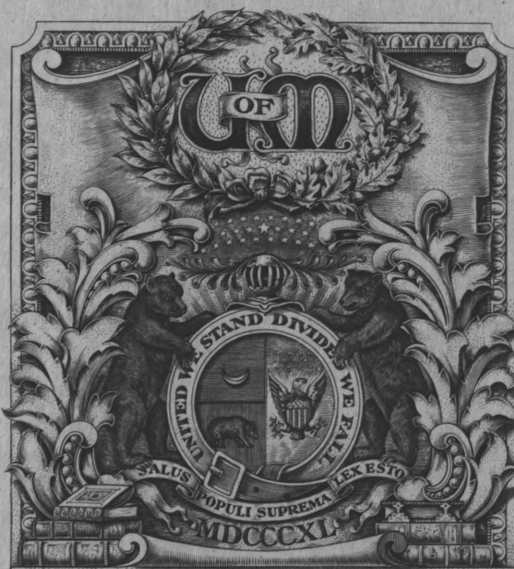
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# ON INTEGRALS OVER SETS OF POINTS

by

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## Bibliography.

Hobson, E.W.: The Theory of Functions of a real Variable; Cambridge University Press, 1907.

Jordan, C.: Cours d'Analyse; Gauthier-Villars, Paris, 1893.

Lebesgue, Henri: Integrale, Longueur, Aire; Annali di Matematica, Serie III, Tomo VIII, 1902.

---------------: Lecons sur l'Integration et la Recherche des Functions Primitives; Gauthier-Villars, Paris, 1904.

Pierpont, James: The Theory of Functions of Real Variables, Vol. I; Ginn and Company, Boston, 1906.

---------------: On Multiple Integrals; Transactions of the American Mathematical Society, Vol. 6, p. 416, 1905.

Vallee-Poussin, Charles de la: Cours d'Analyse Infinitesimale, Vol. I; Gauthier-Villar, Paris, 1903.

Young, H. W.: On General Integration; Transactions of the Royal Society, Section A, Vol. 204, P.221, 1905.

# ON INTEGRALS OVER SETS OF POINTS

## Introduction

The developments of the last twenty years in the theory of sets of points and in the applications of this theory to the theory of functions of real variables, besides leading to a tremendous extension of the ordinary theory of integration(*), have given rise to a number of definitions of the integral for the case of a function defined for a set of points. These may be classified as (a) geometrical and (b) analytical. We may characterize the geometrical definitions by saying that they define the integral as the volume (possibly n-dimensional) or area, as the case may be, of a set of points.(**) As it is not the purpose of this paper to discuss the various ways in which this may be done, we shall content ourselves with saying here that for any geometrical definition the equivalent analytical definition may be found, and conversely. Of the types of analytical definition of the

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(*) Note especially the work of Lebesgue: Integrale, Longueur, Aire; Annali di Matematica, Serie III, Tomo VII, p. 231; and Lecons sur l'Integration.

(**) The words volume, area, and length will be used in place of the word "mesure", defined for any set of points by Lebesgue in Integral, Longueur, Aire, p. 236. The word measure will be used in the sense usually assigned to it.

integral two are fundamental in the theory; (i) those definitions which regard the integral as the inverse of the derivative, and (ii) those which define the integral as the limit of a sum, or set of sums. A third type of definition makes the extension to sets of points by the aid of an auxiliary function, but these definitions are not independent, since they imply a definition of one of the preceding types.

By far the greater number of the definitions which we have are given from the standpoint of the integral as the limit of a sum, or set of sums. The fact that a difference in the way two sums are defined sometimes means a difference in the resulting integrals leads us to the considerations of this paper. It is our purpose to consider all the different kinds of sums which one might reasonably expect to lead to desirable results, to find out which of the resulting integrals are equivalent to integrals already defined, and which of them lead to new integrals of interest. In order to bring the discussion within reasonable limits we shall confine ourselves to what are ordinarily termed proper integrals, that is integrals of limited functions defined for limited sets of points.

Lebesgue has proposed(*) to attach to every limited function defined in a finite interval a number to be called the integral of the function on the interval, and has given the conditions which that number ought to satisfy. For the case of a function of n variables defined for an n-dimensional set of points the conditions which the integral should satisfy will need to be given in a slightly different form, and this will be done in Section IV, Art. 1, of this paper. We shall, following Lesbegue, speak of this set of conditions as the descriptive definition of the integral, and call the definitions which we have referred to as geometrical and analytical, constructive definitions. Each of the constructive definitions which we give will be considered as an attempt to define this number, and criticized from the standpoint of its success or failure to meet the imposed conditions. In some cases we shall state what restrictions on the function and region of integration will enable the integral to satisfy these conditions, and in others we shall give properties of the integral which might be substituted for those demanded in the given set of conditions referred to.

In view of the very large number of possible definitions which we shall consider, and in order to avoid

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(*) Lebesgue: <u>Lecons sur l'Integration</u>, p. 98.

losing the reader in a mass of detail it has seemed best to present the definitions schematically. In order to do this it has been necessary to adopt a rather elaborate system of notation, and what may prove to be a puzzling set of conventions.

## Section I.

### Notations and Conventions.

1. The notation R will be used to denote a region of n-dimensional space defined as follows: A point P of R is determined by n coordinates; $x_1$, $x_2$, ........ $x_n$, such that $a_1 \leq x_1 \leq b_1$, $a_2 \leqq x_2 \leqq b_2$........, $a_n \leqq x_n \leqq b_n$. We shall usually represent P by the single coordinate $x = (x_1, x_2, x_3, \ldots\ldots, x_n)$. If E is any set of the points P of R, and y is a quantity which is determined whenever x is in E, we shall call y a function of x, and write $y = f(x)$. Obviously, y is a function of the n coordinates of the point P. The distance d between two points P and P' will be as usual:

$$d = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + \ldots\ldots\ldots + (x_n - x_n')^2}$$

The limiting point of a set is defined as usual, and R is evidently closed. R will be said to be a closed rectangular region of n-dimensional space.

The maximum diameter of a point set will be the upper limit of the distances between the points of the set.

2. The set of points for which y is determined will be called the domain of definition of the function f(x). We shall consider only cases in which this domain is limited; otherwise we shall arbitrarily restrict the domain. In general we shall call the domain of definition E, except for the special case where E

coincides with R, i. e., is a closed rectangular region.

3. Let C be the complement of E with respect to R. The frontier F of E is the set composed of the limit points of E which belong to C and the limit points of C which belong to E. Obviously F is the frontier of C.

Let $E_1$ and $E_2$ be any two subsets of R. We will call the set formed by taking all the points of the two sets ~~one~~ the Least Common Multiple (L.C.M.) of $E_1$ and $E_2$ , and denote it by $(E_1\ E_2)$. The set of all the points common to both $E_1$ and $E_2$ will be called the Greatest Common Divisor of $E_1$ and $E_2$ (G.C.D.) and denoted by $(E_1, E_2)$.

$E', E'', \ldots\ldots E^p$, will as usual represent the successive derivatives of E, where $E^p$ is the set of limiting points of $E^{p-1}$.

Let us call the set (E,F) the E points of the frontier. The set (E+F) formed by closing E, will be called the <u>completed</u> set of E. We shall frequently have occasion to consider subsets of this set (E+F), and we shall denote them by K.

4. Let $f(x) = 1$ for x in E, and $f(x) = 0$ for x in C. The Upper Riemann Integral of f(x) for the region R will be called the Outer Content of E, the Lower Integral (R) the Inner Content. In case the integral

exists we call the result the content of E. The notations $\overline{c}(E)$, $\underline{c}(E)$, $c(E)$ will denote, upper content, lower content, and content, respectively, of E. If a set of points has content we will call it a metrical set(*). The following properties of content will be needed for reference, and follow readily from the definition given above:

$$\begin{aligned} \overline{c}(E) + \underline{c}(C) &= \underline{c}(E) + \overline{c}(C) \\ &= \underline{c}(E) + \underline{c}(C) + \overline{c}(F) \\ &= c(R) = \text{volume of } R. \qquad (1) \end{aligned}$$

If we suppose E divided into n sets; $e_1, e_2, \ldots\ldots e_n$. the following relations hold:

$$\overline{c}(E) \leq \overline{c}(e_1) + \overline{c}(e_2) + \cdots\cdots + \overline{c}(e_n) \qquad (2)$$

$$\underline{c}(E) \geq \underline{c}(e_1) + \underline{c}(e_2) + \cdots\cdots + \underline{c}(e_n) \qquad (3)$$

5. The measure(**) of a set of points E will be denoted by $m(E)$, the outer measure by $\overline{m}(E)$, the inner measure by $\underline{m}(E)$. A set whose measure exists will be called measureable. We shall make use of the

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(*) cf Pierpont, James: On Multiple Integrals, Transactions of the American Mathematical Society for a discussion of the theory of content. Also Jordan, C: Cour d'Analyse, Ed. 1893, Vol. I. pp. 28-31. The word 'metrical' is used to distinguish the property of having content from the property of having 'measure' in the sense defined by Lebesgue.
(**) Lebesgue: Lecons sur L'Integration, p. 104.

following properties of measure:

$$\overline{m}(E) + \underline{m}(C) = \underline{m}(E) + \overline{m}(C) = m(R)^{(*)} \qquad (1)$$

$$= \text{Volume of } R$$

Let us suppose E divided into a finite or countably infinite number of subsets; $e_1, e_2, e_3, \ldots\ldots\ldots\ldots$

We have the following relations:(**)

$$\overline{m}(E) \leqq \overline{m}(e_1) + \overline{m}(e_2) + \overline{m}(e_3) + \ldots\ldots\ldots \qquad (2)$$

$$\underline{m}(E) \geqq \underline{m}(e_1) + \underline{m}(e_2) + \underline{m}(e_3) + \ldots\ldots\ldots \qquad (3)$$

6. *Methods of subdivision of R.* The subsets of R produced by any method of division will be termed cells. The notation $v_i$ will be used to represent both a cell and its volume. In case the number of cells is countably infinite i will take on the values from 1 to infinity. Two cells will over-lap when they have interior points in common, they will be said to *mix* when the content of their common frontier points is greater than zero. With respect to the points of a set E, a cell will be *full* if every point of the cell is a point of E; *partly full*, if some but not all the points of the cell do not belong to E; and *empty*, if containing no point of E.

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(*) Lebesgue: (loc. cit.) p. 104. This property is one of the fundamental requirements which Lebesgue makes.

(**) Lebesgue: (loc. cit.) p. 103.

Of the many possible types of cell we shall consider only six, the definitions of which we preceed to give.

The largest maximum diameter of the cells produced by a method of division will be called the <u>norm</u> of the division and uniformly denoted by the letter d. In passing we may remark that an alternative choice of norm would be the volume of the largest cell in the division. Some of the consequences of such a choice of norm will be mentioned later.

(1) <u>Rectangular division</u>(D). This method of division is fundamental in the ordinary theory of integration, although it is a special case of methods of division which we shall give later. This division is effected by passing a finite number of (n-1)-dimensional planes through R perpendicular to each of the coordinate axes. The cells produced by this division (D) are rectangular regions similar to R, and like R, will for convenience be considered closed.

(2) <u>Infinite Rectangular Division</u> ($_cD$). Let the division be made as before, with the exception that the number of planes may be countably infinite. The cells will still be taken closed. In a division of this type it is possible that the points of the limiting planes of the division planes may be omitted, that is not

a part of any cell of the division. We shall make the restriction that the content of the omitted points shall be zero. In considering the integrals resulting from this method of division we will show why this requirement is made instead of the broader requirement that the measure of the omitted points vanish.

(3) Over-lapping Rectangular Division($_f\nabla$) Let the cells be rectangular and closed as before, but remove the restriction that they shall not over-lap, keeping the number of cells finite, as in (1). In place of the non-overlapping, let us require ~~the~~ ##### ## the sum of the volumes of the cells to approach the volume of R for every set of successive divisions with norm approaching zero. Two divisions are in succession when a set of cells of one division can be found such that every cell of the other is entirely contained in some one cell of the first.

(4) Over-lapping Infinite Rectangular Division ($_c\nabla$) Let the number of cells be countably infinite, and the content of the omitted points vanish, otherwise the division will be the same as (3).

(5) Metrical Division. ($\triangle$). Let R be divided into a finite number of non-overlapping metrical sets in any way.

(6) Measureable Division ($\mathfrak{D}$). Let R be divided into a finite or countably infinite set of cells in such a way that the measure of the common points of any two cells is zero, and the measure of the omitted points is zero. The cells in divisions (5) and (6) need not be considered

as closed.

6. Methods of division of E. We may regard E as our fundamental set and build our sum for the integral direct from it. We shall denote the subsets of E produced by any method of division by the notation $e_i$, and the norm of the division as usual by d. We will consider the following types of division:

(1) Metrical Division(D) of E. Let E be divided into a finite number of non-overlapping sets in any way, with the restriction that the content of the common frontier points of any two cells is zero.

(2) Measureable Division ($\Delta$) of E. Divide as before, but allow the number of divisions to be finite or countably infinite, and require the measure of the omitted points and the common frontier points to be zero.

(3) Unrestricted Division ($\mathfrak{D}$) of E. We will consider finite or countably infinite divisions of any kind under this type.

7. Methods of division of K. The substitution of K for E in Art. 6 will give the divisions of K which we will consider. The reason for considering K apart from E is that the function may not be defined for some, or all, of the points of K. We will denote the subsets of K by the notation $k_i$.

8. Methods of division of the interval of Variation of f(x).

Let the upper limit of the values of f(x) as x ranges over the domain of definition be M, and the lower limit m.

We shall denote by $V = M-m$ the interval of variation of $f(x)$. We shall consider divisions of V effected in the following ways:

(1) Finite Division ( D). Interpolate between M and m a finite set of values, $y_0 = m < y_1 < y_2 \ldots < y_q = M$. This effects a division of V into a finite set of non-overlapping intervals $d_i$, of length $d_i = y_i - y_{i-1}$. Let d the largest $d_i$ be the norm of the division.

(2) Infinite Division ($_c$D). Let us divide in the same way as before except that the number of intervals will be taken countably infinite. Let the content of the points omitted be zero.

9. Point Maximum and Minimum. About the point x take a sphere of radius d. Let M be the upper limit of the values of f(x) in this sphere. If the function is not defined for any point in this sphere we will take M equal to zero. We will call $M_x$, the lower limit of M as d approaches zero, the maximum at x. Similarly we define the minimum at x, $m_x$.

10. Upper and Lower Limits in a cell. Let $M_i$ denote the upper limit of the values of f(x) and $m_i$ the lower limit in the i'th cell of any method of division of R, E, or K. In case the cell contains no point for which f(x) is defined two courses are open to us; we may either not count that cell in our sum or we may take the upper limit of $M_x$ for $M_i$, and the lower limit of $m_x$

for $m_i$. The first course is the one we shall usually follow, but the consequences of the second will receive consideration in the cases where the result is affected.

It is quite common in definitions of an integral to use, instead of the upper or lower limit of the function in a cell, some value of the function for a point internal to the cell. I propose to adopt the slightly more general form of definition obtained by using, instead of a value of the function, any number $h_i$, for which $(m_i \leqq h_i \leqq M_i)$.

If instead of taking the upper limit of the values of $f(x)$ in a cell we take the upper limit of the values of $M_x$ we obtain the upper integral of $M_x$, the upper semi-continuous function of $f(x)$. This integral is usually equal to the upper integral of $f(x)$, the cases in which it is not will be treated later.(*)

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(*) cf. Young, W.H.: On General Integration, Transactions of the Royal Society, Section A, Vol.204, 1905, p. 220, seq. who gives some interesting theorems regarding the integrals of the upper semi-continuous function.

## Section II.

### Definitions of Integrals.

1. We shall recognize only such differences in definitions which appear to involve some essential point. A consideration of differences due solely to notation and method of presentation could lead to no result with which the present investigation is concerned. The definitions which I shall call Riemann, Lesbegue, etc., will in some respects resemble but little the originals, but will be essentially the same.

2. We shall begin with an analysis of the Riemann definition with a view to determine the essential steps of a definition based on the concept of the integral as the limit of a sum. The Riemann definition contains the following four steps:

I. A division of the region of definition of the function into cells.

II. A formation of product terms, multiplying the volume of each cell by the value of the function for some point in the cell.

III. A formation of a sum from these product terms.

IV. A passage to the limit, if it exists, by successive divisions of norm approaching zero.

We see that the essential operations involved in the

Riemann definition are: I. <u>Division</u>; II. <u>Multiplication</u>; III. <u>Summation</u>; IV. <u>A Limit Process</u>. An examination of all the standard constructive definitions of the type we are considering reveals the fact that these four operations are common to all. We shall take these as the necessary foundation for a constructive definition involving a summation. By operating on different material in different ways we obtain definitions which lead to different results, as is to be expected. The surprising thing is the large number of different definitions which give the same result.

3. The possible definitions of the integral which may be given without passing outside the field of limited functions of real variables and which conform to the general type mentioned above will now be considered. We will consider the different methods of carrying out the four steps mentioned above in order.

I. <u>Division</u>: We may operate on:--

X. the region R in the following ways:--

1. ${}_{f}D$; 2. ${}_{c}D$ ; 3. ${}_{f}\nabla$ ; 4. ${}_{c}\nabla$ ;
5. $\triangle$ ; 6. $\mathfrak{D}$ (cf. Art. 6, p. 8.)

Y. The domain E in the following ways:--

1. $D$ ; 2. $\triangle$ ; 3. $\mathfrak{D}$ (cf. Art. 7, p.11.)

Z. the set K in the following ways:--

1. $D$ ; 2. $\triangle$ ; 3. $\mathfrak{D}$ . (cf. Art. 8, p.11)

I. Division (cont.)

W. the interval of variation V in the following ways:--

1. ${}_{f}D$ ; 2. ${}_{c}D$ . (cf. Art. 9. p. 11.)

II. Multiplication. The method of operation will be the same in all cases, two quantities will be combined to form a product as ordinarily defined. The differences in the things which we may operate on lead us to consider the following nine groups of product terms which may apply to the divisions under I,X,; two groups applying to the divisions under I,Y; two groups applying to the divisions under I,Z; and two groups applying to the divisions under I,W., and indicated accordingly.

X.1. $v_i \quad x \quad \begin{cases} M_i \\ h_i \\ m_i \end{cases}$

2. Let $v_i$ represent any cell of a division partly full of points of E. As before we may form the three product terms:

$$v_i \quad x \quad \begin{cases} M_i \\ h_i \\ m_i \end{cases}$$

3. Let $v_i'$ represent a cell which is full of points of E. We would have:

$$v_i' \quad x \quad \begin{cases} M_i \\ h_i \\ m_i \end{cases}$$

If we substitute K for E in groups 2 and 3 we obtain groups 4 and 5, respectively.

6. Let us denote by $e_i(k_i)$ the points of E (K) in a cell $v_i$. We may form the following terms:

$$\left.\begin{matrix}\bar{c}(e_i) \\ c(e_i) \\ \underline{c}(e_i)\end{matrix}\right\} \quad x \quad \left(\begin{matrix}M_i \\ h_i \\ m_i\end{matrix}\right.$$

Where each quantity on the left is to be combined with each quantity on the right, giving nine kinds of term in the group. In case the sets $e_i$ are not all metrical the terms involving $c(e_i)$ of course cannot be used. The terms of this group will not be used in combination with divisions of type      of R.

7. We shall make an exception to our rule not to combine the measure and content notions in a definition and allow all the methods of division of R to apply to the following group of product terms:

$$\left.\begin{matrix}\bar{m}(e_i) \\ m(e_i) \\ \underline{m}(e_i)\end{matrix}\right\} \quad x \quad \left(\begin{matrix}M_i \\ h_i \\ m_i\end{matrix}\right.$$

8. May be obtained by substituting $k_i$ for $e_i$ in 6.

9. May be obtained by substituting $k_i$ for $e_i$ in 7.

Y. Let $e_i$ be any subset produced by division of E. We may form the following groups of product terms for $e_i$:

Y. (cont.)

1.

$$\begin{matrix} \bar{c}(e_i) \\ c(e_i) \\ \underline{c}(e_i) \end{matrix} \quad \times \quad \begin{matrix} (M_i \\ (h_i \\ (m_i \end{matrix}$$

2.

$$\begin{matrix} \bar{m}(e_i) \\ m(e_i) \\ \underline{M}(e_i) \end{matrix} \quad \times \quad \begin{matrix} (M_i \\ (h_i \\ (m_i \end{matrix}$$

Z. We may obtain the product terms from the divisions of K by substituting $k_i$ for $e_i$ in the terms for divisions of E.

W. We will form the product terms for the divisions of V as follows: let $e_i$ represent the set of points for which $y_{i-1} < f(x) < y_i$, and $\eta_i$ the set for which $f(x) = y_i$. We may have:

1.

$$y_i \quad \times \quad \begin{matrix} \bar{c}(e_i) + \bar{c}(\eta_i) \\ c(e_i) + c(\eta_i) \\ \underline{c}(e_i) + \underline{c}(\eta_i) \end{matrix}$$

$$y_{i-1} \quad \times \quad \begin{matrix} \bar{c}(e_i) + \bar{c}(\eta_{i-1}) \\ c(e_i) + c(\eta_{i-1}) \\ \underline{c}(e_i) + \underline{c}(\eta_{i-1}) \end{matrix}$$

2. A group identical with the above except for the substitution of measure for content (m for c)

Combinations of outer measure with inner measure seem trivial and will not be considered. Sums built up from terms involving $y_i$ will be called upper sums, while those whose terms involve $y_{i-1}$ will be called lower sums.

III. Summation. As all the sums which we shall consider will be absolutely convergent we are at liberty to disregard the order of summation. The sum of the terms for any method of division may be taken in any way.

It is necessary to remark regarding the sums for II, W, that in summing for $y_i$ we go from zero to n, defining $e_0$, to be a set of measure(content) zero, While in summing for $y_{i-1}$ we go from 1 to n+1, defining $e_{n+1}$ to be a null set. These remarks hold for the case in which n is countably infinite.

IV. The Limit Process. We shall consider two methods of defining the limit which we call an integral. In the first case we shall define the integral as the lower (upper) limit for all possible upper(lower) sums of the type considered, a method which cannot be applied to the sums involving $h_i$, and in the second we shall define the integral as the limit approached by the sums for a set of successive divisions of decreasing norm in case the limit exists.

In general we shall require convergence for any set of successive divisions with norm approaching zero as a limit, choosing as our norm the largest maximum diameter in the division. The alternative choice of norm mentioned in Art. 6, Sec. I., P.9, will not be used unless expressly mentioned. In cases

where the Darboux theorem(*) does not hold it would be necessary to give the definition for a particular set of divisions, but it does not seem desirable to consider such definitions.

4. Method of Reference. The dfinitions given above readily fall into the four main classes, X,Y,Z, and W, and these in turn into types determined by the method of division used in the definition. We will in general refer to the definitions by groups as follows: under I,X, we find six methods of division of R, for each of these methods of division we find under II,X several groups of product terms. We will denote the group of definitions which results from the combination of division I,X,2 with product group II, X,3 by the notation X(1,2). The meaning of Z(2,2) for example would mean the group of definitions formed by combining the second method of division of K with the second group of product terms under II,Z. This gives us a simple means of reference to the each of the numerous groups of definitions given us by Art. 3.

Within the group reference will be made in the following way: for integrals of classes X,Y, and Z, Upper(Lower) Sum will mean that $M_i(m_i)$ is used in the definition of the sum. In the definitions which involve content(measure) explicitly, the sum will be

(*) cf. Jordan,C: Cours d'Analyse,1893, Paris, p.33, where generalized proof of the Darboux Theorem is given.

Outer (Inner) when outer(inner) content or measure are used. In the case of the sums of class W the meaning of Upper and Lower Sum has already been given. The conventions for content and measure will be as just stated. A sum involving $h_i$ will be termed a Middle Sum.

In case the Upper and Lower Sums for a group have the same limit the result will be called the integral for the group, except in the case of the groups which explicitly involve outer or inner content (measure). In these cases we may have what we shall call Outer Content(Measure) Integrals, and Inner Content(Measure)Integrals. The Integral for these groups will exist whenever these Outer and Inner Integrals coincide.

## Section III.

### The Definitions of Class X.

1. The definitions of type X,1. The nine groups of definitions involving a rectangular division of R will be considered in some detail, as the treatment of all the definitions we shall consider, to a large extent, reduces to a comparison with the definitions of this type.

The definition group X(1,1). f(x) is supposed defined for all the points of R. This is the group of definitions with which we ordinarily associate the name of Riemann, although the resulting Upper and Lower Integrals are due to Darboux. The Darboux Theorem enables us to take the limits of the Upper and Lower Sums for all possible rectangular divisions as well as for any set of successive divisions with norm approaching zero as a limit. The Upper Integral (R) of f(x) will be represented by the symbol $(R)\overline{\int_R} f(x)dx$ the Lower Integral by $(R)\underline{\int}$, and the Integral by $(R)\int$

Such properties of these integrals as are necessary to this discussion will be assumed familiar to the reader.

The definition group X(1,2). (f(x) defined for E)

The definitions of this group are given by Pierpont(*) and we shall refer the reader to him for the proof that the Darboux theorem holds, and for proofs of other properties of these integrals. The integrals of this group involve the outer content of E, and will be called outer content integrals, and represented by the notation $\overline{\int_{c}}_{E} f(x)dx$, or by $(P)\int_E f(x)dx$ (Upper and Lower Integrals will be indicated in the usual manner.)

The definition group X(1,3). For the case of non-metrical sets this definition is equivalent to a definition given by Jordan(**). In case E is metrical the integral for this group is equal to the integral for the preceding group, and the result will be called the Jordan-Pierpont Integral, Integral (J-P), and also Content Integral. As this definition group differs in form from that given by Jordan we shall give proofs of the Darboux theorem in connection with the proofs for group X(1,4) below. These

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(*) Pierpont, James: <u>On Multiple Integrals</u>; Transactions of the American Mathematical Society, Vol. 6, 1905, p. 416.

(**) Jordan, C: <u>Cours d'Analyse</u>; Paris, 1893, p.31-46. Let $E_n$ be one of a sequence of subsets of E. which are metrical, such that the limit as n approaches infinity of the content of $E_n$ is the inner content of E. Jordan defines the integral over E as the limit of the integral over $E_n$.

definitions involve the inner content, and the resulting integrals will be called Inner Content Integrals. We will use the notations $(J)\int_E f(x)dx$, and $\underset{c}{}\int_E f(x)dx$.

The definition group X(1,4). (Sums taken for cells containing points of K) These definitions are new but lead to definite results, and are of especial interest because by their aid we may show the relations existing between the Pierpont and Jordan Integrals. We shall now prove the Darboux theorem for the Upper and Lower sums.

We will first consider f(x) non-negative in E. Let $v_i$ represent a cell (and its volume) of a rectangular division D of R which contains points of K, and at the same time points of E. The only other cells which contain points of K will contain them on the frontier, furthermore the cells $v_i$ contain all the points of K, for the only K points (cf. Sec. I, Art. 3, p. 6.) which are not E points are limit points of E points, although they may be limit points from one side only. Let $M_i$ be the upper limit of f(x) in $v_i$, and form the sum $\sum M_i v_i$ which is positive or at least not less than zero, and consequently has a lower limit $\overline{S}$ if we consider all possible divisions D. Since $\overline{S}$ is a lower limit there is a division $D_1$ of norm $d_1$ such that

$$\overline{S} \leq \sum M_{1,i} v_{1,i} < \overline{S} + \epsilon/2 \tag{1}$$

Let $D_2$ be any other division of norm $d_2 < d_0$ ($d_0$ to be chosen later). Let $v_{2,i}$ be a cell of $D_2$ of the same type as $v_i$, and let the notation $v_{2,ik}$ represent a cell of $v_2,i$ entirely contained in $v_{1,i}$, and $v'_{2,i}$ one of the remaining cells of the $v_{2,i}$ 's. Let us now choose $d_0$ so small that $\sum v'_{2,i} < \epsilon/4M$. (M is upper limit f(x) in E.) We have

$$\sum M_{2,i}v_{2,i} = \sum M_{2,ik}v_{2,ik} + \sum M'_{2,i}v'_{2,i}$$
$$\leq \sum M_{1,i}v_{1,i} + M\sum v'_{2,i}$$
$$< \sum M_{1,i}v_{1,i} + \epsilon/4 \qquad (2)$$

It follows from (1) and (2) that for any D of norm sufficiently small

$$\overline{S} \leq \sum M_i v_i < \overline{S} + \epsilon \qquad (3)$$

which proves that the limit for a set of successive divisions always exists and is always the same for f(x) non-negative. If we choose f(x) negative we may show $\underline{S}$ the upper limit of the Lower Sum always exists, etc.

Let us now take f(x) any limited function, and let $F(x) = f(x) - m$ (m is the lower limit of f(x) in E.) F(x) will be non-negative, and consequently $\overline{S}_F$ exists.

But $$\overline{S}_F = \lim_{d=0} \sum (M_i - m)v_i = \lim_{d=0} \sum M_i v_i - \lim_{d=0} m \sum v_i$$

That is $$\overline{S}_F = \lim_{d=0} \sum M_i v_i - m\overline{c}(K) \qquad (4)$$

It follows from (4) that $\overline{S}_f$ exists. We will call $\overline{S}_f$ the Upper Outer Content Integral of f(x) over K, and denote it by $\overline{\int}_{\overline{c}\,K} f(x)dx$.

Let now $F(x) = -f(x)$. In the cell $v_i$ of D we will have $M_i(F) = -m_i(f)$, and $\lim_{d=0} \sum m_i(f)v_i - \overline{S}_F = \underline{S}_f$

We will call $\underline{S}_f$ the Lower Outer Content Integral of f(x) over K, and denote it by $\underline{\int}_{\underline{c}\,K} f(x)dx$.

As corollaries of the above we may quote:

Cor. I. The Upper(Lower) Outer Content Integral of $(f(x) + C)$ over K is equal to the Upper(Lower) Outer Content Integral of f(x) over K plus the product of C into the outer content of K, (or, what is equivalent, the Outer Content Integral of C over K.)

Cor. II. The Upper(Lower) Outer Content Integral of f(x) is equal to minus the Lower(Upper) Outer Content integral of $-f(x)$ over K.

If the Upper and Lower Integrals are equal the group Integral exists and will be called the Outer Content Integral over K, and denoted by $\int_{\overline{c}\,K} f(x)dx$.

In case K coincides with the frontier of E we shall speak of the Frontier Integrals of f(x) for E.

We are now in a position to prove the existence of the Upper and Lower Jordan(Inner Content)Integrals over E. For this purpose we will take K equal to F.

Let $v_i$ denote a cell of D which contains points of E, $v_i'$ a cell which is full of points of E, and $v_i''$ a cell which is partly full. The cells $v_i''$ contain all the points of F. We have

$$\sum M_i v_i = \sum M_i' v_i' + \sum M_i'' v_i'' \qquad (5)$$

For proof that $\lim_{d=0} \sum M'v'$ exists see Pierpont, loc. cit. p.418. The proof is not given here because of its similarity to the proof already given for the Upper Outer Content Sum for K. $\sum M''v''$ is the Upper Outer Content Sum for F. It follows that the limits exist for all the sums of (5), and we have by definition groups X(1,2), X(1,3) and X(1,4) the following result:

$$(P)\overline{\int_E} f(x)dx = (J)\overline{\int_E} f(x)dx + \overline{c}\overline{\int_F} f(x)dx. \qquad (6)$$

In a similar way we obtain:

$$(P)\underline{\int_E} f(x)dx = (J)\underline{\int_E} f(x)dx + \overline{c}\underline{\int_F} f(x)dx \qquad (7)$$

It follows from equations (6) and (7) that the Pierpont Integral when it exists is equal to the sum of the Jordan Integral and the Frontier Integral, and that a necessary condition for the coincidence of the Pierpont and Jordan Integrals is the vanishing of the Frontier Integral.

The definition group X(1,5) (Sums taken for cells full of points of K). These definitions correspond for K to the Jordan definitions for E. In the definition of the Upper Sum (Sec. II, Art. 3, p. 17) $v_i'$ was taken to mean a cell of D which was full of points of K, As usual let $v_i$ denote a cell containing K points and E points, and let $v_i''$ denote the cells of $v_i$ which are not $v_i$ cells. We have

$$\sum M_i'v_i' = \sum M_i v_i - \sum M_i''v_i'' \qquad (8)$$

We have shown (p. 24-25) that $\lim_{d=0} \sum M_i v_i$ exists and and have defined it as the Upper Outer Content Integral of f(x) over K. The sum $\sum M''v''$ is the sum for the Upper Frontier Integral for K, i.e. is of the same type as the sum $\sum M_i v_i$. It follows that $\lim_{d=0} \sum M'v'$ exists for any set of successive divisions of norm approaching zero as a limit, f(x) being any limited function. We shall call this limit the Upper Inner Content Integral over K. and denote it by $\underline{c}\overline{\int_K} f(x)dx$

In a similar way we may prove the existence of the limit which we define to be the Lower Inner Content Integral over K, and denote by $\underline{c}\underline{\int_K} f(x)dx$.

A necessary condition for the existence of the limit for the Middle Sum is as usual the coincidence of the Upper and Lower Integrals.

Let us denote the frontier of K by F(K). As remarked above F(K) is a set of the same type as K, and consequently the integrals for F(K) have already been defined. The following equations may be established in a way similar to those for the Pierpont and Jordan Integrals, (6) and (7), p. 27, in fact they follow directly from equation (8), p. 28.

$$ {}_{\bar{c}}\overline{\int}_{K} f(x)dx = {}_{\underline{c}}\overline{\int}_{K} f(x)dx + {}_{\bar{c}}\overline{\int}_{F(K)} f(x)dx \qquad (9)$$

$$ {}_{\bar{c}}\underline{\int}_{K} f(x)dx = {}_{\underline{c}}\underline{\int}_{K} f(x)dx + {}_{\bar{c}}\underline{\int}_{F(K)} f(x)dx \qquad (10)$$

As in the case of the Pierpont and Jordan Integrals, the necessary condition for equality of the Outer and Inner Content Integrals over K is the vanishing of the corresponding Frontier Integral.

<u>The definition group</u> X(1,6). (Sums involving explicit use of outer(inner)content of E points in a cell). We will discuss first the case of non-metrical sets. The middle three definitions of this group do not apply in this case. The Integrals involving outer content are equal to the Pierpont Integral, and those involving inner content to the Jordan Integral, as we shall proceed to show.

We will give the proof for the Upper Pierpont Integral, the equivalence of the other Integrals may be demonstrated in the same way. By the definition of outer content

$$\lim_{d=0} \sum \bar{c}(e_i) = \lim_{d=0} \sum v_i = \bar{c}(E) \qquad (11)$$

Let us choose d so small that

$$\begin{aligned} \sum \bar{c}(e_i) - \bar{c}(E) &< \epsilon/4M \\ \sum v_i - \underline{c}(E) &< \epsilon/4M \end{aligned} \qquad (12)$$

If we consider the difference of the Upper Sums for the two Integrals we have

$$\begin{aligned} \sum M_i (v_i - \bar{c}(e_i)) &\leq M \sum (v_i - \bar{c}(e_i)) \\ &< \epsilon/2 \end{aligned} \qquad (13)$$

by (12) which proves the theorem, since $\sum M_i v_i$ is the Upper Sum for the Pierpont definition.(cf. p.16, under II,X,2.)

<u>The definition group</u> X(1,7) (Sums involve explicit use of outer(inner)measure of E points in $v_i$) The definitions of this group lead to integrals of considerable interest, which for the case of E measureable, are equivalent to an integral defined by Young.(*)

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(*) Young, W.H.; <u>On the General Theory of Integration</u>; Transactions of the Royal Society, London, Vol. 204, 1905, p. 230, Art. 9.

The definition of Young involves the upper and lower semi-continuous functions of f(x), $M_x$ and $m_x$.(cf. Sec. I, Art. 9, p. 12., and will be discussed later.

Let us first consider the case in which E is not measureable. The middle three definitions of the group do not apply to this case as they require the sets formed by the division D of R to be measureable. In this connection we may remark that if E is measure-able a division D effects a division of E into measure-able subsets. This is evidently true, since the content of the common frontier points of any two sets is zero. We shall call the $\lim_{d=0} \sum M_i \overline{m}(e_i)$ the Upper Outer Measure Integral (R) over E, and denote it by $(R)\overline{\int_E}_{\overline{m}} f(x)dx$. The proof that this limit exists and is unique for any set of successive divisions of norm approaching the limit zero will now be given.

Let us first suppose f(x) positive or not negative in E., the Upper Sum will not be less than zero, and consequently has a lower limit for all possible divisions D, say $\overline{S}$ Let $D_1$ be a division of norm $d_1$ such that

$$\overline{S} \leq \sum M_{1,i} m(e_{1,i}) < \overline{S} + \epsilon/2 \tag{14}$$

Let $d_2 \leq d_0$ (to be chosen later) be the norm of a division $D_2$, and $v_{2,ik}$ be the cells of $v_{2,i}$ entirely within $v_{1,i}$, and $v'_{2,i}$ be the remaining cells of $v_i$. Choose $d_0$ so that

$$\sum v'_{2,i} < \epsilon/2M \tag{15}$$

Obviously $\sum \overline{m}(e_{2,i}) < \epsilon/2M$. We have

$$\sum M_{2,i}\overline{m}(e_{2,i}) = \sum M_{2,ik}\overline{m}(e_{2,ik}) + \sum M'_{2,i}\,\overline{m}(e_{2,i})$$

$$< \sum M_{1,i}\,\overline{m}(e_{1,i}) + \epsilon/2 \qquad (16)$$

from (15). It follows from (14) and (16) that

$$\overline{S} \leq \sum M_{2,i}m(e_{2,i}) < \overline{S} + \epsilon \qquad (17)$$

This completes the proof of the existence of the upper integral for a positive or non-negative function. We may show that the integral exists for any function (limited) by the method used in the case of definition group X(1,4), p.24-, and the corresponding theorem for the lower integral. The condition for existence of the limit for the middle sum is the equality of the upper and lower integrals.

The corresponding theorems for the sums involving inner measure can be proven by the simple substitution of inner measure for outer measure in the proofs mentioned above. The corresponding change in notation would be made for the resulting integrals.

The remaining three definitions of the group correspond to the definition given by Young. The difference is that Young divides E directly, and considers all possible divisions into metrical sets. It is necessary to show that for an upper semi-continuous function the upper integral obtained in this way is equal to the integral obtained in accordance with the method of this definition group. It is easy to prove the equality of the upper

integral of the upper semi-continuous function and the upper integral of f(x), since the upper limits of the two functions with respect to the set E, in any $v_i$ containing points of E, aré the same.

Let $\mathcal{D}$ denote a finite, or countably infinite division of E into measureable sets $e_j$, and let $M_j$ denote the upper limit of the values of $M_x$ with respect to E in $e_j$. Let $\overline{S}$ be the lower limit of the upper sum for $\mathcal{D}$ for all conceivable divisions of E into measureable sets, and suppose $\mathcal{D}$ is any division for which

$$\overline{S} \leq M_j m(e_j) < \overline{S} + /2 \tag{18}$$

If we subdivide the sets of $\mathcal{D}$ the upper limit of $M_x$ in a set of the new division will be less than or equal to the upper limit in the set of which it forms a part. If $\mathcal{D}'$ is any division consecutive to $\mathcal{D}$, we have, therefore

$$\sum M_j' m(e_j') \leq \sum M_j m(e_j) \tag{19}$$

Let $d_o$ be chosen so that $M_x$ in every region of R of max. diameter less than or equal to $d_o$ is less than the upper limit of $M_x$ for that region by not more than $\varepsilon/6m(E)$, and let D be any division of R such that $d \leq d_o$. We have then, in every $v_i$ of D

$$\text{upper limit } M_x, M_i, - M_x \leq \varepsilon/6m(E) \tag{20}$$

Let us now choose $d_0'$ so small that for every division $D^*$ of norm $d^* \leq d_0'$ the sum of the cells $v_i^{**}$ which ~~are~~ contain~~ed~~ interior points of more than one cell of D shall be less than $\epsilon/6M$.,i.e.

$$\sum v_i^{**} < \epsilon/6M \qquad M \geq |f(x)| \tag{21}$$

If now $\mathfrak{D}'$ is any division of E consecutive to $\mathfrak{D}$ of norm $d' \leq d_0'/2$ the measure of the subsets of E which are interior to more than one cell of D is less than $\epsilon/6M$. This follows from the fact that if we form for each such set the smallest rectangular cell which will contain it we have a set of cells of the type $v_i^{**}$, whose sum, omitting overlapping parts, by (21) is $< \epsilon/6M$.

Let us now compare the sums $\sum M_i\, m(e_i)$ and $\sum M_j' m(e_j')$. We may denote by $e_{ik}$ any cell of $e_j'$ entirely contained in $e_i$, and by $e_j''$ the remaining cells of $e_j'$. From (21) we have, as $\sum m(e_i) = m(E)$

$$\sum m(e_i) - \sum\sum m(e_{ik}) < \epsilon/6M \tag{22}$$

Let us write

$$\sum M_j' m(e_j') = \sum M_{ik} m(e_{ik}) + \sum M_j'' m(e_j'') \tag{23}$$

Making use of (20) and (21) we may write

$$\sum M_j' m(e_j') < \sum M_i m(e_{ik}) + 2\,\epsilon/6 \tag{24}$$

From (22) we obtain

$$\sum M_i m(e_i) - \sum\sum M_i m(e_{ik}) \leq M(\sum m(e_i) - \sum\sum m(e_{ik}) < \epsilon/6 \qquad (25)$$

From (24) and (25) we obtain

$$\sum M_j' m(e_j') < \sum M_i m(e_i) + \epsilon/2 \qquad (26)$$

Combining (26) and (18) we have for any D of norm $d \leq d_o$

$$\overline{S} < \sum M_i m(e_i) < \overline{S} + \epsilon \qquad (27)$$

completing the proof that the two definitions are equivalent in the case of upper semi-continuous functions. The corresponding theorem for the lower integral of a lower semi-continuous function $m_x$ may be proven in a similar way. This proof in connection with the remark made at the top of p. 33, establishes the existence of the Upper and Lower Measure Integrals for f(x) over E.

Because for the case in which E = R these definitions give us the ordinary Upper and Lower Integrals of Darboux(*) we shall call these Measure Integrals(Riemann) in order to distinguish them from the integrals obtained by a definition which we shall consider later.

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(*) Young, W.H. (loc. cit.).

The definition group X(1,8) (Sums involve explicit use of outer(inner)content of points of K in cells of D) Most of the work under the head of definition group X(1,6) will apply here if we simply substitue K for E. The theorems regarding the existence of the limits of sums for sets of successive divisions will therefore be omitted, similarly for the proofs of equivalence with definition groups X(1,4) and X(1,5). The notation will be the same except for the substitution of K for E.

The definition group X(1,9). (Sums involving explicit use of outer(inner)measure). The same may be said here, For the necessary theorems substitue K for E in the work for group X(1,7). We shall consider the integrals of $M_x$ and $m_x$ in Section IV.

2. The definitions of type X,2. (Infinite Rectangular Division of R). In Section I, Art. 6,(2) p. 9, it was remarked that the infinite set of cutting planes might form an open set,(the case in which is closed offers no difficulties) in which event the cells of $_cD$ would not contain all the points of R, and the restriction, made that the content of the points omitted should be zero. It may be thought that a more natural restriction would be to require the measure of the points external to the cells to be zero. We will proceed to show that the two demands are equivalent.

In the first place, the set of cutting planes must be nowhere dense in R. For, suppose it is dense in $R_1$ a sub-region of R, then $R_1$ is a part of the frontier of the set of planes, and the omitted points in $R_1$ (since the set of cutting planes is countable) have as their measure the volume of $R_1$., thus violating the restriction imposed upon the division. The only remaining possibility is that the set (P) of cutting planes shall be nowhere dense, and the consequence that the set of limit points of P,Q, shall be nowhere dense and of measure zero. But under such circumstances Q is metrical. The proof follows:

Since Q has measure zero, the measure of the complement of Q, C(Q) is the volume of R, but this is nothing more than the sum of the infinite set of cells produced by the division ${}_cD$, since the content of P is zero. Let D be a finite division of R formed in the following way: Let us suppose the cells of ${}_cD$ arranged in order and, represent the sum of the infinite series by $\sum_1^{\infty} v_i$. Take the first N of these and divide the remainder of R into rectangular cells in the usual manner. Let D' be any division successive to D with norm $d' \leq v_N$. Denote by $v_j''$ the cells of D' not contained in any $v_i$ of the sum $\sum_1^{N} v_i$. The cells $v_j''$ contain all the points of Q. Since

$\lim_{N=\infty} \sum v_i = m(R)$, we have $\lim_{\substack{N=\infty \\ d=0}} \sum' v_j^* = 0.$ (Q.E.D.)

A comparison of the sums for the division ${}_cD$ with the sums for the division D' defined above shows us that the integrals resulting from this mode of division are equivalent to those of Article 1, of this Section, and that there is no advantage to be obtained by dividing R into a countably infinite set of cells which are rectangular and non-overlapping.

4. The definitions of type X,4. (Finite Rectangular Overlapping Division of R.) Again we obtain in cases where we are at liberty to take the limit for all possible divisions, the integrals of Article 1.
It is possible to show that if we take our upper(lower) limits in a cell as the upper(lower) limits, not of f(x) but of $M_x(m_x)$ we will obtain as Upper(Lower) Integrals the integrals defined in Article 1, for all possible sets of successive divisions with norm monotonically approaching zero as a limit. In the general case the limit for one set of successive divisions may differ from that for another set, a possibility which forces us to discard this type of definition as unreliable.

Since non-overlapping divisions are a special case of this it is evident that the limits for all possible divisions are as low (high) and also that the limits are not lower(higher).

If we denote by D the non-overlapping division formed by the planes of division of $\nabla$, and by $v_i^*$ a cell of D in which there is no overlapping, by $v_i^{**}$ a cell formed from overlapping parts, it follows from the restriction made upon $\nabla$ that for a small enough norm $\sum v_i^{**} < \epsilon/4M$, i.e. the sum of the overlapping parts is as small as you please. To each $v_i^*$ corresponds a cell $v_i$ of $\nabla$. If we choose our norm so that it satisfies the foregoing condition and so that in addition osc. $M_x(m_x)$ in $v_i$ is less than $\epsilon$/vol. R, we may by a comparison of sums prove the theorems mentioned above for the semi-continuous functions. Our inability to control the oscillation in the general case prevents our proving the corresponding theorems for f(x)

5. <u>The Definitions of type X,5</u>. (Countably Infinite Overlapping Rectangular Division of R). A combination of the methods of Articles 3 and 4 shows us that we again obtain the integrals of Article 1 in the special cases where we may take our limits for all possible methods of division, or define our sums for $M_x$ and $m_x$.

We may remark here that if we took the upper(lower) limit of f(x) in a cell, excepting the values at certain points, and enclosed them separately we might obtain decidedly different results from those usually reached. All that need be said at present on this subject, is that as this method amounts to a division of R into sets of points plus a division into cells it is nothing but a complicated way of doing what Lesbegue and Young

have done, although it is applied to cases which are a slight extension of the work of these men, and is, furthermore a complication with no compensating advantage.

6. The Definitions of Type X,6. (Metrical Division). This type of definition is given because of the analogy to the method of division used by Young, although it leads in general to the same results as other definitions involving the notion of content. The method of passage to the limit will differ in special cases. If we allow the content of a cell to decrease without at the same time requireing the norm to decrease, for successive divisions, the limits for different methods of division may not coincide.(*) For the case of the general function we may take the limit for all possible divisions of type $\triangle$ only for the group X(6,1). In the remaining cases to which this mode of division applies we must require the norm,as already defined, to approach zero as a limit over a set of succèssive divisions. By this mode of definition, as we shall show, we obtain the integrals of Article 1 of this Section.

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(*) Young, W.H., loc. cit. p.226, Example 2.

The proofs that this set of definitions is equivalent to those of X,1, rest upon the following two facts: I. Given any division D a division $\Delta$ may be found such the sum of the cells of $\Delta$ which have points interior to more than one cell of D will be less than any pre-assigned quantity however small; and II. Given any division $\Delta$ we may find a division D such the sum of the cells of D which lie entirely within any one cell of $\Delta$ will differ from the content of that cell by less than $\epsilon/N$, where N is the number of cells in the division $\Delta$. Proofs of these theorems are not given as they follow easily from the properties of content. By the aid of I, and II, we may easily establish the identity of the integrals resulting from this mode of division with those of the type X(1), by giving the proof for a special case and then generalizing after the method of Article 1.

In a way similar to that we have used before, with the aid of II above, and the results of Articles 2, of this section we may show that there is nothing to gain by considering a division of R into a countably infinite set of metrical subsets.

<u>The definitions of Type X,6.</u> (Measureable Division). The case of a function defined for R has been fully discussed by Young(*). The definition given by

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(*) Young, W.H., loc. cit. Note especially Art. 24, p. 243.

Lebesgue(*) is a special case of this. In the case of E measureable this definition differs in form from that given by Young, but can be shown to lead to the same results. Several of the definitions under this type lead to integrals which we have not previously defined and these will be discussed in some detail. Let us consider the various definition groups in order.

<u>The definition group</u> X(6,1) For the case in which f(x) is a summable function Young shows that the result is the Lebesgue Integral. Should f(x) not be summable, we obtain Upper and Lower Integrals which I shall call Upper Integral(Y) of f(x) over R, and Lower Integral(Y),etc., denoting them by $(Y)\overline{\int}_R f(x)dx$, and $(Y)\underline{\int}_R f(x)$.

<u>The definition groups X(6,2,3,4,and5)</u>. These combinations do not lead to satisfactory results except in special cases, and these are covered by the groups X(6,6) and X(6,8). The reason for this is that in these definitions we are considering the sum of a set of cells which is not a constant with varying norm, or varying method of division; a fact which prevents us, in the general case, from defining our limits for all possible methods of division; and the further reason that we are

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(*) Lebesgue, Henri, <u>Lecons sur l'Integration</u>, Paris, 1904, Sections IV and V, Chapitre VII, pp.112-120.

unable to effectively use the method of decreasing norm in defining our limit. In the general case the lower limit for all possible upper sums will lie below the upper limit for all possible lower sums, a highly undesirable result.

We omit the consideration of groups X(6,6) and X(6,8) for the reason that we apparently obtain the integrals of Article 1, and also because the combination is extremely artificial.

<u>The definition group X(6,7)</u> For the case of E measureable the definitions of this group coincide with those of Young already mentioned, provided we make the restriction that the division of R effect a measureable division of E. We will consider the case in which it does not later. To prove this equivalence of definitions it is only necessary to choose the cells of R so that, except for sets of measure zero, the cells containing E points are full of points of E. As remarked before, we shall have to consider our limits for all conceivable methods of division, since we are not able to use the method of decreasing norm.

Let us call the lower limit for all possible divisions of the sum $\sum M_i \overline{m}(e_i)$ ,The Upper Outer Measure Integral (Y) of f(x) over E., and denote it by $(Y)\overline{\int_{\overline{m}}}_E f(x)dx$.

The distinction between these integrals(Y) and the Measure Integrals(R) of Article 1, of this Section should be kept in mind. The relation between them is much the same as the relation between the Riemann and Lebesgue Integrals. Making appropriate changes in the form of the sum we may define the Lower Outer, Upper Inner, and Lower Inner Measure Integrals (Y) over E, which will be represented by a notation corresponding to that given for the Upper Outer Integral (Y).

Let us now consider the case of a measureable E divided into possibly non-measureable subsets by a measureable division of R. It is conceivable, in spite of the fact that the sum of the outer measures of the subsets of E may be greater than the measure of E, that the Upper Outer Integral may lie below the Upper integral defined by Young, and the Lower Outer Integral lie above the Lower Young Integral. The Lower Inner Integral would be expected to lie below the Lower Outer, and the Upper Inner Below the Upper Outer, and it is conceivable that the upper and lower outer integrals might coincide and give a different result from the Upper and Lower Inner Integrals, while the Integral of Young would not exist. Such a state of affairs appears decidedly improbable, but there seems to be

of excluding it from the realm of logical possibility. The definition group X (6,9) Here we encounter a difficulty which we have not had to consider previously. Before this the cells which contained points of K, except as frontier points merely, always contained some point of E, and it was easy to define the upper and lower limits for these cells. Now, however, it is possible to make the divisions in such a way that sets of K points whose measure is greater than zero may be contained in cells of $\mathfrak{D}$ which do not contain a single point of E. Several courses of action are open to us. We may decline to consider the integrals over K, we may neglect those cells which do not contain points of E (which is equivalent to defining $M_i$ and $m_i$ as equal to zero), or define $M_i$ to be the upper limit of $M_x$ in the cell, and $m_i$ the lower limit of $m_x$. Besides these, there are two other possibilities which may be considered, first f(x) may really be defined at the points of K and $M_i$ and $m_i$ could be taken in the usual way, the objection to this being that K loses its special characteristic, becoming a set of type E; and second, we may define the upper limit in any cell con-

taining a K point as the upper limit of the values of $M_x$, instead of f(x), and the lower limit as the lower limit of the values of $m_x$.

In objection to the proposed neglecting of the cells containing no K points we may say that we would obtain, not the integral for K, but the integral for the set of E points in K, in other words an integral of the set produced by definition group X(6,7).

In case we use the upper and lower semi-continuous functions we obtain the integrals defined by definition group X(1,8), the proof of this statement paralelling the proof for semi-continuous functions given for definition group X(1,7), p. 33.

Finally we may define the upper (lower) limit in a cell containing points of K as the upper (lower) limit of the values of f(x) and of $M_x$ ($m_x$) for the points of K which are not points of E. The integrals resulting from this choice of $M_i$ and $m_i$ will be termed the Outer (Inner) Measure Integrals (Y) of f(x) over K.

## Section IV.

## The Definitions Involving Division of E,K, and V.

1. The Definitions of Class Y. The definitions of this class give us nothing that we have not already considered. They are given because they are direct definitions of integrals that we have been defining indirectly through reference to a fundamental set, and because similar definitions have been given previous to this paper by Jordan, Young, and others.

We shall suppose E is not metrical, considering the metrical case later. The condition that the sets produced by the division D of E shall not mix enables us to make use of the principle of decreasing norm in passing to the limit. We shall content ourselves with showing here that we are able to compare sums for different division of E with the corresponding sums based on * rectangular divisions of R. For the case of unmixed sets the outer content of E is the sum of the outer contents of $e_i$'s(*). Let us suppose a division D of E given, and consider a division D of

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(*) Pierpont, James, loc. cit. Theorem 5, p. 422.

R of norm d(d will be chosen later). Some of the cells of D of R will probably have as interior points, points of two or more cells of D of E. By a proper choice of d, however we may make the sum of all such cells as small as we please for any method of division D of R. This is a necessary condition that the sum of the outer contents of the sets of D of E shall equal the outer content of E. If we consider the division of E formed by D of R we see that the sum of the outer(inner)content of the cells of this division which contain points of but one cell of D of E differs from the outer(inner)content of E by as little as we please. Using these facts we may compare sums in the usual way, and prove that the result of the definitions of this type is the same as the result of the definitions of groups X(1;2,3,6, and 7), i.e. the Jordan, Pierpont, Jordan-Pierpont, and(Outer)(Inner) Measure Integrals (R) over E.

The method of division $\triangle$ is given because of the analogy with the method D of E. There is no advantage to be obtained in thus specifying the exact method of division for the reason that, with no greater restriction on the character of the cells than that the measure of the common frontier points shall vanish,

we are unable to prove that a unique limit is approached by the sums obtained for sets of successive divisions with norm approaching the limit zero. Since we are forced to consider the limits for all possible divisions of this type, it seems advisable to cast aside all restrictions on the mode of division, and consider the upper(lower)limits of our sums for all possible divisions $\mathcal{D}$. If we do this we meet the following difficulty. We might find a division of E for which there would correspond no measureable division of R. If we restrict ourselves to divisions of E which have their corresponding measureable division of R, i.e., to divisions which would be effected by a measureable division of R, it is easy to see that we would obtain the integrals of Art. 6, Sec. III, under definition group X(6,7), neglecting the combinations which involve the explicit use of content. We shall also neglect the integrals which we might possibly have from a non-measureable division of R.

In case E is metrical, the division D effects a division into metrical subsets, and we obtain the Jordan-Pierpont Integral for all the definitions of this type.

2. The definitions of Class Z. The direct method of division of K gives rise to a difficulty in connection with the definition of $M_i$ and $m_i$ for a set of the division

D of K which we have not previously had to consider. To fix the ideas an example will be given. Let us consider a perfect nowhere dense set K of outer content greater than zero, formed by the usual method of blackening cells of R with the exception of their frontier. Let E be the blackened region of R plus the frontiers of the blackened cells. Suppose $f(x) = 1$ on the black points of E, and zero on the remaining points of E, the E points of F. Since F is the frontier of E, it is a set of type K, and we may consider the integrals of $f(x)$ with respect to F. Let us first calculate the Upper Outer Content Integral of $f(x)$, using the Upper Sum of definition group X(1,4), p.17. Since by hypothesis F is nowhere dense in R, every cell $v_i$ of D which contains a point of F will contain black points of E, and consequently the upper limit of the values of $f(x)$ in $v_i$ is 1 in every case, the upper sum is exactly $\sum v_i$, and its limit$(d=0)$ is $\bar{c}(F)$. Let us now use the method of division Z,1. The cells of this division contain no points of E that are not in F, and so the upper limit of the values of $f(x)$ in $k_i$ is zero, and the upper integral according to this definition is zero, where for the definition previously given it is $\bar{c}(F)$. It is of interest to note in this connection that for the cases involving inner content this difficulty does not arise, and we obtain the same integrals by division of K as by division of R. For example, in this case, both the integrals are zero.

We shall now show how the definition Z(1,1) may be altered to make it correspond to the definition X(1,4). It will be remembered that the Upper(lower) Outer Content Integrals of definition group X(1,4) possessed useful properties, in connection with the Pierpont and Jordan Integrals, and it seems desirable that the Frontier Integral should retain these properties.

Let us, instead of taking for $M_i(m_i)$ the upper(lower) limit of the values of f(x) in $k_i$, take these quantities to be the upper(lower) limit of the values of $M_x(m_x)$ in $k_i$. We will first show that the Upper Outer Content Integral of $M_x$ over K obtained by direct division of K is the same as that obtained by division of R into rectangular cells (X(1,4). Let $\overline{S}$ be the limit of $\sum M(v_i)v_i$ for a set of successive divisions $D_R$ $(d = 0)$ of R. We shall denote by $D_K$ the division of K produced by $D_R$. $D_K$ is then a division of type Z,1, and consequently $\sum \overline{c}(k_i) = \overline{c}(E)$, where $k_i$ is the set of K points in $v_i$.

Let $D_R$ be any division of R of norm $d$ so small that the following relations hold

$$\left| \overline{S} - \sum M(v_i)v_i \right| < \varepsilon/4 \qquad (1)$$

$$\sum v_i - \overline{c}(E) < \varepsilon/8M \qquad (2)$$

$$M(v_i) - M_x < \varepsilon/8\overline{c}(E) \qquad (3)$$

(1) is possible from the hypothesis on $\overline{S}$, (2) from the definition of outer content, and (3) from the fact that $M_x$ is an upper-semicontinuous function.

Let $M(k_i)$ denote the upper limit of $M_x$ for the points of $k_i$, while $M(v_i)$ denotes the upper limit of $M_x$ for all the E points of $v_i$. We have

from (3) $$M(v_i) - M(k_i) < \varepsilon/8\overline{c}(E) \quad (4)$$

and from (2) $$\sum [v_i - \overline{c}(k_i)] < \varepsilon/8M \quad (5)$$

We obtain the following relation from (4) and (5)

$$\left|\sum M(v_i)v_i - \sum M(k_i)\overline{c}(k_i)\right| \leqq \left|\sum [M(v_i) - M(k_i)] \overline{c}(k_i)\right| + \left|\sum \overline{M}(v_i)[v_i - \overline{c}(k_i)]\right| < \varepsilon/4 \quad (6)$$

Combining (3) and (6) gives us

$$\left|\overline{S} - \sum M(k_i)c(k_i)\right| < \varepsilon/2 \quad (7)$$

Let $D'_R$ be any division consecutive to $D_R$, then $D'_K$ the corresponding division of K will be consecutive to $D_K$, and the relation

$$\overline{S} - \sum M'(k'_i)\overline{c}(k'_i) < \varepsilon/2 \quad (8)$$

is easily established. $\overline{S}$ is therefore the limit of the upper sum for at least one set of successive divisions $D_K$ $(d = 0)$. Supposing $D_K$ is any division of norm d for

which (7) holds, it is now necessary to show that for any division $D_{K,l}$ of norm $d_l$ ( to be chosen later)

$$\left| \bar{S} - \sum M(k_{l,i})\bar{c}(k_{l,i}) \right| < \varepsilon \qquad (8)$$

To obtain (8) we shall choose $d_l$ so that the sets of $D_{K,l}$ which lie interior to more than one cell of $D_R$ have a total outer content less than $\varepsilon/8M$. Since $k_i$ contains all the points of $v_i$, all the cells $k_{ik}$ which are contained in $v_i$ are also in $k_i$, and we have, since $\sum \bar{c}(k_i) = \bar{c}(E)$

$$\sum c(k_i) - \sum_i \sum_k c(k_{ik}) < \varepsilon/4M \qquad (9)$$

Denoting by $k'_{l,i}$ the remaining cells of $D_{K,l}$, we have from (9)

$$\left| \sum M(k_{l,i})\bar{c}(k_{l,i}) - \sum_i \sum_k M(k_{ik})\bar{c}(k_{ik}) \right| < \varepsilon/4 \qquad (10)$$

and from (9) and (1)

$$\left| \sum M(k_i)\bar{c}(k_i) - \sum_i \sum_k M(k_{ik})\bar{c}(k_{ik}) \right|$$

$$\leq \left| \sum_i \sum_k [M(k_i) - M(k_{ik})]\bar{c}(k_{ik}) \right|$$

$$+ \left| \sum_i M(k_i)[\bar{c}(k_i) - \sum_k \bar{c}(k_{ik})] \right|$$

$$< \varepsilon/2 \qquad (11)$$

Combining (7) and (11) gives us (8), thus proving the equivalence of the definitions for $M_x$.

The proof that the lower integrals of $m_x$ are equal for both definitions is similar to the above.

Since in any cell of R the upper limit of f(x) is equal to the upper limit of $M_x$, it follows readily that the Upper Outer Content Integral of any limited function is the same for both methods of definition, and similarly for the Lower Outer Content Integral.

We may mention here that while it was necessary in the case of thegeneral function to take the limit for successive divisions when we used the definition X(1,4) the definition Z(1,1) enables us to take our Upper(lower) limits for all possible Lower(Upper) Sums, giving the same results as those already obtained. The proof of this is simple, being closely analogous to the last part of the precedingproof, beginning with equation (8), and will be omitted.

A further remark will enable us to throw the remainder of the discussion of the integrals over K back to the discussion of the integrals over E. Since $M_x$ and $m_x$ are defined for all the points of K, for these functions K is a set of the same type as E, and any property of an Upper Integral over E will be a property of the

corresponding Upper Integral over K. Similarly for Lower Integrals.

Although the integrals mentioned at the beginning of this article, and to which exception was taken on the ground that they differed from those of Article L, Sec. III, possess points of interest, they are not neccessary to any of the further discussions we shall enter upon, and we shall therefore content ourselves with what we have already said with reference to them.

3. <u>The Definitions of Class W</u>. These definitions involving a division of the interval of variation, V, are essentially a special case of the definitions which we have named after Young, as has already been remarked. In view of this fact we should expect in the general case of non-measureable sets, and functions which not summable to obtain different results.

Lebesgue(*) has called functions for which the set of points determined by the relation $A < f(x) < B$ is measureable <u>summable functions</u>. In case this set

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(*) Lebesgue, Henri, <u>Lecons sur l'Integration</u>, p, 115.

is always metrical we shall say the function is <u>addable</u>. It will be notice on reference to Art. 3, Sec. II,p.17, under the head of II, Multiplication, W, that the use of content in a definition of this type has been suggested. It appears, as one would expect, on examination of the result of such a definition, that for addable functions we obtain the ordinary Riemann Integral, or for a function defined for E, the Jordan-Pierpont Integral. Should the function not be addable,(it will be shown later that this is equivalent to saying that the Riemann Integral does not exist) we find ourselves involved in difficulties when we try to take our limits for successive divisions with norm approaching zero as a limit. Since the sum of the outer (inner) contents of the subsets of E produced by the division of the interval of variation might not approach the outer (inner) content of E as the norm approached zero over a set of successive divisions of V the limit of the sum for these definitions might be a function of the method of successive division of V. We might consider the lower(upper) limit of the Upper (Lower) Sums for all divisions D of V, but these might not be as low(high) as the ordinary Upper(Lower) Integral(R) and for this reason do not seem to be worth considering. We shall therefore consider these definitions restricted to the case in which f(x) is addable.

For reasons similar to those just mentioned we shall restrict the definitions involving measure to the case in which f(x) is summable, thus obtaining the ordinary Lebesgue Integral.

The definitions of type W,2 which employ an infinite division of V may be reduced to the definitions of type W,1 by a method almost exactly the same as that used to reduce the infinite rectangular definitions to the finite rectangular case.(cf. Art.2, Sec. III, p. 36.)

## Section V.

### Conditions for Equality of Upper and Lower Integrals.

1. The principal work of Sections III and IV was the demonstration that (for all the definitions retained) in the case of limited functions the Upper(Lower) Integrals always existed. We found that the Middle Sum could only be considered whenever we could pass to the limit over a set of successive divisions ($d \to 0$). It is our purpose in this section to investigate the conditions under which the Upper and Lower Integrals for the various cases are equal. Obviously the necessary and sufficient condition for the existence of a limit for the Middle sum is the equality of the Upper and Lower Integrals. These conditions are usually called conditions for existence of the integral, and will be so spoken of, since the only condition we shall consider for existence of Upper or Lower Integrals is that the function be limited. The existence conditions for the Riemann and Lebesgue Integrals will be assumed.

2. The Outer Content Integrals over E (K). In the proof that a necessary and sufficient condition for the existence of the Riemann Integral is that the points for which $w_x>0$ form a set of measure zero, R is supposed

closed. Since E may be an open set this condition, while still necessary may no longer be sufficient, as the following example will show. Let n be any odd integer, and let us take R the interval from 0 - 1 on the straight line, (x-axis), E the rational points of R. Let $f(x) = 1$ for $x = m/n$, m and n relatively prime; and $f(x) = 0$, for $x = m/2n$, m and 2n relatively prime. Call these sets $O_1$ and $O_2$, respectively. Since all ~~the limit~~ points of $O_1$ are limit points of $O_2$, and conversely, we have $M_x$ for E is 1, and $m_x = 0$, x in E; i.e., $w_x = 1$ for every point of E. But E has measure zero, although the upper and lower integrals of f(x) are respectively 1 and 0. We may show in the usual manner that the condition, both necessary and sufficient, is that the set for which $w_x > s$ shall have <u>content</u> zero. Let us write:

A necessary and sufficient condition for the existence of the Pierpont (Outer Content) Integral of f(x) over E is that, x in E

$$c(w_x > s) = 0 \tag{1}$$

If E is closed we may write in place of (1)

$$m(w_x > 0) = 0 \tag{2}$$

We saw in Art. 2, Sec. IV, p. 49-, that the upper Outer Integral of f(x) over K was the Upper Outer Integral of $M_x$ over K, and the Lower Integral of f(x) the Lower Integral of $m_x$. The Outer Integral over K will exist; (a) if K is a set of zero content; (b) if $w_x$ is an integrable null (R) function for

x in K, i.e. satisfies condition (1), x in K.

3. The Inner Content (Jordan) Integral over E(K), and the Content Integral. Let I be the points of E not in F. It is very easy to demonstrate the following:

A necessary and sufficient condition for the existence of the Jordan Integral of f(x) over E is that,

$$x \text{ in } I \qquad c(w_x > s) = 0 \qquad (1)$$

or if I be closed, i.e. contain a set of F points of content zero,

$$m(w_x > o) = 0 \qquad (2)$$

If I is the set of interior points of K, then the Inner Content Integral over K will exist if (1) and (2) are satisfied, x in I .

The Jordan-Pierpont (Content) Integral exists if the Outer Content Integral over F vanishes (Art. 1, Sec. III, eqs. (6), (7) p. 27). This will occur for (a) $c(F) = 0$; (b) $m(M_x > 0)$, x in F, since F is closed and the Outer Content Integral over F is supposed to exist.

4. The Outer (Inner) Measure Integral (R) over E (K). (p. 30, X(1,7)). The analogy between the sums for these integrals and the sums for the Outer Content Integrals is so close that we have exactly the same conditions for existence of the integral in each case, and we need only mention that there is no change in the conditions for the case of Inner Measure, as there was in the case of Inner Content.

5. The Outer(Inner)Measure Integral (Y) over E.

There seems to be no way to avoid the logical possibility of the lowwer limit of the Upper Sum being below the Upper limit of the Lower Sum. Let us consider the possibility of $\overline{S} < \underline{S}$. for $f(x) > 0$, $x$ in E. If this is true, there is a number L between $\overline{S}$ and $\underline{S}$ such that the Upper Sum for $\mathcal{D}$, a division of E into non-measureable sets, is less than L, and the Lower Sum for $\mathcal{D}'$ another division of E, is greater than L. Let us combine the divisions $\mathcal{D}$ and $\mathcal{D}'$, dividing the sets of $\mathcal{D}$ into the sets which they have in common with the sets of $\mathcal{D}'$. Let $e_{ik}$ denote the sets of $\triangle$ which lie in $e_i$ of $\mathcal{D}$. We have

$$\sum_k \overline{m}(e_{ik}) \geq \overline{m}(e_i) \qquad (1)$$

The Lower Sum for $\triangle$ will not be less than the Lower Sum for $\mathcal{D}$, since the minimum does not decrease for successive divisions, and we have the relation corresponding for $\mathcal{D}'$ to (1) for $\mathcal{D}$. The problem centers about the behavior of the Upper Sum. Obviously

$$\sum_i \sum_k (M_{ik} - m_{ik})\overline{m}(e_{ik}) \geq 0 \qquad (2)$$

Let $\sum_k \overline{m}(e_{ik}) = \overline{m}(e_i) + \varepsilon_i$, and $M_{ik} = M_i + \eta_{ik}$

We obtain by substitution

$$\sum_i \sum_k M_{ik}\overline{m}(e_{ik}) = \sum M_i \overline{m}(e_i) + \sum M_i \varepsilon_i$$

$$- \sum_i \sum_k \eta_{ik}\, \overline{m}(e_{ik}) \qquad (3)$$

If the Upper sum is to increase we must have

$$\sum M_i \cdot \varepsilon_i > \sum_i \sum_k \eta_{ik} \overline{m}(e_{ik}) \tag{4}$$

Furthermore the left hand side of (4) must be enough larger than the right to include the amount the Upper Sum was originally below the Lower Sum, the increase of the Lower Sum, and also the difference represented by (2). There is no a priori reason for supposing that $\sum \varepsilon_i$ may not be large enough to account for all of these, especially if the other quantities concerned are small. While it seems extremely improbable that examples of such a thing exist, yet as remarked before, we must admit the possibility.

We may always divide E into sets for which the difference $M_i - m_i$, for every one of the sets is small at pleasure. For every such division the difference between the Upper and Lower Sums is as small as we please. Let us now define the Outer Measure Integral of f(x) over E as the lower limit of the Upper Outer Sum for all possible divisions of this type, and the Inner Measure Integral of f(x) as the upper limit of all possible Lower Inner Sums of this type. We shall consider these Integrals later. The Outer(Inner)Measure Integrals(Y) as previously defined will be abandoned as indefinite, and those just defined considered in their place.

In case E is measureable the Measure Integral (Y) will exist whenever f(x) is summable, as is shown by Young(*), who also shows that the upper measure sum is never less than the lower measure sum. We may also show that a necessary and sufficient condition for the existence of the Measure Integral is that there exist divisions of E into measureable subsets such that the sum of the measures of the sets in which the difference $M_i - m_i > 0$ is small at pleasure. Suppose, S is the Integral, and $D$ and $D'$ two divisions of E such that the Upper Sum for D and the Lower Sum for $D'$ are within $\varepsilon/2$ of S. As before consider the division formed by combining $D$ and $D'$. Since the sets are now measureable the sum of their measures is the same for all divisions, and the Upper Sum has not increased, nor the Lower Sum decreased, and consequently we must have, if $v_i$ denote the cells of $\Delta$ :

$$\sum (M_i - m_i) v_i < \varepsilon \qquad (5)$$

from which we easily draw the theorem stated above.

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(*) Young, W. H., loc. cit. p.245, Art. 25, and Art.6, Theorem 4, p. 228.

## Section VI.

### Properties and Relations.

1. We are now in a position to consider the relations which exist between the integrals which we have so far defined and to develop some of the properties which the new types possess. It will be well perhaps to first give a summary of the integrals which we have thought worth considering further.

I. Integrals of f(x) with respect to R.

| | |
|---|---|
| Riemann Integrals | (3) |
| Lebesgue Integral | (1) |
| Young " s | (2) |

II. Integrals of f(x) with respect to E.

| | | | | |
|---|---|---|---|---|
| Pierpont | (Outer Content) | (3) | | |
| Jordan | (Inner Content) | (3) | | |
| Jordan-Pierpont | (Content) | (3) | | |
| Outer Measure | (Riemann) | (3) | | |
| Inner " | ( " ) | (3) | | |
| Measure | ( " ) | (3) | | |
| Outer Measure | (Young) | (3) | | |
| Inner Measure | ( " ) | (3) | | |
| Measure | (Young) | (2), | (Lebesgue) | (1) |

III. Integrals of f(x) with respect to K.

| | | | | |
|---|---|---|---|---|
| Outer Content | | (3) | | |
| Inner Content | | (3) | | |
| Content | | (3) | | |
| Outer Measure | (Riemann) | (3) | (Young) | (3) |
| Inner Measure | ( " ) | (3) | " | (3) |
| Measure | ( " ) | (3) | " | (3) |

Considering Upper and Lower Integrals as attempts to obtain the ideal quantity mentioned in the Introduction we have 60 defined quantities to consider, each of which has some claim to be regarded as an integral, and succeeds or not in so far as it meets or fails to meet the ideal standard which we set up. Of these integrals the following are new: All the Measure Integrals(R) over E; the Outer(Inner) Measure Integral(Y) over E; and all the Integrals over K. We shall assume such properties of the remaining integrals as are necessary to the following discussion.

2. The Integral over an n-dimensional Set(Descriptive Definition). We shall now give the descriptive definition promised in the introduction, generalizing the definition of Lebesgue(*) to the case of a set of points in n-dimensions. Following Lebesgue, we propose to attach to every limited function f(x), defined for a limited set of points S, a finite number $\int_S f(x)dx$, which we shall call the integral of f(x) over S, and which satisfies the following conditions:

I. Let S be any set of points, and $h_1, h_2, \ldots, h_n$, any set of n constants. Let x be any point of S, and x' a point whose coordinates are the coordinates of x increased by $h_1, h_2, \ldots h_n$, respectively. As x runs over S, x' will run over a set which we shall call H. We

(*) Lebesgue, loc.cit. p. 98.

require $$\int_S f(x)dx = \int_H f(x_1-h_1, x_2-h_2, \ldots, x_n-h_n)$$

II. Let $S_1$ and $S_2$ be any two sets with no point in common, then $$\int_{S_1} f(x)dx + \int_{S_2} f(x)dx = \int_{S_1+S_2} f(x)dx.$$

III. $$\int_S [f(x) + F(x)]\,dx = \int_S f(x)dx + \int_S F(x)dx$$

IV. If $f \geq 0$ and vol. $S > 0$, then
$$\int_S f(x)dx \geq 0$$

V. $$\int_S 1\cdot dx = \text{vol. } S$$

VI. If $\lim_{m=\infty} f_m(x) = f(x)$, then
$$\lim_{m=\infty} \int_S f_m(x) = \int_S f(x)dx.$$

We might impose the further condition that the integral be iterable. That this is a condition independent of the six given above is evident from the fact that the Riemann Integral satisfies the above conditions and yet is not iterable. The failure of the Riemann Integral in this respect is due to the fact that non-metrical sets of points exist, i.e. there are functions which are not integrable (R). If we could by a constructive definition obtain an integral which exists for every function and possesses these six properties this integral would inevitably be iterable. This requirement that <u>every</u> function shall have an integral possessing these properties is virtually a <u>seventh</u> condition and implies iterability.

Although it is not proposed to extend this descriptive definition to the case of integrals over K because of the somewhat arbitrary nature of the definitions for K, we shall consider the various integrals with reference to the properties given above.

3. The Content Integrals. Certain properties are possessed in common by all the integrals directly involving content, and we shall give these without proof resting upon the close analogy with the sums for the Riemann Integral. Condition I, is satisfied by all; if we are willing to redefine vol. S as outer or Inner content of S, as the the case may require, IV, and V, are satisfied. VI is satisfied unconditionally by all, proof of this being analogous to the proof for the ordinary case. We shall not discuss the problem of iteration(*). II will be satisfied if the sets $S_1$ and $S_2$ in addition to the restriction imposed, have their common frontier points a set of content zero. It will be noticed that this is the case for the ordinary Riemann Integral, and to ask the removal of this restriction is a considerable generalization of the problem. III, will be satisfied for the integrals which result when Upper and Lower Integrals are equal, the relations which the Upper(Lower)Integrals satisfy will be given for some cases.

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(*) Pierpont, James, loc. cit. p. 428, Art. 6. The question of iteration for the Outer Content(Pierpont) Integral is treated here in some detail.

3. Content Integrals with respect two two or more sets. For simplicity we will consider only two sets $E_1$ and $E_2$, the generalization to any finite number being easily made. The frontiers will be $F_1$ and $F_2$, respectively. Let us consider the Pierpont and Jordan Integrals of f(x) over the two sets $E_1$ and $E_2$. We will give an example to illustrate the way in which the Integrals (P) and (J) fail to satisfy condition II, and show what must take its place if we are to use these definitions. It is at this point that the usefulness of the Frontier Integral appears.

Let $E_1$ be the rational points of R, and $E_2$ the irrational points, and let f(x) = 1, x in $E_1$; f(x) = 1, x in $E_2$, i. e., f(x) = 1 in R. The Integral (P) for $E_1$ and for $E_2$ gives vol. R in both cases. The sum of the integrals over $E_1$ and $E_2$ is then twice the volume of R. The integral over R, the sum of $E_1$ and $E_2$, is, however, exactly the volume of R. Both the Jordan intégrals for $E_1$ and $E_2$ vanish, but the Jordan integral over R is the same as the integral (P), vol. R. Thus, in considering the integrals over two such regions we find that we do not obtain results from either the (P) or (J) integrals which satisfy condition II. If we introduce the frontier integral, we find that the frontier integral is the same for both sets and equal to vol. R. The duplication obtained from the integral (P) is thus seen to be due to counting the frontier

integral twice. We will have the true situation if we write

$$(R)\int_R f(x)dx = \underline{c}\int_{E_1} f(x)dx + \underline{c}\int_{E_2} f(x)dx + \bar{c}\int_F f(x)dx \qquad (1)$$

where F denotes the common frontier points of $E_1$ and $E_2$.

The proof of (1) rests upon formula (6), Sec. III, Art. 1, p.27, and the following general proposition:

Let the Integral(P) of f(x) exist for the set $(E_1 + E_2)$, L. C. M. of $E_1$ and $E_2$. Let K be the G.C.D. of the sets $(E_1+F_1)$ and $(E_2+F_2)$.

Theorem I. _The Outer(Inner)Content Integral of f(x) over K regarded as a subset of $(E_1+F_1)$ is equal to the Outer(Inner)Content Integral of f(x) over K, regarded as a subset of $(E_2+F_2)$._

We will now prove this theorem. Since the values of the function at the points of discontinuity do not affect the integral we shall disregard them. Let us choose a cubical division D of norm d so small that the greatest of the differences $M_i-m_i$ for the cells $v_i$ of D, excepting the points of discontinuity, is less than $\varepsilon/\sum v_i$.
If $M_{1,i}$, and $M_{2,i}$ are the upper limits of f(x), except for the points excluded, in the cell $v_i$ for $E_1$ and $E_2$,

respectively, we have

$$\left|\sum M_{1,i}v_i - \sum M_{2,i}v_i\right| = \left|\sum (M_{1,i} - M_{2,i})v_i\right|$$
$$\leq \text{(largest } M_i - m_i\text{)}.\sum v_i < \varepsilon$$

which proves the theorem, since

$$\lim_{d\,0} \sum M_{1,i}v_i = \overline{c}\int_K f(x)dx.$$

It will be noticed that it is not sufficient for the proof of the above theorem to have the Pierpont Integrals exist separately for $E_1$ and $E_2$. For, suppose $f(x) = 1$ at the rational points of R and, $f(x) = \frac{1}{2}$ at the irrational points. The Integral(P) over $E_1$, the rational points, is 1.vol. R; over $E_2$, $\frac{1}{2}$.vol. R, while $(R)\int_R f(x)dx$ does not exist.

We may write as a corollary of the above theorem:

$$\overline{c}\int_{E_1} f(x)dx + \overline{c}\int_{E_2} f(x)dx = \overline{c}\int_{(E_1\ E_2)} f(x)dx$$
$$+ \overline{c}\int_K f(x)dx \qquad (2)$$

In case the set K does not differ from either $F_1$ or $F_2$ by more than a set of content zero we may have

$$\overline{c}\int_{(E_1+E_2)} f(x)dx = \underline{c}\int_{E_1} f(x)dx + \underline{c}\int_{E_2} f(x)dx + \overline{c}\int_F f(x)dx \qquad (3)$$

where F is either $F_1$ or $F_2$.

This theorem I is true under the slightly more general condition that $w_x$, x in K, be integrable null(R) (cf. Art. 3, Sec. V, p.54.)

4. The Vallee-Poussin Definition.(*) Let f(x) be defined for any limited set of points E, and let C be the complement of E with respect to a set R.(R defined as usual). Let us introduce the auxiliary function F(x) = f(x), x in E, F(x) = 0, x in C. We define as the Integral "par Excess" of f(x) with respect to E the Upper Integral(R) of F(x) over R, and denote it by

$$\int_{E}^{E} f(x)\,ds \qquad (R)\overline{\int_{R}} F(x)\,dx \tag{4}$$

Similarly the Integral "par Defaut" is

$$\int_{E} f(x)\,dx \qquad (R)\underline{\int_{R}} F(x)\,dx \tag{5}$$

This method of definition works very well for sets which are metrical, but for non-metrical sets gives results which are decidedly indefinite, even when the function integrated is constant in sign. In general this definition gives an intégral which fails to satisfy conditions (II) and (III).

The faults of this definition arise from the use of the auxiliary function. In a cell which contains points both of E and C, the Upper(Lower) limit of the values of f(x), when constant in sign, will not be the same as the Upper(Lower) limit of the values of F(x). The following results may easily be established for a positive function:

$$\int_{E}^{E} f(x)\,dx \qquad (P)\overline{\int_{E}} f(x)\,dx \tag{6}$$

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(*) Vallee-Poussin,C, Cours d'Analyse Infinitesimale, Vol. 1, p.225 , Paris, 1903.

and $\int_{E}^{D} f(x)dx \quad (J) \int_{\underline{E}} f(x)dx$ (7)

For a negative function the (P) and (J) would be interchanged in (6) and (7). Evidently, unless f(x) is integrable (R) null on the frontier of E, the Integrals "E" and "D" will not coincide for a non-metrical set.

For a function which is both positive and negative in E:

$$(P) \overline{\int_{E}} f(x)dx \qquad \int_{E}^{E} f(x)dx \tag{8}$$

$$(P) \int_{\underline{E}} f(x)dx \qquad \int_{D\,E} f(x)dx \tag{9}$$

The Integral "E"("D") of f(x) g(x) is not definitely related to the sum of the Integral "E"("D") of f(x) and the Integral "E"("D") of g(x). The addition of a constant even alters the "E" ("D") Integral in a way not to be determined apart from the special character of the function integrated.

5. The Outer(Inner)Measure Integrals (R) over E (K).

Owing to the impracticability of constructing sets of points which are not measureable,the discussion will in the main be confined to measureable sets. Conditions I?IV?V, and VI, may be shown to be satisfied by all these integrals, including Upper and Lower Integrals, it being of course necessary in the case of non-measureable S to substitute outer or inner measure for vol. S. III, will evidently be satisfied whenever F and f are integrable (R) over S. In regard to II, we will prove the following theorem:

Theorem II. *Let* $S_1$ *and* $S_2$ *be any two non-overlapping measureable sets of* R *and* S *their sum, and* f(x) *be integrable* (R) *and defined in* S, *Then the Measure Integral*(R) *of* f(x) *over* S *is the sum of the Measure Integral*(R) *of* f(x) *over* $S_1$ *and the Measure Integral* (R) *over* $S_2$.

$$(R)\int_{m\,S} f(x)dx = (R)\int_{m\,S_1} f(x)dx = (R)\int_{m\,S_2} f(x)dx \qquad (10)$$

We may prove this as follows; Form a rectangular division D of R of norm d such that the $M_i - m_i < \varepsilon/_{m(S)}$ in every cell $v_i$ of D containing points of S, if we except the values at points of discontinuity. Using the fact that a rectangular division divides a measureable set into measureable sets, and denoting by $s_i$, $s_{1,i}$, $s_{2,i}$ the subsets of S, $S_1$, and $S_2$ in $v_i$, we have

$$m(s_i) = m(s_{1,i}) + m(s_{1,i}) \qquad (11)$$

Let us consider the difference

$$\sum M_i(S)m(s_i) - \sum M_i(S_1)m(s_{1,i}) - \sum M_i(S_2)m(s_{2,i})$$

which may be written

$$\sum [M_i(S)-M_i(S_1)]m(s_{1,i}) + \sum [M_i(S)-M_i(S_2)]m(s_{2,i})$$

using (11). Since the upper limits do not include the points of discontinuity we have the above sum less than $\varepsilon$ and Theorem II follows immediately.

Since the outer measure of a set plus the inner measure of its complement gives the measure of the fundamental set, if it is measureable, we may state the above theorem for the case in which $S_1$ and $S_2$ are not measureable in the

form

Theorem III. *If f(x) is integrable (R) in S, a measureable set, $S_1$ is any subset of S, and $S_2$ its complement with respect to S; then the Measure Integral (R) of f(x) over S is equal to the Outer Measure Integral (R) of f(x) over $S_1$ plus the Inner Measure Integral (R) of f(x) over $S_2$. (Outer and Inner may be interchanged).*

The proof of this is evidently parallel to that of Theorem II, and will not be given.

It will be noticed that f(x) may be integrable(R) in $S_1$ and $S_2$ separately but not be integrable (R) in S. There is this difference between these integrals and the ordinary Riemann Integral regarding the way in which condition II is satisfied. In the Riemann case the Integral over S exists if the the Integrals over $S_1$ and $S_2$.

In this connection we may remark that the frontier of a set,being closed is measureable, and that the measure of the frontier is equal to the outer content. From this it follows that, F = frontier of E

$$\begin{array}{ll} {}_{\bar{c}}\overline{\int_F} f(x)dx & (R)\,{}_m\overline{\int_F} (f(x)dx \\ {}_{\bar{c}}\underline{\int_F} f(x)dx & (R)\,{}_m\underline{\int_F} (f]x)dx \end{array} \qquad (12)$$

If we denote by I the points of E which are not points of F, we have, since I is both measureable and metrical,

$${}_{\underline{c}}\overline{\int_E} f(x)dx = {}_c\overline{\int_I} f(x)dx = (R)\,{}_m\overline{\int_I} f(x)dx = (J)\overline{\int_E} f(x)dx \qquad (13)$$

Similarly for the Lower Integrals.

5. The Measure Integrals(y). There is very little to be said about the properties of these integrals beyond what has been shown by Young.(*) The difficulties in connection with the Upper and Lower Outer(Inner) Measure Integrals have been brought out previously, especially in Section V, Art. 5. A tentative definition was given to replace the usual one, that the Integral exists when the Upper and Lower Integrals are equal, but it seems that this definition leads to an integral which is a function of $\varepsilon$ . Let $\Omega$ denote the difference between the Upper and Lower Outer Sums for a division $\mathcal{D}$ of E. If we consider all possible divisions $\mathcal{D}$ for which $\Omega$ is less than $\varepsilon$ and all possible divisions for which $\Omega$ is less than $\varepsilon/2$, there seems to be no reason for supposing that the Lower limit of the Upper Sum is the same for both cases. Since there seems to be nothing which corresponds in the case of these non-measureable sets to the integral of Young, let us define the Integral(Outer Measure) as the Upper Outer Measure Integral, and the Inner Measure Integral to be equal to the Lower Inner Measure Integral. It is not difficult to show by a comparison of sums, that conditions I, IV, and VI are satisfied, and also V, with the usual convention about volume.

(*) Young, W. H. (loc. cit.)

The fact that the sum of the outer(inner) measures may increase(decrease) for sets of successive subdivisions of E prevents our setting up definite relations between the sums for two different methods of division into non-measureable sets, and for this reason we shall omit the consideration of conditions IIand III.for such methods of division and the resulting integrals. We may note, however, that it is always possible to add a constant to a function, obtaining the integral of $f(x) + C$ is equal the integral of $f(x)$, plus the integral of C.

6. Conclusion. As long as there exist sets for which the problem of volume remains unsolved, so long will the problem of the definition of the integral over such sets be unsatisfactorily solved. As we have seen, the attempt to extend the definitions of Young and Lebesgue to non-measureable sets does not result as satisfactorily as the attempt to extend the definition of Riemann, in the sense that the properties of the integrals so obtained are elusive. As long as we restrict ourselves to measureable sets and summable functions the problem of integration over sets of points is completely solved.