)$ is as in (13.31), using the fact that $\gamma > 1$ as well as the estimates in (13.36)-(13.37), for every $\xi \in (0, R]$ we may write

$$\begin{aligned}
\int_0^\xi \omega(t) t^{\gamma-2}\, dt \;&=\; \sum_{k=0}^{+\infty} \int_{2^{-k-1}\xi}^{2^{-k}\xi} \frac{\omega(t)}{t} t^{\gamma-1}\, dt \leq \sum_{k=0}^{+\infty} (2^{-k}\xi)^{\gamma-1} \int_{2^{-k-1}\xi}^{2^{-k}\xi} \frac{\omega(t)}{t}\, dt \\
&\leq\; \eta \sum_{k=0}^{+\infty} (2^{-k}\xi)^{\gamma-1} \frac{\omega(2^{-k-1}\xi)}{2^{-k-1}\xi} 2^{-k-1}\xi = \eta\xi^{\gamma-1} \sum_{k=0}^{+\infty} 2^{-k(\gamma-1)} \omega(2^{-k-1}\xi) \\
&\leq\; \eta\xi^{\gamma-1} \sum_{k=0}^{+\infty} 2^{-k(\gamma-1)} N^{k+1} \omega(\xi) = N\eta\, \xi^{\gamma-1} \omega(\xi) \Big(\sum_{k=0}^{+\infty} 2^{-k(\gamma-1)} 2^{k \log_2 N} \Big) \\
&= \eta N \Big(\sum_{k=0}^{+\infty} 2^{-k(\gamma-1-\log_2 N)} \Big) \xi^{\gamma-1} \omega(\xi) = \frac{\eta N}{1 - 2^{-\gamma+1+\log_2 N}} \xi^{\gamma-1} \omega(\xi).
\end{aligned} \tag{13.38}$$

Thus, the upper-bound for the integral in (13.35) is proved with

$$M := \frac{\eta N}{1 - 2^{-\gamma+1+\log_2 N}} \in (0, +\infty). \tag{13.39}$$

Since the lower bound is a direct consequence of (13.31), this completes the proof of (13.35).

Continuing our series of preliminary matters, let U be an $n \times n$ unitary matrix (with real entries) with the property that $Uh = \mathbf{e}_n$ and define an isometry of $\mathbb{R}^n$ by setting $\mathcal{R}x := U(x - x_0)$ for every $x \in \mathbb{R}^n$. Introduce $\widetilde{\Omega} := \mathcal{R}(\Omega)$. Then if

$$\big(\widetilde{a}^{ij}(y)\big)_{1\leq i,j\leq n} := U\big[\big(a^{ij}(\mathcal{R}^{-1}y)\big)_{1\leq i,j\leq n}\big]U^{-1}, \quad \forall\, y \in \widetilde{\Omega}, \tag{13.40}$$

$$\big(\widetilde{b}^{i}(y)\big)_{1\leq i\leq n} := U\big[\big(b^{i}(\mathcal{R}^{-1}y)\big)_{1\leq i\leq n}\big], \quad \forall\, y \in \widetilde{\Omega}, \tag{13.41}$$

and if we consider the differential operator in $\widetilde{\Omega}$ given by

$$\widetilde{L} := -\sum_{i,j=1}^{n} \widetilde{a}^{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n} \widetilde{b}^{i}\partial_i, \tag{13.42}$$

then $\widetilde{L}$ satisfies properties analogous to L (relative to the new geometrical context), and

$$\widetilde{L}(u \circ \mathcal{R}^{-1}) = (Lu) \circ \mathcal{R}^{-1}. \tag{13.43}$$

Furthermore, $\mathcal{R}(\mathscr{G}^{\omega}_{a,b}(x_0, h)) = \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$ by (7.8). To summarize, given that both the hypotheses and the conclusion in the statement of the theorem transform covariantly under the change of variables $y = \mathcal{R}x$, there is no loss of generality in assuming that, to begin with, x_0 is the origin in $\mathbb{R}^n$ and that $h = \mathbf{e}_n \in S^{n-1}$. In this setting, the transversality condition (13.28) becomes

$$\vec{\ell} \cdot \mathbf{e}_n > 0, \tag{13.44}$$

while the semi-ellipticity condition on L and non-degeneracy condition on L along $h \in S^{n-1}$ read

$$\inf_{x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)} \inf_{\xi\in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq 0 \quad \text{ and } \quad a^{nn}(x) > 0 \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n). \tag{13.45}$$

Going further, for each real number r set

$$[r]_{\oplus} := \max\{r, 0\} \quad \text{ and } \quad [r]_{\ominus} := \max\{-r, 0\}. \tag{13.46}$$

Then, as far as how (13.23)-(13.24) transform under the indicated change of variables, we note that after possibly decreasing the value of R, matters may be arranged so that

$$\sum_{i=1}^{n} a^{ii}(x) \leq \Lambda_0 \frac{|x|\widetilde{\omega}(x_n)}{x_n \omega(|x|)} a^{nn}(x), \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{13.47}$$

$$\sum_{i=1}^{n} [b^i(x)]_{\ominus} \leq \Lambda_1 \frac{\widetilde{\omega}(x_n)}{x_n \omega(|x|)} a^{nn}(x), \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{13.48}$$

$$[b^n(x)]_{\oplus} \leq \Lambda_2 \frac{\widetilde{\omega}(x_n)}{x_n} a^{nn}(x), \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{13.49}$$

for some constants $\Lambda_0, \Lambda_1, \Lambda_2 \in (0, +\infty)$.

We are now ready to begin the proof in earnest. For starters, we note that by eventually increasing the value of $a > 0$ and decreasing the value of $b > 0$ we may assume that

$$\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)} \setminus \{0\} \subseteq \Omega, \quad \forall\, b_* \in (0, b]. \tag{13.50}$$

To proceed, fix $b_* \in (0, b]$ and, with $\gamma \in (1, +\infty)$ as in (13.37) and for two finite constants $C_0, C_1 > 0$ to be specified later, consider the barrier function

$$v(x) := x_n + C_0 \int_0^{x_n} \int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\, dt\, d\xi - C_1 \int_0^{|x|} \int_0^{\xi} \frac{\omega(t)}{t} \Big(\frac{t}{\xi}\Big)^{\gamma - 1} dt\, d\xi, \tag{13.51}$$

for every $x = (x_1, ..., x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}$. Since ω, $\widetilde{\omega}$ are continuous and satisfy Dini's integrability condition, it follows that v is well-defined and, in fact,

$$v \in \mathscr{C}^2\big(\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)\big) \cap \mathscr{C}^0\big(\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}\big). \tag{13.52}$$

Moreover, a direct computation gives that for each $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we have

$$\partial_j v(x) = \delta_{jn} + C_0 \delta_{jn} \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \frac{x_j}{|x|} \int_0^{|x|} \frac{\omega(t)}{t} \Big(\frac{t}{|x|}\Big)^{\gamma-1} dt, \quad 1 \leq j \leq n, \tag{13.53}$$

and, further, for each $i, j \in \{1, ..., n\}$,

$$\partial_i \partial_j v(x) = C_0 \delta_{in} \delta_{jn} \frac{\widetilde{\omega}(x_n)}{x_n} - C_1 \Big[\tfrac{\delta_{ij}}{|x|^\gamma} - \gamma \tfrac{x_i x_j}{|x|^{\gamma+2}} \Big] \int_0^{|x|} \omega(t) t^{\gamma-2} dt - C_1 \frac{x_i x_j}{|x|^2} \frac{\omega(|x|)}{|x|} \tag{13.54}$$

where δ_{ij} is the usual Kronecker symbol. Hence, by combining (13.20) with (13.53) and (13.54), we arrive at the conclusion that

$$(Lv)(x) = I + II + III, \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n), \tag{13.55}$$

where, for each $x = (x_1, ..., x_n) \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we have set

$$I := I' + I'' \ \text{ with } \ I' := C_1 \Big(\sum_{i=1}^{n} a^{ii}(x) \Big) |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \ \text{ and} \tag{13.56}$$

$$I'' := C_1 \Big(\sum_{i,j=1}^{n} a^{ij}(x) \tfrac{x_i}{|x|} \tfrac{x_j}{|x|} \Big) \Big(\frac{\omega(|x|)}{|x|} - \gamma |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \Big), \tag{13.57}$$

$$II := -C_0\, a^{nn}(x) \frac{\widetilde{\omega}(x_n)}{x_n}, \tag{13.58}$$

$$III := b^n(x) + C_0 b^n(x) \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \Big(\sum_{i=1}^{n} b^i(x) \tfrac{x_i}{|x|} \Big) \int_0^{|x|} \frac{\omega(t)}{t} \Big(\frac{t}{|x|} \Big)^{\gamma-1} dt. \tag{13.59}$$

As a preamble to estimating I, II, III above, we make a couple of preliminary observations. First note that since $C_1 \geq 0$, ω is nonnegative, and L is semi-elliptic, we have

$$I'' \leq C_1 \Big(\sum_{i,j=1}^{n} a^{ij}(x) \tfrac{x_i}{|x|} \tfrac{x_j}{|x|} \Big) \frac{\omega(|x|)}{|x|} \leq C_1 \Big(\sum_{i=1}^{n} a^{ii}(x) \Big) \frac{\omega(|x|)}{|x|}, \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{13.60}$$

where the last inequality above is based on Lemma 13.2. Second, for every point $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$, estimate (13.35) used with $\xi := |x| \in (0, R)$ gives that

$$|x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \leq M \frac{\omega(|x|)}{|x|}, \tag{13.61}$$

where the constant $M \in (0, +\infty)$ is as in (13.39). Consequently,

$$I' \leq M C_1 \Big(\sum_{i=1}^{n} a^{ii}(x) \Big) \frac{\omega(|x|)}{|x|}, \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{13.62}$$

In concert with the above observations, formulas (13.56)-(13.59) then allow us to conclude that (recall notation introduced in (13.46)) for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$,

$$I \leq C_1(1+M)\Big(\sum_{i=1}^{n} a^{ii}(x)\Big)\frac{\omega(|x|)}{|x|}, \quad II \leq -C_0 a^{nn}(x)\frac{\widetilde{\omega}(x_n)}{x_n}, \tag{13.63}$$

$$III \leq [b^n(x)]_{\oplus}\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) + C_1 M\Big(\sum_{i=1}^{n}[b^i(x)]_{\ominus}\Big)\omega(|x|), \tag{13.64}$$

where we have also used (13.61) when deriving the last estimate above. Thus, on account of (13.55), (13.62), (13.63), and (13.31), for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we may estimate

$$\begin{aligned}(Lv)(x) \leq\ & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(1+M)\Big(\frac{\sum_{i=1}^n a^{ii}(x)}{a^{nn}(x)}\Big)\frac{x_n\omega(|x|)}{|x|\widetilde{\omega}(x_n)} - C_0\Big\} \\ & + \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{\frac{x_n[b^n(x)]_{\oplus}}{\widetilde{\omega}(x_n)a^{nn}(x)}\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) \\ & \qquad + C_1 M\frac{x_n\omega(|x|)\Big(\sum_{i=1}^n[b^i(x)]_{\ominus}\Big)}{\widetilde{\omega}(x_n)a^{nn}(x)}\Big\}.\end{aligned} \tag{13.65}$$

In turn, (13.65) and (13.47)-(13.49) permit us to further estimate, for each point $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$,

$$\begin{aligned}(Lv)(x) \leq\ & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(1+M)\Lambda_0 - C_0\Big\} \\ & + \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{\Lambda_2\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) + C_1 M\Lambda_1\Big\} \\ \leq\ & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) \\ & \qquad + \Lambda_2 - C_0\Big(1 - \Lambda_2\int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\,dt\Big)\Big\}.\end{aligned} \tag{13.66}$$

We shall return to (13.66) momentarily. For the time being, we wish to estimate the barrier function on the round portion of the boundary of the pseudo-ball. To this end, let us note from (7.2) that if $x = (x_1, ..., x_n) \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x \in \mathbb{R}^n : x_n = b_*\}$

then, given that ω is continuous, we have $x_n = a\omega(|x|)|x|$ which further implies

$$\begin{aligned} x_n + C_0 \int_0^{x_n} \int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\,dt\,d\xi &\leq x_n\Big(1 + C_0 \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\,dt\Big) \\ &= a\omega(|x|)|x|\Big(1 + C_0 \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\,dt\Big). \end{aligned} \tag{13.67}$$

Moreover, since $\omega(t)/t \geq \eta^{-1}\omega(|x|)/|x|$ for every $t \in (0, |x|)$ (cf. (13.31)), we may also write

$$\int_0^{|x|} \int_0^{\xi} \frac{\omega(t)}{t} \Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\,d\xi \geq \eta^{-1} \frac{\omega(|x|)}{|x|} \int_0^{|x|} \int_0^{\xi} \Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\,d\xi = \frac{|x|\omega(|x|)}{2\eta\gamma}. \tag{13.68}$$

Together, (13.51) and (13.67)-(13.68) give that for each

$x \in \partial \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \, x_n = b_*\}$ we have

$$v(x) \leq \Big(a - \frac{C_1}{2\eta\gamma} + aC_0 \int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt\Big)|x|\omega(|x|). \tag{13.69}$$

At this stage, we are ready to specify the constants $C_0, C_1 \in (0, +\infty)$ appearing in (13.51), in a manner consistent with the format of (13.66), (13.69) and which suits the goals we have in mind. Turning to details, we start by fixing

$$C_1 > 2a\eta\gamma \quad \text{and} \quad C_0 > 2\big[C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) + \Lambda_2\big], \tag{13.70}$$

then, using the Dini integrability condition satisfied by $\widetilde{\omega}$, select $b_* \in (0, b]$ sufficiently small so that

$$\int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt < \frac{1}{2\Lambda_2} \quad \text{and} \quad \int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt < \frac{C_1 - 2a\eta\gamma}{2a\eta\gamma C_0}. \tag{13.71}$$

Then (13.66) together with the second condition in (13.70) and the first condition in (13.71) ensure that

$$Lv \leq 0 \quad \text{in} \quad \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{13.72}$$

Furthermore, the second condition in (13.71) is designed (cf. (13.69)) so that we also have

$$v \leq 0 \quad \text{on} \quad \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \setminus \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n = b_*\}. \tag{13.73}$$

Having specified the constants C_0 and C_1 (in the fashion described above) finishes the process of defining the barrier function v, initiated in (13.51). With this task concluded, we proceed by considering the compact subset of Ω given by

$$K := \{x = (x_1, ..., x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)} : x_n = b_*\}, \tag{13.74}$$

and note that (13.27) (and since u is continuous, hence attains its infimum on compact subsets of Ω) entails

$$u(x_0) < \inf_K u. \tag{13.75}$$

Thanks to (13.27), (13.73) and (13.75), we may then choose $\varepsilon > 0$ for which

$$\varepsilon\big(\sup_K |v|\big) < \big(\inf_K u\big) - u(x_0), \tag{13.76}$$

(hence ε depends only on the quantities listed in (13.29)) so that, on the one hand,

$$0 \leq u(x) - u(x_0) - \varepsilon v(x) \quad \text{for every } x \in \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{13.77}$$

On the other hand, from (13.72) and (13.26) we obtain (recall that L annihilates constants)

$$L(u - u(x_0) - \varepsilon v) \geq 0 \quad \text{in} \quad \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{13.78}$$

With the estimates (13.77)-(13.78) in hand, and keeping in mind (13.52) plus the fact that the function u belongs to $\mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}) \cap \mathscr{C}^2(\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n))$, bring in the Weak Minimum Principle presented in Proposition 13.1. This is used in the nonempty, bounded, open subset $\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$ of $\mathbb{R}^n$ and with the vector $\mathbf{e}_n$ playing the role of $\xi^* \in S^{n-1}$ from (13.8). Indeed, granted (13.45), it follows that L is non-degenerate along $\mathbf{e}_n \in S^{n-1}$ in $\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$ and, thanks to (13.49), the analogue of condition (13.8) is valid in the current setting. The bottom line is that Proposition 13.1 applies, and gives

$$u - u(x_0) - \varepsilon v \geq 0 \quad \text{in } \overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}. \tag{13.79}$$

Given that both $u - u(x_0)$ and v vanish at the point $x_0 = 0 \in \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$, this shows that

$$u - u(x_0) - \varepsilon v \in \mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}) \text{ has a global minimum at } x_0 = 0. \tag{13.80}$$

On the other hand, condition (13.44) and the fact that ω continuously vanishes at the origin (see (13.34)) imply the existence of some $t_* \in (0, b_*)$ with the property that $\omega(t) < \vec{\ell}\cdot\mathbf{e}_n/a$ for every $t \in (0, t_*)$. In turn, such a choice of t_* ensures that (cf. (7.2))

$$t\,\vec{\ell} \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \quad \text{ for every } \ t \in (0, t_*). \tag{13.81}$$

In particular, $\vec{\ell}$ points in Ω at x_0 (cf. Definition 13.1), and from (13.1), (13.80)-(13.81) we obtain

$$D^{(\mathrm{inf})}_{\vec{\ell}}(u - u(x_0) - \varepsilon v)(x_0) \geq 0. \tag{13.82}$$

Now (13.53) gives

$$\nabla v(x_0) := \lim_{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)\ni x\to 0} (\nabla v)(x) = \mathbf{e}_n, \tag{13.83}$$

hence

$$(D^{(\mathrm{inf})}_{\vec{\ell}} v)(x_0) = (D^{(\mathrm{sup})}_{\vec{\ell}} v)(x_0) = \vec{\ell}\cdot\nabla v(x_0) = \vec{\ell}\cdot\mathbf{e}_n, \tag{13.84}$$

by (13.52) and the discussion in (13.2). In turn, (13.82)-(13.83) and (13.84) further allow us to conclude that

$$(D^{(\mathrm{inf})}_{\vec{\ell}}u)(x_0) \geq \varepsilon\vec{\ell}\cdot\nabla v(x_0) = \varepsilon\,\vec{\ell}\cdot\mathbf{e}_n > 0, \tag{13.85}$$

where the last inequality is a consequence of (13.44). Choosing $\kappa := \varepsilon\,\vec{\ell}\cdot\mathbf{e}_n > 0$ then yields (13.30), finishing the proof of the theorem. □

We continue with a series of comments relative to Theorem 13.3 and its proof.

Remark 13.1. *(i) As we will discuss in detail later, Theorem 13.3 is sharp. A slightly more versatile result is obtained by replacing Ω by $U\cap\Omega$ in* (13.25)-(13.27)*, where $U\subseteq\mathbb{R}^n$ is some open neighborhood of $x_0\in\partial\Omega$. Of course, Theorem 13.3 itself implies such an improvement simply by invoking it with Ω substituted by $U\cap\Omega$ throughout.*

(ii) Trivially, the last condition in (13.18) *is satisfied if the function $\omega:[0,R]\to[0,+\infty)$ has the property that*

$$\exists\, m\in\mathbb{R} \text{ such that } (0,R]\ni t\mapsto t^m\omega(t)\in(0,+\infty) \text{ is quasi-increasing}, \tag{13.86}$$

hence, in particular, if ω itself is quasi-increasing. Corresponding to the class of function introduced in (1.14)*, the shape function $\omega_{\alpha,\beta}$ satisfies all properties displayed in* (13.18)-(13.19) *for all $\alpha\in(0,1]$ and $\beta\in\mathbb{R}$. However, $\omega_{0,\beta}$ fails to satisfy the Dini integrability condition for $\beta\geq -1$ (while still meeting the other conditions).*

(iii) It is easy to check that if $\omega:(0,R]\to(0,+\infty)$ is such that the map $(0,R]\ni t\mapsto\omega(t)/t\in(0,+\infty)$ is quasi-decreasing and $\sup\limits_{0<t\leq R}\left(\frac{\omega(t/2)}{\omega(t)}\right)<+\infty$,

then for every $c \in (1, +\infty)$ *we also have* $\sup\limits_{0<t\leq R}\Big(\frac{\omega(t/c)}{\omega(t)}\Big) < +\infty$. *Based on this observation, one may then verify without difficulty that if* $\omega : [0, R] \to [0, +\infty)$ *satisfies the conditions in* (13.18)-(13.19) *then, for each fixed* $\theta \in (0, 1)$, *so does the function* $[0, R] \ni t \mapsto \omega(t^\theta) \in [0, +\infty)$. *Furthermore, this function satisfies Dini's integrability condition if* ω *does.*

(iv) *The amplitude parameter* $a > 0$ *used in defining the pseudo-ball* $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ *plays only a minor role since this may, in principle, be absorbed as a multiplicative factor into the shape function* ω *(thus, reducing matters to the case when* $a = 1$*). Nonetheless, working with a generic amplitude adds a desirable degree of flexibility in the proof of Theorem 13.3.*

(v) *It is instructive to note that if* $\omega : [0, R] \to [0, +\infty)$ *satisfies* (13.19) *as well as the first two properties listed in* (13.18), *and is such that* (13.35) *holds for some* $M \in (0, +\infty)$ *and* $\gamma \in (0, +\infty)$, *then actually the last condition in* (13.18) *is also valid. Indeed, using* (13.31), *for each* $\xi \in (0, R]$ *we may estimate*

$$\begin{aligned} M\xi^{\gamma-1}\omega(\xi) &\geq \int_0^{\xi} \frac{\omega(t)}{t} t^{\gamma-1}\, dt \geq \int_0^{\xi/2} \frac{\omega(t)}{t} t^{\gamma-1}\, dt \\ &\geq \eta^{-1}\frac{\omega(\xi/2)}{\xi/2}\int_0^{\xi/2} t^{\gamma-1}\, dt = \frac{1}{\gamma\eta}\frac{\omega(\xi/2)}{\xi/2}\Big(\frac{\xi}{2}\Big)^{\gamma} \\ &= \frac{1}{\gamma 2^{\gamma-1}\eta}\xi^{\gamma-1}\omega(\xi/2), \end{aligned} \tag{13.87}$$

which entails

$$\sup_{0<\xi\leq R}\Big(\frac{\omega(\xi/2)}{\omega(\xi)}\Big) \leq \gamma 2^{\gamma-1} M\eta < +\infty. \tag{13.88}$$

We next prove a technical result (refining earlier work in [10]), which is going to be useful in the proof of Theorem 13.5 below.

Proposition 13.4. *Let $R \in (0,+\infty)$ and assume that $\omega : [0,R] \to [0,+\infty)$ is a continuous function with the property that $\omega(t) > 0$ for each $t \in (0,R]$. In addition, assume that ω satisfies a Dini condition and is quasi-increasing, i.e.,*

$$\begin{array}{l} \int_0^R \frac{\omega(t)}{t}\,dt < +\infty \quad \textit{and} \quad \omega(t_1) \leq \eta\,\omega(t_2) \\ \textit{whenever } t_1, t_2 \in [0,R] \textit{ are such that } t_1 \leq t_2, \end{array} \tag{13.89}$$

for some fixed constant $\eta \in (0,+\infty)$. Consider

$$M := \max\big\{\omega(t) : t \in [0,R]\big\}, \quad t_o := \min\big\{t \in [0,R] : \omega(t) = M\big\}, \tag{13.90}$$

and denote by $\theta_* \in (0,1)$ the unique solution of the equation $\theta = (\ln\theta)^2$ in the interval $(0,+\infty)$.

Then $t_o > 0$ and there exists a function $\widehat{\omega} : [0,t_o] \to [0,+\infty)$ satisfying the following properties:

$$\begin{array}{l} \widehat{\omega} \textit{ is continuous, concave, and strictly increasing on } [0,t_o], \\ \widehat{\omega}(t) \geq \omega(t) \textit{ for each } t \in [0,t_o], \quad \widehat{\omega}(0) = 0, \quad \widehat{\omega}(t_o) = M, \\ \textit{the mapping } \ (0,t_o) \ni t \mapsto \frac{\widehat{\omega}(t)}{t} \in [0,+\infty) \ \textit{ is non-increasing}, \\ \textit{and} \quad \displaystyle\int_0^{t_o} \frac{\widehat{\omega}(t)}{t}\,dt \leq \eta M + \Big(1 + \eta + \frac{\eta(\theta_* + |\ln\theta_*|)}{\theta_*|\ln\theta_*|}\Big)\int_0^{t_o} \frac{\omega(t)}{t}\,dt. \end{array} \tag{13.91}$$

Proof. We start by noting that since ω is continuous at 0 and satisfies a Dini integrability condition, then necessarily ω vanishes at the origin. In turn, this forces $t_o \in (0,R]$ and $M \in (0,+\infty)$. Given that ω is continuous, we also have that $\omega(t_o) = M$. Next, extend the restriction of ω to the interval $[0,t_o]$ to a function $\bar{\omega} : [0,+\infty) \to [0,+\infty)$ by setting $\bar{\omega}(t) := M$ for every $t \geq t_o$, and take $\widetilde{\omega} : [0,+\infty) \to [0,+\infty)$ to be the concave envelope of $\bar{\omega}$, i.e.,

$$\widetilde{\omega}(t) := \sup\Big\{\sum_{j=1}^{N}\lambda_j\bar{\omega}(t_j) \;\;:\;\; N\in\mathbb{N},\, (\lambda_j)_j\in[0,1]^N,\, \sum_{j=1}^{N}\lambda_j=1,$$
$$(t_j)_j\in[0,+\infty)^N,\, \sum_{j=1}^{N}\lambda_j t_j = t\Big\} \tag{13.92}$$

for each $t\in[0,+\infty)$. Then (cf., e.g., the discussion in [78, pp. 35-57]), $\widetilde{\omega}$ is the smallest concave function which is pointwise $\geq\bar{\omega}$, that is,

$$\widetilde{\omega} = \inf_{\substack{\psi\geq\bar{\omega}\text{ on }[0,+\infty)\\ \psi\text{ concave on }\overline{\mathbb{R}}_+}} \psi. \tag{13.93}$$

In particular, $\widetilde{\omega}$ is concave on $[0,+\infty)$, hence continuous on $(0,+\infty)$. Also (as seen from (13.92)), we have

$$\widetilde{\omega}(0)=\omega(0)=0 \quad\text{and}\quad \widetilde{\omega}(t)=M \quad\text{for every}\quad t\geq t_o. \tag{13.94}$$

Moreover, since $\widetilde{\omega},\omega$ are continuous on $(0,R)$, formula (13.93) also entails that

$$\begin{gathered}\forall\, t\in(0,t_o)\;\;\text{with}\;\;\widetilde{\omega}(t)>\omega(t)\Longrightarrow\\ \begin{cases}\exists\, J\text{ open subinterval of }(0,R)\text{ so that }t\in J\\ \text{and such that }\widetilde{\omega}\text{ is an affine function on }J.\end{cases}\end{gathered} \tag{13.95}$$

To proceed, from the fact that $\widetilde{\omega}$ and $\bar{\omega}$ are continuous on $(0,+\infty)$ and (13.94) we deduce that

$$W := \big\{t\in(0,+\infty) : \widetilde{\omega}(t)>\bar{\omega}(t)\big\} \quad\text{is an open subset of}\quad (0,t_o). \tag{13.96}$$

If W is empty, it follows that $\widetilde{\omega}(t)=\bar{\omega}(t)$ for every $t\in(0,+\infty)$, hence ω itself is concave on $(0,t_o)$ As such, we simply take $\widehat{\omega} := \omega\big|_{[0,t_o]}$ and the desired conclusion follows. There remains to study the case when the set W from (13.96) is nonempty. In this scenario, W may be written as the union of an at most countable family of mutually disjoint open intervals (which are precisely the connected components of

W), say

$$W = \bigcup_{i \in I} J_i, \quad \text{where} \;\; J_i := (\alpha_i, \beta_i), \quad 0 \le \alpha_i < \beta_i \le t_o \;\; \text{for each } i \in I. \tag{13.97}$$

Let us also observe that since both $\widetilde{\omega}$ and ω are continuous on $(0, R)$, from (13.94) and (13.96) we may conclude that $\widetilde{\omega}(t) = \omega(t)$ for each $t \in \partial W$. Given the nature of the decomposition of W in (13.96), this ensures that

$$\begin{aligned} &\widetilde{\omega}(t) > \omega(t) \;\; \text{whenever } i \in I \text{ and } t \in (\alpha_i, \beta_i), \\ &\widetilde{\omega}(\alpha_i) = \omega(\alpha_i) \;\; \text{and} \;\; \widetilde{\omega}(\beta_i) = \omega(\beta_i) \;\; \text{for each } i \in I. \end{aligned} \tag{13.98}$$

Moreover, based on this and (13.95) we arrive at the conclusion that

$$\widetilde{\omega}(t) = \frac{t - \alpha_i}{\beta_i - \alpha_i}\big(\omega(\beta_i) - \omega(\alpha_i)\big) + \omega(\alpha_i), \quad \text{if } i \in I \text{ and } t \in [\alpha_i, \beta_i]. \tag{13.99}$$

For further use, let us point out that (13.99) readily entails

$$\widetilde{\omega}(t) \le \omega(\alpha_i) + \frac{\omega(\beta_i)}{\beta_i} t \quad \text{ if } i \in I \text{ and } t \in [\alpha_i, \beta_i], \tag{13.100}$$

since both functions involved are affine on the interval (α_i, β_i) and the inequality is trivially verified at endpoints. Going further, fix $\theta \in (0, 1)$ and partition the (at most countable) set of indices I (from (13.97)) into the following two subclasses:

$$I_1 := \big\{ i \in I : \; \alpha_i > \theta \beta_i \big\}, \quad I_2 := \big\{ i \in I : \; \alpha_i \le \theta \beta_i \big\}. \tag{13.101}$$

Now, the fact that $\widetilde{\omega}$ is concave entails $\widetilde{\omega}\big(\lambda t_1 + (1-\lambda) t_2\big) \ge \lambda \widetilde{\omega}(t_1) + (1-\lambda)\widetilde{\omega}(t_2)$ for all $\lambda \in [0, 1]$ and $t_1, t_2 \in [0, +\infty)$. Pick now two numbers $t'' \ge t' > 0$ and specialize the earlier inequality to the case when $\lambda := t'/t''$, $t_1 := t''$ and $t_2 := 0$ (recall that $\widetilde{\omega}$ vanishes at the origin). This yields $\widetilde{\omega}(t') \ge (t'/t'')\widetilde{\omega}(t'')$, from which we may ultimately conclude that

$$\text{the mapping} \;\; (0, +\infty) \ni t \mapsto \frac{\widetilde{\omega}(t)}{t} \in [0, +\infty) \;\; \text{is non-increasing.} \tag{13.102}$$

For each fixed $i \in I_1$, we necessarily have $\alpha_i > 0$. Keeping this in mind, we may then estimate

$$\begin{aligned}
\int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t}\,dt &\le \frac{\widetilde{\omega}(\alpha_i)}{\alpha_i}(\beta_i-\alpha_i) = \frac{\omega(\alpha_i)}{\alpha_i}(\beta_i-\alpha_i) \\
&\le \eta(\beta_i-\alpha_i)\frac{\beta_i}{\alpha_i}\Big(\inf_{t\in(\alpha_i,\beta_i)} \frac{\omega(t)}{t}\Big) \le \eta\theta^{-1}\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt, \qquad (13.103)
\end{aligned}$$

thanks to (13.102), (13.89), and (13.101). On the other hand, when $i \in I_2$ we may write

$$\begin{aligned}
\int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t}\,dt &= \int_{\alpha_i}^{\beta_i} (\widetilde{\omega}(t)-\omega(\alpha_i))\,\frac{dt}{t} + \int_{\alpha_i}^{\beta_i} \omega(\alpha_i)\,\frac{dt}{t} \\
&\le \int_{\alpha_i}^{\beta_i} \frac{\omega(\beta_i)}{\beta_i}\,dt + \eta\int_{\alpha_i}^{\beta_i} \omega(t)\,\frac{dt}{t} \le \omega(\beta_i) + \eta\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt \\
&\le \frac{\eta}{|\ln\theta|}\int_{\beta_i}^{\beta_i/\theta} \frac{\bar{\omega}(t)}{t}\,dt + \eta\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt, \qquad (13.104)
\end{aligned}$$

by (13.100), (13.89) and the definition of $\bar{\omega}$. At this stage, we proceed to estimate

$$\begin{aligned}
\int_0^{t_o} \frac{\widetilde{\omega}(t)}{t}\,dt &= \int_W \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\widetilde{\omega}(t)}{t}\,dt = \sum_{i\in I}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\omega(t)}{t}\,dt \\
&\le \sum_{i\in I_1}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \sum_{i\in I_2}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \qquad (13.105)
\end{aligned}$$

Note that (13.103) gives

$$\sum_{i\in I_1}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt \le \eta\theta^{-1}\sum_{i\in I_1}\int_{J_i} \frac{\omega(t)}{t}\,dt \le \eta\theta^{-1}\int_0^{t_o} \frac{\omega(t)}{t}\,dt. \qquad (13.106)$$

We continue by observing that

$$\forall\, i,j \in I_2 \ \text{ with } \ i \neq j \implies (\beta_i, \beta_i/\theta) \cap (\beta_j, \beta_j/\theta) = \varnothing. \qquad (13.107)$$

To justify this, fix two different indices $i,j \in I_2$ and, without loss of generality, assume that $\beta_i < \beta_j$. Since (α_i,β_i) and (α_j,β_j) are disjoint connected components of W, it follows that $\beta_i \notin (\alpha_j,\beta_j)$. Hence, $\beta_i < \alpha_j \le \theta\beta_j$ given that $j \in I_2$, which shows that

$\beta_i/\theta < \beta_j$. With this in hand, (13.107) readily follows. Having established (13.107), we next invoke (13.104) in order to estimate

$$\begin{aligned}\sum_{i\in I_2}\int_{J_i}\frac{\widetilde{\omega}(t)}{t}\,dt &= \sum_{i\in I_2}\int_{\alpha_i}^{\beta_i}\frac{\widetilde{\omega}(t)}{t}\,dt \leq \frac{\eta}{|\ln\theta|}\sum_{i\in I_2}\int_{\beta_i}^{\beta_i/\theta}\frac{\bar{\omega}(t)}{t}\,dt + \eta\sum_{i\in I_2}\int_{\alpha_i}^{\beta_i}\frac{\omega(t)}{t}\,dt\\ &\leq \frac{\eta}{|\ln\theta|}\int_0^{t_o/\theta}\frac{\bar{\omega}(t)}{t}\,dt + \eta\int_0^{t_o}\frac{\omega(t)}{t}\,dt. \qquad (13.108)\end{aligned}$$

In concert, (13.105), (13.106) and (13.108) yield

$$\begin{aligned}\int_0^{t_o}\frac{\widetilde{\omega}(t)}{t}\,dt &\leq \left(1+\eta+\eta\theta^{-1}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \frac{\eta}{|\ln\theta|}\int_0^{t_o/\theta}\frac{\bar{\omega}(t)}{t}\,dt\\ &= \left(1+\eta+\eta\theta^{-1}+\frac{\eta}{|\ln\theta|}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \eta M. \qquad (13.109)\end{aligned}$$

Finally, minimizing the right-most hand side of (13.109) over all $\theta \in (0,1)$ gives

$$\int_0^{t_o}\frac{\widetilde{\omega}(t)}{t}\,dt \leq \left(1+\eta+\eta\,\theta_*^{-1}+\frac{\eta}{|\ln\theta_*|}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \eta M. \qquad (13.110)$$

At this point, much of the ground work ensuring that $\widehat{\omega} := \widetilde{\omega}\big|_{[0,t_o]}$ satisfies the properties listed in (13.91) has been done. Two items which are yet to be settled are as follows. First, formula (13.92) shows that $\widehat{\omega}(t) < M$ for $t \in (0,t_o)$. Hence, if $0 \leq t_1 < t_2 \leq t_o$ and if $\lambda := (t_o - t_2)/(t_o - t_1) \in [0,1)$ then, given that $\widehat{\omega}$ is concave, we obtain $\widehat{\omega}(t_2) \geq \lambda\widehat{\omega}(t_1) + (1-\lambda)M > \omega(t_1)$. Consequently, $\widehat{\omega}$ is strictly increasing on $[0,t_o]$. Second, the continuity of $\widehat{\omega}$ at 0 is a consequence of the fact that this function is continuous and increasing on $(0,t_o)$ and satisfies a Dini condition. This concludes the proof of the proposition. $\square$

We are now prepared to present a consequence of Theorem 13.3 in which we impose a more streamlined set of conditions on the shape function (compare (13.111) with

(13.18)-(13.19)). In turn, Theorem 13.5 below readily implies Theorem 1.4 stated in § 1.

Theorem 13.5. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$ and assume that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0. Concretely, assume that* (13.17) *holds for some parameters $a, b, R \in (0, +\infty)$, direction vector $h = (h_1, ..., h_n) \in S^{n-1}$, and a shape function $\omega : [0, R] \to [0, +\infty)$ with the property that*

$$\begin{array}{c}\omega \text{ is continuous, positive and quasi-increasing} \\ \text{on } (0, R], \text{ and } \int_0^R \frac{\omega(t)}{t}\,dt < +\infty.\end{array} \tag{13.111}$$

Also, consider a non-divergence form, second-order, differential operator L which is semi-elliptic and non-degenerate along h (as in (13.20)-(13.21)*) and whose coefficients satisfy*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\sum\limits_{i=1}^{n} a^{ii}(x)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{13.112}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|x-x_0|\Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\,-b^i(x)\big\}\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{13.113}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\,\sum\limits_{i=1}^{n} b^i(x)h_i\Big\}}{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j\Big)} < +\infty. \tag{13.114}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ for which $\vec{\ell}\cdot h > 0$, and suppose that a function $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ satisfies

$$(Lu)(x) \geq 0 \quad \text{and} \quad u(x_0) < u(x) \quad \text{for each} \quad x \in \Omega. \tag{13.115}$$

Then $\vec{\ell}$ points inside Ω at x_0, and there exists a constant $\kappa > 0$ (which depends only on the quantities in (13.29)*) with the property that*

$$(D_{\vec{\ell}}^{(inf)} u)(x_0) \geq \kappa. \tag{13.116}$$

Proof. It may be readily verified that ω continuously vanishes at the origin given that ω satisfies a Dini integrability condition, is continuous and nonnegative on $[0, R]$, as well as quasi-increasing on $(0, R]$. Now, if $\widehat{\omega}$ is associated with the original shape function ω as in Proposition 13.4, properties (13.91) hold. In particular, $\widehat{\omega} \geq \omega$ near the origin and, hence, $\mathscr{G}_{a,b}^{\widehat{\omega}}(x_0, h) \subseteq \mathscr{G}_{a,b}^{\omega}(x_0, h) \subseteq \Omega$. Also, (13.112)-(13.114) imply the versions of (13.23)-(13.24) written with both ω and $\widetilde{\omega}$ replaced by $\widehat{\omega}$. Then Theorem 13.3 applies, with both ω and $\widetilde{\omega}$ in the original statement replaced by $\widehat{\omega}$. From this, the desired conclusion follows. $\square$

Chapter 14

Sharpness of the Boundary Point Principle Formulated in Theorem 13.3

As mentioned previously, Theorem 13.3 is sharp, and here the goal is to make this precise through a series of counterexamples presented as remarks.

Remark 14.1. *The strict inequality in* (13.27) *is obviously necessary, since otherwise any constant function would serve as a counterexample.*

Remark 14.2. *In the context of Theorem 13.3, the nondegeneracy of L along the direction vector h of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ is a necessary condition. A simple counterexample is obtained by taking $n \geq 2$, $\Omega := \mathbb{R}^n_+$, $x_0 := (0, ..., 0) \in \mathbb{R}^n$, $\vec{\ell} := \mathbf{e}_n$, $L := -\partial^2/\partial x_1^2$ and $u(x_1, ..., x_n) := x_n^2$.*

Remark 14.3. *The discussion in § 1 pertaining to* (1.30)-(1.36) *shows that both condition* (13.23) *and condition* (13.24) *in Theorem 13.3 are necessary.*

Remark 14.4. *Fix $\alpha \in (1, 2)$ and in the two dimensional setting consider*

$$\begin{aligned} &\Omega := \{(x, y) \in \mathbb{R}^2 : \, y > (x^2)^{1/\alpha}\}, \\ &L := -\partial_x^2 - \tfrac{2}{\alpha(\alpha+1)} y^{2-\alpha} \partial_y^2 \quad \text{in } \ \Omega, \\ &u(x, y) := y^{1+\alpha} - x^2 y, \quad \forall \, (x, y) \in \Omega. \end{aligned} \tag{14.1}$$

Then $u \in \mathscr{C}^2(\overline{\Omega})$ satisfies $u(0) = 0$, $u > 0$ in Ω, $Lu = 0$ in Ω and $(\nabla u)(0) = 0$. Thus, (13.30) *fails in this case, even though Ω satisfies a pseudo-ball condition at the origin, with shape function $\omega(t) := t^{(2/\alpha)-1}$ satisfying* (13.18)-(13.19)*, and L is (non-uniformly) elliptic in Ω and homogeneous (i.e., L has no lower-order terms). Here, the breakdown is caused by the failure of condition* (13.23) *for a function $\widetilde{\omega}$ as in* (13.24)*. Indeed, since $x^2 + y^2 \leq cy^\alpha$ in Ω,* (13.23) *would imply $\widetilde{\omega}(y)/y \geq c/y$ for all $y > 0$ small, in violation of Dini's integrability condition for $\widetilde{\omega}$. Moreover, varying the parameter $\alpha \in (1, 2)$, this counterexample shows that for any fixed $\varepsilon > 0$ condition* (13.23) *may not be relaxed to*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|x - x_0|^{\varepsilon}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty. \tag{14.2}$$

Remark 14.5. *Here the goal is to show that the conclusion* (13.30) *of Theorem 13.3 may be violated if condition* (13.24) *fails to be satisfied for some $\widetilde{\omega}$ as in* (13.22) *(even though* (13.17)-(13.23) *do hold for some $\widetilde{\omega}$ as in* (13.22)*). We start by making the general observation that if Ω is an arbitrary open set and if $u \in \mathscr{C}^2(\Omega)$ is any real-valued function without critical points in Ω then, obviously,*

$$-\Delta u + \Big(\frac{\Delta u}{|\nabla u|^2}\nabla u\Big) \cdot \nabla u = 0 \quad \text{ in } \Omega. \tag{14.3}$$

This tautology may be interpreted as the statement that u is a null-solution of the second-order differential operator

$$L := -\Delta + \vec{b} \cdot \nabla, \qquad \text{where} \qquad \vec{b} := \frac{\Delta u}{|\nabla u|^2}\nabla u \quad \text{in } \Omega. \tag{14.4}$$

Let us now specialize these general considerations to the case when (in the two-

dimensional setting)

$$\Omega := \mathbb{R}^2_+ \cap B(\mathbf{0}, e^{-1}) \ \ \textit{and} \ \ u(x,y) := y\big[-\ln\sqrt{x^2+y^2}\,\big]^{-\varepsilon} \quad \forall\,(x,y) \in \Omega, \quad (14.5)$$

where $\mathbf{0} := (0,0)$ *is the origin in* $\mathbb{R}^2$*, and* $\varepsilon > 0$ *is a fixed, small number. Clearly,* (13.17)-(13.23) *do hold and* Ω *does satisfy an interior pseudo-ball condition at* $\mathbf{0} \in \partial\Omega$*, if we take* $\omega(t) := \widetilde{\omega}(t) := t^\alpha$ *for some arbitrary, fixed* $\alpha \in (0,1)$*. Note that such a choice guarantees that both* (13.18)-(13.19) *and* (13.22) *are satisfied. Going further, a direct computation in polar coordinates* (r,θ) *shows that*

$$(\nabla u)(r,\theta) = \Big(\varepsilon \sin\theta\cos\theta(-\ln r)^{-\varepsilon-1}\,,\,(-\ln r)^{-\varepsilon-1}(\varepsilon\sin^2\theta - \ln r)\Big), \qquad (14.6)$$

so choosing ε *small enough ensures that* u *does not have critical points in* Ω*. Assuming that this is the case, the drift coefficients* $\vec{b} = (b^1, b^2)$ *of the operator* L *associated with this function may be expressed in polar coordinates* (r,θ) *as*

$$b^1(r,\theta) = \frac{\varepsilon^2\sin^2\theta\cos\theta(2\ln r - 1 - \varepsilon)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}, \qquad (14.7)$$

$$b^2(r,\theta) = \frac{\varepsilon\sin\theta(2\ln r - 1 - \varepsilon)(\varepsilon\sin^2\theta - \ln r)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}. \qquad (14.8)$$

It is then clear from (14.5) *that* $u > 0$ *in* Ω*,* $u \in \mathscr{C}^2(\Omega)$*, and that* u *may be continuously extended to* $\Omega \cup \{\mathbf{0}\}$ *by setting* $u(\mathbf{0}) := 0$*. Furthermore, as is readily seen from* (14.6)*, the fact that* $\varepsilon > 0$ *forces* $\lim\limits_{y\to 0^+}(\partial_y u)(x,y) = 0$*, uniformly in* x*. As a result,* $(D^{(inf)}_{\mathbf{e}_2}u)(\mathbf{0}) = 0$ *which shows that the conclusion in Theorem 13.3 fails. The reason for this failure is the fact that condition* (13.24) *does not hold in the current situation for any choice of* $\widetilde{\omega}$ *as in* (13.22)*. Indeed, if* (13.24) *were to hold, it would then be possible to find a constant* $c > 0$ *with the property that*

$$\frac{\widetilde{\omega}(r)}{r} \geq c\,\max\{0, b^2(r,\tfrac{\pi}{2})\} \geq \frac{c_\varepsilon}{r(-\ln r)} \qquad \textit{for all } r > 0 \textit{ small}, \qquad (14.9)$$

where $c_\varepsilon > 0$ depends only on ε. However, this would then imply that $\widetilde{\omega}$ fails to satisfy Dini's integrability condition since $\int_0^{e^{-1}} \frac{1}{r(-\ln r)}\, dr = \int_1^{+\infty} s^{-1}\, ds = +\infty$ (after making the change of variables $r = e^{-s}$).

The above discussion also shows that condition (13.24) *may not be weakened to*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|\vec{b}(x)|\big|\ln|x-x_0|\big|^{-\delta}}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j\Big)} < +\infty, \quad \text{for some } \ \delta > 0. \tag{14.10}$$

Indeed, in the case of (14.4)-(14.5)*, such a weakened condition would be satisfied for any given $\delta > 0$ by taking, in the notation introduced in* (1.14)*, $\widetilde{\omega} := \omega_{0,-1-\delta}$ i.e., $\widetilde{\omega}(t) = |\ln t|^{-1-\delta}$. However, as already noted, the conclusion in Theorem 13.3 fails for* (14.4)-(14.5)*.*

The same type of counterexample may be easily adapted to the higher-dimensional setting, taking $\Omega := \mathbb{R}^n_+ \cap B(0, e^{-1})$ and $u(x) := x_n(-\ln|x|)^{-\varepsilon}$ in place of (14.5)*. In this case, the drift coefficients continue to exhibit the same type of singularity at the origin as* (14.7)-(14.8)*. In particular, we have*

$$\vec{b} : \Omega \longrightarrow \mathbb{R}^n, \quad |\vec{b}(x)| = O\Big(\frac{1}{|x|\big|\ln|x|\big|}\Big) \quad \text{as } \ |x| \to 0, \tag{14.11}$$

which shows that

$$\vec{b} \in L^n(\Omega). \tag{14.12}$$

This should be compared with the classical Aleksandrov-Bakel'man-Pucci theorem which asserts that the Weak Maximum Principle holds for uniformly elliptic operators in open subsets of $\mathbb{R}^n$ whose drift coefficients are locally in L^n. In this light, the significance of (14.12) *is that, in contrast with the Aleksandrov-Bakel'man-Pucci Weak*

Maximum Principle, the Boundary Point Principle may fail even though the drift coefficients are in L^n. See also [81, Example 1.12], [81, Example 4.1], [65, Remark 3] in this regard.

Remark 14.6. *Here we present another example for which the same type of conclusions (pertaining the singularity of the drift coefficients) as in Remark 14.5 may be inferred. Specifically, consider the domain $\Omega := \{(x, y) \in \mathbb{R}^2 : (x-1)^2+y^2 < 1\} \subseteq \mathbb{R}^2$ and define the function $u : \overline{\Omega} \to \mathbb{R}$ by setting*

$$u(x, y) := x\, e^{-\sqrt{-\ln[(x^2+y^2)/4]}} \quad \textit{for each } (x, y) \in \overline{\Omega} \setminus \{\mathbf{0}\}, \textit{ and } \; u(\mathbf{0}) := 0, \tag{14.13}$$

where, as before, $\mathbf{0}$ denotes the origin in $\mathbb{R}^2$. Then it is not difficult to check that $u \in \mathscr{C}^1\big(\overline{\Omega}\big) \cap \mathscr{C}^\infty(\Omega)$, $u > 0$ in Ω and $(\nabla u)(\mathbf{0}) = 0$. As noted in [25, p. 169], the function u is satisfies the divergence-form, elliptic, second-order equation

$$\partial_x\big(a\, \partial_x u + b\, \partial_y u\big) + \partial_y\big(b\, \partial_x u + c\, \partial_y u\big) = 0 \quad \textit{in } \; \Omega, \tag{14.14}$$

where the coefficients $a, b, c \in \mathscr{C}^0\big(\overline{\Omega}\big) \cap \mathscr{C}^\infty(\Omega)$ are defined as follows:

$$\begin{array}{l} a := \frac{1}{\mu} + \frac{y^2(\mu^2-1)}{(x^2+y^2)\mu}, \;\; b := \frac{xy(1-\mu^2)}{(x^2+y^2)\mu}, \;\; c := \frac{1}{\mu} + \frac{x^2(\mu^2-1)}{(x^2+y^2)\mu} \;\; \textit{in } \overline{\Omega} \setminus \{\mathbf{0}\}, \;\; \textit{where} \\ \mu := 1 + \big(2\sqrt{-\ln[(x^2+y^2)/4]}\,\big)^{-1}, \textit{ and } a(\mathbf{0}) := 1,\, b(\mathbf{0}) := 0,\, c(\mathbf{0}) := 1. \end{array} \tag{14.15}$$

Taking advantage of the differentiability of these coefficients, we may convert (14.14) into the uniformly elliptic, non-divergence form, second-order equation $Lu = 0$ in Ω, where

$$\begin{array}{c} L := -\sum\limits_{i,j=1}^{2} a^{ij}\partial_i\partial_j + \sum\limits_{i=1}^{2} b^i\partial_i \; \textit{with } a^{11} := -a, \; a^{22} := -c, \; a^{12} := a^{21} := -c, \\ \textit{and with } b^1 := -\partial_x a - \partial_y b, \quad b^2 := -\partial_y c - \partial_x b \;\; \textit{in } \; \Omega. \end{array} \tag{14.16}$$

Then the top-coefficients of L are bounded in Ω, while the drift coefficients exhibit the following type of behavior near the origin:

$$b^i(x, y) \;\; \textit{blows up at } \mathbf{0} \textit{ like } \; \frac{1}{\sqrt{x^2+y^2}(-\ln(x^2+y^2))^{3/2}}, \quad i = 1, 2. \tag{14.17}$$

Then (compare with (14.11)*), the same type of conclusions as in Remark 14.5 may be drawn in this case as well.*

Remark 14.7. *The point of the next example is to show that if the Dini condition on $\widetilde{\omega}$ is allowed to fail (while all the other hypotheses are retained) then* (13.30) *is no longer expected to hold, even for such simple differential operators as $L := -\Delta$. To see that this is the case, denote by $\mathbf{0}$ the origin of $\mathbb{R}^2$ and consider the two-dimensional domain*

$$\Omega := \Big\{(x,y) \in B\big(\mathbf{0}, e^{-1}\big) \setminus \{\mathbf{0}\} : \sqrt{x^2+y^2} + y\ln\sqrt{x^2+y^2} < 0\Big\} \subseteq \mathbb{R}^2. \quad (14.18)$$

Then Ω satisfies an interior pseudo-ball condition at $\mathbf{0} \in \partial\Omega$ given that, in fact,

$$\Omega = \mathscr{G}_{1,1}^{\omega_{0,-1}}(\mathbf{0}, \mathbf{e}_2) \quad (14.19)$$

where the shape function $\omega_{0,-1}$ is as in (1.14)*; that is, $\omega_{0,-1}(t) = \frac{-1}{\ln t}$ if $t \in (0, \frac{1}{e}]$ and $\omega_{0,-1}(0) = 0$.*

Next, pick $\varepsilon \in (0, \frac{1}{2})$ and define $u : \Omega \cup \{\mathbf{0}\} \to \mathbb{R}$ by setting for each $(x,y) \in \Omega \cup \{\mathbf{0}\}$,

$$u(x,y) := \begin{cases} \Big(y + \dfrac{\sqrt{x^2+y^2}}{\ln\sqrt{x^2+y^2}}\Big)\big(-\ln\sqrt{x^2+y^2}\big)^{-\varepsilon} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } (x,y) = \mathbf{0}. \end{cases} \quad (14.20)$$

Then, clearly, $u \in \mathscr{C}^0(\Omega \cup \{\mathbf{0}\}) \cap \mathscr{C}^2(\Omega)$ and $u(\mathbf{0}) < u(x,y)$ for every $(x,y) \in \Omega$. Working in polar coordinates (r,θ), an elementary calculation (recall that here $L := -\Delta$) shows that, in Ω,

$$(Lu)(r,\theta) = \frac{1}{r\,(-\ln r)^{\varepsilon+3}} \Big\{ (1 - 2\varepsilon\sin\theta)(\ln r)^2 + (\varepsilon+1)(\varepsilon\sin\theta - 2)\ln r + (\varepsilon+1)(\varepsilon+2)\Big\}. \quad (14.21)$$

Since the squared logarithm in the curly brackets above has a positive coefficient given that $\varepsilon \in (0, \frac{1}{2})$, we infer that $(Lu)(x, y) \geq 0$ at each point (x, y) in Ω. On the other hand, a direct calculation gives that, for each (x, y) in Ω,

$$
\begin{aligned}
&(\partial_y u)(x, y) = \\
&\qquad \Big\{ 1 + \frac{2y}{\sqrt{x^2+y^2}} \frac{1}{\ln(x^2+y^2)} - \frac{4y}{\sqrt{x^2+y^2}} \frac{1}{\big(\ln(x^2+y^2)\big)^2} \Big\} \big(-\ln(x^2+y^2)\big)^{-\varepsilon} \\
&\qquad + \varepsilon \Big\{ \frac{2y^2}{x^2+y^2} + \frac{4y}{\sqrt{x^2+y^2}} \frac{1}{\ln(x^2+y^2)} \Big\} \big(-\ln(x^2+y^2)\big)^{-\varepsilon-1}. \qquad (14.22)
\end{aligned}
$$

Since the two expressions in curly brackets are bounded and $\varepsilon > 0$, it follows that $\lim_{y \to 0^+} (\partial_y u)(x, y) = 0$, uniformly in x. Thus, ultimately, $(D^{(inf)}_{\mathbf{e}_2} u)(\mathbf{0}) = 0$, i.e., the lower directional derivative of u at $\mathbf{0}$ along $\mathbf{e}_2$ is in fact null. As such, the conclusion in Theorem 13.3 fails. The source of this breakdown is the fact that for any continuous function $\omega : [0, R] \to [0, +\infty)$ and any $a, b > 0$ with the property that $\mathscr{G}^{\omega}_{a,b}(\mathbf{0}, \mathbf{e}_2) \subseteq \Omega$, from (14.19) we deduce that $\omega(t) \geq a^{-1}\omega_{0,-1}(t)$ for each $t > 0$ sufficiently small. Granted this and given that $\int_0^{1/e} \frac{\omega_{0,-1}(t)}{t}\, dt = +\infty$, we conclude that ω necessarily fails to satisfy Dini's integrability condition. In concert with (13.32), this ultimately shows that $\widetilde{\omega}$ fails to satisfy Dini's integrability condition.

Remark 14.8. *There exists a bounded, convex domain, which is globally of class $\mathscr{C}^1$ as well as of class $\mathscr{C}^\infty$ near all but one of its boundary points, and with the property that the conclusion in the Boundary Point Principle in Theorem 13.3 fails, even for such simple differential operators as $L := -\Delta$.*

Indeed, it suffices to show that the two-dimensional domain Ω introduced in (14.18) is convex and of class $\mathscr{C}^1$ near the origin $\mathbf{0}$ of $\mathbb{R}^2$, and of class $\mathscr{C}^\infty$ near each point on $\partial\Omega \setminus \{\mathbf{0}\}$ near the origin. With this goal in mind, we seek a representation of $\partial\Omega$

near $\mathbf{0}$ *as the graph of some real-valued function* $f \in \mathscr{C}^1\big((-r,r)\big)$, *for some small* $r > 0$, *which is* $\mathscr{C}^\infty$ *on* $(-r,r)\setminus\{0\}$, *vanishes at* 0, *and such that* $f''(x) > 0$ *for every* $x \in (-r,r)\setminus\{0\}$. *To get started, define* $F : \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}$, *by*

$$F(x,y) := \sqrt{x^2+y^2} + y\ln\sqrt{x^2+y^2}, \quad \forall\,(x,y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{14.23}$$

and note that a point (x,y) *near the origin in* $\mathbb{R}^2$ *belongs to* $\partial\Omega$ *if and only if there holds* $F(x,y) = 0$. *Also, fix an* $r \in (0, e^{-10})$ *and observe that for each* $x \in (-r,r)\setminus\{0\}$ *we have* $F(x,0) = |x| > 0$ *and the equality*

$$F(x, \sqrt{4r^2-x^2}) = 2r + \sqrt{4r^2-x^2}\ln(2r) < 0.$$

Moreover, since the partial derivative in the y*-direction,*

$$\partial_y F(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \ln\sqrt{x^2+y^2} + \frac{y^2}{x^2+y^2}$$

, *is negative in* $B(\mathbf{0},r)$, *it follows that* $F(x,\cdot)$ *is strictly decreasing near* 0. *This analysis shows that if for each fixed* $x \in (-r,r)\setminus\{0\}$ *we define* $f(x)$ *to be the unique number* $y \in (0, \sqrt{4r^2-x^2})$ *such that* $F(x,y) = 0$, *and also set* $f(0) := 0$, *then the upper-graph of* f *coincides with* Ω *near* $\mathbf{0}$ *and* $F(x,f(x)) = 0$ *for every* $x \in (-r,r)$. *Furthermore, since* f *is bounded and*

$$\sqrt{x^2+f(x)^2} + f(x)\ln\sqrt{x^2+f(x)^2} = 0, \quad \forall\, x \in (-r,r)\setminus\{0\}, \tag{14.24}$$

a simple argument shows that $\lim\limits_{x\to 0} f(x) = 0$, *so that* $f \in \mathscr{C}^0\big((-r,r)\big)$. *On the other hand, the fact that* $F(x,f(x)) = 0$ *for every* $x \in (-r,r)$ *gives, on account of the*

Implicit Function Theorem, that $f \in \mathscr{C}^\infty\big((-r,r)\setminus\{0\}\big)$ *and, for each* $x \in (-r,r)\setminus\{0\}$,

$$\begin{aligned} f'(x) &= -\frac{\frac{x}{\sqrt{x^2+y^2}}\frac{1}{\ln\sqrt{x^2+y^2}} + \frac{xy}{(x^2+y^2)}\frac{1}{\ln\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}}\frac{1}{\ln\sqrt{x^2+y^2}} + 1 + \frac{y^2}{(x^2+y^2)}\frac{1}{\ln\sqrt{x^2+y^2}}} \\ &= \frac{xf(x)\big(f(x)+\sqrt{x^2+f(x)^2}\big)}{x^2\sqrt{x^2+f(x)^2}-f(x)^3}. \qquad (14.25) \end{aligned}$$

The first formula above readily gives that $\lim\limits_{x\to 0} f'(x) = 0$. *Based on this and the Mean Value Theorem, we arrive at the conclusion that* f *is differentiable at* 0 *and* $f'(0) = 0$. *Thus, ultimately, we have* $f \in \mathscr{C}^1\big((-r,r)\big) \cap \mathscr{C}^\infty\big((-r,r)\setminus\{0\}\big)$. *Going further, based on the second formula for* f' *in* (14.25) *and* (14.24), *an involved but elementary calculation shows that for each* $x \in (-r,r)\setminus\{0\}$ *we have*

$$\begin{aligned} f''(x) &= \frac{f(x)^2(x^2+f(x)^2)}{\big(x^2\sqrt{x^2+f(x)^2}-f(x)^3\big)^3} \times \\ &\times \Big\{f(x)(2x^2+f(x)^2)\sqrt{x^2+f(x)^2}+x^4+x^2f(x)^2+f(x)^4\Big\}. \qquad (14.26) \end{aligned}$$

In turn, since $x^2\sqrt{x^2+f(x)^2} - f(x)^3 = x^3\big(\sqrt{1+(f(x)/x)^2} - (f(x)/x)^3\big) > 0$ *if* $r > 0$ *is small, thanks to the fact that* $f'(0) = 0$, *we may conclude from* (14.26) *that* $f''(x) > 0$, *as desired.*

Remark 14.9. *Here we strengthen the counterexample discussed in Remark 14.8 by showing that there exists a bounded, convex domain, which is globally of class* $\mathscr{C}^1$ *as well as of class* $\mathscr{C}^\infty$ *near all but one of its boundary points, and with the property that the conclusion in the Boundary Point Principle in Theorem 13.3 fails for* $L := -\Delta$ *even under the assumption that* u *is a null-solution in* Ω *(i.e.,* u *is a harmonic function).*

To see that this is the case, we shall work in the two-dimensional setting and, following a suggestion from [26, p. 35], for every point

$(x,y) \in \mathbb{R}^2 \setminus \big\{(x,0) : x \in (-\infty,0] \cup \{1\}\big\}$ *define*

$$u(x,y) := \mathrm{Re}\Big(\frac{x+iy}{-\ln(x+ix)}\Big) = \frac{-x\ln(\sqrt{x^2+y^2}) - y\,\mathrm{Arg}(x,y)}{(\ln(\sqrt{x^2+y^2}))^2 + (\mathrm{Arg}(x,y))^2}, \tag{14.27}$$

where Arg $: \mathbb{R}^2 \setminus \big((-\infty,0] \cup \times\{0\}\big) \to \big(-\frac{\pi}{2},\frac{\pi}{2}\big)$, *defined as*

$$\mathrm{Arg}(x,y) := \begin{cases} \arctan\Big(\frac{y}{x}\Big), & \text{if } x \geq 0,\ y \in \mathbb{R},\ (x,y) \neq (0,0), \\ \pi + \arctan\Big(\frac{y}{x}\Big), & \text{if } x < 0,\ y > 0, \\ -\pi + \arctan\Big(\frac{y}{x}\Big), & \text{if } x < 0,\ y < 0, \end{cases} \tag{14.28}$$

is the argument of the complex number $z := x + iy \in \mathbb{C}$. *In particular,* Arg *is* $\mathscr{C}^\infty$ *on its domain, and* $\partial_x \mathrm{Arg}(x,y) = -y(x^2+y^2)^{-1}$ *and* $\partial_y \mathrm{Arg}(x,y) = x(x^2+y^2)^{-1}$ *there. Next, consider the open subset of* $\mathbb{R}^2$ *given by*

$$\Omega := \Big\{(x,y) \in \mathbb{R}^2 \setminus \big\{(x,0) : x \in (-\infty,0] \cup \{1\}\big\} : u(x,y) > 0\Big\}. \tag{14.29}$$

Then $u \in \mathscr{C}^\infty(\Omega)$ *and is harmonic in* Ω, *since* u *is the real part of the complex-valued function* $\frac{z}{-\ln z}$, *which is analytic there. Moreover, it is clear that* u *may be continuously extended to* $\mathbf{0}$ *by setting* $u(\mathbf{0}) := 0$. *Also,* $u > 0$ *in* Ω *by design. To proceed, introduce the continuous function*

$$F(x,y) := \begin{cases} \frac{x}{2}\ln(x^2+y^2) + y\,\mathrm{Arg}(x,y) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \big((-\infty,0] \times \{0\}\big), \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{14.30}$$

and note that $F \in \mathscr{C}^\infty\big(\mathbb{R} \setminus \{0\}) \times \mathbb{R}\big)$. *The significance of this function stems from the fact that*

$$\Omega = \Big\{(x,y) \in \mathbb{R}^2 \setminus \big((-\infty,-1) \times \{0\}\big) : F(x,y) < 0\Big\}. \tag{14.31}$$

A careful elementary analysis of the nature of the function F *shows that there exists* $\theta \in (0,\frac{\pi}{2})$ *such that for any* $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \setminus \{0\}$ *the function* $F(\cdot,y) : \{x \in \mathbb{R} : (x,y) \in B(\mathbf{0},e^{-2})\} \to \mathbb{R}$ *is continuous, strictly decreasing, and* *satisfies* $F(0,y) > 0$ *and* $F(\sqrt{e^{-4}-y^2}, y) < 0$*. Consequently, for each* $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \setminus \{0\}$ *there exists a unique number*

$$f(y) \in \big(0, \sqrt{e^{-4}-y^2}\,\big) \quad \textit{such that} \quad F(f(y), y) = 0. \tag{14.32}$$

The Implicit Function Theorem then shows that the function $f : (-e^{-2}\sin\theta, e^{-2}\sin\theta) \setminus \{0\} \to (0, +\infty)$ *just defined is of class* $\mathscr{C}^\infty$*. Moreover, a simple argument based on* (14.32) *gives that* $\lim_{y\to 0} f(y) = 0$*. Therefore, setting* $f(0) := 0$ *extends* f *continuously to the entire interval* $(-e^{-2}\sin\theta, e^{-2}\sin\theta)$*.*

We claim that actually $f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$*. To justify this claim, we first note that, by the Implicit Function Theorem*

$$f'(y) = -\frac{(\partial_y F)(f(y), y)}{(\partial_x F)(f(y), y)} = -\frac{\frac{2f(y)y}{f(y)^2+y^2} + \mathrm{Arg}\,(f(y), y)}{\frac{1}{2}\ln(f(y)^2+y^2) + \frac{f(y)^2-y^2}{f(y)^2+y^2}}, \quad y \neq 0. \tag{14.33}$$

Given that both the numerator of the fraction in the right-hand side of (14.33) *and the expression* $(f(y)^2 - y^2)/(f(y)^2 + y^2)$ *in the denominator are bounded, while the logarithmic factor converges to* $-\infty$ *as* $y \to 0$*, we deduce that* $\lim_{y\to 0} f'(y) = 0$*. In turn, from this and the Mean Value Theorem we may then conclude that* f *is differentiable at* 0*,* $f'(0) = 0$ *and, moreover, it necessarily holds that* $f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$*.*

Moving on, if $U := \{(x,y) \in B(\mathbf{0}, e^{-2}) : |y| < e^{-2}\sin\theta\}$*, the manner in which the function* f *has been designed ensures that*

$$U \cap \Omega = U \cap \big\{(x,y) \in \mathbb{R}^2 : \, y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \;\textit{ and }\; x > f(y)\big\}. \tag{14.34}$$

The latter implies that Ω *is of class* $\mathscr{C}^1$ *near* $\mathbf{0}$ *and of class* $\mathscr{C}^\infty$ *near any point on* $\partial\Omega$ *sufficiently close to* $\mathbf{0}$*. We also claim that* Ω *is convex near* $\mathbf{0}$*. To see this, we make* *use of the fact that* $F(f(y),y)=0$ *and re-write* (14.33) *in the form*

$$f'(y)=\frac{2f(y)^2y+f(y)(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y)}{y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y)+f(y)(y^2-f(y)^2)},\quad y\neq 0. \tag{14.35}$$

Differentiating this and once more making use of the fact that $F(f(y),y)=0$ *then yields (after a lengthy yet elementary calculation)*

$$\begin{aligned}f''(y) \;=\; & \frac{1}{\big(y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y)+f(y)(y^2-f(y)^2)\big)^2}\times\Big\{\Big(5f(y)^4y\,\mathrm{Arg}\,(f(y),y)\\ & +\,2f(y)^3(f(y)^2+y^2)\,\mathrm{Arg}\,(f(y),y)^2\Big)+3f(y)^3(y^2-f(y)^2) \\ & -\,\frac{\big(2f(y)^2y+f(y)(f(y)^2+y^2)\,\mathrm{Arg}\,(f(y),y)\big)^2\big(2f(y)y\,\mathrm{Arg}\,(f(y),y)-3f(y)^2\big)}{y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y)+f(y)(y^2-f(y)^2)}\Big\}\end{aligned} \tag{14.36}$$

for $y\neq 0$*. Note that* $3f(y)^3(y^2-f(y)^2)=3f(y)^3y^2\big(1-(f(y)/y)^2\big)$ *and* $\big(1-(f(y)/y)^2\big)\to 1$ *as* $y\to 0$*. Since the last fraction in* (14.36) *may be written in the form* $f(y)^3y^2\big(-\frac{\pi^2}{4}+o(1)\big)$ *as* $y\to 0$*, this analysis shows that* $f''(y)>0$ *for all* $y\neq 0$ *sufficiently close to* 0*. The bottom line is that* Ω *is convex near* $\mathbf{0}$*.*

However, as it is easily checked from (14.27)*, the inner normal derivative of the function* u *to* $\partial\Omega$ *vanishes at the origin, so the Boundary Point Principle fails even for harmonic functions in this domain.*

A more insightful explanation is offered by the following observation. For any continuous function ω *with the property that* Ω *satisfies an interior pseudo-ball condition at* $\mathbf{0}$ *with shape function* ω*, we necessarily have* $\sqrt{f(y)^2+y^2}\,\omega(\sqrt{f(y)^2+y^2})\geq f(y)$ *for* $y>0$ *small. Hence, if* ω *is slowly growing (say,* $\omega(2t)\leq c\omega(t)$ *for all* $t>0$ *small), then* $\omega(y)\geq c\,f(y)/y$ *for all* $y>0$ *small, for some constant* $c>0$*. As a consequence, if* $R>0$ *is small then*

$$\int_0^R\frac{\omega(y)}{y}\,dy\geq\int_0^R\frac{f(y)}{y^2}\,dy=-R^{-1}f(R)+\int_0^R\frac{f'(y)}{y}\,dy, \tag{14.37}$$

after an integration by parts. However, based on formula (14.33) *and the fact that* $f(y)/y \to 0$, $\mathrm{Arg}(f(y), y) \to \pi/2$ *as* $y \to 0$*, it is not difficult to see that* $f'(y)/y \geq c/(-y \ln y)$ *for all* $y > 0$ *small, where* $c > 0$ *is a fixed constant. Hence,* $\int_0^R \frac{f'(y)}{y}\, dy = +\infty$ *which shows that* Ω *fails to satisfy an interior pseudo-ball condition at* $\mathbf{0}$ *with a shape function for which Dini's integrability condition holds.*

In the context of Theorem 13.3, the significance of this failure is that any function $\widetilde{\omega}$ *for which* (13.23) *holds will, thanks to* (13.32)*, necessarily fail to satisfy Dini's integrability condition, thus contradicting the last condition in* (13.22).

The harmonic function $u(x, y) := xy$ for $x, y > 0$ is a counterexample to the Boundary Point Principle for $L := -\Delta$ when Ω is the first quadrant in the two-dimensional setting. A related counterexample in an arbitrary sector in the plane is presented in [81, Example 1.6]. Compared to these, the counterexamples discussed in Remark 14.8 and Remark 14.9 are considerably stronger since they deal with open sets from the much more smaller class of $\mathscr{C}^1$ domains whose unit normal has a modulus of continuity which fails to satisfy Dini's integrability condition.

We conclude this chapter with a comment pertaining to the nature of the Boundary Point Principle proved by M. Safonov in [82, Theorem 4.3 and Remark 4.4, p. 18]. Specifically, the demands here are that L is uniformly elliptic and that a truncated circular cylinder Q which touches the boundary at x_0 may be placed inside Ω and that the drift coefficients belong to $L^q(\Omega)$ for some $q > n$. What we wish to note here is that there exist vector fields $\vec{b} = (b^1, ..., b^n)$ which satisfy (13.113)-(13.114) for some shape function ω as in (13.111) but for which

$$\vec{b} \notin \bigcup_{q>n} L^q(\Omega). \tag{14.38}$$

For example, one may take $\omega : (0, 1/e) \to (0, +\infty)$ given by $\omega(t) := (\ln t)^{-2}$ for each $t \in (0, 1/e)$, and $\vec{b} : \Omega \to \mathbb{R}$ such that

$$|\vec{b}(x)| \approx \frac{1}{|x - x_0|(\ln |x - x_0|)^2}, \quad \text{uniformly for } x \in \Omega. \tag{14.39}$$

Chapter 15

The Strong Maximum Principle for Non-uniformly Elliptic Operators with Singular Drift

The Strong Maximum Principle (SMP) is a bedrock result in the theory of second-order elliptic partial differential equations, since it enables us to derive information about solutions of differential inequalities without any explicit knowledge of the solutions themselves. In reference to the seminal work of E. Hopf in [39], J. Serrin wrote in [63, p. 9]: "*It has the beauty and elegance of a Mozart symphony, the light of a Vermeer painting. Only a fraction more than five pages in length, it still contains seminal ideas which are still fresh after 75 years.*" The traditional formulation of SMP typically requires the coefficients to be locally bounded (among other things), and here our goal is to prove a version of SMP in which this assumption is relaxed to an optimal pointwise blow-up condition. Specifically, we shall prove the following theorem.

Theorem 15.1. *Let Ω be a nonempty, connected, open subset of $\mathbb{R}^n$, and suppose* *that*

$$L := -\mathrm{Tr}\big(A\,\nabla^2\big) + \vec{b}\cdot\nabla = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i \tag{15.1}$$

is a (possibly non-uniformly) elliptic second-order differential operator in non- divergence form (without a zeroth-order term) in Ω*. Also, assume that for each* $x_0 \in \Omega$ *and each* $\xi \in S^{n-1}$ *there exists a real-valued function* $\widetilde{\omega} = \widetilde{\omega}_{x_0,\xi}$ *satisfying*

$$\widetilde{\omega} \in \mathscr{C}^0\big([0,1]\big), \quad \widetilde{\omega}(t) > 0 \;\; \textit{for each} \;\; t \in (0,1], \qquad \int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < \infty, \tag{15.2}$$

and with the property that

$$\limsup_{\substack{(x-x_0)\cdot\xi>0\\ x\to x_0}} \frac{\Big(\mathrm{Tr}\,A(x)\Big) + \max\Big\{0\,,\vec{b}(x)\cdot\xi\Big\} + \Big(\sum\limits_{i=1}^{n}\max\big\{0\,,\,-b^i(x)\big\}\Big)|x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}\Big((A(x)\xi)\cdot\xi\Big)} < \infty. \tag{15.3}$$

Let $u \in \mathscr{C}^2(\Omega)$ *be a function which satisfies the differential inequality* $(Lu)(x) \geq 0$ *for all* $x \in \Omega$*. Then*

if u *assumes a global minimum value at some point in* Ω*, it follows that* u *is constant in* Ω*.* (15.4)

Remark 15.1. *We wish to emphasize that no assumption on the (Lebesgue) measurability of the coefficients* a^{ij}*,* b^i*, of the operator* L *is made in the statement of the above theorem.*

Proof of Theorem 15.1. The proof proceeds along the lines of the classical Hopf's Strong Maximum Principle (as presented in, e.g., [26, Theorem 3.5, p. 35]), with the Boundary Point Principle established in Theorem 13.5 replacing its weaker, more familiar, counterpart. With the goal of arriving at a contradiction, suppose that $u \in \mathscr{C}^2(\Omega)$ is a non-constant function satisfying $Lu \geq 0$ in Ω and which assumes a global minimum value $M \in \mathbb{R}$ at some point $x_* \in \Omega$. Then if

$U := \{x \in \Omega : u(x) = M\}$, it follows that U is a nonempty, relatively closed, proper subset of the connected set Ω hence, in order to reach a contradiction, it suffices to show that U is open, i.e. that $U \setminus U^\circ = \varnothing$. To this end, reason by contradiction and assume that there exists $y \in U \setminus U^\circ$. Since Ω is open and $y \in \Omega$, one may pick $r > 0$ such that $B(y, r) \subseteq \Omega$. On the other hand, the fact that $y \in U \setminus U^\circ$ implies that $B(y, r/2)$ is not contained in U. Hence, there exists $z \in B(y, r/2) \setminus U$ and we select $x_0 \in U$ with the property that $\operatorname{dist}(z, U) = |z - x_0| =: R > 0$ (since U is relatively closed). In turn, such a choice forces $\operatorname{dist}(z, \partial\Omega) > r/2 > |y - z| \geq \operatorname{dist}(z, U) = R$, hence ultimately

$$B(z, R) \subseteq \Omega \setminus U \quad \text{and} \quad x_0 \in U \cap \partial B(z, R). \tag{15.5}$$

For further use, let us also note here that the fact that $x_0 \in U$ and (15.5) entail, respectively,

$$(\nabla u)(x_0) = 0 \qquad \text{and} \qquad x_0 \in \partial(\Omega \setminus U). \tag{15.6}$$

To proceed, define $h := R^{-1}(z - x_0) \in S^{n-1}$ and let $\widetilde{\omega} : (0, 1) \to (0, +\infty)$ be the function associated with the point $x_0 \in \Omega$ and the vector $h \in S^{n-1}$ as in the statement of the theorem. On account of (15.5) it follows that the open, nonempty set $\Omega \setminus U$ satisfies a pseudo-ball condition at the point $x_0 \in \partial(\Omega \setminus U)$ with shape function $\omega(t) := t$ and direction vector $h = R^{-1}(z - x_0) \in S^{n-1}$. Also, thanks to (15.2)-(15.3), properties (13.23)-(13.24) are satisfied. Since $u(x_0) = M < u(x)$ for each $x \in \Omega \setminus U$, the conclusion in Theorem 13.3 applies with Ω replaced by $\Omega \setminus U$ and, say, $\vec{\ell} := h \in S^{n-1}$. In the current context, this yields

$$0 < (D^{(\mathrm{inf})}_{\vec{\ell}} u)(x_0) = \vec{\ell} \cdot (\nabla u)(x_0), \tag{15.7}$$

which contradicts the first condition in (15.6). $\square$

Remark 15.2. *In the original formulation of the SMP in Hopf's 1927 paper [39], the coefficient matrix of the top-order part of the differential operator L is assumed to be locally uniformly positive definite in Ω, and the drift coefficients locally bounded in Ω. See also [56, pp. 14-15], [76, p. 14]. The version of the SMP given in the reference [77, Theorem 5 on p. 61 and Remark (i) on p. 64] and [26, p. 35] is slightly more general (and natural), in the sense that the conditions on the coefficients of the second- and first-order terms of L are*

$$(A(x)\xi)\cdot\xi > 0 \ \ \textit{for each} \ \ x \in \Omega \ \ \textit{and} \ \ \xi \in S^{n-1}, \quad \textit{and the quantities} \tag{15.8}$$

$$\frac{\operatorname{Tr} A(x)}{\min\limits_{\xi\in S^{n-1}} (A(x)\xi)\cdot\xi} \ \ \textit{and} \ \ \frac{|\vec{b}(x)|}{\min\limits_{\xi\in S^{n-1}} (A(x)\xi)\cdot\xi} \ \ \textit{are locally bounded in } \Omega. \tag{15.9}$$

Compared with the status-quo, our main contribution in Theorem 15.1 is weakening (15.9) *to the blow-up condition for the coefficients formulated in* (15.3)*. Of course, the key factor in this regard, is the more flexible version of the Boundary Point Principle proved in Theorem 13.3.*

Remark 15.3. *Theorem 15.1 readily implies a weak minimum principle of the following form. Let Ω be a nonempty, bounded, open subset of $\mathbb{R}^n$ and retain the same assumptions on L as in the statement of Theorem 15.1. Then, if $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ is a function which satisfies the differential inequality $(Lu)(x) \geq 0$ for all $x \in \Omega$, one has*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{15.10}$$

Theorem 15.1 is sharp, in the sense which we now describe. Fix two numbers $\alpha > 1$, $\beta > 0$ and, for each $i \in \{1, ..., n\}$, define the function $b^i : B(0,1) \to \mathbb{R}$ by setting

$$b^i(x) := \begin{cases} (n+\beta)\dfrac{x_i}{|x|^\alpha} & \text{if } \ x \in B(0,1) \setminus \{0\}, \\ 0 \ \text{ if } \ x = 0. \end{cases} \tag{15.11}$$

Next, consider the differential operator

$$L := -\Delta + \sum_{i=1}^{n} b^i(x)\partial_i \quad \text{in} \quad B(0,1), \tag{15.12}$$

and note that if

$$u : \overline{B(0,1)} \to \mathbb{R}, \quad u(x) := |x|^{2+\beta}, \quad \forall\, x \in \overline{B(0,1)}, \tag{15.13}$$

then

$$\begin{aligned} & u \in \mathscr{C}^2\big(\overline{B(0,1)}\big), \qquad \nabla u(x) = (\beta+2)|x|^\beta x \quad \text{and} \\ & \Delta u(x) = (\beta+2)(n+\beta)|x|^\beta \ \text{ for each } \ x \in \overline{B(0,1)}. \end{aligned} \tag{15.14}$$

Moreover, u is a nonconstant function which attains its global minimum at the origin. More precisely,

$$u \geq 0 \ \text{ in } \ B(0,1), \quad u(0) = 0 \ \text{ and } \ u\big|_{\partial B(0,1)} = 1. \tag{15.15}$$

Furthermore,

$$(Lu)(0) = 0 \ \text{ and for each } \ x \in B(0,1) \setminus \{0\}, \tag{15.16}$$

$$(Lu)(x) = (\beta+2)(n+\beta)|x|^\beta\big[1 - |x|^{2-\alpha}\big], \tag{15.17}$$

which shows that

$$\alpha \geq 2 \iff (Lu)(x) \geq 0 \ \text{ for each } \ x \in B(0,1). \tag{15.18}$$

On the other hand, given a function $\widetilde{\omega} : (0,1) \to (0,+\infty)$ and a vector $\xi \in S^{n-1}$, condition (15.3) entails

$$\limsup_{\substack{x\cdot\xi>0 \\ x\to 0}} \frac{|x|^{-\alpha}\, x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}} < +\infty \tag{15.19}$$

which, when specialized to the case when x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0, +\infty)$ such that $\widetilde{\omega}(t) \geq c\, t^{2-\alpha}$ for all small $t > 0$. In turn, this readily shows that

$$\exists\, \widetilde{\omega} : (0,1) \to (0, +\infty) \text{ such that (15.3) holds} \tag{15.20}$$

$$\text{and } \int_0^1 \frac{\widetilde{\omega}(t)}{t}\, dt < +\infty \iff \alpha < 2. \tag{15.21}$$

The bottom line is that, in the context of the situation considered above, the range of α's for which the conclusion in Theorem 15.1 fails is precisely the complement of the range of α's for which the blow-up condition described in (15.3) is violated (compare (15.18) with (15.20)). Hence, Theorem 15.1 is optimal.

Chapter 16

Applications to Boundary Value Problems

Of course, a direct corollary of the Strong Maximum Principle established in Chapter 15 is the uniqueness in the Dirichlet problem formulated in the geometrical-analytical context considered in Theorem 15.1. We aim at proving similar results for Neumann and oblique type boundary value problems.

In the subsequent discussion, suppose that Ω is a nonempty, open, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu : \partial^*\Omega \to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω (cf. Chapter 9). In addition, consider a second-order, elliptic, differential operator L, in non-divergence form, as in (13.20). In this context, the goal is to assign a concrete meaning to the *conormal derivative associated with the operator* L, which is originally formally expressed (at boundary points) as

$$\partial_\nu^L := -\sum_{i,j=1}^{n} a^{ij}\nu_i\partial_j = \Big(-\sum_{i,j=1}^{n} a^{ij}\nu_i\,\mathbf{e}_j\Big)\cdot\nabla \tag{16.1}$$

where $(\nu_i)_{1\leq i\leq n}$ are the components of ν. To this end, fix a point $x_0 \in \partial^*\Omega$ and assume that

$$\begin{aligned}&L \text{ is uniformly elliptic near } x_0 \text{ and its top-order}\\&\text{coefficients may be continuously extended at } x_0.\end{aligned} \tag{16.2}$$

In this setting, define the vector

$$\mathbf{n} := \mathbf{n}(L, \Omega, x_0) := -\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j \in \mathbb{R}^n \tag{16.3}$$

and note that, since $\nu(x_0) \in S^{n-1}$, we have

$$\mathbf{n} \cdot \nu(x_0) = -\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\nu_j(x_0) < 0. \tag{16.4}$$

In particular, this shows that $\mathbf{n} \neq 0$. Finally, make the assumption that, in the sense of Definition 13.1,

$$\mathbf{n} \text{ points in } \Omega \text{ at } x_0. \tag{16.5}$$

Then, given a function $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^1(\Omega)$, formula (16.3) and the second equality in (16.1) suggest defining

$$\partial_\nu^L u(x_0) := \big(D_{\mathbf{n}}^{(\mathrm{inf})} u\big)(x_0). \tag{16.6}$$

Let us also agree to drop the dependence on L when writing ∂_ν^L in the special case when $L = -\Delta$, in which scenario $\partial_\nu := -\sum_{i=1}^{n} \nu_i \partial_i$ will be simply referred to as the *inner normal derivative to* $\partial\Omega$.

Before concluding this preliminary discussion, we wish to note that

$$\begin{aligned}&\text{if } \Omega \text{ is of locally finite perimeter, satisfying an interior pseudo-ball}\\&\text{condition at } x_0 \in \partial^*\Omega, \text{ and if } L \text{ is as in (16.2) then (16.5) holds.}\end{aligned} \tag{16.7}$$

Indeed, in this scenario Proposition 8.1 shows that $-\nu(x_0) \in S^{n-1}$ is the direction vector for the pseudo-ball at x_0. Then (16.5) follows from this and (16.4), by Theorem 13.3.

Proposition 16.1. *Suppose Ω is a nonempty, open, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu : \partial^*\Omega \to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω. Assume that $x_0 \in \partial^*\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0 for a shape function $\omega : [0, R] \to [0, +\infty)$ satisfying the properties listed in* (13.18)-(13.19) *as well as Dini's integrability condition. Also, suppose that $\vec{\ell} \in S^{n-1}$ is a vector which is inner transversal to $\partial\Omega$ at x_0, in the sense that*

$$\vec{\ell} \cdot \nu(x_0) < 0. \tag{16.8}$$

Next, consider a second-order, differential operator L, in non-divergence form, as in (13.20)*, which is uniformly elliptic near x_0 and whose top-order coefficients, originally defined in Ω, may be continuously extended at the point $x_0 \in \partial\Omega$. In addition, assume that there exists a real-valued function $\widetilde{\omega} \in \mathscr{C}^0([0, R])$, positive on $(0, R]$, satisfying $\int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty$, and with the property that*

$$\limsup_{\substack{\Omega \ni x \to x_0 \\ (x-x_0)\cdot\nu(x_0) > 0}} \frac{\max\Big\{0\,, \vec{b}(x) \cdot \nu(x_0)\Big\} + \Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\, -b^i(x)\big\}\Big)\omega(|x - x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot\nu(x_0))}{(x-x_0)\cdot\nu(x_0)}} < +\infty. \tag{16.9}$$

Finally, suppose that $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ is a real-valued subsolution of L in Ω which has a strict global minimum at x_0 (in the sense of (13.26)-(13.27)*). Then the vector $\vec{\ell}$ points inside Ω at x_0 and*

$$\big(D^{(inf)}_{\vec{\ell}} u\big)(x_0) > 0. \tag{16.10}$$

In particular, with ∂_ν and ∂^L_ν denoting, respectively, the inner normal derivative to $\partial\Omega$, and the conormal derivative associated with L, one has

$$(\partial_\nu u)(x_0) > 0 \quad \textit{and} \quad (\partial^L_\nu u)(x_0) > 0. \tag{16.11}$$

Proof. Proposition 8.1 shows that $-\nu(x_0) \in S^{n-1}$ is the direction vector for the pseudo-ball at x_0. Granted this, the inequality in (16.10) becomes a consequence of (13.30). Then the two inequalities in (16.11) are obtained by specializing (16.10), respectively, to the case when $\vec{\ell} := -\nu(x_0) \in S^{n-1}$, and to the case when

$$\vec{\ell} := -\frac{\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j}{\left|\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j\right|} \in S^{n-1}, \tag{16.12}$$

which is a well-defined unit vector satisfying (16.8) (by the uniform ellipticity of L). □

Corollary 16.2. *With the same background assumptions on the operator L and the function u as in Proposition 16.1, all earlier conclusions hold in domains of class $\mathscr{C}^{1,\omega}$ provided ω satisfies* (7.11), (13.18)-(13.19), *as well as Dini's integrability condition.*

This is sharp, in the sense that there exists a bounded domain of class $\mathscr{C}^1$ (which is even convex and of class $\mathscr{C}^\infty$ near all but one of its boundary points) for which the aforementioned conclusions fail.

Proof. The claim in the first part of the statement is a direct consequence of Theorem 11.3 and Proposition 16.1. Its sharpness is implied by the counterexamples described earlier, in Remark 14.9 and Remark 14.8. □

Theorem 16.3. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, open, connected, bounded set and consider a second-order, elliptic differential operator L, in non-divergence form in Ω, as in* (13.20)*. Also, suppose that there exists a family of real-valued functions $\widetilde{\omega}_{x,\xi} \in \mathscr{C}^0\big([0,1]\big)$, indexed by $x \in \overline{\Omega}$ and $\xi \in S^{n-1}$, each positive on $(0,1)$ and satisfying Dini's integrability condition, such that the following two properties hold:*

(i) for each $x \in \partial\Omega$ there exists $h = h_x \in S^{n-1}$ so that Ω satisfies an interior pseudo-ball condition at x with shape function $\omega = \omega_x$ satisfying the properties listed in (13.18)-(13.19)*, and direction vector h, for which*

$$\limsup_{\substack{\Omega \ni y \to x \\ (y-x)\cdot h > 0}} \frac{\frac{\omega(|y-x|)}{|y-x|}\Big(\mathrm{Tr}\, A(y)\Big) + \max\Big\{0\,, \vec{b}(y)\cdot h\Big\}}{\frac{\widetilde{\omega}_{x,h}((y-x)\cdot h)}{(y-x)\cdot h}\Big((A(y)h)\cdot h\Big)} \quad \textit{and} \tag{16.13}$$

$$\limsup_{\substack{\Omega \ni y \to x \\ (y-x)\cdot h > 0}} \frac{\Big(\sum_{i=1}^{n} \max\big\{0\,,\, -b^i(y)\big\}\Big)\omega(|y-x|)}{\frac{\widetilde{\omega}_{x,h}((y-x)\cdot h)}{(y-x)\cdot h}\Big((A(y)h)\cdot h\Big)} \tag{16.14}$$

are finite;

(ii) for each $x \in \Omega$ and each $\xi \in S^{n-1}$,

$$\limsup_{\Omega \ni y \to x} \frac{\Big(\mathrm{Tr}\, A(y)\Big) + \max\Big\{0\,, \vec{b}(y)\cdot \xi\Big\} + \Big(\sum_{i=1}^{n} \max\big\{0\,,\, -b^i(y)\big\}\Big)|y-x|}{\frac{\widetilde{\omega}_{x,\xi}((y-x)\cdot \xi)}{(y-x)\cdot \xi}\Big((A(y)\xi)\cdot \xi\Big)} < \infty. \tag{16.15}$$

Finally, assume that $\vec{\ell} : \partial\Omega \to S^{n-1}$ is a vector field with the property that

$$\vec{\ell}(x)\cdot h_x > 0 \quad \textit{for each} \ \ x \in \partial\Omega. \tag{16.16}$$

Then for each $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ one has

$$u \textit{ is constant in } \overline{\Omega} \iff \begin{cases} (Lu)(x) \geq 0 \quad \textit{for each} \ \ x \in \Omega, \\ \big(D^{(inf)}_{\vec{\ell}(x)} u\big)(x) \leq 0 \quad \textit{for each} \ \ x \in \partial\Omega. \end{cases} \tag{16.17}$$

In particular, one has uniqueness for the oblique derivative boundary value problem for L in Ω, i.e., for any given data $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$, there is at most one function u satisfying

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ (Lu)(x) = f(x) \quad \textit{for each} \ \ x \in \Omega, \\ \vec{\ell}(x)\cdot(\nabla u)(x) = g(x) \ \ \textit{for each } x \in \partial\Omega. \end{cases} \tag{16.18}$$

As a consequence, if Ω *is also of finite perimeter and has the property that* $\partial^*\Omega = \partial\Omega$*, and if* L *is actually uniformly elliptic and its top-order coefficients belong to* $\mathscr{C}^0(\overline{\Omega})$*, then*

$$u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \quad Lu \geq 0 \ \textit{in} \ \Omega, \ \textit{and} \tag{16.19}$$

$$\partial_\nu^L u \leq 0 \ \textit{on} \ \partial\Omega \Longrightarrow u \textit{ is constant in } \overline{\Omega}. \tag{16.20}$$

Hence, in this setting, one has uniqueness for the Neumann boundary value problem for L *in* Ω*, i.e., for any given data* f, g *there is at most one function* u *satisfying*

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ Lu = f \quad \textit{in} \ \Omega, \\ \partial_\nu^L u = g \ \textit{on} \ \partial\Omega. \end{cases} \tag{16.21}$$

Finally, all these results are sharp in the sense that, even in the class of uniformly elliptic operators with constant top coefficients, condition (16.15) *may not be relaxed to*

$$\limsup_{\Omega \ni y \to x} \big[|x-y||\vec{b}(y)|\big] < +\infty, \quad \forall\, x \in \overline{\Omega}. \tag{16.22}$$

Proof. As a preliminary matter, we note that (16.16) and the fact that, by (i), Ω satisfies an interior pseudo-ball condition at each $x \in \partial\Omega$ with unit direction vector $h_x \in S^{n-1}$, imply that $\vec{\ell}(x)$ points inside Ω for each $x \in \partial\Omega$ (cf. the proof of Theorem 13.3). In particular, $\big(D^{(\inf)}_{\vec{\ell}(x)} u\big)(x)$ is well-defined for each $x \in \partial\Omega$. To proceed, assume that $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ attains a strict global minimum on $\partial\Omega$, i.e., there exists a point $x_0 \in \partial\Omega$ such that $u(x_0) < u(x)$ for all $x \in \Omega$. In this case, granted property (i) in the statement of the theorem, Theorem 13.5 yields $\big(D^{(\inf)}_{\vec{\ell}(x_0)} u\big)(x_0) > 0$, contradicting the second condition in the right-hand side of (16.17). Thus, $u \in \mathscr{C}^0(\overline{\Omega})$

attains its minimum in Ω. In concert with the assumption that Ω is connected, property (ii) in the statement of the theorem, and the fact that $Lu \geq 0$ in Ω, the Strong Maximum Principle established in Theorem 15.1 allows us to conclude u is constant in Ω. This proves (16.17) which, in turn, readily yields uniqueness in the oblique boundary value problem (16.18).

As far as (16.19) is concerned, the fact that $\partial^*\Omega = \partial\Omega$ ensures that the geometric measure theoretic outward unit normal ν to Ω is everywhere defined on $\partial\Omega$. Thus, if the top-order coefficients of L belong to $\mathscr{C}^0(\overline{\Omega})$, we may define

$$\vec{\ell}: \partial\Omega \longrightarrow S^{n-1}, \quad \vec{\ell}(x) := -\frac{\sum_{i,j=1}^n a^{ij}(x)\nu_i(x)\,\mathbf{e}_j}{\left|\sum_{i,j=1}^n a^{ij}(x)\nu_i(x)\,\mathbf{e}_j\right|} \quad \text{for every} \quad x \in \partial\Omega. \tag{16.23}$$

Now (16.19) follows by specializing (16.17) to this choice of a vector field.

Finally, to see that the above results are sharp, take $\Omega := B(0,1) \subseteq \mathbb{R}^n$ and consider the differential operator L and the function $u \in \mathscr{C}^2\big(\overline{B(0,1)}\big)$ as in (1.38). Then

$$(Lu)(x) = 0 \ \text{ for each } \ x \in B(0,1), \quad \text{and} \tag{16.24}$$

$$(\partial_\nu^L u)(x) = -\tfrac{4}{n+2} \leq 0 \ \text{ for each } \ x \in \partial B(0,1) \tag{16.25}$$

which shows that (16.19) fails in this case, precisely because the blow-up of the drift at the origin is of order one, i.e., $|\vec{b}(x)| = |x|^{-1}$ for $x \in B(0,1) \setminus \{0\}$. $\square$

Corollary 16.4. *With the same background assumptions on the operator L and the function u as in Theorem 16.3, all conclusions in this theorem hold in bounded connected domains of class $\mathscr{C}^{1,\omega}$ in $\mathbb{R}^n$ provided ω satisfies* (7.11), (13.18)-(13.19), *as well as Dini's integrability condition.*

Proof. This readily follows from Theorem 11.3 and Theorem 16.3. □

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