FUNCTIONS IN CONTEMPORARY SECONDARY MATHEMATICS TEXTBOOK SERIES IN THE UNITED STATES

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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a candidate for the degree of Doctor of Philosophy,

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ABSTRACT

Textbooks play a central role in US mathematics classrooms (Stein, Remillard, & Smith, 2007) and functions are a key topic in secondary mathematics (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). This study presents results from an analysis of this essential topic in the latest editions of three textbook series: the Glencoe Mathematics series, the University of Chicago School Mathematics Project series, and the Core-Plus Mathematics Project series. In each series, functions were examined in four areas: language used in relation to functions, presence of functions, core features of function examples, and ancillary features of function examples. Language used in definitions generally indicated univalence, arbitrariness, and universal quantification, but beyond these there was little consistency. Function examples were prevalent in all series. Examples were most often represented with equations or formulas and were predominantly polynomial functions, and especially linear. They mostly appeared in homework exercises and in abstract settings. Most examples were not actually identified as functions, explicit recommendations for using technology with examples were relatively infrequent, and opportunities for students to generate function examples or explore non-examples were limited. All series did include multiple representations of functions with examples. Many examples included verbal descriptions, while there were smaller proportions of numeric and graphic representations. These findings can provide valuable information for administrators and teachers using or selecting textbooks and curriculum developers as they plan new or revised textbooks. They also serve as an initial stage for research into how curriculum influences student learning of function.

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Rationale for the Study

In the United States, textbooks have been the center of heated debates about how to help students learn mathematics for over two decades (Schoenfeld, 2004). Some mathematics educators have seen textbooks as a potential vehicle to bring reform into mathematics classrooms (Remillard, 2005). Such educators believe that changes to textbooks can help build and support high quality teaching and bring students a deeper and richer mathematical learning experience. Schoenfeld noted that others have argued that textbooks described as “reform” dilute and distort mathematical content. Thus, these educators claim these books weaken necessary standards of mathematical rigor and only provide students opportunities for developing superficial mathematical understanding. The debates currently continue on academic, political, and personal fronts.

One significant factor in the dispute was the National Council of Teachers of Mathematics’ (NCTM) publication of the *Curriculum and Evaluation Standards for School Mathematics* (1989). The document called for significant changes in school mathematics curricula. The *Standards* (1989) advocated increased attention to active involvement of students in problem solving, mathematical communication, drawing connections, and technology use. As promoted by NCTM, teachers were to use a variety of instructional and assessment techniques. Memorization of facts and procedures, pencil and paper skill work, and teacher exposition were to receive decreased emphasis. As a specific example, consider recommendations related to the concept of function, one of the core concepts for grades 9-12. The *Standards* called for students to generate models of functions from data rich problems and create and explore multiple representations of functions. According to the report, students should be able to use the same type of function to solve a variety of problems but should also be able to use a wide variety of functions flexibly.

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) followed about a decade later. Emphasis was on helping all students develop their ability to solve problems, reason and prove, communicate, make connections, and create and understand representations in key content areas of number and operations, algebra, geometry, measurement, and data analysis and probability. Again, with respect to function, the *Principles and Standards* emphasized the role that functions can play throughout the K-12 curriculum and especially at the secondary level.

The NCTM's *Standards* (1989, 2000) have influenced the development of a wide range of textbooks. Initially, in response to the 1989 *Standards*, the National Science Foundation provided funding for mathematics textbook development projects, including five secondary school projects: the Core-Plus Mathematics Project, the Interactive Mathematics Program, MATH Connections, Systemic Initiative for Montana Mathematics and Science, and Applications/Reform in Secondary Mathematics (Senk & Thompson, 2003). The developers sought to embody the recommendations of the 1989 *Standards* in their textbooks. Because the NCTM called for changes from what was typical, the *Standards*-based textbooks differed significantly from typical textbooks series used in the US at that time (Hirsch, 2007).

However, the five NSF-supported textbook series were not the only mathematics textbooks to be influenced by the *Standards* (1989, 2000). Usiskin (2003) pointed out the close relationships between the *Standards* and the University of Chicago School Mathematics Project textbook series. The NSF has also provided funding more recently for textbook development projects, such as the Educational Development Corporation's series Center for Mathematics Education (CME) Project (2009a, 2009b, 2009c, 2009d). Additionally, authors of other textbook series that have not received funding from the NSF have pointed out their connections to the *Standards*. For example, Glencoe's *Algebra 2* (Holliday, et al., 2005) includes a section detailing alignment between the series and specific sections of the *Principles and Standards* (National Council of Teachers of Mathematics, 2000). Thus, the changes recommended by the *Standards* have impacted textbooks, according to the authors. Debates continue about whether and how these changes in textbooks have influenced student learning.

There are some broad indications that the influence of NCTM standards on textbooks have not as yet led to improvement in student learning in mathematics. In fact, textbooks developed with funding from the NSF have not been widely adopted for use (Resnick, 2008). Thus, few students have experienced a K-12 curriculum designed to embody the recommendations in the *Standards* (1989, 2000). Although numerous students have studied from publisher-generated textbooks, many of which claim to that have been influenced by the *Standards* in some way, US students, especially at the secondary level, have continued to demonstrate relatively flat achievement on international mathematics comparisons such as the Trends in Mathematics and Science Study (TIMSS), as well as national measures such as the National Assessment of Educational Progress (NAEP, Dossey, 2003). Mathematics educators, researchers, and policy makers have sought to find contributing factors and make needed changes. The

TIMSS finding that textbooks in the US are a 'mile wide and inch deep' suggests that they are one major contributing factor (Schmidt, McKnight, & Raizen, 1997).

Although mathematics education researchers currently recognize that mathematics textbooks are not the sole reason for students' low scores and weak understanding of mathematics, researchers have argued repeatedly that textbooks do play an important role in educating mathematics students (McKnight, et al., 1987; Stein, Remillard, & Smith, 2007; Willoughby, 2010). Teachers make significant use of textbooks to guide the mathematical content that they teach and the design of lessons (Grouws & Smith, 2000; Tarr, Chavez, Reys, & Reys, 2006). Students draw upon textbooks as resources and textbooks provide them one of their major opportunities to learn (Reys, Reys, & Chavez, 2004; Stein, et al., 2007).

In addition to the fact that textbooks play a central role in what happens in mathematics classrooms, researchers recognize that the efficacy of any mathematics program cannot be understood without close examination of the written materials. The National Research Council (2004) examined studies of curricular effectiveness and concluded that more high quality studies were needed, arguing that any evaluation of the effectiveness of a mathematics curriculum must include multiple analyses of the textbooks involved. They called for content analyses that provide insight into the quality of the mathematics embodied in the textbook.

The quality of any mathematics textbook is related to how it presents essential topics. Function is a core mathematical concept that plays key roles in both secondary and post-secondary mathematics (National Council of Teachers of Mathematics, 1989, 2009; Oehrtman, Carlson, & Thompson, 2008). Throughout the history of mathematics education in the United States, numerous arguments for the centrality of function in the secondary and post-secondary curriculum have been made. Beginning in the early 1900's, the National Committee on Mathematical Requirements (1923) called for functions to play a focal role in the secondary curriculum. In the School Mathematics Study Group's (SMSG) endeavor to restructure the mathematics curriculum in the 1960's, the concept was considered important enough to create an entire textbook dedicated to function in their secondary curriculum (Allen, et al., 1961). *Everybody Counts* (National Research Council, 1989) called for students' development of function sense, defined as "a familiarity with expressing relations among variables" (p. 51). The NCTM's *Standards* documents (1989, 2000) described function as an important unifying idea in mathematics, especially in the secondary curriculum.

Most recently, the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011) has again emphasized the role of functions in secondary mathematics. Function is one of the six main topics addressed in the standards, which recommend that students be able to interpret and build functions as well as model real world situations with a variety of functions and function representations. The *Common Core State Standards* have been formally adopted by 44 states, and it appears their emphasis on function will have an impact on curriculum, instruction and assessment across the US in future years.

In addition to being an important topic in secondary mathematics, it is also vital for advanced mathematical studies in calculus, algebra, geometry, and topology (Oehrtman, et al., 2008). Tall and Bakar (1992) noted, "The concept of a function permeates every branch of mathematics and occupies a central position in its development…" (p. 39). It follows that understanding how effectively secondary mathematics textbook series help students learn about this central concept and how well these textbooks prepare them for success in post-secondary mathematics requires analyses of textbooks specific to the concept of function.

Numerous approaches to studying functions in secondary textbooks are possible. Certainly the definition of function in textbooks is important. However, Tall and Vinner (1981) argued that the images of a concept that students form also play key roles in how the students understand and use the concept. The examples of functions that students explore heavily influence the concept images of functions that students form (Alcock & Simpson, 2004; Habre & Abboud, 2006). Because students' experiences in the classroom are often significantly based on their textbooks (Grouws, Smith, & Sztajn, 2004; Weiss, Banilower, McMahon, & Smith, 2001) and teachers use textbooks as a major resource as they plan their choice of examples to use in the classroom (Zodik & Zaslavsky, 2008), it is reasonable to assume that the examples of functions that textbooks include or do not include have an influence on the concept image of functions that students develop. Recent research has indicated that the entire collection of examples experienced and explored by students is influential in the development of their broad understanding of mathematical concepts (Bills & Watson, 2008; Goldenberg & Mason, 2008). Also important in students' learning are the examples that they are required to generate (Asghari, 2005; Zaslavsky & Shir, 2005). Thus, an examination of textbooks' examples of functions both provided for and requested of students informs an understanding of the potential impact on their concept images of function.

The development of these concept images is important because limitations in students' concept images of function can restrict their cognition in relation to function. On the other hand, well-developed concept images can provide students powerful insights. For example, Alcock and Simpson (2004) argued that students with concept images of functions that included graphical representations of functions could use visual reasoning to quickly develop insights into certain problems. Other researchers have noted that student concept images based on a rich variety of examples of functions will have a better understanding of the concept (Vinner, 1991; White & Van Dyke, 2006). C. G. Williams (1998) found that a main difference between calculus professors who could deal flexibly with problem situations related to function and students who were limited to algorithmic approaches was the richness and structure of their concept images of function.

In contrast to a rich and nuanced understanding of the function concept, for decades, researchers have found consistently weak understanding of function among secondary mathematics students, even as they transition into collegiate mathematics (Habre & Abboud, 2006; C. G. Williams, 1998). In addition, researchers have documented numerous misconceptions that students' hold in regard to function (for example, see Harel & Dubinsky, 1992). These misconceptions can be examined through the lens of concept images. For example, many students believe that a function must be able to be represented as a single equation or formula (Oehrtman, et al., 2008; Tall, 1990). This suggests that even though the definition of function does not include this requirement, their concept image of function includes symbolic representation as a predominant component. Concept images are heavily influenced by the types of examples studied, suggesting that students with this misconception have mostly studied examples of functions that can be represented by a single equation. Because teachers and students rely heavily on textbooks, the features of the set of examples in a given textbook are of crucial interest. Using the illustration above, if a textbook principally provided examples represented as a single equation, the examples studied in class would likely share this feature, and students' resulting concept images would, in turn, be limited. Thus, a key question is, "What are the features of the set of examples of functions in the textbook?"

This research examines functions in secondary mathematics textbooks in two ways: through analysis of definitions of functions and descriptions of collections of function examples in textbooks and series. The analysis is focused on definitions and examples because research has indicated that students' experiences with these shape their understanding of function. The selection of features of definitions and examples for analysis is also informed by current research in relation to function. Information about functions provided by the analysis are compared to suggestions in research on student learning of function and form a key initial step in the process of analyzing how curriculum influences the development of students' understanding of this core concept.

Purpose of the Study

Textbooks play important roles in secondary mathematics classrooms because they convey concept definitions and are a source of examples that support the development of images of key concepts. Together, the definitions and examples of functions in textbooks provide a perspective and portrait of the concept as well as frame the conceptual space within which functions are considered.

The purpose of this study is to analyze current mathematics textbooks' portrayal of function in light of the significant literature base on students' understanding of function. The research will provide information about how selected secondary mathematics textbooks present this vital concept.

Research Questions

This study will investigate the following research questions:

1. What language do textbook series and individual textbooks analyzed in this study use in relation to function? How does this language compare to recommendations from research?
    a. How are functions defined?
    b. What is the distribution of examples of functions explicitly portrayed as actions, processes, objects, or parts of larger schema?
    c. What is the distribution of general and specific examples of functions?
2. What is the presence of function examples in textbook series and individual textbooks analyzed in this study? How does the presence of function examples compare to recommendations from research?
    a. How prevalent are functions in each textbook?
    b. What proportion of examples are explicitly identified as functions?
    c. What is the distribution of function examples placed in textbook lessons and homework exercises?
    d. How frequently are students asked to generate examples of functions?
    e. How frequently are students provided with non-examples of function?
    f. What errors in function examples are present?

3. How do textbook series and individual textbooks analyzed in this study present core features related to domain, range, representations, and families of functions? How does this presentation compare to recommendations from research?
    a. How frequently are domain and range of specific functions made explicit? What proportion of domains and ranges are numerical?
    b. What is the distribution of different representations of functions? How frequently do students have opportunities to engage with multiple representations of a function example?
    c. What is the distribution of families of functions?
4. How do textbook series and individual textbooks analyzed in this study present ancillary features of functions related to example settings and recommendations for use of technology? How does this presentation compare to recommendations from research?
    a. What is the distribution of abstract and realistic settings for function examples?
    b. How frequently is technology explicitly recommended for use with function examples?

Theoretical Perspective

Tall and Vinner (1981) described a theory of how students comprehend mathematical concepts. They argued that students' cognitive structures related to a given concept, called concept images, are different from concept definitions, which are the words used to specify a given concept. Students' develop concept images through experiences with a concept or concepts over time and maintain them until they experience cognitive conflicts that require them to adapt their images. In relation to function, we can consider contrasts between concept definitions and images. One definition of function could be: a function from a set A to a set B is an object f such that every a in A is uniquely associated with an object $f(a)$ in B. However, the concept image that students associate with function may capture only certain aspects of this definition. For example, a student may have an image of function as any continuous line drawing on a coordinate plane. Although such drawings may represent relationships between two sets of values, some of these drawings do not meet the uniqueness requirement of the concept of function. In addition, other functions, such as functions between two sets of functions, cannot be easily represented as line drawings on a coordinate plane. Thus, students' concept images of function play an important role in the way they think about and use functions.

One influence on the formation of students' concept image is the examples of the concept with which students engage. Watson and Mason's (2005) *example space* construct is similar to Tall and Vinner's (1981) concept image. Example spaces are individual cognitive structures centered around a specific concept. They argued that cognitive spaces are made up of examples representing the concept. Students' example spaces are influenced by the examples that teachers show them when students "gain familiarity with them, internalize them, and integrate them into their example space sufficiently that they come to mind in different situations" (p. 51). In terms of concept images, this suggests that when students engage with examples in meaningful ways, then their concept image is altered by the addition of knowledge about the new example to the image. For example, a student's concept image of function may only include functions without discontinuities. However, if a teacher has the student explore the function $f(x)=\frac{1}{x}, x \neq 0$, the student's concept image may begin to include functions with discontinuities. Given teachers' and students' regular use of textbooks, it seems reasonable to assume that students are likely to engage with the examples of functions a textbook and thus the features of these functions will become incorporated into the student's concept image of function.

In addition to experiences in which examples are provided to students, Watson and Mason (2005) argued that students learn better when they actively construct examples. They explained that when students are asked to construct examples with specific features, they must consider the mathematical structure of their example spaces and thus develop deeper understanding. In terms of concept images of function, this suggests that when students are asked to develop their own examples of functions, they develop richer concept images.

Definitions

The following terms will be used frequently throughout. The definitions draw on commonly accepted mathematical definitions and mathematics education research where applicable in order to facilitate communication and understanding. However, the definitions have been operationally designed by the author to serve the purposes of this study.

General Definitions

Contemporary secondary mathematics textbook series: a collection of textbooks, published no earlier than 2006, designed for consecutive years of mathematics study generally from eighth or ninth through eleventh or twelfth grade and generally including the study of algebra, geometry, and pre-calculus concepts.

Concept image: whatever is evoked by the concept name in a student's memory; a student's mental picture of a concept and a set of properties associated with the concept, including any representations of the concept (Tall & Vinner, 1981).

Concept definition: the words used to specify a given concept (Tall & Vinner, 1981).

Language Used With Function Examples

Action perspective of function: function is described as something one does, that is, a transformation one applies to mathematical elements according to an explicit algorithm. The emphasis is on carrying out the algorithm.

Process perspective of function: function is described as a procedure one has the ability to follow, a procedure that transforms one mathematical element into another. The emphasis is on the ability to carry out the procedure if needed or desired, rather than on the immediate carrying out of the procedure.

Object perspective of function: function is described as something that can be acted upon mathematically. The emphasis is on functions as mathematical elements that can be transformed.

Schema perspective of function: function is described as a part in a larger mathematical perspective. The emphasis is on the role of functions in general or within broader mathematical concepts.

A specific example of a function: an example for which a student is given or can obtain at least one domain element and its corresponding range element. A specific example of a function would be $f(x) = 2x$ with a domain of all real numbers.

A general example of a function: an example for which a student is not given and cannot obtain at least one domain element and its corresponding range element. An example of this would be if they were asked to consider a linear function *f* and determine whether it must have a *y*-intercept.

Core Features of Function Examples

Representations

Symbolic representation: a representation of a function that uses numbers, letters as variables, operation symbols (e.g. plus signs), and/or *f*(x) notation to provide a formula or formulas to generate elements in the range of the function from elements in the domain. An example of a symbolic representation of a function would be $y - x^2 = 1$, with a domain of all real numbers.

***y* = expression:** a symbolic representation of function where an algebraic expression consisting of numbers, letters, and operation symbols, and only utilizing *x* as a variable, is set equal to the variable *y*. For example, $y = x^2 + 1$, with a domain of all real numbers.

Implicit *y* and *x*: a symbolic representation of function where two algebraic expressions consisting of numbers, operations symbols, and the letters *y* and *x* are set equal to each other, and one expression is not simply the variable *y*. For example, $x+y=1$, with a domain of real numbers for *x*.

***f(x)* = expression:** a symbolic representation of function where an algebraic expression consisting of numbers, letters, and operation symbols, and often utilizing *x* as a variable, is set equal to *f(x)* or *f:x*, which acts as a representative of the elements of the range of the function. Note that symbolism utilizing other letters for the function and variable are also considered to be in this category, for example, $g(w) = w^2 + 1$, with a domain of all real numbers.

Recursive representation of function: a symbolic representation in which one correspondence of the function is given and other correspondences are determined from preceding values. For example, $y_1 = 2$, $y_n = y_{n-1} + 3$, where the domain is the natural counting numbers.

Equation with other variables: a symbolic representation of a function where two algebraic expressions consisting of numbers, operation symbols, and letters other than *x* and *y* are set equal to each other

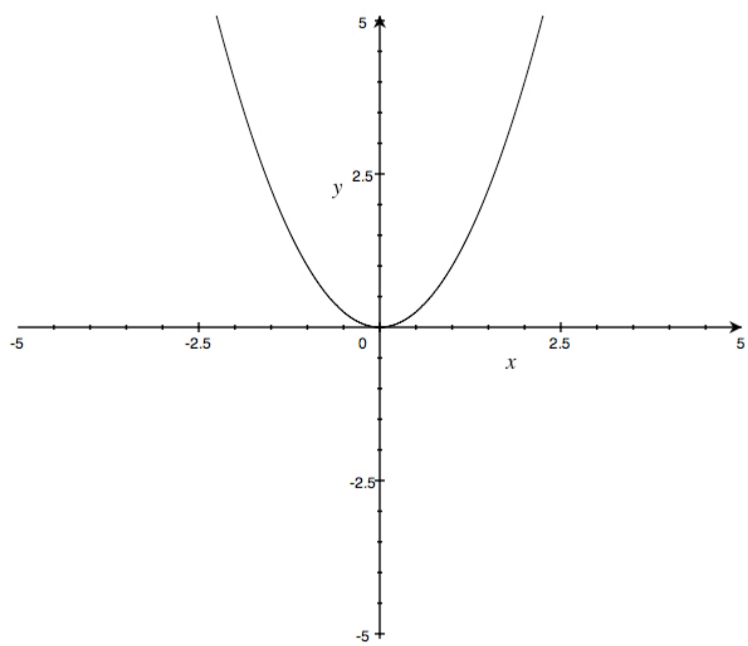


Figure 1.1: A graphic representation of a function.

Graphic representation: a representation of a function that uses points and/or curves in a Cartesian coordinate plane (or corresponding visual representation of higher dimensions) to display some or all of the elements of the domain of the function with their corresponding elements in the range. An example of a graphic representation of a function with a domain of real numbers can be seen in Figure 1.1.

Continuous graph: a graphic representation of a function that is a single unbroken curve without holes or jumps. Note that this definition is somewhat imprecise and is not equivalent to most commonly accepted mathematical definitions of a continuous function (e.g. Given subsets I, D of $\mathbf{R}$, continuity of $f: I \rightarrow D$ at $c \in I$ means that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$: $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$ and a continuous function is a function that is continuous at all $c \in I$). However, it is intended to be an intuitive idea that is related to a conception of continuity prior to an introduction to this definition. Figure 1.1 also shows an example of a continuous graph. An example of a function with a graph that is not continuous at $x = 1$ is $f(x) = \begin{cases} -1, & x \le 1 \\ 1, & x > 1 \end{cases}$, with a domain of all real numbers.

Smooth graph: a continuous graph of a function that has no corners. Corners are defined as points of a graph where, on any arbitrarily small locality, the graph appears to be two straight line segments meeting at a non-straight angle. For example, any polynomial function has a smooth graph. An example of a function with a continuous graph that is not smooth because it has a corner at $(1,1)$ is $f(x) = \begin{cases} x^2, & x \le 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}$, with a domain of all real numbers.

Scatterplot: a graphic representation of a function that is a plot of individual ordered pairs

Numeric representation: a representation of a function listing some or all of the elements of the domain of the function with their associated elements in the range. An example would be $f(1) = 2, f(2) = 4, \text{ and } f(3) = 6$ with a domain of $\{1,2,3\}$.

Tabular representation: a numeric representation in which elements are displayed in a table. A tabular representation of the function given above can be seen in Figure 1.2.

x	$f(x)$
1	2
2	4
3	6

Figure 1.2: A tabular representation of a function.

Ordered pair representation: a numeric representation in which elements are displayed as ordered pairs in parentheses. An ordered pair representation of the function given above is $\{(1,2),(2,4),(3,6)\}$.

***f*(*x*) notation**: a numeric representation in which elements are displayed as $f(x) = y$ for specific x and y. An $f(x)$ notation representation of the function given above is $f(1) = 2$, $f(2) = 4$, and $f(3) = 6$.

Function machine representation: a representation of a function with a diagram suggesting a physical object in which values can be input and which produces corresponding output values, often including a formula for generating output values. An example of a function machine representation can be seen in Figure 1.3.

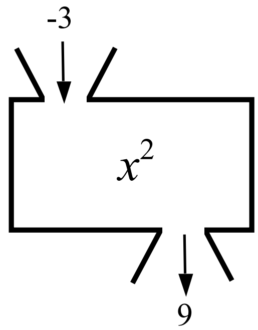


Figure 1.3: A function machine representation of a function.

Mapping diagram representation: a representation of a function with a diagram showing some or all elements of the domain in one part of the diagram, corresponding elements of the range in another part of the diagram, and lines or arrows connecting corresponding elements. An example of a mapping diagram representation of a function can be seen in Figure 1.4.

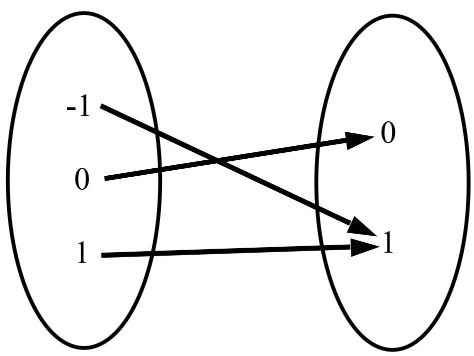


Figure 1.4: A mapping diagram representation of a function.

Verbal description: a representation of a function using words, and potentially numbers, to describe any feature of the relationship between the independent and dependent variables. Verbal descriptions can range from being as simple as stating that a function is increasing to more explicit statements such as, "A function that takes any integer as input and produces twice that integer as output."

Physical representation: a physical model, capable of manipulation by students, that embodies a functional relationship. An example would be a model of a ladder that can slide down a wall, as described by Monk (1992). Note that although textbooks themselves will most likely not include physical representations of functions, they may direct students to produce such representations.

Families

Polynomial function: a function whose symbolic representation is a polynomial of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where *n* is a non-negative integer and $a_i \neq 0$ for some *i*.

Constant function: a polynomial function whose symbolic representation is of the form $f(x) = a_0$ for some a_0. An example of a constant function over all real numbers would be $f(x) = 1$.

Linear function: a polynomial function whose symbolic representation is of the form $f(x) = a_1 x + a_0$ where $a_1 \neq 0$. An example of a linear function over all real numbers would be $f(x) = 2x + 1$.

Quadratic function: a polynomial function whose symbolic representation is of the form $f(x) = a_2 x^2 + a_1 x + a_0$ where $a_2 \neq 0$. An example of a quadratic function over all real numbers would be $f(x) = 3x^2 + 2x + 1$.

Cubic function: a polynomial function whose symbolic representation is of the form $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ where $a_3 \neq 0$. An example of a cubic function over all real numbers would be $f(x) = 4x^3 + 3x^2 + 2x + 1$.

Periodic function: a function where there exists a number k such that $f(x) = f(x+k)$ for all values of x. An example of a periodic function over all real numbers would be $f(x) = (-1)^{\lfloor x \rfloor}$ where $\lfloor x \rfloor$ is the least integer of x.

Trigonometric function: a function whose symbolic representation includes trigonometric ratios (sine, cosine, tangent, cosecant, secant, cotangent) or their inverses. An example of a trigonometric function over all real numbers would be $f(x) = \sin(x)$.

Exponential function: a function with its independent variable as part of an exponent. An example of an exponential function over all real numbers would be $f(x) = 3(2)^x$.

Logarithmic function: a function with its independent variable as part of the argument of a logarithm. An example of a logarithmic function over all real numbers would be $f(x) = \log_{10}(x)$.

Rational function: a function whose symbolic representation is the ratio of two polynomial functions, where the function in the denominator is not a constant function. A zero denominator is undefined and a rational function with any other constant value for a denominator can be classified as a polynomial function. An example of a rational function over all real numbers would be $f(x) = \dfrac{x}{x^2+1}$.

Absolute value function: a function with its independent variable inside an absolute value symbol. An example of an absolute value function over all real numbers would be $f(x) = 3|x+2|$.

Piecewise function: a function whose rule of correspondence for a particular element in its domain is dependent on the value of the element. Thus, a piecewise function may have different rules of correspondence for disjoint subsets of its domain. An example of a piecewise function with a domain of all real numbers would be

$$f(x) = \begin{cases} x^2, & x \le 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}.$$

Ancillary Features of Function Examples

Realistic setting: a function is considered to be in a realistic setting when included with the function is a description of a situation intended to reflect the real world. Note that this setting may not reflect the real world in accuracy or detail, but does include enough information for students to make connections to situations they potentially understand from the real world. An example of a function in a realistic setting would be, "A business can produce 8 cars for $75,000 while the production of 16 cars costs $139,000. If cost is a linear function of the number of cars produced, how much would it cost to produce 20 cars?"

Abstract setting: a function is considered to be in an abstract setting when no realistic setting is included with the function.

Significance of the Study

Results from the textbook analysis have the potential to be useful for researchers in mathematics education seeking to understand how different secondary mathematics textbook series present the concept of function. The results from this research will contribute to the current understanding about different approaches used in mathematics textbooks to foster students' learning of function.

The study will be beneficial for textbook developers who can use the information to guide revisions in future textbooks. Connections between students' conceptions and misconceptions of functions and the examples of functions currently in textbooks can help developers seek ways to improve the types of examples of functions in textbooks to help strengthen students' conceptions of function.

The study may also provide insight into why students develop different conceptions of function through study from various textbook series. Researchers in mathematics education need to continue to develop understanding about how different curricular programs impact students' mathematical understanding (Harwell, et al., 2007). Specifically, there is a need for close analysis of impact on students' learning of core topics such as function. Also, the study may provide a basis for further research into students' study of function in post-secondary mathematics.

The study can help those supporting teachers of mathematics at local sites. School administrators and policy makers can use information about different textbook series as they consider how to guide mathematics education in their school districts and what programs to implement. Secondary mathematics teachers could use findings from the study to better understand the textbooks they are using and to help them address potential weaknesses in their students' learning of function.

Summary

In the US, a wide variety of secondary mathematics textbooks are regularly produced and used in classrooms to guide the planning and implementation of instruction. What students learn and how they learn it is influenced by the content, organization, and quality of these textbooks. Therefore, there is a continuing need to examine textbook content, especially in relation to core topics such as function. The function concept underlies many ideas in the secondary curriculum and a strong understanding of function is essential for the study of post-secondary mathematics. However, decades of research have repeatedly demonstrated continuing student difficulties with the concept. Because of its centrality and because many students struggle with the concept, a large body of research on the learning of function has developed. This research can form the basis for an examination of the treatment of the language, presence, core features, and ancillary features of function in textbooks. Such an examination can provide useful information for current users of textbooks, textbook developers, and mathematics education researchers.

CHAPTER 2: LITERATURE REVIEW

Introduction

This chapter begins with a review of research on the use of textbooks in mathematics K-12 education in the United States, including research on the extent of textbook use by teachers and students and how teachers use textbooks. Then current criticisms of textbooks and their generation and adoption, both in general and specific to mathematics are considered. Following this is a review of research regarding mathematical functions. Centrality of the concept, historical development of the concept and definition, and its historical role in mathematics education and textbooks are reviewed. Research on the learning of function, representations of functions, various families of functions, settings for functions, and technology use in relation to functions is considered. Following this is a review of the extensive research on student misconceptions and difficulties related to function. Based on this review, a conceptual framework related to function is presented. Finally, recent textbook analyses are reviewed in order to inform the proposed analysis.

Use of Mathematics Textbooks

Extent of Use

Textbooks play a pervasive role in education in the United States. Tyson-Bernstein and Woodward (1991) reported, “Textbooks are a ubiquitous part of schooling in the United States. Visit any elementary or secondary school classroom and the chances are that textbooks will be a prominent, if not dominant, part of teaching and learning” (p. 91). They argued that most of time, textbooks define the curriculum, the scope and sequence of topics, and the method of instruction. This claim was based on a review of research studies about textbook use since the first half of the twentieth century, which consistently documented extensive use of textbooks in US classrooms.

One such study was McKnight et al.'s (1987) presentation of major results from the Second International Mathematics Study (SIMS). They found, "Mathematics instruction in US classrooms is clearly textbook-driven" (p. 75), reporting that over 90 percent of teachers in US 8th and 12th grade classrooms used mathematics textbooks as the most common and consistent resource for instruction. Robitaille and Travers (1992), in reviewing international studies of school mathematics achievement, also recognized that SIMS demonstrated the central importance of curriculum in accounting for differences in achievement across countries. They reported that teachers not only in the US, but internationally rely heavily on textbooks for day-to-day teaching. They argued that this is perhaps more characteristic of the teaching of mathematics than any other subject in the curriculum and that, in most countries, mathematics textbooks significantly influence content, instruction, and exercises assigned to students. Similarly, Schmidt, McKnight, and Raizen (1997) studied results from the Third International Mathematics and Science Study (TIMSS) partly to better understand how teachers used their textbooks. They found that most US teachers use textbooks as part of their daily instruction. According to data reported by teachers, they based about 60% of their weekly teaching time on the textbook directly.

Reports from national studies of mathematics education in the US have also demonstrated that textbooks play a noteworthy role in teachers' instruction and students' learning experiences. Grouws and Smith (2000) used data from the 1996 National Assessment of Educational Progress (NAEP) to provide a portrait of teacher characteristics based on responses to a questionnaire given to teachers with at least one student in 4^{th} or 8^{th} grade who took the assessment. They found that teachers of 61 percent of students in grade 4 and 72 percent of students in grade 8 used textbooks daily. Although this is statistically significantly lower than results from the 1992 NAEP survey, it nonetheless suggested widespread use of textbooks in instruction. Braswell et al. (2001) and Grouws, Smith, and Sztajn (2004) examined responses to the 2000 NAEP survey and found similar results. About two thirds of students in 4^{th}, 8^{th}, and 12^{th} grade had teachers who reported that students did problems from their textbooks almost every day. Grouws, Smith, and Sztajn concluded, "… these NAEP data support the assertion that textbooks play a central role in the mathematics instruction that is given to United States students" (p. 251).

Horizon Research conducted two studies of US mathematics education, including an examination of how teachers spent instructional time and what resources they used (Weiss, et al., 2001; Weiss, Pasley, Smith, Banilower, & Heck, 2003). The first was a survey of nearly 3000 K-12 mathematics teachers across the nation. Teachers reported students answering questions from textbooks or worksheets as a "very frequent activity" (p. 70), especially in the higher grades. They also reported having students read from the mathematics textbook in class, although less frequently. Teachers reported that over 85% of their classes were based on a commercially published textbook or program. In addition to a large portion of classes being based on textbooks, teachers reported that they taught a large portion of content from their textbooks. In at least two-thirds of their classes, at least 75% of the textbook was covered in that course. Weiss, Pasley, Smith, Banilower, and Heck followed the survey with observations of and interviews with a national sample of 364 K-12 teachers in order to examine how mathematics instruction looked inside the classroom. For both the content and instructional style of lessons, textbooks were the second greatest influence on teachers. Half of the teachers reported the textbook as influencing the content of their lessons and over 70 percent said the textbook influenced their instruction.

Others have also noted the significant role that textbooks play in mathematics instruction, both for teachers and for students. Seeley (2003) wrote, “Since their use in the nation’s colonial schools, textbooks have been the primary instructional resource in a student’s educational program” (p. 959). She argued that even with increased emphasis on state-provided curricula and learning expectations, teachers continue to rely on their textbooks as the primary determination of what they teach when the classroom door is shut. Finn, Ravitch, and Whitman (2004) claimed textbooks as the *de facto* curriculum in many US schools. They argued that surveys of teachers’ use of textbooks, which have generally reported high amounts of use of textbooks, “most likely *understate* teachers’ and students’ true dependence on textbooks” (p. 1). Reys and Reys (2006) reported that students use textbooks daily in class and Reys, Reys, and Chávez (2004) argued that textbooks are a staple in US mathematics classrooms. They noted that one demonstration of the importance of the role of textbooks is that in 2001-2002, K-12 school districts spent more than $4 billion on textbooks. This is one of the largest school expenditures after teacher salaries.

Teachers’ Use of Mathematics Textbooks

It is clear that both students and teachers frequently use textbooks in many US mathematics classrooms, but it is also necessary to examine how teachers are using textbooks. For example, McKnight et al. (1987) argued from the SIMS data that while textbooks were used heavily in US classrooms, closer analysis revealed that teachers might not be using every strategy or content representation in the textbook. However, they found that if a strategy or content representation was not in the textbook, then teachers were almost certain not to use it. Thus, in US classrooms, the textbooks tended to define boundaries for instruction.

Freeman and Porter (1989) also acknowledged, "…textbooks play a central role in most American classrooms" (p. 403). They pursued a deeper understanding of how teachers use textbooks by conducting year-long case studies of four elementary school teachers regarding their use of mathematics textbooks and instruction. Through analysis of teacher logs of their instructional activities and weekly interviews, they determined three styles of textbook use: (a) textbook-bound, (b) focus on basics, and (c) focus on district objectives. The textbook-bound teacher followed the textbook approach closely, although she did not use about 40% of the lessons in the book. The teachers focusing on the basics used the textbook as a resource for lessons centered on the basic skills they thought were important. The teacher focusing on district objectives used the textbook lessons aligned with these objectives and other lessons as enrichment for advanced students. Despite this variety in textbook use, the authors found all teachers tended to emphasize in class what was emphasized by the textbook and that teachers' sequence of instruction generally followed the order in the textbook.

Sosniak and Stodolsky (1993) conducted a similar study with four 4th grade teachers across several subject areas and also found variation in the way teachers used textbooks, even within the same teacher across subjects. They found, just as Robitaille and Travers (1992) argued, that teachers used textbooks most frequently in mathematics class, and that even teachers who tended not to use textbooks in other subjects relied on their mathematics textbook heavily. One teacher described the textbook as an authority and relied on the decisions made by the textbook developers. In contrast, the teacher who used the textbook the least nonetheless used it for problem sets. Across teachers, they did not find all teachers following the sequence of the mathematics textbook as Freeman and Porter (1989) had. They also found a variety of influences on teachers, such as teacher beliefs and district guidelines, which created a larger agenda for their instruction and textbook use. Their findings led them to conclude that teachers used textbooks selectively and adaptively in ways that emphasized content and tended to ignore pedagogical strategies in textbooks.

Schmidt, McKnight, and Raizen (1997) argued that the topics, activities, and tasks in textbooks are resources that help shape what actually happens in classrooms. They wrote, "… for many US mathematics… teachers, textbooks at a 'micro' level *are* the curriculum that guides mathematics… instruction" (p. 53). Even for those that move beyond the textbook as the curriculum, Schmidt, McKnight, and Raizen argued that teachers regard textbooks as indicating what is acceptable, and must often settle for textbook-based default choices. Teachers often plan lessons beginning with a selected segment of the textbook on which they build activity plans for their classroom. The authors believed that even when teachers draw on other sources for planning, such as past experiences and collegial support, textbooks play a core role. They concluded,

> "[Commonly] textbooks are, by default and overwhelming demand, the backbone of 'micro' organization for classroom activities. They provide the fine details of

curriculum… Textbooks define the domain of *implementable* day-to-day curricular possibilities. Without restricting what teachers *may* choose to do, they drastically affect what US teachers are *likely* to do…" (p. 53)

Remillard (2005) noted that mathematics teaching has a long history of being textbook-driven, but that teachers can misinterpret, subvert, or ignore textbooks. Through a review of studies of textbook use, she found not only a variety of ways that teachers may use textbooks, but also four theoretical perspectives of how teachers interact with and use curriculum materials: (a) following or subverting, (b) drawing on, (c) interpreting, and (d) participating with. She argued that future research should view teachers as interpreting or participating with curricula as teachers must employ creative intelligence to continually redesign their activities in the act of practice, even if their goal is to follow the textbook as closely as possible. In addition to teacher factors contributing to use of textbooks such as teachers' beliefs and identities, Remillard argued that researchers of textbook use need to consider aspects such as: representations of concepts, material objects and representations, representations of tasks, structures, voice, and look. These aspects are significant elements in the enactment of curricula in the classroom.

Tarr, Chávez, Reys, and Reys (2006) sought to examine the role of the teacher and textbook through analysis of findings from a study of the impact and influence of two types of textbooks (NSF-funded and publisher-generated) on students' opportunity to learn middle school mathematics. They gathered data from a sample of teachers in eleven schools in six states through surveys, teacher diaries, classroom observations, and interviews over the course of a school year. Teachers had strong intentions to use the textbook; over half of the teachers expected to use textbooks more than 90% of instructional days, and almost 90% planned to use textbooks at least three fourths of the time. Teacher diaries and observations showed that both teachers and students did use the district-adopted textbooks both in the classroom and for homework assignments. In addition, interviews with teachers indicated that, for about half of the teachers, the textbook was a strong determinant of the mathematics content they taught as well as the order of presentation. Also, over 80% of the teachers used their textbook as primary resource for planning and teaching mathematics. They concluded, "These data suggest that the district-adopted textbook strongly influences both *what* and *how* mathematics is taught to middle school mathematics students… textbooks likely impact students' mathematics experience in important ways" (p. 200).

McNaught (2009) used data collected from a sample of teachers in schools from five different states over two years to examine their use of secondary mathematics textbooks produced by the Core-Plus Mathematics Project (Coxford, et al., 1998a, 1998b). Teachers reported using the textbooks without any supplementation in roughly two-thirds of their lessons and covering slightly less than two-thirds of the content of the textbooks over the course of the year. Observations supported this, as the content of the observed lessons was primarily attributable to the textbook. However, the textbook had less influence on the manner in which the observed lessons were taught. McNaught concluded, "As these results indicate, teachers relied heavily on the textbook when teaching the content, but they did not use the textbook in its entirety or exclusively" (p. 103). In two related publications, McNaught and her colleagues (McNaught, Tarr, & Grouws, 2008; McNaught, Tarr, & Sears, 2010) reported similar findings. Teachers did not teach all of the content in textbooks, with percentage of coverage in the first study ranging from about 75% to 78%, and in the second study coverage ranging from about 61% to 77%. However, of the lessons that were taught, most were influenced by the textbook. The percentage of lessons taught primarily from a source other than the textbook ranged in the first study from 9% to 14% and in the second study from 13% to 21%.

Teachers in the US tend to draw on textbooks to guide their mathematics instruction. Although they generally don't teach all the content in textbooks, rarely do they introduce content that is not in their textbooks. Often they follow the sequence in their textbooks and use their books as they plan instruction. Thus, textbooks form a key influence on instruction and their content is of interest.

Criticism of US Mathematics Textbooks

General Criticism

Although textbooks are used heavily by teachers in a variety of ways, many criticisms of textbooks and the processes through which they are created and selected for use in the United States have been raised. McKnight et al. (1987) noted that in SIMS, US students generally performed at or below the international average and argued that the curriculum and textbooks shaped by the curriculum to be central to the problems of school mathematics. They characterized the curriculum and resultant textbooks as being unfocused. They found that contents and topics were so fragmented that they could be taught in one or two lessons and following lessons would be dedicated to different topics. The intended upward spiraling of content over an entire curriculum was found to be closer to a "spiral of almost constant radius" (p. 99) that repeated content without significantly increasing depth or challenge. This contrasted with curriculum and textbooks in other countries such as Japan, where large blocks of lessons were focused on deep development of single topics and the constant forward movement in the curriculum. They argued that reform of mathematics textbooks was critical for improvement.

Schmidt, McKnight, and Raizen (1997) used results from the Third International Mathematics and Science Study (TIMSS) to find elements that contributed to the “splintered vision” in US mathematics education. They argued that one of the main causes was textbooks that they characterized as bloated, unfocused, and undemanding. They found that, compared to most other TIMSS countries, US mathematics textbooks included many more topics, that the emphasized topics accounted for less space in textbooks, and that performance expectations emphasized in textbooks are generally low level. They concluded that the textbooks were unfocused and labeled them “a mile wide and an inch deep” (p. 62). They noted that US mathematics education has a long history of fragmenting content so that students can master smaller skills or topics and then combine these into larger ideas and argue that this tradition has contributed to the fragmentation they found in US mathematics textbooks.

In a notable exception, Schmidt, McKnight, and Raizen (1997) found US Algebra 1 textbooks were more focused, dropping much of the content from previous books and devoting a substantial portion of space to the topic of equations and formulas. In addition, although they noted that these textbooks reflected some recommendations of documents such as the NCTM *Standards* (1989), generally performance expectations were still not highly demanding.

Tyson-Bernstein and Woodward (1991) examined the process of textbook selection in the United States and its effects on textbooks. They argued that textbook selection is plagued by a host of false assumptions: (a) textbook series can and do match state curricula, (b) textbooks provide foolproof means of ensuring successful teaching, (c) textbooks closely aligned to state tests can ensure school success, (d) readability formulas can ensure proper reading levels, (e) new textbooks are better than old textbooks, (f) good textbooks are organized for standard class periods, and (g) centralized textbook selection is more efficient. These assumptions led to many problems in the production of textbooks. Publishing the same textbook for many states has led to textbooks that appeal to all users' requests, even if only superficially. Adoption committees often do not take the time and effort to look beyond superficial agreement with state curricula and assessments, and often rely only on examination of tables of contents and indexes for their judgments. Publishers also use increased aesthetic appeal to lure adoption committees, teachers, and students rather than working to improve textbooks support for instruction or learning. Complimentary items, large author groups, and long-term relationships with publishers often sway committees rather than quality of books. They concluded that, given all these issues, "…it is not surprising that textbooks are not written for students or for learning" (p. 97).

Seeley (2003) also reviewed textbook adoption in the US and demonstrated how this process has led to inferior mathematics textbooks. She noted several factors in textbook adoption that negatively impact the development of textbooks. Publishers must attempt to match textbooks to a wide variety of state curriculum frameworks and often have a short time frame in order to do this. Policymakers, special interest groups, parents, and teachers all express differing desires for textbooks. A few states, such as Texas and California, hold such a large share of the market that they can essentially make their own textbook demands that must be met. The result is that publishers often work toward the lowest common denominator, include more and more material in order to please everyone, make only cosmetic changes to textbooks, and lose focus on how to best help students learn. In addition, innovative materials and smaller companies cannot compete for a place in the market. She concluded, "… because of the system of production and adoption, textbooks did not move nearly as far forward as curriculum reformers might have wished" (p. 968).

Finn, Ravitch, and Whitman (2004) were highly critical of US textbooks. They noted that every review of American textbooks led to the conclusion that books were dumbed-down, full of disconnected and sometimes erroneous or misleading content, incoherent, and unstimulating. Like the authors above, they also trace these problems to several elements in the process of state textbook adoption. Textbook publishers face strong pressure from special interest groups and a few large states. This leads to the inclusion of more and more content dealt with in a shallow manner and contributes to incoherence. Often anonymous authors who are not experts in the area hurriedly assemble textbooks in "chop shops" (p. 5) in order to meet demands. Small publishers and innovative materials simply cannot compete. In addition, textbooks are commonly judged through checklists of surface features rather than through an in-depth analysis of content. Reviewers often only have time or willingness to flip through textbooks quickly before granting approval for adoption. Rarely do adopters attempt to study the role or effectiveness of textbooks for improving students' learning. Thus, books are judged for adoption on surface features rather than on quality and effectiveness. Finn, Ravitch, and

Whitman (2004) offer a number of recommendations, including more and better textbook reviews.

Project 2061 (AAAS, 2000, 2002) reviewed both middle school and Algebra 1 mathematics textbooks and found most lacking in support for students' learning. Reviewers examined textbook presentations of content and instruction in relation to a small set of standards identified as important by national organizations such as the National Council of Teachers of Mathematics. None of the twelve middle school textbooks series attempted to address all of the ideas and skills contained in the six benchmarks they identified and the series that did receive overall excellent reviews were not the most popular. Most of the series provided weak instructional support for students and teachers and poor development in sophistication of mathematical ideas over the series. They concluded that a majority of middle school textbooks did not provide a purpose for learning mathematics, take account of student ideas, or promote student thinking. The analysis of twelve Algebra 1 textbooks or the algebra content of integrated textbook series had similar, although slightly more positive, results. The majority of textbooks had some potential to help students learn, but each also had serious weaknesses, especially in building on students' existing ideas about algebra, overcoming misconceptions, and providing assessments for instructional decisions. Again, the most widely used textbooks were rated so inadequate that they lacked potential for student learning.

Reys, Reys, and Chávez (2004) determined that U. S. mathematics textbooks are physically larger and conceptually less coherent than most other countries, including ones that typically perform better on international studies. They noted problems in the development process that led to textbooks of lower quality. One main problem is that there are no national standards for developers to use. Instead, each state has their own unique standards at each grade level so that textbooks at each level must be made to meet a wide variety of standards. Reys and Reys (2006) added that because of the high level of competition in the textbook market, most new textbooks avoid the risk of being different, and therefore reflect qualities similar to the marker leading textbooks. Furthermore, no publishers can afford to produce a book that does not at least superficially meet a state's standards and risk losing that share of the market. Also, textbook publishers do not generally field-test and revise textbooks on the basis of evidence of effectiveness before publication. Instead, the testing of new materials generally focuses on physical design of book rather than content or pedagogical approach. Because they often face tight timelines for production, in-depth academic research of the textbooks is not conducted. However, textbook publishers often conduct market research and tout their textbooks as research-based. In addition, in order to meet deadlines, textbooks are often mainly written by authors in development houses hired to produce them quickly without any long-term commitment to the books. Reys and Reys also recognized many of the issues with textbook adoption as noted above by Tyson-Bernstein and Woodward (1991) and Finn, Ravitch, and Whitman (2004). Ultimately, development of many textbooks appears to be focused more on superficial aspects and less on the needs of students.

Criticism of US Mathematics Textbooks Specific to Function

A number of researchers have criticized textbooks specifically in relation to the approach used for the function concept. Kaput (1992) wrote, "… we are not surprised that [students] do not understand much about functions because they have little opportunity to learn about functions in the traditional algebra courses" (p. 315). Schmidt, McKnight, and Raizen (1997) found US Algebra 1 textbooks dedicated only about 5% of content to the topic of patterns, relations, and functions. More specifically, Carlson and her colleagues (1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) have argued that curriculum developers underestimate the complexity of learning about the function concept and textbook approaches do not promote understanding of the covariational aspects of function. Similarly, Goldenberg, Lewis, and O'Keefe (1992) argued that the typical approach to function in textbooks did not help students grasp the dynamic aspect of function and that dynamic representations of functions should be provided for students before the traditional graphs.

Numerous researchers have criticized definitions of functions in secondary mathematics textbooks as overly formal. Markovits, Eylon, and Bruckheimer (1986) noted that the typical textbook definition at the time of their writing was based on set theory and was far too abstract for students to learn with understanding. They provided numerous arguments against using it: (a) the fundamental idea that functions are relations between variables is too obscured using this definition, (b) a definition describing functions as relationships between variables is used in sciences and applied mathematics, (c) the historical development of the function definition and the development of the student argue against using the set definition, and (d) the set definition is not essential until the study of analysis and topology. They concede that the set definition allows use of non-numerical sets, although rarely do students need this by the end of high school. Vinner and Dreyfus (1989) also saw problems with using the formal set definition in secondary textbooks. They felt that the formal definition should not be introduced until it is needed, perhaps even after secondary school. Sierpinska (1992) argued that the formal set definition is so abstract that students will ignore it when dealing with functions. Even (1993) agreed that students' understanding of function is not determined by the set definition when it is presented to them, but rather by the examples with which they engage. Sfard (1992) proposed that it is the lack of algorithmic nature of the set definition that challenges students. She argued that new concepts should not be introduced in structural terms and that in fact a structural conception should not be required of students as long as they can do without it. Recently, Zaskis and Leikin (2008) warned,

> "The importance of mathematical definitions in teaching and learning mathematics is irreducible. The definition of a concept, once determined in a curriculum, influences the approach to teaching mathematics, the learning sequence, the set of theorems and proofs. Consequently, definitions… shape the relationship between a concept image and a concept definition, forming an essential part of one's knowledge structure that affects the learner's thinking processes." (pp. 132-3)

Several researchers have also criticized textbooks for limitations in the examples of functions they provide for students. Vinner and Dreyfus (1989) noted that examples of functions in textbooks are generally given as symbolic formulas, which leads students to have difficulties understanding other representations of functions. Even (1993) noted that the graphs of functions included in textbooks almost always lack discontinuities, sharp corners, and restricted domains. She argued that the few examples with uncommon features are not enough. Schwarz and Hershkowitz (1999) noted that generally examples of linear and quadratic functions are focused on systematically, whereas other families of functions only get brief attention. Both White and Van Dyke (2006) and Oehrtman, Carlson, and Thompson (2008) recognized these limitations in textbooks' presentations of function and argued that student misconceptions are an unsurprising result. Dahlberg and Housman (1997) concluded that examples given to students need to be thought through carefully because of the persistence of the student's sometimes erroneous initial conception of function and the lack of persistence of teachers' interventions intended to change these misconceptions.

Although Project 2061 (AAAS, 2002) was not designed only examine the concept of functions in textbooks, the project did review twelve Algebra textbooks and algebra content from integrated textbook series in relation to their potential to help students learn to represent and model functions. They rated textbook content and support for instruction in seven categories and found wide variation between textbooks, although no textbook had satisfactory scores in every category. In general, textbooks received better scores in engaging students and developing mathematical ideas and weakest scores in promoting student thinking, assessing student progress, and especially in building on student ideas about mathematics. Very few textbooks supported teachers by alerting them to student ideas and common misconceptions about functions that are discussed in more detail below.

A few researchers have recommended that textbooks be created with different approaches. Fey and Good (1985) argued that the traditional approach to function in textbooks was abstract and missed out on opportunities to motivate students through connections to real world situations. They argued that functions could become the core of the algebra curriculum, with other algebra concepts being motivated by a study of function families. Schwartz and Yerushalmy (1992) argued similarly, noting that the current approach to algebra in textbooks does not take advantage of the power of an approach to algebra based on function for organization and pedagogy.

Function

Historical Role of Function in Mathematics and Secondary Textbooks

Centrality in Mathematics and Secondary Mathematics Education

Function is currently widely recognized as a core concept in mathematics and many claim that a strong understanding of function is necessary for success in secondary and postsecondary mathematics (Akkoc & Tall, 2005; Carlson, 1998; Carlson, et al., 2002; DeMarios & Tall, 1999; Ferrini-Mundy & Lauten, 1994; Leinhardt, Zaslavsky, & Stein, 1990; National Council of Teachers of Mathematics, 1989, 2009; Oehrtman, et al., 2008; Schwartz & Yerushalmy, 1992; Schwingendorf, Hawks, & Beineke, 1992; Selden & Selden, 1992; Tall & Bakar, 1992; White & Van Dyke, 2006). Already in 1989, the National Council of Teachers of Mathematics noted that function is a unifying idea in all of school mathematics from arithmetic operations through calculus. The National Research Council (NRC, 1989) argued, "… if it does nothing else, undergraduate mathematics should help students develop function sense – a familiarity with expressing relations among variables" (p. 51). Schwartz and Yerushalmy (1992) argued that the secondary algebra curriculum should be centered around function because it is "… a mathematically powerful and pervasive idea" (p. 262) that provides a simple, coherent view of algebra. Tall and Bakar (1992) argued that the concept of function permeates and holds a central position in the development of every branch of mathematics. DeMarois and Tall (1999) noted that function is a central concept in schools and university around the world. In 2006, the College Board (2006) published the *College Board Standards for College Success: Mathematics and Statistics*. These standards for middle and high school mathematics include the function concept significantly throughout, for example, in Algebra 1, three of the four major standards are centered on the function concept. Most recently, two documents have conveyed the importance of function. The National Governors Association (NGA) Center for Best Practices and Council of Chief State School Officers (CCSSO) (2011) published their *Common Core State Standards*, which 44 states and provinces have currently adopted. One of the six core standards for secondary mathematics is the function concept. The National Council of Teachers of Mathematics (2009) also noted the centrality of function in the secondary curriculum and called for students to have repeated and varied experiences with functions. The next two sections review the development of this fundamental concept in the field of mathematics as well as in secondary mathematics textbooks.

Historical Development of the Function Concept

Sfard (1991, 1992) offered a historical perspective of the development of the concept of function. The official use of function as a mathematical term began near the end of the 17th century as the result of a long search for a mathematical model of physical phenomena involving variable quantities. Initially, the concept was closely connected to algebraic symbolism and denoted by Bernoulli as “a quantity composed in any manner whatever of [a] variable and constant” (1991, p. 14). This definition leaned heavily on the still vague concept of a variable, and later Euler suggested a quantity should be called a function only if it depends on another quantity so that the two quantities covary. This formulation of a definition of function was explicitly operational in nature, and mathematicians continued to seek a definition that relied on mathematical structure rather than on algorithm. Difficulties with an algorithmic approach were still often connected to the lack of a clear definition of variable. Attempts were made to formulate a definition relying on a graphic or geometric understanding of function as a two-dimensional curve; however, mathematicians found problems with this approach as well. Eventually, Dirichlet rejected an algorithmic approach and suggested a definition close to the structural ordered pair definition proposed by Bourbaki and currently accepted by many mathematicians (Selden & Selden, 1992; Sfard, 1992; Sierpinska, 1992).

Hight (1968) identified four conceptions of function used from early in the twentieth century up to the 1960's. Initially, function was commonly used synonymously with dependent variable, quantity that varies, expression, or formula. Generally, one variable was considered a function of the other, and thus the concept of function was associated with the dependent variable. Hight notes that by the 1930's, the concept had shifted toward the correspondence between the variables. Phrases such as "functional relation" or "functional correspondence" were commonly used. By 1960, functions were associated with rules that assigned a member of the domain to a member of the range. Finally, Hight identified as current in 1968 the "set definition," which is a version of the Dirichlet-Bourbaki definition. Hight argued that each of the previous conceptions were inadequate and demonstrated that the set definition allows one to prove that with this definition, the set of functions have the structure of a commutative ring with unity.

Selden and Selden (1992) noted that the main difference between Dirichlet and Bourbaki's definitions is that Dirichlet's was understood to apply to sets of real numbers, where Bourbaki's can be used with any sets. Despite the difference, both are commonly credited with the current definition. Even (1993) offered one formulation of what is often referred to as the Dirichlet-Bourbaki definition: "a function f from A to B is defined as any subset of the Cartesian product of A and B, such that for every $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in f$" (p. 95). According to Even, there are two key features of the modern definition: arbitrariness and univalence. Arbitrariness refers to the fact that sets can be any sets and the relationship between the sets does not have to exhibit any regularity connected to a symbolic or graphic representation. Univalence refers to the fact that for each element in the domain there is exactly one element in the range. Even claimed that, in the historical development of the concept of function, univalence was not originally required, but was added later in order to make mathematical work with functions more reasonable.

Sierpinska (1992) provided a detailed examination of the fundamental acts of understanding and obstacles faced in the development of the concept of function over history and argues that this is paralleled in the development of an individual's conception of function. Sierpinska's analysis focused on the origin of the concept in humans' desires to understand and model the surrounding world. When relationships inherent in practical problems was believed to be worthy of mathematical study, then the concept of function began to develop. Generalizing the notion of number and gaining an understanding of variables was another key development in the process. Functions then needed to be associated with mathematical laws but also separated from the tools and definitions used to describe them. Then the function concept could be distinguished from related concepts, synthesized with representations of functions, and refined to the current conception.

Sierpinska argued that this process led to the current conception of function is a synthesis of the ideas of law, value, domain, and codomain.

Role of Function in School Mathematics Textbooks

For over a century, there have been calls for function to play a more central role in the study of mathematics. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) noted repeated calls to increase the emphasis on functions in school curricula since the late 1800's. One of the earliest calls they point out was in 1883, when Felix Klein called for an increased emphasis on function in school curricula. After the turn of the century, the National Committee on Mathematical Requirements (1923) proposed that the mathematics curriculum focus on functional relation as the central unifying theme of school mathematics. They argued that the function concept would not only unify the content areas of algebra, geometry, and trigonometry; but also unify school mathematics with the students' world. They claimed that if students learn mathematics through the unifying idea of function, secondary school mathematics would have new meaning for them, and they would be better equipped to confront problems in their own lives. In fact, they argued that a study of function must arise from real world relationships between varying quantities from students' experience.

Decades later, the Commission on Mathematics (1959) renewed calls for function to play a greater role in school mathematics. They argued that changes were needed in the secondary mathematics curriculum and encouraged judicious use of function as a unifying idea. They specifically recommended that 12^{th} grade curriculum be focused on families of functions, such as polynomial, exponential, logarithmic, circular. A few years later, the *Goals for School Mathematics* (Eduational Services Inc., 1963) promoted an ideal curriculum in which the study of function began in second grade.

Several researchers have noted that several of the "New Math" textbooks developed around the time of these reports promoted a different approach to function. The approach was based heavily on the set theory and the Dirichlet-Bourbaki definition. Markovits, Eylon, and Bruckheimer (1986) called this the "formal set definition" in the "modern" approach and noted that textbooks in the School Mathematics Study Group curriculum in the United States and the School Mathematics Project in England both used it. They argued that by the middle 1980's "In almost all school curricula the function is now defined by [the formal set definition]" (p. 18). They described this "modern" approach as more abstract than approaches to function in earlier textbooks. Schwingendorf, Hawks, and Beineke (1992) wrote, "The new math of the sixties and of the seventies introduced the function concept via sets, in terms of domain, range, and a rule relating each element in the first set with a unique element in the second. Such a general notion of function using sets proved to be too difficult for most students to comprehend and appreciate" (p. 133). Eisenberg (1991) was even more harsh in his criticism of the formal approach to functions in the new math textbooks, calling it a "total failure" (p. 141).

In 1989, the National Council of Teachers of Mathematics published their influential *Curriculum and Evaluation Standards for School Mathematics*. This document called for many changes in mathematics curricula, and it took a strong position on the role function should play in the school mathematics. These *Standards* encouraged the study of functional relationships as early as 4^{th} grade and continuing through secondary school. At the secondary level, function takes its place as a content standard along with the other core content areas of algebra, geometry, trigonometry, statistics, probability, discrete mathematics, and calculus. The NCTM recommended that function play an important role in school mathematics and that curricula reflect this.

Around the time that the *Standards* (National Council of Teachers of Mathematics, 1989) were released, the focus on function in mathematics education research was growing. Vinner and Dreyfus (1989) stated that the Dirichlet-Bourbaki approach was frequently presented in textbooks and curricula. However, this contradicted most examples of functions in the books, which were presented as rules of correspondence given by formulas. Tall (1990) agreed that textbooks generally begin with simple examples of functions that led to limitations in students' concept image of function that later contradict the definition. Leinhardt, Zaslavsky, and Stein (1990) noted debates at that time about how function should be introduced in curriculum, centered around whether the Dirichlet-Bourbaki definition should be used or whether function should be defined as a rule-based relationship between two interconnected numerical variables. They noted criticisms of the formal definitions as too abstract and not merging well with the more dynamic conception of function. They offered three possibilities for beginning teaching function: (a) discover the rule, (b) generate data and plot, and (c) interpret qualitative graphs of situations. They argued that most textbooks generally do either the first or second or a combination of these, but typically do not use the third approach. Goldenberg, Lewis, and O'Keefe (1992) claimed that an ideal introduction to function should not begin with traditional, static graphic representations of functions.

Instead, dynamic graphic representations should come first so that students can better understand the dynamism implicit in traditional graphs.

In 2000, the National Council of Teachers of Mathematics again emphasized the role of function in the school curriculum in their *Principles and Standards for School Mathematics*. They again called for the study of functional relationships as early as 3rd grade and included the study of functional relationships and patterns of change as significant parts of their standards for algebra at all grade levels. They suggest all students develop a comprehensive and flexible understanding of function by the end of secondary school. Since the publication of NCTM's *Standards* (1989, 2000) documents, curriculum developers have created textbooks that they claim are based on recommendations in the *Standards*. Publishers of other textbook series have also claimed to meet many of the recommendations in the *Standards* (see for example Holliday, et al., 2005). In essence, many mathematics textbooks publishers have claimed that their textbooks are at least influenced by the *Standards*, and thus, one would expect that function would play a growing role in textbooks published since 1989. Despite this, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) argued that it was not clear to what extent mathematics curricula had responded to calls for increased attention to function. In addition, the NRC's (2004) comprehensive review of textbook analyses did not find any that were focused on the role of function in secondary mathematics textbooks.

Perspectives on Learning About Function

Approaches that researchers have used to explore how students learn about function have been organized below into five main perspectives: (a) concept definition and concept image, (b) historical and individual development and the Action-Process-

Object-Schema perspective, (c) covariation, (d) prototype theory, and (e) role of examples. Main concepts and examples of research from each perspective are examined because they suggest key aspects for the analysis of the function concept in textbooks.

Concept Definition and Concept Image

Tall and Vinner (1981) described a perspective of thinking about mathematics that has influenced research on learning about function since its publication. The words used to define a concept are accepted and used by the larger mathematical community to specify a mathematical concept. In contrast, a concept image is an individual's cognitive structures associated with the given concept. They argued that only portions of the concept image are evoked at a given time by elements of the individual's experience. Individuals' concept images can be in conflict with the concept definition and other concept images. A given aspect of an individual's concept image can be in conflict with other aspects of the concept image. When an individual becomes aware of these conflicts, they can become confused. If these conflicts are not resolved, individuals may think or act in ways inconsistent with the concept definition.

Tall and Vinner (1981) employed their perspective in an analysis of textbook presentations of sequences, limits, and continuity of functions as well as student responses to questionnaires on the same topics. They found that in relation to continuous functions, School Mathematics Project (1967) textbooks tended to provide informal discussions of continuity rather than build on the formal concept definition. Unsurprisingly, they found the 41 secondary students who responded to their questionnaire had concept images of continuity that had potential conflicts with the definition of continuity. For example, some students believed functions must be represented by a single formula in order to be continuous.

Vinner (1983) applied the perspective of concept image and concept definition directly to the concept of function. He gave questionnaires to 146 10th and 11th grade students in high schools in Jerusalem who had studied function as a correspondence between elements of two sets. He found students provided a variety of definitions for function, including function as a rule, as a formula or an equation, as an element of a mental picture (for example, a curved line on a graph), and a definition that mostly matched the textbook definition. He also observed six main concept images among students: (a) a function should be given by one rule, (b) a function is given by several rules on disjoint intervals, but not at points, (c) functions must have a name and approval by mathematicians, (d) the graph of function should be in some way "reasonable" (e) a function should have an element in its domain for each element in its range, and (f) a function is a one-to-one correspondence. These various concept images could lead to potential difficulties with functions that conflict with the images. Vinner's analysis demonstrated that the concept image and concept definition perspective could help provide insight into why students have difficulties thinking about function. Tall (1990) noted limited concept images similar to Vinner's, such as functions must be given by a single formula or graphs should be "smooth" (p. 57). He argued that improving concept definitions would not improve concept images, and that instead, student should experience a variety of examples of functions.

Similar to Vinner's (1983) study, Vinner and Dreyfus (1989) used questionnaires to examine 281 college freshmen and 36 junior high school teachers' definitions and concept images of function. They found similar categories of definitions as in Vinner's study, but also definitions they categorized as general dependence rules and as an operation. Aspects of subjects' concept images that they used to determine whether relations were functions included attending to discontinuities, exceptional points, split domains, and whether the relation assigned exactly one value for each element in its domain. They examined conflicts between subjects' definitions and concept images and found that they did not necessarily use the definition they provided, but rather they often relied on their concept images.

Vinner (1991) expanded on this finding by proposing several possible ways for thinking to include concept definitions and concept images. He argued that, ideally, when dealing with a given mathematical concept, an individual would use the concept definition. He suggested three models: (a) purely formal deduction using only the definition, (b) intuition leading to deduction involving first using the concept image, then the definition, and (c) interplay between intuition and deduction involving moving back and forth between the definition and the concept image. However, he argued that often students find definitions unclear and confusing, do not follow one of these three models, and instead rely only their concept image. When this happens, students may be misled by conflicts between their concept image and the concept definition.

Vinner (1992) used the concept image and concept definition framework to focus on conflicts in knowledge that he termed "compartmentalization" (p. 201). He argued that results from the studies noted above indicated that students had conflicts in their knowledge of which they were not aware and that the stability of concept images allows these conflicts to endure for long periods unless they are confronted directly. This led students to providing incorrect or irrelevant reasoning when dealing with functions.

Tall (1992) continued to use the framework of concept image and concept definition to review research on student learning of function. His focus was on the transition to advanced mathematics, and argued that the perspective was particularly applicable at this stage because students' earlier learning would form concept images that students would bring to bear on study of the formal definition of function. He ultimately argued that the function concept "remains a large and complex schema of ideas requiring a broad range of experience to grasp in any generality" (p. 501).

The concept image and concept definition perspective suggests that both definitions and examples of functions are key in students' learning of the function concept. Features of the examples students study influence their concept images of function, which must then be compared to, and interact with, the definitions of functions they attempt to understand. Therefore, features of function examples and definitions of functions are worthy of careful attention in an analysis of function in textbooks.

Historical and Individual Development and the Action-Process-Object-Schema Perspective

Sierpinska (1992) argued that students' development of understanding of the function concept would include fundamental acts of understanding and cognitive obstacles that mirrored those experienced in the historical development of the concept. She offered a detailed review of the development of the function concept, highlighting 35 acts of understanding and obstacles determined to be important. She emphasized that, both in history and for students, the concept of function must arise from a need to solve practical problems in the surrounding world. She wrote, "… if the notion of function does not appear to the students as one of the possible tools in trying to answer their questions about the variability of the world then it may remain meaningless for them outside the mathematics classroom" (p. 56). Sierpinska's fundamental acts of understanding trace an arc from beginnings in real world problems through developing appropriate conceptions of variables, generalizing understanding of number, understanding how functions can model relationships, synthesizing the concept of function as an object, distinguishing function from its representations, and returning to a more nuanced understanding of functions and causal relationships in the surrounding world. As students' thinking develops, Sierpinska argued that they must overcome numerous obstacles. These are noted later in a discussion on common student misconceptions about function.

Sfard (1991, 1992) also traced the historical development of the concept of function in order to inform her theory of the development of the function concept in individuals. She argued that functions can be conceived of either operationally or structurally and that the function concept is initially operational and moves towards a structural conception through three phases: interiorization, condensation, and reification. The function concept is initially tightly connected to computational processes. Interiorization involves a process being performed on familiar mathematical objects and thus for functions, this involves the idea of variable being learned and students using a formula to find values for the dependent variable. Condensation occurs when this process is then turned into an autonomous entity. Sfard argued that for functions, this involves students using the mapping of a function as a whole with less attention to individual values; however, students are still tightly connected to a specific function at this point.

Finally, reification involves integrating the process into a new, object-like entity. For functions, this involves mathematical actions such as working in situations in which "unknowns" are functions, talking about processes performed on functions, and recognizing that a rule is not necessarily needed to work with a function. For Sfard, transition through the stages from processes to objects enhances an individual's sense of understanding the function concept.

Several researchers have developed a theory of mathematical learning similar to Sfard's (1991, 1992) called the Action-Process-Object-Schema (APOS) perspective and have applied it to the study of how students learn about the function concept. Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Dubinsky and Harel (1992) discussed the first three parts of the theory in addition to a level prior to action. An individual with a prefunction conception has little to no understanding of the function concept and cannot engage with problems about functions in any meaningful way. In contrast, an individual with an initial understanding of function views it as an action. An action is any repeatable physical or mental manipulation that transforms mathematical objects to obtain objects. An example of an individual with an action conception of function would be able to plug numbers into an algebraic expression and calculate an output value. An action can become interiorized to a process. A process is an action that can take place entirely in the mind of a subject without the individual necessarily running through all the specific steps of the action. An example of an individual with a process conception of function would be able to transform functions according to repeatable means, for example, by composing two functions. A process can be encapsulated to become a mathematical object, which is a process that can be transformed by an action.

An example of an individual with an object conception of function would understand a transformation of a function without needing a specific formula for the function. Generally, an individual would progress from an action to process to object conception of function, but Breidenbach et al. (1992) argued that often it is vital that an individual be able to move from an object conception back to a process conception.

Asiala et al. (1996) provided a detailed discussion of the APOS perspective. Foremost, they add the concept of a mathematical schema, which consists of the collection of processes and objects related to the concept and their interconnections. They noted that schema themselves can be treated as objects and included in higher-level schemas. For example, schema related to the set of linear functions can be conceived of and operated on as a function space. The authors also highlighted the difference between the external nature of actions and the internal nature of processes. Students with action conceptions of function tend to be governed by external features of the situation. For example, when they see $f(x)$ notation, they know they must substitute some number for x in the expression and compute a value. They do not understand the function beyond a kind of set of instructions to follow. In contrast, a student with a process conception of function can understand that a function is a process that accepts inputs and produces outputs without being compelled to carry out this process. Asiala, Cottrill, Dubinsky, & Schwingendorf (1997) used the APOS perspective to analyze 41 calculus students' graphical understanding of function. Through interviews with the students, they found a consistent picture of students struggling to construct a process conception of function. They concluded that APOS was a useful framework for analyzing students' thinking, and suggested that students might learn and understand more from course designed on APOS.

The historical and individual development and APOS perspective suggests several areas worthy of careful attention in an analysis of function in textbooks. Sierpinska's (1992) argument that students need to see the function concept as a solution to real problems suggests that the settings in which function examples are placed is important. Also, in both definitions and examples, the language used to describe functions is important. Do textbooks provide language that encourages students to view functions as actions, processes, or objects? Whether textbooks provide specific or general examples of functions is important, as engagement with general examples force students away from an action conception of function toward process or object conceptions. Finally, because they can encourage students to conceive of functions as actions, processes, or objects, the representations used for function examples deserve attention in the analysis.

Covariation

Carlson and her colleagues (Carlson, 1998; Carlson, et al., 2002; Oehrtman, et al., 2008) have largely been responsible for developing the covariational perspective on students' learning of function. Carlson (1998) reported on results from a 25-item exam given to three groups of college students and follow up interviews with 5 students from each group. The exam provided a wide array of questions related to function, which allowed her to gain evidence of a variety of student misunderstandings, including several that related to the dynamic nature of function. For example, students had difficulty representing and interpreting graphs of functions in a dynamic sense and interpreting rates of change. She found persistence of weaknesses in understanding and concluded, "…function constructs develop slowly" (p. 137). As Carlson continued her work, she sought to develop a framework to examine the different rates at which she observed students acquiring function concepts.

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) developed their Covariational Framework to further Carlson's previous work. They argued that students' ability to view function dynamically as a process that accepts inputs and produces outputs is key for the development of a mature understanding of function. This ability to coordinate two varying quantities while attending to ways they change in relation to each other forms the heart of the Covariational Framework, which consists of five hierarchical mental actions and five related levels. The actions are: (a) coordinating the dependence of one variable on another, (b) coordinating the direction of change in one variable with changes in the other variable, (c) coordinating the amount of change in one variable with changes in the other, (d) coordinating the average rate of change in one variable with uniform increments of change in the other, and (e) coordinating the instantaneous rate-of-change of a function with changes in the input over an entire domain. The five levels are Coordination, Direction, Quantitative Coordination, Average Rate, and Instantaneous Rate. Each level corresponds to the mental action with the same number, but includes all mental actions at lower levels, and thus is hierarchical. For example, a student classified at the Direction level would demonstrate both the first and second actions.

Carlson et al. (2002) used this framework to examine results from a five item questionnaire given to 20 students who had completed a second semester of calculus along with follow up interviews with six of these students. Their framework allowed them to categorize students in relation to each problem, and found most of these advanced students operating consistently at best at the Quantitative Coordination level.

They found most students "… had difficulty constructing images of a continuously changing rate…" (p. 372) despite studying calculus for two semesters.

Oehrtman, Carlson, and Thompson (2008) reviewed results of prior research and applied the Covariational Framework to make recommendations for strengthening students' understanding of function and ability to reason about covariation. They suggested asking students questions related to the five actions of the framework and pursuing students' understanding of rate of change in various contexts and representations. This perspective on learning about function thus can inform teaching that promotes students' understanding of functions as dynamic processes that accept inputs and produce outputs. The authors argued this will allow students to work with functions in more general ways, understand graphical representations of functions as dynamic mappings, and view functions as dynamic in more meaningful ways.

The covariation perspective suggests three areas worthy of careful attention in an analysis of function in textbooks. Whether textbooks provide language that encourages students to consider the covariation involved with functions should be examined. Representations used for function examples are again important because students need to come to understand the covariation captured in the representation. Finally, because technology can represent covariation in dynamic ways, whether textbooks encourage students to use technology with function examples is of interest.

Prototype Theory

A number of researchers have used a theory of prototypes to study students' learning of function. The theory fits well with concept image and concept definition theory, and most researchers interested in prototypes make connections to this perspective. Prototype theory adds structure to the process of concept image formation. Schwarz and Hershkowitz (1999) argued that for a given concept, for each student, some special examples of that concept are more central to learning than others. These examples, called prototypes, are important in categorical and conceptual judgment. Prototypes are examples that are most highly correlated to other examples of a concept and therefore tend to reflect the structure of the concept as a whole better than other examples. Goldenberg and Mason (2008) noted that students don't necessarily think in terms of equivalence classes, where one example is equivalent to any other valid example in the class, but often think in radial classes where some examples, serving as prototypes, are more representative and central to the class and more easily understood. Students thus use prototypical examples as cognitive reference points in other examples that are judged according to how similar or different they are from the prototype. During initial learning of a concept, often an individual's concept image includes only prototypes. Tsamir, Tirosh, and Levenson (2008) argued that protoypes can support the initial formation of the concept; however, prototypes may not capture all aspects of a concept and thus may hinder cognitive formation of the concept and reasoning based on the prototype. Ideally, examples that serve as prototypes change as learning occurs and concept images begin to include a wider variety of examples. Thus individuals have a broader conceptual base from which to make judgments. However, prototypical judgment often relies on visual comparisons to a prototype or judgment based on attributes of a prototype. When the visual or attribute comparison is based on critical properties common to all examples of the concept and identified by the definition of the concept, this type of judgment is successful (Tsamir, et al., 2008). However, when it is based on attributes that are unique to the prototype rather than applicable to all examples in the concept, prototypical judgments often lead to erroneous conclusions (Goldenberg & Mason, 2008; Schwarz & Hershkowitz, 1999). Students may try to inappropriately impose the unique attributes of the prototype on other examples, and when they cannot do so, they incorrectly reject the examples as instances of the concept.

For example, Schwarz and Hershkowitz (1999) argued that research has demonstrated that quadratic and especially linear functions serve as prototypes of function for many students. Suppose a student is asked to judge whether the graph of the least integer function could represent a function. The student might judge this image against the image of a line or a parabola. Since it is not visually continuous, whereas lines and parabolas are, the student might incorrectly conclude that it cannot be a function. Thus, the prototypes students have for function can play a key role in how they reason about functions, and Schwarz and Hershkowitz maintained that they form an important part of students' concept image of function. They studied the function concept images of 103 ninth grade students who had studied from algebra curricula of two different types. One was a mainly traditional algebra course, and the other had students using interactive computer software to explore algebraic, graphical, and numerical representations of functions. They gave students conceptual judgment tasks to assess their concept image of function. One aspect of this concept image they called prototypicality, which they defined as students' choice of which functions are used, how they are used, and whether they are used exclusively. Amongst all students they found strong draws toward linear prototypes; however, they concluded that the students that had used computer software were less constricted by linear prototype and used a broader base of prototypes more appropriately.

Tall and Bakar (1992) also took a prototype perspective, reasoning that students build up interconnected ideas of prototypes against which they compare new examples that they experience. These interconnected ideas are developed through experiences with the function concept in mathematics class. They argued, "… the collection of activities [in mathematics class] inadvertently colours the meaning of the function concept with impressions that are different from the mathematical meaning which, in turn, can store up problems for later stages of development" (p. 42). They used written questionnaires with 137 secondary and college students to examine students' function prototypes. They found no students with a coherent function concept and observed numerous inappropriate prototypes, such as a function is any expression in *x* set equal to *y*. They argued that students' initial experiences with examples and non-examples of functions had influenced their function prototypes in ways that conflicted with the formal definition.

The prototype perspective suggests that the features of all examples of functions provided in textbooks are worthy of careful attention in the analysis. Students' understanding of the function concept centers around the prototype they hold, which is influenced by the examples they study. The features of these examples, such as representations used, shape the prototype that develops, and thus are of interest. The research noted above also suggests that the families of functions from which examples are drawn are of key importance, as linear and quadratic functions often figure prominently in students' prototypes.

Role of Examples

Role of examples in learning mathematics. Each of the four perspectives above suggests that examples play a role in students' learning about function. Researchers using both the APOS and covariation perspectives claim that students' engagement with certain types of examples can help them move from one level to another. For example, Oehrtman, Carlson, and Thompson (2008) recommended that students engage with examples of functions in which they must attend to changes in direction of one variable in relation to the other so that they can move beyond merely recognizing the dependence of one variable on the other. Researchers coming from a concept image and concept definition perspective have claimed that examples students experience impact concept image. For example, Vinner (1992) wrote, "… very often, the concept image is entirely shaped by some examples" (p. 198). He claimed that knowing examples that students had experienced and the way teachers presented them would allow researchers to predict student concept images. Finally, researchers using a prototype perspective believe examples are core to the establishment of prototypes that students' continually use to make conceptual judgments about functions. For example, Tsamir, Tirosh, and Levenson (2008) argued that examples play an important role in the complex process of concept formation because students' prototypes emerge from the examples with which they engage.

Recently, a number of researchers have taken as a focus the role that examples play in mathematical learning (Bills & Watson, 2008). Watson and Mason (2002) wrote, "It has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques" (p.

378). They argued that in order to learn mathematical concepts, students need a variety of examples to compare for common features and generalities. Thus, students form collections of examples related to concepts in specific contexts. They labeled these collections example spaces and argued that mathematical learning is growth and adaptation of personal, situated example spaces. Competence is then seen as the development of complex, interconnected, accessible example spaces.

Several researchers have built upon Watson and Mason's (2002) concept of example spaces. Zazkis and Leikin (2008) noted several kinds of example spaces fulfilling specific functions, including (a) situated personal example spaces actually serving an individual in a context, (b) personal potential example spaces that an individual might draw on in a situation, (c) conventional example spaces as generally understood by mathematicians and as displayed in textbooks, and (d) collective and situated example spaces that a group such as a class would share in a particular context. They further refined conventional example spaces into expert and instructional example spaces. Expert spaces reflect a rich variety of expert knowledge and include unconventional examples, whereas instructional spaces are ones generally used in instruction and displayed in textbooks. They proposed that example spaces, such as the collection of examples presented in a textbook, can be evaluated in terms of accessibility, correctness, richness, and generality.

Goldenberg and Mason (2008) claimed that example spaces are "inescapable components of the experience of learners" (p. 190). For the authors, learning means expanding example spaces through more experiences with a variety of examples and by means of example construction, including students generating more examples themselves. Example spaces thus flourish when conjecturing and generating are part of classroom norms rather than when teaching focuses on a few paradigmatic or generic examples. Repeated opportunities to generate examples provides additional connections made between examples and concepts, and therefore these examples become more likely to come to mind in the future. Other researchers have also noted that example generation is cognitively complex and can support concept learning in powerful ways (Asghari, 2005; Zaslavsky & Shir, 2005). What is key for learners, according to Goldenberg and Mason, is that learners are aware of what features of an object make it an example, what features can be varied, and over what range they can vary. This variation in examples helps students understand what are essential and what are incidental features. Thus, examining the distribution of features in the examples of functions provided in textbooks is critical.

Other researchers also emphasize the importance of students' attending to variation between examples (Watson & Shipman, 2008; Zodik & Zaslavsky, 2008). Zodik and Zaslavsky argued that examples and non-examples should be carefully chosen and sequenced to highlight the critical and non-critical features of a concept, as this may facilitate or impede students' learning. They noted that, especially at the secondary level, the selection of examples is complex and that examples can be incorrect in a number of ways. Examples may not satisfy conditions to qualify as an example, may not actually exist as possible examples, and if claimed as a counterexample, may not logically contradict the claim. Thus, understanding the correctness and sequencing of examples that students experience can provide insight into what they could learn from engaging with the examples.

Tsamir, Tirosh, and Levenson (2008) also noted the importance of examples in concept formation by specifying two roles. First, because they take a prototype perspective of learning, they argued that examples are the building blocks of concept formation. Thus, they believe students initially accept certain examples as prototypes and compare attributes of other examples against these prototypes. Second, they argued that examples are also the outcomes of concept acquisition. When a student learns a mathematical concept, they have more examples they can draw on when dealing with the concept and have the ability to generate more examples. The more fully they understand the concept, the greater the number of examples they can draw on and generate. The authors also noted the importance of non-examples in concept formation. In addition to positive prototypes to compare new examples against, students also need to be able to compare against non-examples and understand why they are not examples of a certain concept.

Role of examples in learning about function. Researchers have also recognized the role that examples can play in the learning and teaching of function for decades. For example, in their review of research on functions and graphs, Leinhardt, Zaslavsky & Stein (1990) wrote, "The selection of examples is the art of teaching mathematics… Helping teachers to identify and construct focused examples on which to build explanations is an obvious and overlooked area for intervention" (p. 52). Numerous other researchers have also noted the role of examples in the learning of function and comments fall into four main categories: (a) use of examples in relation to the definition of function, (b) variety of examples provided, (c) use of non-examples, and (d) student generation of examples.

Vinner (1991) argued that students should be encouraged to always turn to the concept definition in order to identify examples and non-examples, because their tendency is to rely on their concept images. As Vinner and Dreyfus (1989) wrote, “The student… does not necessarily use the definition when deciding whether a given mathematical object is an example or nonexample of the concept. In most cases, he or she decides on the basis of a concept image…” (p. 356). Leinhardt, Zaslavksy, and Stein (1990) noted similar issues with students’ tendency to rely on examples they remembered rather than the definition. The danger, as noted by Eisenberg (1991), is that the mathematical objects determined by the concept image are not necessarily the same as the mathematical objects determined by the concept definition. A way to potentially decrease problems due to this is through development of richer concept images through interaction with a wider variety of examples.

Vinner (1983) argued that students’ concept images that do not match concept definitions might be a result of the specific set of examples given to the students. If this set is too limited, or students’ attention is not directed to relevant features of the functions, students’ resulting understanding may reflect this limitation if they do not turn to the definition of function. Tall (1990) agreed with this position and criticized most textbooks for using a limited set of simple examples when first introducing the concept of function and claimed this led to limitations in students’ understanding of function. Leinhardt, Zaslavksy, and Stein (1990) also argued that students’ conception of function may be limited because of a lack of variety of instructional examples. They noted that most examples of function students experience in secondary school have explicit formulas and have easily recognizable patterns when graphed, thus it is unsurprising when students balk at functions with graphs that are less regular. Similarly, Even (1993) and White and Van Dyke (2006) claimed that students develop limited function concepts through school experiences because they have mostly experienced familiar equations and smooth graphs. Tall and Bakar (1992) wrote, "When the function concept is introduced initially, the examples and non-examples which become prototypes for the concept are naturally limited in various ways, producing conflicts with the formal definition… we must attempt to develop an approach which makes the prototypes developed by the students as appropriate as possible" (p. 50). Vinner argued that, instead of only experiencing a small number of types of functions, students need continual exposure to a variety of examples that constantly reinforce the key elements of the function concept. Schwarz and Hershkowitz (1999) recommended use of a wider variety of function families beyond linear and quadratic functions so that students' are not limited to thinking about functions of these types. For example, Habre and Abboud (2006) found that students assumed a U-shaped graph must represent the function $f(x) = x^2$ even though they were not provided enough information to make that conclusion and did not need to make that assumption to solve the problem.

Vinner (1991, 1992) gave his view of how examples should be used to learn about function. First, a variety of examples and non-examples of functions should be provided to students. He argued that appropriate pedagogies, before suggesting definitions to the students, suggest examples and opportunities for experiences with the concept. These initial selections of examples can be critical, as Dahlberg and Housman (1997) noted, "… a student's initial understanding of a concept often persists despite the presence of examples and information which conflict with this initial understanding" (p. 283). From these initial examples, students will form the beginning of a concept image of function. Then, Vinner argued, the concept definition should be provided for students, and they should be asked to compare their idea of function to the definition and identify conflicts. As students progress, they should "deeply discuss the weird examples" (Vinner, 1991, p. 80) in order to further enrich their concept image. Carlson (1998) also argued that students needed to repeatedly engage with examples that represented extreme cases of the function concept.

In addition to dealing with a wide variety of examples of function, students should also examine non-examples, which are mathematical objects that do not possess the critical attributes in order to meet the requirements of the definition in order to be categorized as a function. Vinner (1991) argued that for enhanced development of concept images of function, non-examples of functions should be provided for students to evaluate. Similarly, Leinhardt, Zaslavksy, and Stein (1990) wrote, "A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (p. 6). Eisenberg (1991) argued that the reason to include non-examples is to help students solidify their understanding of the exclusive nature of the concept. Examples of functions help students understand how the definition can be used to determine that an example should be categorized as a function. Students also need experiences seeing how the definition can be used to determine that a non-example should not be categorized as a function. This helps students gain a more complete understanding of what a function is and is not. Just as examples should be carefully constructed for students to examine invariants, non-examples should be carefully chosen to help students understand specific aspects of the concept.

Rather than focusing only on careful selection of examples and non-examples of functions to present to students, Dahlberg and Housman (1997) made the case that a more powerful approach may be to have students generate examples of functions. The authors conducted interviews with 10 college mathematics majors and one computer science major. In the interviews, students were presented with a definition of a type of function new to them and were asked to engage with a variety of problems about it. They found four basic learning strategies being used by students. The students that generated their own examples in order to answer the questions and think about the problems learned more, were more successful, had more examples in their evoked concept image, and were better able to use examples appropriately. The authors argued that the generation and reflection on examples provided powerful stimuli for learning. They concluded, “Our findings suggest that it may be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing them with examples and commentary” (p. 297-298).

The examination of the role of examples in learning about functions suggests a number of areas worthy of careful attention in an analysis of function in textbooks. Not only examples, but also non-examples of functions should be examined. A number of researchers also emphasize the importance of the variety of representations and families provided in examples. Finally, the work of Dahlberg and Housman (1997) suggests that whether examples are provided for students or students are asked to provide examples is a valuable feature to examine.

Representations of Functions

Types of Representations

Much of the research on students' learning about functions has focused on the different representations for functions, how students learn about functions through the representations, and how students make connections between various representations. In fact, because the concept of function must be approached through some representation, all research on learning of function addresses representation, at least tangentially. However, some researchers have focused more closely on the impact of the specific representations in students' learning of function.

What different representations of function have been studied and promoted by researchers? Much of the research has been on the symbolic and graphical representations of function, as evidenced by Leinhardt, Zaslavsky, and Stein's (1990) review of research on functions, graphs, and graphing and Romberg, Fennema, and Carpenter's (1993) collection of research on graphical representations of function. No standard names have been agreed upon for each representation: symbolic representations are often also referred to as algebraic or analytic representations and graphical representations are referred to as geometric or visual representations.

Researchers have also recognized or recommended that students experience a wide variety of representations. Leinhardt, Zaslavsky, and Stein (1990) noted that most instruction focuses first on symbolic and then graphic representations, but they also recognized the use of ordered pairs and verbal descriptions of functions. Eisenberg's

(1991, 1992) focus was on students' ability to imagine or generate graphs from symbolic representations; however, he also mentioned arrow diagrams, tables, black input-output boxes, and ordered pairs. Selden and Selden (1992) also listed arrow diagrams and input-output function machines as representations of functions that would encourage different understandings of function in students. Schwingendorf, Hawks, and Beineke (1992) claimed that the function machine representation is common in most secondary mathematics textbooks, but again their focus was mainly on symbolic and graphic representations. In addition to graphs and equations, Sfard (1991) argued that algorithmic representations, which provide a sequence of steps to move from an input value to an output value, would tend to produce a different type of understanding of function in students. Sfard (1992) claimed that experiencing tables, symbols, and graphs may support students' reification of function and wrote, "exposing students to many kinds of representations may be helpful in uprooting [mis]conceptions" (p. 79). Although focusing mainly on graphs, Monk (1992) suggested that students think differently about functions when they are represented by a physical model. For example, he had students use a model of a ladder sliding down a wall as they thought about the functional relationship between the distance of the bottom of the ladder from the wall and the height of the ladder. Carlson et al. (2002) found similar results with the same problem when students spontaneously created models with objects around them. The authors claimed, "The use of physical enactment appeared to provide a powerful representational tool that assisted these students in reasoning…" (p. 372). Tall (1996) also argued that visio-spatial representations of functions that students can interact with provide opportunities for students to develop unique intuition about functions. Kaput (1992) provided students with not only graphical information about functions, but also numerical information in tables and had students provide symbolic representations. DeMarois and Tall (1999) examined students' understanding related to seven different representations: graphic, symbolic, numeric, written, verbal, notational, and colloquial. The difference between written and verbal representations was that the former were descriptions written down and the latter were spoken. Their term numeric referred to tables of input and output values. The notational representation was a symbolic representation specifically using the *f*(x) notation. They used colloquial to refer to what is often called a function machine. In related work, Tall, McGowen, and DeMarois (2000; 2000) focused on this function machine representation and its potential to be an initial representation students use to learn about function that can be expanded and used later in their studies as well. Akkoc and Tall (2005) presented problems to students using not only graphic and symbolic representations, but also ordered pair and mapping representations, which they called set correspondence diagrams. Oehrtman, Carlson, and Thompson (2008) recommended curricula and instruction provide a wide variety of function types and multiple representations to promote a more flexible and robust understanding of function. The National Council of Teachers of Mathematics (1989, 2000) has advocated students' exposure to and use of multiple representations of function. Most recently (National Council of Teachers of Mathematics, 2009), they have argued that one of the key elements in reasoning with and making sense of functions is using multiple representations of functions, including tabular, graphic, symbolic (both explicit and recursive formulas), visual, and verbal representations. The NGA and the CCSSO (2011) have also recently emphasized as a core skill students' ability to describe qualitative behavior using expressions, graphs, and tables.

Most of the above research includes function representations that are fairly common. Two researchers have proposed arguments for use of unique representations. Although Goldenberg, Lewis, & O'Keefe (1992) recognized that graphical representations of function could support students' conception of function as an object, they saw weaknesses in the exclusive use of graphical representations of function. They argued that dynamic representations of functions are needed for students to develop understanding of the dynamic nature of function. They had students use computer software that allowed them to drag the mouse over a domain number line while the software simultaneously generated values corresponding to the function output on a separate number line representing the range of the function. They also restricted this representation so that it did not provide quantitative information to the students. They found that through this representation students tended to develop a more holistic and dynamic understanding of function.

Bridger and Bridger (2001) also argued that traditional graphic representations of functions do not support students' understanding of the dynamic mapping nature of functions. They recommended use of a modified version of mapping diagrams as a supplementary representation. Rather than typical mapping diagrams, which generally use bubbles to represent domain and range, Bridger and Bridger use parallel vertical number lines for domain and range. Lines are drawn from elements in the domain to elements in the range corresponding to the function rule. Through an example of using the representation with a student, they showed that the representation is natural for the student to use and promotes understanding function as a mapping.

Issues with Specific Representations

In addition to Goldenberg, Lewis, & O'Keefe (1992) and Bridger and Bridger's (2001) claims that their unique representations would support specific understandings in students, several researchers have emphasized the value of particular representations for supporting different types of understanding in students. Monk (1992) argued, "Any representation system one chooses will shape the students' responses in its own particular way" (p. 177). Sfard (1991) wrote that graphs would help students understand functions as objects, algorithmic representations such as verbal descriptions of functions would help students understand functions as processes, and symbolic representations would support both understandings. She expected that representations could play a key role in helping students progress from a process to an object conception of function. In a similar vein, Schwartz and Yerushalmy (1992) asserted that a symbolic representation promotes viewing function as a process and a graphical representation promotes viewing function as an object. They also argue that symbolic and graphic representations also help students understand different operations on function more readily. Eisenberg (1991, 1992) argued that thinking visually about functions is a core to understanding function, but that symbolic representations are needed in order to use functions. Ferrini-Mundy and Lauten (1994) also argued that visual thinking is powerful, especially when using functions to explore calculus concepts. Kaput (1992) had 46 high school algebra students use a computer program called *Guess My Rule*, which had students provide domain values and the computer would produce range values corresponding to a function unknown to the student. Then students would try to guess the rule being used by the computer. Through analysis of students' input, he determined that in their interactions with this software, which mainly used numerical representations, students tended to demonstrate a process conception of function. Schwarz and Dreyfus (1995) argued that numerical representations usually involve ambiguity because they only provide some of the ordered pairs of the function. They noted that graphical representations involve ambiguity because of the limited intervals drawn for domain and range as well as the limits on the accuracy of the drawing. Finally, they argued that symbolic representations often have ambiguity because the domain is often not specified and because any function has an infinite number of possible equivalent symbolic representations. Thus, each type involves ambiguities, but of slightly different natures. Dahlberg and Housman (1997), through interviews with 11 college students about a type of function new to them, found that the students that used visualization had the greatest expansion of their concept image of the new type of function. McGowen, DeMarois, and Tall (2000) and Tall, McGowen, and DeMarois (2000) discussed strengths of the function machine representation for introducing students to the concept of function. They argued that its strengths include that it is visual, iconic, strongly process oriented, and it allows simple interpretations of profound ideas. They claimed that it is good to use before examination of specific types of function, but can be expanded on to build future understanding of specific functions and therefore can have long-term meaning. As they used function machine representations in a course with college students, they found through surveys, interviews, and student work that students had improved understanding of functions, and especially functions as processes. White and Van Dyke (2006) supported this position and bemoaned the fact that often the function machine representation is dropped quickly from the curriculum.

Although specific representations can support students' understanding in particular ways, several researchers have reported students having difficulty with certain representations of functions, especially the graphic representation. Eisenberg (1991, 1992) argued that students as well as teachers tend to resist thinking about functions visually; however, he also noted that many students have difficulties with notational complexities of symbolic representations. White and Van Dyke (2006) also argued that many students have difficulties with functional notation in symbolic representations. Eisenberg proposed three reasons for avoiding visualizing functions: (a) it is more cognitively difficult, (b) it is harder to teach, and (c) it is believed to be less mathematical. Schwingendorf, Hawks, & Beineke (1992) also found that most of the calculus students in their study preferred symbolic representations and that students in a traditional calculus course especially had difficulties with graphical representations. Although not working with students, Goldenberg, Lewis, and O'Keefe (1992) argued that students commonly have difficulty understanding graphs as representations of function mappings. Kaput (1992) found that most students in his study relied heavily on numeric representations of functions and ignored graphical representations that were provided, even when encouraged to use them. They also had difficulty with symbolic representations. In contrast, Tall (1996) argued that students tend to cling fairly strongly to their preferred representation of function, but that few students prefer numeric representation. Asiala, Cottrill, Dubinsky, and Schwingendorf (1997) found that of their 41 calculus students, most of them sought a symbolic representation to work with rather than use the graphical representation provided. Similarly, Dahlberg and Housman (1997) found that students had a strong tendency to think about symbolic representations, even when encouraged to think graphically. DeMarois and Tall (1999) found that in each of their three cases, the students had most difficulty with the graphic representation of function. Bridger and Bridger (2001) argued that many students do not understand what information a graphical representation of a function is providing. Gonzalez-Martin and Camacho (2004) found that of the 31 calculus students who took their questionnaire, many tended not to use graphical representations and preferred to work with symbolic representations. Habre and Abboud (2006) used classroom observations and interviews with 89 students in a first semester calculus course emphasizing multiple representations of function to study changes in their understanding of function. They found that many students initially resisted using graphical representations, but that by the end of the course, most students had begun to think visually about functions more often.

Connections Between Representations

Beyond understanding functions through a specific representation in isolation of other representations, many researchers have argued that students' ability to recognize connections and translate between representations is key to improved understanding of the function concept. Researchers have recognized this as a major difficulty experienced by students (Akkoc & Tall, 2005; Artigue, 1992; Eisenberg, 1991; Gonzalez-Martin & Camacho, 2004; Markovits, et al., 1986; Schwarz & Dreyfus, 1995; Sfard, 1992; Sierpinska, 1992; Tall, 1996; White & Van Dyke, 2006). Vinner (1992) found that many students have the misconception that functions are embodied by only one type of representation, for example, that all functions are equations. Leinhardt, Zaslavsky, &

Stein (1990) recognized the limitations of only using graphical representations of function, argue that symbolic and graphical representations illuminate each other, and examine research that reveals how students both recognize and construct different representations of the same function. Eisenberg (1991, 1992) focused on students' ability to visualize functions, that is, create a graphical representation of a function, either in their mind or on paper, from other representations of function. He argued that it is the connections between representations that should be stressed. Schwingendorf, Hawks, & Beineke (1992), in a study of 56 first year calculus students in either a traditional course or a course utilizing computer programming, found that the students using the computer programming made more connections between graphs and symbolic representations of functions and thus developed a greater sense of function than the traditional students. Goldenberg, Lewis, and O'Keefe (1992) argued that use of symbolic, graphical, and tabular representations together should provide students with an improved understanding of function. Kaput (1992) found that the students who struggled most had difficulty writing appropriate symbolic representations that matched the verbal rules they generated from numeric representations. Thompson (1994) suggested that students must see the invariants between different representations in order to understand the concept of function. Schwarz and Dreyfus (1995) examined students' ability to recognize these invariants when working with symbolic, graphic, and numerical representations of function. They gave questionnaires to 140 ninth grade students, half of which had studied in a traditional algebra course, and the other half of which had studied functions using computer software that dynamically produced all three representations of functions. They found students who used the software were better able to identify invariants between representations and could coordinate their actions between representations better. DeMarois and Tall (1999) examined the ability of three college algebra students to translate between function machine, tabular, symbolic, and graphic representations. They found limitations in each student's ability to flexibly move between representations, but the student who struggled most had the weakest conception of function. Bridger and Bridger (2001) note the importance of students analyzing functions analytically, graphically, and numerically. The National Council of Teachers of Mathematics (2009) argued that students need to make connections between representations in order to gain different perspectives on functional relationships and improve their understanding of functions.

Function Families

Whereas many researchers have categorized learning about functions by the representation used, fewer have focused on studying specific function families. Families of functions are important to researchers who use a prototype perspective to examine how students learn about functions. For example, Tall and Bakar (1992) sought to describe concept images of function of 28 high school students and 109 first year college students. They believed that students would have developed certain prototypes of functions based on the examples of functions that they had experienced. Through examination of students' responses to questionnaires, they found that they tended to develop concept images of function that aligned with the families of functions with which they were most familiar. Students tended to classify graphs that looked like parabolas or other familiar polynomials as functions. However, they did not consider representative of functions the graphs of functions that did not fit into function families they had studied. In addition, they tended not to recognize constant functions as functions.

Other researchers have also found that students' exposure to certain families of functions, especially linear functions, tends to produce prototypical images of function closely tied to that family. Sierpinska (1992) argued that students' difficulties in post-secondary mathematics courses can be connected to the fact that they have mostly experienced functions that are linear, quadratic, or trigonometric. She wrote, "…specific beliefs and prototypes are linked with the kind of introductory examples a student has been exposed to" (p. 47).

Markovits, Eylon, and Bruckheimer (1986) noted what they described as a "student addiction to linear functions" (p. 28). They investigated how junior high students began to understand function by giving problems related to functions to over 400 ninth graders. They found that examples generated by students tended to be familiar examples and concluded these students had a limited repertoire of examples of functions, especially outside the family of linear functions. Leinhardt, Zaslavsky, and Stein (1990) and Even (1993), in reviewing research on student understanding of function, also both noted that students tend to produce linear examples, define functions as linear relationships, and generally hold a linear prototypic image of functions. Schwarz and Hershkowitz (1999) argued that both linear and quadratic functions typically serve as prototypes of functions for students, but linear functions seemed to be especially core to students' conception of function. De Bock, Verschaffel, and Janssens (2002), through examination of questionnaires from nearly 700 secondary school students, found that students tended to rely on linear reasoning even when warned in advance of the possibility of nonlinear situations and when provided with visual representations of nonlinear situations. Recently, Oehrtman, Carlson, and Thompson (2008) found that students tend to unwarrantedly assume functions are linear or quadratic. They argued that this finding is unsurprising because typical curricula introduce functions through these families. Collectively, the above findings suggest that it is important that students experience types of functions outside of the linear and quadratic families.

Even more extreme than merely adding examples of functions from a wider array of function families, several authors have suggested an entire reorganization of secondary algebra. Fey and Good (1985) argued that burgeoning technology would allow a restructuring of algebra around major families of elementary functions such as polynomial, trigonometric, exponential, and algebraic. For each family of functions, students would explore real-world situations, work to recognize and mathematize relationships, answer questions about the functions they perceive, and move toward formal organization and verification of their understanding. They argued that this focus on function families would allow students to initially work more intuitively until their understanding about each family of functions is crystallized and organized into a body of mathematical results that can be used for further study. Schwartz and Yerushalmy (1992) described such a technology- and function-based algebra curriculum organized around studying invariants within function families. They claimed that such an approach was "… a far more sophisticated and potentially far more rewarding activity for algebra students" (p. 288). Haimes (1996) also described a portion of an algebra curriculum centered on function families including linear, quadratic, exponential, reciprocal, and periodic functions, where function was the unifying theme of curriculum. Tall (1996) mentioned one common approach in secondary algebra curricula, which is to study function families by first focusing on linear functions, then quadratic, polynomial, rational, trigonometric, exponential, and finally logarithmic functions. The main difference in a typical approach is that functions and properties of functions in the families are not the focus, rather particular procedures that can be used with functions in each family are the focus. Despite the differences in focus, it appears that linear, quadratic, polynomial, rational, periodic or trigonometric, exponential, and logarithmic families are important in both approaches. The *Principles and Standards* (NCTM, 2000) recommend that students "understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions" (p. 296). Most recently, the National Council of Teachers of Mathematics (2009) has recommended that students need to "perceive functions as a larger algebraic structure in which different families of functions (e.g., linear, quadratic, exponential, and periodic) behave in particular ways and possess unique characteristics." (p. 40). They argued that when students learn to model realistic data with function families, they learn to appreciate the usefulness of mathematics. The NGA and the CCSSO (2011) presented a similar argument, placing an understanding of function families, especially linear and exponential functions, as a central for students.

Function Settings: Abstract Versus Realistic

Traditionally, secondary mathematics textbooks have been dominated by abstract symbol manipulation and procedure (Oehrtman, et al., 2008). However, as noted previously, many have criticized an overly abstract approach to function in many textbooks. Additionally, for over two decades, calls have been made for increased use of realistic contexts for examples and problems in general as well as for functions in particular. Fey and Good (1985) stressed the value of using situations in which functions are used to model the structure of real-life situations. The first recommendation in the 1989 *Standards* (National Council of Teachers of Mathematics) related to function is that all students learn to "model real-world phenomena with a variety of functions" (p. 154). Similarly, the 2000 *Standards* (National Council of Teachers of Mathematics) recommended that students be able to use functions to represent relationships arising from various contexts. Oehrtman, Carlson, and Thompson (2008) also claimed that an approach that emphasizes abstract symbol manipulation and procedure does not support development of students' understanding of foundational function concepts. Most recently, the National Council of Teachers of Mathematics (2009) argued that students need opportunities to model realistic data with functions.

Some researchers have emphasized a need for striking a balance between using abstract and realistic settings to study function. Schwartz and Yerushalmy (1992) recognized the need for learning about both abstract functions and functions that are descriptive of phenomena in the world for modeling. Leinhardt, Zaslavsky, and Stein (1990) recognized that settings of functions can be "more or less contextualized or abstract" (p. 20); however, they warned that realistic contexts are not always straightforwardly beneficial for students. When students are familiar with a context, their preconceptions may make their understanding of functions more difficult. For example, when students examine the distance a car has traveled graphed against time, they may be tempted to interpret the line as the actual path the car has followed and thus have difficulty interpreting the representation. The NGA and the CCSSO (2011) argued that a coherent understanding of function involves both ability to manipulate and make sense of abstract, symbolic representations of functions as well as make connections between functions and the real world. Ultimately, it is important that students have experiences with functions in both abstract and realistic contexts.

Technology and Learning About Functions

A comprehensive review of research on the influence of technology on students' learning of mathematics in general and functions in particular was not conducted, as it would have comprised an entire study in itself. However, one can hardly examine research on students' learning of function without reading about studies using a range of technologies to help students learn about the concept and examine results of such approaches.

Researchers have recognized that the computational power of most technologies has the potential to change the way students engage with functions. Fey and Good (1985) argued that computers give students the ability to explore functional relationships in real world data that were previously inaccessible because of the time involved in doing "messy" computations by hand. Asiala et al. (1997) studied calculus students who had studied in a reform calculus course that involved examining calculus concepts with a Computer Algebra System (CAS) and computer programming. The CAS was able, for example, to compute output of functions without requiring the students to do any computing. They found that these students had made progress toward a process conception of function, whereas students who took a more traditional calculus course maintained an action conception of function for the most part. Zbiek and Heid (2009) argued that the computational power of a CAS has the potential to allow students to focus on the “big ideas” about functions.

The computational power of technology is related to the potential for technology to better represent the dynamic nature of functions. Goldenberg, Lewis, and O’Keefe (1992) described software that dynamically represented function output in a graphical representation as students provided input values on a parallel graphic representation. They noted that this technology allowed students to see and explore ideas about functions that many students never have the opportunity to examine, such as ways that functions compress or expand. They argued, “… if the concepts of mathematics should be expressed first by their simplest and most natural representation, then dynamic representation of functions arguably should precede the graphs of traditional advanced analysis” (p. 259).

Several researchers have noted the ability for various technologies to provide more representations of functions for students and the positive impact this tends to have on their understanding of function. Schwartz and Yerushalmy (1992) proposed that students should use software that represents functions graphically and symbolically rather than use a traditional approach of expressing particular functions symbolically and then graphically as an afterthought. In this way, families of functions can be studied for invariant properties. Technology allows the representations to be presented simultaneously. They argued this approach to algebra is far more sophisticated, pedagogically powerful, and potentially more rewarding for students. Zbiek and Heid (2009) also argued that CAS allows curricular approaches to function to differ because it has the capability to help students understand “There are many types of functions, some of which have multiple representations that need not be limited to graphs, symbols, and numbers" (p. 544). Kaput (1992) noted that results from using software that incorporates numeric, graphic, and symbolic representations of function with students studying polynomial families of functions showed promise. Schwarz and Dreyfus (1995) studied students who participated in a course in which they used a computer microworld that provided dynamic views of numeric, graphic, and symbolic representations. They compared these students' understanding of function against students from more traditional classes. They found the students who had engaged with the computer software could cope with arbitrary elements of function situations better, coordinate actions among different representations better, and recognize invariants in these situations better. Schwarz and Hershkowitz (1999) found similar results with students who used interactive computer software to explore multiple representations of a wide variety of functions. Compared to students who did not, these students were less constricted by linear prototype, tended to consider the context of the problem more, integrated more considerations in their reasoning, selected functions and graphs more appropriately in their justifications, and relied less on simple visual comparison to parabolas or lines as prototypes.

The additional capability that technology brings to the study of the function concept has several implications for how students might learn about functions. Because technology allows students to generate more representations of functions more rapidly and dynamically, students are more able to conduct their own exploration of examples of functions (Goldenberg, et al., 1992). With technology, students are also able to explore a wider variety of function families and representations (Zbiek & Heid, 2009). As Fey and

Good (1985) noted, technology opens “a fast track to the polynomial, trigonometric, exponential, and algebraic functions that model interesting phenomena” (p. 48). Fey and Good and Schwartz and Yerushalmy (1992) argued that technology is powerful enough to enable an entire reorganization of the algebra curriculum, centering it on the study of families of functions.

Along with the promise of technology for supporting students’ learning of the function concept, researchers have recognized limitations in this area as well. Students do not always understand what the technology is doing or the information that it is providing. For example, Goldenberg, Lewis, and O’Keefe (1992) argue that students may not understand the traditional, static graphic representations of functions provided in many technologies. Kaput (1992) actually found that students tended to ignore computers’ graphic representations of functions, even when prompted by the interviewer to use them. Kaput also found that the *Guess My Rule* software he was using encouraged a procedural conception of function and that a different approach would be needed to help students move toward an object conception. Asiala et al. (1997) found that despite an entire semester using a CAS, calculus students largely still had an action conception of function. Despite these limitations, most researchers appear positive about the potential for technology to positively impact students’ study of the function concept.

Technology provides the capability for students to engage with function examples in new ways, which means there is the potential for them to learn about functions in different ways. An analysis of the relationships between functions, technology, curriculum, and students would require its own study. For this study, an analysis of how often students are encouraged to use technology to study function examples can provide initial, basic information about these relationships.

Student Difficulties and Misconceptions about the Function Concept

Much of the research on students' learning of function has focused on students' difficulties or misconceptions. Some of the common difficulties have been discussed above in the context of perspectives on student learning of function. For example, students' troubles with translating between representations and their tendency to think only about linear functions have already been noted. In this section, how the various perspectives treat difficulties and misconceptions is examined. Then results of research on student difficulties and misconceptions are presented.

Perspectives on Difficulties and Misconceptions

Researchers have sought to probe students' understanding of function and study their difficulties and misconceptions through a variety of perspectives. Previously, several theoretical approaches to the study of students' learning of function were discussed. These approaches have guided researchers to frame student difficulties and misconceptions differently. Vinner (1983, 1991, 1992; Vinner & Dreyfus, 1989) and other researchers who have taken a concept image and concept definition perspective viewed student difficulties and misconceptions as aspects of their individual concept images that create conflict when compared to the concept definition. For example, a students' concept image may only include functions with continuous graphs. This is a misconception because the definition of function does not require this. Difficulties and misconceptions are overcome when conflicts are confronted explicitly and students have opportunity to revise their concept images.

Sierpinska (1992) traced the development of the concept of function over time and argued that this development is mirrored in individual development. Therefore, individual difficulties and misconceptions are obstacles to individual development mirrored by historical obstacles to the development of the concept of function. Dubinsky and Harel (1992) and other researchers using an APOS perspective took similar views to Sierpinska, but add that difficulties and misconceptions are developmental obstacles particular to a certain level of the APOS framework. For example, a student may believe a function must have a numerical domain because they cannot conceive of a function as a mathematical object and require numbers on which to carry out a process. This misconception then disappears when the students moves to the Object level of the framework.

Researchers who have taken a prototype theory perspective, such as Tall and Bakar (1992), have argued that difficulties and misconceptions arise because of prototypical thinking based on limited prototypes. For example, students might consider $y = \pm\sqrt{x}$ to be a function because their prototype of symbolic functions includes any equation where y is equal to an expression in x. Difficulties and misconceptions are overcome when students adjust their prototypes to better match the definition of function.

A few researchers have taken unique views of student difficulties and misconceptions. For example, Markovits, Eylon, and Bruckheimer (1986) considered student difficulties and misconceptions to be largely a direct function of the content and presentation of the curriculum. These difficulties and misconceptions would be overcome by changing the curriculum. Monk (1992) viewed misconceptions as arising from broad global rules that students held. For example, students appeared to believe that many aspects of the problem situation they were dealing with were symmetric. This rule was largely correct, but misled some students to believe the function had a point of inflection at the midpoint of the domain. In this view, difficulties and misconceptions would be overcome through examination of the appropriateness of the global rule related to the problem.

Categories of Student Difficulties and Misconceptions Related to Function

Despite the variety of perspectives on student misconceptions related to function, common difficulties and misconceptions emerge from the numerous studies. As discussed above, underlying the history of the development of the concept of function is a struggle between understanding function as dynamic or static. Various definitions used throughout its history have emphasized one or the other, but the current definition accepted by mathematicians implies a static viewpoint of function. Thus it is unsurprising that researchers have found that students struggle with understanding function also as a dynamic relationship (Ferrini-Mundy & Lauten, 1994; Goldenberg, et al., 1992; Leinhardt, et al., 1990; Monk, 1992). Carlson and her colleagues (Carlson, 1998; Carlson, et al., 2002; Oehrtman, et al., 2008) have especially focused on function as covariation between two quantities as key and have found that students have difficulty understanding function in this way. From an APOS perspective, researchers have found students predominantly at the Action level, with some students at the Process level, and almost no students at an Object level, even after studying calculus (Asiala, et al., 1996; Dubinsky & Harel, 1992; Eisenberg, 1991, 1992; Habre & Abboud, 2006; Kaput, 1992; Sfard, 1992; White & Van Dyke, 2006; C. G. Williams, 1998). Thus, most students can only use functions to carry out operations to produce images. They have difficulty thinking about this function process itself, and generally cannot think about functions as objects to act on. These findings suggest that in addition to difficulties understanding the dynamic nature of function, students also have troubles with understanding the static nature of functions in ways that are useful in higher levels of mathematics.

In addition to the foundational nature of function as static and dynamic, several researchers note students' difficulties with the two aspects of function that Even (1993) maintained are fundamental to the concept: univalence and arbitrariness. Univalence refers to the requirement that a function map exactly one element in the range to each element in the domain. Students have been found to have problems reversing the direction of this relationship (Leinhardt, et al., 1990; Markovits, et al., 1986; Vinner, 1983), requiring the relationship to hold in both directions so that only one to one relationships are considered functions (Dubinsky & Harel, 1992; Leinhardt, et al., 1990; Vinner, 1983), and generally confusing the roles of dependent and independent variables or preimages and images (Carlson, 1998; Eisenberg, 1991; Markovits, et al., 1986; Oehrtman, et al., 2008). Arbitrariness refers to the fact that functions can be any mapping and need not have a specific algebraic rule or algorithm relating preimages to images. Often students do not classify functions as such unless they do have a formula (Even, 1993; Vinner, 1983).

Domain and range are also core to the concept of function. Students often demonstrate a general lack of attention to the domain and range, which leads them to make mistakes when dealing with functions (Markovits, et al., 1986; Schwingendorf, et al., 1992; Sfard, 1992). Dubinsky and Harel (1992) also found that many students believe that the domain and range of all functions must consist only of numbers. Tall and Bakar

(1992) found evidence that students tend to believe that the domain must be complete in some sense, and not intervals or subsets of a larger set. For example, they found many students rejected a function defined over rational numbers because it was not defined for irrationals, and thus the domain was not all real numbers.

This result hints at another, less clearly defined, misconception that several researchers have suggested. Often students have been found to decide functions are not functions or experienced other difficulties with functions because they are not in some way familiar, reasonable, or well known (Baker, et al., 2000; Even, 1993; Leinhardt, et al., 1990; Tall & Bakar, 1992; Vinner, 1983; White & Van Dyke, 2006). For example, Baker, Cooley, and Trigueros, through interviews with 41 college calculus students, found that they had difficulties with a function that had a cusp because they had little experience with such functions. The opposite, that students accept familiar objects as functions even if they are not, has also been observed. Tall and Bakar found that students believed a graph of a circle was a function because they knew its name and could write an equation to represent it. These same students rejected as representations of functions graphs that had an "unreasonable" appearance that included many changes in direction. Another related finding is that some students believe a function must have some type of approval from a source of mathematical authority (Even, 1993; Leinhardt, et al., 1990; Vinner, 1983). For example, a student might believe that a function with a name such as "the step function" qualifies as a function, even if he or she rejects other functions with similar properties such as disconnected graphical representations. Exactly what will be familiar, reasonable, or approved to students is highly individual and researchers currently have not attempted to clarify this source of misconceptions any further.

Because most research has been conducted utilizing symbolic and graphic representations of functions, researchers have found numerous difficulties and misconceptions specific to each of them. With regard to symbolic representations, students often have difficulty with understanding variables (Eisenberg, 1992; Kaput, 1992; Sierpinska, 1992). Often they confuse the role of a letter such as *x* in a symbolic representation of a function as a varying quantity with the role of a letter such as *x* in an algebraic representation of an equation as an unknown fixed value. Similarly, students often have trouble with common function notation in which a letter such as *f* is different than other letters in that it plays the role of the name of the function (Carlson, 1998; Sierpinska, 1992; White & Van Dyke, 2006). Even when students understand this notation, some students can only understand and treat symbolic representations of functions as algorithms to follow rather than as representations of mathematical objects (Dubinsky & Harel, 1992; Habre & Abboud, 2006; Kaput, 1992; White & Van Dyke, 2006). For these students, discussion of general functions without specifying an algebraic formula is impossible. Tall and Bakar (1992) and White and Van Dyke (2006) also report that some students believe these algebraic formulas must be in the form of an expression in *x* set equal to *y*. Other researchers report similar, but less rigid, beliefs. For example, some students believe a function must be able to be written as a formula (Habre & Abboud, 2006; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Sierpinska, 1992; Vinner, 1983), and many believe that functions can have only one formula (Carlson, 1998; Even, 1993; Ferrini-Mundy & Lauten, 1994; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Schwingendorf, et al., 1992; Sfard, 1992; Tall, 1990; Vinner, 1983). These students would reject piecewise functions as functions because they are defined by multiple formulas. They would also believe that two different but equivalent formulas are actually two different functions, for example, they would regard $f(x) = 2(x+1) - 3$ and $g(x) = 2x - 1$ to be two different functions despite the fact that $f(x) = 2(x+1) - 3 = 2x - 1 = g(x)$.

Researchers have also identified numerous difficulties and misconceptions that many students have in regard to graphic representations of functions. Just as some students believe a function is a formula, some believe it must be a graph or have a graph (Sierpinska, 1992; Vinner, 1983). Other students have the opposite difficulty and believe graphs are somehow disconnected or peripheral to functions, and therefore see no value in attempting to visualize functions (Eisenberg, 1991). Even when presented with graphic representations of functions, some students have a general lack of understanding of what is being presented to them (Bridger & Bridger, 2001; Sierpinska, 1992; White & Van Dyke, 2006). White and Van Dyke reported students being confused about the axes generally included to orient the image of the function. Some students believed they were part of the function itself and without them the graph was not a function. They also reported a result found by several other researchers (Baker, et al., 2000; Carlson, 1998; Monk, 1992), that students have more difficulty interpreting information about intervals of graphs as opposed to dealing with individual preimages and images. Other students tend to look at graphs holistically but misinterpret them as physical pictures related to the context of the problem (Goldenberg, et al., 1992; Leinhardt, et al., 1990; Monk, 1992). For example, a graph representing a car's speed over time might go up and then down, representing an increase and then decrease in speed. However, some students will have the misconception that the graph is showing that the car went over a hill because the graph looks like the cross-section of a hill. Other students misinterpret graphs to represent functions with formulas they are familiar with, even when this is not warranted (Habre & Abboud, 2006; Oehrtman, et al., 2008; Tall & Bakar, 1992). For example, Habre and Abboud provided students with a U-shaped graph without an accompanying formula and almost all students assumed it represented $f(x) = x^2$, even though it could have realistically represented $f(x) = x^{2n}$ for a range of positive integer values for *n*. Finally, many researchers report that students require functions to be visually "regular" in a variety of somewhat vaguely defined ways. Many students reject functions as functions when their graphs are not continuous (Carlson, 1998; Dubinsky & Harel, 1992; Markovits, et al., 1986; Sfard, 1992). Here continuous does not refer to the rigorous mathematical definition, but rather something akin to the informal definition of being able to draw the graph without lifting the pencil. Thus, these students would reject as representations of function graphs with visual holes, gaps, or jumps. Some students also do not consider graphs to represent functions unless the graphs meet some level of smoothness (Even, 1993; Tall, 1990; Tall & Bakar, 1992; White & Van Dyke, 2006). These students would reject as representations of functions graphs with sharp corners or a plethora of changes of direction, even if they are continuous in the sense above. Researchers have not pursued what level of smoothness students might desire, and this certainly varies from student to student. Finally, Tall and Bakar (1992) found that some students expected graphs of functions to be visually "complete" in some sense. For example, students rejected the graph of a quarter circle centered at the origin and in the first quadrant and the graph of a semicircle centered at the origin and above the x-axis as representations of functions, but they did accept the graph of a circle centered at the origin as the representation of a function. These students claimed the graphs of first two were incomplete. Researchers have not currently pursued this finding to clarify what different students believe about this requirement for completeness.

In addition to the many findings in relation to the symbolic and graphic representations, several researchers have specifically examined students' understanding of constant functions. Some students do not believe they are functions (Even, 1993; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Tall & Bakar, 1992). This may be because the image of the function does not vary or the symbolic representation does not include a letter as a variable. Markovits, et al. (1986) found students with difficulties thinking about various aspects of constant functions, such as the preimages, images, domains, and ranges of these functions.

A final area of difficulty identified by researchers is related to the contexts of the functions. When functions are embedded in practical, realistic settings, students struggle with connections between the function and their understanding of the context (Sierpinska, 1992; C. G. Williams, 1998). They especially have difficulty generating function representations of real world functional relationships (Carlson, 1998; Eisenberg, 1992). These difficulties may be related to those noted above, such as troubles understanding the covariational nature of function.

Conceptual Framework

Drawing on the research related to function, a conceptual framework has been developed to guide the analysis of secondary mathematics textbooks' approach to function (see Figure 2.1). The framework includes four aspects of function in textbooks: language, presence, core features, and ancillary features. For each function in a given textbook, these four aspects will be considered. The language used with function definitions and examples will be analyzed. Elements related to language include how function is defined; whether a function is described as an action, process, object, or part of a schema; and whether examples of function are general or specific. Together, these aspects provide information about the textbook authors' use of language about functions.

The presence of functions in textbooks will also be considered. Elements related to the presence of function include how prevalent examples of function are in textbooks, how frequently examples are identified as functions, whether examples are present in textbook lessons or exercises, how frequently textbooks request that students provide examples, and errors in examples present in textbooks. Collectively, these elements provide insight into the presence of functions in a textbook.

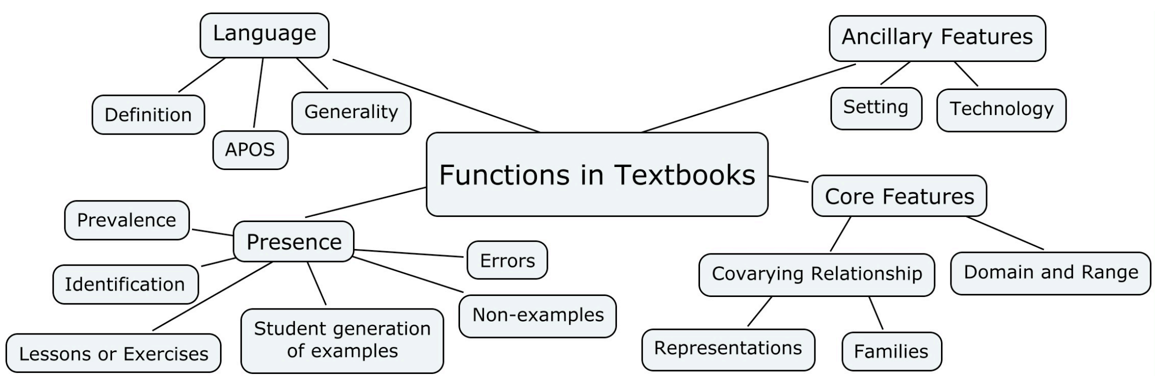


Figure 2.1: A conceptual framework to guide the analysis of function in textbooks.

Core features of function include the domain and range of the function and the covarying relational nature of the function. Aspects of this covariance are captured by the ways the authors choose to represent functions as well as the families of functions that authors choose to include. These are core features because these features cannot be changed without changing the function. Note that although many representations of a function are possible, any representation must represent aspects of the same covarying relationship. In this sense, this aspect cannot be changed without changing the function.

Ancillary features of a function include the setting for the function and technology recommended to examine the function. The setting refers to whether the function is in a relatively abstract setting or whether it is embedded in a more realistic context. These aspects are considered ancillary because textbook authors could make a variety of changes to these features without changing the function. For example, one textbook might simply present a symbolic formula for a function and ask students to generate a graphic representation, whereas another might give the same symbolic formula, but present it as part of a realistic situation and ask students to use graphing calculators to create a graph. Both approaches deal with the same function but in different ways.

The features of the framework can support comparisons of the results of the analysis to recommendations from research on how students learn about functions. For example, researchers have found students often have difficulty translating between representations of functions and have suggested that students have opportunities to experience a variety of representations and be asked to translate between them. This framework will guide the analysis of textbooks to focus on function representations and whether students will be asked to produce different representations.

Textbook Analyses

Mathematics textbooks play a prominent role in US classrooms (Tyson-Berstein & Woodward, 1991); however, they do not contain the same content or approaches. Therefore, textbook analyses play a key role in mathematics education research. A number of recent analyses are described below in order to review methodologies that have been used as well as to demonstrate the need for a textbook analysis focused on the concept of function in secondary textbooks in the US.

Jones (2004) analyzed the probability content of popular and alternative middle grade mathematics textbook series from four eras of mathematics education in the US: the New Math (1957-72), Back to Basics (1973-83), a focus on Problem Solving (1984-93), and advent of the NCTM Standards (1994-04). He analyzed the extent and nature of the treatment of probability and structure of probability lessons from these textbooks. He used both market-share data and a "professional consensus" to determine which textbooks to select. Within each textbook, Jones determined the number of pages that contained probability tasks and their placement in the textbooks, checked whether content in later books was new or repeated, tracked the introduction and development of probability concepts, determined the level of cognitive demand required by each probability task, and selected archetypal tasks to illustrate each era. In order to check reliability of his coding, he trained two additional coders, had them code 60 tasks independently, and compared their codes to his own. In addition, he recoded the 60 selected tasks and did a test-retest comparison with his own codes.

Jones (2004) used his data to compare treatment of probability within series across grades, between series of an era, and across eras. He found that the Standards era books contained far more probability tasks than other eras and a wider variety of learning expectations. The majority of tasks in all series had low levels of cognitive demand and archetypal activities were remarkably similar across eras. Exceptions were the Standards alternate series and the Back to Basics alternate series, which had more tasks with higher levels of cognitive demand and a broader range of levels of cognitive demand.

Probability lessons generally occurred at the ends of textbooks, and lessons tended to have a similar format, beginning with descriptions of probabilistic situations, providing definitions focused on theoretical probability rather than applied, and then providing student exercises related to probability. Again, an exception was in the Standards alternate series, which involved students throughout the lesson and provided more applied situations.

As in Jones' (2004) work, the analysis presented in this study also focuses on a particular content area, uses market-share data to inform textbook selection, examines prevalence of the concept in the textbooks, uses archetypal examples to illustrate findings, and draws comparisons within and between series. Similar to the way Jones tracked the introduction and development of a concept, this study provides information about how definitions and examples of functions are provided initially in each series, and how their features change over the course of the series. Rather than coding examples according to levels of cognitive demand, this study draws on frameworks used specifically to research student learning of function for the coding design. In contrast to Jones' study, older textbooks were not selected, as one main goal of the study was to provide information about textbooks currently being used in classrooms.

Dingman (2007) examined two popular elementary and two popular middle grades textbook series for alignment to ten states' grade-level Learning Expectations (LEs) related to fraction concepts. States were selected from two categories, those with statewide textbook adoption policies and those without such policies. The five most populous states from each were selected for inclusion. For textbook selection, Dingman used market share data from the 2004-05 school year to select the two most popular elementary and middle grade textbook series. He constructed a generalized set of LEs from a sample of states' LEs in addition to research on rational numbers. This resulted in 50 generalized LEs in nine categories, which he used to create a standardized grain size in order to make comparisons across states and against textbook instructional segments related to fraction. He coded each states' LEs with these generalized LEs. For textbooks, he analyzed five types of instructional segments related to fraction concepts: lessons, pre-lessons, end-of-lesson extra features, end-of-chapter features, and games. For each segment, he tracked the pages on which it occurred, the related learning objectives, and the matching generalized LEs. Dingman checked the reliability of his coding by having three doctoral students code instructional segments from one textbook and checked the percent of agreement on their coding of the primary LE of each segment as well as agreement of all codes with his codes.

Dingman (2007) used his generalized LE codes for both state LEs and textbooks to identify levels of agreement between the two. He concluded that there were considerable differences in alignment. Generally, he found many instructional segments in books that did not correspond to states' LEs, but that generally textbooks covered the majority of states' LEs. He compared alignment in the three most populous textbook adoption states against the rest of the states. The "Big Three" aligned well with elementary LEs, but the alignment was stronger with the other seven at the middle grade level. Overall, elementary series aligned better than middle grade series and publisher-generated textbooks aligned better than NSF-funded books.

As in Dingman's (2007) work, the analysis presented in this study also focuses on a particular content area, uses market-share data to inform textbook selection, examines prevalence of the concept in the textbooks, and establishes reliability of the coding procedure. Rather than comparing textbooks to state LEs, this study compares findings to recommendations from research on student learning. Dingman also used a larger unit of analysis than this study, focusing on instructional segments. The units chosen for this analysis were function definitions and examples, because concept image and concept definition perspective informing the study indentifies these as key constructs in students' learning.

The American Association for the Advancement of Science (AAAS, 2000, 2002) through Project 2061 has produced two analyses of mathematics textbooks. The first was an evaluation of six important mathematics concepts and skills and seven categories of instructional strategies in twelve middle school series. A small, carefully chosen number of benchmarks related to number, geometry, and algebra were used to profile strengths and weaknesses of the entire series. Six independent pairs of evaluators were trained and reliability of scoring was established. Evaluators compared each textbook activity, which consisted of a lesson or a part of a lesson, to mathematics learning goals from AAAS's Benchmarks for Science Literacy and corresponding standards from the NCTM (1989) and provide ratings from 0 to 3 for each criterion. The goal of the evaluation was to establish whether the series were useful in classrooms where mathematics literacy is the goal for all students.

Evaluators found a few excellent series, although none were among the best-selling series. The review allowed evaluators to examine in-depth content and quality instructional support in these series, although even these did not receive excellent scores across all of the ideas and skills examined. Over all the series, generally content areas of number and geometry as well as instructional areas of student engagement and development of mathematical ideas were satisfactorily addressed. However, conceptual benchmarks were less well met, development of sophistication over a series was generally deficient, and textbooks were generally characterized as inconsistent. Most of the textbooks were particularly lacking in providing a purpose for learning mathematics, taking account of student ideas, and promoting student thinking.

Project 2061 also produced analyses of Algebra 1 textbooks and portions of integrated mathematics series related to algebra. The analysis was similar to their first, including using criteria based on national standards, examining both content and instructional aspects of the materials, and providing profiles for each textbook or series. Trained reviewers examined materials' presentation of function, variables, and operation with equations in seven categories: sense of purpose, building on student ideas, engaging students, developing ideas, promoting student thinking, assessing progress, and enhancing the learning environment.

Reviewers generally found the materials to be adequate, but none were rated highly across all areas. Overall, the materials did an acceptable job in relation to providing a variety of contexts, giving students firsthand experiences with concepts and skills, representing ideas, demonstrating content, and providing appropriate practice. However, they were especially weak in building on students' previous knowledge, overcoming misconceptions, and providing assessments for teachers to adapt instruction based on what students understand. Three of the most popular textbooks were deemed to lack potential for supporting student learning.

The Project 2061 textbook analyses were broader than the analysis provided in this study, as six concepts and skills and seven instructional strategies were examined. The Project 2061 analyses used benchmarks to profile strengths and weaknesses and compared findings to learning goals and standards, whereas in this study, a similar purpose is accomplished through comparison to recommendations from research. Similar to the Project 2061 study, the reliability of textbook coding was carefully established.

Harel (2009), working for the Washington State Board of Education, reviewed four secondary mathematics series: Holt's in-house series, Key Curriculum's *Discovering* series, Glencoe's in-house series, and Glencoe's *Core-Plus* series. He focused on three topics central to secondary mathematics: (a) forms of linear functions and equations, (b) forms of quadratic functions and equations, and (c) parallel lines and the Triangle Sum Theorem. He examined textbooks for the mathematical soundness of content in relation to coherence, completeness, correctness, and strength of foundation for further study. In order to judge the soundness of each series, he used three criteria to judge content: mathematical justification, symbolism and structure, and language. In addition, he judged whether topics were introduced through non-contrived problems and homework problems for whether there were enough non-trivial and holistic problems. He reviewed mainly relevant units from each text, for example, chapters about linear function, but also other portions of the text important for the development of each idea.

Harel (2009) provided mainly qualitative results from his review, but also gave a judgment for the algebra and geometry content in each series on the three criteria as sound, minimally sound, or unsound. Holt received the best review, with the *Core-Plus* and *Discovering* series next. All series were mathematically unsound in algebra in justification and symbolism and structure. Only Holt was minimally sound in geometry in the same two areas. All series except Glencoe were sound in language. In addition, homework problems and topic introduction in each series were categorized as satisfactory or unsatisfactory. No series was satisfactory in provision of practice problems and only the *Core-Plus* and *Discovering* series were satisfactory on introduction of new concepts. Harel admitted his focus was on a small number of topics, but he argued they are central to secondary mathematics and beyond. He concluded, “The four programs failed to convey critical mathematical concepts and ideas that should and can be within reach for high school students” (p. 29).

As in Harel’s (2009) critique, the analysis presented in this study also focuses on a critical concept in secondary mathematics textbooks, although he included three important concepts. However, rather than providing judgments of the textbooks in broad categories, this study provides information about what the textbooks contained and comparisons of these results to recommendations from research.

Olson (2010) conducted a page-by-page analysis of Articulated Learning Trajectories (ALTs) related to the development of algebraic thinking in 12 textbooks from four middle grade series. He had two primary components in his findings: an examination of the structure of and mathematics content within the ALTs; and a description of the ALTs in relation to the disciplinary perspectives of clarity, comprehensiveness, accuracy, depth of mathematical inquiry and reasoning, organization, and balance.

Olson (2010) found that the ALTs had combinations of both ladder-like and branching structures that spanned different portions of the textbooks. His analysis revealed that none of the textbook series clearly defined sequences in a way that delineated them from patterns and established their fundamental connection to functions, and two did not provide an appropriate definition of function. Thus, none of the series comprehensively developed the concepts of sequence and function and their relationship. Olson also found, “serious shortcomings related to the accuracy of the mathematical development of [sequence and function] concepts across all four of the textbook series” (p. 364). Two series, Saxon and Glencoe, were especially limited in their opportunities related to depth of mathematical inquiry and reasoning because they mostly provided sequences with only numerical contexts and far fewer sequences with geometric or realistic contexts. Ultimately, Olson concluded that the different series provided qualitatively different learning opportunities for students; however, each textbook did incorporate the central activity of algebra, that is, the symbolic generalization of patterns, structures, and processes.

As in Olson’s (2010) work, the analysis presented in this study also focuses on a key topic in several current mathematics textbook series. Both studies examine definitions of key concepts; however, Olson utilized ALTs to provide information about the development of his topic, whereas this study illuminates development of the function concept through analysis of changes in features of function examples. In both studies, these results are used to compare textbooks within and between series.

The NRC (2004) did not themselves conduct mathematics textbook analyses, but did evaluate 36 reviews that had been done and provided recommendations for carrying out quality work in this area. Although they did recognize that textbook analyses are “a form of connoisseurial assessment” (p. 72), they determined through their review that a more rigorous model for the planning, execution, and evaluation of analyses is clearly needed. In order to support this, they recommended characteristics of quality textbook analyses in three dimensions. The first is the disciplinary dimension and includes an examination of the mathematical clarity, comprehensiveness, accuracy, depth of inquiry and reasoning, organization, and balance of materials. The second is the learner dimension and includes an analysis of the way the textbook engages students, provides timely support for diverse student needs, and provides means of assessment. The final dimension is related to teachers and resources. It includes a review of support for pedagogy, expectations of designers for professional development, and resources provided for teachers.

This study follows the recommendations of the NRC (2004) by providing a carefully developed model for the execution and evaluation of analyses of a key topic in secondary mathematics textbooks. More specifically, the study is designed to directly address two of the recommended dimensions and provide limited information related to the third.

The study provides information about related to the disciplinary dimension through analysis of both definitions and examples of functions. Although it does not provide rankings of definitions in terms of clarity, comprehensiveness, accuracy, organization, and balance, the qualitative analysis of definitions in aspects identified as important by researchers supplies information about each of these. For example, analysis of how explicitly key aspects such as univalence and arbitrariness are provided in definitions provides a measure of clarity, comprehensiveness, and accuracy of these definitions. Similarly, the specification of distributions of essential features of function examples also supplies information in the disciplinary dimension described by the NRC (2004). For example, the percent of graphic representations provided in each textbook in a series speaks to the organization and balance of materials in the textbooks and series.

The study is also designed to illuminate the learner dimension in the areas described by the NRC (2004). Information about the types of definitions provided, the representations used for examples, the settings of function examples, and inclusion of technology all give insight into ways learners are engaged. Analysis of the types of definitions, representations and families of functions used for examples, and settings for examples provide a sense of how textbooks support the needs of diverse learners. Finally, analysis of function examples in exercises supplies information about means of assessment in the textbook.

Although the study is not designed to primarily provide information about resources for teachers, portions of the analysis can give insight into the dimensions noted by the NRC (2004). Analysis of function definitions and distributions of key features and types of functions used in the textbooks speaks to the support for pedagogy and resources provided. For example, if the collection of function examples in a textbook is especially robust for a key feature, this would indicate strong support and sufficient resources for teachers in this regard.

Conclusion

The review of research included in this chapter provides information supporting the analysis that follows. Numerous studies suggested that teachers and students use textbooks heavily in mathematics K-12 education in the US. Teachers tend to use textbooks for planning their instruction as well as in the classroom, although they usually do not teach the textbook in its entirety. Instead, textbooks generally provide boundaries and rarely do teachers teach material that is not in the textbook.

These findings are cause for concern as many criticisms of US mathematics textbooks have been leveled. In general, most textbooks include a vast array of disconnected and poorly sequenced material creating a presentation of topics with vast breadth and little depth. Analyses of textbooks such as those conducted by Project 2061 have found many textbooks to be of such poor quality that they cannot be expected to support student learning. Additionally, many researchers have criticized textbooks in relation to function, especially for not presenting functions as dynamic, providing a definition that is too abstract, not presenting a sufficient variety of examples, and not capitalizing on the potential to structure content around the function concept.

A review of research regarding mathematical functions demonstrated the centrality of the concept in mathematics and secondary mathematics education. Despite this, as well as numerous calls for the structuring of secondary algebra courses around function, there is no evidence that the majority of textbooks have made significant changes in this direction.

Research on major perspectives on the learning of function, representations of functions, various families of functions, settings for functions, and technology use in relation to functions, and student misconceptions and difficulties related to function informed a conceptual framework that guides the study. This framework includes the language used in function definitions and examples, the presence of functions in textbooks, the core features of function examples, and the ancillary features of function examples.

Finally, recent textbook analyses suggest appropriate methods to use in an analysis. Especially pertinent are the NRC's (2004) recommendations to attend to disciplinary, learner, and teacher dimensions. The focus of the study is on features of textbooks related to the learner and the discipline, although the analysis will also provide limited information related to the teacher dimension.

CHAPTER 3: RESEARCH DESIGN AND METHODOLOGY

Introduction

This chapter outlines the textbook and analysis choices made. For each choice, supporting arguments are provided. These arguments are largely based on the analysis of research regarding the concept of function and the resulting perspective and framework described in the preceding chapter. The analysis of the data is also described in this chapter. The data collected and analysis of those data have been designed to illuminate the research questions guiding the study:

1. What language do textbook series and individual textbooks analyzed in this study use in relation to function? How does this language compare to recommendations from research?
    - a. How are functions defined?
    - b. What is the distribution of examples of functions explicitly portrayed as actions, processes, objects, or parts of larger schema?
    - c. What is the distribution of general and specific examples of functions?
2. What is the presence of function examples in textbook series and individual textbooks analyzed in this study? How does the presence of function examples compare to recommendations from research?
    - a. How prevalent are functions in each textbook?
    - b. What proportion of examples are explicitly identified as functions?
    - c. What is the distribution of function examples placed in textbook lessons and homework exercises?
    - d. How frequently are students asked to generate examples of functions?

e. How frequently are students provided with non-examples of function?

f. What errors in function examples are present?

3. How do textbook series and individual textbooks analyzed in this study present core features related to domain, range, representations, and families of functions? How does this presentation compare to recommendations from research?

    a. How frequently are domain and range of specific functions made explicit? What proportion of domains and ranges are numerical?

    b. What is the distribution of different representations of functions? How frequently do students have opportunities to engage with multiple representations of a function example?

    c. What is the distribution of families of functions?

4. How do textbook series and individual textbooks analyzed in this study present ancillary features of functions related to example settings and recommendations for use of technology? How does this presentation compare to recommendations from research?

    a. What is the distribution of abstract and realistic settings for function examples?

    b. How frequently is technology explicitly recommended for use with function examples?

Selection of Textbook Series

In the business of textbook publishing in the United States, recent trends have been toward the merging and amalgamation of companies. Currently, three major textbook publishers dominate the market: Pearson, McGraw-Hill, and Houghton

Mifflin/Harcourt (Usiskin, 2010). Each publisher carries multiple secondary mathematics series (M. Cooney, 2008). Generally, one series is named after the publisher or the subsection of the publisher dedicated to secondary textbooks. Publishers also offer alternative series that are often produced by a group different than the publisher and generally bear the name of the group.

In order to examine a cross section of current secondary textbooks, a publisher-named series and two alternate textbooks series were selected. Market share data were obtained through Education Market Research (2009) and used to inform the selection of the publisher-named series. No textbooks from alternate series held a large enough market share to appear in the Education Market Research report. Rationale for the selection of each series and brief descriptions of each series are as follows.

Glencoe Mathematics

The McGraw-Hill series, *Glencoe Mathematics*, includes four textbooks: *Algebra 1* (Carter, Cuevas, Day, Malloy, Holliday, et al., 2010), *Geometry* (Carter, Cuevas, Day, Malloy, Cummins, et al., 2010), *Algebra 2* (Carter, Cuevas, Holliday, et al., 2010), and *Advanced Mathematical Concepts: Precalculus with Applications* (*Precalculus*) (Holliday, Cuevas, McClure, Carter, & Marks, 2006). The series holds the greatest market share collectively and for each textbook relative to other textbooks of the same type. The textbooks are organized into chapters made up of a number of lessons. Each lesson includes some exposition, definitions and theorems, a number of examples of problems with solutions provided for students, and a large collection of exercises. In addition to lessons, chapters often include activities specifically designed for students to explore a concept or use a specific technology, such as graphing calculators. Chapters also dedicate a number of pages to summaries of content and practice assessments for students. Selection of this series ensures that the study includes textbooks that many U.S. students are using.

University of Chicago School Mathematics Project (UCSMP)

The UCSMP textbook series was originally developed in the 1980's (Usiskin, 2003) and is currently in its third version. Similar to the Glencoe series, the UCSMP series includes *Algebra* (Brown, et al., 2008), *Geometry* (Benson, et al., 2009), *Advanced Algebra* (Flanders, et al., 2010), and *Precalculus and Discrete Mathematics* (*PCDM*) (Peressini, et al., 2010); however, an additional book, *Functions, Statistics, and Trigonometry* (*FST*) (McConnell, et al., 2010) is also included in the series. Also, these textbooks have an integrated design, drawing on concepts from many areas of mathematics in each textbook (Usiskin, 2007). The textbook authors also claim that the series differs from other secondary mathematics textbooks by including more reading, greater use of technology, greater use of geometric transformations throughout the series, and careful sequencing of mathematical applications (Usiskin, 2007). The UCSMP textbooks do not command a greater market share than any publisher–named series; however, Usiskin reported in 2003 that over 2 million UCSMP textbooks were sold from 1989 to 1998. The series is also designed to follow *Everyday Mathematics*, which is the most popular elementary mathematics textbook series in the US (Stanton, 2011). This series has a presence in the market, and this claim is supported by the fact that UCSMP is now in its third edition. Inclusion of the UCSMP series also offers perspective on a series that has been designed to be intentionally different from competing series (Usiskin, 2007).

The Core-Plus Mathematics Project (CPMP)

The CPMP textbooks are significantly different from the Glencoe and UCSMP series in several regards. The textbooks are not organized by subject; rather, content related to algebra, geometry, statistics, and discrete mathematics are all included in each of four textbooks. Topics are arranged over the series not only to build concepts, but also to provide what the authors considered the most important mathematics at each grade level. For example, to determine content for *Course 1* (Hirsch, et al., 2008a), the authors asked themselves what mathematics students need to learn if they never take another formal mathematics class after *Course 1*.

The textbooks look markedly different from the Glencoe and UCSMP textbooks: lessons mainly consist of a series of questions that lead students to investigate, reflect on, and summarize mathematical concepts. Much more reading is expected of students, and the textbooks tend to include fewer, longer examples and exercises that provide a greater emphasis on mathematical modeling of real world situations. Technology use is integrated into these longer exercises (Fey & Hirsch, 2007). CPMP textbooks do not hold a greater market share than any Glencoe textbooks; however, their continued support for the recent revision suggests that the textbooks do have a market presence.

The focus of this analysis will be on student textbooks. The goal of the study is to better understand the way textbooks portray functions to students, and therefore focuses on what students experience rather than on teachers' experiences or resources. The study is based on two assumptions: first, that students are likely to use their textbooks in class and for homework and therefore be exposed to function concepts and examples in their textbooks, and second, that students have a higher probability of experiencing material in the student textbook than material that is not in their textbook.

Analysis of Textbooks

Example Codes

Every instructional page of each textbook was analyzed for information pertaining to functions. Instructional pages were considered to begin at the introduction to the first unit or chapter, and end with the last page of the last unit or chapter. Tables of content, appendices, glossaries, and other end material were not examined, with the exception of definitions of function in glossaries. Functions were only coded when they appeared with one of the anticipated representations: symbolic, graphic, numeric, function machine, mapping, verbal, or physical. For example, a rigid transformation of a geometric figure would not be analyzed as a function unless the transformation was explicitly noted as a function or represented using one of the representations noted above. Also note that, while an equation such as $y = 2x + 1$ may not be explicitly identified as a function, it would be coded as a function because it uses a common representation for functions.

Many pages contained more than one function. For example, the page with exercises for students in Figure 3.1 had eight examples of functions on it. Other pages did not have any examples of functions on them, such as many of the pages in geometry textbooks. Some functions were represented in multiple ways and even on multiple pages. For example, in Figures 3.2 and 3.3, CPMP *Course 2* (Hirsch, et al., 2008b) provided an example of a function on one page, and then several pages later, asked students to reconsider the earlier example. This example was considered as one function example for the analysis.

b. When a club was planning its Halloween party at the Fun House, the planners figured the cost per person using $C = \frac{225}{n}$, where n is the number of club members who would attend.

c. The number of tickets sold to a charity basketball game is a function of the price charged with rule $N = 4{,}000 - 50p$.

d. When a doctor or nurse gives an injection of medicine like penicillin, the amount of active medicine t hours later can be estimated by a function like $M = 300(0.6^t)$.

e. The circumference of a circle is related to the radius by the formula $C = 2\pi r$.

f. When a football punt leaves the kicker's foot, its height above the ground at any time in its flight is given by a function like $h = -16t^2 + 80t + 4$.

g. The average speed for the Daytona 500 race is a function of the time it takes to complete the race with rule $s = \frac{500}{t}$.

6 Using your data plots from Problems 2 and 3, experiment with function graphs to find function rules that seem to be good models for the relationships between:

a. roll time T and ramp length L for each platform height you tested in Problem 2.

b. roll time T and platform height H for each ramp length you tested in Problem 3.

Basic Variation Patterns The situations in this investigation involved a variety of functions relating dependent and independent variables. Several examples involved special patterns called *direct* and *inverse variation*.

6 UNIT 1 • Functions, Equations, and Systems

Figure 3.1: One page containing a eight function examples (Hirsch, et al., 2008, p. 6).

Families of Variation Patterns The data patterns and graphs that show how roll time depends on ramp length and platform height may remind you of other relationships between variables that you have seen in prior mathematical studies.

5. Decide which of the following functions have table and graph patterns that:
   - are similar to the (*ramp length, time*) relationship.
   - are similar to the (*platform height, time*) relationship.
   - are different from those relationships.

   Be prepared to explain your decisions.

   a. The sales tax on store purchases in Michigan is a function of the purchase price and can be calculated with the formula $T = 0.06p$.

LESSON 1 • Direct and Inverse Variation **5**

Figure 3.2: First page introducing function example (Hirsch, et al., 2008, p. 5).

10. Identify the inverse variation relationships in Problem 5. For each:
   a. Explain how the value of the dependent variable changes as the value of the independent variable increases steadily.
   b. Express the relationship between the variables in two different but equivalent symbolic forms.
   c. Describe the relationship of the variables involved by completing a sentence like this: "The variable ________ is inversely proportional to ________, with constant of proportionality ________."

11. Examine the tables below, each of which describes a relation between x and y.

Table I

x	25	50	60	100	150
y	8	4	3.33	2	1.33

Table II

x	10	15	25	40	100
y	6	9	15	24	60

Table III

x	5	15	30	64	80
y	9.6	3.2	1.6	0.75	0.6

   a. Which relations involve direct variation? What is the constant of proportionality in each case?
   b. Which relations involve inverse variation? What is the constant of proportionality in each case?

8 **UNIT 1** • Functions, Equations, and Systems

Figure 3.3: Later page referring to earlier function example (Hirsch, et al., 2008, p. 8).

Codes for each example (see Appendix A) were recorded in one row of a spreadsheet. First, whether the example represented a function or was a non-example of a function was recorded. Functions were never coded as non-examples. Non-functions were only recorded as non-examples when either they were identified by the textbook as non-examples of functions or students were asked to identify whether they were functions and they were not actually functions.

Next, for each function, codes related to presence of functions in the textbook were recorded. The page number, a brief description for identification purposes, the size of the example, whether the example was identified as a function, whether the function occurred in the body of a lesson or as a part of student exercises, and any errors were recorded. An example was coded as identified as a function either if the text included the word "function" with the example or if the example was part of a group of examples that were identified as functions. The example was coded as students deciding if it was a function when the textbook provided directions for students to indicate whether the example was a function. Next, in order for examples to be coded as part of the lesson or exercises, for each textbook, a description of the differences between the body of the lesson and the student exercises was generated to support reliability of the coding and to inform an understanding of the differences between these two sections for each textbook (see Appendix B). Finally, descriptions of any errors in the example were recorded.

The next codes shown in Appendix A were in relation to the language used in the example. Whether a function was identified as an action, process, object, or part of a schema was coded. A function only received one of these codes when the textbook included an explicit description of the function in one of these categories. For example, in

Glencoe *Precalculus* (Holliday, et al., 2006), the statement was made, “To solve the profit problem, you can subtract the cost function $c(x)$ from the revenue function $r(x)$.” This describes carrying out an operation on two mathematical objects, functions $c(x)$ and $r(x)$, so each of these functions received an “object” code.

In addition, functions were coded as general or specific. A function was coded general when no specific element in its domain and the corresponding element in the range could be identified. If any corresponding pair could be identified, it was coded as specific. An example of a specific function is one for which an equation is provided. In contrast, examples of general functions would not have equations provided, or, if an equation is provided, parameters were not specified. For example, $y = mx + b$ was coded as a general linear function.

The next codes were related to the core features of function presented in the textbook. First, each function’s representation was coded. Codes included *Symbolic*, *Graphic*, *Numeric*, *Function Machine*, *Mapping Diagram*, *Verbal Description*, *Physical*, *Multiple*, and *Other*. In addition, if a function was coded as *Symbolic*, it was also coded as *y = Expression*, *Implicit y and x*, *f(x) = Expression*, *Recursive*, *Equation With Other Variables*, or *Other*. If a function was coded as *Graphic*, it was also coded as *Continuous*, *Smooth*, *Scatterplot*, or *Other*. If a function was coded as *Numeric*, it was also coded as *Table*, *Ordered Pair*, *f(x) Notation*, or *Other*. All codes but *Multiple* and *Other* were defined in the first chapter. Note that many functions received multiple representation codes. The *Multiple* code was applied when more than one representation was involved, either being provided to, or requested of, the student or a combination of both. *Other* was reserved for function representations that did not match any code. If the *Other* code was used, a description of the representation was included in the cell. These descriptions were then available for closer analysis of examples coded as *Other*.

Next, as part of core features, each function's family was coded. Codes included *Polynomial*, *Periodic*, *Exponential*, *Logarithmic*, *Rational*, *Absolute Value*, *Piecewise*, and *Other*. In addition, if a function was coded as *Polynomial*, it was coded as *Constant*, *Linear*, *Quadratic*, *Cubic*, or *Other*. If a function was coded as *Periodic*, it was coded as *Trigonometric* or *Other*. All codes but *Other* were defined in the first chapter. In addition, *Other* was reserved for function families that did not match any code. If the *Other* code was used, a description of the representation was included in the cell. These descriptions were then available for closer analysis of examples coded as *Other*.

Finally, as part of the core features, each function was coded regarding its domain and range. If the domain and range of the function were not specified in the text for the function, no code was entered. If it was specified, whether the domain and range was numerical or not numerical was recorded. When both contained only numbers, the function received the numerical code. An example of a function where the range was identified and was not numerical from page 248 of Glencoe *Algebra 2* (Carter, Cuevas, Holliday, et al., 2010) was a diagram indicating that the domain element 1 mapped to the range element D, the domain element 2 mapped to the range element B, and the domain elements 3 and 4 mapped to the range element C.

Example 3

A rule for the function graphed at the right is $y = 2^x - 4$. Find the domain and range of the function.

Solution The domain is the set of x-values for which $2^x - 4$ is defined, which is the set $\mathbb{R}$ of all real numbers. From the graph, the range appears to be the set of all real numbers greater than –4, which can be written as $\{y \mid y > -4\}$.

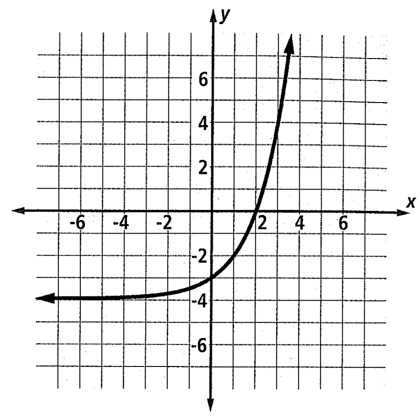


Figure 3.4: An abstract setting (McConnell, et al., 2010p. 83).

The final area coded was the ancillary features of the functions. First, whether the function was set in a realistic or abstract context, as defined in the first chapter, was coded. Although contexts in mathematics textbooks may never be entirely realistic, researchers have maintained that students need to work with functions in contexts that have connections to situations outside of mathematics (Oehrtman, et al., 2008). Even though settings coded as "realistic" did not include all the detail and idiosyncrasies of real situations, they were distinguished from purely mathematical contexts for the purposes of coding. For example, the example in Figure 3.4 was coded as abstract, whereas the example in Figure 3.5 was coded as realistic.

Example 2

$2500 is invested in an account with a 5.3% annual yield.

a. **What is the yearly growth factor?**

b. **What is the equation for the balance A after t years?**

c. **What is the balance after 7 years?**

d. **What value is associated with $t = -2$? What does $t = -2$ mean?**

Solution

a. $r = 0.053$. So the yearly growth factor is $(1 + r) =$ ___?___.

b. The initial value is ___?___ and the growth factor is ___?___, so $A =$ ___?___.

c. $A = 2500($___?___$)^7 =$ ___?___.

d. $A = 2500($___?___$)^{-2} =$ ___?___. If the account had been started 2 years ago, an investment of ___?___ would have produced $2500 this year.

Figure 3.5: A realistic setting (McConnell, et al., 2010p. 104).

Finally, whether technology was recommended for use with the function was coded. Certainly a variety of ways to use technology are possible for nearly any specific function example; therefore, this code was used to capture instances when the textbook explicitly recommended that students use a technological tool or demonstrated the use of a technological tool in conjunction with the function. This occurred both for individual functions and as a blanket recommendation for a group of functions, such as in Glencoe *Precalculus* (Holliday, et al., 2006), where a Graphing Calculator Exploration included the initial directions, “Use a graphing calculator to explore the sum of two functions” (p. 13). In contrast, an example of a recommendation for an individual function from the same textbook was when students were directed in part c. of exercise 31 in Lesson 1-7, “Use a graphing calculator to find an equation for the regression line for the data” (p. 51).

Each code for each function was recorded as provided for the student, requested from the student, or a combination of these, except for page number, description, size, placement in lessons or exercises, and error. Some functions had all aspects provided for a student, such as examples provided for students to illustrate a point, but without any explicit request for students to do anything beyond consider the example. Conversely, many functions included both provisions of certain aspects for students and requests for students to provide certain aspects. For example, many examples included symbolic representations of a function and requested that students provide a graphic representation of the function.

Definitions

In addition to analyzing examples of functions in each textbook, definitions of functions were also examined. Definitions were noted and recorded during the page-by-page analysis. In addition, the index was used to check that all definitions of function provided in the textbook were found. Finally, the textbook glossaries were checked for definitions of function.

Reliability

Reliability of coding was established through two procedures. First, two additional coders were trained and coded a section of one textbook from each series in order to establish inter-coder reliability. The two coders read this chapter and the first chapter in order to develop an understanding of the purpose of the coding process. In addition, a concise description of each code and guide for the coding process was provided for coders to refer to as they work (see Appendix A). Sections from each book were used for practice coding, and coders had opportunities to ask questions.

Once the practice coding was completed satisfactorily, each coder independently coded one section from each textbook series, purposefully selected to include examples of functions, for a total of three sections. Codes from these sections were then compared to the author's codes for percent agreement. First, agreement on where examples and non-examples of functions occurred in the section was examined. Disagreements existed when the author included an example and the coder did not and vice versa. Percent of agreement out of the total number of examples coded by the author were examined. Second, agreement on each of the 55 codes described above was examined by comparing columns of data, with any examples where coders disagreed removed. Where pair-wise agreements between codes were low (less than 80%), then discussions were held regarding the examples and the coding scheme. Once discussions lead to agreement on what the codes should have been, the coding process was repeated until satisfactory levels of agreement on all codes were achieved.

The section coded in the Glencoe series was Lesson 9-7 from *Algebra 1* (Carter, Cuevas, Day, Malloy, Holliday, et al., 2010), entitled "Growth and Decay." It included 44 examples of functions, and agreement on coding the number of examples was 93.2%. Of the 55 codes related to each example, 48 had over 90% agreement. Only three codes were below 85%: the *Representation: Numeric: Other* code (81.8%), the *Representation: Verbal Description* code (75%), and the *Representation: Multiple* code (75%). Because the *Multiple* code is based on the rest of the representation codes, other disagreements caused this code to be lower. The *Verbal Description* code was lower due to one block of six of the same type of homework problems on which the coder and author disagreed. In essence, one disagreement caused these six errors. Without these, the *Verbal Description* code would have been at 88.6% agreement. See Table 3.1 for percent agreement on all codes.

The section coded in the UCSMP series was Lesson 7-1 from *PCDM* (Peressini, et al., 2010), entitled "Power Functions." It included 30 examples of functions, and agreement on coding the number of examples was 96.7%. Of the 55 codes related to each example, 51 had over 90% agreement. No codes were below 85%. See Table 3.1 for percent agreement on all codes.

The section coded in the CPMP series was Unit 1, Lesson 3, Investigation 1 from *Course 2* (Hirsch, et al., 2008b), entitled "Systems of Linear Equations." It included 41 examples of functions, and agreement on coding the number of examples was 92.7%. Of the 55 codes related to each example, 48 had over 90% agreement. Three codes were slightly below 85%: the *Size:3 rows – 1/4 pg* code (84.2%), the *Representation: Numeric: Other* code (84.2%), and the *Representation: Verbal* code (84.2%). See Table 3.1 for percent agreement on all codes.

In addition, the author recoded the same three sections at least one month after the initial coding as a check of test-retest reliability. In the Glencoe section, agreement on coding the number of examples was 95.5%. Only 3 of 55 codes were below 90%: the *Representation: Symbolic: Equation with Other Variables* code (86.4%), the *Representation: Verbal Description* code (88.6%), and the *Representation: Multiple* code (88.6%). The *Multiple* code was lower because it was based on the other representation codes. See Table 3.2 for percent agreement on all codes.

Table 3.1

Percent agreement between coders

Code	Percent Agreement		
	Glencoe	UCSMP	CPMP
Non-example	100.0	100.0	100.0
Size: <1 row	100.0	100.0	97.4
Size: 1-2 rows	95.5	93.1	100.0
Size: 3 rows – 1/4 pg	95.5	93.1	84.2
Size: 1/4 – 1/2 pg	100.0	96.6	86.8
Size: 1/2 pg – 1 pg	100.0	96.6	100.0
Size: +1 pg	100.0	100.0	100.0
Function Label: Labeled as Function	93.2	100.0	86.8
Function Label: Student Decides if Function	100.0	100.0	100.0
Placement: Lesson	100.0	100.0	100.0
Placement: Exercises	95.5	96.6	100.0
APOS: Action	100.0	100.0	100.0
APOS: Process	100.0	100.0	100.0
APOS: Object	100.0	100.0	100.0
APOS: Schema	100.0	100.0	100.0
Generality: Specific	95.5	96.6	100.0
Generality: General	95.5	96.6	100.0
Representation: Symbolic: y = expression	88.6	96.6	94.7
Representation: Symbolic: Implicit y and x	100.0	100.0	100.0
Representation: Symbolic: $f(x)$ = expression	100.0	100.0	100.0
Representation: Symbolic: Recursive	100.0	100.0	100.0
Representation: Symbolic: Equation With Other Variables	86.4	86.2	97.4
Representation: Symbolic: Other	97.7	86.2	100.0
Representation: Graphic: Continuous	86.4	96.6	89.5
Representation: Graphic: Smooth	86.4	96.6	94.7
Representation: Graphic: Scatterplot	93.2	100.0	94.7
Representation: Graphic: Other	100.0	100.0	100.0
Representation: Numeric: Table	97.7	100.0	100.0
Representation: Numeric: Ordered Pair	100.0	100.0	89.5
Representation: Numeric: $f(x)$ Notation	100.0	100.0	100.0
Representation: Numeric: Other	81.8	96.6	84.2
Representation: Function Machine	100.0	100.0	100.0
Representation: Mapping Diagram	100.0	100.0	100.0
Representation: Verbal Description	75.0	89.7	84.2
Representation: Physical	100.0	100.0	100.0
Representation: Multiple	75.0	89.7	92.1
Representation: Other	100.0	100.0	100.0
Family: Polynomial: Constant	100.0	100.0	100.0
Family: Polynomial: Linear	100.0	100.0	100.0
Family: Polynomial: Quadratic	100.0	96.6	100.0
Family: Polynomial: Cubic	100.0	100.0	100.0
Family: Polynomial: Other	100.0	93.1	100.0
Family: Periodic: Trigonometric	100.0	100.0	100.0
Family: Periodic: Other	100.0	100.0	100.0
Family: Exponential	100.0	96.6	100.0
Family: Logarithmic	100.0	100.0	100.0
Family: Rational	100.0	100.0	100.0
Family: Absolute Value	100.0	100.0	100.0
Family: Piecewise	100.0	100.0	100.0
Family: Other	100.0	100.0	100.0
Specified Domain and Range: Numerical	100.0	93.1	100.0
Specified Domain and Range: Not Numerical	100.0	100.0	100.0
Setting: Abstract	97.7	96.6	94.7
Setting: Realistic	97.7	96.6	97.4
Technology Recommended	93.2	100.0	100.0

Table 3.2

Percent agreement for test-retest reliability

	Percent Agreement		
Code	Glencoe	UCSMP	CPMP
Non-example	100.0	100.0	100.0
Size: <1 row	97.7	100.0	100.0
Size: 1-2 rows	100.0	100.0	100.0
Size: 3 rows – 1/4 pg	95.5	96.6	94.7
Size: 1/4 – 1/2 pg	97.7	96.6	94.7
Size: 1/2 pg – 1 pg	100.0	100.0	100.0
Size: +1 pg	100.0	100.0	100.0
Function Label: Labeled as Function	100.0	96.6	100.0
Function Label: Student Decides if Function	100.0	100.0	100.0
Placement: Lesson	100.0	100.0	100.0
Placement: Exercises	95.5	100.0	100.0
APOS: Action	100.0	100.0	100.0
APOS: Process	100.0	100.0	100.0
APOS: Object	100.0	100.0	100.0
APOS: Schema	100.0	100.0	100.0
Generality: Specific	97.7	100.0	100.0
Generality: General	97.7	100.0	100.0
Representation: Symbolic: y = expression	97.7	96.6	94.7
Representation: Symbolic: Implicit y and x	100.0	100.0	94.7
Representation: Symbolic: $f(x)$ = expression	100.0	100.0	100.0
Representation: Symbolic: Recursive	100.0	100.0	100.0
Representation: Symbolic: Equation With Other Variables	86.4	100.0	100.0
Representation: Symbolic: Other	90.9	100.0	100.0
Representation: Graphic: Continuous	100.0	100.0	94.7
Representation: Graphic: Smooth	100.0	100.0	94.7
Representation: Graphic: Scatterplot	100.0	100.0	100.0
Representation: Graphic: Other	100.0	100.0	100.0
Representation: Numeric: Table	100.0	100.0	100.0
Representation: Numeric: Ordered Pair	100.0	96.6	100.0
Representation: Numeric: $f(x)$ Notation	100.0	100.0	100.0
Representation: Numeric: Other	95.5	100.0	89.5
Representation: Function Machine	100.0	100.0	100.0
Representation: Mapping Diagram	100.0	100.0	100.0
Representation: Verbal Description	88.6	86.2	84.2
Representation: Physical	100.0	100.0	100.0
Representation: Multiple	88.6	93.1	97.4
Representation: Other	100.0	100.0	100.0
Family: Polynomial: Constant	100.0	100.0	100.0
Family: Polynomial: Linear	100.0	100.0	100.0
Family: Polynomial: Quadratic	100.0	100.0	100.0
Family: Polynomial: Cubic	100.0	100.0	100.0
Family: Polynomial: Other	100.0	100.0	100.0
Family: Periodic: Trigonometric	100.0	100.0	100.0
Family: Periodic: Other	100.0	100.0	100.0
Family: Exponential	93.2	100.0	100.0
Family: Logarithmic	100.0	100.0	100.0
Family: Rational	100.0	100.0	100.0
Family: Absolute Value	100.0	100.0	100.0
Family: Piecewise	100.0	100.0	100.0
Family: Other	93.2	100.0	100.0
Specified Domain and Range: Numerical	100.0	89.7	100.0
Specified Domain and Range: Not Numerical	100.0	100.0	100.0
Setting: Abstract	97.7	100.0	94.7
Setting: Realistic	97.7	100.0	94.7
Technology Recommended	100.0	100.0	100.0

In the UCSMP section, agreement on coding the number of examples was 100%. Only two codes were below 90% agreement: the *Representation: Verbal Description* code (86.2%) and the *Domain and Range: Numerical* code (89.7%). See Table 3.2 for percent agreement on all codes.

In the CPMP section, agreement on coding the number of examples was 100%. Only one code was below 90% agreement: the *Representation: Numeric: Other* code (89.5%) See Table 3.2 for percent agreement on all codes.

Analysis of Data

Analysis of the data was designed to address the research questions guiding the study listed at the beginning of this chapter.

Language Used in Relation to Functions

Definitions

In order to address the perspective of function presented by textbooks and series, definitions of function were examined qualitatively, in the following dimensions drawn from findings and recommendations in the literature as noted in the previous chapter. First, definitions were classified by type as variable, relationship of correlation, or set. Second, whether wording did not restrict or suggest the type of functions possible was analyzed. Third, the univalence feature of definitions, indicating that each element of the domain corresponds to exactly one element of the range, was examined. Fourth, how explicitly definitions included concepts of domain and range was analyzed. Fifth, definitions were also examined for how clearly they conveyed the notion that a function encapsulates covariation between two variables. Finally, the universal quantification facet of definitions, concerning the clarity of expression that each and every domain element must be associated with a range element, was analyzed. Each definition was examined for whether these features were explicitly present, suggested, not present, or contradicted.

In addition, consistency and development of definitions over the course of each textbook and series was examined. Changes in type and features were noted and compared to research claims about students' conceptual development of function. Comparisons across series were conducted by examining differences between definitions at each grade level. Finally, a broad qualitative description of current secondary mathematics textbook definitions of function was developed.

APOS and Generality

With respect to language used in function examples, areas identified for analysis included the generality of function examples and the presentation of functions as actions, processes, objects, or part of larger schema. The general approach used to analyze the distribution of these features was to examine the relative frequency of codes pertaining to each feature. In order to obtain these frequencies, the number of examples of function in a given textbook was tracked by the number of rows in the spreadsheet associated with the textbook, and the number of examples with each code was then divided by the total. For example, the total number of functions coded as general was divided by the total number of examples in order to obtain the relative frequency of general functions in the textbook. The relative frequencies were then used in three comparisons: (a) between books at the same grade level, (b) within all books in one series, and (c) against recommendations from research.

Presence of Functions in Textbooks

With respect to presence of functions in textbooks, areas identified for analysis included the prevalence of function examples in the textbook, whether examples were identified as functions, placement of examples in lessons or exercises, whether students asked to generate examples, and errors related to function examples. The same approach was used to analyze the distribution of most of these features, namely, the relative frequency of codes pertaining to each feature was found by dividing the number of examples with each code by the total. The relative frequencies were again used in three comparisons: (a) between books at the same grade level, (b) within all books in one series, and (c) against recommendations from research.

A slightly different approach was used to analyze the prevalence of function examples in a textbook. First, the total number of function examples in each textbook was calculated and compared. These totals were then divided by the number of instructional pages in the textbook in order to obtain the number of functions per page in the textbook. These rates were then examined over the course of a series and between textbooks at each grade level in the three series and thus provided a relative indication of how prevalent function examples were in the textbooks.

Descriptions of errors were collected and examined for common types. For example, many of the errors that occurred were instances of mismatches between two representations of the function. Types of errors were added through analysis of the descriptions so that each error could be categorized by type.

Core Features of Function

Within core features of function examples, areas identified for analysis included representations of functions, function families, and identification of domain and range. Each of these areas included a number of codes (see Appendix A). The same approach was used to analyze the distribution of these features, namely, the relative frequency of codes pertaining to each feature was found by dividing the number of examples with each code by the total. Codes for each core feature were also analyzed for frequency of whether examples were provided for students or requested of students. For example, the percent of symbolic representations provided for students were compared to the percent requested of students. The relative frequencies were again used in three comparisons: (a) between books at the same grade level, (b) within all books in one series, and (c) against recommendations from research.

Ancillary Features of Function

Within ancillary features of function examples, areas identified for analysis included whether settings for examples were abstract or realistic and whether technology was recommended for use with the examples. Again, the same approach was used to analyze the distribution of these features, namely, the relative frequency of codes pertaining to each feature was found by dividing the number of examples with each code by the total. The relative frequencies were again used in three comparisons: (a) between books at the same grade level, (b) within all books in one series, and (c) against recommendations from research.

Archetypes

After relative distributions of the various features of function examples were examined, archetypes of functions for each textbook were constructed based on this information. These examples were designed to embody the most common features of functions included in a given textbook and thus provide an example of the kinds of functions students would commonly encounter through use of the textbook. Descriptions were also generated indicating salient features of the archetypical examples and how they demonstrate findings.

Limitations

This study has several limitations. The first regards the textbooks used for analysis. Not every mathematics textbook was analyzed in the study. Popular textbooks and a few alternatives were selected; however, there are other textbooks in use with different collections of examples and definitions of function. In addition, in less than a decade most of these textbooks will probably not be in use in many classrooms. Selecting the most recent editions of textbooks and the more popular series for analysis was an attempt to reduce this limitation.

Another limitation of the study was related to the use of the textbooks by teachers and students. Analysis was conducted with student textbooks, but no study of how these materials are used in classrooms or by students was conducted. Thus, the findings regarding the contents of a textbook can only be understood as a potential and indirect influence on student learning. For example, results might indicate that a given textbook provides a wide variety of representations of function. However, if the teacher only focuses on one type of representation, then students' experiences may not be closely aligned to the textbook content.

A third limitation is in relation to the extent of the analysis. Only the function concept was analyzed, despite a variety of other topics in the textbooks. Definitions and examples of functions were only analyzed in regard to the specific features noted above. Certainly textbook treatments of functions include other features that were not analyzed or compared. The centrality of the function concept to the secondary curriculum addresses this limitation. Also, the literature review supports a claim that the features included in this analysis are features that researchers have found to be important in students' learning of the function concept.

Conclusion

The most current popular and alternative textbook series were selected for analysis from each of the three major US textbook publishers. In these textbooks, every page of all chapters and sections were examined for examples of function. Features of these examples related to textbooks' perspective of function, core features of function, and ancillary features of function were coded. In addition, definitions of functions were identified and examined. Analysis of the features of function examples consisted of examination and comparisons of percents of occurrence of the features coded, and archetypical examples of functions were created from these analyses.

The sample selection, data collection, and data analysis described in this chapter were designed to address the research questions guiding the study. Descriptions of the methodology used are intended to support the validity and reliability of the findings of the study. The study was designed to provide insight into textbooks currently impacting students' secondary mathematics education in the United States.

CHAPTER 4: FINDINGS

Introduction

The following are results from the research design and analyses described in the previous chapter. The results are organized according to the research questions they address. First, results are presented that provide insight into the language used in relation to functions in the textbook series and individual textbooks included in the study. This includes analysis of definitions of functions; portrayal of functions as actions, processes, objects, or parts of schema; and the distribution of specific and general functions. Second, results are presented that provide insight into the presence of functions in textbooks and series, including analysis of the prevalence of functions, labeling of functions, presence of functions in lessons or exercises, requests for students to provide functions, provision of non-examples, and errors in function examples. Third, results are presented that provide insight into the core features of function examples. This includes analysis of domains and ranges provided with function examples, distribution of function representations, and distribution of function families. Fourth, results are presented that provide insight into the ancillary features of functions. This includes analysis of settings for examples and recommendations for technology use with function examples. Finally, as a way to summarize and illustrate these results, archetypes of function examples are presented for each textbook in each series, based on the most commonly occurring features.

Language Used in Relation to Function

Function Definitions

Definitions of function from instructional pages and glossaries from each textbook were recorded and analyzed according to a number of key features. Each definition was categorized as one of the following types: (a) variable, (b) relationship of correspondence, or (c) set of ordered pairs. A variable type of definition indicates that the function is actually the output variable, often designated as y. In contrast, a set definition identifies a function as a set of ordered pairs, including values of both the input variable and output variable, and relating each specific input with an output. The relationship of correspondence type of definition does not describe the function as the ordered pairs themselves, but instead as the relationship that generates the ordered pairs.

Definitions were examined for their inclusion of the concept of univalence, which means that a given input value of a function is mapped to only one output value. For set definitions, univalence could also be expressed through the idea that the set of ordered pairs does not include different ordered pairs (x, y) that have the same values for x.

Also key to the concept of function is the arbitrariness of the definition. Except for the univalence requirement, an appropriate definition cannot restrict functions to any particular type or relationship or form of expression. When considering functions as sets of ordered pairs, arbitrariness means that a function can be represented by any set of ordered pairs, as long as the univalence requirement is met. Thus using a particular type of representation such as an equation to define functions is problematic, because not all ordered pairs can be represented by equations. For example, when the output of a function is nonnumeric, such a function cannot be represented by a numeric equation.

Another key concept examined in each definition was universal quantification. This is the idea that a function must assign an output value for every element in its domain. Defining function as a set of ordered pairs implicitly suggests this concept, because an ordered pair (x, y) always connects a y value to an x value.

The concepts of domain and range are key to the concept of function, and all definitions were examined for these ideas. The domain of a function is the set of elements that the function can take as input. When considering function as a set of ordered pairs (x, y), this is all possible values of x. The range is all possible outputs of a function, or, as a set of ordered pairs (x, y), all possible values of y.

Finally, definitions were also examined for how clearly they indicated the covariance between input values and output values that functions create. Although this concept is not required to be explicit in a rigorous definition of function, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) argued that it is vital for students to understand this feature of functions and that students often do not grasp it because of how functions are defined and presented to them. Because Carlson and her colleagues emphasize the importance of students grasping covariation, definitions were analyzed for this concept. Covariance is indicated by explanation that as input values of a function change, this causes changes in the output values of the function.

The following analysis provides the function definitions from each series in a table. Definitions from each series are analyzed by textbook, and then a summary of the series is presented. Finally, a comparison between series is made.

Definitions in the Glencoe Series

Definitions for functions in the Glencoe series are provided in Table 4.1. Glencoe *Algebra 1* (Carter, Cuevas, Day, Malloy, Holliday, et al., 2010) provided two definitions of functions in the instructional pages and one in the glossary. All three defined function as a relationship of correlation, clearly identified univalence, and provided universal quantification. The second definition and the glossary definition also explicitly indicated domain and range, while the first definition did not refer to domain and range. All three definitions suggested arbitrariness, but did not provide any indication of the covariance that a function creates.

Table 4.1

Definitions of function in the Glencoe series

Textbook	Page	Definition
Algebra 1	45	"A **function** is a relationship between input and output. In a function, there is exactly one output for each input."
Algebra 1	45	"A function is a relation in which each element of the domain is paired with *exactly* one element of the range."
Algebra 1	R103	"A relation in which each element of the domain is paired with exactly one element of the range."
Algebra 2	P4	"A **function** is a [set of ordered pairs] in which each element of the domain is paired with *exactly one* element of the range."
Algebra 2	61	"Recall that a function is a relation in which each element of the domain is paired with exactly one element in the range."
Algebra 2	R137	"A relation in which each element of the domain is paired with exactly one element in the range."
Precalculus	6	"A function is a relation in which each element of the domain is paired with exactly one element in the range."
Precalculus	7	"An alternate definition of a function is a set of ordered pairs in which no two pairs have the same first element."
Precalculus	A74	"[A] relation in which each element of the domain is paired with exactly one element in the range"

The Glencoe *Geometry* (Carter, Cuevas, Day, Malloy, Cummins, et al., 2010) textbook did not provide any definitions of function. However, *Algebra 2* (Carter, Cuevas, Holliday, et al., 2010) included two definitions in the instructional pages and one in the glossary. All three defined function as a set of ordered pairs or as a relation, which was defined as a set of ordered pairs. They also clearly identified univalence, indicated domain and range, and provided universal quantification. All three definitions suggested arbitrariness, but did not provide any indication of the covariance that a function generates.

Glencoe *Advanced Mathematical Concepts: Precalculus with Applications* (*Precalculus*) (Holliday, et al., 2006) included two definitions in the instructional pages and one in the glossary. The first defined function as a relation of correspondence, while the second defined function as a set of ordered pairs. The glossary definition called function a relation, which was defined elsewhere in the glossary as a set of ordered pairs. All three definitions clearly identified univalence and provided universal quantification. The first definition and the glossary definition also explicitly indicated domain and range, while the second definition did not refer to domain and range. All three definitions suggested arbitrariness, but did not provide any indication of the covariance that a function generates.

An examination across the Glencoe series reveals that functions were initially defined as relationships of correspondence, but by *Algebra 2*, statements predominantly defined functions as sets of ordered pairs, with one exception in *Precalculus*. All definitions across the series consistently indicated univalence and universal quantification clearly. Indication of domain and range was inconsistent across definitions in the series, with only the *Algebra 2* textbook clearly including these in all definitions. All definitions did suggest the arbitrary nature of function, but no definitions suggested the covariance that functions create. However, in the *Precalculus* textbook, this covariance was suggested in a section that described some conceptions of functions proposed by a number of mathematicians, including Descartes, Leibniz, Bernoulli, Euler, Lagrange, Dirichlet, and Cantor. These descriptions suggested these mathematicians were wrestling with ways to capture the concept of covariance in a formal mathematical definition.

Definitions in the UCSMP Series

Definitions for functions in The University of Chicago School Mathematics Project (UCSMP) series are provided in Table 4.2. UCSMP *Algebra* (Brown, et al., 2008) provided one definition of function in the instructional pages and one in the glossary. The first defined function as a relationship of correlation, but in the glossary, function was defined as a set of ordered pairs. Both clearly identified univalence and provided universal quantification. Both definitions also suggested arbitrariness, but did not provide any indication of domain and range nor the covariance that a function creates.

The UCSMP *Geometry* (Benson, et al., 2009) textbook did not provide any definitions of function. However, *Advanced Algebra* (Flanders, et al., 2010) included three definitions in the instructional pages and one in the glossary. The first definition indicated a relationship of correspondence, whereas the rest defined function as a set of ordered pairs or as a relation, which was defined as a set of ordered pairs. All but the first clearly identified univalence. The second and glossary definitions provided universal quantification, but the third definition only implied universal quantification. The first definition did not provide any indication of univalence or universal quantification, and none of the four suggested the covariance generated by a function nor indicated domain and range. All did, however, suggest arbitrariness.

Table 4.2

Definitions of function in the UCSMP series

Textbook	Page	Definition
Algebra	426	“A **function** is a correspondence in which each value of the first variable (the **input**) corresponds to *exactly one* value of the second variable (the **output**), which is called a **value of the function**.”
Algebra	S91	“A set of ordered pairs in which each first coordinate corresponds to *exactly one* second coordinate.”
Advanced Algebra	14	“Functions are the mathematical models of relationships between two variables.”
Advanced Algebra	15	“Definition of Function: A **function** is a set of ordered pairs (x,y) in which each first component x of the pair is paired with exactly one second component y.”
Advanced Algebra	15	“Definition of Function (reworded): A **function** is a relation in which no two ordered pairs have the same first component x.”
Advanced Algebra	S91	“A set of ordered pairs (x,y) in which each first component x of the pair is paired with exactly one second component y. A relation in which no two ordered pairs have the same first component x.”
FST	80	“When each value of the independent variable determines exactly one value of the dependent variable, the relation is called a *function*.”
FST	80	“A **function** is a set of ordered pairs (x,y) in which each first component (x) is paired with exactly one second component (y).”
FST	81	“A **function** is a correspondence between two sets A and B in which each element of A corresponds to exactly one element of B.”
FST	142	“Any set of ordered pairs is a relation. **Functions** are those relations in which no two ordered pairs have the same first component. A function can also be viewed as a correspondence between two sets A and B, which relates each element of A (the function's **domain**) to exactly one element of B.”
FST	199	“Recall that a function can be considered as a set of ordered pairs in which each first element is paired with exactly one second element.”
FST	S77	“A set of ordered pairs (x,y) in which each value of x is paired with exactly one value of y. A correspondence between two sets A and B in which each element of A corresponds to exactly one element of B.”
PCDM	6	“A function is a relation in which each element of the domain is paired with exactly one element in the range.”
PCDM	7	“An alternate definition of a function is a set of ordered pairs in which no two pairs have the same first element.”
PCDM	A74	“[A] relation in which each element of the domain is paired with exactly one element in the range”

The UCSMP *Functions, Statistics, and Trigonometry* (*FST*) (McConnell, et al., 2010) textbook included five definitions of function in the instructional pages and one in the glossary. All but the third defined function as a set of ordered pairs or as a relation, which was defined as a set of ordered pairs. The third defined function as relationship of correspondence. All definitions explicitly identified univalence and universal quantification, and the fourth definition also included explicit reference to the function's domain. None of the rest of the definitions referred to domain or range, and none of the definitions provide any indication of the covariance that a function creates. All five definitions suggested arbitrariness, but did not explicitly state it.

The UCSMP *Precalculus and Discrete Mathematics* (*PCDM*) (Peressini, et al., 2010) textbook included two definitions in the instructional pages and one in the glossary. The first statement and the glossary statement defined function as a relation of correspondence, while the second defined function as a set of ordered pairs. All three definitions clearly identified univalence and provided universal quantification. The first and second definitions suggest the concepts of domain and range, while the glossary definition explicitly indicated domain and range. All three definitions suggested arbitrariness, but did not provide any indication of the covariance that a function generates.

An examination across the UCSMP series reveals that functions were defined as either relationships of correspondence or as sets of ordered pairs in each textbook in the series. Thus, students are not consistently moved toward the set definition over the series. All definitions across the series consistently indicated univalence clearly, and all but one definition clearly identified universal quantification. This exception did at least suggest universal quantification; however, the wording did not make it explicit. Attention to domain and range increased over the course of the series, with no indication of domain and range made in definitions in the first books of the series. One of five definitions in *FST* included reference to domain. Finally, the *PCDM* textbook made the strongest reference, with one definition explicitly including domain and range and the other two suggesting it. All definitions did suggest the arbitrary nature of function, but no definitions suggested the covariance that functions create. However, covariance is highlighted once by the statement in *Advanced Algebra*, "Functions exist whenever the value of one or more variables determines the value of another variable" (p. 5).

Definitions in the CPMP Series

Definitions for functions in the Core-Plus Mathematics Project (CPMP) series are provided in Table 4.3. CPMP *Course 1* (Hirsch, et al., 2008a) provided two definitions of functions in the instructional pages and one in the glossary. All three defined function as a relationship of correlation, clearly identified univalence, and provided universal quantification. While all three definitions suggested arbitrariness, none referred to domain and range nor provided any indication of the covariance that a function creates.

CPMP *Course 2* (Hirsch, et al., 2008b) provided two definitions of functions in the instructional pages and one in the glossary. The first and glossary statements defined function as a relationship of correlation. The second statement was less clear. It used the phrase "*y* is a function of *x*" (p. 329), suggesting the variable type of definition; however, it also included the phrase "in a relationship between two variables" (p. 329), suggesting a relationship of correspondence. All three clearly identified univalence and provided universal quantification. None of the definitions referred to domain and range. The second and glossary definitions did not provide any indication of covariance, but they did suggest the arbitrary nature of function. The wording of the first definition subtly contradicted the arbitrariness by including the phrase “in such cases” (p. 327). Although this phrase appears to be intended to indicate relations with univalence, it could be understood to mean relations where *y* depends on *x* as in the context of the luge run. The first definition also suggests the covariance created by the function.

Table 4.3

Definitions of function in the CPMP series

Textbook	Page	Definition
Course 1	69	“In mathematics, relations like these - where each possible value of one variable is associated with exactly one value of another variable - are called **functions**.”
Course 1	69	“The only condition required for a relationship to be called a function is that each possible value of the independent variable is paired with one value of the dependent variable.”
Course 1	595	“Function (in one variable) (p. 69) A relationship between two variables in which each value of the independent variable corresponds to exactly one value of the dependent variable.”
Course 2	327	“The graph and table below show the dependence of run time *y* on vertical drop *x* in a luge run. For each *x* value, there is exactly one corresponding value of *y*. In such cases, we say that *y* **is a function of** *x* or that the relationship between run time *y* and vertical drop *x* is a **function**.”
Course 2	329	“Recall that in a relationship between two variables *x* and *y*, *y* is a function of *x* when there is exactly one *y* value corresponding to each given *x* value.”
Course 2	595	“Function (p. 161) A relationship between two variables in which each value of the independent variable *x* corresponds to exactly one value of the dependent variable *y*. The notation $y = f(x)$ is often used to denote that *y* is a function of *x*.”
Course 3	543	“A mathematical function *f* sets up a correspondence between two sets so that each element of the domain *D* is assigned exactly one image in the range *R*.”
Course 4	337	“A relationship between two variables in which each value of the independent variable *x* corresponds to exactly one value of the dependent variable *y*. The notation $y = f(x)$ is often used to denote that *y* is a function of *x*.”

CPMP *Course 3* (Fey, et al., 2009) provided one definition in the instructional pages, which defined function to be a relationship of correspondence. The definition explicitly identified univalence, universal quantification, and domain and range. The arbitrary nature of function was suggested, but the covariance created by function was not indicated.

CPMP *Course 4* (Hirsch, et al., 2010) provided one definition in the glossary, which defined function to be a relationship of correspondence. The definition explicitly identified univalence and universal quantification. The arbitrary nature of function was suggested, but the domain and range and the covariance created by function were not indicated.

An examination across the CPMP series reveals that functions were always defined as relationships of correlation, with one exception. In *Course 2*, one definition included phrases that suggested both the variable and relationship of correlation types of definition. All definitions across the series consistently indicated univalence and universal quantification clearly. No definitions referenced domain and range except the only definition in *Course 3*. All definitions suggested the arbitrary nature of function, except for one in *Course 2*, for which the phrasing was potentially confusing and could contradict the arbitrary quality of function. No definitions suggested the covariance that functions create except one in *Course 2*. Additionally, in *Course 1*, the following statement is used to clearly indicate this covariance, "A key feature of any function is the way the value of the dependent variable changes as the value of the independent variable changes” (p. 153).

Comparison of Definitions Between Series

Each series had gaps in presenting students with the definition of function. In the *Geometry* textbooks in both the Glencoe and UCSMP series, function was not defined in either the instructional pages or the glossaries, despite these textbooks including hundreds of examples of functions. The CPMP series did not provide a definition of function in the glossary in *Course 3* and did not provide a definition of function in the instructional pages in *Course 4*, despite one half of the units in the *Course 4* textbook including the word "function" in the unit titles.

Each series had different trends in how they defined function across the series. The Glencoe series introduced functions in *Algebra 1* as relationships of correspondence, but then switched to using set definitions almost exclusively in the latter half of the series. The CPMP series used the relationship of correspondence definition for all but one definition, which suggested this type but was unclear. The UCSMP series was least consistent, using both the relationship of correspondence and set definitions in the same books throughout the series.

With few exceptions, all definitions in all series made the univalence and universal quantification properties of functions explicit, and suggested the arbitrariness of the function concept. Exceptions to these tended to occur in textbooks that provided multiple definitions, so that students still had opportunities to see these properties in definitions elsewhere; however, inconsistencies between definitions in textbooks could also be confusing to students. For example, in CPMP *Course 2*, one definition of function provided wording that could suggest functions are not completely arbitrary. The other definitions in the textbook and the rest of the series suggested that functions are arbitrary, but did not indicate this explicitly. Thus, overall, this property of functions may not be clear to students.

In all the series, the greatest inconsistency was in the reference to domain and range in function definitions. In the Glencoe series, only *Algebra 2* included domain and range concepts in every definition. The UCSMP series did not include domain and range in definitions of function until the *FST* textbook, which only included reference to domain in one of five definitions. However, in all of the definitions in *PCDM*, the concepts of domain and range were at least suggested. These concepts were only included in one definition in the CPMP series. However, for this analysis, the definitions of domain and range were not identified or recorded. Therefore, the possibility exists that in instances where the function definition did not include reference to domain and range, these terms were defined after function and made reference back to the function definition. In this way, domain and range would be connected to the function concept in a way that was not captured in this analysis.

Of all the series, only one definition in CPMP *Course 2* suggested the covariance between input and output created by functions. Each series also provided some support for understanding this covariance through statements that were not definitions. The CPMP series provided the greatest potential for students to understand covariance through a clear statement of this covariance in *Course 1*, a definition that suggests this covariance in *Course 2*, and consistent use of the relationship of correspondence type of definition throughout the series.

Summary of Definitions

No series provided completely consistent definitions of function over the entire series; however, the CPMP series did provide almost all relationship of correspondence definitions of function and the Glencoe series used set definitions almost exclusively later in the course. Most definitions in all series did explicitly include the key properties of univalence and universal quantification and imply arbitrariness. References to domain and range were highly inconsistent. Almost no references to covariance were made, although the CPMP series did include more reference to this than other series.

Portrayal of Functions as Actions, Processes, Objects, or Parts of Larger Schema

The total number of examples explicitly portrayed as actions, processes, objects, or parts of larger schema over all series was extremely minimal. Of nearly 14,000 examples in the Glencoe series, 3 were depicted as actions, 7 as processes, and 4 as objects. The UCSMP series included over 10,000 function examples, but only 1 was described as an action, 8 as processes, and 1 as an object. In over 5,000 examples of functions in the CPMP series, only three were presented as processes.

General and Specific Examples of Functions

The proportions of specific and general function examples are provided in Table 4.4. The overwhelming majority of function examples in all the textbooks were specific examples. The Glencoe series included over 95% specific examples, and the UCSMP and CPMP series both had almost 90% specific examples. The first book of each series had very similar proportions of specific examples, with around 97% specific examples in each. The Glencoe *Geometry* book had a slightly higher percent specific, while the UCSMP *Geometry* book and the CPMP *Course 2* book each had around 90% specific examples. The UCSMP *Advanced Algebra* and *FST* books and the CPMP *Course 3* book also had around 90%. In contrast, the Glencoe *Algebra 2* book had nearly 98% specific examples. The final book in all three series had smaller percents of specific functions than any other books in the respective series.

Table 4.4

Distribution of general and specific function examples in each textbook

Textbook	Total number of function examples	Percent of specific examples (p/r*)	Percent of general examples (p/r*)
Glencoe			
Algebra 1	3937	97.3% (94.8/2.5)	2.7% (2.7/0)
Geometry	768	98.2% (96.6/1.6)	1.8% (1.8/0)
Algebra 2	5124	97.7% (95.2/2.5)	2.3% (2.2/0.1)
Precalculus	4046	92.9% (88.5/4.4)	7.1% (6.9/0.2)
Series Total	13875	96.2% (93.2/3)	3.8% (3.7/0.1)
UCSMP			
Algebra	2029	96.3% (90.4/5.9)	3.7% (3.7/0)
Geometry	528	89.8% (87.7/2.1)	10.2% (10.2/0)
Advanced Algebra	2595	89.2% (84.8/4.4)	10.8% (10.8/0)
FST	2382	90.3% (85.5/4.8)	9.7% (9.6/0.1)
PCDM	2796	83.7% (78/5.8)	16.3% (16.3/0)
Series Total	10330	89.4% (84.4/5)	10.6% (10.6/0)
CPMP			
Course 1	1145	96.7% (89.8/6.9)	3.3% (3.2/0.1)
Course 2	1125	91.6% (77.2/14.5)	8.4% (8.1/0.3)
Course 3	1124	91.3 % (85.9/5.4)	8.7% (8.7/0)
Course 4	2211	83.7% (78.9/4.8)	16.3% (16.2/0.1)
Series Total	5605	89.5% (82.2/7.3)	10.5% (10.4/0.1)

* p = provided, r = requested

Comparison of Language Use to Research Recommendations

Researchers have indicated a number of features of the function concept that their work suggests are important for how functions are defined for students. Those focusing on the historical development of the concept of function argue that each student's understanding of function should follow a similar development (Sfard, 1991, 1992; Sierpinska, 1992). Sierpinska's careful description of this historical development suggests that an appropriate order of types for function definitions would begin by defining function as a variable, then as a relationship of correspondence, and finally as a set of ordered pairs. This would also align with Sfard's (1991) argument that students need to move from viewing function as a computational process toward a structural conception and with researchers who argue that students first view function as an Action, then as a Process, and then as an Object (Asiala, et al., 1996; Breidenbach, et al., 1992; Dubinsky & Harel, 1992; Eisenberg, 1991, 1992; Habre & Abboud, 2006; Kaput, 1992; Sfard, 1992; White & Van Dyke, 2006; C. G. Williams, 1998).

Only once, in CPMP *Course 2*, was a definition included that even suggested understanding function as a variable. The research noted above would intimate that students could be better served by beginning with a variable definition of function. The difficulty with this is that defining function as a variable is mathematically questionable, as it leads to problems with consistency and rigor when using the function concept at higher levels. Each of the textbook series does begin by defining function as a relationship of correspondence before defining function as a set, which is consistent with the above recommendations from the research. The Glencoe series made the clearest shift to defining function as a set later in the textbook series, which would be supported by these research recommendations. The CPMP series used relationship of correspondence definitions almost exclusively, and the UCSMP series tended to include a mixture of both types. The research of Tall and Vinner (1981) on concept images and concept definitions supports the CPMP approach over the UCSMP approach as they argued that conflicts between concept definition and concept image create confusion and difficulties for students. Students studying from CPMP textbooks at least have a consistent definition of function, whereas students studying from UCSMP textbooks will see different definitions in each book and even within the same book. However, because this analysis was limited to only the definitions of function, the possibility exists that additional text surrounding definitions could either support or fail to support students' understanding of function along the trajectory proposed in the research.

A number of researchers have also argued that the set definition of function is so abstract that students are rarely able to use it in any meaningful way (Even, 1993; Markovits, et al., 1986; Sierpinska, 1992; Vinner & Dreyfus, 1989). This type of definition was most frequently used in the Glencoe series, which included it predominantly in *Algebra 2* and *Precalculus*. The UCSMP series included the set type of definition in each textbook except *Geometry*, which did not include any definitions of function. However, relationship of correspondence definitions were also included in each textbook. The CPMP series did not use the set definition in any textbook.

In addition to the type of function definition, other key aspects of function definitions were analyzed. Even (1993) argued that two are fundamental to the concept of function: univalence and arbitrariness. Recall that univalence means that a function maps a given element of the domain to only one element of the range. All definitions in all textbooks clearly indicated univalence. Even also found that arbitrariness, which is the idea that a function may be any set of paired values, was key and that students who did not understand this concept tended to believe functions needed a formula in order to truly be functions. Arbitrariness was at least suggested in most definitions; however, it was rarely explicitly stated.

Also key to the understanding of function are the concepts of domain and range (Markovits, Eylon, & Bruckheimer, 1988). Researchers have found students with difficulties appropriately attending to and understanding domain and range, understanding how they make up important aspects of each function, and understanding what they can consist of (Dubinsky & Harel, 1992; Markovits, et al., 1986; Schwingendorf, et al., 1992; Sfard, 1992; Tall & Bakar, 1992). References to the domain and range in textbook definitions were inconsistent. The Glencoe series included some references to domain and range throughout the series, but in the UCSMP series only definitions in the final two textbooks referred to domain and range. The CPMP series was the weakest, with only one definition in the entire series including reference to domain and range. However, it is possible that text surrounding the definitions in these textbooks included discussion of domain and range that was not captured in this analysis. Connected to this is the concept of universal quantification, which states that functions must assign an element of the range to every element of the domain. Although definitions did not necessarily include this concept in terms of domain and range, every definition in each book did explicitly include it, usually in connection with the expression of univalence.

Finally, with respect to definitions, Carlson and her colleagues (Carlson, et al., 2002; Oehrtman, et al., 2008) have argued that students must understand the covariation created by functions or they will face difficulties in using the concept. Other researchers have also found that students struggle with understanding function as a dynamic relationship (Ferrini-Mundy & Lauten, 1994; Goldenberg, et al., 1992; Leinhardt, et al., 1990; Monk, 1992). No definitions explicitly identified function as a dynamic relationship, and only one definition, in CPMP *Course 2*, suggested such covariance. However, it must be noted again that surrounding text may have suggested the dynamic aspects of function that the definitions did not.

With regard to language use in function examples, researchers who have argued that students' individual understanding of function tends to mirror the historical development of the concept (Sfard, 1991, 1992; Sierpinska, 1992) or use the Action-Process-Object-Schema (APOS) framework (Asiala, et al., 1997; Breidenbach, et al., 1992; Dubinsky & Harel, 1992) argue that students' thinking moves from carrying out actions with specific functions toward being able to think about functions more generally as objects that can be acted upon. This suggests that as students' conception of function develops, it would be appropriate for them to engage in more general examples of function. Each textbook series did include a larger proportion of general examples in later textbooks. The UCSMP and CPMP series each included just over 16% general examples in their final textbooks, and had general examples for about one in ten examples over the entire series. The Glencoe series, despite having more than double the proportion of general examples in *Precalculus* than in any other textbook, had less than 4% general examples. Also, textbooks included almost no direct use of APOS terminology with function examples to help develop more sophisticated understandings of function.

Summary of Language Used in Relation to Function

The above analysis provides insight into the language used in relation to functions in each textbook and series. This language was examined through analysis of function definitions provided; portrayal of functions as actions, processes, objects, or parts of larger schema; and distribution of general and specific function examples.

Definitions of functions were not consistent within any series, although almost all definitions explicitly included the key features of univalence and universal quantification and implied arbitrariness. A number of researchers have noted students' inattention to domain and range, and references to these in definitions were only made occasionally. The most consistent series was CPMP, which used almost all relationship of correspondence definitions. The series, more than Glencoe or UCSMP also suggested the covariance created by functions, which Carlson and her colleagues (Carlson, et al., 2002; Oehrtman, et al., 2008) argue is vital to understand. The Glencoe series moved from using correspondence definitions early to using set definitions later, and the UCSMP series used both throughout the series.

Each series included only a few examples with text portraying functions as actions, processes object, or parts of larger schema. Each series included mostly specific examples, with no series including more than 11% general examples. These results suggest textbooks may not be providing explicit support for students to move toward more sophisticated conceptions of functions according to the APOS perspective.

Presence of Functions

Prevalence of Function Examples

Number of Function Examples

The number of function examples and function examples per page for each textbook series are provided in Table 4.5. The Glencoe series had the highest total number of function examples, with a total of nearly 14,000 examples across four textbooks. In five textbooks, the UCSMP series had about 4,000 fewer examples. In four textbooks, the CPMP series had less than half as many as Glencoe, with 5,605 examples.

Because the length of textbooks in each series differed, comparing the number of examples per page provides another meaningful measure. Again, the Glencoe series provided the most examples per page, averaging nearly 4 examples per page. UCSMP and CPMP had very similar numbers of examples per page, both around 2.4.

Table 4.5

Number of function examples in each textbook

Textbook	Total number of function examples	Total number of instructional pages	Number of examples per page
Glencoe			
Algebra 1	3937	801	4.92
Geometry	768	953	0.81
Algebra 2	5124	933	5.49
Precalculus	4046	983	4.12
Series Total	13875	3670	3.78
UCSMP			
Algebra	2029	833	2.44
Geometry	528	880	0.60
Advanced Algebra	2595	937	2.77
FST	2382	837	2.85
PCDM	2796	865	3.23
Series Total	10330	4352	2.37
CPMP			
Course 1	1145	588	1.95
Course 2	1125	588	1.91
Course 3	1124	604	1.86
Course 4	2211	623	3.55
Series Total	5605	2403	2.33

Comparing the first textbook in each series, Glencoe *Algebra 1* had nearly 4,000 examples in 800 pages, averaging nearly 5 examples per page. UCSMP *Algebra* had only about half as many examples (2,029) in slightly more pages (833), and thus had an average of about half as many examples per page compared to Glencoe *Algebra 1*. CPMP *Course 1* had about half as many as UCSMP *Algebra*, but also had just under 600 pages, averaging just under 2 examples per page.

The second textbook in the Glencoe and UCSMP series were *Geometry* textbooks and had far fewer examples of functions. Glencoe *Geometry* had only 768 examples in 953 pages, averaging less than one example per page. UCSMP *Geometry* had just over 500 examples in 880 pages, averaging only 0.6 examples per page. CPMP *Course 2* was not exclusively *Geometry* content, and it had numbers very similar to *Course 1*: 1,125 examples in 588 pages, again yielding an average of just under two examples per page.

The *Algebra 2* textbook in the Glencoe series had the most function examples and highest number of examples per page of any book in the series, with 5,124 and an average of 5.49 per page in 933 pages. The UCSMP *Advanced Algebra* book had only a few pages more, but only included 2,595 examples for slightly less than 3 examples per page. The CPMP *Course 3* book was again the shortest, with just over 600 pages, and again had the fewest examples of functions, with 1,124 examples, resulting in less than 2 examples per page.

In Glencoe *Precalculus*, over 4,000 examples were provided in nearly 1,000 pages, resulting in just over 4 examples per page. The fourth CPMP book, *Course 4*, was designed for "the preparation of students for success in college mathematics with a focus on preparation for calculus" (Hirsch, et al., 2010, p. xii), and this resulted in a number of function examples (2,211) and number of examples per page (3.55) quite different from the other three books in the series. The UCSMP series offered two books for the fourth year: *Functions, Statistics, and Trigonometry* (*FST*), and *Precalculus and Discrete Mathematics* (*PCDM*). The *FST* book provided fewer examples, with 2,382 examples in 837 pages, yielding 2.85 per page. In contrast, the *PCDM* book provided 2,796 examples in 865 pages, resulting in 3.23 examples per page.

A general trend in all series was to provide a greater number of examples of functions in later textbooks. Only the Glencoe series did not have the most examples in the final book, with the *Algebra 2* book containing more than the *Precalculus* book. This trend was present in the Glencoe and UCSMP series even without considering the *Geometry* textbooks. For both series, the *Geometry* textbooks contained far fewer examples of functions, reducing the number of examples per page. Without these textbooks, the Glencoe series averaged 4.86 examples per page and the UCSMP series averaged 2.85 examples per page. The CPMP series did not include a textbook with mainly *Geometry* content and also did not follow a strictly increasing number of functions. Instead, the first three books had very similar numbers of functions, and the fourth book included many more examples.

Lengths of Function Examples

The analysis of the prevalence of functions in the textbooks can also be supported by an analysis of the length of the function examples in the books (see Figure 4.1). Each function example was coded with one of six length codes. The shortest examples took up less than one row of text in the textbook and included other examples or text on the same typographical row. A common example of this was functions in the homework exercise sections in the Glencoe series, where multiple homework exercises were placed in a block with several in a row. The next length coded was for examples that were on one to two typographical rows in the textbook. Even longer were examples that ranged from three rows up to one quarter of a full page of text. Examples that were over one quarter of one page, but shorter than one half of one page of the text were coded with the next code. The next code was for examples over one half of one page, but did not extend over one page of text, and the final code was reserved for function examples that took up more than one page of text.

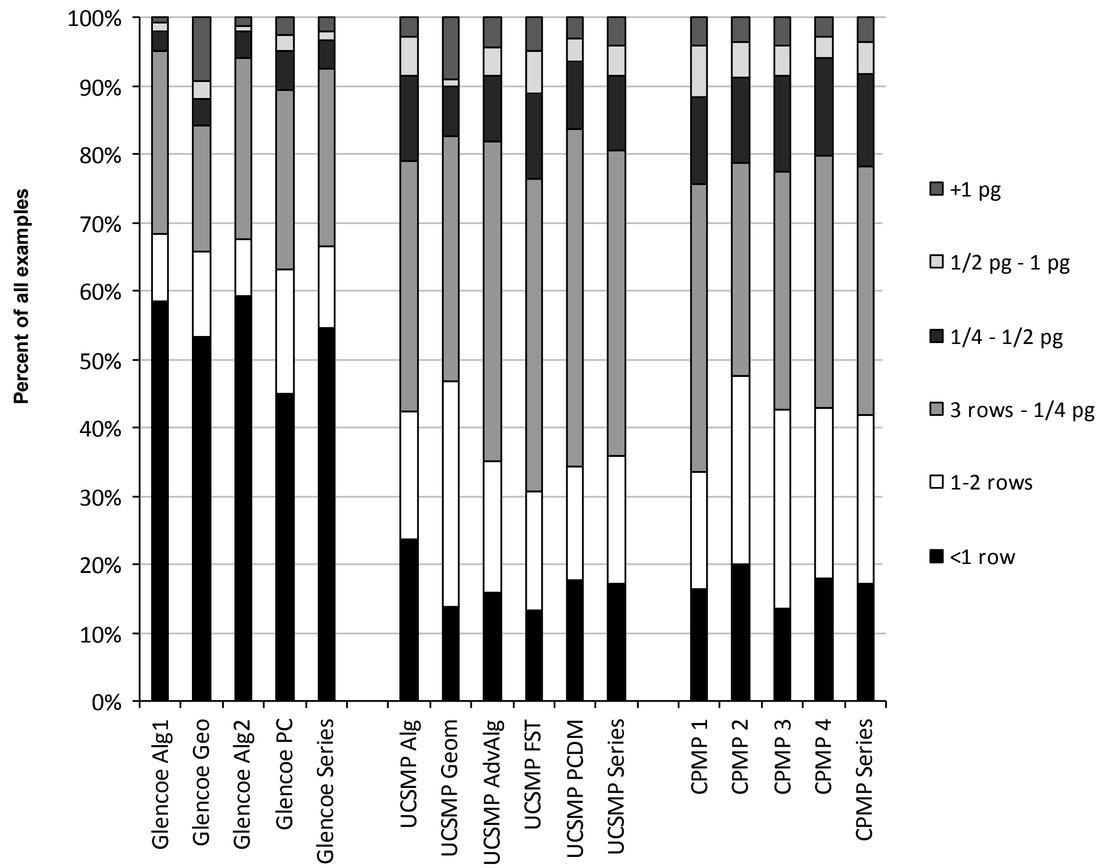


Figure 4.1: Lengths of function examples.

In the Glencoe series, the predominant length for function examples was less than one row of text, with over 50% of all examples in this category. The next largest category was examples between three rows of text and one quarter of a page long. In contrast, in the UCSMP series and the CPMP series, examples with lengths between three rows of text and one quarter of a page were much more prevalent, with approximately 40% of examples in each series being of this length. These series also had similar proportions of examples longer than one quarter of a page, and these proportions were higher than the

Glencoe series. However, the smallest categories for all three series were examples over one half of a page long.

The CPMP series showed similar patterns of length among all four textbooks. The Glencoe and UCSMP series had a few exceptions to similar patterns across textbooks. In both *Geometry* books, the proportion of examples longer than one page was higher than in any other books in the series. In the UCSMP *Geometry* book, the proportion of examples that were one to two rows long was higher than any other in the series; whereas, this held true in the Glencoe *Precalculus* book for that series. Besides these small exceptions, the distribution of lengths of examples in these two series was largely similar across books.

Summary of Prevalence of Function Examples

In combination with the number of examples, we can see that while the Glencoe series tended to include the most function examples, a large portion of these examples were very short. The CPMP series had far fewer examples of functions, but these examples tended to be longer. The UCSMP had more examples than CPMP and these examples tended to be longer than examples in Glencoe. The result was that functions tended to be more prevalent in the UCSMP and CPMP series. For example, consider that both the UCSMP and CPMP series had around 2.35 examples of functions per page compared to Glencoe's 3.78 examples per page. However, more than half of these Glencoe examples were less than one row of text, suggesting four typical examples per page fitting on only one or two rows of text. In contrast, two typical examples in UCMSP and CPMP would take up between 6 rows and one half of a page.

Explicit Identification of Functions

The proportions of function examples labeled as functions and examples with requests for students to label as functions are provided in Table 4.6. Most textbooks labeled less than half of examples as functions, except CPMP *Course 3*, which identified over 60% of its examples. The CPMP series labeled over 45% of its function examples as such, which was a higher percent than the other two series. The Glencoe series labeled just over one quarter of its examples as functions, and the UCSMP series labeled less than 18% of its examples as functions.

Table 4.6

Distribution of function examples labeled as functions and examples for students to label

Textbook	Total number of function examples	Percent of examples labeled as functions (p/r/c*)	Percent of examples for students to label (p/r/c*)
Glencoe			
Algebra 1	3937	16.1% (16.0/0.1/0)	0.6% (0.6/0/0)
Geometry	768	1.6% (1.6/0/0)	0.0% (0/0/0)
Algebra 2	5124	33.1% (32.4/0.7/0)	0.1% (0.1/0/0)
Precalculus	4046	28.6% (28.6/0/0)	0.8% (0.8/0/0)
Series Total	13875	25.2% (24.9/0.3/0)	0.4% (0.4/0/0)
UCSMP			
Algebra	2029	7.0% (6.9/0.1/0)	0.2% (0.2/0/0)
Geometry	528	0.6% (0.6/0/0)	0.0% (0/0/0)
Advanced Algebra	2595	17.0% (16.8/0.2/0)	2.1% (1.8/0.2/0)
FST	2382	21.0% (21.0/0/0)	1.0% (1/0/0)
PCDM	2796	26.3% (26.3/0/0)	0.3% (0.3/0/0)
Series Total	10330	17.6% (17.6/0.1/0)	0.9% (0.8/0.1/0)
CPMP			
Course 1	1145	37.6% (36.7/1/0)	2.0% (1.9/0.1/0)
Course 2	1125	36.9% (36.9/0/0)	0.6% (0.6/0/0)
Course 3	1124	61.3 % (61.3./0/0)	0.0% (0/0/0)
Course 4	2211	45.5% (45.5/0/0)	0.1% (0.1/0/0)
Series Total	5605	45.4% (45.2/0.2/0)	0.6% (0.6/0/0)

* p = provided, r = requested, c = combined

The clear outlier in both the Glencoe and UCSMP series was the *Geometry* books in each, with each book labeling less than 2% of examples as functions. The *Algebra 1* books in both series had the next fewest examples labeled as functions, with about 16% of examples in Glencoe *Algebra 1* and 7% of examples in UCSMP *Algebra* being identified as functions. The remaining three books in the UCSMP series increase from 17% up to about 26% of examples being labeled as functions. Glencoe *Algebra 2* labeled a full third of its examples as functions, and *Precalculus* labeled slightly under 30% as functions.

All textbooks in the CPMP series had over 35% of their examples identified as functions. Courses 1 and 2 had about 37% and *Course 4* had over 45% of its examples labeled as functions.

Far fewer examples called for students to identify whether they were functions or not. The UCSMP series included the most, with almost 1% of its functions, which was bolstered by the *Advanced Algebra* book that requested students to decide over 2% of its examples as functions. CPMP *Course 1* requested students identify 2% of its examples as functions. Besides these two textbooks, the rest of the books requested students identify examples as functions for around 1% or less of the included examples.

Function Examples in Lessons and Exercises

The percent of function examples presented in lessons and in homework exercises contributes to an understanding of the presentation of function in textbooks (see Table 4.7). All textbooks in all series had more than half of their function examples in the exercises. Because some examples were included in both lessons and exercises, percents totaled more than 100%, and two textbooks also had more than 50% of examples in lessons: CPMP *Course 3* and *Course 4*. The CPMP series had nearly half of the function examples in lessons, and slightly over 55% of examples in exercises. In contrast, the UCSMP series had just over one out of every four of examples in lessons and over three out of every four in exercises, and Glencoe had only about one in five examples in lessons and over 4 out of every five in exercises.

Table 4.7

Distribution of function examples in lessons and exercises

Textbook	Total number of function examples	Percent of examples in lessons	Percent of examples in exercises
Glencoe			
Algebra 1	3937	19.6%	81.4%
Geometry	768	25.3%	86.1%
Algebra 2	5124	20.2%	81.8%
Precalculus	4046	20.1%	82.6%
Series Total	13875	20.3%	82.2%
UCSMP			
Algebra	2029	23.7%	77.8%
Geometry	528	34.7%	75.2%
Advanced Algebra	2595	25.0%	81.9%
FST	2382	29.2%	76.9%
PCDM	2796	30.8%	74.0%
Series Total	10330	27.8%	77.5%
CPMP			
Course 1	1145	42.5%	59.7%
Course 2	1125	42.9%	60.2%
Course 3	1124	53.4%	50.5%
Course 4	2211	50.2%	52.7%
Series Total	5605	47.8%	55.2%

Excepting *Geometry*, the other three Glencoe textbooks had nearly the same distribution of examples, with 20% of examples in lessons and 80% in exercises. The *Geometry* book had greater overlap, with about 25% of examples in lessons and over 85% in exercises.

In contrast, UCSMP *Geometry* had about 35% of examples in lessons and about 75% in exercises. UCSMP *Algebra* and *Advanced Algebra* had greater overlap than the parallel books in Glencoe, with about 80% of examples in exercises, but about 25% of examples in lessons. The final two books in the series include an even greater percent of examples in the lessons, about 30%, and slightly less in the exercises, with about 75%.

The CPMP series included a much larger portion of function examples in the lessons. *Course 1* and *Course 2* were nearly the same, with over 40% of examples in lessons and about 60% in exercises. *Course 3* and *Course 4* were also similar, with just over 50% of examples both in lessons and exercises.

Requests for Students to Generate Function Examples

The numbers of requests for function examples are provided in Table 4.8. The UCSMP series requested that students generate more examples of functions than either of the other two, asking for a total of 523 examples. The Glencoe series included 430 requests for examples of functions. The CPMP series only asked students to generate examples of functions 417 times. However, because this series included fewer examples of functions, this is a larger proportion of examples than the other two series; 7.4% as opposed to 3.1% in Glencoe and 5% in UCSMP.

In the Glencoe and UCSMP series, the requests for functions were lowest in the *Geometry* books, with just over 10 requests, and highest in the *Precalculus* books, with over 160 requests in each. The *Algebra* textbooks in both series and the *FST* book in UCSMP each had around 100 requests each. In the CPMP series, Courses 1 and 3 had the fewest requests, with 82 and 61 requests respectively. *Course 4* had just over 100 requests, and *Course 2* had the most, with 166 requests.

Table 4.8

Number of requests for function examples in each textbook

Textbook	Total number of function examples	Requests for specific examples	Requests for general examples
Glencoe			
Algebra 1	3937	100	0
Geometry	768	12	0
Algebra 2	5124	128	5
Precalculus	4046	178	7
Series Total	13875	418	12
UCSMP			
Algebra	2029	119	0
Geometry	528	11	0
Advanced Algebra	2595	114	1
FST	2382	114	3
PCDM	2796	161	0
Series Total	10330	519	4
CPMP			
Course 1	1145	81	1
Course 2	1125	163	3
Course 3	1124	61	0
Course 4	2211	106	2
Series Total	5605	411	6

Non-Examples

In addition to examples of functions, non-examples of functions were also coded when textbooks explicitly identified them as not being functions or asked students to determine whether they were functions. For instance, a textbook could have provided a set of ordered pairs with at least two different pairs that had the same x value. If this set was identified as not being a function or if students were asked to determine whether the set was a function, the set would have been coded as a non-example. These non-examples help students determine boundaries of the example space of functions provided by the textbook (Watson & Mason, 2005). In Table 4.9, the total number of coded non-examples is provided, as well as the number provided, requested, and combined. Non-examples were coded as provided when information about the non-example was only given to students with any requests of students to provide any part of the non-example. Non-examples were coded as requested when students were requested to provide a non-example without any part of the non-example provided to them. Non-examples were coded as combined when some of the non-example was provided to the student and they were requested to provide some of the non-example.

Table 4.9

Distribution of non-examples of functions

Textbook	Total number of function examples	Number of non-examples of functions (p/r/c*)
Glencoe		
Algebra 1	3937	27 (27/0/0)
Geometry	768	0 (0/0/0)
Algebra 2	5124	35 (33/2/0)
Precalculus	4046	31 (30/1/0)
Series Total	13875	93 (90/3/0)
UCSMP		
Algebra	2029	8 (8/0/0)
Geometry	528	0 (0/0/0)
Advanced Algebra	2595	44 (41/3/0)
FST	2382	22 (21/1/0)
PCDM	2796	16 (16/0/0)
Series Total	10330	90 (86/4/0)
CPMP		
Course 1	1145	2 (2/0/0)
Course 2	1125	13 (12/1/0)
Course 3	1124	1 (1/0/0)
Course 4	2211	4 (4/0/0)
Series Total	5605	20 (19/1/0)

* p = provided, r = requested, c = combined

Both the Glencoe and UCSMP series identified, or had students identify, about 90 non-examples of functions, whereas the CPMP series offered only 20. In fact, the only

CPMP book that identified more than 4 non-examples was *Course 3*, which included 65% of the non-examples in the entire series. Glencoe *Geometry* and UCSMP *Geometry* did not identify any non-examples of functions. The rest of the Glencoe books were balanced, with about 30 non-examples each. The UCSMP series began with only 8 non-examples in *Algebra*, peaked in *Advanced Algebra* with 44 non-examples, and included about half as many as *Advanced Algebra* in the final two books.

Errors in Function Examples

Table 4.10

Number of errors in function examples in each textbook

Textbook	Total number of function examples	Number of errors
Glencoe		
Algebra 1	3937	7
Geometry	768	1
Algebra 2	5124	10
Precalculus	4046	8
Series Total	13875	26
UCSMP		
Algebra	2029	1
Geometry	528	1
Advanced Algebra	2595	2
FST	2382	5
PCDM	2796	7
Series Total	10330	16
CPMP		
Course 1	1145	0
Course 2	1125	0
Course 3	1124	0
Course 4	2211	2
Series Total	5605	2

The numbers of errors in function examples from each textbook are provided in Table 4.10. The errors in all of the textbook series were minimal; however, the CPMP series had only two errors over four books in contrast to 26 errors in the four Glencoe books. The UCSMP series fell in between these, with 16 errors in five books. In UCSMP, the *PCDM* book had the most errors, with 7. Three of the Glencoe books had at least this many, with *Geometry* being the only exception. In contrast, the first three CPMP books were error-free.

The two errors in CPMP *Course 4* were both mismatches between representations of functions, in these cases, graphs did not match the equations provided. In the UCSMP series, half of the errors were also mismatches between representations of functions. These errors were related to a number of different representations, including symbolic, graphic, numeric, and verbal. For example, in one example on page 61 in *Algebra*, the text described the function that can be represented symbolically as $y = 5 + |3n|$, but an accompanying table provided values for the function that can be represented as $y = |5 + 3n|$. Another five errors were instances where textbooks didn't provide enough information for students. For example, on page 166 in *Advanced Algebra*, students are asked to write an equation based on the graphic representation of the function, but no graph is provided. The final three errors were apparently typographical errors, for example, on page 599 in *Geometry*, one equation used y for both variables, rather than an x and y.

Of the 26 errors in the Glencoe series, seven were typographical errors, and another seven were instances when the text labeled an example as a function when it was not. For example, on page 324 in *Algebra 2*, the inequality relation represented by $y > x^2 + 3x + 2$ is described as a function. Six errors resulted from the textbooks not providing enough information for students. For example, on page 42 in *Algebra 1*, students are asked to provide ordered pairs for points on a scatterplot, but the scale is not provided for students. The series had four instances of mismatches between representations of functions, such as on page 258 in *Precalculus*, where one function is described as a cubic function, but the graph represents a quadratic function. Finally, there were two examples with inaccurate verbal descriptions. For example, on page 65 in *Algebra 1*, the textbook provides a verbal description of a function and a realistic setting, but the domain and range provided do not match the setting.

Ultimately, the number of examples of function with errors in each series was a minuscule proportion of all examples. However, these errors could be confusing or misleading to students. The Glencoe series had the greatest number of errors, the UCSMP series had very few errors in the first three textbooks but more in the last two, and the CPMP series was nearly free of errors, except two in *Course 4*.

Comparison of Presence of Functions to Research

There have been repeated calls for increased attention to functions in the high school mathematics curriculum (Commission on Mathematics, 1959; National Committee on Mathematical Requirements, 1923; National Council of Teachers of Mathematics, 1989, 2000). The textbooks in these series certainly include a large number of examples of functions, with only Glencoe *Geometry* and UCSMP *Geometry* including fewer than 1000 examples. Every other textbook averaged more than two examples of functions for each page of text. However, the large number of functions in textbooks does not necessarily mean the textbook was directing students' attention to the examples as functions. Less than half of the examples in each series were identified as functions.

Often researchers have found that students have difficulty deciding whether something is a function or not, especially when the functions are not in some way familiar, reasonable, or well known (Baker, et al., 2000; Even, 1993; Leinhardt, et al., 1990; Tall & Bakar, 1992; Vinner, 1983; White & Van Dyke, 2006). These textbook series included thousands of examples of functions, but the majority of these examples were not identified as functions for students, and thus opportunities for students to make connections between the function concept and examples of function could often be missed. The UCSMP series had the smallest percent identified as functions and included over 8500 examples that were not identified as functions. The Glencoe series had a slightly higher percent identified as functions, but included over 10,000 examples of functions that were never identified as such. The CPMP series had the highest proportion of examples identified as functions, but still left over half unidentified. In addition, the proportion of examples for which students were to determine whether they were functions was less than 1% in each series.

Dahlberg and Housman (1997) argued that students should also have opportunities to construct their own examples of functions. Such requests encourage students to wrestle with the meaning of the concept. Again, in comparison to the considerable number of examples provided for students, there were relatively few requests or students to create examples of functions in each of the series. There were, on average, about 100 such requests in each textbook, although, not surprisingly, the two geometry textbooks were far below this average.

A number of researchers have pointed out the importance of non-examples in students' formation of a mathematical concept (Eisenberg, 1991; Leinhardt, et al., 1990; Tall & Bakar, 1992; Tsamir, et al., 2008; Vinner, 1991; Watson & Shipman, 2008; Zodik & Zaslavsky, 2008). Students need carefully chosen and sequenced non-examples to accompany examples so that they can make comparisons and better understand what functions are by examining what they are not. In contrast to the vast number of examples of functions for students in the textbook series, there were only a handful of non-examples for students. The Glencoe and UCSMP series had less than 100 each, and the CPMP series had only 20 non-examples total.

Summary of Presence of Function

The above analysis provides insight into the presence of functions in each textbook and series. This presence was examined through analysis of the prevalence of function examples, proportions of examples identified as functions, distribution of function examples in lessons and exercises, requests for students to provide examples, and errors in function examples.

The Glencoe series included many more examples of functions than either the UCSMP series or the CPMP series, with nearly triple the number in the CPMP series and nearly half again as many as the UCSMP series. The Glencoe series included nearly twice as many examples per page as both the UCSMP and CPMP series. However, the examples in the Glencoe series tended to be much shorter, with over half of the examples in the entire series being less than 1 row of text. Thus, because the UCSMP and CPMP series had larger proportions of longer examples that took up to half of a page, a larger portion of these textbooks was function examples. However, in the CPMP series, almost half of these examples were labeled as functions, while in the UCSMP series, less than 20% were labeled. The Glencoe series, with a smaller portion of textbooks being function examples, also only labeled about one in every four examples as functions. In the Glencoe and UCSMP series, these examples tended to be predominantly in the exercises, with much smaller proportions in the lessons. In contrast, in the CPMP series, only slightly over half of all function examples were in the exercises.

A number of aspects related to the presence of functions in each series occurred infrequently. Almost all of the examples were provided for students, and requests for students to provide general examples were scarce, with 12 requests in the Glencoe series being the most of any series. Each series had a relatively small number of non-examples provided, and almost no non-examples requested of students. A minimal number of examples had errors in them. Common errors included mismatches between representations of functions, insufficient information provided, typographical errors, and mislabeling relations as functions. The Glencoe series had the most errors, while the CPMP series had only two errors in the entire series.

Ultimately, each series tended to provide a consistent presence of functions by including a large number of examples of specific functions, mostly provided in exercises for students. Main differences in the series were that the Glencoe series provided a large number of short examples, while the UCSMP and CPMP series provided fewer, but longer, examples of functions.

Core Features of Function Examples

Domain and Range

Domains and Ranges in Examples

The proportions of function examples with domains and ranges provided or requested are provided in Table 4.11. Only a small percent of the examples of functions in each textbook included or requested specified domain and range. The Glencoe series provided the most examples, with over 6% of their nearly 14,000 examples including or requesting an explicit domain or range, for a total of nearly 900 examples. Almost three-quarters of these were requests for students to provide a domain or range. The UCSMP series had a similar total percent as Glencoe, but only about 50% of these requests were for students to provide a domain or range. The CPMP series provided less than 200 examples with explicit domain or range: only 3.5% of the 5625 examples. Over 70% of these examples were requests for students to provide a domain or range.

Table 4.11

Distribution of function examples with domains specified and with non-numerical domains specified in each textbook

Textbook	Total number of function examples	Percent of examples with domains specified (p/r/c*)	Percent of examples with non-numerical domains specified (p/r/c*)
Glencoe:			
Algebra 1	3937	8.6% (1.3/7.3/0)	0.1% (0.1/0/0)
Geometry	768	0.0% (0/0/0)	0.0% (0/0/0)
Algebra 2	5124	7.3% (1.7/5.6/0)	0.0% (0/0/0)
Precalculus	4046	3.2% (1.8/1.3/0)	0.0% (0/0/0)
Series Total	13875	6.1% (1.5/4.5/0)	0.0% (0/0/0)
UCSMP:			
Algebra	2029	2.7% (1.3/1.3/0.1)	0.0% (0/0/0)
Geometry	528	0.0% (0/0/0)	0.0% (0/0/0)
Advanced Algebra	2595	7.7% (1.8/4.8/1.0)	0.0% (0/0/0)
FST	2382	6.8% (2.9/3.4/0.5)	0.0% (0/0/0)
PCDM	2796	7.0% (3.5/3.2/0.3)	0.1% (0.1/0/0)
Series Total	10330	5.9% (2.3/3.1/0.5)	0.0% (0/0/0)
CPMP:			
Course 1	1145	0.6% (0.2/0.4/0)	0.0% (0/0/0)
Course 2	1125	1.2% (0/1.2/0)	0.0% (0/0/0)
Course 3	1124	6.9 % (0.5/6/0.4)	0.0% (0/0/0)
Course 4	2211	4.4% (1.7/2.4/0.3)	0.0% (0/0/0)
Series Total	5605	3.5% (0.8/2.5/0.2)	0.0% (0/0/0)

* p = provided, r = requested, c = combined

Compared to UCSMP *Algebra* and CPMP *Course 1*, Glencoe *Algebra 1* included more requests for students to provide domains or ranges. In fact, the 7.3% of such examples in Glencoe *Algebra 1* was higher than any other textbook. CPMP *Course 1* had the fewest examples either providing or requesting domains or ranges, with less than 1% of the book's examples having this feature. UCSMP *Algebra* had just over 1% of its examples provide domains or ranges, and the same percent of requests.

Neither *Geometry* book had any examples where domain or range were explicitly provided or requested. CPMP *Course 2* also did not include any examples where domain or range were provided, but did have just over 1% of its examples request that students provide the domain or range.

In comparison to the rest of the textbooks in the series, UCSMP *Advanced Algebra*, and CPMP *Course 3* included a greater percent of examples where domain or range were made explicit or requested of students, with 7.7% and 6.9%, respectively. *Algebra 2* had the second highest percent in the Glencoe series, behind *Algebra 1*.

In the final books in all three series, the percents of examples where domain or range were made explicit or requested of students were lower. Both Glencoe *Precalculus* and CPMP *Course 4* had around 4% of such examples, and UCSMP *FST* and *PCDM* each had approximately 7% of such examples.

There were almost no examples of functions with non-numerical domains specified in any textbook. The CPMP series did not include any such examples. The Glencoe series provided three such examples and the UCSMP series provided four such examples and requested one.

Comparison of Domain and Range to Research

A number of researchers have noted a general lack of attention toward domain and range on the part of students (Markovits, et al., 1986; Schwingendorf, et al., 1992;

Sfard, 1992). Only a small proportion of the examples in each series included explicit reference to domain and range, with about 6% in the Glencoe and UCSMP series and 3.5% in the CPMP series. For the remaining thousands of examples in each series, students were not directed to attend to the domain and range by the textbooks. Generally, the assumption to be made is that the domain and range are all real numbers, which aligns with Tall and Bakar's (1992) finding that students tend to believe that the domain must be complete in some sense, and not intervals or subsets of a larger set and with Dubinsky and Harel's (1992) finding that many students believe that the domain and range of all functions must consist only of numbers. Also in relation to Dubinsky and Harel's finding, almost no examples of functions with non-numerical functions were present in any textbook in any series.

Representations

Based on the literature review, codes for seven different function representations were developed: *Symbolic*, *Graphic*, *Numeric*, *Function Machine*, *Mapping Diagram*, *Verbal Description*, and *Physical*. A representation was coded as (a) *Symbolic* if it included an equation, (b) *Graphic* if it used points or curves to represent the function, (c) *Numeric* if it provided some or all of the ordered pairs of the function, (d) *Function Machine* if it provided a diagram suggesting an input-output machine, (e) *Mapping Diagram* if it provided a diagram depicting elements in the domain and range with lines or arrows connecting them, (f) *Verbal Description* if it used words to describe the function, or (g) *Physical* if it provided a physical model or a description of a physical model that embodied a functional relationship. Note that each example of a function received a code for at least one of these representations, or was coded as *Other*. In addition, many examples received multiple codes. For example, an exercise that requested that students create a graph of a function based on its equation would have received a code for both *Symbolic* and *Graphic* representations. Therefore, the sum of the representation codes for each row of the table does not equal the total number of examples.

Table 4.12

Distribution of representations of function examples in each series

Series	Symbolic	Graphic	Numeric	Function Machine	Mapping Diagram	Verbal Description	Physical	Other
Glencoe								
Provided	7439	1617	2596	0	34	3888	3	3
Requested	2010	2540	2359	0	15	2319	19	16
Combined	863	48	1652	0	0	982	0	0
UCSMP								
Provided	4925	1486	1623	0	8	3526	30	0
Requested	1605	1369	1212	0	0	1795	8	16
Combined	691	108	1765	0	0	1831	1	0
CPMP								
Provided	2124	575	582	0	2	1723	32	0
Requested	1330	1148	758	0	0	986	22	9
Combined	493	54	873	0	1	1810	0	0

The numbers of function examples coded as each representation are provided in Table 4.12. Function representations were also coded as either provided, requested, or combined. They were coded as provided when a specific type of representation was given to students and there was no request for students to generate any part of that type of representation. They were coded as requested when students were asked to create a type of representation for the function and the textbook did not provide any part of that type of representation for the student. They were coded as combined when part of a certain kind of representation was provided and students were asked to add to it.

Of the seven codes, almost no examples used *Function Machine, Mapping Diagram*, or *Physical Representations*. There were no *Function Machine* representations in any of the series. Out of nearly 14,000 examples in the Glencoe series, only 34 *Mapping Diagrams* were presented and 15 such representations were requested of students. Almost all of these examples were split evenly between *Algebra 1* and *Algebra 2*. In over 10,000 examples in the UCSMP series, only 8 examples of *Mapping Diagrams* were provided, six in *FST* and two in *PCDM*. The CPMP series provided about 5,600 examples, of which two included *Mapping Diagrams* and one provided a partial diagram that students were requested to complete. All three examples occurred in *Course 3*. The CPMP series included the most examples with *Physical* representations, with 32 provided and 22 requested of students. The vast majority of these were in *Course 1*. The UCSMP series provided 30 examples with *Physical* representations, requested 8 such examples, and had one combined representation for students to complete. The Glencoe series had the fewest examples with *Physical* representations, with 3 provided and 19 requested representations in examples scattered among books in the series.

A small number of examples in each series were also coded as *Other*. The majority of these in all series were examples where students were requested to provide representations of the functions, and the type of representation was not specified. Therefore, these examples were not coded as one of the specific representations. The Glencoe series included several examples with representations provided that were either general descriptions that did not fit other codes or specific representations not anticipated by codes. For example, in *Algebra 1*, one function example is a solution to a system of inequalities. The inequalities are represented symbolically, but the function itself is not.

However, relatively few examples had *Function Machine*, *Mapping Diagram*, *Physical*, or *Other* representations. Therefore, further analysis focuses on the remaining representations: *Symbolic*, *Graphic*, *Numeric*, and *Verbal Representations*.

Symbolic Representations

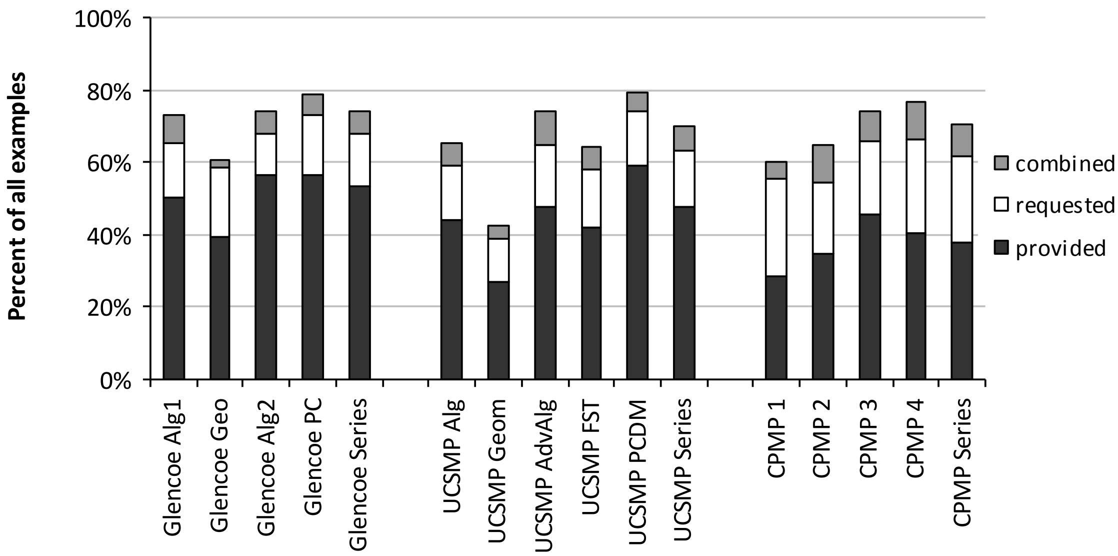


Figure 4.2: Function examples with *Symbolic* representation.

The proportions of function examples with *Symbolic* representation are provided in Figure 4.2. Examples of function more often involved *Symbolic* representations than any other type of representation in both the Glencoe and UCSMP series, and were second only to *Verbal Descriptions* in the CPMP series. In all three series, nearly three out of every four examples of function involved a *Symbolic* representation. In the Glencoe and UCSMP series, the majority of these examples had a representation provided and about one out of every five of the *Symbolic* examples requested that students provide a representation. In these two series, approximately one of every ten examples with a *Symbolic* representation provided only part of a *Symbolic* representation and asked students to complete the representation. In the CPMP series, slightly less than half of the examples involving *Symbolic* representations had them provided, about one third requested that students provided them, and about 12% requested that students complete them. These proportions between *Symbolic* representations being provided, requested, or a combination remained relatively constant across each series, with CPMP books consistently requesting students to produce or complete more *Symbolic* representations than Glencoe or UCSMP textbooks.

The examples in the *Geometry* textbooks in the Glencoe and UCSMP series tended to involve *Symbolic* representations less frequently. UCSMP *Geometry* was the only textbook with less than half of the examples involving a *Symbolic* representation, and Glencoe *Geometry* had only about 60%. Discounting these two books, the general trend throughout series was an increasing percent of examples to involve *Symbolic* representations. The *Precalculus* textbooks and CPMP *Course 4* each had the highest percent in their series. One exception to this trend was UCSMP *FST*, which only had more than the *Geometry* textbook in the series. The increasing trend was mainly a result of increased number of examples with *Symbolic* representations provided over the series. In contrast, the proportions of examples with *Symbolic* representations requested or combined remained moderately constant over the course of each series. Besides UCSMP *FST* and the *Geometry* textbooks, the exception to this pattern is CPMP *Course 4*, which has less provided for students than *Course 3*, but more requested of students, resulting in a net increase in the number of examples with a *Symbolic* representation involved. Overall, in each series, the books tend to represent most functions symbolically and do so increasingly more over the series.

The examples involving S*ymbolic* representations were coded in five categories: (a) *y = Expression*, (b*) Implicit y and x*, (c) *f*(*x*) = *Expression*, (d) *Recursive*, or (e) *Equation With Other Variables*. A representation was coded as (a) *y = Expression* if it was an *Algebra*ic expression in *x* set equal to *y*, (b) *Implicit y and x* if it was an *Algebra*ic equation with variables *y* and *x* such that one side of the equation is not only *y*, (c) *f*(*x*) = *Expression* if it was an *Algebra*ic expression set equal to the name of a function in *f*(*x*) notation, (d) *Recursive* if it was a rule was provided to obtain one correspondence or ordered pair of the function from preceding values, or (e) *Equation With Other Variables* if it was an *Algebra*ic equation with variables other than *y* and *x*. Any example that did not fit one of these categories but used numbers, letters as variables, and operation symbols to provide a formula to generate ordered pairs was coded as *Other*. The following graphs and discussion provide the distributions of each of these codes as a percent of the examples with *Symbolic* representations, rather than as a percent of all examples.

y = Expression. The proportions of function examples with *y = Expression* representation are provided in Figure 4.3. The Glencoe series included a higher portion of examples involving this representation than the UCSMP series, and the CPMP series included the smallest proportion. In Glencoe, when examples included a *Symbolic* representation, more than one out of every four were *y = Expression*. Almost all of these were provided for students, with very few requested of students and almost none combined. The UCSMP and CPMP series had similar distributions of examples with provided, requested, and combined *y = Expression* representations, but with smaller overall proportions of this representation.

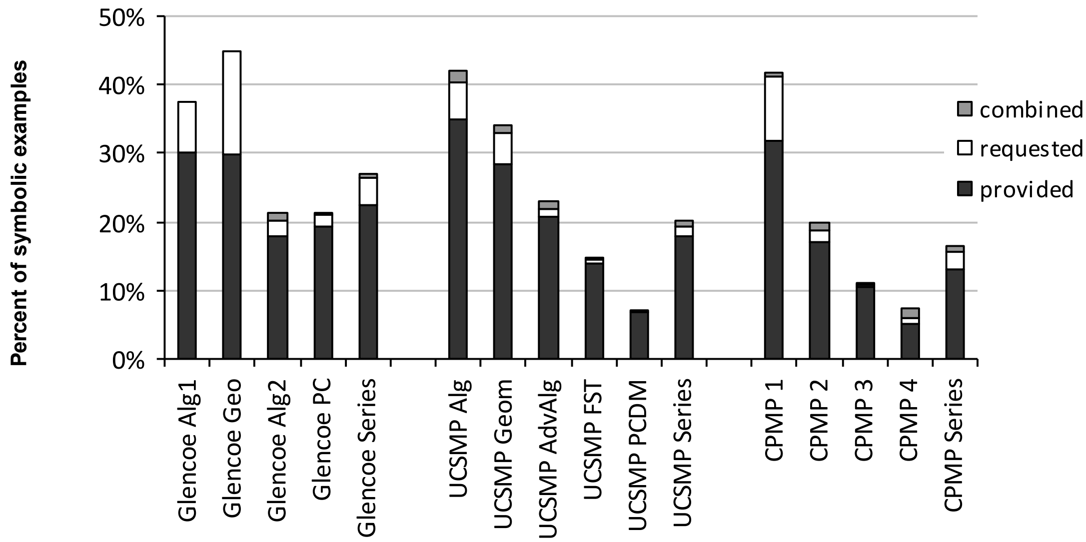


Figure 4.3: Function examples with *y = Expression* representation as a percent of functions with *Symbolic* representation.

Both the UCSMP and CPMP series had distinct downward trends in percent of *y = Expression* representations, so students would tend to see fewer *Symbolic* representations that were *y = Expression* over the course of the series. The Glencoe series was divided into two halves, with *Algebra 1* and *Geometry* including many more of these examples, and *Algebra 2* and *Precalculus* including fewer. This again suggests that students would tend to see fewer *Symbolic* representations that were *y = Expression* later in the series.

Implicit. The proportions of function examples with *Implicit y and x* representation are provided in Figure 4.4. As with the *y = Expression* representation, the Glencoe series included a higher portion of examples involving an *Implicit y and x* representation than the UCSMP and CPMP series, with a percent more than twice as large. For all three series, almost all of these representations were provided for students, with very few requested of students.

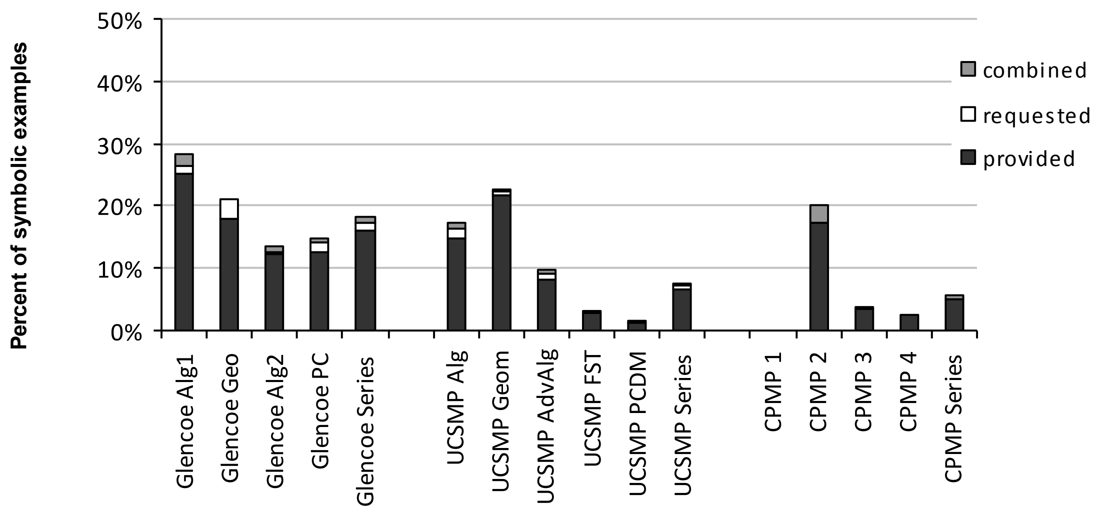


Figure 4.4: Function examples with *Implicit y and x* representation as a percent of functions with *Symbolic* representation.

The general trend within each series was smaller percents of *Implicit y and x* representations later in the series. In the Glencoe series, the *Algebra 2* and *Precalculus* textbooks had the smallest percents, with *Precalculus* slightly higher. In the UCSMP series, the *Geometry* textbook did have a higher portion than *Algebra*, but the rest of the series showed a downward trend. In the CPMP series, *Course 1* had almost no examples with *Implicit y and x* representations, *Course 2* had the most, and percents declined over the rest of the series. These trends suggest that students would tend to see fewer *Symbolic* representations that were *Implicit y and x* later in the series.

f(x) = Expression. Overall, the three series had similar distributions of examples with the $f(x)$ =*Expression* representation, with about one out of every four *Symbolic* examples including this representation (see Figure 4.5). Almost all of these examples had the representation provided. However, in the Glencoe series, more $f(x)$ *=Expression* representations were requested than combined, and in the other two series, combined representations occurred more frequently than requested.

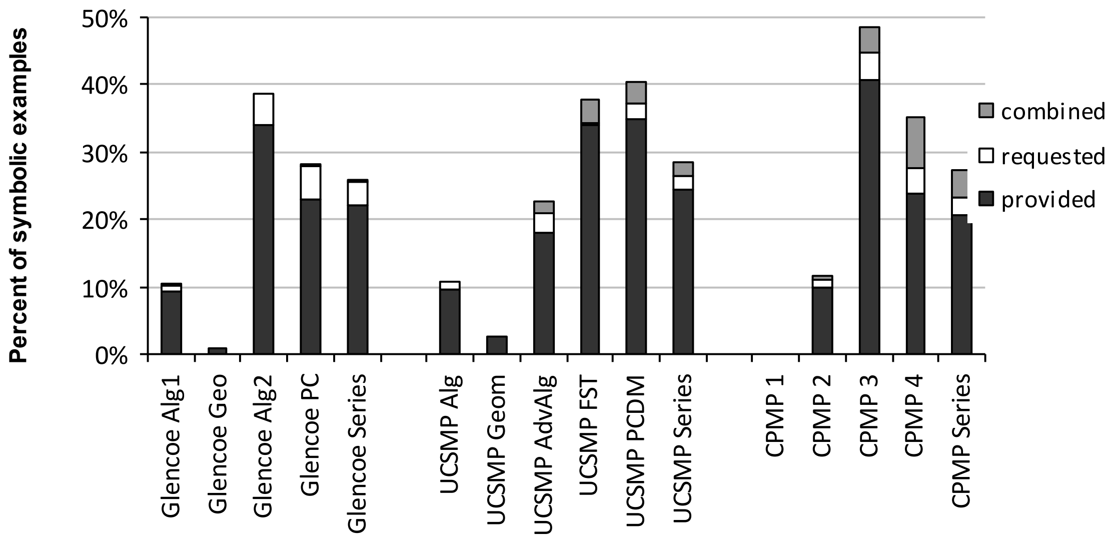


Figure 4.5: Function examples with $f(x)$ *=Expression* representation as a percent of functions with *Symbolic* representation.

In all series, the first two books contained far fewer examples with $f(x)$ *=Expression* representations than the later textbooks. In the Glencoe and UCSMP series, the *Geometry* textbooks had almost no examples of this type. However, in Glencoe, *Algebra 2* had the highest proportion, whereas in UCSMP, *PCDM* had the most. In CPMP, *Course 1* had virtually no examples of this type, and *Course 3* had the highest proportion.

Recursive. The proportions of function examples with *Recursive* representation are provided in Figure 4.6. Relatively few examples in any series included *Recursive* representations. The Glencoe series had only 64 such examples. The UCSMP series had about 250 *Recursive* examples, of which about two thirds were provided for students. The

CPMP series included a slightly higher number of examples with this representation, which resulted in the percent out of *Symbolic* examples being almost twice as large as UCSMP. In CPMP, almost five out of every six examples including a *Recursive* representation requested students to generate the representation. In CPMP, the majority of these examples were in *Course 1*, although *Course 3* also included just over 10%.

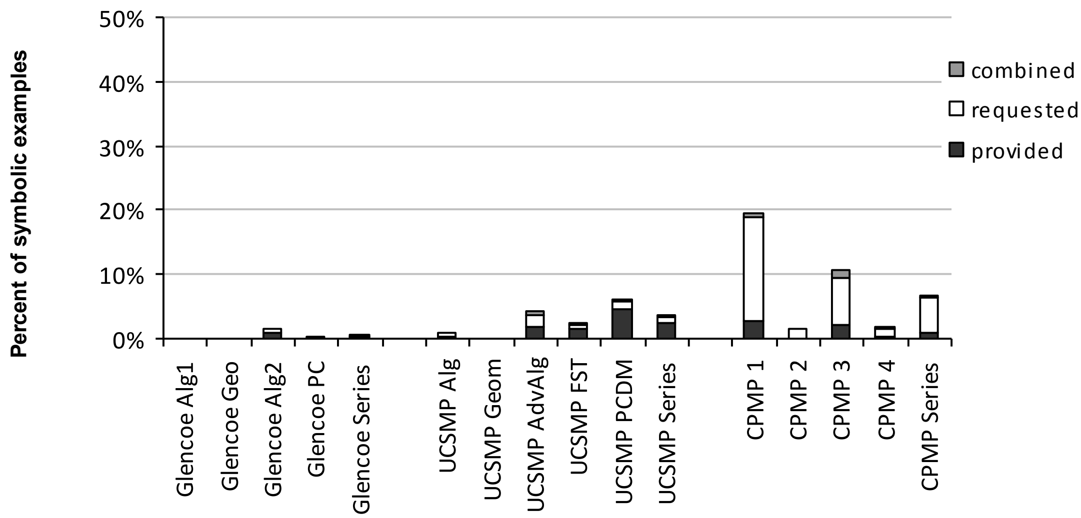


Figure 4.6: Function examples with *Recursive* representation as a percent of functions with *Symbolic* representation.

Equation With Other Variables. The proportions of function examples with *Equation With Other Variables* representation are provided in Figure 4.7. All series had a similar proportion of this representation, with about 20% of all *Symbolic* representations being *Equation With Other Variables*. In the Glencoe series, these were almost all provided for students. The UCSMP series had a slightly higher proportion of requests for students to generate or complete this type of representation, and the CPMP series had the highest proportion of requests for students to generate *Equation With Other Variables* representations.

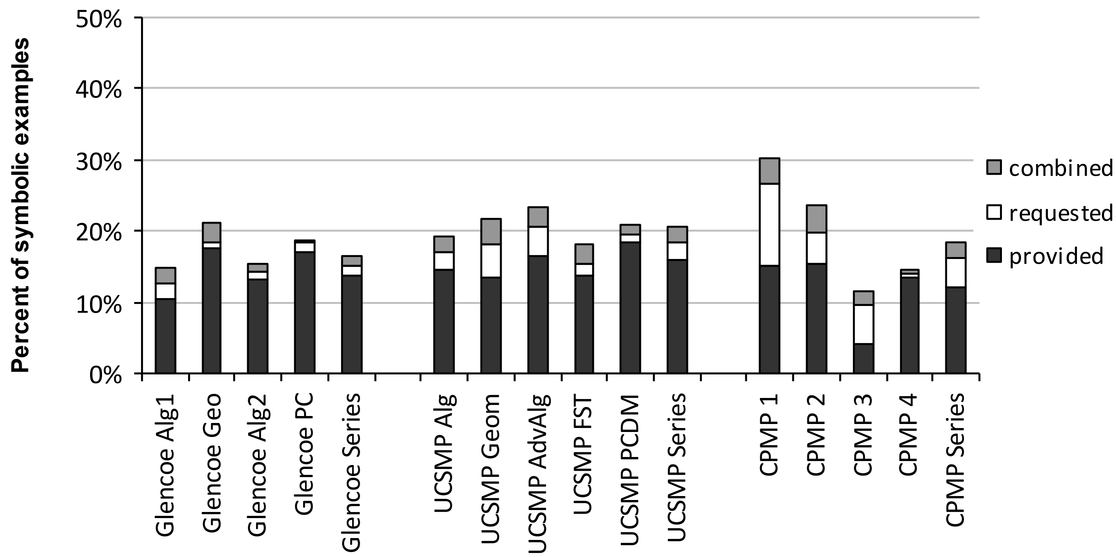


Figure 4.7: Function examples with *Equation With Other Variables* representation as a percent of functions with *Symbolic* representation.

Throughout the Glencoe series, proportions of this representation remained relatively constant, with *Geometry* including a slightly higher proportion than the other books. Similarly, in the UCSMP series, overall proportions remained relatively constant; however, the proportion of requests for the *Equation With Other Variables* representation declined after *Algebra 2*. Each textbook in the CPMP series was different with respect to this representation. *Course 1* had the highest overall proportion, boosted by a large number of requests for students to generate the *Equation With Other Variables* representation. *Course 2* had fewer requests for the representation, and *Course 3* had far fewer examples with this representation provided. However, *Course 4* had almost the same proportion provided as the first two books, but almost no requests for students to generate or complete *Equation With Other Variables*.

Other Symbolic representations. The proportions of function examples with *Symbolic: Other* representation are provided in Figure 4.8. Each series included a relatively large proportion of such representations. The Glencoe series had about 20%, the UCSMP series had almost 30%, and the CPMP series had over 35%, with almost half of the *Symbolic* codes in *Course 4* including the *Other* code. In each series, the majority of these representations were requested from students. Almost all of these requests were coded as *Other* because the request did not specify the form that students should provide. Such requests were for equations, systems of equations, functions, formulas, rules, models, asymptotes, directrixes, inverses of functions, or derivatives. Because students could provide a number of *Symbolic* representations to meet these requests, it was impossible to code these requests as one of the other *Symbolic* codes.

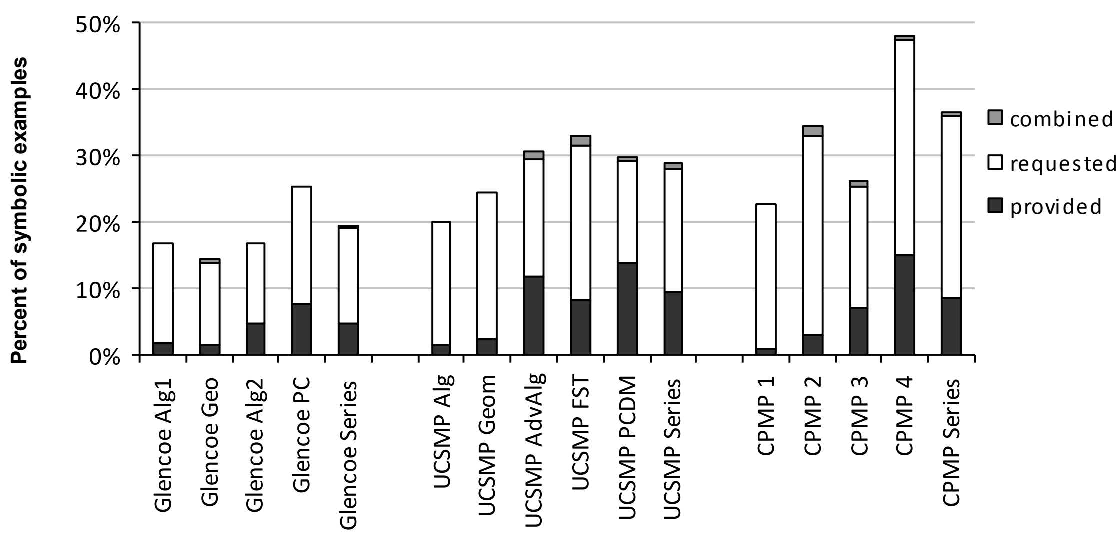


Figure 4.8: Function examples with *Other* representation as a percent of functions with *Symbolic* representation.

A smaller proportion of these representations were provided for students. These representations included equations with words, sigma notation, matrices, equations using vectors, and integral notation. However, the largest proportion of these was general symbolic notation. For example, a textbook may have defined functions f and g with equations, but then referred to the new function $f + g$ without providing an equation for it.

Summary of Symbolic representations. Symbolic representations were included more frequently than any other type of representation in the Glencoe and UCSMP series, and in every 7 out of 10 examples in the CPMP series. A majority of these representations were provided for students. The *Geometry* textbooks in Glencoe and UCSMP series and UCSMP *FST* included fewer *Symbolic* representations, but besides these books, the general trend was for an increasing proportion of symbolic representations over the course of each series.

In all the series, the *Symbolic* representations initially tended to be *y = Expression*, but later the $f(x)$ =*Expression* became more predominant. The proportion of *Implicit* representations also tended to decrease later in each series. *Equation With Other Variables* remained relatively stable through the Glencoe and UCSMP series, but was quite varied in the CPMP series. Each of these representations were predominantly provided for students, whereas the *Recursive* and *Other* were mainly requested of students. Very few textbooks included many *Recursive* representations, except for CPMP *Course 1* and *Course 3*. Many more *Other* representations were included, and these mainly consisted of requests for students to provide equations without specifying the format of the equation.

Graphic Representations

The proportions of function examples with *Graphic* representation are provided in Figure 4.9. In all three series, examples of function involved *Graphic* representations less than *Symbolic*, *Numeric*, or *Verbal Description*, with only about 30% of all examples involving such a representation. In both the Glencoe and CPMP series, students were more often requested to generate *Graphic* representations than provided representations, and there were very few examples in which students were to complete these representations. In the UCSMP series, the balance between provided and requested *Graphic* representations was nearly even, with about 1% more provided over the entire series. *PCDM* had the greatest difference, with about 4% more provided than requested.

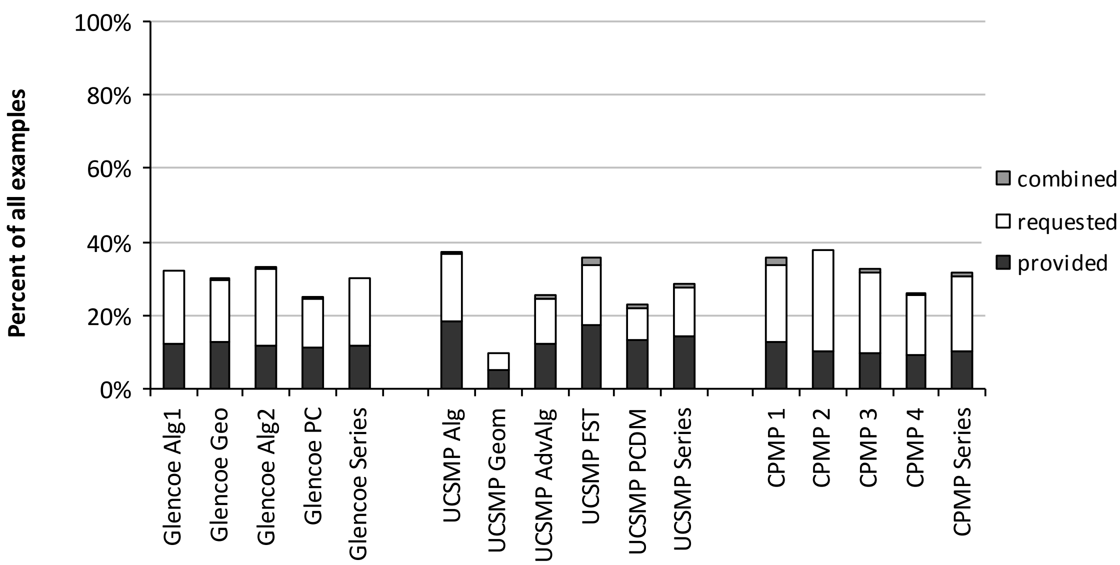


Figure 4.9: Function examples with *Graphic* representation.

Throughout both the Glencoe and CPMP series, the proportions noted above remained fairly consistent. In both series, however, the final textbook in the series had a smaller proportion of examples with *Graphic* representations. In the UCSMP series, the split between provided and requested *Graphic* representations remained fairly even, but the overall proportion was quite different between textbooks. *Algebra* and *FST* included the highest proportion, while *Geometry* had only 26 examples with *Graphic* representations included.

The examples involving *Graphic* representations were coded in three categories: (a) *Continuous*, (b) *Smooth*, or (c) *Scatterplot*. A representation was coded as (a) *Continuous* if it was a single unbroken curve without holes or jumps, (b) *Smooth* if it was a graph with no pointed corners, or (c) *Scatterplot* if it was a plot of individual ordered pairs. Any example that did not fit one of these categories but used points or curves in a Cartesian coordinate plane (or corresponding visual representation of higher dimensions) to display some or all of the elements of the domain of the function with their corresponding elements in the range was coded as *Other*. The following graphs and discussion provide the distributions of each of these codes as a percent of the examples with *Graphic* representations, rather than as a percent of all examples.

Continuous and Smooth. The distributions of examples with *Continuous* and *Smooth* representations were nearly identical, so they are examined together (see Figure 4.10 and Figure 4.11). In all series, over 75% of the examples with *Graphic* representations were *Continuous* and *Smooth*. In the Glencoe and CPMP series, the majority of these were requested of students and very few were combined. In the UCSMP series, there were more *Continuous* and *Smooth* representations provided than requested of students, but also had very few combined.

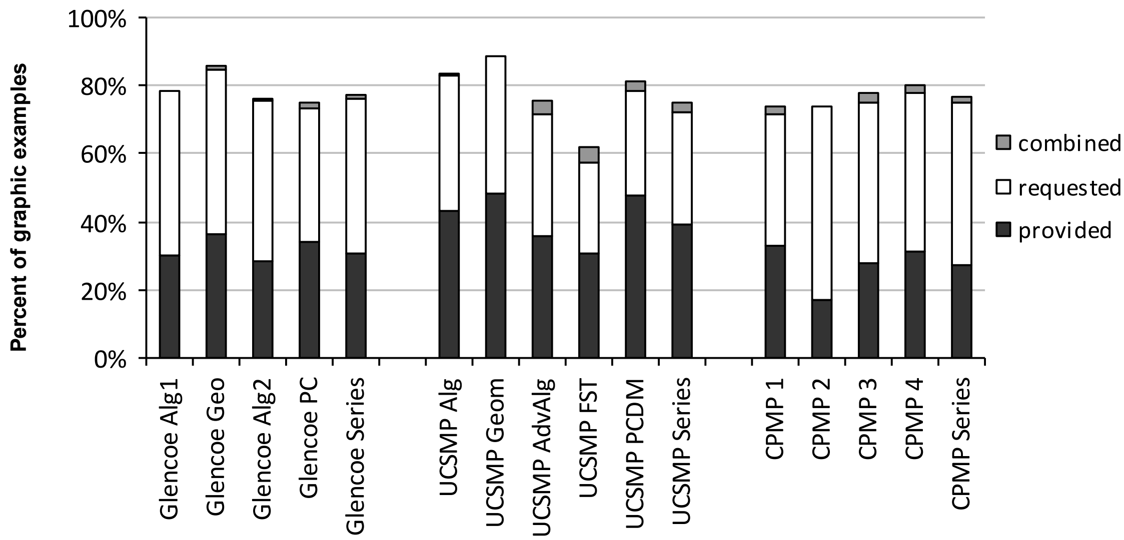


Figure 4.10: Function examples with *Continuous* representation as a percent of functions with *Graphic* representation.

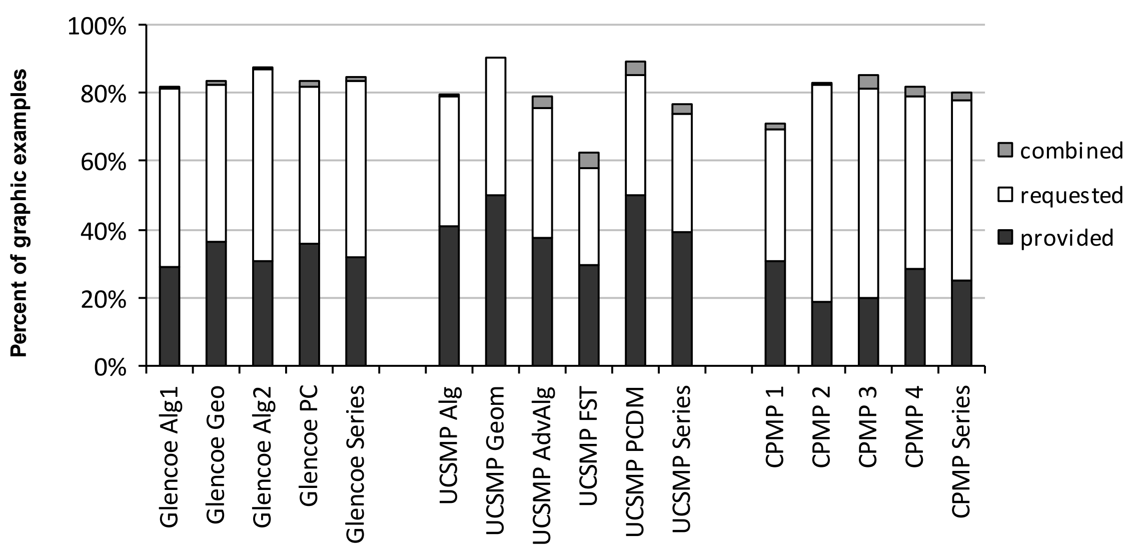


Figure 4.11: Function examples with *Smooth* representation as a percent of functions with *Graphic* representation.

In the Glencoe and CPMP series, the proportions noted above remained fairly consistent across textbooks. However, in Glencoe *Algebra 2* and *Precalculus*, there were slightly fewer requests for *Continuous* representations than the first two textbooks in the series as well as when compared to *Smooth* representations. Also, in the CPMP series, *Course 1* included fewer requests for *Continuous* and *Smooth* representations. Across the UCSMP series, the relative proportion of provided and requested representations remained fairly constant; however, the total proportion of these representations varied. Of all *Graphic* representations in the series, *FST* had the smallest proportion of *Continuous* and *Smooth* representations, with only about 60%. In contrast, *Geometry* had the largest proportion, with about 90%. The other three textbooks in the UCSMP series each had about 80%, with the exception of *Smooth* representations in *PCDM*, which occurred in almost 90% of *Graphic* representations in the book.

Scatterplot. Relatively few *Graphic* examples included *Scatterplot* representations (see Figure 4.12). In the Glencoe series, only about 7% included this type of representation, and in the UCSMP and CPMP series, only about 12%. The majority of these examples requested that students provide the *Scatterplot* representation, with almost no requests for combined representations. This trend was the most extreme in UCSMP *Geometry*, which had only requests for *Scatterplot* representations, and CPMP *Course 1*, in which there were 10 times as many requests as provided representations. UCSMP *PCDM* had very few requests, which resulted in it being the only book to contain more presented *Scatterplot* representations than requests. In the rest of the textbooks in each series, the distribution of provided, requested, and combined representations was similar to the overall distribution for the series.

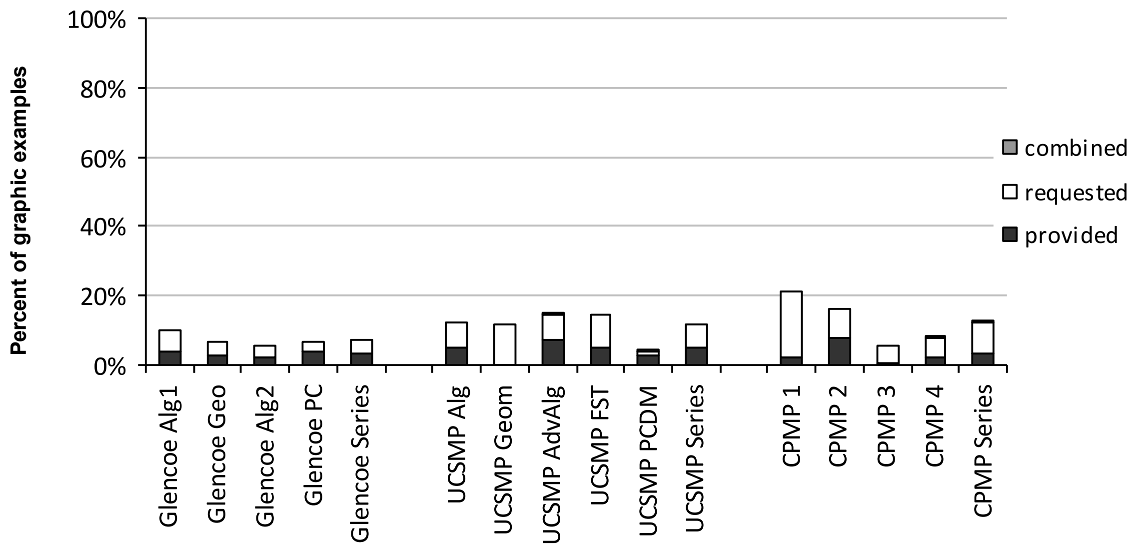


Figure 4.12: Function examples with *Scatterplot* representation as a percent of functions with *Graphic* representation.

Other. There were very few examples with *Graphic* representations coded as *Other* (see Figure 4.13). Of these examples, most provided representations were connected to sets of data, such as histograms, dot plots, bar graphs, or circle graphs. While these types of graphs alone were not coded as examples of functions, when these graphs were connected to an example of function that included another common representation of function, they were coded as *Graphic: Other*. For example, when a table of data was provided which indicated a function's pairs of data and students were asked to create a histogram, this was identified as an example of function because of the table, and the example would receive a code for both the table and the histogram. Because of the statistics component of UCSMP *FST*, this textbook included many examples where students were provided or requested to produce histograms, and thus this textbook has a higher proportion of *Other* representations.

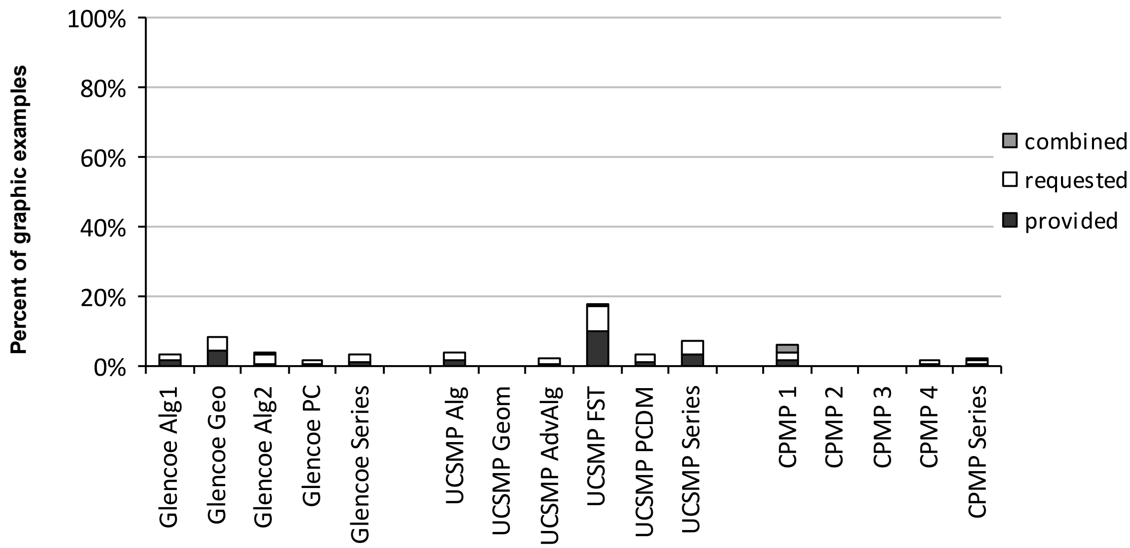


Figure 4.13: Function examples with *Other* representation as a percent of functions with *Graphic* representation.

Some provided *Other* representations were not graphs of data, but instead were G*raphic* representations that were not *Continuous*, *Smooth* or *Scatterplots*. For an example, see the graph of the function in Figure 4.14.

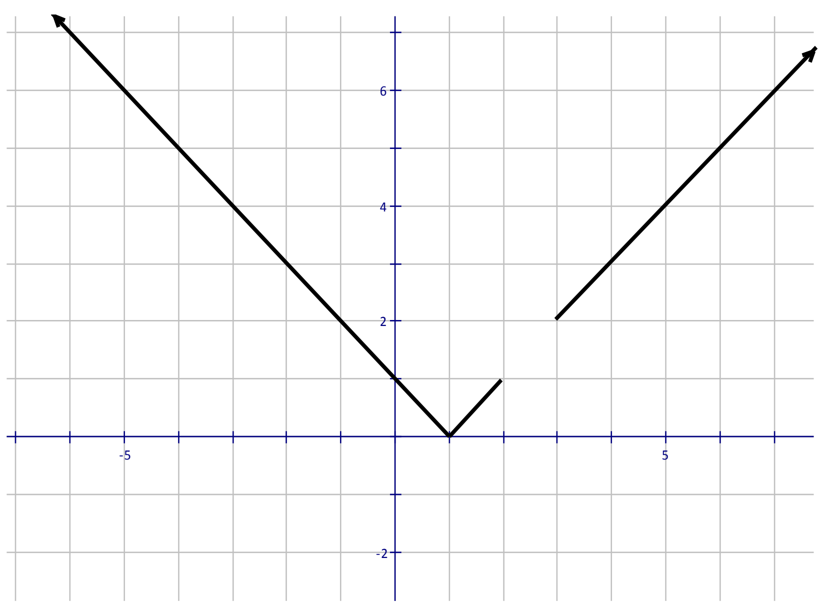


Figure 4.14: A *Graphic* representation that is not *Continuous, Smooth,* or a *Scatterplot.*

The most common requested *Graphic* representation coded as *Other* was a general request for a graph that allowed many possibilities, so that the example could not be coded as *Continuous*, *Smooth*, or *Scatterplot*. There were also some requests for students to create a histogram, dot plot, bar graph, or circle graph. Most examples where students were asked to complete an *Other* representation were of this type. A few requests for representations also involved functions such as the one represented in Figure 4.14.

Summary of Graphic representations. Across all series, *Graphic* representations were less common than *Symbolic*, *Numeric*, or *Verbal Descriptions*. When included, *Graphic* representations were most often requested of students, with this occurring somewhat more frequently in the Glencoe and CPMP series than the UCSMP series. A common type of example that included requested *Graphic* representations was the provision of a *Symbolic* representation and request that a student graph it. Of all the *Graphic* representations, most were both *Continuous* and *Smooth*. Very few *Scatterplot* representations were included, although these were slightly more common in the UCSMP and CPMP series. The *Other* representations tended to be graphs such as histograms or general requests for graphs.

Numeric Representations

The proportions of function examples with *Numeric* representation are provided in Figure 4.15. Examples of function involved *Numeric* representations more often than *Graphic*, but less frequently than *Symbolic* and *Verbal Descriptions*. Overall, in all three series, about 40% of the examples of function involved a *Numeric* representation, with the CPMP series having the least. The distribution of representations provided, requested, and combined was split roughly evenly in all three series, although Glencoe had a slightly higher percent presented, while combined was the highest percent for the UCSMP and CPMP series. The CPMP series had the smallest proportion of provided representations.

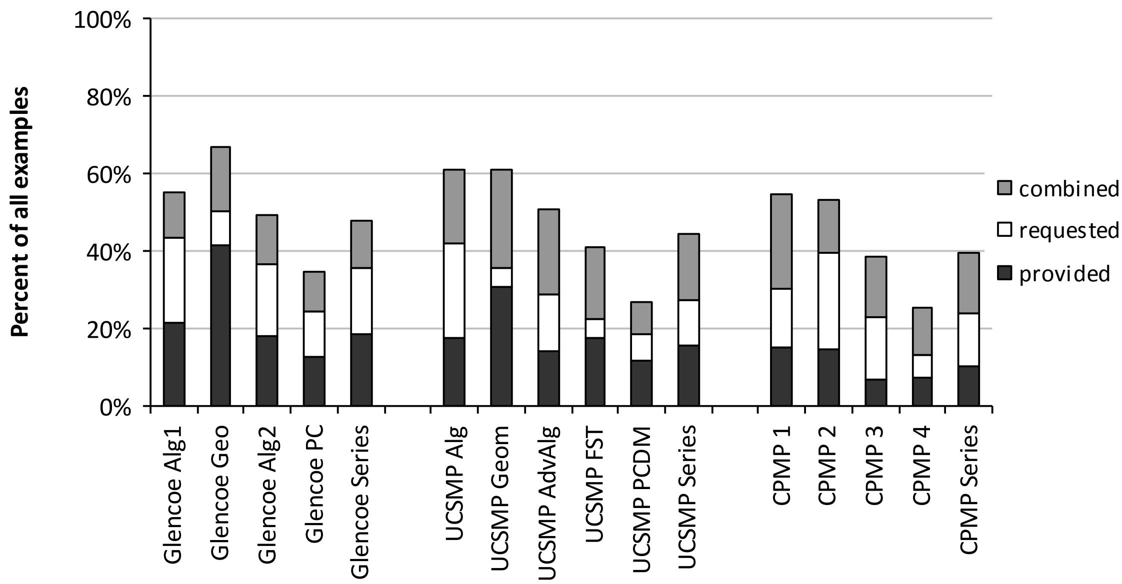


Figure 4.15: Function examples with *Numeric* representation.

One trend was a smaller proportion of *Numeric* representations over the course of each series. The only exception was Glencoe *Algebra 1*, which had a slightly smaller proportion than *Geometry*. Also, the distribution of provided, requested, and combined representations did not remain constant over each series. In the Glencoe series, all of the books were balanced among these except *Geometry*, which had a higher proportion of provided *Numeric* representations. UCSMP *Geometry* also had a higher proportion of provided as well as combined representations and very few requested. *FST* followed a similar pattern. The remaining UCSMP textbooks had a roughly even split between provided, requested, and combined. The only CPMP textbook with such a balance was

Course 1. *Course 2* had a higher proportion of requested, *Course 3* had few provided, and *Course 4* had few requested *Numeric* representations.

The examples involving *Numeric* representations were coded in three categories: *Table*, *Ordered Pair*, or $f(x)$ *Notation*. A representation was coded as *Table* if paired elements of the function were displayed in a table, as *Ordered Pair* if elements of the domain and range of the function were displayed in ordered pairs, or as $f(x)$ *Notation* if paired elements of the function were displayed in $f(x)$ notation. Any example that did not fit one of these categories but was a listing of some or all of the elements of the domain of the function with their associated elements in the range was coded as *Other*. The following graphs and discussion provide the distributions of each of these codes as a percent of the examples with *Numeric* representations, rather than as a percent of all examples.

Table. The proportions of function examples with *Table* representation are provided in Figure 4.16. The distribution of *Table* representations was similar in the Glencoe and UCSMP series, but quite different in the CPMP series. In *Numeric* representations in both the Glencoe and UCSMP series, about 20% of them were coded as *Table*. The majority of these were provided for students. In contrast, almost 40% of the *Numeric* representations in the CPMP series were coded as *Table*, and there was a rough balance among provided, requested, and combined representations. Thus, the CPMP series provided about the same proportion of *Table*, but included a larger proportion of requested and combined *Table* representations.

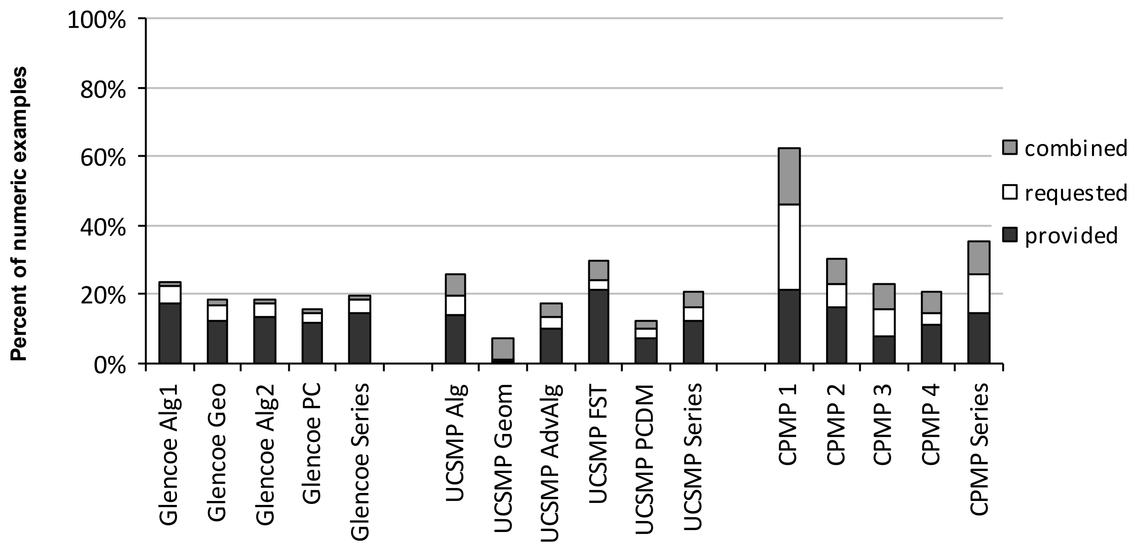


Figure 4.16: Function examples with *Table* representation as a percent of functions with *Numeric* representation.

Each series had different distributions across its textbooks. The Glencoe series was fairly stable over the textbooks, with a large portion of *Table* representations provided and only a small decrease in the total proportion over the series. The CPMP series also showed a decline in the proportion of *Table* representations over the series, but *Course 1* had over 60%, which was more than twice as many as any other textbook. The proportion of requested *Table* representations was highest in *Course 1* and *Course 3*, but much smaller in *Course 4*. The relative proportion of provided, requested, and combined representations in the UCSMP series remained roughly constant, except in *Geometry*, which was almost all combined. The rest of the textbooks in the series were dominated by provided representations; however, the total proportion of *Table* representations varied. *Algebra 1* and *FST* included the most, while *Geometry* and *PCDM* had the least. Thus, the only consistent pattern across series, with the exception of CPMP *Course 1*, was the relatively low occurrence of *Table* representations.

Ordered Pair. The proportions of function examples with *Ordered Pair* representation are provided in Figure 4.17. The Glencoe series had the highest proportion of *Ordered Pair* representations, with about 40% of all *Numeric* representations being of this type, in contrast to less than 25% in the UCSMP series and just over 20% in the CPMP series. In the Glencoe and CPMP series, *Ordered Pair* representations were requested a majority of the time. In the UCSMP series, the proportion of requested was slightly lower than that of provided. All series had a very small proportion of combined *Ordered Pair* representations.

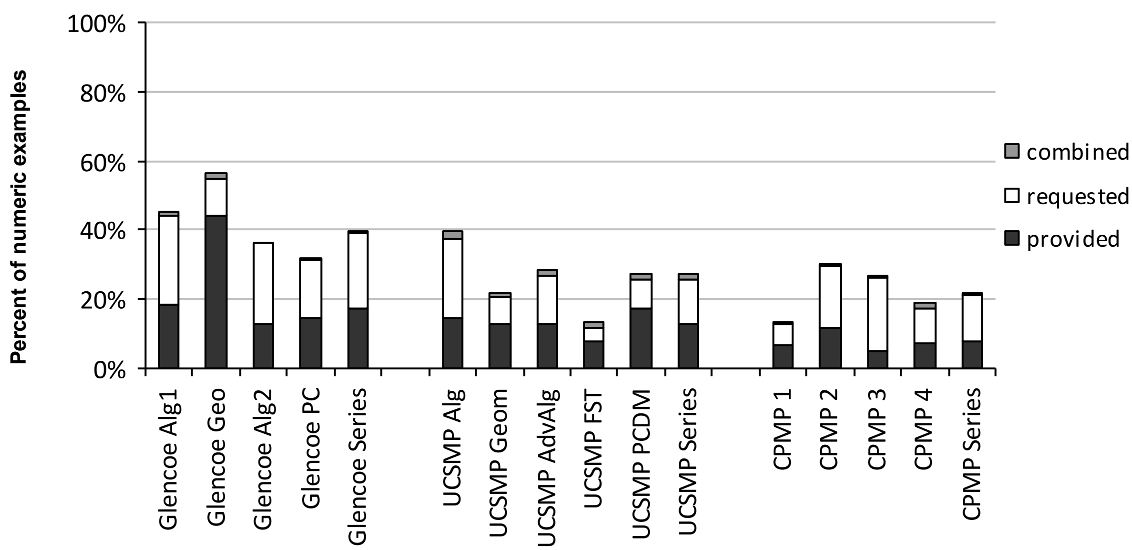


Figure 4.17: Function examples with *Ordered Pair* representation as a percent of functions with *Numeric* representation.

There was a general trend of decreasing proportions of *Ordered Pair* representations over the course of each series, although there were exceptions in each series. In the Glencoe series, *Geometry* actually had the highest proportion because of a large amount of provided *Ordered Pair* representations. The rest of the textbooks in the Glencoe series had fairly consistent proportions of provided representations, with slightly decreasing proportions of requested representations. In the CPMP series, *Course 1* did not follow this pattern, with the smallest proportion of *Ordered Pair* representations of any textbook in the series. In USCSMP, both *Geometry* and *FST* did not follow the trend of decreasing *Ordered Pair* representations over the series by having the two lowest proportions in the series. However, both of these books matched the trend in the series to provide more *Ordered Pair* representations than request them.

f(x) Notation. Overall, a very small proportion of *Numeric* representations were *f*(*x*) *Notation*, and there were almost no requests for these representations (see Figure 4.18). The Glencoe series had very few such representations in any textbook. In the CPMP series, only *Course 4* had more than 7% *f*(*x*) *Notation* representations, and almost all of these were combined.

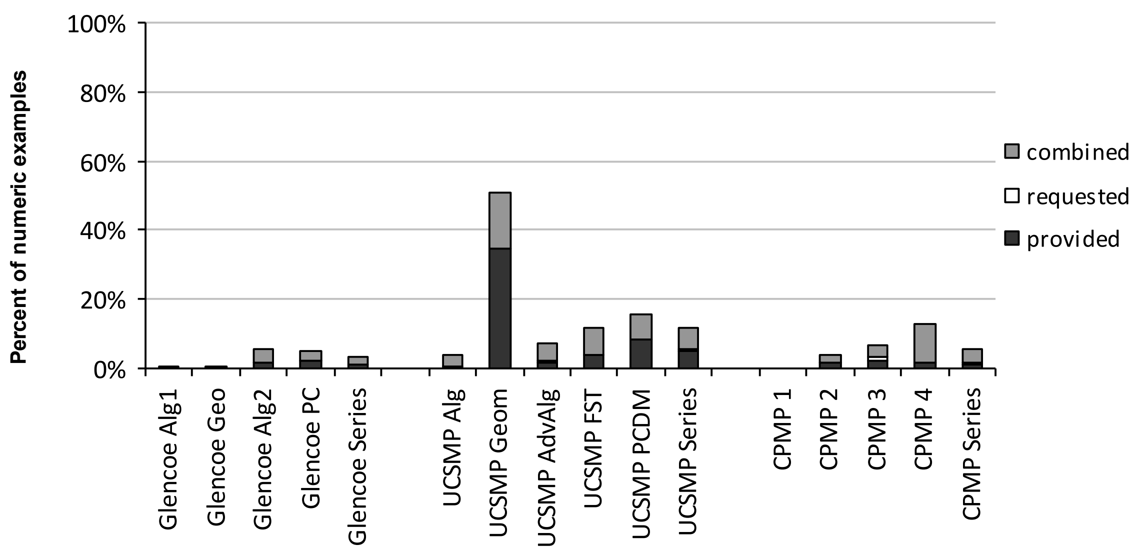


Figure 4.18: Function examples with *f(x) Notation* representation as a percent of functions with *Numeric* representation.

The UCSMP series had a higher proportion of *f*(*x*) *Notation* representations, due to a large proportion in *Geometry*. About 35% of all *Numeric* representations in

Geometry were provided *f*(*x*) *Notation*, and an additional 16% were combined, far higher than any other textbook. In the series, besides *Geometry*, there was a general trend of increasing total proportion, ending with just over 10% in *PCDM*. However, besides UCSMP *Geometry*, the proportion of *f*(*x*) *Notation* representations was small in every textbook.

Other. A relatively large proportion of *Numeric* representations were coded as *Other*, with over 40% in the Glencoe series and approximately half in the UCSMP and CPMP series (see Figure 4.19). Most were combined representations, and the general trend in the Glencoe and CPMP series was increasing proportions of *Other* representations later in the series, although Glencoe *Geometry* had a smaller proportion than *Algebra 1* because it had almost no requested *Other* representations. The UCSMP *Geometry* had by far the fewest *Other* representations, of which almost all were combined. The proportion in the UCSMP series peaked in *Advanced Algebra*.

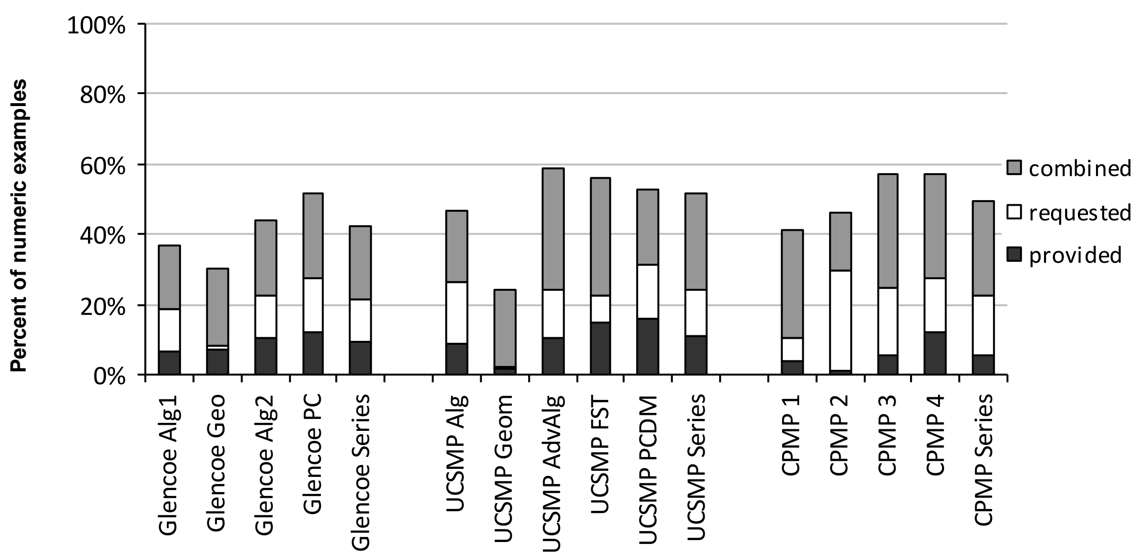


Figure 4.19: Function examples with *Other* representation as a percent of functions with *Numeric* representation.

The *Other* representations that were provided for students tended to be corresponding elements from the domain and range of the function that were not given as a table, ordered pairs, or with $f(x)$ notation. These elements were often given verbally, but were not coded as *Verbal Description* because the words were not used to describe the function, but to indicate the pairs of elements that were representing the function. There were more requests for students to generate *Other* representations than provided. These requests were often not specific about the format and thus could not be coded as *Table*, *Ordered Pair*, or $f(x)$ *Notation*. The other common type of request for *Other* representations was a request for an *x*- or *y-intercept*. The most common *Other* representation was combined, and most examples coded this way provided an element in the domain and asked students to provide the corresponding element in the range, but did not use one of the formats matching the other codes.

Summary of Numeric representations. Numeric representations were somewhat common, but less so over the course of each series and overall included in less than half of the function examples in each series. Provided, requested, and combined representations were present in roughly equal amounts overall in each series, but not within each textbook. Glencoe *Geometry*, UCSMP *Geometry* and *FST*, and CPMP *Course 4* each had relatively small proportions of requested *Numeric* representations.

These representations were also coded as *Table*, *Ordered Pair*, $f(x)$ *Notation*, or *Other*. The first three of these made up a relatively small proportion of all *Numeric* representations, with a few exceptions. Over 60% of the *Numeric* representations in CPMP *Course 1* were *Table* representations. Around 50% in Glencoe *Algebra 1* and *Geometry* were *Ordered Pair*. UCSMP was the only textbook with a large amount of $f(x)$

Notation representations with over 50%. All the series did have a larger portion of *Other* representations. These were mainly combined representations providing an element of the domain for students and requesting that they provide an element of the range, but not using or specifically requesting one of the coded formats.

Verbal Description Representations

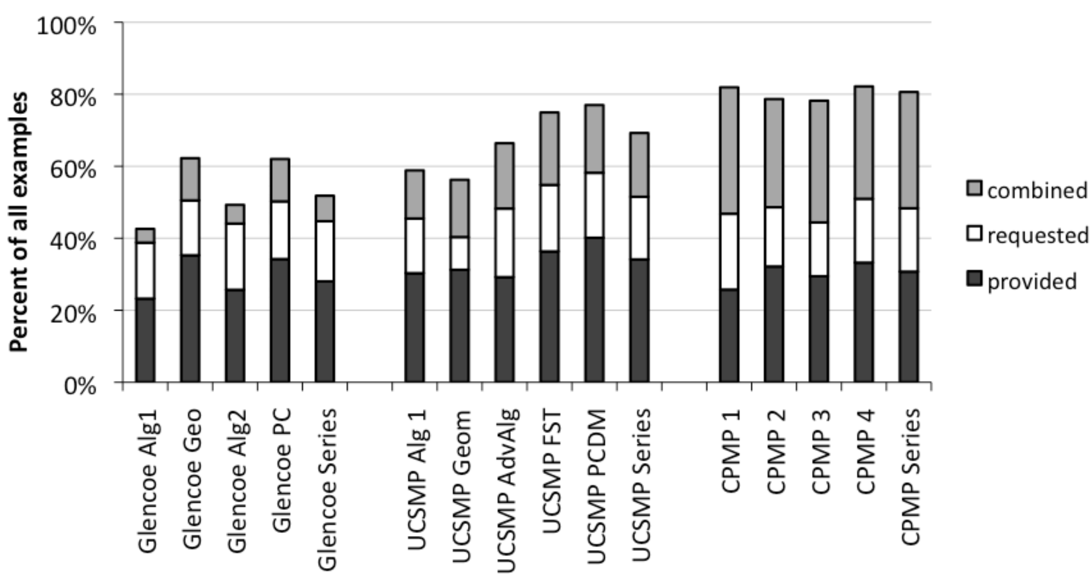


Figure 4.20: Function examples with *Verbal Description* representation.

The proportions of function examples with *Verbal Description* representation are provided in Figure 4.20. In the CPMP series, more examples included *Verbal Description* representation than any other type, and in the Glencoe and UCSMP series, this representation was second only to *Symbolic*. In the Glencoe series, just over half of all examples included *Verbal Description*, with over 25% of all examples having such a representation provided. Fewer *Verbal Descriptions* were requested, and only about 7% were combined. The UCSMP series had similar proportions of provided, requested, and combined representations, although the requested and combined were closer to equal proportions. Of all examples in the series, nearly 70% included a *Verbal Description*. In the CPMP series, over 80% of all examples included this representation, with about 30% each provided and combined. The requested *Verbal Descriptions* made up the smallest proportion, with only about 18%.

Somewhat different patterns were present across each of the series. Over the CPMP series, the proportion of provided, requested, and combined *Verbal Description* representations remained consistent in each textbook. Over the UCSMP series, the overall proportion of these representations grew. Requested and combined representations remained relatively consistent, and the proportion of provided *Verbal Descriptions* was higher in the later books. The exception to these patterns was *Geometry*, which had a slightly smaller proportion than *Algebra* due to fewer requested representations. In the Glencoe series, the two *Algebra* textbooks had smaller proportions of *Verbal Descriptions*, due to fewer provided and combined representations. *Geometry* and *Precalculus* had nearly identical proportions of these representations. Overall, each series included *Verbal Descriptions* with a large proportion of function examples, with a tendency to include more provided representations in the Glencoe and UCSMP series, and more provided and combined representations in the CPMP series.

Multiple Representations

The proportions of function examples with multiple representations are provided in Table 4.13. For each series, over 75% of the examples included multiple representation, with CPMP having the highest percent at just over 80%. Only a small portion of these examples had multiple representations only provided, and an even smaller portion had multiple representations only requested. The majority of examples had some representations provided for students and requested students to provide others, so these were designated as combined representations. The CPMP series had the highest percent of examples with combined representations, nearly 75%. The Glencoe and UCSMP each had about 62% of examples with combined representations. All three series had very few examples with only requested representations; all were less than half of one percent. The UCSMP series had the highest percent of examples with only provided representations, with just over 15%. The CPMP series had less than half this percent, with about 6% of all examples having multiple representations only provided.

Table 4.13

Number of function examples in which students engage with multiple representations in each textbook

Textbook	Total number of function examples	Percent of examples with multiple representations (p/r/c*)
Glencoe		
Algebra 1	3937	76.3% (12.3/0.9/63.1)
Geometry	768	77.3% (17.8/0/59.5)
Algebra 2	5124	79.0% (14.0/0.1/64.9)
Precalculus	4046	75.3% (15.7/0/59.6)
Series Total	13875	77.0% (14.2/0.3/62.5)
UCSMP		
Algebra	2029	82.6% (14.9/0.2/67.4)
Geometry	528	54.9% (12.3/0/42.6)
Advanced Algebra	2595	78.1% (10.7/0.7/66.7)
FST	2382	79.6% (14.9/0/64.6)
PCDM	2796	78.6% (20.4/0/58.2)
Series Total	10330	78.3% (15.2/0.2/62.8)
CPMP		
Course 1	1145	80.3% (6.5/0.4/73.4)
Course 2	1125	83.2% (5.6/0/77.6)
Course 3	1124	80.0% (7.5/0/72.5)
Course 4	2211	80.0% (6.1/0/73.9)
Series Total	5605	80.7% (6.3/0.1/74.3)

* p = provided, r = requested, c = combined

Textbooks within both the Glencoe and CPMP series had nearly the same percent of examples with multiple representations throughout the series. Each of the Glencoe books had about 75% and each of the CPMP books had about 80%. For the CPMP series, the distribution of provided, requested, and combined representations also remained fairly constant with about 6%, 0%, and 74% respectively. There was a similar pattern in the Glencoe series, with all books except *Geometry* containing about 14% provided, 0% requested, and 62% combined. The *Geometry* book had a higher percent only provided, with almost 18% provided, and thus less than 60% combined.

The UCSMP series had a greater variety in its distribution. *Algebra* had the highest percent of examples with multiple representations, with about 82%. It also had the highest percent of examples with combined representations, with about 67%. In contrast, the *Geometry* book had the lowest in both of these areas, both in the series and among all textbooks, with only about 55% of examples including multiple representations and about 43% of examples having a combination of provided and requested representations. *Advanced Algebra*, *FST*, and *PCDM* each had around 78% of examples that included multiple representations. The percent of examples with only provided representations increased from around 10% to around 20% over the three books, while the percent with combined representations decreased from around 66% to around 58%.

Comparison of Function Representations to Research

Many researchers have recommended that students experience a wide variety of representations to promote a more flexible and robust understanding of function. (Akkoc & Tall, 2005; Carlson, et al., 2002; DeMarios & Tall, 1999; Eisenberg, 1991, 1992; Kaput, 1992; Leinhardt, et al., 1990; Monk, 1992; National Council of Teachers of

Mathematics, 1989, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011; Oehrtman, et al., 2008; Schwingendorf, et al., 1992; Selden & Selden, 1992; Sfard, 1992; Tall, 1996; Tall, et al., 2000). Recommended representations include symbolic, graphic, numeric, physical, verbal descriptions, mapping diagrams, and function machines. The textbooks in all of the series tended to use *Symbolic* representations and *Verbal Descriptions* most heavily. A smaller proportion of *Graphic* and *Numeric* representations were also used, but very few *Physical* representations, *Mapping Diagrams*, and *Function Machines* were used. The proportions were similar among all three series, with no series clearly including a wider variety of representations.

Some researchers have pointed out a tendency of students to prefer *Symbolic* representations or even believe functions must have *Symbolic* representations and that these formulas actually are the functions (Asiala, et al., 1997; Dahlberg & Housman, 1997; Gonzalez-Martin & Camacho, 2004; Habre & Abboud, 2006; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Sierpinska, 1992; Vinner, 1983). *Symbolic* representations were the most common type in the Glencoe and UCSMP series, and the second most common in the CPMP series. About 70% of all function examples included *Symbolic* representations in all series, with Glencoe having the highest proportion of nearly 75%. Despite this, other researchers have pointed out that students often have trouble with $f(x)$ notation (Carlson, 1998; Sierpinska, 1992; White & Van Dyke, 2006). Of the *Symbolic* representations in the textbooks, $f(x)$ = *Expression* notation was one of the most common; however, it still made up less than one third of *Symbolic* representations in each series.

A number of researchers have argued that graphic representations of functions are potentially powerful representations in helping students build understanding of function (Dahlberg & Housman, 1997; Eisenberg, 1991, 1992; Ferrini-Mundy & Lauten, 1994; Schwartz & Yerushalmy, 1992; Sfard, 1991). However, Eisenberg (1991) argued that many students believe graphs are peripheral to functions and do not value visualizing functions. Many other researchers have found that students have a general lack of understanding of what graphic representations of functions mean (Bridger & Bridger, 2001; DeMarios & Tall, 1999; Goldenberg, et al., 1992; Schwingendorf, et al., 1992; Sierpinska, 1992; White & Van Dyke, 2006). The proportions of *Graphic* representations were similar in all three textbook series and less common than *Symbolic*, *Numeric*, and *Verbal Descriptions*. Only about 30% of function examples in each series included *Graphic* representations. Researchers have pointed out that students often reject examples as functions when their graphs are not continuous (Carlson, 1998; Dubinsky & Harel, 1992; Markovits, et al., 1986; Sfard, 1992) or when the graphs do not meet some level of smoothness (Even, 1993; Tall, 1990; Tall & Bakar, 1992; White & Van Dyke, 2006). The *Graphic* representations in all three series were almost all *Continuous* and *Smooth*. Opportunities to engage with functions that do not have *Continuous* or *Smooth* graphs were few and far between in all of the textbooks.

Many researchers have argued that students' ability to recognize connections and translate between representations is key to their improved understanding of the function concept, but they have also recognized this as a major difficulty experienced by students (Akkoc & Tall, 2005; Artigue, 1992; Eisenberg, 1991; Goldenberg, et al., 1992; Gonzalez-Martin & Camacho, 2004; Markovits, et al., 1986; National Council of

Teachers of Mathematics, 2009; Schwarz & Dreyfus, 1995; Sfard, 1992; Sierpinska, 1992; Tall, 1996; Thompson, 1994; White & Van Dyke, 2006). All of the textbook series provide many opportunities for students to make connections between function representations. In every textbook except UCSMP *Geometry*, around 80% of all examples involved multiple representations of functions, with the vast majority of these providing a representation for the students and requesting that they provide a different representation.

Summary of Function Representations

The most common representation for functions in the Glencoe and UCSMP series was *Symbolic*. This representation was also common in the CPMP series, and in each series about 3 out of every 4 examples included this representation. In the initial books in each series, the majority of these were y = *Expression* representations, but they became increasingly $f(x)$ = *Expression* and *Other* representations. A large portion of these *Other* representations were requests for *Symbolic* representations, without specification of the format. Other types of *Symbolic* representations tended to be provided for students.

Verbal Descriptions were the most common representation in the CPMP series, with over 80% of all examples including this representation. Over half of all examples in the Glencoe series, and nearly 70% of all examples in the UCSMP series included *Verbal Descriptions*. A larger portion of these representations were requested or combined.

Numeric representations were the next most common, with slightly less than 50% of examples in each series including these representations. There was a balance among provided, requested, and combined representations. With a few exceptions, most of these representations were coded as *Other*, and many of these were examples where students were provided with one element and asked to produce the corresponding element, without specifying the format. *Ordered Pair* representations were also somewhat common, especially early in the Glencoe series.

Graphic representations were less common than *Numeric*, but more common than *Function Machine*, *Mapping Diagram*, *Physical*, and *Other* representations, which occurred infrequently. Almost all *Graphic* representations were *Continuous* and *Smooth*. There were slightly more requests than provided representations in the Glencoe and CPMP series, but the reverse was true in the UCSMP series. Very few combined *Graphic* representations were included in any series. However, ultimately, a large proportion of the function examples in each textbook did include multiple representations, and most of these were combinations of provided and requested representations.

Function Families

Codes for seven different function families were developed: *Polynomial*, *Periodic*, *Exponential*, *Logarithmic*, *Rational*, *Absolute Value*, and *Piecewise.* A representation was coded as (a) *Polynomial* if its symbolic representation was a polynomial, (b) *Periodic* if its graph could be mapped onto itself with a horizontal translation, (c) *Exponential* if in its symbolic representation its independent variable was an exponent, (d) *Logarithmic* if in its symbolic representation its independent variable was a logarithm, (e) *Rational* if its symbolic representation was a ratio of polynomials, (f) *Absolute Value* if in its symbolic representation its independent variable was inside an absolute value symbol, or (g) *Piecewise* if the rule of correspondence to generate an image for an element of the domain of the function was dependent on the value of the element. Note that each example of a function received a code for exactly one of these representations, or was coded as *Other*. Each example was coded as either provided or requested of the student. No examples received multiple codes and the sum of the percents of the codes for all function families in a given textbook or series is 100%, although small differences may appear due to rounding.

The numbers of function examples coded in each family are provided in Table 4.14. Function representations were also coded as either provided or requested. They were coded as provided when at least one pair of corresponding values of the function or a method to obtain such a pair of values was given to students. They were coded as requested when students were asked to create a function example and the textbook did not provide any part of the function.

Of the seven families, few examples were *Logarithmic, Absolute Value*, or *Piecewise*. Out of nearly 14,000 examples in the Glencoe series, there were only 107 *Logarithmic*, 185 *Absolute Value*, and 200 *Piecewise* functions. *Logarithmic* examples only appeared in *Algebra 2* and *Precalculus*, and the other two families were split among *Algebra 1*, *Algebra 2*, and *Precalculus*. Almost all of these examples were provided, with only four requested *Absolute Value* examples. In over 10,000 examples in the UCSMP series, there were slightly less than 100 each of *Logarithmic*, *Absolute Value*, and *Piecewise* functions. The *Logarithmic* and *Piecewise* examples were mainly in *Advanced Algebra*, *FST*, and *PCDM*. The *Absolute Value* functions were mainly in *Algebra*, *Advanced Algebra*, and *FST*. Almost all of the examples were provided, with only four *Logarithmic* and four *Piecewise* functions requested. Of the 5605 function examples in the CPMP series, there were only 21 *Logarithmic* examples, 50 *Absolute Value*, and 38 *Piecewise*. The *Logarithmic* functions were spread evenly over *Course 2*, *Course 3*, and

Course 4. However, the majority of *Absolute Value* and *Piecewise* examples were in *Course 4*, with only a few in the other textbooks. Nearly all of these examples were provided, with only one *Absolute Value* and one *Piecewise* function requested. Because there were such few examples of *Logarithmic*, *Absolute Value*, and *Piecewise* functions, further analysis focuses on the remaining families: *Polynomial, Periodic, Exponential, Rational*, and *Other*.

Table 4.14

Distribution of families of function examples in each series

Series	Polynomial	Periodic	Exponential	Loga-rithmic	Rational	Absolute Value	Piece-wise	Other
Glencoe								
Provided	8345	603	711	107	796	181	200	2502
Requested	223	73	10	0	10	4	0	110
UCSMP								
Provided	4808	558	677	94	535	93	91	2951
Requested	230	20	16	4	10	0	4	239
CPMP								
Provided	2523	358	608	21	381	49	37	1211
Requested	214	10	29	0	35	1	1	127

Polynomial

By far, the most common function family was *Polynomial* (see Figure 4.21). The Glencoe series had a higher proportion of these examples, with over 60% of all function examples in the series being *Polynomial*. However, almost half of all examples in the UCSMP and CPMP series were also *Polynomial*. Almost all *Polynomial* functions in all series were provided; the CPMP series had the greatest percent of requests, with about 4%.

In each series, the later textbooks tended to have smaller proportions of *Polynomial* examples. In the Glencoe series, the *Geometry* textbook had the most, with over 75%. *Algebra 2* and *Precalculus* dropped to less than 60% in each. In the UCSMP series, *Algebra* had the highest proportion of *Polynomial* examples by far, with over 75%. In contrast, *FST* and *PCDM* had about 25% and 40%, respectively. The CPMP series peaked in *Course 2*, with about 60% *Polynomial* examples. *Course 2* also had the highest percent of requests of any textbook, with nearly 10%. The overall percent of *Polynomial* examples dropped to about 40% by *Course 4*.

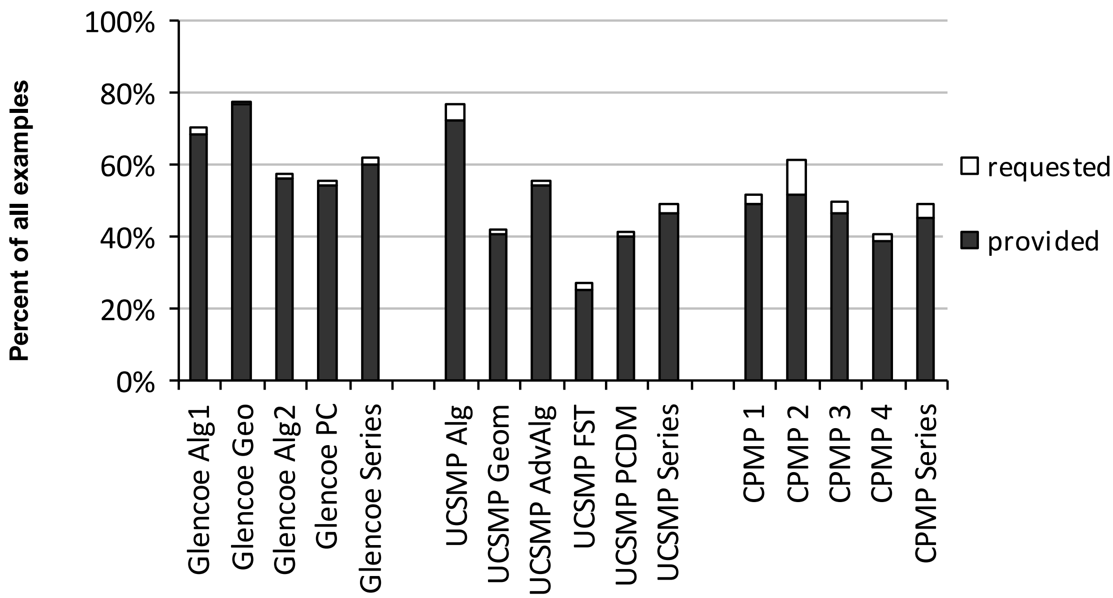


Figure 4.21: Polynomial function examples as a percent of all functions.

The *Polynomial* examples were coded in four categories: (a) *Constant*, (b) *Linear*, (c) *Quadratic*, or (d) *Cubic*. An example was coded as (a) *Constant* if it was a *Polynomial* with a range that consisted of a single number, (b) *Linear* if it was a first degree *Polynomial* , (c) *Quadratic* if it was a second degree *Polynomial*, or (d) *Cubic* if it was a third degree *Polynomial*. Any example that did not fit one of these categories but was *Polynomial* was coded as *Other*. The following graphs and discussion provide the distributions of each of these codes as a percent of the *Polynomial* examples, rather than as a percent of all examples.

Constant. There were very few *Constant* function examples in any textbook, and virtually no requests for these examples (see Figure 4.22). The Glencoe series had the smallest proportion, with about 2% of *Polynomial* functions being *Constant*. In the UCSMP and CPMP series, about 3% were *Constant*. UCSMP *Geometry* had the highest percent, with about 8%, all of which were provided. No textbook had more than two requested *Constant* examples.

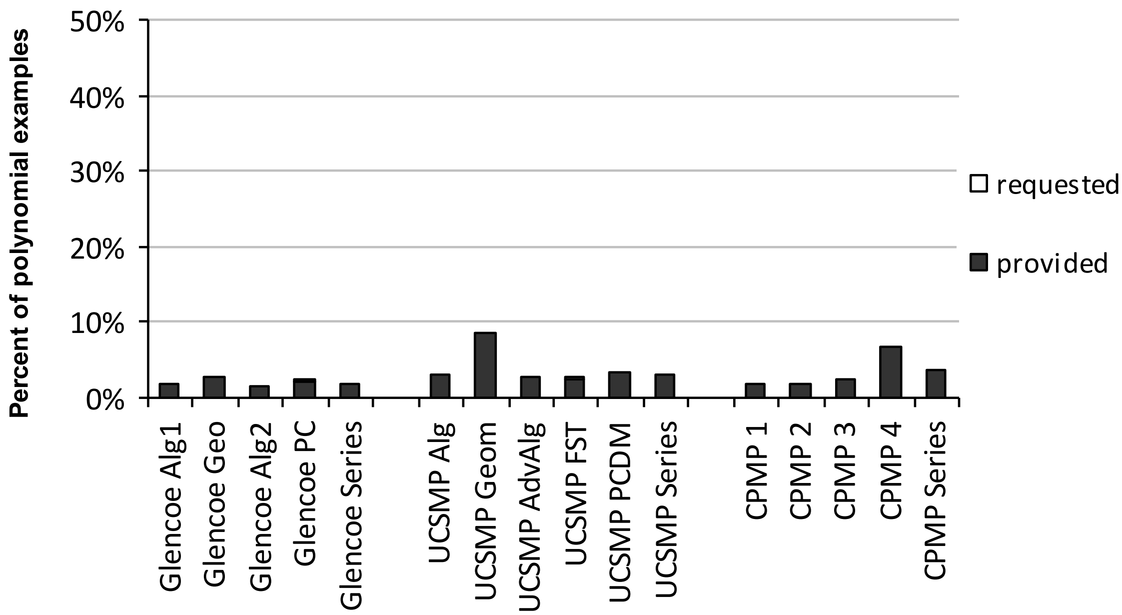


Figure 4.22: Constant function examples as a percent of *Polynomial* functions.

Linear. The majority of *Polynomial* examples in all series were *Linear* (see Figure 4.23). The Glencoe series had the highest proportion of these, with nearly 70% of all *Polynomial* examples being *Linear*. Both the UCSMP and CPMP series had less than 60% each. Almost all *Linear* examples were provided for students. The CPMP series had the highest percent requested, with about 4%.

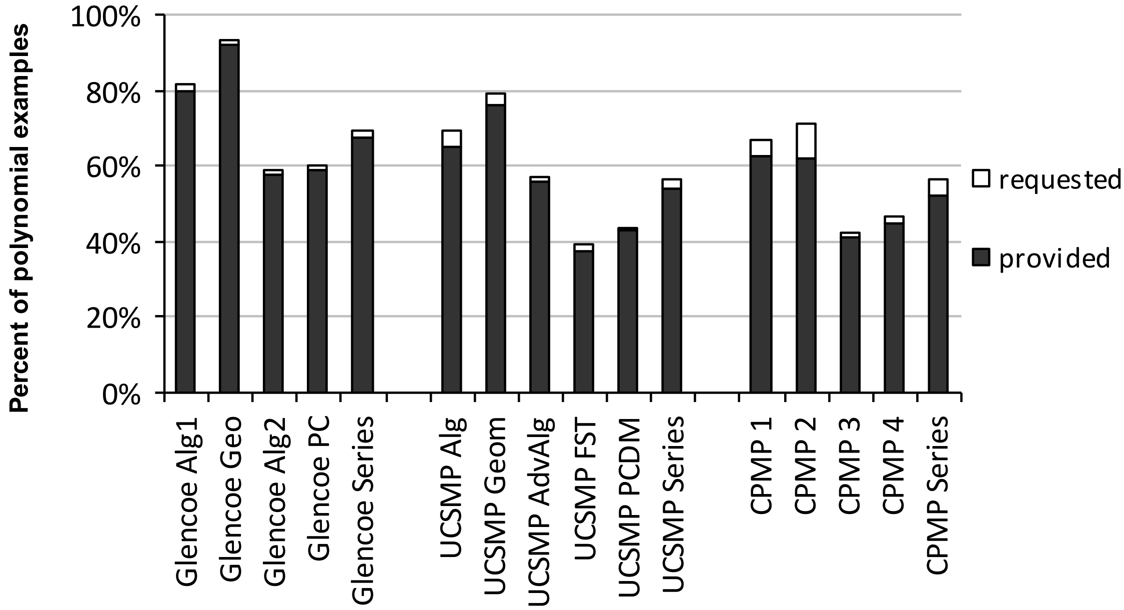


Figure 4.23: Linear function examples as a percent of *Polynomial* functions.

In each series, the final two textbooks tended to have less *Linear* examples. In the Glencoe series, *Algebra 1* had over 80% and *Geometry* had over 92% of all *Polynomial* examples being *Linear*. In contrast, *Algebra 2* and *Precalculus* had about 60%. In the UCSMP series, again *Geometry* had the highest percent *Linear*, with nearly 80%. *Advanced Algebra* had under 60%, and *FST* and *PCDM* had approximately 40% each. In CPMP, *Course 1* and *Course 2* each had about 70% of *Polynomial* examples being *Linear*. *Course 3* and *Course 4* each had about 45%. Also, despite making few requests for *Linear* examples overall, there was a trend in the UCSMP and CPMP series for there to be fewer requests in later textbooks. In UCSMP, the highest percents of requests were in *Algebra* and *Geometry*, with about 4% and 3% respectively. In CPMP *Course 1* about 4% were requested, and in *Course 2* nearly 9% were requested, as contrasted with less than 2% each in the later textbooks.

Quadratic. *Quadratic* examples were the most common *Polynomial* functions after *Linear*, but were far less frequent than *Linear* (see Figure 4.24). The Glencoe series had the smallest proportion, with only about 20% of all *Polynomial* examples being *Quadratic*. Virtually all of these were provided. In the UCSMP series, just over 20% were *Quadratic*, and again almost all were provided. The CPMP series had the highest overall proportion, with nearly 30% *Quadratic*, and the most requests, with almost 3%.

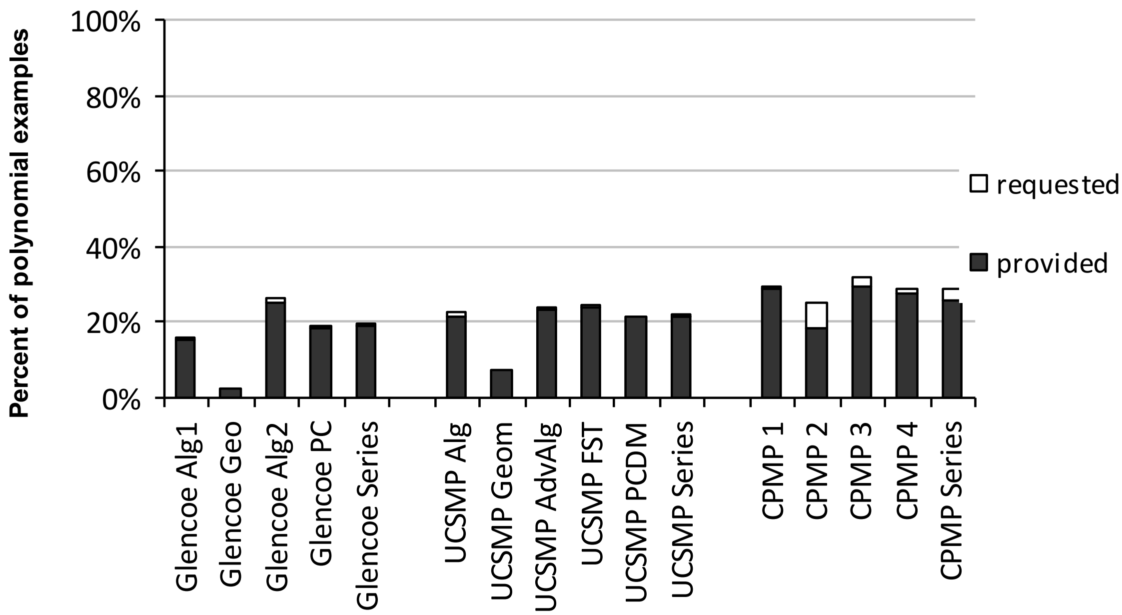


*Figure 4.24: Quadratic f*unction examples as a percent of *Polynomial* functions.

In the Glencoe and UCSMP series, the proportion of *Quadratic* examples was roughly constant over the series, except in the *Geometry* textbook in each series, which each had far fewer than other books in the series. In the CPMP series, the overall proportion of *Quadratic* examples remained relatively constant; however, the proportion of requests varied. *Course 1* and *Course 4* each had about 1% requested, while *Course 3* had over 2% and *Course 2* had nearly 7% requested.

Cubic. In each series, only a small proportion of *Polynomial* examples were *Cubic* (see Figure 4.25). The UCSMP series had the highest proportion, with over 7%, while the Glencoe and CPMP series had about 5%. There were almost no requests for *Cubic* examples in any series. In each series there was a tendency for later textbooks to include higher proportions of *Cubic* examples, with the highest percents in each series in Glencoe *Precalculus*, UCSMP *FST*, and CPMP *Course 3*.

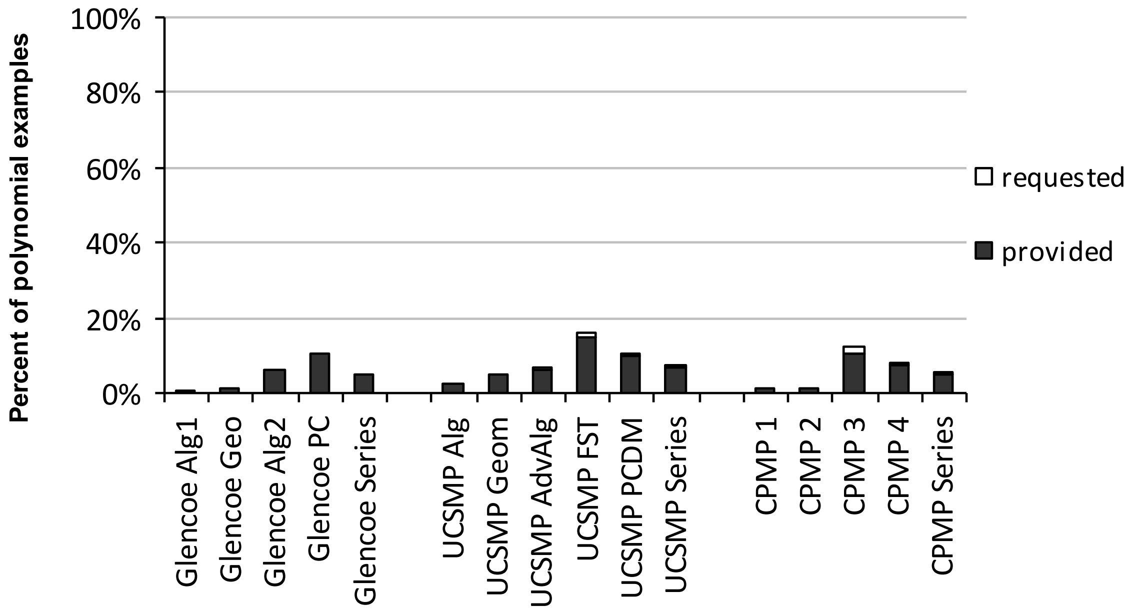


Figure 4.25: Cubic function examples as a percent of *Polynomial* functions.

Other. A relatively small portion of examples of *Polynomial* functions were coded as *Other* (see Figure 4.26). The UCSMP series had the highest proportion, with just over 10%. The proportions in both the Glencoe and CPMP series were about half as large. Almost all *Other* examples were provided, except in UCSMP *FST*, in which almost 3% of all *Polynomial* examples were requests for *Other* examples. There was a trend in each series for the first two books to have fewer *Other* examples than later textbooks. In the Glencoe series, *Algebra 2* and *Precalculus* both had the highest proportion, with about

7%. In the UCSMP series, *PCDM* had the highest proportion, with nearly one out of every five *Polynomial* examples being *Other*. In the CPMP series, *Course 3* had the highest proportion, with over 11%.

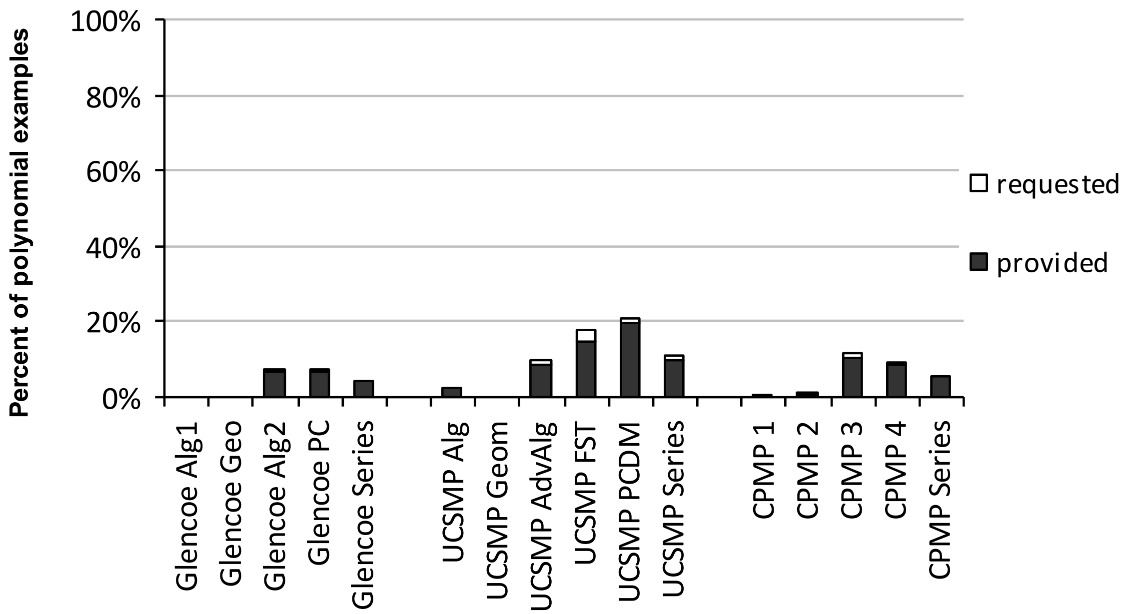


Figure 4.26: Other function examples as a percent of *Polynomial* functions.

Most *Other* examples were *Polynomial* functions with degree higher than three. The most common type was quartic functions, but examples ranged from degree four to 200. Generally, the higher degree examples had fewer occurrences. There were also nearly 200 general examples in which the degree was a variable. Requests for *Other* examples followed a similar pattern, with the most requests being for quartic functions or *Polynomial* examples where degree was not specified. There were fewer requests for higher degree functions, with degree 10 being the highest request.

Summary of Polynomial. Several patterns prevail in the distribution of *Polynomial* examples of functions. The first is that most *Polynomial* functions were linear in each series and in at least half of the textbooks in each series. The proportion of *Linear* functions dropped over the course of each series, and the proportion of higher degree *Polynomial* examples increased in the later books in each series. However, excepting *Constant* functions, there was an inverse relationship between the degree of the *Polynomial* examples and the frequency of occurrence. For example, there were fewer *Quadratic* examples than *Linear*, and fewer *Cubic* than *Quadratic*. Almost all of these examples were provided for students.

Periodic

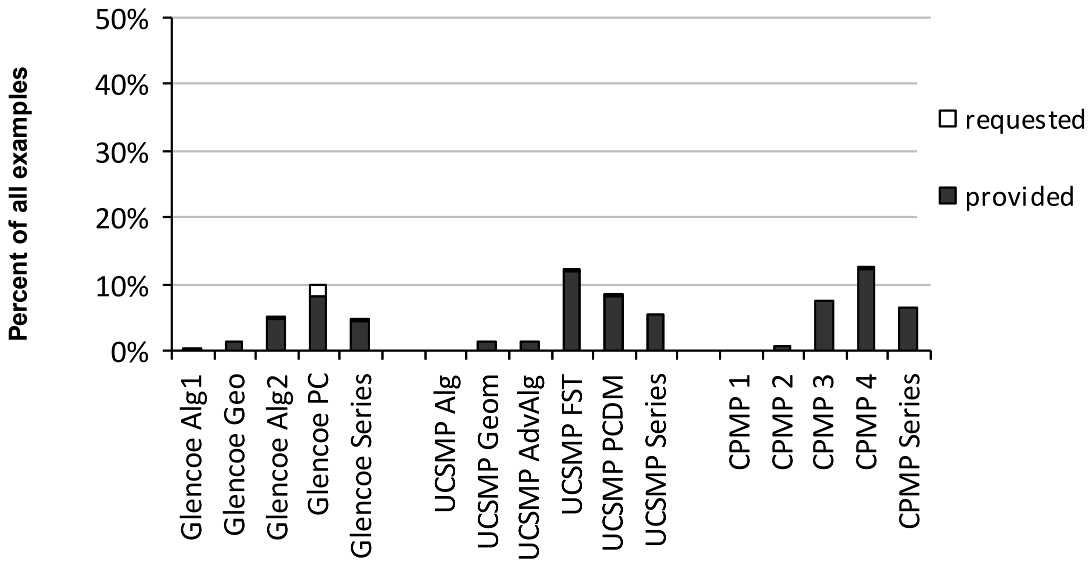


Figure 4.27: Periodic function examples as a percent of all functions.

Overall, there were few *Periodic* functions in any series (see Figure 4.27). The CPMP series had the highest proportion, with over 6%. The UCSMP series had over 5%, and the Glencoe series had just under 5%. Nearly all *Periodic* examples were presented, as Glencoe *Precalculus* had the highest proportion at just under 2% requested. The final two textbooks in each series tended to have higher proportions of *Periodic* examples. In the Glencoe and CPMP series, the proportions increased to the final textbooks. Glencoe

Precalculus had just under 10%, and CPMP *Course 4* had over 12%. In the UCSMP series, *FST* had the highest percent, with about 12%, but *PCDM* also had nearly 10%.

The *Periodic* examples were coded as *Trigonometric* or *Other*. Examples of function that would include a trigonometric expression in their symbolic representation were coded as *Trigonometric*. Any example that did not use a trigonometric expression but was *Periodic* was coded as *Other*. The following graphs and discussion provide the distributions of each of these codes as a percent of the *Periodic* examples, rather than as a percent of all examples.

Trigonometric. Almost all *Periodic* examples were *Trigonometric* in each series, and in each textbook (see Figure 4.28). CPMP *Course 1* was an exception, but it only contained one *Periodic* function, which was not *Trigonometric*. UCSMP *Advanced Algebra* had the next lowest proportion of *Trigonometric*, with slightly over 75%. All other textbooks had over 90%. Few textbooks requested *Trigonometric* examples. The

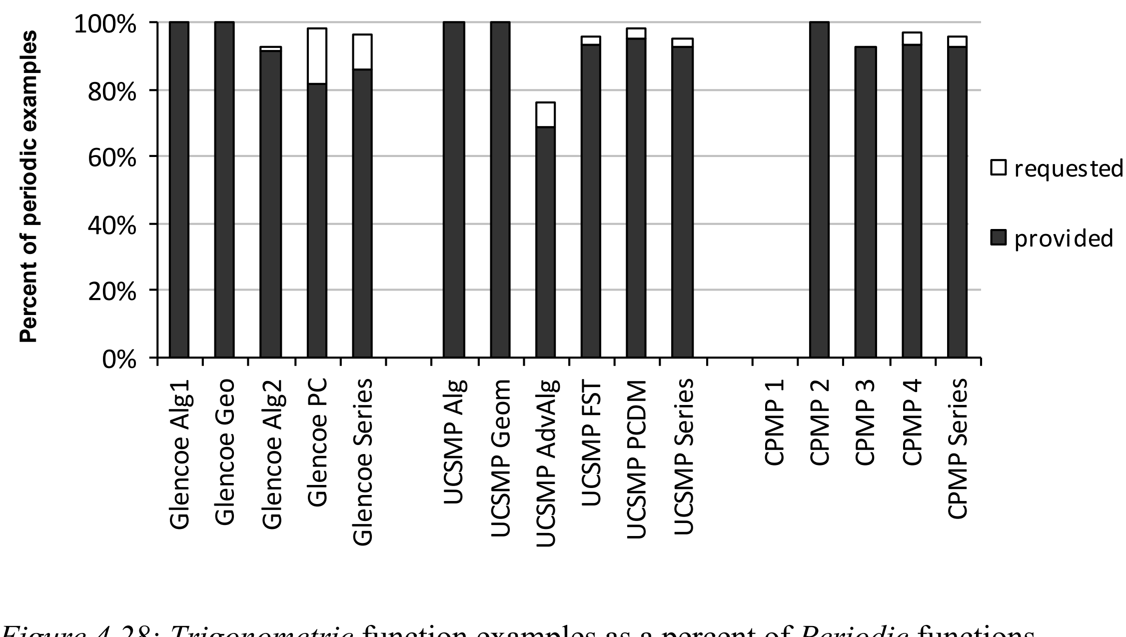


Figure 4.28: Trigonometric function examples as a percent of *Periodic* functions.

highest proportion was in Glencoe *Precalculus*, with just over 15% of a *Periodic* examples being requested *Trigonometric* functions. The only other textbook with more than 4% requested was UCSMP *Advanced Algebra*, with about 8%.

Other. The number of *Periodic* examples coded as *Other* was very small (see Figure 4.29). No series had more than 30 such examples total. In the first two textbooks in each series, there was only one *Other* example. This function, in CPMP *Course 1*, was the only *Periodic* example, and therefore was 100% of these examples in the textbook; however, it was only one example. The textbook with the next highest proportion was UCSMP *Advanced Algebra*, where nine *Other* examples made up almost one quarter of *Periodic* examples. The remaining textbooks had very few *Other* examples. Only three requests were made for such examples, and all in the Glencoe series.

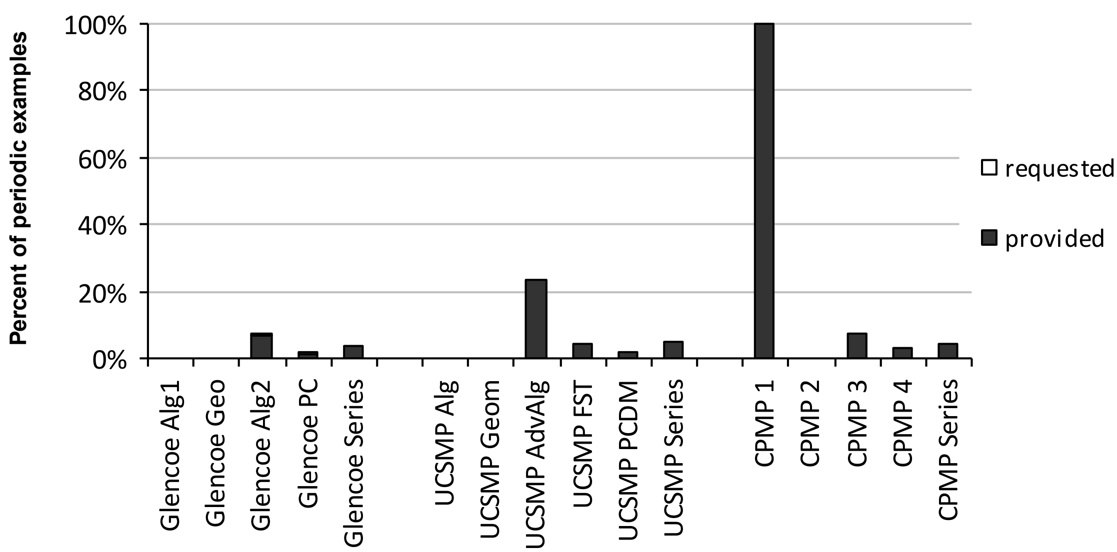


Figure 4.29: Other function examples as a percent of *Periodic* functions.

The types of examples coded as *Other* varied greatly. Some were described as types of waves such as saw-tooth waves, square waves, and triangle waves. Some represented other mathematical situations, such as decimal places of repeating decimals. A few general examples were also provided. The three requests were also general requests for *Periodic* functions.

Summary of Periodic. The *Periodic* examples in all three series were mostly provided *Trigonometric* functions. There were very few *Periodic* examples coded as *Other*, and only a small proportion of *Periodic* examples were requested.

Exponential

The proportions of function examples from the *Exponential* family are provided in Figure 4.30. There were relatively few *Exponential* examples in any series, although the proportion in the CPMP series, which was just over 11%, was about twice as high as the Glencoe or UCSMP series. Virtually all *Exponential* functions were provided to students in every series. In the Glencoe and UCSMP series, the proportion was relatively constant across textbooks, except for the *Geometry* books. Glencoe *Geometry* had only seven examples of *Exponential* functions, and UCSMP *Geometry* had no examples. In the CPMP series, *Course 1* had the highest proportion, with more than one out of every five examples of function being *Exponential*. *Course 2* had the fewest, with about 4%. *Course 3* and *Course 4* had similar proportions, with about 12% and 10%, respectively.

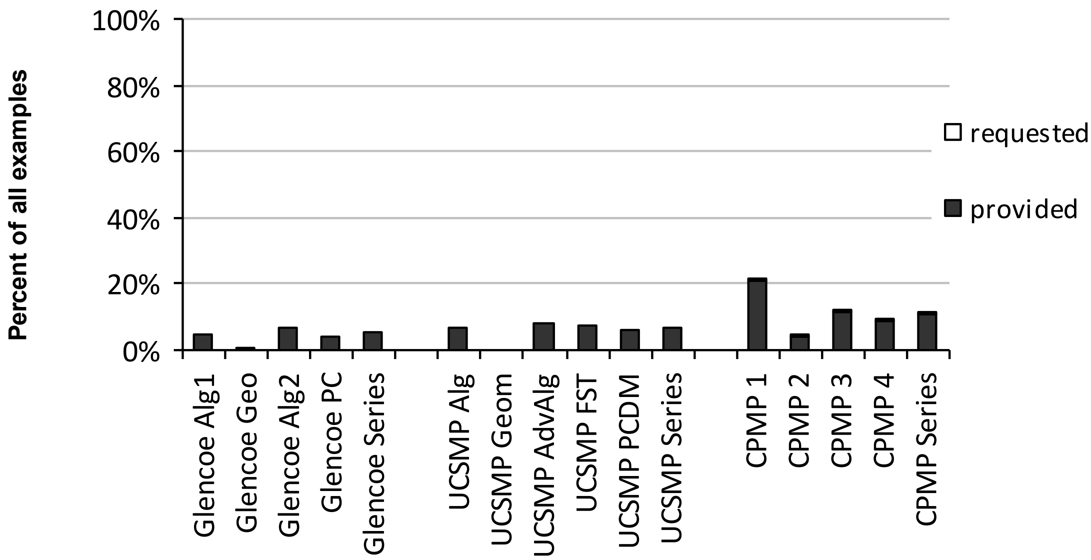


Figure 4.30: Exponential function examples as a percent of all functions.

Rational

There were relatively few *Rational* functions in any series, with no series having more than 8% (see Figure 4.31). The CPMP series had the highest proportion, with over 7%, and the Glencoe and UCSMP series had under 6% each. Almost all *Rational* examples were provided. CPMP *Course 2* was the exception, with 2.5% of all function examples being requests for *Rational* functions. Over the Glencoe and UCSMP series, the proportion of *Rational* functions remained relatively constant, except for the *Geometry* textbooks, which had virtually no such functions. In the CPMP series, the proportions peaked in *Course 3*, which had 13%.

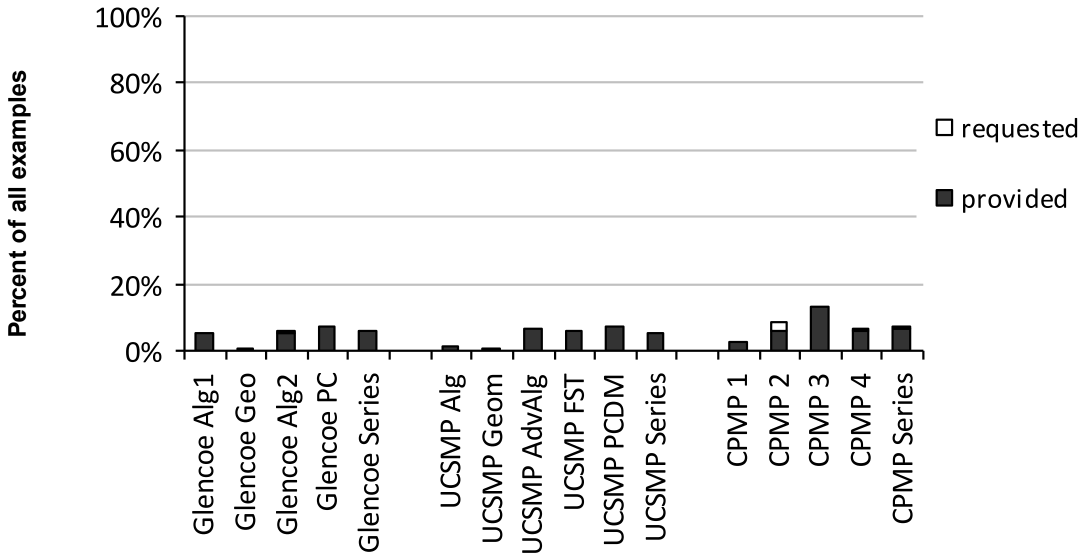


Figure 4.31: Rational function examples as a percent of all functions.

Other

Other functions were the second most common type of example after *Polynomial* functions (see Figure 4.32). The UCSMP series had the highest proportion of these examples, with more than 3 out of every 10 examples coded as *Other*. The CPMP series had just less than 25%, and the Glencoe series had the smallest proportion, with less than 20% of all examples coded as *Other*. Nearly all of these examples in each series were provided, with less than 1% in the Glencoe series and just over 2% in both the UCSMP and CPMP series.

In each textbook, the Glencoe series had proportions of *Other* examples consistent with the overall proportions in the series. The CPMP series was also relatively consistent across the series, with only *Course 3* having a slightly smaller proportion of *Other* examples. In contrast, in the UCSMP, there was a wide variety of proportions across textbooks. *Algebra* had the smallest proportion, with about 13%, and *Geometry* had the highest, with over half of the examples being *Other*. *Advanced Algebra* had the second smallest proportion, with about one quarter of function examples. *FST* had around 43% and *PCDM* had over one third of the examples coded as *Other*.

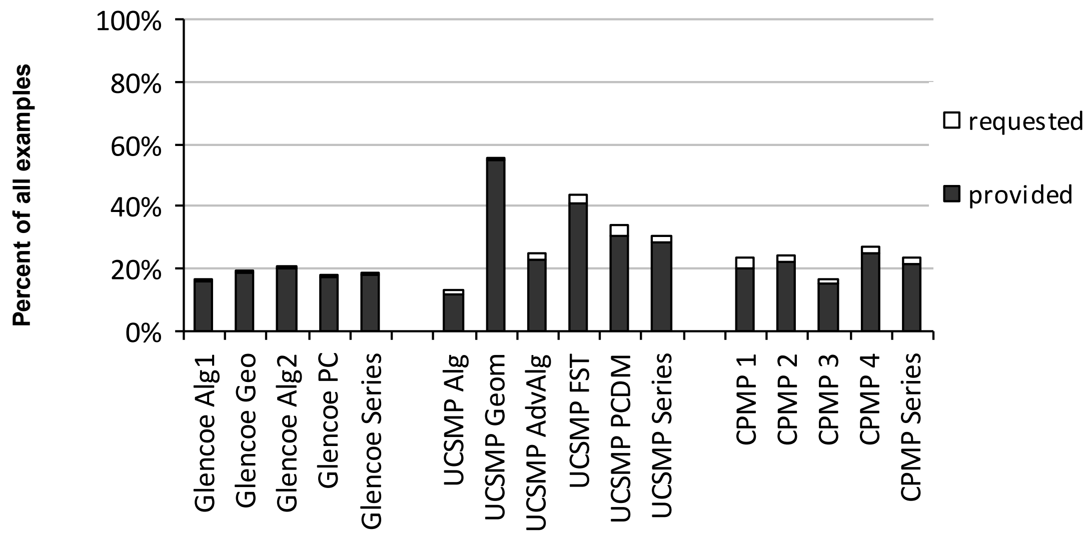


Figure 4.32: Other function examples as a percent of all functions.

Each series had slightly different types of functions that were coded *Other*. In the Glencoe series, there were two common types of provided *Other* examples. One was sets of ordered pairs provided without a rule that would connect elements from the domain to elements in the range, and not necessarily in a format using parentheses to show each pair. For example, a table of data that provided years in one column and data for each year in another would have been labeled this way. Also common early in the series were functions of multiple variables. For example, the formula for the area of a rectangular figure gives area as a function of two variables: the length and the width. Of *Other* function examples in all textbooks in the series, both of these kinds of functions were provided most frequently. The most common *Other* request in all textbooks in the series was a general request for an example that could be in a variety of families, and therefore could not be coded as one specific family.

In UCSMP *Algebra*, the most common provided *Other* examples were also sets of ordered pairs and functions of multiple variables. The most common requests were general requests and also requests for sets of ordered pairs. However, in *Geometry*, by far the most common type of provided and requested *Other* functions were for transformations, such as translations, rotations, or reflections. This type of example remained common through the rest of the series, along with functions of multiple variables. Also, in *FST*, because of the statistics portion of the textbook, sets of ordered pairs were commonly provided. In *PCDM*, many of the *Other* examples were polar functions or general functions without any identifiable pairs of input and output. Also in *PCDM*, about 13% of *Other* examples were not easily placed into one category. For example, many were represented by equations that combined several features of families such as trigonometric functions, exponents, and rational expressions. Requests for *Other* functions in the later textbooks in the series tended to be general requests.

In the CPMP series, sets of ordered pairs were among the most common provided *Other* examples up to *Course 4*. In addition, many functions of multiple variables were provided; these were the most frequently provided *Other* examples in *Course 4*. Sets of ordered pairs were also the most prevalent request for *Other* functions in *Course 1* and *Course 2*. In the final two textbooks, general requests that could be met with many different families of functions were the most common.

In addition to these most common *Other* examples, a variety of additional types of *Other* functions were recognized in the analysis. These included functions with square roots and other roots, functions defined recursively, relative frequencies, probability distributions, functions of data sets, and functions combining features of several families. Ultimately, these represented a very small proportion of all examples coded.

Comparison of Function Family to Research

Several researchers have pointed out that students tend to understand best the function families they experience the most (Sierpinska, 1992; Tall & Bakar, 1992). In fact, many have documented that students tend to associate the concept most strongly with linear functions (Even, 1993; Leinhardt, et al., 1990; Markovits, et al., 1986) or linear and quadratic functions (Oehrtman, et al., 2008; Schwarz & Hershkowitz, 1999). In contrast, many students have difficulty even recognizing constant functions as functions (Even, 1993; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Tall & Bakar, 1992) or other unfamiliar functions as functions (Baker, et al., 2000; Even, 1993; Leinhardt, et al., 1990; Tall & Bakar, 1992; Vinner, 1983; White & Van Dyke, 2006). This suggests that it is important that students experience types of functions outside of the linear and quadratic families. Numerous researchers have recommended that students not only study linear and quadratic functions, but also other polynomial, exponential, periodic, rational, and logarithmic functions (Fey & Good, 1985; Haimes, 1996; National Council of Teachers of Mathematics, 2000; Tall, 1996).

The majority of examples in most of the textbooks were *Polynomial*, and the vast majority of the *Polynomial* examples were *Linear* or *Quadratic*. However, the proportions of *Linear* and *Quadratic* examples tended to decrease over the course of each series, while proportions of higher *Polynomial* functions tended to increase. Most textbooks included at least a small proportion of *Exponential*, *Periodic*, and *Rational* functions, and the later books in each series included a small proportion of *Logarithmic* functions. In all textbooks, the percent of *Constant* functions was minute. The Glencoe series included the highest proportions of *Linear* and *Quadratic* examples, with more than 65% of all functions of these types in *Algebra 1* and *Geometry*. Even *Precalculus* included more of these examples than the overall averages in the UCSMP and CPMP series. In the UCSMP series, both *Algebra* and *Advanced Algebra* included large proportions of *Linear* and *Quadratic* examples, but the other textbooks included fewer, especially *FST*, which only had about 16%. In all textbooks in the CPMP series, less than half of the examples were *Linear* or *Quadratic.* Notably, *Course 1* had over 20% *Exponential* examples, which was more than any other textbook. However, over one third of all examples in the CPMP and UCSMP series and over one half of all examples in the Glencoe series were *Linear* or *Quadratic*.

Summary of Function Family

Function examples in all three series were predominantly provided *Polynomial* examples. A large proportion of these were *Linear*, although there were increasing proportions of higher degree *Polynomial* functions in later textbooks in each series. The next largest proportion of functions was examples coded as *Other*. These examples were mostly provided sets of ordered pairs and functions of multiple variables. In the UCSMP series, transformations were also common. Three other families of functions also constituted more than 2% of all functions in each series: *Periodic*, *Exponential*, and *Rational*. These examples were mostly provided for students and tended to make up slightly larger proportions of functions in later textbooks in each series. One exception was over 20% of functions in CPMP *Course 1* were *Exponential*. However, over half of the examples in *Course 1* were *Polynomial*, as was the case with many other textbooks. Those with smaller proportions of *Polynomial* examples tended to have larger proportions of *Other* examples, such as UCSMP *Geometry*, in which most functions were transformations, and *FST*, in which many functions were sets of ordered pairs.

Summary of Core Features

Core features of function examples had both similarities and differences among the three textbook series. The proportion of examples with specified domains or ranges was relatively small in all series, and there were almost no examples with non-numerical domains or ranges specified. The proportion of examples with domain or range specified in the CPMP series was only half as large as the proportions in the Glencoe and UCSMP series. In all three series, a larger number of examples requested domains or ranges than provided them for students.

Symbolic representations were the most common function representations included in the Glencoe and UCSMP series, and second most common in the CPMP series. *Verbal Descriptions* were the most common in the CPMP series and also very frequently included in the UCSMP series; however, they were less so in the Glencoe series. *Symbolic* representations tended to be provided for students, whereas there were more requests for or combined *Verbal Descriptions*. In all three series, *Numeric* and *Graphic* representations were less common, but included much more often than *Function Machine*, *Mapping Diagram*, *Physical*, and *Other* representations. Examples in each textbook predominantly included multiple representations.

A large portion of examples in all three series were *Polynomial* functions, especially in the Glencoe series. Most *Polynomial* examples were provided for students.

Many of these were *Linear*, again, especially in the Glencoe series, but this proportion tended to decrease in later textbooks in each series. Also, the proportion of all *Polynomial* functions decreased over each series, and the proportions of *Periodic* and *Rational* functions tended to increase. Examples coded as *Other* were also a significant presence in each series, especially in the UCSMP series. There were a variety of different types of functions coded as *Other* in each series, but many were sets of ordered pairs or functions of multiple variables. The UCSMP series also included many transformations throughout most of the textbooks. However, ultimately, a much larger proportion of functions were *Polynomial* than *Other* in each series.

Ancillary Features of Function Examples

Abstract and Realistic Settings

The proportions of abstract and realistic function examples are provided in Table 4.15. All but one textbook presented the majority of function examples in abstract settings. The only exception, CPMP *Course 1*, provided realistic settings for nearly 60% of the examples in the book. The CPMP series also had the largest overall percent of examples in realistic contexts, with just over 38%. The Glencoe and UCSMP series both provided realistic settings for about one quarter of the examples in the series.

Despite providing a higher percent of examples in abstract settings, the first books in Glencoe and UCSMP were similar to CPMP *Course 1* in that they provided realistic contexts for the highest percent of examples of any textbook in the respective series. The CPMP and Glencoe series had decreasing trends in percents of examples placed in realistic settings over the series. In UCSMP, the *Geometry* book was markedly different than the others, with less than 10% of examples in realistic contexts, which was the lowest percent of any book in any series. Both UCSMP *Advanced Algebra* and *FST* contained about 30% of examples in realistic settings, while *PCDM* had only about 16% in realistic settings.

Table 4.15

Distribution of abstract and realistic function examples in each textbook

Textbook	Total number of function examples	Percent of abstract examples (p/r/c*)	Percent of realistic examples (p/r/c*)
Glencoe			
Algebra 1	3937	72.8% (72.6/0.2/0)	27.4% (26.9/0.5/0)
Geometry	768	84.9% (84.9/0/0)	24.2% (24.2/0/0)
Algebra 2	5124	78.2% (78.2/0/0)	22.7% (22.4/0.3/0)
Precalculus	4046	82.8% (82.8/0/0)	18.2% (18.0/0.2/0)
Series Total	13875	78.4% (78.3/0.1/0)	22.8% (22.5/0.3/0)
UCSMP			
Algebra	2029	65.7% (65.6/0.1/0)	34.9% (34.3/0.6/0)
Geometry	528	96.4% (96.4/0/0)	9.3% (9.3/0/0)
Advanced Algebra	2595	73.0% (73.2/0/0)	28.2% (27.6/0.5/0)
FST	2382	71.6% (71.6/0/0)	30.8% (30.4/0.4/0)
PCDM	2796	84.5% (84.5/0/0)	16.1% (15.9/0.2/0)
Series Total	10330	75.6% (75.5/0/0)	25.9% (25.5/0.4/0)
CPMP			
Course 1	1145	43.1% (43.1/0/0)	58.3% (57.3/1/0)
Course 2	1125	62.4% (62.4/0/0)	39.5% (39.4/0.1/0)
Course 3	1124	70.2% (70.2/0/0)	31.0% (30.5/0.4/0)
Course 4	2211	69.9% (69.9/0/0)	30.6% (30.6/0.1/0)
Series Total	5605	63.0% (63/0/0)	38.1% (37.8/0.3/0)

* p = provided, r = requested, c = combined

Technology Recommendations

The percent of examples for which textbooks explicitly recommended the use of technology was less than 10% for most books (see Table 4.16). Both the CPMP and UCSMP series explicitly recommended the use of technology with about 10% of their examples. In contrast, Glencoe only recommended its use for approximately 6% of its examples.

Table 4.16

Distribution of function examples with technology explicitly recommended for use

Textbook	Total number of function examples	Percent of examples with technology recommended (p/r/c*)
Glencoe		
Algebra 1	3937	5.0% (5/0/0)
Geometry	768	2.7% (2.7/0/0)
Algebra 2	5124	5.6% (5.6/0/0)
Precalculus	4046	7.6% (7.6/0/0)
Series Total	13875	5.9% (5.9/0/0)
UCSMP		
Algebra	2029	13.6% (13.6/0/0)
Geometry	528	3.8% (3.8/0/0)
Advanced Algebra	2595	10.8% (10.7/0.1/0)
FST	2382	11.7% (11.7/0/0)
PCDM	2796	6.2% (6.2/0/0)
Series Total	10330	10.0% (9.9/0/0)
CPMP		
Course 1	1145	12.1% (12.1/0/0)
Course 2	1125	8.2% (8.2/0/0)
Course 3	1124	9.2 % (9.2/0/0)
Course 4	2211	11.2% (11.2/0/0)
Series Total	5605	10.4% (10.4/0/0)

* p = provided, r = requested, c = combined

The *Geometry* books in the Glencoe and UCSMP series had the lowest percents of examples where technology was recommended, with about 3% in each. Excepting the *Geometry* book, the percent of examples with technology recommended increased over the course of the Glencoe series, from 5% in *Algebra 1*, up to nearly 8% in *Precalculus*. In contrast, again excepting the *Geometry* book, the UCSMP series has decreasing percents of examples where technology is recommended, from over 13% in *Algebra* down to about 6% in *PCDM*. In the CPMP series, *Course 1* has the highest percent of examples recommending the use of technology, with about 12%. *Course 2* drops to the lowest percent, at about 8%. *Course 3* increases to about 9%, and finally about 11% of examples in *Course 4* recommend use of technology.

Comparison of Ancillary Features to Research

A number of researchers have been critical of an overly abstract approach to the function concept (Eisenberg, 1991; Markovits, et al., 1986; Oehrtman, et al., 2008; Schwingendorf, et al., 1992), although recent research has asserted that high school textbooks continue to present functions in predominantly abstract settings (Oehrtman, et al., 2008). There have been numerous calls for increased use of realistic contexts for functions for many years (Fey & Good, 1985; National Council of Teachers of Mathematics, 1989, 2000), and several researchers have noted student difficulties with using functions in realistic settings (Carlson, 1998; Eisenberg, 1992; Sierpinska, 1992; C. G. Williams, 1998). However, calls have been made for a balanced use of both abstract and realistic settings (Leinhardt, et al., 1990; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011; Schwartz & Yerushalmy, 1992). None of the textbook series provided an even balance between abstract and realistic settings, but instead favored abstract settings heavily. The UCSMP series had about three functions in abstract settings for each one in a realistic setting. The Glencoe series had even more abstract, and the CPMP series had somewhat less abstract. The CPMP *Course 1* textbook was the only one with more realistic settings than abstract. The later books in the series had only 30-40% of examples in realistic settings.

Researchers have recognized that the computational power of most technologies has the potential to change the way students engage with functions. (Asiala, et al., 1997; Fey & Good, 1985; Goldenberg, et al., 1992) Technology has the potential to help students engage with more representations of functions (Schwartz & Yerushalmy, 1992; Schwarz & Dreyfus, 1995; Schwarz & Hershkowitz, 1999; Zbiek & Heid, 2009), direct their own exploration of functions (Goldenberg, et al., 1992), and explore a wider variety of function families (Zbiek & Heid, 2009). Some have noted limitations in studying functions through technology; however, their encouragements have been to improve technology or instruction rather than to keep students from using technology to study functions (Asiala, et al., 1997; Goldenberg, et al., 1992; Kaput, 1992). In light of these recommendations, the three textbook series have relatively limited explicit directives for students to use technology with function examples. The weakest series was Glencoe, with less than 6% of all examples including recommendations for students to use technology. Both the UCSMP and CPMP had directions for student to use technology in about 10% of the function examples in the series. Although it is possible that these series included broader encouragement to use technology, it remains that in the best case, nearly nine out of every ten examples of function did not include an explicit recommendation that students use technology to engage with those specific examples.

Summary of Ancillary Features

Each series tended to present most examples of functions in abstract settings, and the proportion of realistic settings tended to decrease over the course of each series. The CPMP series had the highest proportion of realistic settings, while the Glencoe series had the smallest proportion. The Glencoe series only recommended the use of technology with about 6% of its examples, whereas the UCSMP and CPMP series included such recommendations with about 10% of each of their examples.

Archetypes

Based on the above results, archetypical examples of functions have been developed to embody the most common features of function examples in each textbook. These examples are not taken directly from each book, but rather have been created in an effort to capture all of the common features in one example. Each archetype most likely does not appear in the textbook, but many examples with similar features are present in the textbook, and thus the example serves as an archetype. In Watson and Mason's (2005) terminology, the archetype captures the most common features of the example space, and thus represents the core of the space. Certainly each textbook includes a wide variety of examples, many of which do not share the features of the archetype; however, the archetype provides an example containing the most common features. In the descriptions of these features in each textbook, parentheses are included after each feature providing the approximate percent of all examples for which the feature appeared.

Glencoe Series

The archetypical function examples for the Glencoe series are provided in Table 4.17. In Glencoe *Algebra 1*, most function examples were specific examples (97.3%) that occurred in the exercises (81.4%) and took up less than one row of text (58.4%). The textbook most often provided *y = Expressions* (21.9%) and *Verbal Descriptions* (23.2%). The most common requested representations were *Continuous* (15.5%) and *Smooth* (16.7%) graphic representations and *Verbal Descriptions* (15.5%). Most examples had a combination of different representations provided and requested (76.3%). Regardless of the representation, most examples were provided linear functions (56.1%) and generally in abstract settings (72.6%). In addition, examples were frequently labeled as functions (16.0%).

Table 4.17

Archetypes for the Glencoe series

Textbook	Archetype
Algebra 1	Graph the linear function: $y = 3x + 1$. Find the slope.
Geometry	The linear equation $y = 3x + 1$ contains (1,4). Graph it and find the slope.
Algebra 2	Graph the linear function, label 3 points, and find the slope: $f(x) = 3x + 1$.
Precalculus	Graph the linear function: $f(x) = 3x + 1$. Find the slope. Then write the equation in another form.
Series	Graph the linear function: $y = 3x + 1$. Find the slope.

In Glencoe *Geometry*, most function examples were again specific examples (98.2%) that occurred in the exercises (86.1%) and took up less than one row of text (53.1%). The textbook most often provided *Verbal Descriptions* (35.3%) and *Ordered Pairs* (29.4%). *y = Expressions* (18.1%) were commonly provided, but slightly less often than in *Algebra 1*. *Continuous* (14.6%) and *Smooth* (13.9%) graphic representations were also requested slightly less often than in *Algebra 1*, but there were about the same proportion of requests for *Verbal Descriptions* (15.2%). Most examples again had a combination of different representations provided and requested (77.3%). More examples were linear functions (72.4%) and abstract (84.9%) than *Algebra 1*.

In Glencoe *Algebra 2*, most function examples were again specific examples (97.7%) that occurred in the exercises (81.8%) and took up less than one row of text (59.2%). Two common representations were included in similar proportions: *f*(*x*) = *Expression* (25.2%) and *Verbal Descriptions* (25.7%). The percentage of requested

Continuous (15.6%) and *Smooth* (18.7%) graphic representations were similar to *Algebra 1*, but the proportion of requests for *Verbal Descriptions* (18.4%) was slightly higher. Most examples again had a combination of different representations provided and requested (79.0%). A much smaller proportion of examples were linear functions (33.7%) than either *Algebra 1* or *Geometry*; however it was the most common function family included. The percent of abstract (78.2%) examples in *Algebra 2* was similar to the previous two textbooks. In addition, nearly one-third of all examples were labeled as functions (32.4%).

In Glencoe *Precalculus*, most function examples were again specific examples (92.9%) that occurred in the exercises (82.6%). Examples tended to be somewhat longer than the other textbooks in the series; however, nearly half took up less than one row of text (45.0%). Note that in order to include all of the most common features, the archetypical example does not take up less than one row of text in Table 4.17 above. For the most common provided representations, the proportion of $f(x)$ = *Expression* (18.0%) representations was slightly less than *Algebra 2*, but the percent of *Verbal Descriptions* (34.2%) was higher. The most common requested representations were *Verbal Descriptions* (16.0%), *Symbolic: Other* (13.9%), and *Continuous* (9.8%) and *Smooth* (11.5%) graphic representations. Most examples again had a combination of different representations provided and requested (75.3%). As in *Algebra 2*, a smaller proportion of examples were provided linear functions (32.7%), but this family remained the most common included in the textbook. The proportion of abstract (82.8%) examples was again high, and the percent of examples labeled as functions (28.6%) was similar to *Algebra 2*.

Over the entire Glencoe series, most function examples were specific examples (96.2%) that occurred in the exercises (82.2%) and took up less than one row of text (54.5%). The series most often provided *y = Expressions* (16.6%) or *f*(*x*) = *Expressions* (16.5%) and *Verbal Descriptions* (28.0%). The most common requested representations included *Verbal Descriptions* (16.7%) and *Continuous* (13.8%) and *Smooth* (15.8%) graphic representations. Most examples had a combination of different representations provided and requested (77.0%). More examples were linear functions (42.6%) than any other family and were in abstract settings (78.4%). Over one-quarter of all examples were labeled as functions (25.2%).

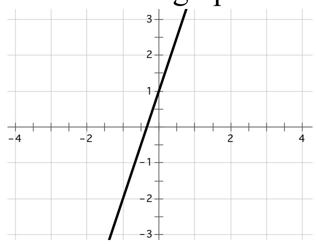
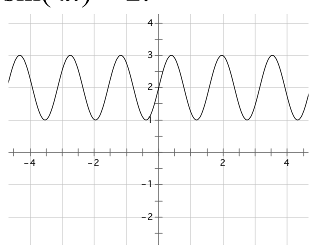
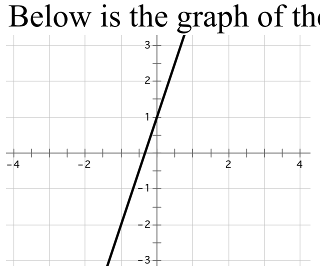
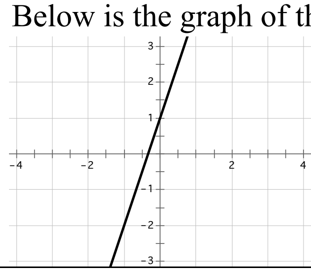
The most striking feature of the archetypes for the Glencoe series is their similarity. All are brief examples that provide a *Linear* equation and request students to create a graph and provide *Verbal Descriptions*. These textbooks did include other function representations and families; however, these archetypes make clear the fact that the most common core features of function examples had little variation over the entire series.

UCSMP Series

The archetypical function examples for the UCSMP series are provided in Table 4.18. In UCSMP *Algebra*, most function examples were specific examples (96.3%) that occurred in the exercises (77.8%). The most common length was examples that took up between three rows of text and one-quarter of a page (36.5%). The textbook most often provided *y = Expressions* (22.9%) and *Verbal Descriptions* (30.3%), but *Continuous* (16.2%) and *Smooth* (15.1%) graphic representations were also commonly provided.

Table 4.18

Archetypes for the UCSMP series

Textbook	Archetype
Algebra	Below is the graph of the linear equation $y = 3x + 1$.  Recreate the graph on a graphing calculator and use it to find 3 distinct ordered pairs that are on the line.
Geometry	Suppose triangle ABC has the vertices $A(2,3)$, $B(4,1)$, and $C(3,5)$. Let R_x be a reflection such that $R_x(A) = (1,3)$ and $R_x(B) = (-1,1)$. Find the image for C under the reflection. Then describe the line of reflection.
Advanced Algebra	Use the following linear function to complete each: $y = 3x + 1$ When $x = 2$, then $y =$ ________ When $y = 2$, then $x =$ ________ The slope of the function is: ________
Functions, Statistics, and Trigonometry	Below is a graph of the sine wave that represents the function $f(x) = \sin(4x) + 2$.  Recreate the graph on a graphing calculator and use it to find y values when x is π and when x is 2π. Then find the period and amplitude of the function.
Precalculus and Discrete Mathematics	Below is the graph of the linear function $f(x) = 3x + 1$.  Write the equation of the line in another format. Then find the slope of the line.
Series	Below is the graph of the linear function $f(x) = 3x + 1$ generated by a graphing calculator.  Write the equation of the line in another format. Then find the output when x is 2 and find the slope of the line.

Continuous (14.6%) and *Smooth* (14.2%) representations were also among the most frequently requested representations, along with *Ordered Pairs* (14.2%). Most examples had a combination of different representations provided and requested (82.6%). Regardless of the representation, examples were mostly linear functions (53.3%) and generally in abstract settings (65.7%). *Algebra* also included recommendations for use of technology frequently (13.6%).

In UCSMP *Geometry*, most function examples were again specific examples (89.8%) that were placed in the exercises (75.2%) and the most common length again was examples that took up between three rows of text and one quarter of a page (35.8%). The most common provided representations were *Verbal Descriptions* (31.3%) and *Numeric: f(x) Notation* (21.2%). *Verbal Descriptions* (15.9%) and *Numeric: Other* (13.3%) were the most common combined representations. Many examples had a combination of different representations provided and requested (54.9%). The most common function examples were geometric transformations (44.9%) and all examples were almost exclusively in abstract settings (96.4%).

As with *Algebra* and *Geometry*, in UCSMP *Advanced Algebra*, most function examples were specific examples (89.2%) that occurred in the exercises (81.9%) and even more examples took up between three rows of text and one quarter of a page (47.2%). The textbook most often provided *y = Expressions* (15.4%) and *Verbal Descriptions* (29.2%). As with *Geometry*, the most common requested representation was *Verbal Descriptions* (19.0%) and the most common combined representations were *Verbal Descriptions* (18.2%) and *Numeric: Other* (17.6%). Most examples had a combination of different representations provided and requested (78.1%). As with

Algebra, the most common function family was *Linear* (31.7%), although the proportion was much smaller. Most examples were again in abstract settings (73.0%), and more were labeled as functions (17.0%) than in the previous two textbooks.

In UCSMP *FST*, most function examples were again specific examples (90.3%) that were placed in the exercises (76.9%). Nearly half of the examples were between three rows of text and one quarter of a page long (46.0%). Several representations were commonly provided, including $f(x)$ = *Expressions* (21.7%), *Verbal Descriptions* (36.4%), and *Continuous* (11.0%) and *Smooth* (10.5%) graphic representations. *Symbolic: Other* (14.8%) and *Verbal Descriptions* (18.5%) were among the most frequently requested representations. The most common combined representations were *Numeric: Other* (13.8%) and *Verbal Descriptions* (20.1%). Most examples had a combination of different representations provided and requested (79.6%). Three function families were nearly equally represented and made up the largest proportions: trigonometric (11.4%), sets of ordered pairs (11.9%), and transformations (11.2%). Examples were mostly in abstract settings (71.6%). About one in every five examples was labeled as functions (21.0%), and over one-tenth of examples included recommendations for technology use (11.7%).

Similar to the other textbooks in the series, in *PCDM*, most function examples were specific examples (83.7%) that occurred in the exercises (74.0%). Almost half of the examples were between three rows of text and one quarter of a page long (49.5%). As with *FST*, several representations were commonly provided, including $f(x)$ = *Expressions* (27.6%), *Verbal Descriptions* (40.1%), and *Continuous* (11.0%) and *Smooth* (11.6%) graphic representations. The most common requested representations were *Symbolic: Other* (12.0%) and *Verbal Descriptions* (18.1%). Most examples had a combination of different representations provided and requested (78.6%). As with *Algebra* and *Advanced Algebra*, the most common function family provided was *Linear* (17.8%), although the proportion was much smaller than in these two textbooks. Most examples were again in abstract settings (84.5%), and more were labeled as functions (26.3%) than in any other textbook in the series.

Overall in the UCSMP series, most function examples were specific examples (89.4%) that occurred in the exercises (77.5%). The most common length for examples was between three rows of text and one-quarter of a page (44.8%). Several representations were commonly provided throughout the series, including *f*(*x*) = *Expressions* (17.1%), *Verbal Descriptions* (34.1%), and *Continuous* (11.2%) and *Smooth* (11.2%) graphic representations. The most common requested representations were *Symbolic: Other* (12.8%) and *Verbal Descriptions* (17.4%), and the most common combined representations were *Numeric: Other* (12.2%) and *Verbal Descriptions* (17.7%). Most examples had a combination of different representations provided and requested (78.3%). The most common function family was *Linear* (27.5%), and most examples were in abstract settings (75.6%). Examples were also commonly labeled as functions (17.6%) and included recommendations for technology use (10.0%).

The archetypes for the UCSMP series illustrate the variety of common function examples in the series. Examples tended to be longer than in the Glencoe series, *Graphic* representations tended to be both provided and requested, technology tended to be involved more frequently, and although linear functions were common, there were other types of functions more common in two of the textbooks. However, just as in the Glencoe textbooks, *Symbolic* representations were very common, except in *Geometry*.

CPMP Series

The archetypical function examples for the CPMP series are provided in Table 4.19. In CPMP *Course 1*, most function examples were specific examples (96.7%) that occurred in the exercises (59.7%). The most common length was examples that took up between three rows of text and one-quarter of a page (42.2%). The textbook most often provided *y = Expressions* (19.2%) and *Verbal Descriptions* (25.8%). The most common requested representations included *Continuous* (13.9%) and *Smooth* (13.6%) graphic representations, *Tables* (13.7%), and *Verbal Descriptions* (21.0%). *Verbal Descriptions* (35.1%) and *Numeric: Other* (16.7%) were the most frequently combined representations. Most examples had a combination of different representations provided and requested (80.3%). Regardless of the representation, the most common function family was linear (34.5%). Just over half of the examples were in realistic settings (58.3%), and over one third of the examples were labeled as functions (37.6%). *Course 1* also recommended use of technology (12.1%) with more than one in every ten examples.

In CPMP *Course 2*, most function examples were specific examples (91.6%) that occurred in the exercises (60.2%). The most common length was examples that took up between three rows of text and one-quarter of a page (31.3%), although this was less common than in *Course 1* and a larger proportion were shorter. The textbook most often provided *Verbal Descriptions* (32.2%). The most common requested representations included *Continuous* (21.3%) and *Smooth* (24.0%) graphic representations, *Symbolic: Other* (19.5%), *Verbal Descriptions* (16.4%), and *Numeric: Other* (15.2%). Combined *Verbal Descriptions* (30.0%) were also common. Most examples had a combination of different representations provided and requested (83.2%). The most common function family was again linear (43.3%). In contrast to *Course 1*, the majority of the examples were in abstract settings (62.4%). Again, over one-third of the examples were labeled as functions (36.9%).

Table 4.19

Archetypes for the CPMP series

Textbook	Archetype
Course 1	The high school drama production planned to charge \$3 per ticket sold for their performance. The relationship between number of tickets sold and income from tickets can be represented by the linear function $y = 3x$. a) Use a graphing calculator to graph the function. b) What is the slope of the linear function? c) Create a table of values relating number of tickets sold and income. d) What would the income be if they sold 145 tickets?
Course 2	Consider the linear function that intercepts the y-axis at 1 and in which the y-value increases by 6 for each increase of 2 in the x-value. a) Create a graph of the function. b) Find the slope of the function. c) Provide an equation for the function. d) Find one corresponding x and y value.
Course 3	Consider the linear function $f(x) = 3x + 1$. a) Create a graph of the function. b) Find the corresponding output when x is 5. c) Find the slope of the line and explain what it tells you about the function.
Course 4	Consider the linear function $f(x) = 3x + 1$. a) Use a graphing calculator to create a graph of the function. b) Find the slope of the function. c) Provide another equation for the function.
Series	Consider the linear function $f(x) = 3x + 1$. a) Use a graphing calculator to create a graph of the function. b) Provide another equation for the function. c) Find the corresponding output when x is 5. d) Find the slope of the line and explain what it tells you about the function.

CPMP *Course 3* had a vast majority of function examples that were specific examples (91.3%), but most were in lessons (53.4%). The most common length was again examples that took up between three rows of text and one quarter of a page (34.9%). The textbook most often provided *f*(*x*) *Expressions* (30.0%) and *Verbal Descriptions* (29.4%). *Continuous* (15.6%) and *Smooth* (20.1%) graphic representations were the most frequently requested representations. *Verbal Descriptions* (33.8%) and *Numeric: Other* (12.4%) were the most common combined representations. Most examples had a combination of different representations provided and requested (80.0%). The most common function family was again linear (20.9%), although this was a much smaller proportion than in the first two textbooks of the series. The majority of the examples were in abstract settings (70.2%), and the majority of the examples were labeled as functions (61.3%).

CPMP *Course 4* had a majority of function examples that were specific examples (83.7%), and most were in exercises (52.7%). The most common length was again examples that took up between three rows of text and one-quarter of a page (36.7%). The textbook most often provided *f*(*x*) *Expressions* (18.3%) and *Verbal Descriptions* (33.2%), and requested *Continuous* (12.2%) and *Smooth* (13.3%) graphic representations, *Verbal Descriptions* (17.7%), and *Symbolic: Other* (24.8%). *Verbal Descriptions* (31.2%) were also the most common combined representation. Most examples had a combination of different representations provided and requested (80.0%). The most common function family provided was again linear (19.1%); however, this was the smallest proportion in the entire series. The majority of the examples were in abstract settings (69.9%), almost half of the examples were labeled as functions (45.5%), and technology was recommended (11.2%) in better than one in every ten examples.

Overall, the CPMP series had a majority of function examples that were specific examples (82.1%) in the exercises (55.2%). The most common length was examples that took up between three rows of text and one-quarter of a page (36.4%). The textbook most often provided *f*(*x*) *Expressions* (14.5%) and *Verbal Descriptions* (30.7%). Most common requested representations were *Continuous* (15.0%) and *Smooth* (16.9%) graphic representations, *Symbolic: Other* (19.1%), and *Verbal Descriptions* (17.6%). In addition, *Verbal Descriptions* (32.3%) and *Numeric: Other* (10.6%) were the most frequent combined representations. Most examples had a combination of different representations provided and requested (80.7%). The most common function family was linear (27.5%). The majority of the examples were in abstract settings (63.0%), almost half of the examples were labeled as functions (45.4%), and technology was recommended (10.4%) in over ten percent of the examples.

The archetypes for the CPMP series illustrate the fact that the most common features of function examples remained fairly constant over the entire series. Examples tended to be longer than the Glencoe series, but like the Glencoe series, they tended to include *Symbolic* and *Verbal Description* representations and request *Graphic* representations. *Linear* functions were also the most common, although in smaller proportions than the Glencoe series. The CPMP textbooks also tended to request *Symbolic* representations and recommend the use of technology.

Comparison of Archetypes Between Series

The archetypes highlight several similarities between series. All three series most commonly provided verbal and symbolic representations of linear functions in the majority of their textbooks. Symbolic representations tended to be in function notation more frequently in later textbooks in each series. Requests for verbal representations were typical in most textbooks in each series, and requests for graphic representations were also common, especially in the Glencoe and CPMP series. Most of the examples were in abstract settings and part of homework exercises.

The archetypes also highlight several differences between series. Examples in the Glencoe series were typically much shorter than in the UCSMP or CPMP series. The UCSMP series tended to provide graphic representations more frequently and was the only series with textbooks where linear functions were not the predominant function family. The UCSMP and CPMP series included requests for numeric representations more often than the Glencoe series. The CPMP series included the only textbook with a majority of examples in a realistic setting (*Course 1*) and textbook with a majority of examples in lessons (*Course 3*).

Summary

The results presented in this chapter were based on the qualitative examination of definitions of functions and quantitative analysis of features of function examples provided or requested in the textbooks included in the study. The analysis was designed to provide information about the questions guiding the study concerning the language used in relation to function in textbooks, the presence of functions in textbooks, the core features of function examples, and ancillary features of function examples. The insights gained about textbooks and series related to each question are reviewed below.

Each series had some inconsistency in language used to define functions. The most constant features of definitions throughout all books were explicit indication of univalence and universal quantification and language that implied arbitrariness. References to domain and range in definitions were made infrequently, and language indicating the covarying relationship created by a function was almost nonexistent. Textbook series included two types of definitions: relationship of correspondence definitions and set definitions. The CPMP series used relationship of correspondence definitions almost exclusively, while the Glencoe series used the set definition exclusively in later textbooks, and the UCSMP series included both types of definition throughout the series. The CPMP series suggested the covariance created by functions slightly more strongly than the Glencoe or UCSMP series.

Language portraying functions as actions, processes object, or parts of larger schema was manifestly absent from all textbooks. Each series included predominantly specific examples, with limited provision of general examples. Thus the language used in these textbook series does not provide explicit support using APOS terminology for students to move toward more sophisticated conceptions of functions, such as understanding functions as mathematical objects.

Functions were present throughout all textbooks in each series. The Glencoe series included a vast number of examples of functions, with nearly 14,000, and had nearly twice as many examples per page as the UCSMP and CPMP series. However, these two series also averaged more than two examples per page over all textbooks in the series. Examples in these series were most often between 3 rows of text long and one-quarter of a page long, whereas Glencoe series tended to be much shorter. Thus, larger portions of the CPMP and UCSMP textbooks consisted of function examples. In the CPMP series, almost half of these examples were labeled as functions, while in the UCSMP series, less than 20% were labeled. The Glencoe series, with a smaller portion of textbooks being function examples, also only labeled about one in every four examples as functions. In the Glencoe and UCSMP series, these examples were largely in the homework exercises. In contrast, in the CPMP series, nearly half of all function examples were in the lessons.

Several features of function examples related to the presence of functions in each series occurred rarely. Few examples were requested of students, and of these, almost none were general examples. Each series provided few non-examples, and requested almost none. Only a small number of examples with errors were present in each series, especially in CPMP. Common errors included mismatches between representations of functions, insufficient information provided, typographical errors, and mislabeling relations as functions.

Core features of function examples had both similarities and differences among the three textbook series. All series provided a small proportion of examples with specified domains or ranges and virtually no examples with non-numerical domains or ranges specified. Each series did include a larger proportion of examples with requests for domains or ranges, but this still occurred less than 5% of the time. The Glencoe and UCSMP series included explicit references to domain and range almost twice as often as the CPMP series.

Symbolic representations and *Verbal Descriptions* were the most common function representations in the three series. The CPMP series tended to draw on *Verbal Representations* more frequently than the other series, and the Glencoe series relied more heavily on *Symbolic* representations. In all three series, there were more requests for or combined *Verbal Descriptions*, while *Symbolic* representations tended to be provided for students more frequently. *Numeric* and *Graphic* representations were less common in each series, but *Function Machine*, *Mapping Diagram*, *Physical*, and *Other* representations were virtually absent. However, the vast majority of examples in each textbook did include multiple representations for the functions.

In all three series, provided examples were prevailingly *Polynomial* functions. This was especially in the Glencoe series, in which about 60% of all examples were *Polynomial*. In each series, *Polynomial* examples were predominantly *Linear*, and again, especially in the Glencoe series. However, the proportion of *Linear* as well as *Polynomial* examples tended to decrease in later textbooks in each series, while the proportions of *Periodic* and *Rational* functions tended to increase. Examples coded as *Other* were also a noteworthy presence in each series, especially in the UCSMP series. In each series, among the variety of types of functions coded as *Other*, many were sets of ordered pairs or functions of multiple variables. Throughout the UCSMP series many functions coded as transformations were also included. However, ultimately in each series, a much larger proportion of functions were *Polynomial* than *Other*.

In relation to ancillary features of function examples, in each series most examples of functions were in abstract settings, with this tendency increasing over the course of each series. The Glencoe series had the smallest proportion of realistic settings, while the CPMP series had the highest proportion. No series included explicit directions for the use technology with more than 11% of function examples, and the Glencoe series included such directions least often, with about 6% of its examples.

Archetypes were developed to facilitate comparison of common features of function examples within and across textbook series. The examples developed illustrate the findings indicated above.

CHAPTER 5: SUMMARY, DISCUSSION, RECOMMENDATIONS, AND LIMITATIONS

Summary of the Problem and Research Questions

Textbooks have long been a center of debate in mathematics education (Reys & Reys, 2006; Schoenfeld, 2004; Willoughby, 2010). Poor performance by United States students on national and international mathematics assessments have lead to some condemning textbooks as the cause, while others have called for development and implementation of other types of textbooks (Schmidt, et al., 1997; Schoenfeld, 2004). Despite disagreements, few deny a need for improvement in mathematics instruction in the US, and many have argued that textbooks are potential vehicles for change in classrooms (Lloyd & Pitts Bannister, 2010; Remillard, 2005). Changes in textbooks may not be the only necessary change, but because both teachers and students use textbooks frequently in the US, they play an important role in the quest for improvement (Stein, et al., 2007; Tarr, et al., 2006). Indeed, understanding the composition of textbooks is vital to comprehending and evaluating the efficacy of any mathematics program (National Research Council, 2004).

Changes for improvement must be focused on key mathematical topics, and at the high school level, functions play a central role (Oehrtman, et al., 2008). There has been a long history in mathematics education in the US of calls for function to be emphasized in secondary mathematics (Commission on Mathematics, 1959; National Committee on Mathematical Requirements, 1923; National Council of Teachers of Mathematics, 1989). However, researchers have consistently found weak understanding of function and numerous misconceptions about function among students (Habre & Abboud, 2006; Harel

& Dubinsky, 1992). Current recommendations from the National Council of Teachers of Mathematics, while not explicitly calling for a restructuring of secondary mathematics around function, do emphasize the importance of function (T. J. Cooney, Beckmann, & Lloyd, 2010; 2000). There have been some indications of the influence of the NCTM's recommendations on current mathematics textbooks (Senk & Thompson, 2003), but debates continue about whether and how these changes have influenced student learning. Nevertheless, because of the importance of the function concept and the significant role that textbooks play in US classrooms, an understanding of the way that functions are treated in textbooks is vital.

The purpose of this study was to analyze functions in three current mathematics textbook series. The examples and definitions of functions presented in these textbooks influence how students come to understand the concept (Tall & Vinner, 1981), and therefore these were the aspects of the textbook series addressed in this analysis. The following questions guided the analysis:

1. What language do textbook series and individual textbooks analyzed in this study use in relation to function? How does this language compare to recommendations from research?
    a. How are functions defined?
    b. What is the distribution of examples of functions explicitly portrayed as actions, processes, objects, or parts of larger schema?
    c. What is the distribution of general and specific examples of functions?

2. What is the presence of function examples in textbook series and individual textbooks analyzed in this study? How does the presence of function examples compare to recommendations from research?

    a. How prevalent are functions in each textbook?

    b. What proportion of examples are explicitly identified as functions?

    c. What is the distribution of function examples placed in textbook lessons and homework exercises?

    d. How frequently are students asked to generate examples of functions?

    e. How frequently are students provided with non-examples of function?

    f. What errors in function examples are present?

3. How do textbook series and individual textbooks analyzed in this study present core features related to domain, range, representations, and families of functions? How does this presentation compare to recommendations from research?

    a. How frequently are domain and range of specific functions made explicit? What proportion of domains and ranges are numerical?

    b. What is the distribution of different representations of functions? How frequently do students have opportunities to engage with multiple representations of a function example?

    c. What is the distribution of families of functions?

4. How do textbook series and individual textbooks analyzed in this study present ancillary features of functions related to example settings and recommendations for use of technology? How does this presentation compare to recommendations from research?

a. What is the distribution of abstract and realistic settings for function examples?

b. How frequently is technology explicitly recommended for use with function examples?

Methodology

For this study, three high school mathematics textbook series were examined: Glencoe Mathematics, the University of Chicago School Mathematics Project (UCSMP), and the Core-Plus Mathematics Project (CPMP). In the Glencoe series, four textbooks were analyzed: *Algebra 1* (Carter, Cuevas, Day, Malloy, Holliday, et al., 2010), *Geometry* (Carter, Cuevas, Day, Malloy, Cummins, et al., 2010), *Algebra 2* (Carter, Cuevas, Holliday, et al., 2010), and *Advanced Mathematical Concepts: Precalculus with Applications* (*Precalculus*) (Holliday, et al., 2006). In the UCSMP series, five textbooks were examined: *Algebra* (Brown, et al., 2008); *Geometry* (Benson, et al., 2009); *Advanced Algebra* (Flanders, et al., 2010); *Functions, Statistics, and Trigonometry* (*FST*) (McConnell, et al., 2010); and *Precalculus and Discrete Mathematics* (*PCDM*) (Peressini, et al., 2010). In the CPMP series, four textbooks were analyzed: *Core-Plus Mathematics: Contemporary Mathematics in Context, Course 1* (*Course 1*) (Hirsch, et al., 2008a), *Core-Plus Mathematics: Contemporary Mathematics in Context, Course 2* (*Course 2*) (Hirsch, et al., 2008b), *Core-Plus Mathematics: Contemporary Mathematics in Context, Course 3* (*Course 3*) (Hirsch, et al., 2009), and *Core-Plus Mathematics: Contemporary Mathematics in Context, Course 4* (*Course 4*) (Hirsch, et al., 2010).

Every definition in each textbook was recorded and analyzed according to the following key features: type, arbitrariness, univalence, domain and range, covariation, and universal quantification. In addition, on every instructional page of each textbook every example of function that used one of the anticipated representations was coded. Coded features were in four main categories: language used in relation to functions, presence of functions, core features of functions, and ancillary features of functions. Data collected to provide information about the language used in relation to functions included function definitions, inclusion of Action-Process-Object-Schema terminology (Breidenbach, et al., 1992) with examples, and distribution of general and specific examples. Data collected to provide information about the presence of functions were the number of function examples, lengths of examples, the identification of examples as functions, placement of examples in lessons or exercises, number of non-examples included, frequency of requests for students to generate examples, and number of errors related to function examples. For the core features of functions, examples were coded for inclusion of domain and range, types of representations provided and requested, and families of functions used in examples. The codes capturing ancillary features of functions included the distribution of abstract and realistic settings for examples and the recommendation for technology to be used with examples.

The general approach to analysis of these data was to examine the distribution of various features through the use of relative frequencies of codes. The distributions of features were compared between books at the same grade level, between series, within books of a given series, and against research recommendations. For example, the percent of all functions that included *Symbolic* representations was calculated for each textbook. These percents were then compared across Glencoe *Algebra 1*, UCSMP *Algebra* and CPMP *Course 1*. The percents for *Algebra 1* were also compared to the other textbooks in the Glencoe series. The overall percents for the series were also compared to each other, and both textbook and series percents were considered in light of recommendations from research on functions in school mathematics.

Finally, archetypes were created to embody common features in each textbook and series. These archetypes provide further insight into findings and allow additional comparisons within and between series.

Findings

Language Used in Relation to Function

Definitions

The most consistent collection of definitions was in the CPMP series. The type of definition used consistently described functions as relationships of correspondence. All definitions explicitly indicated univalence and universal quantification while implying arbitrariness. The CPMP series also included stronger indications of covariance than the Glencoe or UCSMP series. The Glencoe series mainly included set definitions, especially later in the series. The definitions in this series also included the key properties of univalence and universal quantification and implied arbitrariness. The UCSMP series included both relationship of correspondence and set definitions in most textbooks, but almost all explicitly indicated univalence and universal quantification and suggested arbitrariness. References to domain and range in definitions were inconsistent in each of the three series, but the Glencoe series included the most references to them.

Inclusion of Action-Process-Object-Schema Terminology

There were virtually no explicit references to Action-Process-Object-Schema (APOS) terminology in examples of functions in the three textbook series. The Glencoe series had 14 references, the UCSMP series had 10 references, and the CPMP series had three references. The majority of these references in each series were to functions as processes, with only a few references to functions as actions or objects.

Distribution of General and Specific Examples

Examples were overwhelmingly specific in each series, with about 90% specific in the UCSMP and CPMP series and over 96% in the Glencoe series. Specific examples were examples in which at least one pair of corresponding elements could be determined, for instance, if the equation $y = 3x + 1$ was provided for students. In contrast, for general examples, no ordered pairs could be determined, for instance, if a textbook referred to a linear function without providing any information that could be used to find corresponding elements of the domain and range. Within each series, the proportion of specific examples decreased in later textbooks, and the last textbook in each series included the largest proportion of general examples. Glencoe *Precalculus* had approximately 7% general examples and UCSMP *PCDM* and CPMP *Course 4* included over 16% general examples each.

Presence of Functions

Number of Function Examples and Lengths of Examples

Nearly 30,000 examples of functions were coded for this study. The Glencoe series had almost 14,000 examples, but a large proportion of these took up less than one row of text in the book. *Algebra 2* had the most examples of any textbook, with over 5,000, *Algebra 1* and *Precalculus* had about 4,000 each, and *Geometry* had less than 800. Similarly, UCSMP *Geometry* had about 500 examples, but each of the other textbooks in the series had between 2,000 and 3,000, for a total of over 10,000 examples. Larger portions of these examples were longer than the Glencoe series. There were about 5,600 examples in the CPMP series. These examples were also generally longer than the Glencoe series, most often taking up between three rows of text and one-quarter of a page. The first three textbooks had about 1,000 each, and *Course 4* had just over 2,000. The CPMP and UCSMP series averaged about 2.3 examples per page, and many of these examples took up between 3 rows of text and one quarter of a page. In contrast, the Glencoe series averaged nearly 4 examples per page, but these tended to take up less than one row of text each.

The Identification of Examples as Functions

The proportions of function examples that were explicitly identified as functions ranged from nearly half in the CPMP series to less than 20% in the UCSMP series, with about one in every four examples identified as a function in the Glencoe series. In all three series, the proportion tended to be higher in later textbooks, and almost no examples were labeled as functions in either *Geometry* textbook. CPMP *Course 3* was the only textbook that labeled over half of its examples as functions. In all series, students were infrequently asked to decide if examples were functions, only reaching as high as approximately 2% in UCSMP *Advanced Algebra* and CPMP *Course 1*.

Placement of Examples in Lessons or Exercises

The majority of examples in each series were in the exercises. This majority was stronger in the Glencoe and UCSMP series, with nearly 80% of all examples being presented in exercises in these series. In the CPMP series, about 55% of examples were in the exercises. Different textbooks within each series had the highest proportion of examples in exercises: *Geometry* in Glencoe, *Algebra 2* in UCSMP, and both *Course 1* and *Course 2* in CPMP.

Frequency of Requests for Students to Generate Examples

A small proportion of examples were requests for students to generate examples in each textbook series. In the Glencoe and UCSMP series, all textbooks requested between 100 and 200 examples except the *Geometry* textbooks, which requested about 10. The number of requests in the CPMP series ranged from 61 in *Course 3* to 163 in *Course 2*. There were virtually no requests for general examples in any series, with seven requests in Glencoe *Precalculus* being the highest total.

Number of Non-examples Included

There were relatively few identified non-examples of functions provided in each series. The Glencoe and UCSMP series each had about 90, with none provided in the respective *Geometry* textbooks. The CPMP series had 20 non-examples, with 13 of these in the *Course 2* textbook.

Number of Errors Related to Function Examples

There were a few errors in function examples in each series. The CPMP series had two in the series, both in *Course 4*. The Glencoe series had 26 total, with ten occurring in *Algebra 2*. The UCSMP series had 16, with seven in *PCDM*. Errors tended to be mismatches between representations of functions, insufficient information provided, labeling examples functions inappropriately, and apparent typographical errors, although the current analysis cannot indisputably determine their cause.

Core Features

Domain and Range

Only a small proportion of function examples specified domain and range or requested that students indicate the domain and range, and in virtually all of these examples, domains and ranges were numerical. About 6% of the Glencoe and UCSMP series and about 4% of the CPMP series specified or requested numerical domains and ranges. In the CPMP series, the percent increased in each textbook. In the Glencoe and UCSMP series, the *Geometry* textbooks did not specify or request any domains or ranges, but there was no other clear trend of changes in proportions.

Representations

In each series, about 75% of all examples included *Symbolic* representations. For the Glencoe and UCSMP series, this represented the most common representation, while in the CPMP series, *Verbal Descriptions* was slightly more common. The *Symbolic* representations provided for students were most often y = *Expression* in the earlier textbooks in all of the series, but they became increasingly $f(x)$ = *Expressions* in later textbooks in the series.

Also very common were *Verbal Descriptions*, which were included with over 80% and over 70% of all examples in the CPMP and UCSMP series, respectively. The Glencoe series included fewer *Verbal Descriptions*, with just over half of all examples. In the CPMP and UCSMP series, more than half of the *Verbal Descriptions* were either requested of students or a combination of provided and requested. In the Glencoe series, just over half of the *Verbal Descriptions* were provided for students. For instance, an example in the Glencoe series would be more likely to tell students that the function is a linear function, thereby providing a verbal description, whereas an example in UCSMP or CPMP would ask students to identify the type of function or to describe whether it was increasing or decreasing.

Numeric representations were less common in each of the series than *Symbolic* or *Verbal Descriptions*, and least common in the CPMP series, in which less than 40% of all examples included this representation. The other two series included *Numeric* representations in nearly half of all examples. The tendency over each series was toward fewer *Numeric* representations in later textbooks. In each series there was also a balance between these representations being provided, requested, and a combination of these. *Ordered Pair* representations were somewhat common in earlier textbooks in each series, but *Other* representations were the most frequent. These were typically provisions of one element and requests that students provide the corresponding element. An example of such a situation would be, "For the function $y = 3x + 1$, find y when x is 2."

Graphic representations were less common than *Numeric*, but more common than *Function Machine*, *Mapping Diagram*, *Physical*, and *Other* representations, which were rare. The proportions of *Graphic* representations were consistently around 10% in each textbook in the Glencoe and CPMP series. The UCSMP *Algebra* and *FST* textbooks had nearly 20%, but *Geometry* had less than 5%, while the other textbooks had around 12%. The vast majority of all of these representations were *Continuous* and *Smooth*, for example, graphs of polynomial functions. In the Glencoe and CPMP series, there were more requests for *Graphic* representations, but in the UCSMP series, more were provided. None of the series included many combined *Graphic* representations in which part of the graph was provided and students were asked to complete the graph.

Families of Functions

By far the most common function family used in examples was *Polynomial*. These examples tended to be *Linear*, especially in earlier textbooks. The Glencoe series had the highest proportions, with over 50% of functions in every textbook being *Polynomial*, and over half of these being *Linear* in each textbook. In Glencoe *Geometry*, over 70% of all function examples were *Linear*. These proportions were not as high in the other two series; however, at least 45% of all functions in the UCSMP and CPMP series were *Polynomial*. Over the course of each series, the proportion of *Polynomial* and *Linear* functions tended to decrease. The UCSMP series had two textbooks that included smaller proportions of *Polynomial* functions: *Geometry* and *FST*. In both of these, *Other* functions were more common, with transformations being common in *Geometry* and sets of ordered pairs being common in *FST*. Only three other function families were represented in more than 2% of all examples in each series: *Periodic*, *Exponential*, and *Rational*. These families tended to make up slightly larger proportions of examples in later textbooks in each series. A notable exception was CPMP *Course 1*, which had over 20% *Exponential* functions.

Ancillary Features

Abstract and Realistic Settings for Examples

In all but one textbook, the majority of examples were in abstract settings. CPMP *Course 1* provided realistic contexts in nearly 60% of its examples. In both the Glencoe and UCSMP series, about 75% of all examples were in abstract settings. In contrast, in the CPMP series, about four of every ten examples had realistic settings. For instance, in CPMP *Course 2*, students were asked to determine the function family for the following example, “The number of tickets sold to a charity basketball game is a function of the price charged with rule $N = 4{,}000 - 50p$” (p. 6). In contrast, the Glencoe and UCSMP series would have been more likely to asked students what type of function was represented by the equation $N = 4{,}000 - 50p$ without the description of tickets being sold for a basketball game. With exception of the *Geometry* textbooks, which provided nearly all examples in abstract settings, the trend in each series was for more examples to be in abstract contexts in later textbooks.

The Recommendation for Technology to be Used with Examples

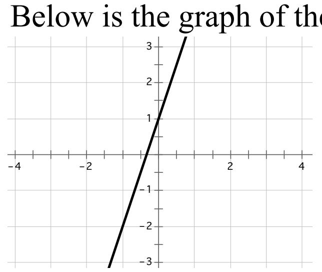
The Glencoe series included the smallest percent of examples with explicit recommendations for students to use technology, with less than 6% of all examples. The UCSMP and CPMP series recommended students use technology with at least 10% of examples. For instance, it would be slightly more common for examples in the UCSMP and CPMP series to direct students to use a graphing calculator to generate the graph of a function, as compared to the Glencoe series, which was more likely to simply request that students graph the function without explicit directions to use a graphing calculator. For the CPMP series, the percent was fairly consistent across textbooks, but in the Glencoe and UCSMP series, the *Geometry* textbooks included fewer recommendations.

Archetypes

The archetypical example created for the overall Glencoe series is in Table 5.1. This archetype highlights the fact that examples in the Glencoe series tended to be less than one row of text in abstract settings, identify examples as functions, provide linear equations and verbal representations, and request students to create graphs. Although the archetype shows that y = *Expression* representations were most common in the series, later textbooks in the series did use $f(x)$ = *Expression* representations more often.

Table 5.1

Archetypes for the each series

Series	Archetype
Glencoe	Graph the linear function: $y = 3x + 1$. Find the slope.
UCSMP	Below is the graph of the linear function $f(x) = 3x + 1$ generated by a graphing calculator.  Write the equation of the line in another format. Then find the output when x is 2 and find the slope of the line.
CPMP	Consider the linear function $f(x) = 3x + 1$. a) Use a graphing calculator to create a graph of the function. b) Provide another equation for the function. c) Find the corresponding output when x is 5. d) Find the slope of the line and explain what it tells you about the function.

The archetypical example created for the UCSMP series in Table 5.1 highlights differences from the Glencoe series. Both *Symbolic* and *Graphic* representations of functions were both provided for students and requested from them frequently. *Numeric* representations were also common in four of the five textbooks. Examples tended to be longer than in the Glencoe series and included requests for the use of technology more frequently. Although the archetype includes a linear function as the most common family for the series, transformations and trigonometric functions were actually more common than linear functions in *Geometry* and *FST*, respectively.

In the CPMP series, the most common features of function examples remained relatively consistent over the series. These features are evident in the archetype for the series in Table 5.1. Again, in comparison to the Glencoe series, examples in the CPMP series were longer and included more recommendations for the use of technology. Over the entire CPMP series, functions were labeled as such more frequently than in the other series. Like the Glencoe series, examples tended to be in abstract settings, provide linear equations, provide *Verbal Descriptions*, and request *Graphic* representations most often. However, *Symbolic* representations and *Verbal Descriptions* were requested more often.

Discussion

Language Used in Relation to Function

The collections of definitions in each series had a number of similar features. They shared the strengths of clearly identifying the univalence and universal quantification concepts as well as at least implying the arbitrary nature of functions. These features are vital to the function concept (Even, 1993). In contrast, no series consistently made connections to domain and range concepts in function definitions. A number of researchers (Dubinsky & Harel, 1992; Markovits, et al., 1986; Schwingendorf, et al., 1992; Sfard, 1992; Tall & Bakar, 1992) have found that students are confused about, or simply do not give any attention to, domain and range of functions. When this finding is coupled with the finding that few function examples included explicit reference to domain and range, it becomes clear that in most instances, functions are presented to students in the textbook in a way that does not encourage them to think about domain and range. In addition, almost no reference to the covariation created in function relationships was made. Carlson and her colleagues (Carlson, et al., 2002; Oehrtman, et al., 2008) as well as other researchers (Ferrini-Mundy & Lauten, 1994; Goldenberg, et al., 1992; Leinhardt, et al., 1990; Monk, 1992) have noted that students struggle to understand functions as creating dynamic covariation. Based on these textbook definitions, it can be argued that they could be a contributing factor to students' lack of understanding.

Definitions were categorized by type: variable, relationship of correspondence, or set of ordered pairs. Each textbook series used different types of definitions, and each approach had potential strengths and weaknesses. After providing relationship of correspondence definitions in *Algebra 1*, the Glencoe series provided all set definitions but one. Some researchers have harshly criticized the use of the set definitions with all but the most advanced secondary students (Markovits, et al., 1986; Sfard, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989). However, researchers who claim students' development should mimic historical development of the function concept (Sfard, 1992; Sierpinska, 1992) and researchers using the APOS framework (Asiala, et al., 1996; Breidenbach, et al., 1992; Dubinsky & Harel, 1992; Eisenberg, 1991, 1992; Habre & Abboud, 2006; Kaput, 1992; Sfard, 1992; White & Van Dyke, 2006; C. G. Williams, 1998) might suggest that moving from a relationship of correspondence definition to a set definition follows the anticipated trajectory. A key question would then be how students are supported along this trajectory. If the type of definition is changed without further support or changed before students are ready, a move to the set definition would not be appropriate. This study was limited to the definitions themselves, and did not examine supporting text or additional support that a teacher might provide. Therefore it is not appropriate to judge how the switch to the set definition made in the Glencoe textbook impacts student learning. Instead, the findings only highlight the fact that there was a shift.

In contrast, in the CPMP series, almost all definitions are the relationship of correspondence type, and students are not introduced to the set definition. The advantages of this collection of definitions are the consistency that it provides students while avoiding the criticisms of use of the set definitions. However, one must ask whether students could be prepared to make sense of the set definition by the fourth course in the series designed to prepare them for calculus (Hirsch, et al., 2010). Again, this is a question that cannot be answered in this study, rather, this study points out the need to consider this question when implementing the curriculum.

Finally, the UCSMP series used a combination of the relationship of correspondence and set definitions throughout the series. A potential weakness of this approach is the lack of consistency for students. For example, a student using the *Algebra* textbook could read in the text that a function is a correspondence, but then look in the glossary and read that it is a set of ordered pairs. Without an understanding of how these two conceptions of functions are both accurate and important, a student may be confused. On the other hand, if teachers know that both types of definition are provided for students, this may provide opportunities for teachers and students to discuss both definitions and come to an understanding of how they work together.

There were few general examples of functions, especially in the Glencoe series, where less than 4% of examples were general. Examples were considered general when no corresponding domain and range elements could be identified, for instance, if a textbook described function f as an even polynomial function, but did not provide a formula or other means to actually find ordered pairs for the function. Researchers proposing the developmental trajectory of students' conception of function through the APOS framework suggest that students should work with general functions as their development of the concept progresses (Asiala, et al., 1997; Breidenbach, et al., 1992; Dubinsky & Harel, 1992). Although each series did tend to increase the percent of general functions in later textbooks, the proportions remained relatively small. In addition, there was virtually no use of APOS terminology to support students' development of the function concept. Thus, it appears that in all three series, opportunities to provide students with explicit language related to functions that could support more sophisticated understandings of the concept are limited. Almost all examples were specific functions for which students could continue to hold an Action or Process conception, rather than progress toward an Object conception and integrate the function concept into larger Schema.

Presence of Functions

There continue to be arguments made for the importance of the function concept in secondary mathematics (T. J. Cooney, et al., 2010; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011). With several thousand examples in each textbook series and averages of about two to four examples per page for each series, it is evident that the potential for developing the concept is certainly present in these textbooks. Even the *Geometry* textbooks in the Glencoe and UCSMP series included more than 700 and 500 examples, respectively. However, the question remains whether the textbooks are directing students' attention to the function concept and attempting to teach them about functions through these examples. In the Glencoe and UCSMP series, only about one-quarter of the examples were actually identified as functions. In the CPMP series, closer to half of the examples were labeled as functions. In addition, much research has been conducted related to students' difficulty with determining whether an example is a function or not. (Baker, et al., 2000; Even, 1993; Leinhardt, et al., 1990; Tall & Bakar, 1992; Vinner, 1983; White & Van Dyke, 2006). Thousands of opportunities to explicitly connect the function concept with examples were missed in each series. The function concept could be an even more significant element in these series if a higher proportion of the examples provided were labeled so that students would be encouraged to consider them as functions.

In order to better understand the boundary between functions and non-functions, researchers have argued for the importance of student engagement with non-examples to make comparisons and develop deeper understanding of a concept (Eisenberg, 1991; Leinhardt, et al., 1990; Tall & Bakar, 1992; Tsamir, et al., 2008; Vinner, 1991; Watson & Shipman, 2008; Zodik & Zaslavsky, 2008). However, in each of these series, students are presented with very few non-examples of functions. Dahlberg and Housman (1997) have also argued that students should construct their own examples of functions, but there were relatively few requests for students to provide examples in each of the series.

Core features

Domain and Range

As noted above, only a small proportion of functions included explicit reference to domain and range. For other examples, it appeared that students were to assume domain and range were all the real numbers allowed or suggested by the function representation. Almost no functions had domain or range elements that were non-numerical. Tall and Bakar (1992) found students tended to believe the domain must be complete in some sense, and Dubinsky and Harel (1992) found that students usually believe domain and range elements must be numbers. Students studying from any of these series would seldom see examples that would help them attain understandings different from either of these findings.

Representations

One of the most frequent recommendations in research on student learning of function is that students experience a wide variety of representations. (Akkoc & Tall, 2005; Leinhardt, et al., 1990; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011; Oehrtman, et al., 2008; Sfard, 1992; Tall, et al., 2000). Each of the textbook series provided multiple representations of functions with over 75% of the examples provided. Inclusion of multiple representations could be as simple as provision of a graph of a function with an ordered pair labeled, or as complex as provision for a table of values and requests from which students should create an equation, a graph, and a description of the function.

Both *Symbolic* representations and *Verbal Descriptions* were the most common representations used in each of the textbook series. Often *Verbal Descriptions* were as simple as noting that a function was, for example, linear. *Verbal Descriptions* were more predominant in the CPMP series, and *Symbolic* representations were most common in the Glencoe series, although the UCSMP series had nearly the same proportion, and the CPMP series also included *Symbolic* representations in about 70% of all examples. The proportions of *Symbolic* representations also tended to increase over each series. Several researchers have noted students' tendency to prefer *Symbolic* representations and even believe functions must have or are the same as *Symbolic* representations (Asiala, et al., 1997; Dahlberg & Housman, 1997; Gonzalez-Martin & Camacho, 2004; Habre & Abboud, 2006; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Sierpinska, 1992; Vinner, 1983). The collection of examples presented to students in these textbook series would certainly support student preference for *Symbolic* representations.

Smaller proportions of *Numeric* representations were used in the textbook series, and these tended to decrease over each series. Even fewer *Graphic* representations were included. The series had remarkably similar distributions of these representations, especially the Glencoe and CPMP series. In each, only about one third of the examples included *Graphic* representations, and the majority of these were requested from students. A number of researchers have argued that students should study and use graphs of functions regularly (Dahlberg & Housman, 1997; 1991, 1992; Ferrini-Mundy & Lauten, 1994; Schwartz & Yerushalmy, 1992; Sfard, 1991). However, many have found that students have a general lack of understanding of what graphic representations of functions mean (Bridger & Bridger, 2001; DeMarios & Tall, 1999; Goldenberg, et al., 1992; Schwingendorf, et al., 1992; Sierpinska, 1992; White & Van Dyke, 2006). Other researchers have pointed out that students often reject non-continuous and non-smooth graphs as functions (Carlson, 1998; Dubinsky & Harel, 1992; Even, 1993; Markovits, et al., 1986; Sfard, 1992; Tall, 1990; Tall & Bakar, 1992; White & Van Dyke, 2006). Graphs in all three series were almost all *Continuous* and *Smooth*. There is potential for providing students greater opportunity to develop deeper understanding of the function concept through *Graphic* representations through inclusion of more graphs, and especially of functions that do not have *Smooth* or *Continuous* graphs.

Each textbook series included very few *Physical* representations, *Mapping Diagrams*, and *Function Machines*. A number of researchers have pointed out various potentialities for each of these representations (Akkoc & Tall, 2005; DeMarios & Tall, 1999; Kaput, 1992; Leinhardt, et al., 1990; Monk, 1992; Selden & Selden, 1992; Tall, 1996; Tall, et al., 2000); however, students studying examples in these textbooks will seldom have the opportunity to experience them.

Families of Functions

Polynomial functions were predominant in each series and most textbooks, but especially in the Glencoe series. Of these functions, *Linear* functions were by far the most common, with *Quadratic* functions the second most frequent. The proportions of *Polynomial* and *Linear* functions tended to decrease across the books in each series; however, hundreds of *Linear* functions are still provided in the final textbooks in each series. Many researchers have found that students tend to associate the function concept most strongly with linear and quadratic functions (Even, 1993; Leinhardt, et al., 1990; Markovits, et al., 1986; Oehrtman, et al., 2008; Schwarz & Hershkowitz, 1999). Others have argued that presenting students with a limited variety of function examples as they initially learn about functions can impede their understanding of the concept (Even, 1993; Leinhardt, et al., 1990; Tall, 1990; Vinner, 1983; White & Van Dyke, 2006). For example, several researchers have found that students have difficulty understanding *Constant* examples as functions (Even, 1993; Leinhardt, et al., 1990; Oehrtman, et al., 2008; Sfard, 1992; Tall & Bakar, 1992), a function family that was only minimally represented in each textbook series. Carlson (1998) claimed that students need to repeatedly study examples that represent extreme cases of the function concept. It is clear that the textbooks in these series are presenting many examples of *Polynomial*, and especially *Linear*, functions. However, textbooks also provide examples from a variety of other families. The question of whether students experience a sufficient variety of functions cannot be addressed through the research presented here and warrants further study.

Ancillary Features

Abstract and Realistic Settings

Taking an overly abstract approach to the function concept has been criticized by a number of researchers who argue that students need connections to realistic settings in order to make sense of the concept and be motivated to learn about it (Eisenberg, 1991; Fey & Good, 1985; Markovits, et al., 1986; Oehrtman, et al., 2008; Schwingendorf, et al., 1992). Researchers have also noted student difficulties with using functions in realistic settings (Carlson, 1998; Eisenberg, 1992; Sierpinska, 1992; C. G. Williams, 1998). In contrast, almost all of the textbooks reviewed included far more functions in abstract than realistic settings. The Glencoe series had the highest percent, with over 75% of all examples in abstract settings. The CPMP series had the lowest percent, but was still over 60%. In each series, the trend was for larger proportions of function examples to be in abstract settings later in the series. These findings agree with Oehrtman, Carlson, and Thompson's (2008) claim that high school textbooks continue to present functions in predominantly abstract settings.

Technology Recommendations

Researchers have recognized the power for technology to support students' learning of functions in a variety of ways (Asiala, et al., 1997; Fey & Good, 1985; Goldenberg, et al., 1992; Schwartz & Yerushalmy, 1992; Schwarz & Dreyfus, 1995; Schwarz & Hershkowitz, 1999; Zbiek & Heid, 2009). A comprehensive analysis of the relationships between technology and function in these textbooks was not done. Instead, only explicit recommendations for use of technology were coded, and such recommendations were limited in all three series. Only about 10% of examples in UCSMP and CPMP included recommendations to use technology, and less than 6% of examples in the Glencoe series called for technology use explicitly. It is certainly possible that students are encouraged to use technology to study functions in other ways, for example, by broad recommendations for use of technology in the textbooks or by encouragement from the teacher. These possibilities fall outside the scope of this study. What can be determined from these results is that when looking at specific function examples, even in the series with higher proportions of recommendations for technology use, students are still only explicitly encouraged to use technology with function examples 10% of the time.

Cognitive Demand

For this study, attempts to find Action-Process-Object-Schema (APOS) language in examples were intended to illuminate the cognitive demand of function examples. Language associated with higher APOS levels would indicate higher levels of cognitive demand. For instance, examples encouraging students to consider functions as mathematical objects would be more demanding than ones encouraging students to understand functions as processes. However, use of such language was rare in function examples in all series, and therefore this approach provided little direct evidence about how cognitively demanding function examples might be. This finding suggests that the textbooks are not using one potential tool for helping students develop more sophisticated understandings of functions. For example, anticipating that students in Algebra 1 generally hold an Action conception of function, examples could include language that would explicitly challenge this understanding and encourage students to think about functions as a process that they can describe without actually computing output values for the function. Later textbooks could further encourage students to think about how examples of functions could actually be thought of as mathematical objects that they could operate on by, for example, adding two functions together. It was certainly the case that textbooks included examples where students were asked to add functions; however, the APOS language that could support students' development was seldom included with these examples.

Although examining cognitive demand through APOS language was not possible with these series, some other coded features of function examples can be connected to the APOS framework in ways that suggest levels of cognitive demand. The use of general examples suggests higher cognitive demand because students would not be able to deal with these examples with an Action conception. Similarly, requests for students to generate examples that meet certain criteria would also force students to consider the features of functions as processes rather than simply using the functions to carry out an action. Examination of non-examples of functions also encourages students to consider the properties of the functions in a general way, rather than holding an Action conception of function. Inclusion of general examples and non-examples and requests for students to generate examples of functions were relatively rare in all three series, and therefore opportunities to encourage students to move beyond an Action conception of function in these ways were limited. The proportions of general examples did increase later in each series, but such a pattern was not present for non-examples and requests for students to generate examples. Additional use of these three features throughout each series would likely increase cognitive demand.

Research has not clearly indicated that specific representations are closely associated with specific aspects of the APOS framework. For example, students often view *Symbolic* representations through an Action conception as they use the equation or formula to compute output values of a function, but *Symbolic* representations also allow students to understand functions as mathematical objects that they can add together, for example. Similar arguments have been made for the other typical representations. Nevertheless, research suggests that a cognitively demanding task for students is to translate from one function representation into another. From the APOS perspective, such a task would encourage students to attend to the general aspects of the function rather than on carrying out the action of computing output values. In this respect, each textbook series provided students many opportunities to translate function representations. In Glencoe and UCSMP, over 60% of examples included both provision of and requests for function representations, and in CPMP, nearly 75% of examples made a similar request. This suggests that cognitive demand for most examples encourages students to at least move beyond an Action conception of function.

Research also suggests that functions in realistic settings can be more cognitively demanding for students. Connections between realistic settings and APOS theory have not been discussed extensively in APOS research, but Carlson and her colleagues (Carlson, 1998; Carlson, et al., 2002; Oehrtman, et al., 2008) have suggested that realistic settings can encourage students to grapple with understanding the covariational nature of the function example. This appears especially true when they are asked to use functions to create models of realistic situations and data. The CPMP series provided realistic settings for a larger proportion of examples than the Glencoe or UCSMP series, and thus may be providing greater cognitive demand in more examples. However, this finding is restrained by the fact that the coding used for this analysis did not capture how students were asked to make connections between the function and the setting. The realistic settings for some examples likely serves as little more than extraneous adornment that students can simply ignore as they engage with the example, whereas in other examples, students must make significant connections between setting and function.

Overall, although ranking of examples by levels of cognitive demand is not possible, information about the cognitive demand of examples can be indirectly examined. Low levels of cognitive demand was suggested in all three series as none use general examples, non-examples of functions, or requests for students to generate examples of functions extensively. However, inclusion of requests for students to translate between function representations implied that cognitive demand was not at the lowest levels in over 60% of examples in Glencoe and UCSMP and over 75% of examples in CPMP. Realistic settings, also possibly suggesting higher levels of cognitive demand, were also more prevalent in CPMP. However, the relative consistent percents of each of these features across series as well as no deliberate use of APOS language in the series suggests that no series is consistently using these opportunities to provide students with function examples with increasing levels of cognitive demand.

Implications for Future Curriculum Development

The results of this analysis should be useful for future curriculum development. With respect to definitions, most included key features of univalence, universal quantification, and arbitrariness, and these need to continue to be clearly included in definitions of functions. Attention to domain and range within definitions was mixed. Given that many students do not attend to this core feature of functions, curriculum developers should consider ways to demonstrate the importance of domain and range to the function concept by consistently including them in definitions of function. Research has also demonstrated that students have difficulty making sense of the set definition of functions. An abrupt switch to use of the set definition, or use of both relationships of correspondence and set definitions could be confusing to students. The analysis presented in this study can support the intentional shaping of definitions over a series in a way that can help students move from an intuitive understanding of connections between two varying quantities toward a formal conception of function.

The findings regarding the collections of examples of functions in each textbook and over series can similarly support the creation of example spaces in textbooks that support students' development of rich conceptions of function. The collections of examples in the three series examined in this study were predominantly specific, abstract, and polynomial. Most examples lack description of domain and range, and very few used representations outside of *Symbolic*, *Verbal*, *Numeric*, or *Graphic*. There were relatively few opportunities for students to generate examples or explore non-examples of function. Research suggests that students could benefit from changes to these features. Students need opportunities to grapple with general functions and with functions in realistic settings. Students too often associate functions with the features of polynomial functions represented with equations and graphs. They need to explore functions that cannot be represented easily with equations or with graphs that are not *Smooth* and *Continuous*. As Vinner (1991) argued, they need to explore "weird examples" (p. 80) of functions. Students' attention needs to be directed to domains and ranges more often. They need to be asked to generate examples of functions and to wrestle with non-examples of functions more frequently. This study also found almost no explicit use of language reflecting student thinking about function as proposed in APOS research (Breidenbach, et al., 1992). Curriculum developers could consider intentionally using the findings of this body of work to encourage student progression toward a schematic understanding of function.

The collections of examples in the textbooks did have features that should serve students well as they learn about functions, and curriculum developers should seek to maintain these features. One strength across all series was the inclusion of multiple representations for the majority of function examples. Research has demonstrated that students need to engage in the process of moving between function representations in order to understand better what the representations mean as well as gain deeper understandings of the function concept. Each series presented students with ample opportunities for this. In addition, although textbook series were dominated by *Linear* functions in early textbooks, there was a greater mix of function families in later textbooks in each series. Curriculum developers need to continue to present students with examples of functions from diverse families and explicitly confront students' concept images of function that are too often intensely linear.

Each series did have unique strengths as well. The vast number of function examples in the Glencoe series means that it is probable that students studying from these textbooks will daily explore functions. However, the majority of these examples were not labeled as functions. Curriculum developers for Glencoe should consider making the fact that students are regularly working with functions more explicit to the students. One strength of the UCSMP series was its framing of transformations as functions and use of transformations throughout the series. Repeatedly studying transformations as functions provides students examples of functions that are clearly not *Polynomial*, which supports the formation of richer concept images of function. Although the Glencoe and UCSMP series had a large majority of examples in abstract settings, the CPMP series offered a more balanced approach, with nearly half of the examples being in realistic settings. These realistic settings should especially support students' development of understanding the covariance embodied in a function relationship.

Summary of Discussion

Review of the collections of examples in the three series and comparison to research recommendations and theory highlights several key findings from the study:

- Each series contained many function examples; however, the majority were not identified as functions.
- Domain and range were not consistently included in definitions nor specified for most function examples.

- Functions were overwhelmingly represented with *Symbolic* representations; however, most examples included multiple representations, and thus *Verbal Descriptions* were common as well.
- Function examples were predominantly *Polynomial*, and especially *Linear*.
- The majority of functions were presented in abstract settings.

These findings indicate key areas that need to be addressed in future curriculum revision and development. Collectively, these findings also suggest that although many problems have more than minimal cognitive demand, the textbooks do not take advantage of numerous opportunities to challenge students with cognitively demanding examples.

Recommendations for Future Research

There are a number of additional valuable analyses possible with the data collected for this study. Function examples could be separated into examples that occurred in lessons and examples that occurred in exercises, and the same analyses used in this study could be conducted on these two types of examples. Findings would provide insight into similarities and differences between the coded features of functions in lessons and exercises in each series. For instance, many function examples in the Glencoe series were found to be very short, but it is possible that most of these examples were in exercises, and that lessons tended to include longer examples. Function examples could also be separated by abstract and realistic settings and analyzed in the same manner.

Using a similar approach, the examples from each textbook that were explicitly identified as functions could be separated and analyzed. This analysis would be useful because for these examples labeled as functions, there is a higher probability that students would recognize they are dealing with an example of a function, and therefore the example would have a greater potential to impact their concept image of function.

Further analysis of function representations could also be conducted with the data collected for this study. Results reported in this study indicated only the percent of examples that included multiple representations of function; however, the data collected would allow an analysis of the relative frequency of occurrence of various combinations of representations. For instance, the percent of function examples that had a *Symbolic* representation provided and a *Graphic* representation requested could be compared to the reverse situation, in which *Graphic* representations were provided and *Symbolic* representations requested. Such an analysis would provide a more nuanced picture of how multiple representations of functions are employed in the series.

Several aspects of this dissertation also warrant further research that would require additional data collection. For instance, although comparison of several features coded in this study to research recommendations afforded insight into the cognitive demand of examples, these examples were not coded with a framework specifically designed to rank the levels of cognitive demand of examples. Similarly, although explicit recommendations for use of technology were coded, a more thorough analysis of how textbooks encourage and expect students to use technology would be valuable. Such an analysis would need to be informed by the growing body of research on how students learn about mathematics, and specifically functions, using technology.

The framework proposed by the Center for the Study of Mathematics Curriculum (2011), shown in Figure 5.1 provides a useful way to consider additional studies that could build from the research presented in this dissertation. In this model, the textbook curriculum, intended curriculum, and assessed curriculum influence teachers' decisions realized in the implemented curriculum, which then has a direct impact on the curriculum learned by the student. This study, being focused on a part of the textbook curriculum, provides information about part of a sequence several steps removed from student learning. Additional research is needed in each of the curricula noted in the model.

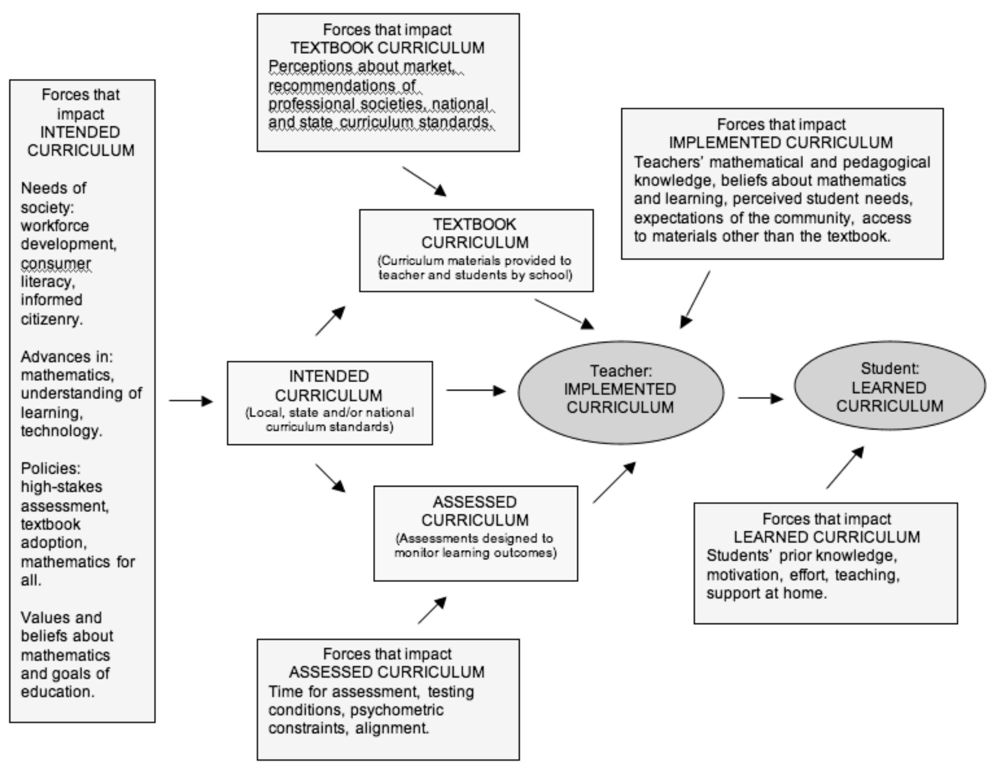


Figure 5.1: Center for the Study of Mathematics Curriculum framework.

Other textbook curricula also need to be analyzed for their treatments of the function concept and they could be studied using the methodology provided here and compared to results from the three series in this study. Recently, the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011), which place a strong emphasis on function, have been published and are being widely adopted. The textbooks in this study were not influenced by these standards, but soon their influence will become apparent in textbooks, and analyses of functions in these new textbooks will be necessary. Future textbook analyses also need to consider movements toward digital publication of textbooks. As publishers create digital and online versions of textbooks and more teachers and students use them, it will be important to include these versions in analyses and compare functions in these resources as well.

In addition, the approach used here could be developed for other mathematical content and used to study textbooks. Intended curricula in the form of state standards and learning expectations could also be examined for their treatment of function with an appropriate alternate methodology. National standards, such as the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011) should also be analyzed in similar fashion. With this information, comparisons could be made between textbooks and standards. Such a comparison would be valuable to everyone involved in trying to help students meet standards and learning expectations. Assessments designed to monitor learning outcomes also need to be examined to determine the knowledge and skills related to functions on which students are being assessed. This information would again be valuable to many constituents, including curriculum developers, teachers, administrators, and assessment designers.

The results of this study also need to be complemented with study of the implemented curriculum. This analysis provided detailed information about the definitions and examples of functions in the textbooks that teachers can draw on. A key question is how teachers actually use their textbooks as they plan and carry out instruction about functions. This can greatly impact the potential for student learning. For example, in homework exercises in Glencoe *Algebra 1* (Carter, Cuevas, Day, Malloy, Holliday, et al., 2010), function examples were mostly very short examples of linear functions, represented by an equation, and in an abstract context. However, a minority of exercises included longer examples from other function families, drew on other representations, had realistic contexts, and asked students to use technology to engage with the example. One teacher might choose to use many of these types of examples, while another might choose to only use the short, linear, abstract equation examples. This would result in two very different implemented curricula, which would likely have differing impacts on student learning. Thus, further study is needed of the how teachers draw on the collection of definitions and examples analyzed in this study and their eventual impact on student learning.

Ultimately, the learned curriculum needs to be a central focus of research. The assumption cannot be made that the textbook curricula described in this study will have a direct impact on students' learning. Armed with more information about the implemented curriculum, careful analysis of students' understanding of and skills related to functions needs to be conducted and connected back to the textbook curriculum they experienced. Without such work, at best we can only make suppositions about relationships between what is in textbooks and what students learn.

Limitations

A limited number of textbooks were analyzed compared to available choices. Each of the three major publishers provides multiple series, and alternatives outside the major publishers also exist. Of these options, only three series were analyzed, and in fact, all textbooks examined for this study came from the same publisher, McGraw-Hill. These textbooks were selected to represent both a popular series and two alternatives to this. Similar analysis of other textbook series is warranted and would provide a more complete picture of how function definitions and examples are reflected in mathematics textbooks.

The analysis of these textbooks was limited to only the function concept in each textbook. Certainly the presentation of other mathematical concepts and topics in these textbooks are worthy of study, and in order to have more comprehensive understandings of these series, additional analyses are needed. However, the importance of function at the secondary level and beyond guided the selection of this concept, and the complexity of the topic restricted this analysis from including other content.

The analysis of the presentation of the function concept in the textbooks was also limited. Other appropriate analyses of the function concept in these textbooks could provide additional useful information. The perspective selected for this study grew from multiple claims by researchers that students develop understanding of mathematical concepts through exposure to, and study of, both definitions and examples of that concept (Bills & Watson, 2008; Leinhardt, et al., 1990; Tall & Vinner, 1981). Features of the definitions and examples were also drawn from a review of research on student learning of function. However, other textbook content related to function and other aspects of functions are potentially worthy of analysis. For example, this analysis found few references to domain and range in function definitions, but it is certainly possible that textbooks made connections between functions and domain and range outside of function definitions that this analysis did not capture. Another example would be that although this analysis captured one perspective of the prevalence of functions through the number and length of examples, it did not attempt to provide an estimate of the proportion of each textbook that was focused on the function concept.

The study was limited to an analysis of textbooks and did not examine how textbooks are used by teachers or students. Teachers do not typically use all content in textbooks (McNaught, Tarr, & Grouws, 2008), and therefore the findings from this study do not necessarily represent the collection of examples students would experience in a classroom. For example, CPMP *Course 1* was unique in that it provided a majority of function examples in realistic settings. However, a teacher may decide to use and assign a subset of examples that are mostly in abstract settings. Therefore, the results presented here provide information about the potential collection of examples that teachers may draw on. Further research is warranted regarding the examples that teachers actually choose to use.

Finally, the study was limited temporally. The most recent editions of textbooks were selected for analysis, but these series will not be used in schools indefinitely. In fact, new editions of the Glencoe series, with 2012 copyrights, will be available the summer of 2011 (Mcgraw Hill, 2011). Also, these textbooks were developed before the release of the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2011). Where these standards are accepted and implemented, there is the likelihood that immediate changes will be forthcoming. These limitations point to the need for continuing analysis as new textbooks are produced.

Summary

Textbooks play a key role in mathematics education in the US, both for teachers and students. High quality analyses of textbooks in relation to central concepts provide vital information and form an initial phase in research seeking to make connections between curriculum and student learning. The function concept is widely recognized as a core concept in high school mathematics and bridge into advanced mathematics. This study provided an analysis of function definitions and examples in three contemporary secondary mathematics textbook series. The analysis was informed by a review of extensive research on how students learn about functions. The latest editions of the Glencoe Mathematics, University of Chicago School Mathematics Project, and Core Plus Mathematics Project series were included in the study. Results provided information about the language used in relation to functions, presence of function examples, core features of function examples, and ancillary features of function examples.

Functions in all three textbook series had much in common. Thousands of examples were provided in each series, with most textbooks averaging two or more functions per page. Most examples were in homework exercises and included both provision of representations for the functions as well as requests for students to create representations. *Symbolic* and *Verbal Description* representations were most common, with some use of *Numeric* and *Graphic* representations. Functions were in abstract settings and were predominantly drawn from *Polynomial* families, especially *Linear*, although the proportions of non-*Polynomial* functions increased over the course of each series. Most examples did not include explicit mention of domain or range, did not direct

students to use technology, and did not include language to encourage students to think of functions as actions, processes, or objects or consider them in general terms.

Despite many similarities, findings did indicate key differences between series. The Glencoe series, especially in the later textbooks, provided set definitions of function, whereas the CPMP series provided relationship of correspondence definitions, and the UCSMP series provided some of each type throughout the series. The Glencoe series included many more short examples and used *Symbolic* representations more often, while the CPMP series tended to include longer examples with *Verbal Descriptions* more frequently. The UCSMP series also provided longer examples, included *Graphic* representations slightly more often, and tended to use *Linear* functions less frequently, drawing on *Trigonometric* functions and transformations considered as functions more often than the other series. The UCSMP and CPMP series also tended to include slightly more directions for the use of technology than Glencoe, and the CPMP series had higher proportions of examples in realistic settings.

Comparison of findings to research into how students learn about functions suggests both strengths and weaknesses in the textbook series. The use of set definitions, as was the case in both the Glencoe and UCSMP series, has been condemned by a number of researchers; however, neither the CPMP series nor the Glencoe or UCSMP series provide language explicitly based on the APOS framework designed to help students develop more sophisticated conceptions of function. Aligned with current recommendations for emphasis on functions, students are provided with many examples of functions and are directed to both examine and generate multiple representations of functions. However, large portions of these functions are never identified as such, and thus students are not explicitly directed to connect their experience with the function concept. In addition, by far the most common function family drawn on was *Polynomial*, and the majority of these functions were *Linear*. A number of researchers have warned that when students have limited experience outside of *Linear* and *Polynomial* functions, their concept image of function tends to only include these types of functions. However, in each series, the proportions of non-*Polynomial* functions tended to increase over the course of the series. A number of researchers have also suggested the potential to support student learning about functions with the use of realistic settings and technology. None of the series included these features in the majority of examples; however, the CPMP series included more realistic settings and the UCSMP and CPMP series directed students to use technology more often than the Glencoe series.

These findings are useful for a number of constituents. Certainly further analysis of other textbook series and other mathematical content is needed, and future analyses of textbooks can draw on the methodology presented in this study. Teachers and administrators can better understand function definitions and examples in each of these textbook series. Increased understanding can support classroom instruction as teachers can draw on strengths of their textbook series and attend to weaknesses and can also support future textbook selection. The findings can also be useful for future curriculum development, as textbook authors can intentionally maintain strengths of the series while seeking to address weaknesses. Finally, the findings are important as vital research into the connections between curriculum and student learning moves forward. Since functions play a critical role in secondary mathematics and into advanced mathematical topics, we must continue to seek understanding of how students learn about functions and why they learn what they do. Mathematics textbooks play a central part in their secondary mathematics learning. The findings presented here give a clearer picture of features of this major resource used by teachers and students. Now we must use this knowledge to inform studies of how teachers and students use the definitions and examples analyzed here, and what students ultimately learn about functions.

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APPENDIX A: FUNCTION EXAMPLE CODING

Coding for a textbook will proceed through each page of each textbook. Coders will read each page and identify examples of functions according to two guidelines:

1) All functions explicitly identified as functions will be included in the analysis.
2) All functions represented by any of the following:
    a) an equation with two or more variables, function notation, or a recursive formula;
    b) a graph in the Cartesian coordinate plane;
    c) a table or list of ordered pairs;
    d) a function machine;
    e) a mapping diagram;
    f) a verbal description of a function; or
    g) a physical model of a functional situation.

Non-examples of functions will also be identified for analysis according to two guidelines:

1) Any example described as a relation but is not a function or identified as a non-example of a function, or
2) Any example that is not a function and for which students are asked to decide whether it is a function

For each example or non-example, codes will be recorded in the Textbook Coding Scheme. A description of each code is provided below.

I. **Non-example**: Record a "p" here if the entry pertains to a non-example of a function that the textbook provides and specifically indicates is not a function or asks students to decide whether it is a function. Record an "r" here if the textbook requests that students provide an example that is not a function.

II. **Presence of Function**
   A. **Page:** Record the page number or numbers on which the function example appears.
   B. **Description**: Provide a description of where the function can be located on the page. An example or exercise number can be used if available. If these cannot be easily used to identify the function example, write a brief description of the function.
   C. **Size**: Note: unless initial directions provide information necessary to know what the function is, they should not be included in the amount of space taken up by an example.
      1. **<1 row**: Record a 1 here if the function example does not take up more than one row and includes other information in addition to that example in that row.
      2. **1-2 rows**: Record a 1 here if the function example takes up one to two rows.
      3. **3 rows – 1/4 pg**: Record a 1 here if the function example takes up more than two rows but not more than one quarter of one page of text. For each textbook, a template indicating the size of one quarter of one page will be provided.
      4. **1/4 - 1/2 pg**: Record a 1 here if the function example takes up more than one quarter of one page but not more than one half of one page of text. For each

textbook, a template indicating the size of one quarter and one half of one page will be provided.

5. **1/2 pg – 1 pg**: Record a 1 here if the function example takes up more than one half of one page but not more than one page of text. For each textbook, a template indicating the size of one half of one page will be provided.
6. **+1 pg**: Record a 1 here if the function example takes up more than one page of text. Note that this does not only mean that the example is on more that one page, but that the content related to the example should take up more space than would fit on one page.

D. **Function Label**:

1. **Labeled as Function**: Record a "p" here if an example of a function provided for a student is explicitly labeled as a function or if a non-example is explicitly labeled as being a non-example of function. Record an "r" here if the student is requested to provide a function and the request uses the term "function" or the student is requested to provide a non-example of function and the request makes it clear it is to be a non-example of function. These labels can be included in blanket statements intended for groups of functions or non-examples.
2. **Student Decides if Function**: Record a "p" here if students are asked to decide whether an example provided in the textbook is a function. Record an "r" here if the student is requested to create an example and decide whether it is a function.

E. **Placement:**

1. **Lesson:** Record a 1 here if the function example appears in the main body of the lesson. For each textbook, a description for how to identify the main body of the lesson will be provided.
2. **Exercises:** Record a 1 here if the function example appears in the student exercises associated with the lesson. For each textbook, a description for how to identify the student exercises will be provided.

F. **Error**: If an example of a function has a mathematical error, or an example is labeled as a function when it is not a function, include a description of the error here.

III. **Language Used in Relation to Function Examples**:

A. **APOS**:

1. **Action**: Record a "p" here if the example is provided for the student along with text that describes the example as a transformation one applies to mathematical elements according to an explicit algorithm with an emphasis on carrying out the algorithm. Record an "r" if such a description is provided and an example is requested of a student.
2. **Process**: Record a "p" here if the example is provided for the student along with text that describes the example as a procedure one has the ability to follow, a procedure that transforms one mathematical element into another. The emphasis is on the ability to carry out the procedure if needed or desired. Record an "r" if such a description is provided and an example is requested of a student.

3. **Object**: Record a “p” here if the example is provided for the student along with text that describes the example as something that can be acted upon mathematically with an emphasis on functions as mathematical elements that can be transformed. Record an “r” if such a description is provided and an example is requested of a student.
4. **Schema**: Record a “p” here if the example is provided for the student along with text that describes the example as a part in a larger mathematical perspective. The emphasis is on the role of functions in general or within broader mathematical concepts. Record an “r” if such a description is provided and an example is requested of a student.

B. **Generality**:

1. **Specific**: Record a “p” here if the example is provided for the student such that at least one domain element and its corresponding range element is given or can be obtained. Record an “r” if such an example is requested of a student.
2. **General**: Record a “p” here if the example is provided for the student such that no domain elements and corresponding range elements are given or can be obtained. Record an “r” if such an example is requested of a student.

IV. **Core Features**:

A. **Representation:**

1. **Symbolic:** A representation of a function is symbolic if it uses numbers, letters as variables, operation symbols (e.g. plus signs), and/or *f*(x) notation to provide a formula or formulas to generate elements in the range of the function from elements in the domain.
   a. ***y* = expression**: Record a “p” here if a symbolic representation of a function where an algebraic expression consisting of numbers, letters, and operation symbols, and only utilizing x as a variable, is set equal to the variable y is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such an equation and asked to provide another form or other parts of the equation.
   b. **Implicit *y* and *x*:** Record a “p” here if a symbolic representation of a function where two algebraic expressions consisting of numbers, operations symbols, and the letters y and x are set equal to each other, and one expression is not simply the variable y, is provided for the student. Record an “r” here if such an example is requested of a student. Record a “c” here is students are provided with a form or part of such an equation and asked to provide another form or other parts of the equation.
   c. ***f*(*x*) = expression**: Record a “p” here if a symbolic representation of a function where an algebraic expression consisting of numbers, letters, and operation symbols set equal to $f(x)$ or $f{:}x$ is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such an equation and asked to provide another form or other parts of the equation. Note that symbolism utilizing other letters for the function and variable are also counted in this category, for example $g(w)$ instead of $f(x)$.

d. **Recursive**: Record a “p” here if a symbolic representation in which one correspondence of the function is given and other correspondences are determined from the preceding values is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation and asked to provide another form or other parts of the representation.

e. **Equation With Other Variables**: Record a “p” here if a symbolic representation of a function where two algebraic expressions consisting of numbers, operation symbols, and letters other than *x* and *y* are set equal to each other is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation and asked to provide another form or other parts of the representation.

f. **Other**: Record a “p” here if a symbolic representation is provided for the student but cannot be coded as one of the three other symbolic representations. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation and asked to provide another form or other parts of the representation. Include a description of the representation.

2. **Graphic:** A representation of a function is graphic if it uses points and/or curves in a Cartesian coordinate plane (or corresponding visual representation of higher dimensions) to display some or all of the elements of the domain of the function with their corresponding elements in the range.

a. **Continuous**: Record a “p” here if a graphic representation of a function that is a single unbroken curve without holes or jumps is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation beyond axes or a grid and asked to provide another form or other parts of the representation.

b. **Smooth**: Record a “p” here if a continuous graph of a function that has no corners is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation beyond axes or a grid and asked to provide another form or other parts of the representation. Corners are defined as points of a graph where, on any arbitrarily small locality, the graph appears to be two straight line segments meeting at a non-straight angle.

c. **Scatterplot**: Record a “p” here if a graphic representation of a function that is a plot of individual ordered pairs is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation beyond axes or a grid and asked to provide another form or other parts of the representation.

d. **Other**: Record a “p” here if a graphic representation is provided for the student but cannot be coded as one of the two other graphic representations. Record an “r” if such an example is requested of a student.

Record a “c” here if students are provided with a form or part of such a representation beyond axes or a grid and asked to provide another form or other parts of the representation. Include a description of the representation.

3. **Numeric:** A representation of a function is numeric if it is a listing of some or all of the elements of the domain of the function with their associated elements in the range.
   a. **Table**: Record a “p” here if a numeric representation in which elements are displayed in a table is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a table with values in at least one cell and asked to provide values for other cells of the table.
   b. **Ordered pair**: Record a “p” here if a numeric representation in which elements are displayed as ordered pairs in parentheses is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with at least one value in one ordered pair and asked to provide other values in ordered pairs.
   c. ***f*(*x*) Notation**: Record a “p” here if a numeric representation in which elements are displayed as $f(x) = y$ is provided for the student. Note that letters other than *f, x,* and *y* may be used. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with at least one value in one pair and asked to provide other values.
   d. **Other**: Record a “p” here if a numeric representation is provided for the student but cannot be coded as one of the two other numeric representations. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with part of such a representation that includes some values of the function and then are asked to provide other parts of the representation. Include a description of the representation.
4. **Function Machine**: Record a “p” here if a representation of a function with a diagram suggesting a physical object in which values can be input and which produces corresponding output values is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation beyond the diagram of the machine itself and then are asked to provide another form or other parts of the representation.
5. **Mapping Diagram**: Record a “p” here if a representation of a function with a diagram showing some or all elements of the domain in one part of the diagram, corresponding elements of the range in another part of the diagram, and lines or arrows connecting corresponding elements is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation beyond the parts of the diagram showing where elements of the domain and range could be placed and then are asked to provide another form or other parts of the representation.

6. **Verbal Description**: Record a “p” here if a representation using words, and potentially numbers, to describe any feature of the relationship between the independent and dependent variables is provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with some verbal description and asked to provide additional verbal description.
7. **Physical**: Record a “p” here if a physical model, capable of manipulation by students, that embodies a functional relationship, or directions for the creation of such a model are` provided for the student. Record an “r” if such an example is requested of a student. Record a “c” here if students are provided with a form or part of such a representation and asked to provide another form or other parts of the representation.
8. **Multiple**: Record a “p” here if representations of multiple types (symbolic, graphic, numeric, function machine, mapping diagram, verbal description, or physical) are provided for the student. Record an “r” here if students are requested to generate an example of a function and provide at least two different representations of it. Record a “c” here if students are provided at least one representation and are requested to provide at least one other type of representation.
9. **Other**: Record a “p” here if a representation is provided for the student but cannot be coded as one of the other representations described. Record an “r” if such an example is requested of a student. Include a description of the representation.

B. **Family:**

1. **Polynomial:** A polynomial function is a function whose symbolic representation is a polynomial of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $a_i \neq 0$ for at least one *i*.
   a. **Constant**: Record a “p” here if a polynomial function whose symbolic representation is of the form $f(x) = a_0$ for some a_0 is provided for the student. Record an “r” if such an example is requested of a student.
   b. **Linear**: Record a “p” here if a polynomial function whose symbolic representation is of the form $f(x) = a_1 x + a_0$ where $a_1 \neq 0$ is provided for the student. Record an “r” if such an example is requested of a student.
   c. **Quadratic**: Record a “p” here if a polynomial function whose symbolic representation is of the form $f(x) = a_2 x^2 + a_1 x + a_0$ where $a_2 \neq 0$ is provided for the student. Record an “r” if such an example is requested of a student.
   d. **Cubic**: Record a “p” here if a polynomial function whose symbolic representation is of the form $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ where $a_3 \neq 0$ is provided for the student. Record an “r” if such an example is requested of a student.
   e. **Other**: Record a “p” here if a polynomial function is provided for the student but cannot be coded as one of the other functions described. Record an “r” if such an example is requested of a student. Include a description of the function.

2. **Periodic:** A function is periodic if there exists a number k such that $f(x) = f(x+k)$ for all values of x.
   a. **Trigonometric**: Record a "p" here if a function whose symbolic representation includes trigonometric ratios (sine, cosine, tangent, cosecant, secant, cotangent) or their inverses is provided for the student. Record an "r" if such an example is requested of a student.
   b. **Other**: Record a "p" here if a periodic function is provided for the student but cannot be coded as a trigonometric function. Record an "r" if such an example is requested of a student. Include a description of the function.
3. **Exponential**: Record a "p" here if a function with its independent variable as part of an exponent is provided for the student. Record an "r" if such an example is requested of a student.
4. **Logarithmic**: Record a "p" here if a function with its independent variable as part of the argument of a logarithm is provided for the student. Record an "r" if such an example is requested of a student.
5. **Rational**: Record a "p" here if a function whose symbolic representation is the ratio of two polynomial functions, where the function in the denominator is not the constant function is provided for the student. Record an "r" if such an example is requested of a student.
6. **Absolute Value**: Record a "p" here if a function with its independent variable inside an absolute value symbol is provided for the student. Record an "r" if such an example is requested of a student. If no symbolic representation is provided, code a function as absolute value if the function can be modeled by two linear functions with opposite slopes.
7. **Piecewise**: Record a "p" here if a function whose rule of correspondence for a particular element in its domain is dependent on the value of the element is provided for the student. Record an "r" if such an example is requested of a student. Note that if a function is coded as an absolute value function, it should not be coded as a piecewise function.
8. **Other**: Record a "p" here if a function is provided for the student but cannot be coded as one of the other function families described. Record an "r" if such an example is requested of a student. Include a description of the function.

C. **Specified Domain and Range:**

1. **Numerical**: Record a "p" here if the domain or range of a provided function is specified and if the domain and range of the function consist only of numbers. Record an "r" here if the student is requested to provide a function with only numerical values in the domain and range and is explicitly directed to specify the domain of the function or if the student is provided a function with only numerical values in the domain and range and is directed to specify the domain of the function, range of the function, or both. Record a "c" here if the student is provided with either the domain or range and is requested to provide the other.
2. **Not Numerical**: Record a "p" here if the domain or range of a provided function is specified and any element of the domain or range is not a number. Record an "r" here if the student is requested to provide a function with any domain or range element that is not a number and is explicitly directed to

specify the domain of the function or if the student is provided a function with elements in the domain or range that are not numbers and is directed to specify the domain of the function, range of the function, or both. Record a “c” here if the student is provided with either the domain or range, is requested to provide the other, and one of the two contains non-numerical elements.

V. **Ancillary Features:**

A. **Setting:**

1. **Abstract**: Record a “p” here if a function with no description of a situation intended to reflect the real world in which the function is embedded is provided for the student. Record an “r” if such a function without a realistic situation is requested of a student.
2. **Realistic**: Record a “p” here if a function and a description of a situation intended to reflect the real world in which the function is embedded is provided for the student. Record an “r” if such a function and description are requested of a student.

B. **Technology Recommended**: Record a “p” here if along with an example of a function provided for a student, the text includes explicit directions to the student to use a technology tool. Record an “r” here if a student is directed to use a technological tool to create an example or representation of a function or use a technological tool to study an example he or she has created. These directions can be included in blanket statements intended for groups of functions.

APPENDIX B: DESCRIPTIONS OF LESSON AND EXERCISE SECTIONS FOR EACH TEXTBOOK SERIES

Series	Code	Description
Glencoe	not coded	pages prior to Ch. 0 Pretest
	exercises	Ch 0. pretest
	lesson	chapter introductory page
	lesson	examples in "Get Ready for Chapter X"
	exercises	QuickCheck exercises in "Get Ready for Chapter X"
	lesson	"Get Started on Chapter X"
	lesson	pages from Lesson X-X up to but not including section titled "Check Your Understanding"
	exercises	sections titled "Check Your Understanding," "Practice and Problem Solving," "H.O.T. Problems," "Standardized Test Practice," "Spiral Review," and "Skills Review"
	exercises	Mid-Chapter Quizzes
	lesson	Lab sections labeled "Activity" (Graphing Technology, Algebra, Spreadsheet)
	exercises	Lab sections labeled "Analyze the Results," "Model and Analyze,""Model," and "Exercises" (Graphing Technology, Algebra, Spreadsheet)
	lesson	Chapter Study Guide and Review: Chapter Summary, Key Vocabulary, examples in Lesson-by-Lesson Review
	exercises	Chapter Study Guide and Review: Vocabulary Check, exercises in Lesson-by-Lesson Review
	exercises	Chapter Practice Test
	lesson	sections of Chapter Preparing for Standardized Tests up to section titled "Exercises"
	exercises	section titled "Exercises" in Chapter Preparing for Standardized Tests
	exercises	Chapter Standardized Test Practice

Glencoe Precalculus		
	not coded	pages prior to Unit 1 description, p. 2
	lesson	Unit introductory pages
	lesson	chapter introductory pages
	lesson	pages from (lesson) X-X up to but not including section titled "Check For Understanding"
	exercises	sections titled "Check For Understanding," "Exercises
	lesson	sections titled "Career Choices"
	lesson	sections titled "Graphing Calculator Exploration" up to but not including sections titled "Try These" and "What Do You Think?"
	exercises	sections within GCE's titled "Try These" and "What Do You Think?"
	lesson	sections titled "History of Mathematics" up to but not including "Activities"
	exercises	sections within HoM's titled "Activities"
	exercises	Mid-Chapter Quizzes
	lesson	Chapter Study Guide and Assessment: Vocabulary, Skills and Concepts - Objectives and Examples
	exercises	Chapter Study Guide and Assessment: Understanding and Using the Vocabulary, Skills and Concepts - Review Exercises, Applications and Problem Solving, Alternative Assessment
	exercises	Chapter SAT & ACT Preparation
	lessons	career sections
UCSMP	not coded	pages prior to Ch. 1 introductory pages
	lesson	chapter introductory pages
	lesson	pages from Lesson X-X up to but not including section titled "Questions" (lesson includes Mental Math and Quiz Yourselves)
	exercises	sections under "Questions" titled "Covering the Ideas," "Applying the Mathematics," "Review," and "Exploration"
	exercises	Chapter Projects
	lesson	Chapter Summary and Vocabulary
	exercises	Chapter Self-Test
	exercises	Chapter Review
CPMP	not coded	pages prior to Unit 1 introductory pages
	lesson	Unit introductory pages
	lesson	pages beginning at "Lesson X" through each "Investigation X" (including Think About This Situation, Summarize the Mathematics, and Check Your Understanding) and ending at title "On Your Own"
	exercises	pages beginning at "On Your Own" (including all problems in Applications, Connections, Reflections, Extensions, and Review) ending at title "Lesson X"

VITA

Daniel J. Ross is currently an Assistant Professor of mathematics at Maryville College in Maryville, TN. He earned a B.S. in Education from Martin Luther College in New Ulm, MN (2000), a M.S.T. in Mathematics from the University of Missouri in Columbia, MO (2007), and a Ph.D. in Learning, Teaching, and Curriculum with an emphasis in Mathematics Education from the University of Missouri in Columbia, MO (2011). Dan's research interests include mathematics curriculum, student learning of mathematics and especially the function concept, and student engagement and motivation in mathematics learning.

Dan taught high school mathematics in Crete, IL for one year and Waco, NE for five years prior to graduate studies. During doctoral studies, he instructed undergraduate mathematics and mathematics education courses and conducted research in high schools. As a graduate research assistant, he was actively involved in a longitudinal study of mathematics curriculum, curriculum implementation, and student learning in schools in multliple US states funded by the National Science Foundation. His work included classroom observation, assessment development and scoring, and data entry and analysis.