AN EXAMINATION OF HOW TEACHERS USE CURRICULUM MATERIALS FOR THE TEACHING OF PROOF IN HIGH SCHOOL GEOMETRY

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

AN EXAMINATION OF HOW TEACHERS USE CURRICULUM MATERIALS FOR THE TEACHING OF PROOF IN HIGH SCHOOL GEOMETRY

presented by Ruthmae Sears

a candidate for the degree of Doctor of Philosophy

and hereby certify that, in their opinion, it is worth of acceptance.

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ABSTRACT

This case study examined how three high school geometry teachers used their geometry textbooks (*Prentice Hall Geometry* and *McDougal Littell Geometry*) to teach proof. More specifically, this study examined the following: How subject-specific curriculum materials present proofs related to parallel and perpendicular lines, angles, and congruent triangles? How do geometry teachers use curriculum materials to facilitate students learning to prove? What factors influences teachers' decision to deviate or not from curriculum materials? Data were collected via a classroom observation protocol, teacher artifacts, audio and video classroom recording, and teacher interviews. A conceptual analytical framework, which consisted of three dimensions, comprised of the Mathematical Tasks Framework (Henningsen & Stein, 1997) and proof schemes framework (Harel & Sowder, 1998) was used to analyzed the data. The first dimension focused on task features, the second on levels of cognitive demands and the third considered the proof schemes utilized.

To inform the classroom observations and the data analysis, a textbook analysis was conducted of proof and proof-related tasks. This analysis considered the frequency of proof and proof-related tasks, types of proof representations used, real world or abstract context of proofs, use of figures, occurrences of fill in the blank and multiple choice tasks, and the extent to which tasks were composed of multiple parts. Additionally, the levels of cognitive demand of tasks were evaluated. During classroom observations, attention was given to what constitutes convincing proof arguments, and how curriculum materials were utilized.

The data analysis showed that the geometry curriculum materials used by the teachers in this study provided few opportunities to prove, and that there were differences between textbook series in the tasks' features and the levels of cognitive demand of the proof tasks included. Additionally, the teachers in this study enacted proof tasks generally by promoting *memorization* or *procedures without connections*. Moreover, whenever lower levels cognitive demand tasks were posed *external conviction proof schemes* were more evident; while *analytical proof schemes* appeared more frequently when higher-level cognitive demand tasks were posed. Furthermore, teachers' beliefs, experience, desire to make mathematics "easy", professional community, and assessment were factors that contributed to how proof was taught.

CHAPTER I: INTRODUCTION AND RATIONALE FOR THE STUDY

"No way of thinking or doing, however ancient, can be trusted without proof".

Henry David Thoreau ~

Proof in Mathematics and School Mathematics

Throughout the history of mathematics, the methods of proof, and even the idea of what proof is, have evolved. Regardless of the nature of the evolution, mathematicians have mostly used proofs to share knowledge and convince others of the merit of their assumptions and their results, as well as to reflect on the possibilities of conceived notions. Western mathematics, and therefore our current way of understanding proof, is strongly influenced by the conception of proof developed by the Greeks, who used logic and a deductive presentation of arguments to substantiate claims made rather than just the results of empirical findings (Reid & Knipping, 2010).

Proof can be understood as one way to communicate mathematical understanding. This communication can take different forms and reach different levels of formality, but it can generally be assumed that among mathematicians, proof is a deductive argument that follows certain accepted rules. In school mathematics, a broader notion of *proof* as a convincing argument is frequently needed. For example, utilizing multiple examples to justify claims may be a sufficient argument at the elementary level; at the high school level, however, the generalizability of the argument may be a decisive factor on the validity of the proof. Nevertheless, there is no doubt that proof is important in school mathematics, at all grade levels, because it represents the essence of "doing mathematics".

Proof is a vital element in mathematics teaching and learning (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Yackel & Hanna, 2003). Proofs can verify, explain, systematize, communicate, explore, discover and incorporate mathematical ideas, as well as provide an intellectual challenge (De Villiers, 1999). Recent reform initiatives suggest that proof be taught in K-12 mathematics, and not restricted to the discipline of geometry (NCTM, 2000). There is an expectation that all teachers of mathematics should incorporate proof in their instructional practices and make argumentation part of the classroom sociomathematical norms. This is the result of a changing perspective, considering that historically proof was taught primarily in high school geometry, and it seldom appeared in other K-12 mathematics courses (Harel & Sowder, 2007; Herbst, 2002b; Stylianou, Blanton, & Knuth, 2009; Yackel & Hanna, 2003).

In mathematics courses, but particularly in geometry, "doing proofs" embodies various actions by teachers and students which are influenced by stated or implicit norms of what work is valued, the structure in which proof ought to be presented, the time allocation for proving, and the responsibility of students and teachers while "doing proofs" (Herbst et al., 2009). Sociomathematical norms in a classroom can influence how proof is taught. There is a reflexive relationship between students' perceptions of their role, the teacher's role, classroom social norms and what is deemed mathematical activity (Yackel & Cobb, 1996). According to Yackel and Cobb (1996), "…normative understanding of what counts as mathematically different, mathematically efficient, and mathematically elegant in a classroom are sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm" (p.461). Simon and Blume (1996) acknowledged that the class conceptualization of mathematics can affect their acceptance of correct justification. Since there exist variation in instructional practices for teaching proof (Harel & Sowder, 2007) which may impact students learning of proof (G.J. Stylianides, Stylianides, & Philippou, 2007), careful consideration must be given to the context and factors that influence how proof is taught and learned.

Teachers' conceptions and knowledge of proof may be factors that hinder their ability to effectively teach proof (Knuth, (2002a). Numerous studies have documented students' difficulties to construct proofs and provide appropriate reasoning for geometrical conjectures (Chazan, 1993; Clements & Battista, 1992; Healy & Hoyles, 2000; Mullis et al., 1998; Senk, 1985). It is important to note, as Harel and Sowder (2007) found, that students weak proof schemes may be due to limited opportunities to engage in proving activities in mathematics courses. NAEP results revealed that 8th grade students, whose teachers allocated a larger portion of instructional time for reasoning and analytical ability, performed considerably better than their counterparts (Silver & Kenney, 2000). Clement and Battista (1992) suggested that students' poor performance on geometrical proof tasks may be due to the fact that elementary and middle school geometry curriculum merely expose students to geometrical shapes, constructions and measurement, without placing an emphasis on higher-level thinking activities. Thus, when students enter high school geometry they may not be adequately prepared to succeed.

Proof in Curriculum Materials

There exist variations in how proof is treated in curriculum materials (Harel & Sowder, 2007; Hoyles, 1997). The attention given to proof varies in textbooks around the world. In lower secondary schools in Japan, geometry textbooks "set out to develop students' deductive reasoning skills through the explicit teaching of proof" (Fujita & Jones, 2003, p. 220), while in the United Kingdom geometry textbooks accentuate constructing diagrams, and finding measurements and angles, while presenting very minuscule opportunities to prove (Fujita & Jones, 2003). Furthermore, Sweden's upper secondary schools geometry textbooks place little if any attention to proof (about 2%), when compared to other routine tasks and applications (Nordström & Löfwall, 2005). However, Hanna and de Bruyn (1999) examination of 12th grade advance level mathematics textbooks in Ontario found that primarily, only geometry textbooks provide opportunities for students to prove and visualize various types of proofs. Herbst (2002b) historical study documented that over time, proof in US geometry textbooks evolved from requiring students to repeat given geometrical proofs to encouraging students to construct original proofs and to providing exercises for students to engage with proofs. Hence, curriculum materials vary in how proof is represented; nevertheless teachers are challenged to facilitate students learning to prove.

Few studies focus on teachers' relative to proof in mathematics curriculum (Mariotti, 2006). Bieda (2010) conducted one of the few studies that have examined curriculum materials during the enactment of proof-related tasks during instruction. Bieda (2010) studied how proof-related tasks are enacted in middle school mathematics classrooms. She found that 71% of the textbook tasks that were categorized as proof tasks during the curriculum analysis were implemented as such and that time constraints hindered the implementation of some of these tasks. Furthermore, Bieda's results highlighted that when an opportunity to prove arose, students did not provide adequate justification approximately half of the time; and that 42% of the time teachers did not provide a response, 34% of the time teachers sanctioned students conjectures, and 24% of the time teachers requested the input of the class. She acknowledged that teachers were likely to provide positive feedback for non-proof arguments as if it were general arguments. Bieda concluded that "teachers in the classrooms observed did not provide sufficient feedback to sustain discussions about students' conjectures and/or justifications…[and] when a teacher provided feedback to students' justifications, it was not sufficient to establish standards for proof in a mathematics classroom" (Bieda, 2010, p. 377).

Cirillo (2009) documented that a challenge in teaching authentic proof is that textbooks emphasize applications of theorems rather than their proofs. She recommended that greater emphasis be placed on the curriculum materials and objectives that teachers are given to facilitate the teaching of proof. Therefore, not only the treatment of proof differs across geometry curriculum materials, but also how teachers' use their curriculum materials can impact how proof is presented and what aspects of mathematical proof are emphasized. Hence it is important to consider not only the curriculum materials used, but also how they are used during instruction.

Statement of the Problem

There are numerous studies in the mathematics education literature about how proof is conceptualized by students (Battista, 2009; Harel & Sowder, 1998, 2007; Senk, 1989), about students' poor performance on proof (Battista, 2007; Clements & Battista, 1992; Healy & Hoyles, 2000; Reiss, Hellmich, & Reiss, 2002; Reiss, Klieme, & Heinze, 2001; Senk, 1985), and intervention strategies used to enhance the learning of proof

(Herbst, 2002a; Herbst & Brach, 2006; Herbst & Chazan, 2003; Jones, 2000b; Mariotti, 2000; Reiss, Heinze, Renkl, & Gross, 2008). However, little is known about how proof is taught in the classroom in relation to curriculum materials (Mariotti, 2006). Considering that curriculum materials are a major investment for school districts (Reys & Reys, 2006; Reys, Reys, & Chavez, 2004) and that proof has been identified as a core process of mathematics, it would be indeed beneficial to document how proof is bridged from the geometry textbook to the teacher, and from the teacher to the class. I addressed this gap in the literature by studying how high school geometry teachers use curriculum materials to teach proof.

Research Questions

The need to gain a better perspective of how proof is taught and how teachers use curriculum materials to support the teaching of proof requires an examination of the differences between the written curriculum and the enacted curriculum with regard to proof. In particular, I considered the case of proof in geometry, which led me to pose the following research question: How do teachers use curriculum materials to facilitate the teaching of proof in geometry? More specifically, what is the nature of the differences between how proof is presented in the written curriculum and how it is reflected in the enacted curriculum in a high school geometry course? To narrow the focus of my study, I looked at specific topics in specific geometry curriculum materials (namely, *McDougal Littell* which in recent time is known as *Holt McDougal*, and *Prentice Hall*) that have been identified by previous literature as topics that more readily presented students the opportunity to prove (Donoghue, 2003; Herbst, 2002b). Hence, I focused my attention on geometric content within the realms of Euclidean geometry pertinent to congruency, perpendicular and parallel lines, and segments and angles. These topics are also core content the authors of the textbooks expect students to learn. For instance, according to the authors of *McDougal Littell Geometry*, “In *Geometry*, students will develop reasoning and problem solving skills as they study topics such as congruence and similarity, and apply properties of lines, triangles, quadrilaterals, and circles”(Larson, Boswell, Kanold, & Stiff, 2007, p. T2).

Therefore, under the overarching question, which examined the nature of the differences between how proof is presented in the written curriculum and how it is reflected in the enacted curriculum in a high school geometry course, I addressed the following sub-questions:

1. How do *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher’s Editions present proof for segments and angles, parallel and perpendicular lines, and congruent triangles to facilitate students learning to prove?
2. To what extent do geometry teachers use *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher’s Editions to teach proof for segments and angles, parallel and perpendicular lines, and congruent triangles to facilitate students learning to prove?
3. What influences teachers’ decisions to deviate or not from the *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher’s Editions implied or explicit instructions and lesson plans?

Definitions

To promote a common understanding of the meaning of key terms, the following operational definitions are used.

Proof — The meaning given to proof varies among researchers (Reid & Knipping, 2010). Some researchers perceive proof as proof-text: what students write (Duval, 1990; Fischbein, 1982; Fischbein & Kedem, 1982). Others may perceive the meaning for proof in terms of discourse (Balacheff, 1991a, 1991b; Fawcett, 1938; Mariotti, 1997, 2000), while others consider the meaning as reasoning (Harel & Sowder, 1998; Reid, 1995). "For Harel and Sowder a proof must be convincing but not necessarily deductive or semi-formal. For Reid a proof must be deductive, but not necessarily convincing or semi-formal" (Reid & Knipping, 2010, p. 52). Hence, there is no clear consensus of what proof is in mathematics education. Therefore, for the purposes of this study, I used the definition of proof in school mathematics developed by Stylianides (2007). According to Stylianides, "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; 2. It employs form of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of the classroom community" (p.291).

Curriculum Materials – Curriculum materials are printed resources (such as textbooks), which are published with the intent of being used before, during, and after classroom instruction by students and teachers (Remillard, 2005; Remillard, Herbel-Eisenmann, & Lloyd, 2009; Stein, Remillard, & Smith, 2007).

Enacted Curriculum - is the implementation of the curriculum, via the usage of tasks from the curriculum materials, during classroom instruction (Stein et al., 2007), which is "jointly constructed by teachers, students, and materials in particular contexts…" (Ball & Cohen, 1996, p. 6).

Proof Tasks – are tasks designed to have students write proof arguments, and complete skeletal proofs (such as fill in the blank type proof, or matching statements to appropriate reasons) in which the finish product illustrates complete proofs.

Proof-related Tasks are tasks that are related to a proof in the sense that are meant to provide students with an opportunity to perform a step that may be used in later proofs and are not necessarily proof tasks by themselves. For example, identifying corresponding sides in a triangle, identifying the congruence criterion that must be used in a given proof, etc.

Significance of the Study

More is known about how students learn proof than about how proof is taught (Stylianou et al., 2009). Ball et al. (2002) identified three research areas that need attention pertinent to the teaching of proof. The first research area identified refinement of the functions, perceptions and role of proof. The second research area highlights the need for empirical research documenting the challenges of learning to prove. Finally, the last research area promotes "the development, implementation, and evaluation of effective teaching strategies along with carefully designed learning environments that can foster the development of the ability to prove in a variety of levels…" (p. 908). This study addresses the third primary need for research on the teaching of proof, because it seeks to examine how geometry teachers use curriculum materials in teaching proof.

Thus my study assists with unraveling the complexity of teaching proof in high school geometry courses.

Furthermore, the literature on teachers' use of curriculum material focuses primarily on primary and middle grades and seldom focuses on a particular content strand. Thus, focusing my attention on how teachers' use curriculum materials in the teaching of proof in geometry at the high school level would indeed address gaps in the current literature on how teachers interact with curriculum materials.

Finally, my study is worthwhile because it sheds light on the nature of proof tasks posed, and what constitutes a convincing argument in high school geometry classrooms. Although tasks vary in the level of cognitive demands and the possibility of engaging students in doing proofs, teachers' actions can stifle or influence the extent tasks are enacted. Furthermore, geometry teachers facilitate discussions of what constitute a proof while teaching the concept of proof. Thus, my study captures the nature of convincing proof arguments deemed acceptable in the geometry classroom; such information can inform researchers of classroom sociomathematical norms pertinent to proof.

CHAPTER II: REVIEW OF RELATED LITERATURE

In this chapter I review the literature pertinent to proof, the teaching of proof, and how mathematics curriculum materials are used. More specifically, I discuss what constitute a proof in school mathematics, teachers' conceptions of proof, how proof is taught, representations of proof in written materials, teachers usage of curriculum materials, and the possible roles proof can play during instruction. Furthermore, I highlight research on how geometry is learned and understood, since it may potentially influence how geometry teachers teach proof.

Proof

Defining proof is not a simple task, and most researchers agree that the definition of proof is subjective (Harel & Fulller, 2009; Harel & Sowder, 2007). Depending on the researchers' perspective, it may be useful to define proof in different ways, so that the definition fits the researcher's focus. For instance, NCTM (2000) suggests that a proof reflects specific forms of justification and reasoning; while Fischbein and Kedem (1982) emphasized the universal validity of statements and logical rules.

Within classroom environments, the norms, argumentation, and didactical contracts can influence what constitutes a proof. Harel and Sowder (2007) suggest that a proof establishes truth for a community or an individual. Whether proof is socially constructed or a statement that validates a universal truth, researchers have tried to characterize proof in a way that is consistent with the historical development of mathematics and that is flexible enough to be of use in mathematics education. Stylianides (2007) identified as the essential elements of proof in school mathematics a *set of accepted statement*, *modes of argumentation* and *modes of argument representation*. The full definition is as follows:

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community (p.291).

Thus, in school mathematics, proof is not restricted to written text, but is reflective of acceptable classroom justification, appropriate reasoning, and supportive representation.

Herbst and Balacheff (2009) discussed three notions of proof that may be visible in mathematics classrooms. The first notion refers to classroom discourse used for verification purposes, the second notion (C-proof) occurs when students fit solutions to proof tasks based on their conception, and the third notion (K-proof) considers the role proof plays as students acquire new knowledge. Conception refers to unwavering ways students relate to the arrangement and structure of their mathematical milieu during precise moments (Balacheff, 1994). According to Herbst and Balacheff (2009), "for a conception to have a regulatory structure that deserves the name of C-proof, the system of representation must be capable of representing not just the referents in the problem but also the means of operating on the objects" (p. 53). For example, if an elementary student was asked to add 18 + 59, and the child produces the correct solution of 77 by adding 20 and 57, the child would be demonstrating a C-proof. Furthermore, "K-proof, addresses the possible role of proof in coming to know new things: it describes how the metaphorical mapping of a conception already known by the class onto a new conception can shape and warrant plausible knowledge" (Herbst & Balacheff, 2009, p. 41). A K-proof maps an established C-proof to a new conception of a C-proof (Herbst & Balacheff, 2009).

Kilpatrick, Swafford, & Findell (2001) included proof in the mathematical proficiency strand: adaptive reasoning. "Adaptive reasoning [is the] capacity for logical thought, reflection, explanation, and justification" (Kilpatrick, Swafford, & Findell, 2001, p. 5). The researchers noted that justification is the skill of providing adequate reasoning; and that although all justification may not necessarily be a proof, a proof is indeed a form of justification. This viewpoint aligns with A. J Stylianides (2007) framework of practices that cultivate proof during instruction. Stylianides framework begins with the classroom engaging in a proof activity, which branches out to either the base argument qualifying as a proof or not a proof, which influences whether the ensuing argument qualifies as a proof or not a proof. The base argument is the dominant argument students present at the beginning of the proof activity (A. J. Stylianides, 2007). Stylianides acknowledged that, although proofs are result of base arguments, not all base arguments may result in an arguments that are deemed proofs in school mathematics.

Argumentation

Argumentation within a classroom can evolve into a proof, but in some instances it remains just an argument (McClain, 2009). McClain refined Toulmin's (1958) scheme for argumentation by considering sociomathematical norms in a 7th grade mathematics classroom. McClain found that "there were continual shifts in the norm of *what counts as an acceptable mathematical argument*" (p. 233). She concluded that arguments could be characterized as *argument for defending*, *argument for disagreement*, *argument for justification*, and *argument for refinement*. Mathematical argumentation, if used purposefully, can potentially enhance students understanding of the subject matter.

Heinze and Reiss (2009) studied the development of proof and argumentation of secondary students (more specifically 7th, 8th, 12th and 13th grade students), using a pretest-posttest design. They propose using hierarchical levels to represent the development of students' argumentation abilities: Level 1, knowledge of basic rules; Level 2,simple argumentation; and Level 3, complex argumentation. Students at Level 2 provided one step reasoning, while student at Level 3 provide multiple reasoning steps. The results found that there exist statistically significant differences between the performance of students in high-achieving and low achieving classrooms. The results found that mathematics classrooms in which students were enrolled influenced their individual performance. Heinze and Reiss (2009) found that the 7th and 8th grade students performance on the pretest and posttest was correlated more strongly at the classroom level ($r=0.604$, $p<0.001$) than to the individual ($r=0.435$, $p<0.001$).

Furthermore, although the 12th and 13th grade students had declarative knowledge (content knowledge needed to solve the posed problem), they were deficient in procedural knowledge (a chain which links idea to form a correct proof) that was needed for completing proof, and exhibited heuristic strategies similar to that of 7th and 8th grade students. Additionally, although 12th and 13th grade students were able to construct two-step argumentations, only high achieving students were able to construct argument with more than two steps. The researchers concluded that students' proof and argumentation skills are weak across secondary grade levels.

Proof Schemes

Harel and Sowder (1998) conducted teaching experiments with college students and a case study of one high school student enrolled in geometry and calculus in order to explore students' appreciation, production, and understanding of proof. Harel and Sowder (1998) considered students' proofs relative to vector spaces, systems of equations, matrices, properties of complex numbers, algebra, circle theorem, and Euclidean geometry. They found that there are three major proof schemes (*external conviction, analytical*, and *empirical*) that encompass sub-groups of schemes (*ritual*, *authoritarian*, *symbolic*, *transformation*, *axiomatic*, *inductive*, and *perceptual*).

The *external conviction proof schemes* are schemes where outside sources influence students' conceptions of proof; and embody *ritual*, *authoritarian* and *symbolic* schemes. According to Harel and Sowder (1998), "When formality is introduced prematurely students come to believe that ritual and form constitute mathematical justification" (p. 243). Students utilizing *ritual* proof scheme may accept a mathematical argument based on its initial appearance, and assert that mathematical symbols and traditional mathematical representation (such as two-column proof) are needed to validate a proof argument. For the *authoritarian* proof scheme, students rely on teachers and textbooks to establish truth claims, and do not consider intrinsic arguments on the merit of the proof. "The…most common expression of this proof scheme is students insistence on being told the procedure to solve their homework problems, and when homework are emphasized they expect to be told the proof rather than take part in its construction" (Harel and Sowder, 1998, p. 247). Conversely, the *symbolic* proof scheme utilizes symbolical representation to prove a theorem. The richness of a symbolic proof can vary significantly in mathematical depth.

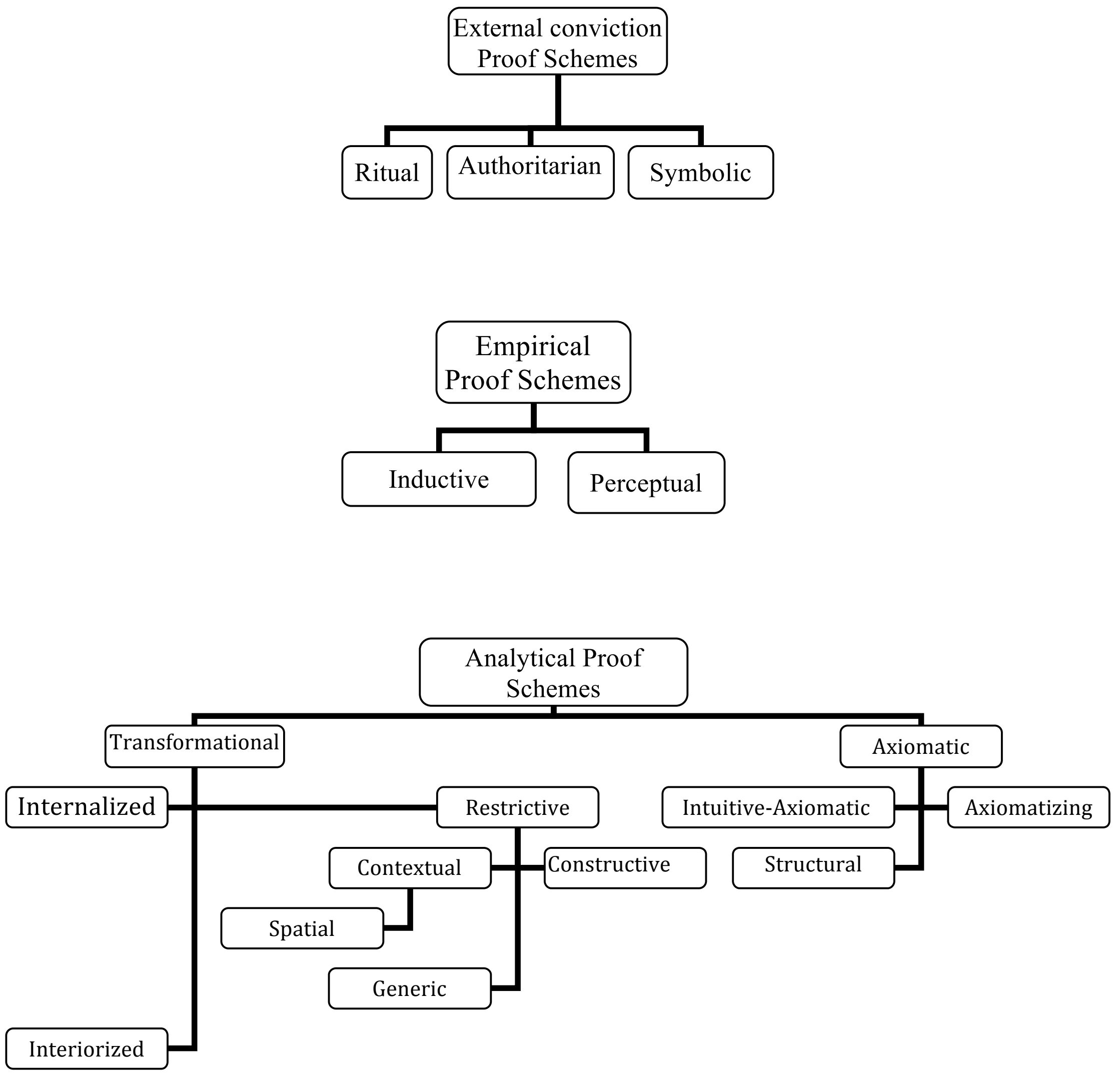
The second major proof scheme is the *empirical proof schemes,* in which conjectures are disproven or proven utilizing facts and visual perceptions; it includes the *inductive* and *perceptual* proof schemes. Students exhibit *inductive* proof scheme by quantitatively evaluating the truth of an argument in effort to convince themselves or persuade others of the validity of their proof via counterexamples and examples. There are potential issues with "proof by examples", hence the researchers documented that students were provided with examples that may be true in some instances and false in others. This method sought "to convince the students that inductive verifications are insufficient to validate a conjecture. Although… students seemed to understand the limitations inherent in the inductive method, their subsequent behavior was not consistent with this impression" (Harel and Sowder, 1998, p. 253). The authors acknowledged using inductive evidence is a natural inclination. Conversely, students demonstrate *perceptual* proof scheme by visualizing images in the mind as to how a proof ought to be constructed, but are unable to transform the proof or consider the possibility of the result if the proof was transformed.

The third major proof scheme is *analytical proof schemes*. *Analytical proof schemes* are evident when students use logical deduction to evaluate the merit of proposed conjectures. *Analytical proof schemes* comprise *transformational* and *axiomatic* proof schemes (*intuitive- axiomatic, structural* or *axiomatizing*).

Students demonstrate a *transformational* proof scheme when they can change an argument and can consider the result that the transformation will have on the proof argument. "The *transformational* proof scheme is characterized by (a) consideration of the generality aspect of the conjecture, (b) application of mental operations that are goal oriented and anticipatory, and (c) transformations of images as part of a deductive process" (Harel and Sowder, 1998, p.261). *Transformational* proof scheme can be *internalized, interiorized,* and *restrictive* (that encompasses *contextual, spatial, generic,* and *constructive* proof schemes). *Internalized* proof scheme utilize heuristic arguments; an *interiorized* proof scheme is evident when students can reflect on an internalized proof scheme, such that they become aware as to why their proposed argument is truth. However, *restrictive* proof scheme occur when students place limits on conjectures and justifications. Students demonstrate *contextual* proof scheme when they emphasize that the context of the proof is vital for constructing a proof argument. A *spatial* proof scheme (which is a subclass of the *contextual* proof scheme) may be used in considering location on a geometrical plane, and g*eneric* proof scheme is evident when students utilize basic argument for a particular context. Students that use *construction* proof scheme construct objects to eliminate doubt of a proof argument.

Axiomatic proof scheme is also a sub-category of the *analytical proof schemes*. An *axiomatic* proof scheme exists when students know that a proof must be underpinned by mathematical axioms and subsequent theorems. A person possessing *axiomatic* proof scheme is "aware of the distinction between the undefined terms, such as "point" and "line" and defined terms such as "square" and "circle", and between statements accepted without proof and ones that are deducible from other statements" (Harel and Sowder, 1998, p.273). Conversely, *intuitive*- *axiomatic* proof scheme occurs when a student can see the merit of a proof based on intuition such as the commutative property of addition. Alternatively, students possessing *structural* proof scheme understand that an axiom can be applied in different context because the structures share certain properties and subsequently certain axioms. Finally, the last proof scheme described by Harel and Sowder (1998), is *axiomatizing. Axiomatizing* occurs when students can examine a set of axiom across various field of mathematics. Harel and Sowder proof schemes are represented in Figure 1.

Figure 1. Proof Schemes (Harel & Sowder, 1998, p. 245).



Different Roles of Proof

Proof can play various roles in mathematics classrooms. Proof can be used to verify, explain, systematize, communicate, explore, discover and incorporate mathematical ideas, as well as provide an intellectual challenge (de Villiers, 1999). Verification presents a way to demonstrate the validity of a mathematical statement.

Explanation sheds light as to why a statement is true. Systematization presents results in an organized format, such that the connections between known results and previous results are made apparent. Communication is the social sharing of meaning and knowledge of mathematics; and intellectual challenge is the self satisfaction and realization one obtains from the construction of a proof (De Villiers, 1999). Furthermore, proof can be used to explore mathematical conjectures and statements, discover new mathematical results, and incorporate known facts into novel frameworks in an effort to consider alternative perspectives (Hanna, 2000).

Representations of Proof in Geometry

Curriculum materials often use various representations of geometrical proofs to facilitate students' development of proof skills. Among these representations we can enumerate two-column proof, a flow proof, a proof tree, and a paragraph poof (Cirillo & Herbst, 2010).

For many years, two-column proofs have been a standard means for teaching proof. Herbst (2002b) historical study, which describes the evolution of proof during the period 1850-1910, documented that two-column proofs was instrumental in changing how proof was taught in geometry. This teaching strategy changed the students' role from being solely learners of mathematical proof to doers of mathematical proof in high school geometry (Herbst, 2002b). Considering the popularity of two-column proofs, students are often perplexed about the validity of a proof when written in alternative form, such as paragraph proof (McCrone & Martin, 2009). Two-column proofs have received criticism in recent times because they appear rather austere, and can potentially bias students to believe that the deductive process is a linear process, thus concealing the excitement of doing proof (Cirillo & Herbst, 2010; Lakatos, 1976; Schoenfeld, 1986). Herbst (2002a) presented theoretical arguments of the challenges teachers experience in efforts to facilitate students doing proofs using the two-column proof (such as the careful set up of the proof task, and didactical contracts established during the implementation of the task**)**, and suggest that challenges are necessary considering the responsibilities of students and teacher while doing proofs. The didactical contracts for doing proofs may include: *the declared purpose of the task is to produce a proof*, *what is to be proved is a proposition that is explicitly given at the outset*, *the proposition is stated in premises and conclusions identified and separated as given and prove*, and *a diagram is given with the task and lettered in an alphabetical sequence supporting a circular reading*. Herbst pointed out that two-column proofs put conflicting demands on geometry teachers as to how to develop students proof skills (such as the promotion of rich encounters with doing proofs and the reduction of the difficulty level of the proof tasks); he suggested it may be advantageous to teachers to consider alternative forms of doing proofs.

A flow proof is similar to the two-column proof because it makes explicit the connection between each statement and its justification and it uses arrows to connect logical statements, although students are not expected to provide reasons to support every statement (as is the case in the two-column proof) (Cirillo & Herbst, 2010). Flow proof provides a mean for students to understand and analyze information that is given, and to consider information that is needed, so that they can construct logical arguments to prove what needs to be proved (Brandell, 1994).

A proof tree is an outline of a proof (Anderson, 1983; Cirillo & Herbst, 2010). Anderson (1983) noted that a proof tree provides students an opportunity to search forward or backward through logical statements that link the given statement to supporting statements and what needs to be proven. Considering a proof tree links various ideas, Anderson (1983) characterizes the proof tree as “an *abstract* specification of a proof” (p.193). Cirillo and Herbst (2010) suggest that a proof tree can be used for scaffolding to assist students to construct a geometrical proof.

A paragraph proof uses sentences to state logical statements (Cirillo & Herbst, 2010). Paragraph proof use everyday writing for an explanation and is not as structured as other proof representations (Education Development Center, 2009). Cirillo (2008) documented that the teacher in her case study was concerned about the appropriateness of paragraph proof at the school level, since students who use paragraph proof often deduce incorrect conclusions and frequently neglected to include supportive reasoning. Nevertheless, paragraph proof provide a mean for students to develop proficiency in writing proof (Cirillo & Herbst, 2010).

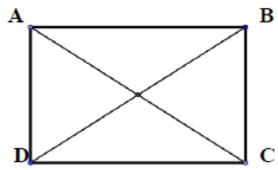
The various representations of proof may be visible in textbooks, and can be used to facilitate students’ development of proof skills. In fact, in the *McDougal Littell Geometry* (2007) Teacher’s Edition requires students to write two-column proofs, paragraph proofs, and flow proofs as a means to foster students doing proofs. Hence, it is important to be mindful of the types of representations textbooks use to write proofs, because it may significantly influence the type of representations of proofs encouraged during the teaching of proof. Figure 2 depicts various representations that be can used to write a proof for a given task.

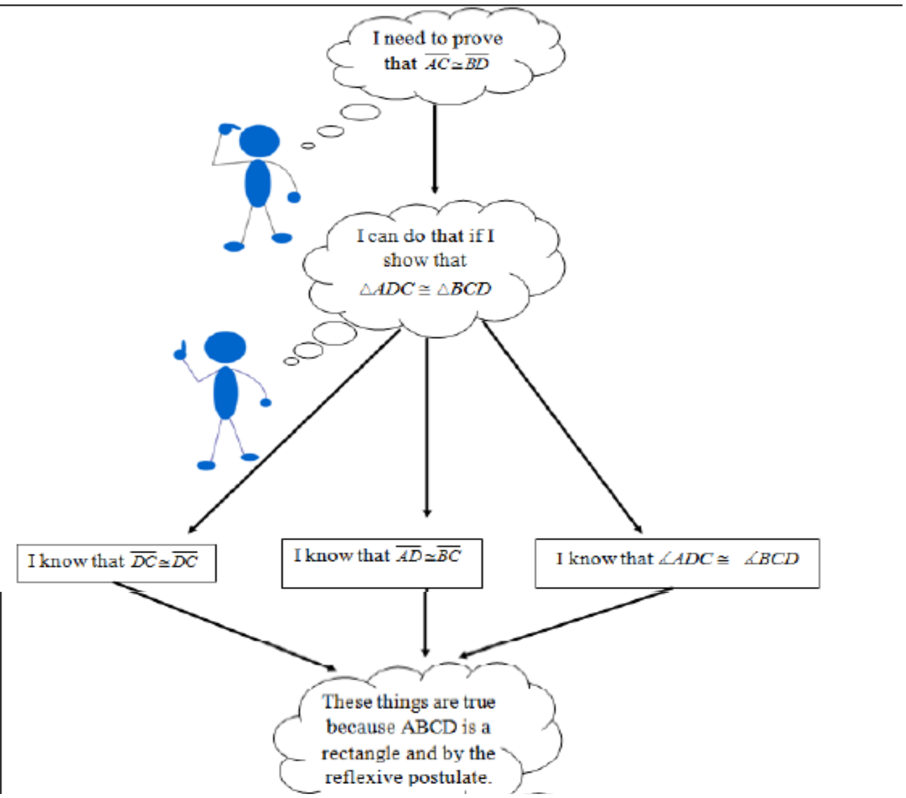
Figure 2. Various representations of proof (Cirillo & Herbst, 2010, pp. 17-18).

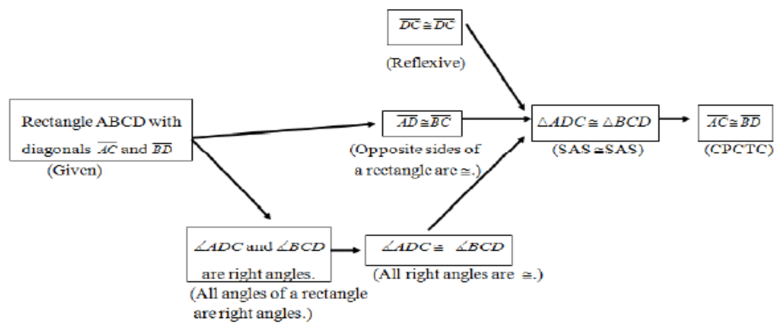
THEOREM: If a parallelogram is a rectangle, then the diagonals are congruent.

Given: Rectangle *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$

Prove: $\overline{AC} \cong \overline{BD}$



Proof Form	Proof
A **proof tree** is an outline or plan of action that specifies a set of geometric rules that allows students to get from the givens of the problem, through intermediate levels of statements, to the to-be-proven statement. (Adapted from Anderson, 1983)	

<table>
<tr><td>A flow proof uses the same statements and reasons as a two-column proof, but the logical flow connecting the statements is indicated by arrows. Depending on whether it is the plan or the proof itself, students may or may not choose to write the reasons beneath the statements.</td><td>
</td></tr>
<tr><td>A two-column proof lists the numbered statements in the left column and a reason for each statement in the right column.</td><td>Statements
1. Rectangle ABCD with diagonals $\overline{AC}$ and $\overline{BD}$
2. $\overline{AD} \cong \overline{BC}$
3. $\overline{DC} \cong \overline{DC}$
4. $\angle ADC$ and $\angle BCD$ are right angles.
5. $\angle ADC \cong \angle BCD$
6. $\triangle ADC \cong \triangle BCD$
7. $\overline{AC} \cong \overline{BD}$

Reasons
1. Given
2. Opposite sides of a rectangle are congruent.
3. Reflexive Postulate
4. All angles of a rectangle are right angles.
5. All right angles are congruent.
6. Side-Angle-Side ≅ Side-Angle-Side
7. Corresponding Parts of Congruent Triangles are Congruent (CPCTC)</td></tr>
<tr><td>A paragraph proof describes the logical argument with sentences. It is more conversational than a two-column proof.</td><td>Since ABCD is a rectangle with diagonals $\overline{AC}$ and $\overline{BD}$, then $\overline{AD} \cong \overline{BC}$ because opposite sides of a rectangle are congruent. By the reflexive postulate $\overline{DC} \cong \overline{DC}$. Since all angles in a rectangle are right angles, then $\angle ADC$ and $\angle BCD$ are right angles. Thus, $\angle ADC \cong \angle BCD$. By Side-Angle-Side, $\triangle ADC \cong \triangle BCD$. Thus, $\overline{AC} \cong \overline{BD}$.</td></tr>
</table>

Teaching of Proof in Geometry

According to Battista and Clements (1995), efforts to improve students performance of proof by teaching it in creative ways have been ineffective based on the multiple research findings that suggests students continues to perform poorly on proof tasks. Numerous studies have documented how high school and beginning college students fail to perform well when doing proofs (Healy & Hoyles, 2000; Recio & Godino, 2001; Senk, 1985). For example, Senk (1985) found that only 30% of students enrolled in geometry for a full year were able to construct proofs at a mastery level of 75%.

The instructional responsibilities of geometry teachers are vast, and teaching proof is just one of them. According to Jones (2002), "Teaching geometry well involves knowing how to recognize interesting geometrical problems and theorems, appreciating the history and cultural context of geometry, and understanding the many and varied uses to which geometry is put" (p. 122). Jones (2000a) acknowledged that there are many things in the geometry curriculum that have to be taught so teachers have to consider: aspects of proofs that need to be accentuated, potential pedagogical strategies that can bridge deductive thinking and geometric insight, and tools that can facilitate students' learning to prove. Fussell (2005) discussed the role of teachers and curricula for teaching proof. She noted teachers need to assist students in writing logical statements; teachers should be flexible in their instructional practices such that students can share their ideas about a proof before attempting to construct a proof independently, and teachers should provide students opportunities to prove during instructional time. Unfortunately, due to time constraints, teachers often choose to provide students with examples, and neglect to prove theorems (Fussell, 2005).

Herbst et al. (2009) discussed 25 classroom norms that occurs while "doing proofs", which included specific responsibilities for teachers and students. Teachers' responsibilities may include providing students with the tasks, ensuring that sufficient details are provided (for example, the given and what needs to be proven) such that the students can attempt to construct proofs, activating students thinking by making suggestions about how to construct proofs, and ensuring reasons are provided for statements made. Conversely, the students' responsibilities may include constructing proofs and marking the diagrams without altering them.

Similarly, Martin, McCrone, Bower, and Dindyal (2005) conducted an interpretivist classroom study about how geometry is taught and learned. The researchers' captured data (video and field notes) of the interaction between students and their teacher over a 4-month period in an honors geometry course. The researchers classified teachers' actions into the following primary categories: *select task, revoice or rebound, request explanation/reasoning, model proof-related skills, evaluate student responses,* and *value students' ideas (coaching)* (Martin et al., 2005, pp. 103-104).

Martin and McCrone (2001) reported year 1 results of a 3 year study of students ability to construct proof arguments and beliefs about proof, in which two geometry teachers of a midwestern high school participated. Martin and McCrone (2001) documented teachers pedagogical practices and sociomathematical norms and concluded that teachers were the authority of mathematics within the classroom setting, little time was allocated for students to provide answers to questions posed, scarce opportunities were provided for students to engage in sense-making activities, and teachers' mathematics content knowledge limited their pedagogical practices when teaching proof. Furthermore, it was not apparent that the problems posed required to be proved.

McCrone, Martin, Dindyal, and Wallace (2002) conducted a study with four teachers and found that their pedagogical practices when teaching proof were rather similar. Two of the teachers used geometrical proofs as application of theorems that resulted in students' learning proof via rote learning. The other two teachers used geometrical proofs to introduce new concepts. The students of teachers who taught via rote learning were explicitly encouraged to memorize geometrical facts (definition, theorems, postulates, etc.) and to readily identify the given and what needed to be proved;

although the students of the other classes were not explicitly required to memorize geometrical facts it was expected that they learn the geometrical facts due to regular use. Furthermore, it appeared that the teachers who did not explicitly emphasize memorization of geometrical facts emphasized the creative form or "elegance" of the proof, while the teachers who emphasized memorization of geometrical facts emphasized the amount of detail needed for a proof. The four classroom teachers encouraged students to make marks on the diagrams. Three of the teachers reinforced the perception that tasks can be solved quickly, in a relatively short time span. The researchers concluded that the teachers' pedagogical actions significantly influenced students' perception of what constitutes a proof.

Teaching Proof with Curriculum Materials

Proof-related tasks in curriculum materials are often implemented as such during the enacted curriculum (Bieda, 2010). Bieda conducted a multiple case study of six middle school teachers who used Connected Mathematics Project (CMP) curriculum and their enactment of proof-related tasks. Bieda found that 71% of the tasks identified as proof-related tasks during the curriculum analysis were implemented as such by the teachers, 21% of the proof-related tasks were not implemented as proof-related tasks due to time constraints, and 8% of the proof-related tasks were not implemented because of teachers' discretion. Bieda's results indicate that students' opportunities to prove decreased as the grade levels increased. Overall, the enacted proof tasks did not fully develop students' understanding of what is considered a complete mathematical justification, due in part to the fact that "when a teacher provided feedback to students' justifications, it was not sufficient to establish standards for proof in a mathematics classroom" (Bieda, 2010, p. 377).

McCrone, Martin, Dindyal, and Wallace (2002) acknowledged that the four teachers in their study followed the textbook rather closely to structure the enacted lesson on proof, as well as for allocating homework assignments pertaining to proof, and used technology or hands on investigation activities sparingly. Admittedly, it may not always be ideal for teachers to follow the curriculum rather closely. For example, Schoenfeld (1988) conducted a year long study of teaching and learning in a 10th grade geometry course. He found that, although the teacher exhibited "good teaching", the teacher's actions might have had a negative impact on students' perceptions of proofs. He suggested that the teacher's strict adherence to the curriculum might have caused students to differentiate between constructive and deductive geometry, consider the form of the mathematical argument to be paramount, and view doing proofs as a quick activity.

Considering that teachers rely on curriculum materials to teach proof, thoughtful considerations must be given to the guidance that curriculum materials provide for teachers. Knuth, Choppin, and Bieda (2009) recommended that curriculum materials ought to facilitate both teachers' and students' development of proof skills. The researchers noted, "Given the inextricable link between teachers' instructional practices and the curricula they implement…Curricular materials not only must provide opportunities for students to engage in proving activities, but must also support teachers' efforts to facilitate such engagement" (Knuth et al., 2009, p. 169).

Teachers' Conceptions about Proof

Teachers' beliefs and subject matter knowledge can significantly influence their classroom practices (Borko & Putnam, 1996). Hence, teachers' conceptions of proof may influence the way they teach it. Knuth (2002a) conducted semi-structured interviews with 16 in-service high school mathematics teachers about their conceptions of proof in mathematics. He found that 75% of teachers considered that the role of proof was to communicate mathematics, 50% of teachers considered that it was to systematize mathematical ideas and construct new knowledge, 18.75% considered proof as a means of explaining (answering why), and no teacher considered proof a means of explanation that can promote understanding. Additionally, Knuth (2002a) found that 37.5% of teachers believed a proof became "invalid" if there was a contradictory statement, 31.25% of teachers were hesitant to accept a counterexample as a proof, and that 31.25% of teachers believed that unusual cases of counterexamples ought to be tested. Furthermore, Knuth (2002) showed that teachers identified characteristics of a convincing proof primarily in terms of concrete features (81.25% of teachers), familiarity (62.5%), generality (56.25%) and amount of details (50%). Knuth's results emphasized that teachers consider proof as a means to communicate mathematics, and that concrete features of proof are of utmost importance. We can assume that these different conceptions of proof will affect how teachers teach it.

Furthermore, knowledge of proof can have implications on how proof is taught. According to Stylianides, Stylianides and Phillippou (2007), "If teachers' knowledge of proof is fragile, that is, it is shaky and yields to attempts to inject contradictions into it … it is likely that teachers will teach proof poorly or will not teach proof at all" (p.146).

Knuth (2002a) found that although 93% of teachers in his study were successful in identifying proof statements, one third of the teachers identified non-proofs as proofs. As a matter of fact, all teachers in his study identified at least one non-proof as a proof. Similarly, Stylianides et al. (2007) found that preservice teachers had difficulties identifying and conceptualizing correct inductive proof methods. Arguably, teachers who have difficulties to identify proofs will not be able to teach proof successfully.

Geometrical Cognitive Development

In examining how geometry teachers use curriculum materials for the teaching of proof, it is important to take into account research on students' development of geometric cognitive development. Thus, I discuss literature related to how students develop a conceptual understanding of geometrical constructs (Battista, 2007, 2009; van Hiele, 1959/1985).

Van Hiele Model of Geometrical Thinking

The van Hiele model of geometrical thinking is a hierarchical model of students' geometrical development (van Hiele, 1959/1985). The van Hiele model consists of five ascending levels: Level 0-visualization, Level 1-analysis, Level 2- abstraction, Level 3- deduction, and Level 4-rigor. According to the van Hiele model at the visualization level students make claims based on their perception rather than on mathematical reasoning. Students who are on the analysis level recognize objects as a collection of mathematical properties, but are unable to differentiate which properties are useful in conducting a proof and experience difficulty in recognizing relationship between properties (van Hiele, 1959/1985). At the abstraction level, students are able to observe relationship between geometrical figures, but are unable to deduce mathematical principles. Students at the deduction level should be able to construct proof for Euclidean geometry using mathematical theorems and postulates appropriately. At the rigor level, students can perform multiple forms of proofs using deductive and inductive reasoning and conduct proofs for geometries that are not Euclidean (van Hiele, 1959/1985). According to the van Hiele model, students' cannot regress or be between levels (Senk, 1989).

Battista (2007, 2009) refined the van Hiele model to reflect students conceptual development of 2D-geometrical shapes, to a formal understanding of shapes and geometrical constructs. The refined model still consists of five ascending levels however the labeling of the levels has changed. The refined van Hiele classification is as follows: Level 1- Visual-Holistic Reasoning; Level 2- Descriptive-Analytic Reasoning, Level 3- Relational-Inferential Reasoning, Level 4- Formal Deductive Proof and Level 5-Rigor. At the Visual-Holistic Reasoning level, students can identify and reason about shapes based on the appearance of the shape as a whole. At the Descriptive-Analytic Reasoning level, students are able to observe how the structural parts of a shape are related, and can appropriately apply geometrical constructs in examining relationships that may exist between various parts of a shape. Students at the Relational-Inferential level can speak with certainty of the relationships that exists among the shape geometrical properties. Students may use empirical evidence or logical inference rather than re-presenting the structure being discussed. At the Formal-Deductive level students are able to interpret and construct complete geometrical proofs. Battista (2009) concluded that at the Rigor level students can analyze and use multiple axiomatic systems (such as Euclidean and non-Euclidean geometries), which is more visible in tertiary level mathematics courses.

Abstraction

The abstractionist model suggests that learners repeatedly reflect and abstract to develop more advance mental models of phenomena (Battista, 2007). Battista (2009) explained, "Abstraction is the process by which the mind registers objects, actions and ideas in consciousness and memory. Once objects, actions, and ideas have been abstracted at a sufficiently deep level, they themselves can be mentally operated on (e.g., compared, decomposed, analyzed)" (p.94). *Spatial structuring* and *mental model* are special forms of abstraction pertinent to the learning of geometry. *Spatial structuring* is the mental process of arranging sets of objects or an object, based on the identification of relationships that exist among the object(s) components (Battista, 2009). *Mental models* entail nonverbal mental accounts of situations, which embody the form of the situations they signify. Hence, how an individual abstracts a geometrical proposition can significantly influence how he or she conceptualizes it.

Concept Learning and the Objects of Geometric Analysis

Finally, concept learning involves two primary concepts: *natural concepts,* and *formal concepts* (Battista, 2009; Pinker, 1997). *Natural concepts* occur in everyday life and seldom, if ever, require concept definitions; however *formal concepts* always require the usage of definitions. Furthermore, in an effort to learn geometric concepts, students reason about three forms of geometrical objects: *physical objects* (things), *concepts* (mental representations that an individual abstracted for groups of similar objects) and *concept definitions* (verbal or symbolic formal mathematical requirement for an object) (Battista, 2009).

Diagrams and Representations

The theory of Diagrams and Representation (Battista, 2009) considers whether a diagram is representing an object or a concept. " On the one hand, physical objects can be thought of as the input for geometric conceptualization... On the other hand, physical objects, including diagrams are often used to represent formal geometric concepts" (Battista, 2009, pp. 96-97). In many instances, students have difficulties differentiating between what is being represented and the representing object, such as a diagram (Battista, 2009; Clements & Battista, 1992).

Proofs in Curriculum Materials

In a review of geometry textbooks during the 20th century, Donoghue (2003) noted that congruent polygons and Pythagoras' Theorem were prime topics for the teaching of proof across textbooks. Nevertheless, the pedagogical recommendations, content, and context for the teaching of proof included in these textbooks varied. More recent textbooks also differ in their approach to proof. For example, the *University of Chicago School Mathematics Project (UCSMP)* geometry textbook (Coxford, Usiskin, & Hirschhorn, 1993), which is aligned with the NCTM *Standards*, embedded proof into the "properties" dimension of geometrical understanding. According to Senk (2003) "in each UCSMP course a multidimensional view of understanding mathematics is emphasized. Secondary school mathematics has at least four dimensions: Skills, Properties, Uses and Representation…designed to illustrate the breath of mathematics…" (p. 431).

Furthermore, although NCTM (2000) recommended that proof be taught across grade bands and various mathematical strands, the visibility of proof in mathematics textbooks outside the realms of geometry is not quite apparent (G. J. Stylianides, 2008).

Stylianides examined the recommendations for teaching proof in the Teacher's Edition of the *Connected Mathematics Project* for geometry, algebra and number theory units. He found that 5% of the 4, 578 tasks could be considered proof tasks. Of the task that were considered proof tasks, 10% of them provided teachers with the solutions and guidance, and the remaining 90% provided the solutions only.

Curriculum Materials

Curriculum materials vary by embedded pedagogies, mathematical depth and emphases, contexts, and style. Curriculum materials are printed resources (such as textbooks), which are published with the intent of being used before, during, and after classroom instruction by students and teachers (Remillard, 2005; Remillard et al., 2009; Stein et al., 2007). In some instances, researchers refer to textbooks and curriculum materials in an interchangeable manner, however practitioners often differentiate between them (Stein et al., 2007).

During instruction teachers rely primarily on textbooks as a resource for the teaching of mathematics (Grouws & Smith, 2000; Grouws, Smith, & Sztajn, 2004). Chávez (2003) found that most teachers use textbooks as a resource of mathematical tasks, and as a "lesson plan". In a study in middle school classrooms, Tarr, Chávez, Reys, and Reys (2006) found that teachers used 60-70% of the lesson guides for instructional purposes. Furthermore, most teachers using commercially published textbooks were less likely to vary in their textbook usage when compared to teachers that used NSF-funded textbooks (Chval, Chávez, Reys, & Tarr, 2009). Additionally, approximately three fourths of eight graders reported that they used textbooks daily in mathematics (Grouws & Smith, 2000), and 89% of students reported using their mathematics textbooks as a resource for mathematics problems (Lindquist, 1997). As a result, there is a great likelihood that content not presented in the curriculum materials, may ultimately not be covered (Stein et al., 2007).

Curriculum Use

Although, teachers may use similar textbooks for teaching proof, there is no guarantee that teachers will use the textbooks in the same way. Researchers have documented how teachers use curriculum materials in different ways. These may be influenced by their conceptions about the curriculum, teaching, and teacher-curriculum relationships (Lloyd, Remillard, & Herbel-Eisenmann, 2009; Remillard, 2005, 2009). For example, Cirillo (2008) found that the novice teacher in her case study supplemented curriculum materials with more proof tasks as a consequence of attending professional development training.

Teachers' use of curriculum materials embodies various pedagogical actions; teachers' can exhibit a reliance on curriculum materials for lesson planning, and the enacted lesson, or interact with curriculum materials as a form of resources (Lloyd et al., 2009). Manouchehri and Goodman (1998) study of 66 middle school teachers found that teachers varied in the extent curriculum materials were used during instruction time, expectations of students, and effort to build a classroom culture that promoted the use of reform materials. Porter (2002) discussed factors that could potentially influence content being taught in elementary mathematics curriculum. He found that teachers' receive messages about what to be taught from different sources: curriculum materials, professional development, assessments, school hierarchy, parents, other teachers, administrators, district supervisors, and from their own experiences. Hence teachers' draw on curriculum materials for insight as to what to teach, but curriculum materials are not the only variable that contributes to what is taught. Considering that teachers use curriculum materials in various ways for the planning and implementation of lessons, thought must be given as to how teachers use curriculum materials for teaching specific topics and mathematical processes, such as proof.

In *McDougal Littell Geometry* (2007), the authors provided a "Pacing the Course Guide" that illustrates the number of instructional days which ought to be allotted for each chapter. For example, Chapter 2 which is entitled "Reasoning and Proof" is allocated 14 instructional days, however there is no guarantee that teachers will exhibit complete fidelity to textbook recommendations (McNaught, Tarr, & Sears, 2010).

Descriptions of teachers' use of curriculum materials are based frequently on the degree of closeness to which teachers follow them (e.g., fidelity of implementation), but also on the more complex relationship between teacher and textbook. For example, Remillard (2005) and Stein et al. (2007) have described teachers' use of curriculum materials as *following or subverting*, *drawing on*, *interpretation*, and *participating with*. Researchers that study curriculum as *following or subverting* argue that fidelity between what is written and subsequently enacted is possible, a rather positivist perspective (Stein et al., 2007). Researchers that consider curriculum use as *drawing on* the text, focus on teacher agency, and consider the text as a possible resource that can be used during the enacted curriculum (Remillard, 2005). Individuals with this perspective vary in their viewpoints about the feasibility of fidelity. Researchers adhering to an interpretive perspective believe it is impossible to have fidelity between a teacher's actions during the enacted curriculum and the content written in the text, because teachers' experience and beliefs facilitate the meanings teachers derive from what is written, and influences how teachers interpret the written text in the curriculum materials (Remillard, 2005). Moreover, curriculum use can also be described as *participating with*. This perspective emphasizes that there is a dynamic interrelationship between curriculum materials and teachers. Although similar to the interpretive perspective, the difference between the viewpoints exist in the analysis of data (Stein et al., 2007). "In other words, the distinguishing characteristics of this perspective is its focus on the activity of using or participating with the curriculum resource and on the dynamic relationship between the teacher and curriculum" (Stein et al., 2007, p. 345).

Brown (2009) suggested that curriculum can be used for *offloading*, *adapting* and *improvising*. When teachers offload a curriculum, they rely heavily on, and strongly adhere to curriculum materials for pedagogical guidance, resources such as worksheets, and tasks. Teachers that adapt the curriculum make adjustments to the curriculum materials to fit the context of the learning environment, thus the adapted usage of curriculum materials reflects contributions from teachers and from suggestions in the curriculum materials. Finally, when teachers improvise the curriculum, the agency shifts from the curriculum materials to the teachers. Teachers who improvise may utilize alternative strategies (rather than what is recommended in the curriculum materials) to facilitate classroom-learning episodes.

For the purposes of curriculum comparison studies, Chval et al. (2009) suggest a way of measuring textbook integrity, the degree to which teachers use textbooks as a main resource for teaching mathematics. They propose to document three main variables: how much of the textbook teachers use for teaching mathematics, how often they use it, and the extent to which mathematics teachers' instructional practices align with the embedded pedagogy of the textbook.

Stein and Kim (2009) studied how curriculum materials supported teacher learning. The authors' data analysis considered teachers interaction with curriculum materials, and the influence of different organizational conditions. They found that textbooks varied in their cognitive demand of instructional tasks, and the likelihood of teachers' change varies depending on social and human capital of their schools and subsequent districts.

Theoretical Framework- Conceptual Model of Teacher-Curriculum Interactions and Relationships

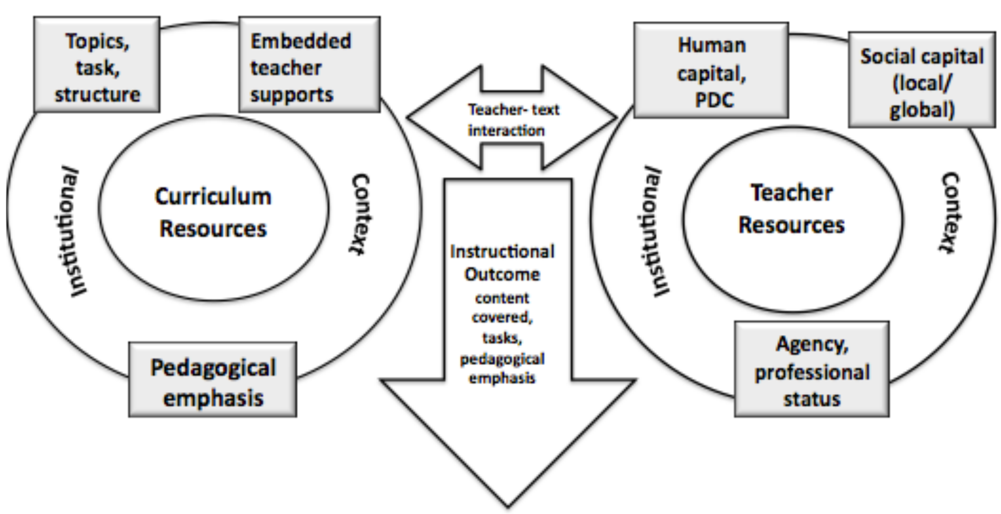
Remillard (2009) conceptualization of the teacher-text interaction and instructional outcomes is used as the overarching theoretical framework for my study. Remillard's framework considers the potential interplay among curricular resources, teachers and instructional outcomes.

Remillard's (2009) "conceptual model of teacher-curriculum interaction and relationships" (Figure 3) incorporates Brown's (2009) Design Capacity Enactment (DCE) framework, McClain, Zhao, Visnovska, and Bowen's (2009) curriculum use in regards to professional status and agency, Stein and Kim's (2009) analysis in respect to human and social capital, and Chval et al. (2009) consideration of embedded pedagogies.

Brown's (2009) DCE framework considers the bi-directional relationship of types of use (offload, adapt, improvise) between curriculum resources (such as representations of physical objects, procedures, and concepts), and teacher resources (subject matter knowledge, goals, beliefs and pedagogical content knowledge) that can influence instructional outcomes. Physical objects are materials that are recommended but not included in the curriculum materials, which may include blueprints to assemble objects. The tasks representation includes directions, written materials, and explicit procedures for teachers and students about the intent of the lesson and how it ought to be enacted. The domain representation of concepts can be illustrated via diagrams, analogies, explanations, models, sequence, or descriptions. The teacher resources comprise the teachers' subject matter knowledge, teachers' pedagogical content knowledge, and teacher's beliefs and goals.

McClain et al. (2009), discussing the role of curriculum materials in facilitating or hindering teacher change, consider teaching as a *social practice*, and the instructional practices teachers' use is situated within schools and subsequently within districts. They suggest that the relationship between teachers and text consist of three constructs, namely teacher's *instructional reality, locus of agency,* and *teachers' professional status*. McClain et al. (2009) defined "agency as having authority over both the mathematics that is taught and the sequencing and presentation of that content" (p.63).

Figure 3. Conceptual model of teacher-curriculum interactions and relationships (Remillard, 2009, p. 89).



The model suggests that *teacher-text interaction* with *curriculum resources* and *teacher resources* can influence *instructional outcomes*. Although my study focused on how geometry teachers use curriculum materials for the teaching of proof, I could not ignore the environment in which teachers' work, since it can potentially influence teachers' decision making as to how to use curriculum materials. Hence, all construct of Remillard's (2009) framework is considered to be of equal importance for this study.

Summary

There exist a need to improve the teaching of proof. Studies document that students perform poorly on proof tasks. To date researchers have documented how proof is learned, outlined how proof is taught, and acknowledged that teachers use curriculum materials for teaching proof. The literature is scarce, however, in explicitly describing how teachers use curriculum materials to teach proof and in promoting a particular proof scheme. Table 1 depicts highlights of major themes in the review of the literature relative to how proof is taught and how curriculum materials are used.

Table 1. Highlights of major themes in the review of the literature pertinent to how proof is taught and how curriculum materials are used

Theme	Critical point(s)
Teaching of Proof in Geometry	•Efforts to improve students' performance on proof tasks by teaching proof in creative ways have been proven to be ineffective (Battista & Clements, 1995).
Teaching Proof with curriculum materials	•Proof-related tasks in curriculum materials are often implemented as such during the enacted curriculum (Bieda, 2010). •Teachers follow the textbook rather closely to structure the enacted lesson on proof, as well as for allocating homework assignment pertaining to proof, and sparingly used technology or hands on investigation activities (McCrone, Martin, Dindyal, & Wallace, 2002).
Teachers' conception (knowledge and belief) about proof	•Teachers believed the roles of proof are to communicate mathematics, systematize mathematical ideas, and construct new knowledge (Knuth, 2002). •Characteristics of a convincing proof includes concrete features, familiarity, generality and amount of details (Knuth, 2002). •Teachers are sometimes challenged to differentiate between proof and nonproof tasks (Knuth, 2002).
Proofs in curriculum materials	•Proofs were represented predominantly in geometry textbooks for grades K-12 mathematics (Donoghue, 2003).
Representation of Proof in Geometry	•Common representation of proof includes: Two-column proof, flow proof, proof tree, and paragraph proof (Cirillo & Herbst, 2010).
Curriculum Materials	•Curriculum materials vary by embedded pedagogies, mathematical depth and emphases, contexts and style (Stein, Remillard, & Smith, 2007).
Curriculum Use	•Teachers may use curriculum materials as following or subverting, drawing on, interpretation, and participating with (Stein, Remillard, & Smith, 2007). •Curriculum can be used for *offloading*, *adapting* and *improvising* (Brown, 2009).
Geometrical cognitive development	Perspectives on how students develop an understanding of geometry includes: •van Hiele Model of geometrical thinking (Van Hiele, 1959/1985) •Abstraction •Concept Learning and the Objects of Geometric Analysis •Diagrams and Representations (Battista, 2009)

Table 1 highlights that teachers enact proof tasks as such, and teachers have various conceptions about proof in mathematics. Furthermore, it suggests that teachers may use the curriculum to provide a structure for the enacted lesson pertinent to proof, and that proofs are more prevalent in geometry textbooks for grades K-12 . Admittedly, although many studies were done pertaining to proof, the field has not reached saturation in describing how proof is taught. My study adds to the literature by describing how geometry teachers use curriculum materials to teach proof, by taking into consideration the levels of cognitive demands of the written and enacted tasks and identifying the dominant proof scheme encouraged.

CHAPTER III: METHODS

This study employs qualitative methods to investigate how geometry teachers use curriculum materials for teaching proof. According to Stein, Remillard and Smith (2007) researchers who examine curriculum use from an interpretive view hold that "teachers bring their own beliefs and experiences to their encounters with curriculum to create their own meanings, and that by using curriculum materials teachers interpret the intentions of the authors" (Stein et al., 2007, p. 344). From this perspective, I utilized a multiple case study research design (Creswell, 2008; Stake, 1995, 2006; Yin, 2009) to examine how teachers use curriculum materials for teaching proof. A curriculum use theoretical framework, and a mathematical tasks and proof schemes analytical framework was used to address the broader issue of how teachers use curriculum materials to facilitate the teaching of proof in geometry and, more specifically, answer the following research questions:

1. How do *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions present proof for segments and angles, parallel and perpendicular lines, and congruent triangles to facilitate students learning to prove?
2. To what extent do geometry teachers use *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions to teach proof for segments and angles, parallel and perpendicular lines, and congruent triangles to facilitate students learning to prove?
3. What influences teachers' decisions to deviate or not from the *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions implied or explicit instructions and lesson plans?

In this chapter, the conceptual analytical framework used, and the data collection and data analysis procedures employed is described. In addition, justification is provided for the use of a case study research design. In the previous chapter, an overarching theoretical framework for curriculum use that situates my study was presented; however, a conceptual analytical framework is needed for my data analysis in order to address the specific attributes of proof in mathematics textbooks and as enacted in the classroom. Hence, in this chapter, the conceptual analytical framework is discussed.

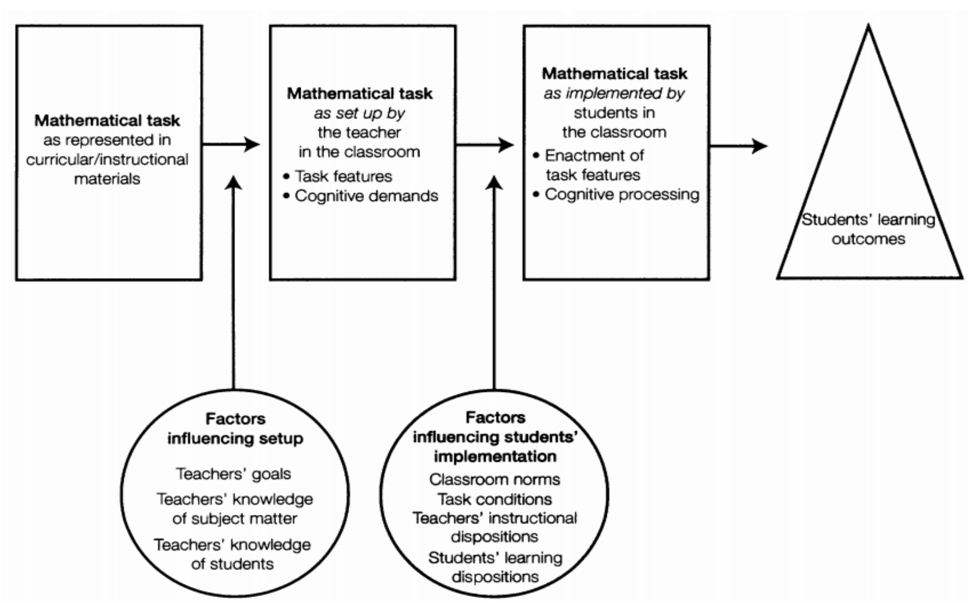
Conceptual Analytical Framework

The Mathematical Tasks Framework [MTF] (Henningsen & Stein, (1997), and a proof schemes framework (Harel & Sowder, (1998) were used as the composites of my conceptual analytical framework. The MTF provided a means to analyze the level of cognitive demands of mathematical tasks written in curriculum materials, teachers enactment of mathematical tasks, and students' implementation of mathematical tasks. It was used it to examine proof-related tasks, in particular. Harel and Sowder (1998) framework considers the nature of the convincing arguments provided for proof tasks as enacted in the classroom. Considering that my operational definition of proof in school mathematics (A. J. Stylianides, 2007) emphasized that "*Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim" (A. J. Stylianides, 2007, p. 291), it was also important to consider the nature of the arguments deemed convincing. Therefore, it was vital that to draw on two frameworks such that I considered the nature of the mathematical tasks as well as the nature of the proof arguments. Hence, my conceptual analytical framework is conceptualized as consisting of three dimensions: the first two dimensions are drawn from MTF, and the last dimension utilizes Harel and Sowder's (1998) proof schemes. The first dimension represents the task features; the second dimension, the cognitive demands of the tasks; and the third dimension, the proof schemes emphasized to complete the tasks.

Mathematical Tasks Framework

The MTF suggests that mathematical tasks go through three phases: *mathematical task as represented in curriculum materials, mathematical task as set up by the teacher in the classroom*, and *mathematical task as implemented by students in the classroom*. The framework identifies mathematical task variables that results in students learning mathematics, and factors that can influence the relationship between the mathematical task variables (Stein, Grover, & Henningsen, 1996). Figure 4 shows Henningsen and Stein Mathematical Tasks (1997) Framework.

Figure 4. Mathematical Tasks framework - "Relationship among various task-related variables and students' learning outcomes" (Henningsen & Stein, 1997, p. p.528).



The two dimensions of mathematical tasks are: task features (first dimension) and cognitive demands (second dimension). "Task features refer to aspects of tasks that mathematics educators have identified as important considerations for the development of mathematical understanding, reasoning and sense making. These features include multiple solution strategies, multiple representations, and mathematical communication" (Henningsen & Stein, 1997, pp. 528-529). Task features for the set up phase refers to the amount of encouragement teachers provide to students for using multiple representations and strategies to justify and explain their responses. For the implementation phase, task features refer to the degree students uses multiple representations, and strategies. The second dimension considers the cognitive demand needed to complete the task.

> Cognitive demands, refers to the kind of thinking processes entailed in solving the task as announced by the teacher (during the set-up phase) and the thinking processes in which students engage (during the implementation phase). These thinking processes can range from memorization to the use of procedures and algorithms (with or without attention to concepts, understanding, or meaning) to complex thinking and reasoning strategies that would be typical of "doing mathematics" (e.g., conjecturing, justifying or interpreting). (Henningsen & Stein, 1997, p. 529)

The task features and cognitive demands of mathematical tasks can be changed between phases. For example, written proof tasks may require high cognitive demand, however, when it is implemented by teachers it may require low cognitive demand of students due to teachers explicitly telling the students what needs to be done to construct a complete proof, or illustrating how to prove it.

The framework also identifies factors that can contribute to the set up and implementation of mathematical tasks. Factors that can influence the set up of task include teachers' pedagogical content knowledge and goals. *Classroom norms* are the conventional practices and expectations of the academic accountability, and quality (Henningsen & Stein, 1997). *Task conditions* consider the appropriateness of a task based on students' prior knowledge, and sufficient time allotted for completing the task. *Teacher and student dispositions* refer to the disposition of learning and pedagogy that influence how students and teachers approach events within a classroom.

Although Henningsen and Stein (1997) did not discuss in detail factors that influence the set up, other researchers (Stein & Baxter, 1989; Stein, Smith, & Silver, 1999) have confirmed that teacher beliefs and knowledge of subject matter can influence how tasks are set up by teachers.

Proof Schemes

Harel and Sowder's (1998) theoretical framework (as described in the literature review) was used as a lens to describe the nature of the convincing arguments used by teachers and students during the enactment of proof-related tasks in the classroom. This proof scheme classification includes three categories, which are not necessarily independent to each other: *external conviction proof schemes, empirical proof schemes* and *analytical proof schemes*.

According to Harel and Sowder (1998, 2007), these proof schemes are psychologically grounded and subjective. "In short, people in different times, cultures, and circumstances apply different methods to remove doubts in the processes of ascertaining and persuading. Accordingly: A person's proof scheme consist of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, pp. 243-244). Hence, Harel and Sowder (1998) proof schemes provided a useful perspective to examine how teachers' facilitate students' construction of proofs.

Case Study

Examining multiple cases present opportunities to characterize and compare, thus illuminating the issue being studied (Creswell, 2008). Case study design is an empirical inquiry that studies real-world phenomena (Merriam, 1998; Yin, 2009). It is suitable to use a case study research design when addressing research questions that emphasize "how" and "why", in instances in which the researcher cannot control the events being observed, and when the phenomenon being observed occurs in a real world context (Yin, 2009). Since using a case study design is suitable for descriptive or explanatory research question (Yin, 2003, 2006), it was an appropriate means to answer my research question that sought to describe how geometry teachers use curriculum materials for teaching proof. Admittedly, teaching is a complex system, and proof is a mathematical process that is not always fully conceptualized (Healy & Hoyles, 2000; Knuth et al., 2009), hence multiple case studies can be used to describe how teachers use curriculum materials for teaching proof because it takes into account the uniqueness of each teacher's instructional practices and the variation that may exist among them (such as differences in districts, school, and class structure). Furthermore, according to Patton (2002), "Well-constructed case studies are *holistic* and *context sensitive*" (p. 447). Holistic in the sense that it is mindful of the complex system in which the phenomenon exist; it is context sensitive to the phenomenon being observed such that data is not over interpreted (Patton, 2002). Acknowledging that teaching of geometry goes beyond proofs and embodies the development of geometrical concepts, using a case study provided a holistic and context sensitive means to examine how geometry teachers use curriculum materials to teach proof. I observed the geometry lesson as a whole as well as focused my attention to the proof tasks used during instruction.

Because geometry teachers' use of curriculum materials for teaching proof was described individually and collectively, an *intrinsic multiple-descriptive* case study was conducted. An intrinsic case occurs when seeking to gain insight into a unique case, that may have no bearing on other cases (Stake, 1995); multiple cases consist of two or more cases, and descriptive cases utilizes multiple data sources to shed light on significant attributes of a phenomenon (Yin, 2003, 2009). Hence, utilizing an intrinsic case study, I identified how each teacher used proof tasks for my analytical framework which consisted of three dimensions. Furthermore utilizing multiple-descriptive cases, I compared and contrasted how tasks were used, as well as adequately described the extent as to how the tasks were used.

Additionally, case study research design must have boundaries or carefully identified unit of analysis (Hatch, 2002; Stake, 1995). For my study, the unit of analysis is the classroom lessons taught by the case study teachers. I focused my attention on how proof tasks are used during the enacted lessons. The unit of analysis "is typically a system of action rather than an individual or group of individuals" (Tellis, 1997, p. 5), and is of utmost importance, because it embodies the phenomenon that would be described in the results of the study (Tellis, 1997).

To build a coherent justification of my results, data was triangulated by utilizing multiple data sources: interviews, physical artifacts, and observations (Yin, 1994, 2009).

Furthermore I engaged in pattern matching, which is a useful strategy that could be used to analyze case study data (Yin, 1994). Yin recommends that when using pattern matching for descriptive studies, the predicted patterns be identified prior to data collection. Hence, I matched teachers' use of proof tasks to mathematical task features, cognitive demands, and proof schemes, the dimensions of my conceptualized analytical framework.

Although case studies can provide insight into the cases being studied, there are limitations regarding generalizable claims of occurrence in other environments (Shavelson & Towne, 2002). Therefore, the findings of this study would be limited to similar contexts to those in which it were conducted.

Textbooks Selection

This study was conducted within the midwest region of the United States, with 3 high school geometry teachers who teach in rural and urban school districts that have adopted *McDougal Littell Geometry* or *Prentice Hall Geometry*. In February 2011, I contacted geometry teachers from various districts in the Midwest to inquire about the textbooks that they currently use for teaching proof. In most schools, teachers use primarily subject-specific textbooks to teach geometry (*Prentice Hall Geometry*, *McDougal Littell Geometry*, and *Glencoe Geometry*). Since these subject-specific textbook series have similar organizational structures (Tarr et al., 2010), the *McDougal Littell* and *Prentice Hall* series were selected, based on proximity of the school districts. In recent times, McDougal Littell publishing merged with Holt publishing, hence modern version of the *McDougal Littell Geometry* curriculum is referred to as *Holt McDougal Geometry*.

More particularly, my observation focused on chapters 2, 3, and 4 of Larson et al. (2007) "*McDougal Littell Geometry"* [1] and Bass et al. (2004) *"Prentice Hall Geometry"* Teacher's Edition. In both books Chapter 2 (which is entitled "Reasoning and Proof") was selected because it is devoted to fostering the development of proof skills. Chapter 3 and 4 (which is entitled "Perpendicular and Parallel Lines" and "Congruent Triangles" respectively) were selected because a review of the literature emphasizes that these topic readily present opportunities for students to prove in geometry (Donoghue, 2003; Herbst, 2002b). The content of Bass et al. (2004) Chapters 2 through 4 are rather similar to Larson et al. (2007). Hence, although the two districts used different subject specific geometry curriculum, the mathematical content and organization are relatively the same.

Both Larson et al. (2007) and Bass et al. (2004) textbooks are structured such that every chapter in the textbook has big ideas, postulate and theorems, key concepts, vocabulary, and mixed review. Additional resources (such as skill review handbook, postulates and theorems, additional proofs, worked out solution, and selected answers) are available in the back of the book.

To ensure that teachers are familiar with the organizational structure of the curriculum materials of the *McDougal Littell Geometry* and *Prentice Hall Geometry*, only teachers who have used the curriculum for at least 3 years were invited to participate in the study.

[1] For this study, references to *McDougal Littell Geometry* Teacher's Edition (Larson et al, 2007) will be simply stated as *McDougal Littell Geometry.* Likewise *Prentice Hall Geometry* (Bass et al., 2004) Teacher's Edition will be stated as *Prentice Hall Geometry* .

Participants

Purposeful (yet convenient) sampling, was used to identify participants (Creswell, 2008). It was purposeful since participants had to use the textbook for at least three years to ensure that they were familiar with the organizational structure of the textbook. It was also convenient since they had to be willing to participate. Three teachers were asked to participate in the study. The three participants provided opportunities to document variation and kept the sample size manageable, such that I conducted observations of similar lessons in all three classrooms. The high schools geometry teachers selected were representative of regular geometry classes from a rural and urban school district. Teachers' that agreed to participate received a $50.00 VISA gift card.

Initially, permission was requested from the school districts to conduct research within their respective district. Once the school districts granted permission, teachers were identified to participate in the study. Once teachers in the two sites consented to participate, I scheduled a minimum of six lesson observations with each of them.

Data Collection

To answer my research questions I utilized multiple data collection instruments: teacher artifacts, classroom observations, and interviews.

Teacher Artifacts

To document the nature of planned proof tasks for the enacted lesson, teachers' reflection of proof tasks after enacted, and teachers' perceptions of what constitutes students learning of proof, I collected teacher artifacts. I used a modified version of the artifact packet for reasoning and proof developed by Horizon Research, Inc. for the Cases of Reasoning and Proving in Secondary Mathematics Project[2]. The artifact packet requires teachers to provide pertinent information based on their planned and enacted lessons on proof-related tasks. Teachers were asked to provide copies of: *task cover sheet-before implementation*, *task reflection sheet- after implementation*, *copy of the original task*, *copy of the modified task (if applicable)*, samples of student work *(that meet/exceeded expectation, demonstrated progress,* and *struggles)*, and *copies of additional material (e.g., class notes, homework assignments, etc)*. The *task cover sheet-before implementation* asked teachers to describe the nature of tasks, source of tasks, goal of tasks and decide whether student engaged in proofs, mathematical arguments, identifying patterns, and making conjectures, etc. For the *task cover sheet- after implementation* teachers were asked to describe segments of the lesson in which students work on the tasks, rationale for altering the tasks (if applicable), directions given to students and communications they had with students about the tasks. I modified the *task cover sheet-before implementation*, *and task reflection sheet- after implementation* (Appendix B and Appendix C), by only using a portion of the instrument that aligned with my research questions. Furthermore, teachers reflected on the nature of students' mathematical arguments, conjectures and what they believed students learned. Teachers' were also asked to provide copies of additional materials that were used to supplement the lesson (if applicable).

[2] Horizon Research, Inc. developed artifact packet for the Cases of Reasoning and Proving in Secondary Mathematics Project, with funding from the National Science Foundation (Award No. DRL-0732798)

Interviews

Teachers were interviewed about their conceptions of proof, the importance of proof in school mathematics, and the frequency of proof-related tasks in their instructional practices. The interview protocol included the following questions:

- What does the notion of proof mean to you? (Knuth, 2002b, p. 67)
- What does it mean to prove something? (Knuth, 2002a, p. 383)
- What constitutes proof in secondary school mathematics? (Knuth, 2002b, p. 67)
- What purpose does proof serve in mathematics? (Knuth, 2002a, p. 383)
- Why teach proof in secondary school mathematics? (Knuth, 2002b, p. 67)
- When should students encounter proof? (Knuth, 2002b, p. 67)
- How does an argument become a proof? (Knuth, 2002a, p. 383)
- Do proofs ever become invalid? (Knuth, 2002a, p. 383)
- What are some of the things you look for when you're reading a student's proof? (Pettey, 2011, email correspondence)
- How frequently is proof integrated into the teaching of geometry?
- What are the strengths and limitation of using the textbook for teaching proof in geometry lessons?

Utilizing a protocol provided a means to compare and contrast responses across participants (Creswell, 2008), but I also asked follow-up questions either to clarify responses to the questions above or to gain a better understanding of the teachers' views on proof.

Teachers were also shown various sets of mathematical arguments and were asked to determine if the argument is a convincing proof, if it was not then the teacher was asked to provide feedback on how they would help students identify their errors and improve their arguments. The tasks used were drawn from COSMIC Test E Question 7 (which requires students to construct a geometrical proof) (Chávez, Papick, & Ross, 2009), Healy and Hoyles (2000) and Knuth (2002a) studies of conceptions about proof.

Furthermore, at the end of observed lessons, I asked teachers about specific actions observed during the class. For instance, I might have asked a teacher to explain why a particular feedback comment was provided for a student's written/ oral response. The interviews were generally audio recorded and were used to triangulate teachers written responses submitted via the artifact packets. The instances in which the teachers' comments were not audio recorded included some conversations during breaks, when they spoke freely about events that have transpired in their teaching.

Classroom Observations

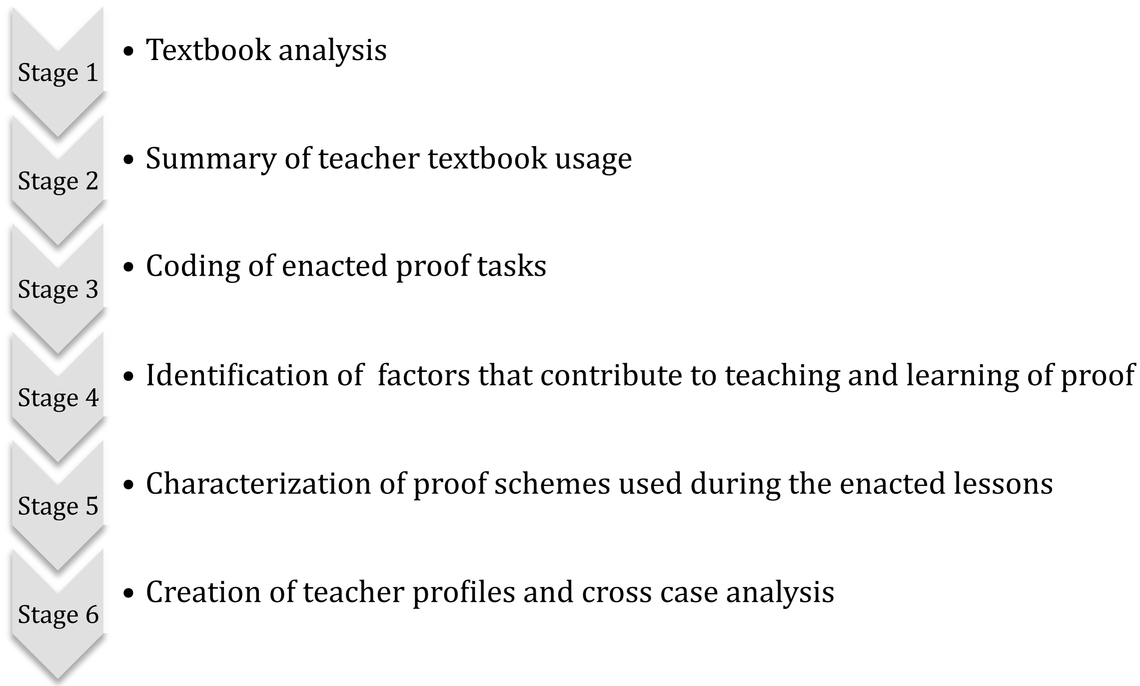
Finally, I observed a minimum of 6 lessons per teacher as a non-participant observer (Creswell, 2008). I used a classroom observation protocol, adapted from an instrument developed by Horizon Research, Inc. for the Cases of Reasoning and Proving (CORP) in Secondary Mathematics Project) that focused on proof-related activities and helped me to document how teachers use the proof tasks during the lesson. I acknowledge that I excluded portions of the protocol that did not readily align with my research questions, considering that the instrument was designed for a much broader study. The portion of the protocol used include the following sections: *background information, context and nature of the lesson, students, outline of the lesson*, *classroom culture*, *use of instructional tools*, *student tools*, *facilitation of the tasks*, and *cognitive demand of the tasks.* Additionally, I added a section to document proof schemes observed during the lesson. The background information requires the observer to report their name, date of the observation, the teacher being observed and the school name for which the teacher is employed. For the *context and nature of the lesson* section the observer is required to report the instructional material used and the mathematical strand emphasized during the lesson (which is geometry). The section devoted to *students* asks for the grade level of students and the total number of students within the classroom. The *outline of the lesson* requires a description of: the goals, structure and flow, and how reasoning and proof was integrated. For the *classroom culture* the observer measures the extent the classroom learning environment and mathematical norms provides students an opportunity to learn the mathematical objective of the lesson. The scale used ranged from 1-greatly inhibited to 4-greatly facilitated. Similarly, the *use of instructional tools* and *facilitation of the tasks* gauged the extent the tools teachers used facilitate students' opportunity to learn. In the *cognitive demand* section of the protocol, the observer measured the intellectual potential and engagement of the tasks. The scale ranges from 0 to 4: "0-No academic thinking required, 1-Memorization, 2- Use of procedures without connection to meaning, concept or understanding, 3-Use of procedures with connection to meaning, concepts or understanding, [and] 4-Engaging in the thinking practices of the discipline" (Horizon Research Inc., 2011). The proof schemes section of the protocol required the identification of proof schemes observed during the enacted lesson with supportive rationale for claims made. The lessons were video-recorded / audio-recorded for triangulation of written entries on the observation protocol. The video recording sought to capture teachers' actions and students' board work.

Data Analysis

The data were analyzed primarily qualitatively. I utilized a six-stage analytical process. During the first stage descriptive statistics was used to summarize the number of tasks that could be considered proof-related tasks in the observed lessons. Additionally, I conducted a curriculum analysis using Henningsen and Stein's (1997) Mathematical Tasks Framework of tasks in chapters 2, 3, and 4, while focusing on the proof tasks. This analysis provided the basis for a comparison between the written curriculum and the enacted curriculum.

During the second stage, I summarized the extent to which teachers used their textbooks to teach proof, based on the data collected in the teacher artifacts. For the third stage I coded proof-related tasks enacted in the classroom using the MTF. For the fourth stage, I coded and classified factors influencing the set-up phase and students' implementation during the enacted lesson; I drew on information obtained from teacher artifacts, classroom observation protocol, and interviews. For the fifth stage, I characterized significant proof schemes used during the lesson. At the sixth stage a profile of each case was constructed to represent the 3-dimensional conceptual analytical framework, in an effort to conduct a within case analysis, as well as a cross case analysis (Creswell, 2008; Patton, 2002; Stake, 1995). Additional researchers assisted with the analysis for reliability purposes, and an additional independent observer visited classrooms to validate consistency in the data entries of the observation protocols. Coders were asked to code textbook tasks independently, and subsequently compare responses. If there was a disagreement, coders discussed discrepancy, and if they were unable to come to an agreement, another coder was asked to provide additional feedback. Figure 5 illustrates how I progressed during the data analysis.

Figure 5. Six-stage data analysis outline.



Coding of Written Tasks

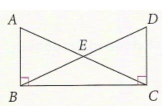
It is difficult to find examples of classification of high school geometry tasks according to the MTF, so a careful process of coding the tasks in the textbooks and those used during the observed lessons was fundamental to establish valid conclusions. Although the inter-rater reliability was an average of 89%, higher than the reliability average reported by Henningstein and Stein (1997) of 79%, upon further examination of these codes by other researchers, they suggested that a more conservative approach to the proof tasks would be more appropriate. As a result, I coded the tasks one more time, aided by a colleague with experience teaching college level mathematics. We first sorted tasks into low-level and high-level and subsequently coded them using the four levels of the MTF. During this process we agreed to code a proof task as "*doing* *mathematics*" if it required writing a complete proof, that was not similar to previous tasks and examples, and could change the context, utilized a different representation, and was not algorithmic. We further agreed to use the more conservative coding for the proof tasks in cases of disagreement, and if there were objections an alternative coder would be asked to contribute to the discussion. Consequently, I recoded the lessons observed. Most of the tasks were classified at the same level as the original coding. The primary changes were in proof tasks originally coded as *doing mathematics* that were generally recoded as *procedures with connections*.

It is imperative that I state that the coding of task features and cognitive demand is based on the content of the Teacher's Editions (answers, instructional suggestions, tasks, etc), which may not necessarily be visible in students' version of the textbooks. Considering that teachers uses textbook as an instructional resource, and can read the guidance provided by textbook authors, it is highly likely that they would encourage students to follow the paths recommended by textbook authors. Hence, the coding of tasks were based on the written tasks, as well as the answers provided for the tasks. Task features were coded for the following: whether or not the tasks were proof or proof-related tasks; what proof representations were used; whether answers were provided in isolation or with reasoning; were the tasks "challenge", were group collaboration required to complete the tasks, were tasks in realistic context or abstract context; were figures / diagrams presented; are the tasks fill in the blanks, or multiple choice; and whether or not the tasks were composed of multiple parts. The cognitive demand of the tasks were coded as *memorization*, *procedures without connections*, *use of procedures with connections*, and *doing mathematics* (Henningsen & Stein, 1997). Examples 1 and 2 illustrate how questions were coded using dimensions 1 (task features) and 2 (cognitive demand) of the conceptual framework. Neither examples required group collaboration. Since dimension 3 (proof schemes) considers the nature of persuasion for a person within a community, proof schemes (Harel & Sowder, 1998) were coded based on the didactics of the enacted lesson in relation to tasks. Although the examples represents coding of the written tasks, I was mindful that when enacted the levels of cognitive demands of the tasks may change.

Example 1 is from *Prentice Hall Geometry* Section 4.7 Question 28 (Figure 6) (Bass et al., 2004, p. 229); it is an example of how a task was coded. A question of this nature requires limited cognitive demand and mirrors the examples within the lesson notes. Hence, the cognitive demand of the task was coded as *memorization.* As it relates to the task features, the task is deemed a proof task, utilizing a two-column proof representation, with only the answer provided, it comprises of a figure, is abstract in nature requires students to fill in the blank and is composed of multiple parts.

Figure 6. Example 1 – Illustrating coding of a proof task (Bass et al., 2004, p. 229).[3]

28. Given: $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$, $\overline{AC} \cong \overline{DB}$
Prove: $\overline{AE} \cong \overline{DE}$



Statements	Reasons
1. $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$	a. ? Given
2. $\angle ABC$ and $\angle DCB$ are right angles.	b. ? Def. of $\perp$
3. $\triangle ABC$ and $\triangle DCB$ are right triangles.	c. ? Def. of rt. $\triangle$
4. $\overline{AC} \cong \overline{DB}$	d. ? Given
e. ? $\cong$? $\overline{BC} \cong \overline{BC}$ 28f. Reflexive	f. ? Property of Congruence
6. $\triangle ABC \cong \triangle DCB$	g. ? HL
7. $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DC}$	h. ? CPCTC
i. $\angle AEB \cong \angle$? DEC	j. ? Vert. $\angle$s are $\cong$.
9. $\triangle ABE \cong \triangle DCE$	k. ? AAS
l. ? $\cong$? $\overline{AE} \cong \overline{DE}$	m. ? CPCTC

Example 2 is Section 4.7 Question 33 in Bass et al. (2004) (Figure 7). The task is a proof

[3] From *Prentice Hall Mathematics Geometry* Teacher's Edition by Bass, et al. Copyright 2004 © Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

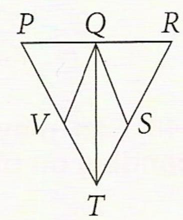
task, which uses a paragraph proof representation, it is labeled "challenge" by the textbook, only one solution strategy is used, it is abstract in nature, and has a figure represented. The task provides no explicit guidance of how to proceed, and presents an opportunity for students to reflect and construct an original proof. Due to the rich nature of the task, and its feasibility to facilitate students engaging in doing proofs, the cognitive demand of the task was coded as *doing mathematics.*

Figure 7. Example 2 – Illustrating coding of a proof task (Bass et al., 2004, p. 229).[4]

33. Given: $\overline{QT} \perp \overline{PR}$, $\overline{QT}$ bisects $\overline{PR}$, $\overline{QT}$ bisects $\angle VQS$.

Prove: $\overline{VQ} \cong \overline{SQ}$

See margin.



33. $\overline{PQ} \cong \overline{RQ}$ and $\angle PQT \cong \angle RQT$ by Def. of $\perp$ bisector. $\overline{QT} \cong \overline{QT}$ so $\triangle PQT \cong \triangle RQT$ by SAS. $\angle P \cong \angle R$ by CPCTC. $\overline{QT}$ bisects $\angle VQS$ so $\angle VQT \cong \angle SQT$ and $\angle PQT$ and $\angle RQT$ are both rt. ∠s. So $\angle VQP \cong \angle SQR$ since they are compl. of ≅ ∠s. $\triangle PQV \cong \triangle RQS$ by ASA so $\overline{QV} \cong \overline{QS}$ by CPCTC.

Figure 8 and 9 illustrates flow proofs as represented in *McDougal Littell Geometry*. Section 4.5 Question 25 required students to fill in missing responses to complete the proof tasks, while Section 4.5 Question 35 request students to write a flow proof.

[4] From *Prentice Hall Mathematics Geometry* Teacher's Edition by Bass, et al. Copyright 2004 © Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved

Figure 8. Section 4.5 Question 25 in *McDougal Littell Geometry* (Larson et al., 2007 p. 254)–Illustrating flow proof.[5]

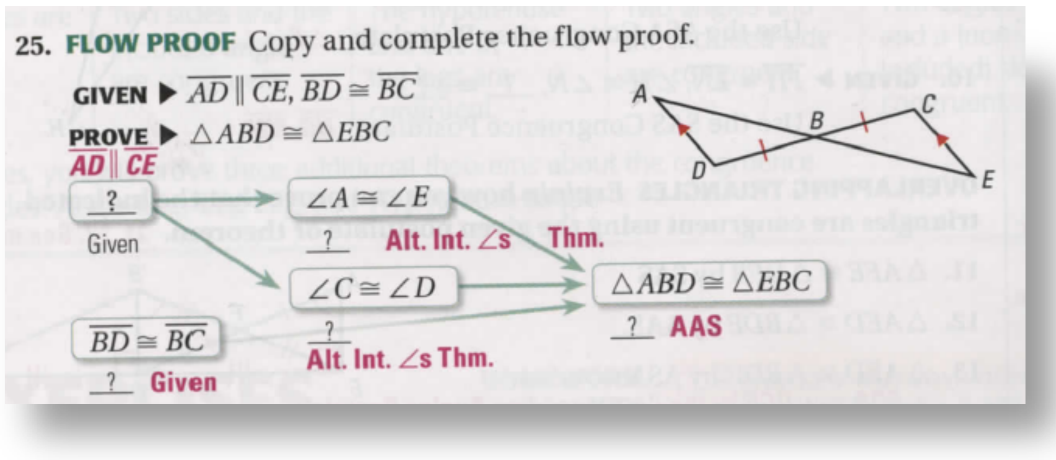
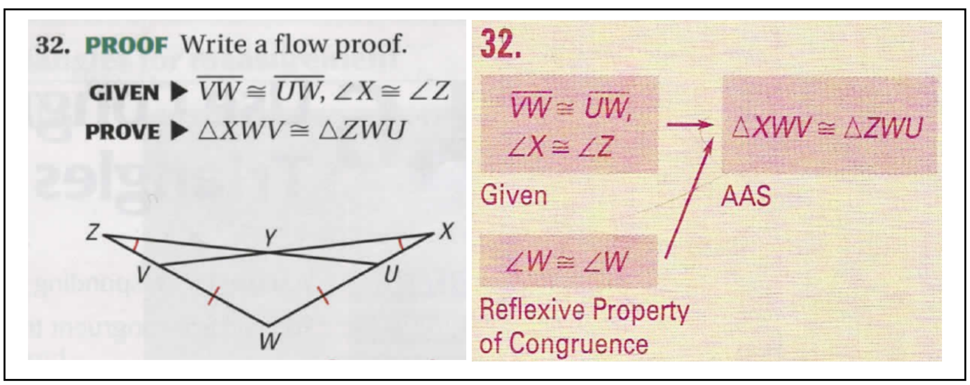


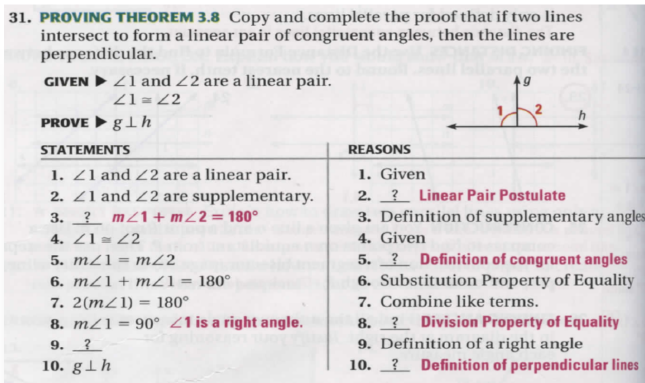
Figure 9. Section 4.5 Question 32 in *McDougal Littell Geometry* (Larson et al., 2007 p. 255)–Illustrating flow proof. [6]



Considering that a conservative approach was used to classify proof tasks relatives to levels of cognitive demands, Figures 10 -18 illustrate proof tasks classified as *memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics*.

[5] [6] From *McDougal Littell Geometry*, Teacher's Edition, by Larson, et al. Copyright © 2007 by McDougal Littell. All rights reserved. Reprinted by permission of Houghton Mifflin Harcourt Publishing Company.

Figure 10. Lower-level demand task (*memorization*) from Section 3.6 Question 31 in *McDougal Littell Geometry* (Larson et al., 2007 p. 196). [7]



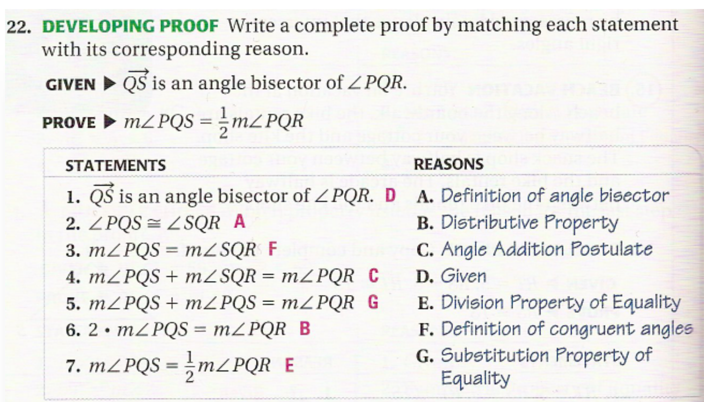
31. **PROVING THEOREM 3.8** Copy and complete the proof that if two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.

GIVEN ▶ $\angle 1$ and $\angle 2$ are a linear pair.
$\angle 1 \cong \angle 2$

PROVE ▶ $g \perp h$

STATEMENTS	REASONS
1. $\angle 1$ and $\angle 2$ are a linear pair.	1. Given
2. $\angle 1$ and $\angle 2$ are supplementary.	2. ? **Linear Pair Postulate**
3. ? **$m\angle 1 + m\angle 2 = 180°$**	3. Definition of supplementary angles
4. $\angle 1 \cong \angle 2$	4. Given
5. $m\angle 1 = m\angle 2$	5. ? **Definition of congruent angles**
6. $m\angle 1 + m\angle 1 = 180°$	6. Substitution Property of Equality
7. $2(m\angle 1) = 180°$	7. Combine like terms.
8. $m\angle 1 = 90°$ **$\angle 1$ is a right angle.**	8. ? **Division Property of Equality**
9. ?	9. Definition of a right angle
10. $g \perp h$	10. ? **Definition of perpendicular lines**

Figure 11. Lower-level demand task (*procedures without connections*) from Section 2.6 Question 22 in *McDougal Littell Geometry* (Larson et al., 2007 p. 118). [8]



22. **DEVELOPING PROOF** Write a complete proof by matching each statement with its corresponding reason.

GIVEN ▶ $\overrightarrow{QS}$ is an angle bisector of $\angle PQR$.

PROVE ▶ $m\angle PQS = \frac{1}{2} m\angle PQR$

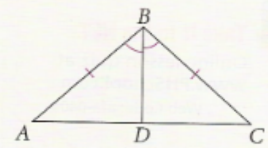
STATEMENTS	REASONS
1. $\overrightarrow{QS}$ is an angle bisector of $\angle PQR$. **D**	A. Definition of angle bisector
2. $\angle PQS \cong \angle SQR$ **A**	B. Distributive Property
3. $m\angle PQS = m\angle SQR$ **F**	C. Angle Addition Postulate
4. $m\angle PQS + m\angle SQR = m\angle PQR$ **C**	D. Given
5. $m\angle PQS + m\angle PQS = m\angle PQR$ **G**	E. Division Property of Equality
6. $2 \cdot m\angle PQS = m\angle PQR$ **B**	F. Definition of congruent angles
7. $m\angle PQS = \frac{1}{2} m\angle PQR$ **E**	G. Substitution Property of Equality

Figure 12. Lower-level demand task (*procedures without connections*) from Section 4.4 Question 24 in *Prentice Hall Geometry* (Bass et al., 2004 p. 207).[9]

24. Use the plan to write a paragraph proof. **See left.**

Given: $\overline{BA} \cong \overline{BC}$, $\overline{BD}$ bisects $\angle ABC$.

Prove: $\overline{BD} \perp \overline{AC}$, $\overline{BD}$ bisects $\overline{AC}$.

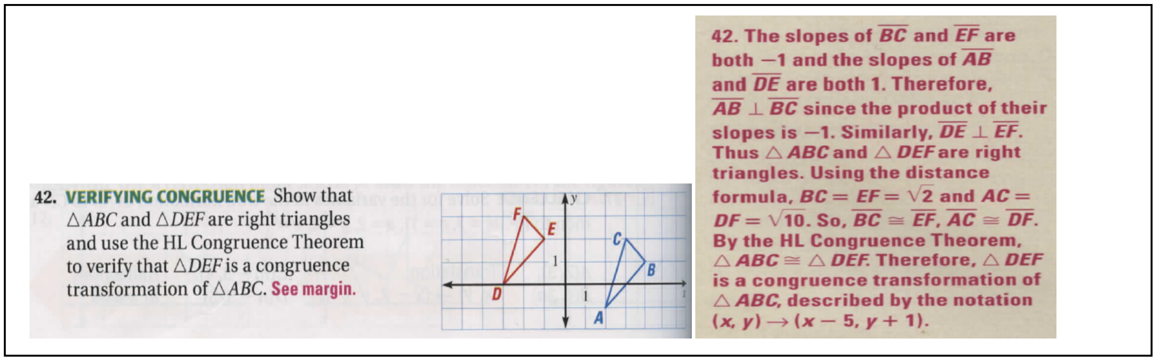


Plan: To show $\overline{BD} \perp \overline{AC}$, you can show that $\angle BDA \cong \angle BDC$ and use the fact that congruent supplementary angles are right angles. To show that $\overline{BD}$ bisects $\overline{AC}$, you can show that $\overline{AD} \cong \overline{CD}$. The desired congruent angles and segments are corresponding parts of $\triangle ABD$ and $\triangle CBD$. So, first show that $\triangle ABD \cong \triangle CBD$.

$\overline{BA} \cong \overline{BC}$ is given; $\overline{BD} \cong \overline{BD}$ by the Reflexive Prop. of $\cong$ and since $\overline{BD}$ bisects $\angle ABC$, $\angle ABD \cong \angle CBD$ by Def. of an $\angle$ bisector; thus, $\triangle ABD \cong \triangle CBD$ by SAS; $\overline{AD} \cong \overline{DC}$ by CPCTC so $\overline{BD}$ bisects $\overline{AC}$ by Def. of a bis.; $\angle ADB \cong \angle CDB$ by CPCTC and $\angle ADB$ and $\angle CDB$ are supp.; thus, $\angle ADB$ and $\angle CDB$ are right $\angle$s and $\overline{BD} \perp \overline{AC}$ by Def. of $\perp$.

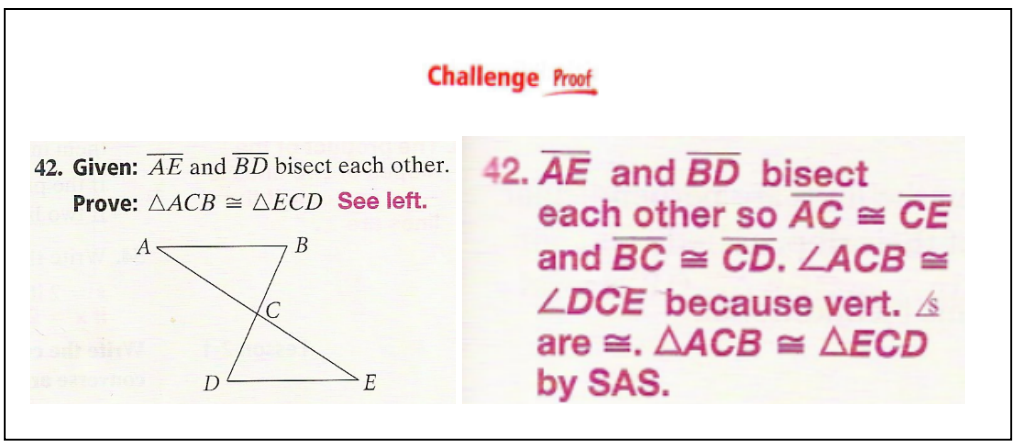
[9] From *Prentice Hall Mathematics Geometry* Teacher's Edition by Bass, et al. Copyright 2004 © Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

Figure 13. Higher-level demand task (*procedures with connections*) from Section 4.8 Question 42 in *McDougal Littell Geometry* (Larson et al., 2007 p. 278). [10]


42. **VERIFYING CONGRUENCE** Show that $\triangle ABC$ and $\triangle DEF$ are right triangles and use the HL Congruence Theorem to verify that $\triangle DEF$ is a congruence transformation of $\triangle ABC$. **See margin.**

42. The slopes of $\overline{BC}$ and $\overline{EF}$ are both −1 and the slopes of $\overline{AB}$ and $\overline{DE}$ are both 1. Therefore, $\overline{AB} \perp \overline{BC}$ since the product of their slopes is −1. Similarly, $\overline{DE} \perp \overline{EF}$. Thus $\triangle ABC$ and $\triangle DEF$ are right triangles. Using the distance formula, $BC = EF = \sqrt{2}$ and $AC = DF = \sqrt{10}$. So, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$. By the HL Congruence Theorem, $\triangle ABC \cong \triangle DEF$. Therefore, $\triangle DEF$ is a congruence transformation of $\triangle ABC$, described by the notation $(x, y) \rightarrow (x - 5, y + 1)$.

Figure 14. Higher -level demand task (*procedures with connections*) from Section 4.2 Question 41 in *Prentice Hall Geometry* (Bass et al., 2004, p.191), which is also labeled as “challenge”.[11]


Challenge Proof

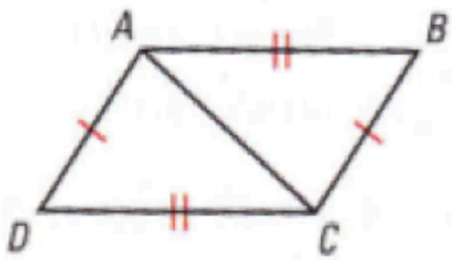
42. Given: $\overline{AE}$ and $\overline{BD}$ bisect each other.
Prove: $\triangle ACB \cong \triangle ECD$ **See left.**

42. $\overline{AE}$ and $\overline{BD}$ bisect each other so $\overline{AC} \cong \overline{CE}$ and $\overline{BC} \cong \overline{CD}$. $\angle ACB \cong \angle DCE$ because vert. ∡ are ≅. $\triangle ACB \cong \triangle ECD$ by SAS.

Figure 15. Higher-level demand task (*procedures with connections*) from Section 2.5 Question 29 in *McDougal Littell Geometry* (Larson et al., 2007 p.109).[12]

PERIMETER **In Exercises 28 and 29, show that the perimeter of triangle ABC is equal to the perimeter of triangle ADC.** 28–29. See margin.

29.



29. Equation (Reason)

$AD = CB$, $DC = BA$ (Given)

$AC = AC$ (Reflexive Property of Equality)

$AD + DC = CB + DC$ (Addition Property of Equality)

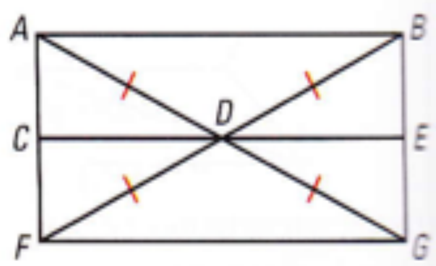
$AD + DC = CB + BA$ (Substitution)

$AD + DC + AC = CB + BA + AC$ (Addition Property of Equality)

Figure 16. Higher-level demand task (*doing mathematics*) from Section 4.6 Question 26 in *McDougal Littell Geometry* (Larson et al., 2007 p. 260). [13]

PROOF **Use the information given in the diagram to write a proof. 23–26.**

26. PROVE ▶ $\overline{AC} \cong \overline{GE}$



26. Statements	Reasons
1. $\overline{AD} \cong \overline{GD} \cong \overline{FD} \cong \overline{BD}$	1. Given
2. $\angle ADC \cong \angle GDE$, $\angle FDC \cong \angle BDE$	2. Vertical Angles Congruence Theorem
3. $m\angle ADC = m\angle GDE$, $m\angle FDC = m\angle BDE$	3. Definition of angle congruence
4. $m\angle ADC + m\angle FDC = m\angle ADF$, $m\angle BDE + m\angle GDE = m\angle GDB$	4. Angle Addition Postulate
5. $m\angle ADC + m\angle FDC = m\angle GDB$	5. Substitution Property of Equality
6. $m\angle ADF = m\angle GDB$	6. Transitive Property of Equality
7. $\angle ADF \cong \angle GDB$	7. Definition of angle congruence
8. $\triangle ADF \cong \triangle GDB$	8. SAS
9. $\angle FAD \cong \angle BGD$	9. Corr. parts of ≅ ⧋ are ≅.
10. $\triangle ADC \cong \triangle GDE$	10. ASA
11. $\overline{AC} \cong \overline{GE}$	11. Corr. parts of ≅ ⧋ are ≅.

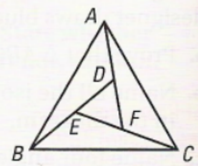
[13] From *McDougal Littell Geometry*, Teacher's Edition, by Larson, et al.

Figure 17. Higher-level demand task (*doing mathematics*) from Section 4.7 Question 49 in *McDougal Littell Geometry* (Larson et al., 2007 p. 270).[14]

49. PROOF Write a proof. **See margin.**

GIVEN ▶ $\triangle ABC$ is equilateral, $\angle CAD \cong \angle ABE \cong \angle BCF$.

PROVE ▶ $\triangle DEF$ is equilateral.



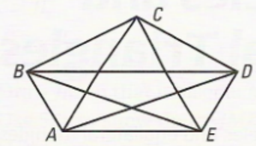
49.

Statements	Reasons
1. $\triangle ABC$ is equilateral, $\angle CAD \cong \angle ABE \cong \angle BCF$.	1. Given
2. $m\angle CAD = m\angle ABE = m\angle BCF$	2. Definition of angle congruence
3. $m\angle CAD + m\angle DAB = m\angle CAB$, $m\angle ABE + m\angle EBC = m\angle ABC$, $m\angle BCF + m\angle FCA = m\angle BCA$	3. Angle Addition Postulate
4. $m\angle CAB = m\angle ABC = m\angle BCA$	4. Corollary to the Base Angles Theorem
5. $m\angle CAD + m\angle DAB = m\angle ABE + m\angle EBC = m\angle BCF + m\angle FCA$	5. Substitution
6. $m\angle CAD + m\angle DAB = m\angle CAD + m\angle EBC = m\angle CAD + m\angle FCA$	6. Substitution Property of Equality
7. $m\angle DAB = m\angle EBC = m\angle FCA$	7. Subtraction Property of Equality
8. $\angle DAB \cong \angle EBC \cong \angle FCA$	8. Definition of angle congruence
9. $\triangle ACF \cong \triangle CBE \cong \triangle BAD$	9. ASA
10. $\angle BEC \cong \angle ADB \cong \angle CFA$	10. Corr. parts of $\cong$ △s are $\cong$.
11. $\angle BEC$ and $\angle DEF$, $\angle ADB$ and $\angle EDF$, $\angle CFA$ and $\angle DFE$ are linear pairs and are supplementary.	11. Definition of linear pair
12. $\angle DEF \cong \angle EDF \cong \angle DFE$	12. Congruent Supplements Theorem
13. $\triangle DEF$ is equiangular.	13. Definition of equiangular triangle
14. $\triangle DEF$ is equilateral.	14. Corollary to the Converse of Base Angles Theorem

[14] From *McDougal Littell Geometry*, Teacher's Edition, by Larson, et al.

Figure 18. Higher-level demand task (*doing mathematics*) from Section 4.6 Question 40 in *McDougal Littell Geometry* (Larson et al., 2007 p. 263). [15]

40. CHALLENGE In the diagram of pentagon $ABCDE$, $\overline{AB} \parallel \overline{EC}$, $\overline{AC} \parallel \overline{ED}$, $\overline{AB} \cong \overline{ED}$, and $\overline{AC} \cong \overline{EC}$. Write a proof that shows $\overline{AD} \cong \overline{EB}$. **See margin.**



40. Statements	Reasons
1. $\overline{AB} \parallel \overline{EC}$, $\overline{AC} \parallel \overline{ED}$, $\overline{AB} \cong \overline{ED}$, $\overline{AC} \cong \overline{EC}$	1. Given
2. $\angle DEC \cong \angle ECA$, $\angle ECA \cong \angle BAC$	2. Alternate Interior Angles Theorem
3. $\angle DEC \cong \angle BAC$	3. Transitive Property of Angle Congruence
4. $\triangle DEC \cong \triangle BAC$	4. SAS
5. $\overline{BC} \cong \overline{CD}$, $\angle BCA \cong \angle DCE$	5. Corr. parts of $\cong$ △s are $\cong$.
6. $m\angle BCA = m\angle DCE$	6. Definition of congruent angles
7. $m\angle BCA + m\angle ACE = m\angle DCE + m\angle ACE$	7. Addition Property of Equality
8. $m\angle BCE = m\angle DCA$	8. Angle Addition Postulate
9. $\angle BCE \cong \angle DCA$	9. Definition of congruent angles
10. $\triangle BCE \cong \triangle DCA$	10. SAS
11. $\overline{AD} \cong \overline{EB}$	11. Corr. parts of $\cong$ △s are $\cong$.

[15] From *McDougal Littell Geometry*, Teacher's Edition, by Larson, et al.

Coding of Enacted Lesson

All audio recording of the lessons were transcribed, and subsequently imported into NVivo 9 (a qualitative software program) for analysis. Considering that teachers transported the audio recorder with them as they visited various groups, the audio recording might have captured conversations that might not be heard on the video recording of the lessons. Scanned copies of the observation protocols and other artifacts were also imported into the qualitative software. The MTF (Henningsen & Stein, 1997) and proof schemes framework (Harel & Sowder, 1998) were used to provide the primary codes of the enacted lessons. Hence, the enacted lessons were coded for tasks features, cognitive demand of enacted tasks, factors influencing the teaching of proof and visible proof schemes. For the task features the general codes were: mathematical communication, multiple representations, and multiple solution strategies. The cognitive demands of the tasks were coded as *memorization*, *procedures without connections*, *procedures with connections*, or *doing mathematics*. For the factors influencing teaching of proof the initial codes considered factors that influence set up and factors that influence students implementation of the task (Henningsen & Stein, 1997); however new categories emerged based on the frequency of words and phrases used during the lessons. Hence the codes used for the factors influencing the teaching of proof were: assessment, classroom norm, community (professional environment), making mathematics easy, proof and mathematical tasks should be short, task conditions, students disposition, teachers' beliefs, teachers decision to adapt or improvise the curriculum, teachers' knowledge of students and or students learning, and teachers use of textbook and tools. Harel and Sowder's (1998) proof schemes classification were used to code proof schemes.

Credibility of Qualitative Inquiry

The criteria for a credible qualitative inquiry involves three unique, yet related elements: " (1) rigorous techniques and methods for gathering high-quality data that is carefully analyzed, with attention to issues of validity, reliability, and triangulation; (2) the credibility of the researcher…; and (3) philosophical belief in the phenomenological paradigm..." (Patton, 1990, p. 461).

To promote rigorous research practices, I utilized multiple coders to verify validity in the coding, and funded-research instruments that focuses on proof and reasoning tasks (Horizon Research Inc., 2011) to increase the reliability of the data collected. Initially two coders piloted the coding of tasks. During this time, it was decided only the exercises would be coded rather than the examples in the textbooks. Considering there was a consistency in the initial coding, three additional coders were asked to code exercise tasks in Chapters 2-4 using the description of levels of cognitive demand (Smith & Stein, 1998). After refining the *doing mathematics* proof tasks, another researcher coded the tasks as well, the consistency was relatively the same. Furthermore, triangulation was built into my data collection. Data triangulation involves the usage of multiple data source to substantiate claims made (Creswell, 2008; Patton, 2002). For my study, I utilized multiple data sources—Teacher's Editions of textbooks, teacher interviews, classroom observations, and teacher artifacts—such that I can adequately support emergent themes. An additional researcher accompanied me for approximately a quarter of the classroom lessons observed. A comparison of our observation protocols revealed consistency in the rating of the lessons. Considering the nature of my research questions, I believe that case study research design is appropriate for capturing the uniqueness of each case and for comparison among cases.

CHAPTER IV: RESULTS

This chapter summarizes how subject-specific materials introduce students to proofs in geometry, describes how teachers use curriculum materials to develop students' learning to prove, and identifies factors that influences how proof is taught. Initially, a textbook analysis of Chapter 2-4 of *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions was conducted, in which frequency of task features and levels of cognitive demands of tasks were coded. Subsequently, data collected from teacher interviews, observation protocols, transcriptions of enacted lessons and teacher artifacts were analyzed, using the conceptual framework, which embodies Mathematical Task Framework (Henningsen & Stein, 1997) and proof schemes framework (Harel & Sowder, 1998), to describe how teachers use curriculum materials to teach proof and identify factors that influences how proof is taught. Therefore, the organization of this chapter is as follow: summarized descriptive statistics of textbook analysis of *McDougal Littell Geometry* and *Prentice Hall Geometry,* a description of how teachers used curriculum materials to teach proof, and identification of factors that contributes to how proof is taught.

How Textbooks Present Segments and Angles, Parallel and Perpendicular Lines to Facilitate Students Learning to Prove

To answer the first research question, how do *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions present proof for segments and angles, parallel and perpendicular lines and congruent triangles to facilitate students learning to prove*,* a textbook analysis was conducted on the task features, and levels of cognitive demand for mathematical tasks in the exercises of Chapters 2-4 in both textbooks.

Afterwards, an analysis was conducted on proof tasks. Multiple researchers coded the tasks for levels of cognitive demands. In most instances (89% inter-rater reliability), the coders agreed. The results suggest that although the textbooks have similar organizational structure, there exist difference in the frequency of proof tasks, levels of cognitive demands of proof tasks, and proof representation used. To present the results as to how *McDougal Littell Geometry* and *Prentice Hall Geometry* facilitate students learning to prove, I discus each textbook tasks features and levels of cognitive demand individually, afterwards I conclude by comparing the proof tasks within the textbooks to highlight possible similarities and differences between them.

Analysis of McDougal Littell Chapters 2-4

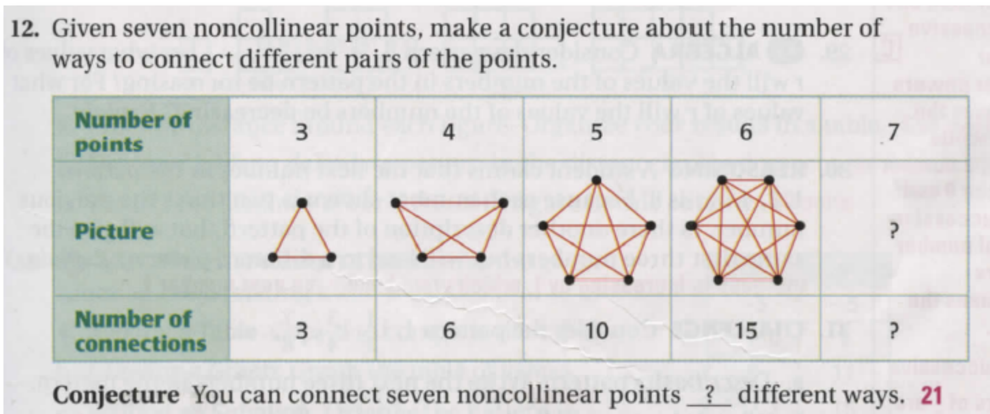
The *McDougal Littell Geometry* seeks to foster students' engagement with adaptive reasoning (Larson et al., 2007, see page T2). Of the 21 sub-sections in Chapters 2-4, 33% of the titles have the word "prove" in their titles, which suggest the authors intend students to have the opportunity to engage in proofs. An analysis of Chapters 2-4, which contains content relevant to reasoning and proof, parallel and perpendicular lines and congruent triangles, revealed that the textbook illustrates various representations of proofs (such as two-column proof, paragraph proof, flow proof, and other), and had tasks that reflected all levels of cognitive demands (Henningsen & Stein, 1997).

Tasks within *McDougal Littell Geometry* were coded for task features and levels of cognitive demand. The classifications for task features and proof representations were based on the tasks and the answers in the Teacher's Edition. Since teachers see the answers, it is highly likely that they will steer students towards that particular path. Hence, the classifications were based on the very explicit and visible features of the problem that both the teachers and students can see, or in the solution provided for teachers, even if the students cannot see it. Solutions in the textbook often used particular proof representations, and stated if the solutions may vary (multiple solution strategy). Thus, the decision to analyze the task in addition to the solution is because of the possibility that teachers may use the textbook as a structural resource for their enacted lessons. In many instances, the two-column proof representation was presented with the statement adjacent to the reason in parenthesis; therefore, although this form was not a standard two-column proof (with a clear line dividing statement and reason), these tasks were still coded as two-column proof. Excluding the mixed review practice exercises, 977 tasks (with solutions) were analyzed using descriptive statistics for task features, and cognitive demand of the tasks. Generally, the tasks posed were abstract in nature and required low-level command demand (*memorization* or the use of a *procedures without connections*). Table 2-4 summarizes the frequency of task features and cognitive demand for each chapter and across chapters using percentages.

In coding tasks, a distinction was made between proof tasks and proof-related tasks. Proof tasks are tasks designed to have students write a proof argument, or complete skeletal proof (such as fill in the blank type proof, or matching statements to appropriate reasons) in which the finished product illustrates a complete proof. Proof- related tasks are tasks that are related to a proof in the sense they are meant to provide students with an opportunity to perform a step that may be used in later proofs and are not necessarily proof tasks by themselves. For example, identifying corresponding sides in a triangle, identifying the congruence criterion that must be used in a given proof, etc. An example of a task coded a proof-related task, and not a proof task is evident in Section 2.1

Question 12 (Figure 19) (Larson et al., 2007, p. 75). This task presents an opportunity for students to pose a conjecture that may result in a proof; however, it does not explicitly require students to construct a proof.

Figure 19. Section 2.1 Question 12 in *McDougal Littell Geometry* (Larson et al., 2007 p. 75).[16]


12. Given seven noncollinear points, make a conjecture about the number of ways to connect different pairs of the points.

Number of points	3	4	5	6	7
Picture					?
Number of connections	3	6	10	15	?

Conjecture You can connect seven noncollinear points _?_ different ways. 21

Chapter 2, entitled "Reasoning and proof", is comprised of seven sections: Use inductive reasoning, Analyze conditional statements, Apply deductive reasoning, Use postulate and diagrams, Reason using properties from algebra, Prove statements about segments and angles, and Prove angle pair relationships. I analyzed 320 tasks. Considering the nature of the title it was not surprising that this Chapter provided a greater opportunity for students to engage in proofs or proof-related activities when compared to the other two chapters. In fact, 77.2% of the tasks were proof or proof-related tasks, 17.2% of which were deemed proof tasks. The proof tasks included paragraph proofs, two-column proofs, and other representation formats. Despite the opportunity to prove, most of the tasks require lower-levels of cognitive demand (86.9%

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as displayed in Table 4).

Chapter 3 focuses on “Parallel and perpendicular lines”, and encompasses six sections: Identify pairs of lines and angles, Use parallel lines and transversal, Prove lines and parallel, Find and use slopes of lines, Write and graph equation of lines, and Prove theorems about perpendicular lines. This chapter presented the least opportunity for students to engage in proofs of the three chapters analyzed. Most of the 306 tasks analyzed were not explicitly proof tasks (since only 5.9% of tasks were proofs as displayed in Table 2). When the opportunity to prove was provided, two-column proof representation, and paragraph proof representation were used in the solutions for the proof tasks.

Congruent Triangles was the primary focus of Chapter 4. This chapter was divided into eight sections: Apply triangle sum properties, Apply congruence and triangles, Prove triangles congruent by SSS, Prove triangles congruent by SAS and HL, Prove triangles congruent by ASA and AAS, Use congruent triangles, Use isosceles and equilateral triangles, and Perform congruence transformations. Within this chapter, two-column proof representations were more readily utilized for proof tasks. Furthermore, this chapter introduced students to flow proof. Despite opportunities for students to prove, memorized facts can be used to solve 31.6% of the tasks, and 40.2% of tasks can be solved using *procedures without connections* as shown in Table 4.

Task Features of Chapters 2-4

Most of the tasks in the textbook, 96.1%, had only one solution strategy. None of the tasks required group collaboration. Graphical illustrations of some kind (e.g. pictures, tables, or figures) accompanied 57.5% of tasks, and 17.2% of the tasks were composed of multiple parts. Thus, students were provided opportunity to visualize mathematical ideas within a particular context, and observe how a mathematical idea can be composed of various components. Most of the tasks were situated in an abstract context (81%), rather than a realistic context (19%). In some instances, the difficulty of tasks was reduced when broken into multiple parts. Although the textbook includes solutions for teachers, the reasoning behind the solutions was not always present. For 34.1% of the tasks, the textbook included answers and the corresponding reasoning; for the rest, only the exact answer appeared. Although 7.6% of the tasks explicitly required students to fill in the blank, and 3.2% of the task required students to select a correct response from multiple-choice items, the tasks posed generally required worked solutions and very few tasks (7.3%) were labeled as "challenge".

Proof and proof-related tasks were common in the geometry textbook. Of the 977 tasks analyzed, 57.2% of the tasks were deemed as proof or proof-related. More specifically, 13.1% of the tasks were classified as proof tasks. Within this group of tasks, the various representations observed in either the tasks themselves or the solutions provided for the tasks included: flow proof, paragraph proof, two-column proof, and other (which did not reflect any of the stated proof representation). Two-column proof representation was most common (see Table 3). Table 2 depicts tasks features in *McDougal Littell Geometry* Chapters 2-4.

Table 2. *McDougal Littell Geometry* Chapters 2-4 (excluding mixed review exercises) tasks features.

Task Features	*McDougal Littell Geometry* Chapters 2-4 N =977	Chapter 2- Reasoning and Proof N= 320	Chapter 3- Parallel and Perpendicular Lines N= 306	Chapter 4- Congruent Triangles N=351
Proof tasks or proof-related tasks	57.2%	77.2%	39.5%	54.4%
Answers				
Answer with reasoning	34.1%	36.9%	19.9%	43.9%
Answer only	65.9%	63.1%	80.1%	56.1%
Proof tasks	13.1%	17.2%	5.9%	15.7%
Labeled as "challenge"	7.3%	8.4%	7.8%	5.7%
Solution strategies				
One solution strategy	96.1%	90.3%	99%	98.9%
Multiple solution strategies	3.9%	9.7%	1.0%	1.1%
Explicitly encourage group collaboration	0.0%	0.0%	0.0%	0.0%
Context				
Real world context	19.0%	23.8%	16%	17.4%
Abstract context	81.0%	76.3%	84%	82.6%
Pictures/tables/ or figures provided	57.5%	38.8%	59.5%	72.9%
Fill in the blank	7.6%	10.6%	5.2%	6.8%
Multiple choice	3.2%	3.1%	2.9%	3.4%
Composed of multiple parts	17.2%	25%	15.4%	11.7%

Proof Representations

Most tasks were not explicitly proof tasks as Table 3 illustrates. Of the proof tasks, two-column proof representation was more readily used. Additionally, Chapter 3, Parallel and perpendicular lines, had the least amount of proof tasks when compared to the other two chapters. Paragraph proofs were more common in later chapters, and flow proof was only used in Chapter 4.

Table 3. Types of proof representations used in Chapter 2-4 of *McDougal Littell Geometry*.

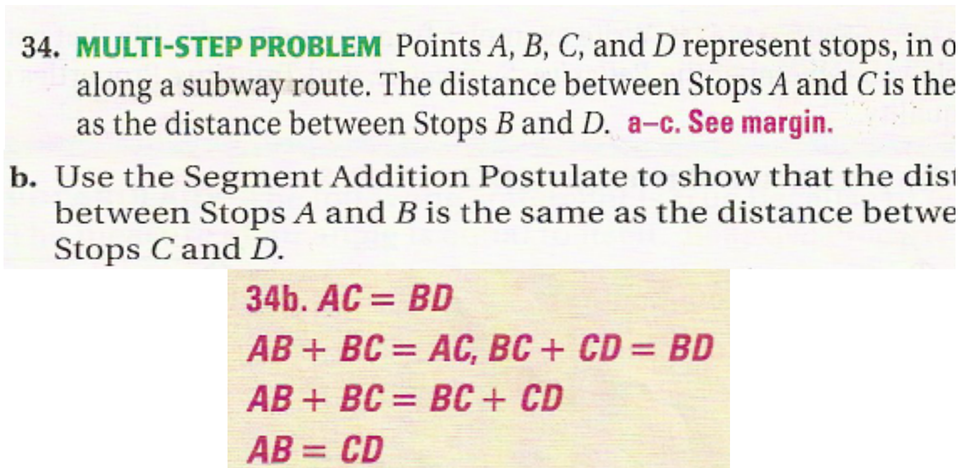
Types of Proof Representation	*McDougal Littell Geometry* Chapters 2-4 N =977	Chapter 2- Reasoning and Proof N =320	Chapter 3- Parallel and Perpendicular Lines N=306[17]	Chapter 4- Congruent Triangles N=351
Flow	0.5%	0.0%	0.0%	1.4%
Paragraph	2.9%	1.9%	1.6%	4.8%
Two-column	9.5%	15%	4.2%	9.1%
Other	0.2%	0.3%	0.0%	0.3%
Not applicable (non-proof tasks)	86.9%	82.8%	94.1%	84.3%

An example of a task deemed "other" is Section 2.5 question 34b (Larson et al., 2007, p. 110). Students are expected to use the segment addition postulate to calculate the distance between multiple points. Since the solution cannot be classified as flow proof,

[17] Percent may not sum to 100% due to rounding.

paragraph proof, or two-column proof, I classified it as "other". This task and its solution appear in Figure 20.

Figure 20. Section 2.5 Question 34b in *McDougal Littell Geometry* (Larson et al., 2007 p. 110) - Task reflecting "other" proof representation.[18]



34. MULTI-STEP PROBLEM Points *A*, *B*, *C*, and *D* represent stops, in o along a subway route. The distance between Stops *A* and *C* is the as the distance between Stops *B* and *D*. **a–c. See margin.**

b. Use the Segment Addition Postulate to show that the dis between Stops *A* and *B* is the same as the distance betwe Stops *C* and *D*.

34b. $AC = BD$
$AB + BC = AC, BC + CD = BD$
$AB + BC = BC + CD$
$AB = CD$

Cognitive Demand of Tasks

Most of the tasks in the textbook require lower-levels of cognitive demand. The tasks generally promote *memorization* of mathematical facts and the use of procedures. Three other coders and I coded tasks, with respect to levels of cognitive demands. We had an inter-rater reliability agreement of 89%. The coders discussed their disagreements and a consensus was always reached. A secondary analysis of the tasks were conducted by myself and a new coder due to the refinement of the *doing mathematics* proof tasks. Due to the conservative approach as to what constitute *doing mathematics* proofs, some of the proof tasks, although still viewed as requiring higher cognitive demand, were reduced from *doing mathematics* to *procedures with connections*. Nevertheless, the secondary analysis were rather consistent overall with the level of cognitive demands of mathematics tasks posed. Of the 977 tasks, 34.2% of tasks required *memorization*, 47.3% required the use of *procedures without connections*, 15.8% required the *use of procedures with connections*, and the remaining 2.8% of the tasks was classified as *doing mathematics* (Table 4).

Table 4. Levels of cognitive demands of mathematical tasks in Chapter 2-4 of *McDougal Littell Geometry*.

Levels of Cognitive Demands	*McDougal Littell Geometry* Chapters 2-4 N =977[19]	Chapter 2- Reasoning and Proof N =320	Chapter 3- Parallel and Perpendicular Lines N=306	Chapter 4- Congruent Triangles N=351
Lower-level demands (*Memorization*)	34.2%	38.8%	32.4%	31.6%
Lower-level demands (*Procedures without connections*)	47.3%	48.1%	54.6%	40.2%
Higher-level demands (*Procedures with connections*)	15.8%	10.3%	12.4%	23.6%
Higher-level demands (*doing mathematics*)	2.8%	2.8%	0.7%	4.6%

Proof Tasks

In Chapters 2-4 of *McDougal Littell Geometry*, 128 tasks were classified as explicitly proof tasks. Reasoning was provided for 88.3% of the proof tasks. Providing an explanation, justification or supportive reasoning was readily encouraged when doing

[19] Percent may not sum to 100% due to rounding.

proofs. In many instances, the tasks required students to prove theorems known to be true to them, rather than present novel and innovative situations in which original proofs were required.

Most of the proof tasks (95.3%) were situated in an abstract context rather than a realistic context (Table 5). All proof tasks admitted only one solution strategy. While almost two-thirds (64.1%) of the proof tasks included a picture, table or graphic of some sort. Table 5 summarizes the task features of proof tasks in these chapters.

Table 5. Task features of proof tasks in Chapters 2-4 of *McDougal Littell Geometry*.

Task Features	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128	Chapter 2- Reasoning and Proof N= 55	Chapter 3- Parallel and Perpendicular Lines N= 18	Chapter 4- Congruent Triangles N=55
Answers				
Answer with reasoning	88.3%	81.8%	88.9%	94.5%
Answer only	11.7%	18.2%	11.1%	5.5%
Labeled as "challenge"	11.7%	10.9%	11.1%	12.7%
Solution strategies				
One solution strategy	100.0%	100.0%	100.0%	100.0%
Multiple solution strategies	0.0%	0.0%	0.0%	0.0%
Explicitly encourage group collaboration	0.0%	0.0%	0.0%	0.0%
Context				
Real world context	4.7%	5.5%	0.0%	5.5%
Abstract context	95.3%	94.5%	100%	94.5%
Pictures/tables/ or figures provided	64.1%	43.6%	66.7%	83.6%
Fill in the blank	10.2%	14.5%	11.1%	5.5%
Multiple Choice	0.8%	1.8%	0.0%	0.0%
Composed of multiple parts	18.8%	25.5%	33.0%	7.3%

The 128 proof tasks, proof representations, were classified as: flow, paragraph, two-column, or other. Most of the tasks used two-column proof representation. Table 6 summarizes the type of proof representation presented in the tasks, or implied by the solutions to the tasks, as presented in the Teacher's Edition of the textbook.

Table 6. Proof representations used for proof tasks in Chapter 2-4 of *McDougal Littell Geometry*.

Types of Proof Representation	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128[20]	Chapter 2- Reasoning and Proof N =55	Chapter 3- Parallel and Perpendicular lines N=18	Chapter 4- Congruent Triangles N=55
Flow	3.9%	0.0%	0.0%	9.1%
Paragraph	21.9%	10.9%	27.8%	30.9%
Two-column	72.7%	87.3%	72.2%	58.2%
Other	1.6%	1.8%	0.0%	1.8%

Almost two-thirds (64.8%) of the proof tasks were classified as high cognitive demand (*procedures with connections* or *doing mathematics*), as indicated in Table 7. In fact, half of the proof tasks (50%) were deemed *procedures with connections*, whereas 14.8% required students to engage in the thinking practice of the discipline. Proof tasks classified as *procedures without connections* occurred 25%, while the remaining 10.2% of proof tasks were *memorization* tasks. The introduction to proof tasks was primarily low-level in Chapter 2, but the difficulty increased in Chapters 3 and 4. Table 7 summarizes the level of cognitive demand of proof tasks for individual chapters as well as for the total of three chapters.

[20] Percent may not sum to 100% due to rounding.

Table 7. Levels of cognitive demands of proof tasks in *McDougal Littell Geometry*.

Levels of Cognitive Demands	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128	Chapter 2- Reasoning and Proof N= 55	Chapter 3- Parallel and Perpendicular Lines N= 18	Chapter 4- Congruent Triangles[21] N=55
Lower-level demands (*Memorization*)	10.2%	14.5%	11.1%	5.5%
Lower-level demands (*Procedures without connections*)	25.0%	45.5%	16.7%	7.3%
Higher-level demands (*Procedures with connections*)	50.0%	34.5%	61.1%	61.8%
Higher-level demands (*doing mathematics*)	14.8%	5.5%	11.1%	25.5%

Analysis of Prentice Hall Geometry Chapters 2-4

The developers of *Prentice Hall Geometry* (Bass et al., 2004, p. ii) sought to meet content standards of national and state curriculum. With the exclusion of mixed review exercises, 1066 tasks in Chapters 2-4 were analyzed. As in other textbooks, the classification considered both the tasks and solutions. Since teachers can view the solutions, it is likely that they will encourage students to present their responses in a similar way. Of the 19 units within the identified chapters only 10.5% of the titles had the word “proving”. Most of the tasks posed were abstract in nature, had one solution strategy, and required lower-level cognitive demand. Paragraph proof and two-column

[21] Percentages may not sum to 100% due to rounding.

proof representation were used more frequently to represent proof tasks. Almost half of the proof tasks required students to fill in the blank.

Chapter 2, entitled “Reasoning and proof ”, consists of five sections: Conditional statements, Biconditionals and Definitions, Deductive Reasoning, Reasoning in Algebra, and Proving angles congruent. Of the 283 tasks analyzed in this chapter, many (71%) were considered proof tasks or proof-related tasks; 5.3% of the tasks were explicitly proof tasks (as indicated in Table 8). Two-column proof and paragraph proof representation were used within this chapter to depict proof tasks (Table 9). Realistic tasks were posed more frequently in Chapter 2 when compared to the other two chapters. Almost two-thirds (64.3%) of the task required *procedures without connections* (Table 10).

Chapter 3, “ Parallel and perpendicular lines”, is comprised of seven sections: Properties of parallel lines, Proving lines parallel, Parallel lines and the triangle angle-sum theorem, The polygon angle-sum theorems, Lines in the coordinate plane, Slope of parallel and perpendicular lines, and Constructing parallel and perpendicular lines. Proofs was not a primary focus of this chapter. Of 449 tasks analyzed, approximately one quarter (25.6%) of tasks were proof tasks or proof-related tasks (Table 8). In fact, only 3.6% of tasks were proof tasks. Paragraph proof representation was used more frequently in this chapter. Most of the tasks required lower levels of cognitive demands (Table 10).

Chapter 4, “Congruent triangles”, has seven sections: Congruent Figures, Triangles congruence by SSS and SAS, Triangle congruence by ASA and AAS, Using congruent triangles: CPCTC, Isosceles and equilateral triangles, Congruence in right triangles, Using corresponding parts of congruence triangle. Of the 334 tasks analyzed for this chapter, most of the tasks were proof tasks or proof-related tasks (72.8% as displayed in Table 8). Most of the tasks were accompanied by a picture, figure, or table, and were abstract in nature. Approximately two-thirds (65.9%) of the tasks required lower-levels of cognitive demands (*memorization*, or *procedures without connections*) (Table 10). Chapter 4 had more fill in the blank tasks than Chapters 2 and 3.

Task Features of Chapters 2-4

Prentice Hall Geometry often included tasks in an abstract context that required one solution strategy. Slightly more than a tenth (11.3%) of the tasks were labeled as "challenge", and only 7.4% of tasks require students to construct or complete proofs (Table 8). None of the tasks required students to collaborate with peers. Few of the task formats were multiple-choice (3.9%) and fill in the blank (6.2%). A picture, table, or figure accompanied more than one- half (51.3%) of the tasks. Table 8 summarizes the tasks features of *Prentice Hall Geometry*.

Table 8. Prentice *Hall Geometry* Chapters 2-4 (excluding mixed review exercises) tasks features.

Task Features	*Prentice Hall Geometry* Chapters 2-4 N = 1066	Chapter 2- Reasoning and Proof N = 283	Chapter 3- Parallel and Perpendicular lines N = 449	Chapter 4- Congruent Triangles N = 334
Proof tasks or proof-related tasks	52.4%	71.0%	25.6%	72.8%
Answers				
Answer with reasoning	23.0%	21.9%	22.3%	24.9%
Answer only	77.0%	78.1%	77.7%	75.1%
Labeled as "challenge"	11.3%	8.5%	13.6%	10.5%
Task explicitly require students to prove or complete a proof	7.4%	5.3%	3.6%	14.4%
Solution strategies				
One solution strategy	94.9%	90.5%	96.0%	97.3%
Multiple solution strategies	5.1%	9.5%	4.0%	2.7%
Explicitly encourage group collaboration	0.0%	0.0%	0.0%	0.0%
Context				
Real world context	16.7%	33.6%	9.4%	12.3%
Abstract context	83.3%	66.4%	90.6%	87.7%
Picture/tables and or figures provided	51.3%	32.5%	42.5%	79%
Fill in the blank	6.2%	6.7%	2.4%	10.8%
Multiple choice	3.9%	3.2%	4.0%	4.5%
Composed of multiple parts	18.6%	12.7%	20.5%	21.0%

Proof Representations

Paragraph proof representation occurred just as often as two-column proof representation. While in Chapter 2 two-column proof was more prevalent, in later chapters paragraph proof was a more common representation for proof tasks (Table 9). The textbook also used flow proof and other proof representation formats. Table 9 depicts types of proof representation used in *Prentice Hall Geometry*.

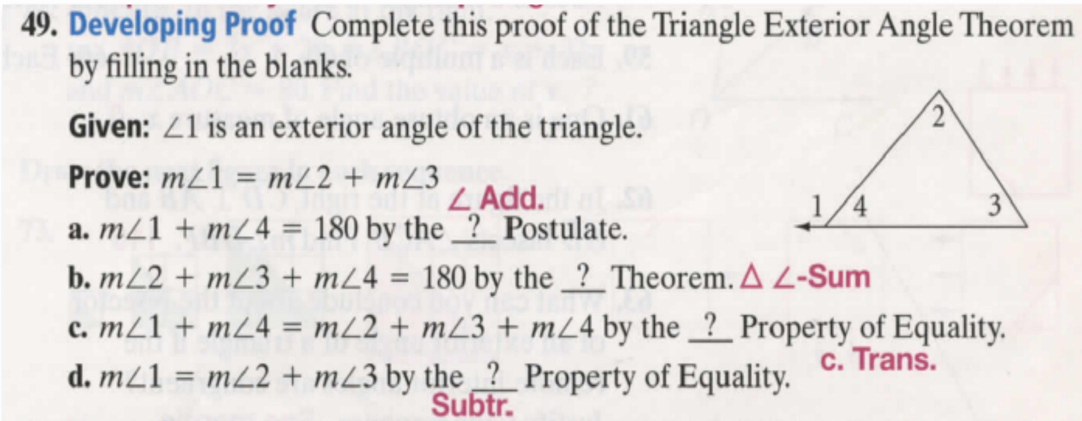
Table 9. Types of proof representations used in Chapter 2-4 of *Prentice Hall Geometry*.

Types of Proof Representation	*Prentice Hall Geometry* Chapters 2-4 N = 1066	Chapter 2- Reasoning and Proof N = 283	Chapter 3- Parallel and Perpendicular lines N = 449[22]	Chapter 4- Congruent Triangles N = 334
Flow	1.3%	0.0%	1.1%	2.7%
Paragraph	3.0%	1.8%	1.3%	6.3%
Two-column	3.0%	3.5%	0.9%	5.4%
Other	0.1%	0.0%	0.2%	0.0%
Not applicable (non proof tasks)	92.6%	94.7%	96.4%	85.6%

The only proof task to be classified as other in *Prentice Hall Geometry* Chapters 2-4, was Section 3.3 Question 49 (Figure 21) (Bass et al., 2004, p. 137) reflecting "other" proof representation. Students have to complete the proof by filling in the blanks.

[22] Percentages may not sum to 100% due to rounding.

Figure 21. Section 3.3 Question 49 in *Prentice Hall Geometry* (Bass et al., 2004, p. 137) - Tasks reflecting "other" proof representation.[23]



49. Developing Proof Complete this proof of the Triangle Exterior Angle Theorem by filling in the blanks.

Given: $\angle 1$ is an exterior angle of the triangle.

Prove: $m\angle 1 = m\angle 2 + m\angle 3$

a. $m\angle 1 + m\angle 4 = 180$ by the __?__ Postulate. ∠ Add.

b. $m\angle 2 + m\angle 3 + m\angle 4 = 180$ by the __?__ Theorem. △ ∠-Sum

c. $m\angle 1 + m\angle 4 = m\angle 2 + m\angle 3 + m\angle 4$ by the __?__ Property of Equality. c. Trans.

d. $m\angle 1 = m\angle 2 + m\angle 3$ by the __?__ Property of Equality. Subtr.

Cognitive Demand of Tasks

Most of the tasks in *Prentice Hall Geometry* Chapters 2-4 required lower level cognitive demands (*procedures without connections*, and *memorization*). Only 2.1% were classified *doing mathematics*. Nevertheless, Chapter 4 included more high-level tasks than the previous two chapters. Table 10 indicates the level of cognitive demand of tasks in Chapter 2-4 in *Prentice Hall Geometry*.

[23] From Prentice Hall Mathematics Geometry Teacher's Edition by Bass, et al. Copyright 2004 © Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

Table 10. Levels of cognitive demands of mathematical tasks in Chapter 2-4 of *Prentice Hall Geometry*.

Levels of Cognitive Demands	*Prentice Hall Geometry* Chapters 2-4 N =1066	Chapter 2- Reasoning and Proof N = 283	Chapter 3- Parallel and Perpendicular lines N = 449	Chapter 4- Congruent Triangles N = 334
Lower-level demands (*memorization*)	17.6%	17.0%	15.8%	20.7%
Lower-level demands (*procedures without connections*)	59.8%	64.3%	67.7%	45.2%
Higher-level demands (*procedures with connections*)	20.5%	18.7%	15.8%	28.4%
Higher-level demands (*doing mathematics*)	2.1%	0.0%	0.7%	5.7%

Proof Tasks

Prentice Hall Geometry had very few proof tasks (N =79). For slightly over a half of the proof tasks (51.9%), the textbook included reasoning for the solutions. Less than half of the proof tasks required students to fill in the blanks (46.8%). In these skeletal proofs students had to complete some steps for the proof. The number of complete proofs that students are asked to write (from start to finish) is very small (N=38). All proof tasks admitted one solution strategy, most (98.7%) of which were abstract in nature. A picture, figure, or table accompanied the majority (87.3%) of the proof tasks. More than half of the proof tasks were composed of multiple parts (55.7%), whereas slightly larger than a quarter (27.8%) were labeled as "challenge". Table 11 indicates the task features of *Prentice Hall Geometry* proof tasks.

Table 11. Task features of proof tasks in Chapters 2-4 of *Prentice Hall Geometry*.

Task Features	*Prentice Hall Geometry* Proof tasks Chapters 2-4 N = 79	Chapter 2- Reasoning and Proof N = 15	Chapter 3- Parallel and Perpendicular Lines N = 16	Chapter 4- Congruent Triangles N = 48
Answers				
Answer with reasoning	51.9%	40.0%	56.3[24]%	54.2%
Answer only	48.1%	60.0%	43.8%	45.8%
Labeled as "challenge"	27.8%	20.0%	50.0%	22.9%
Solution strategies				
One solution strategy	100.0%	100.0%	100.0%	100.0%
Multiple solution strategies	0.0%	0.0%	0.0%	0.0%
Explicitly encourage group collaboration	0.0%	0.0%	0.0%	0.0%
Context				
Real world context	1.3%	0.0%	0.0%	2.1%
Abstract context	98.7%	100.0%	100.0%	97.9%
Picture/tables and or figures provided	87.3%	80.0%	93.8%	87.5%
Fill in the blank	46.8%	60.0%	43.8%	43.8%
Multiple choice	0.0%	0.0%	0.0%	0.0%
Composed of multiple parts	55.7%	73.3%	62.5%	47.9%

[24] Percentages may not sum to 100% due to rounding.

The textbook included various proof representations: flow, other, paragraph, and two-column. Over the three chapters, two-column proof representation and paragraph proof representation appeared just as frequently. Nevertheless, paragraph proof became more common than two-column proof in Chapters 3 and 4. Table 12 summarizes the type of proof representations in *Prentice Hall Geometry* proof tasks.

Table 12. Proof representations used for proof tasks in Chapter 2-4 of *Prentice Hall Geometry*.

Types of Proof Representation	*Prentice Hall Geometry* Chapters 2- 4 Proof tasks N = 79	Chapter 2- Reasoning and Proof N = 15	Chapter 3- Parallel and Perpendicular Lines N = 16[25]	Chapter 4- Congruent Triangles N = 48
Flow	17.7%	0.0%	31.3%	18.8%
Paragraph	40.5%	33.3%	37.5%	43.8%
Two-column	40.5%	66.7%	25.0%	37.5%
Other	1.3%	0.0%	6.3%	0.0%

A little more than half (58.2%) of the tasks were characterized as lower-level cognitive demands. Chapter 4 had more proof tasks that presented opportunities for *doing mathematics* than any of the two previous chapters. Table 13 summarizes how the proof tasks in this textbook were classified according to their level of cognitive demand.

25 Percentages may not sum to 100% due to rounding.

Table 13. Levels of cognitive demands of proof tasks in *Prentice Hall Geometry*.

Levels of Cognitive Demands	*Prentice Hall Geometry* Chapters 2-4 Proof tasks N =79	Chapter 2- Reasoning and Proof N = 15	Chapter 3- Parallel and Perpendicular Lines N = 16[26]	Chapter 4- Congruent Triangles N = 48
Lower-level demands (*memorization*)	46.8%	60.0%	43.8%	43.8%
Lower-level demands (*procedures without connections)*	11.4%	20.0%	25.0%	4.2%
Higher-level demands (*procedures with connections*)	29.1%	20.0%	31.3%	31.3%
Higher-level demands (*doing mathematics*)	12.7%	0.0%	0.0%	20.8%

Comparison of McDougal Littell Geometry and Prentice Hall Geometry Proof Tasks

A comparison of *McDougal Littell Geometry* and *Prentice Hall Geometry* facilitation of students learning to prove revealed that despite similarities in organizational structures of the textbooks, there exist variation between them relative to task features and levels of cognitive demands of proof tasks. Tables were used previously to describe the textbooks task features and levels of cognitive demands individually for proofs, while taking into account the three chapters studied. However, Table 14 reports the proof tasks features, Table 15 illustrates proof representations and Table 16 displays levels of cognitive demands of both textbooks together, to emphasize noticeable differences between them relative to proof tasks.

[26] Percentages may not sum to 100% due to rounding.

Table 14. Comparison of tasks features in *McDougal Littell Geometry* and *Prentice Hall Geometry.*

Task Features	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128	*Prentice Hall Geometry* Proof tasks Chapters 2-4 N = 79
Answers		
Answer with reasoning	88.3%	51.9%
Answer only	11.7%	48.1%
Labeled as "challenge"	11.7%	27.8%
Solution strategies		
One solution strategy	100.0%	100%
Multiple solution strategies	0.0%	0%
Explicitly encourage group collaboration	0.0%	0%
Context		
Real world context	4.7%	1.3%
Abstract context	95.3%	98.7%
Pictures/tables/ or figures provided	64.1%	87.3%
Fill in the blank	10.2%	46.8%
Multiple Choice	0.8%	0%
Composed of multiple parts	18.8%	55.7%

Table 14 highlights that *McDougal Littell Geometry* posed more proof tasks, provided more reasoning for proofs, situated proofs in realistic setting more frequently, and used less pictorial illustration than *Prentice Hall Geometry* for the chapters analyzed. On the other hand, almost half (46.8%) of *Prentice Hall Geometry* proof tasks required students to fill in the blanks, whereas only 10.2% of the proof tasks required students to fill in the blanks. Additionally, more proof tasks were labeled "challenge" in *Prentice Hall Geometry* (27.8%) when compared to *McDougal Littell Geometry* (11.7%). Despite differences, there existed similarities as well. For example, most proof tasks in both textbooks could have been solved using one solution strategy, and did not explicitly require students to work in groups. Nevertheless, proof tasks in geometry textbooks varied in task features.

As it relates to proof representations, Table 15 reports that *McDougal Littell Geometry* was more likely to include proof tasks that required students to use two-column proof representation than *Prentice Hall Geometry*. Conversely, *Prentice Hall Geometry* utilized two-column proof representation just as frequently as paragraph proof representation. Furthermore, flow proof representation was more visible in *Prentice Hall Geometry* than *McDougal Littell Geometry* for the chapters studied.

Table 15. Comparison of proof representations in *McDougal Littell Geometry and Prentice Hall Geometry* for proof tasks.

Types of Proof Representation	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128[27]	*Prentice Hall Geometry* Chapters 2- 4 Proof tasks N = 79
Flow	3.9%	17.7%
Paragraph	21.9%	40.5%
Two-column	72.7%	40.5%
Other	1.6%	1.3%

Table 16 suggests that *Prentice Hall Geometry* was more likely to pose proof tasks that required lower-levels of cognitive demands, whereas *McDougal Littell Geometry* posed more proofs that required higher-levels of cognitive demands. Admittedly, *McDougal Littell Geometry* required students to write complete proofs for more than half of the proofs. On the other hand, almost half of the *Prentice Hall Geometry* proof tasks required students to complete a proof by providing missing information to skeletal proof arguments.

[27] Percent may not sum to 100% due to rounding.

Table 16. Comparison of levels of cognitive demands of proof tasks in *McDougal Littell Geometry and Prentice Hall Geometry.*

Levels of Cognitive Demands	*McDougal Littell Geometry* Chapters 2-4 Proof Tasks N = 128	*Prentice Hall Geometry* Chapters 2-4 Proof tasks N =79
Lower-level demands (*Memorization*)	10.2%	46.8%
Lower-level demands (*Procedures without connections*)	25%	11.4%
Higher-level demands (*Procedures with connections*)	50%	29.1%
Higher-level demands (*Doing mathematics*)	14.8%	12.7%

Hence, the comparison of proof tasks suggest that geometry textbooks can vary in the attention given to facilitating students learning to prove. The results indicates that *McDougal Littell Geometry* was more likely to provide more tasks that required higher-levels of cognitive demands than *Prentice Hall Geometry*.

Geometry Teachers' Use of Curriculum Materials to Teach Proof

To answer my second research question, to what extent do geometry teachers use *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions to teach proof for segment and angles, parallel and perpendicular lines and congruent triangles to facilitate students learning to prove, I conducted a qualitative analysis of the data collected from the three participants in the study. Data were coded according to the Mathematical Task Framework (MTF) (Henningsen & Stein, 1997) and the proof schemes framework (Harel & Sowder, 1998). Due to the refinement of a more conservative definition of *doing mathematics* proof tasks, after the data were collected, a secondary coding of the enacted lesson was conducted. To present those results, each teacher's teaching practices related to proofs, the level of cognitive demand of tasks used, and the proof schemes observed were summarize. At the end, the results of all the teachers were taken collectively, to examine a possible relationship between the levels of cognitive demand and the subsequent proof schemes utilized.

The three teachers studied, taught mathematics for at least three years, with a specialized concentration in geometry. Mrs. Davis and Mrs. Bethel (which are pseudonyms) taught at an urban school that used *Prentice Hall Geometry*, and were on an 8 weeks schedule to teach Chapters 1- 4, while Mr. Walker taught from *McDougal Littell Geometry* at a rural school, in which he took 12 weeks to complete the same chapters. The number of observed lessons varied among teachers due to scheduling limitations. All of the teachers taught at the same time and I was only able to observe lessons twice a week. Mrs. Davis and Mrs. Bethel's lessons were allocated 88 minutes, and Mr. Walker's lessons were allocated 75 minutes. Since, I observed Mrs. Davis's lessons 6 times, Mrs. Bethel's lessons 8 times, and Mr. Walker's lesson 13 times, in total, I observed high school geometry teachers, teach proof or proof-related content for 2,207 minutes, which is approximately 37 hours.

The teachers in this study used their geometry textbooks to facilitate students' learning to prove. Students were encouraged to memorize a list of reasons to fill in the missing part of skeletal proofs. Students were also encouraged to remember procedures to complete proofs. For instance, students knew the given should be on the first line of the proof, and what needs to be proven should be on the last line, while key words and phrases suggested what should be in the middle. Using instructional practices for teaching proof that required low cognitive demand was not due to randomness, but rather due to teachers' decisions to provide students an opportunity to achieve success while doing proofs. The teachers in this study were mindful that most students had limited, if any, experience with proofs before their geometry class. Hence, memorizing a list of reasons and procedures provided a means to introduce students to mathematical proofs, while ensuring students would not be frustrated. In many instances, tasks that were originally classified as requiring higher-level cognitive demand (namely, *procedures with connections*) were enacted as *procedures without connections*. None of the teachers posed tasks that reflected *doing mathematics.* In most of the observed lessons, teachers used the textbook to pose low-level cognitive demand tasks, which facilitated students to develop *external conviction proof schemes*. Although teachers had good intentions (which was to facilitate learning), the guidance they offered during whole class discussion often reduced the level of cognitive demand of potentially richer tasks. Admittedly, teachers talked less and allowed students to work independently, when enacted tasks reflected higher-levels of cognitive demand. Whenever higher cognitive demand tasks were posed, there was a greater likelihood that *analytical proof schemes* were evident. Generally, the classroom norms facilitated that students had the opportunity to learn the mathematical content. Supportive reasoning was often emphasized for each step of the proof, and the classroom community often created norms about how to communicate definitions, postulates, and theorems.

Mr. Walker

Mr. Walker taught mathematics, more particularly, geometry for five years. He pursued an undergraduate degree in statistics, and subsequently earned a Master's degree in Mathematics Education. This is the only school he has worked at, and he expressed satisfaction in teaching geometrical concepts to students.

Due to my observations, Mr. Walker is a teacher who desires to improve his practice as an educator. At the end of most lessons, he shared with me his reflections on his instructional practice, and the means to improve students' disposition toward proofs and geometry as a whole. He expressed a keen interest in understanding the proof schemes framework. He acknowledged that he looks for additional resources in the Internet, or creates his own tasks. His drive to improve his practice is encouraged by the principal, who Mr. Walker claims often challenge teachers to facilitate critical thinking within their respective discipline. Mr. Walker is the chair of the mathematics department and often collaborates with his fellow math educator on writing worksheets and assessment goals. He acknowledged on multiple occasions that he and his colleague writes and share proof tasks to supplement the proof tasks in their textbook.

Mr. Walker desired for his students to learn to reason effectively, and emphasized that the order matters in how a proof argument is presented. He gave students a list of 28 reasons and regularly quizzed students about the content on the list. The list included definitions, properties of basic operations, properties of equality (such as reflexive and symmetric), theorems about congruent, and segment and angles postulates. He stated, " I think the biggest thing is just kind of staying on them and pressing them to explain their reasoning. You got to do that over and over again" (November 3, 2011- Follow up interview at the end of the lesson). Furthermore, he wanted students to recognize patterns while doing proofs. According to Mr. Walker,

> So I want them to see patterns. Seeing patterns is not a bad thing, because you're still thinking. They still have to think on their own and it's not completely mindless … I want to know how they look at things like, okay, when I see that line, they share. It reflex…I can always use that when I see vertical angles. I can always prove those congruent so a lot of the steps repeat themselves throughout the proofs and I want them to pick up those patterns. (November 10, 2011- Follow up interview at the end of the lesson)

The classroom routine began with a warm-up activity, homework review, introduction to the lesson, and subsequently have students work in groups on a practice activity. In many instances, the supplementary tasks Mr. Walker posed increased opportunities for students to engage in higher cognitive thinking. Students were first required to prove a theorem, before they could use it as supportive reasoning in a future proof. Mr. Walker stated, " I explained that once we prove a theorem is true we can then use it whenever we see fit" (September 29, 2011- Task reflection sheet-after implementation-sent via email). He asked students to work in groups to construct proof arguments for proof on cards, or organize shuffle proof arguments to create logical arguments to exchange with other groups. For example, on October 6, 2011 (at the beginning of the lesson), he told students "so we're going to have seven problems… because if you've got no card, if somebody else is done, they're waiting. So make sure you are working". Mr. Walker assigned a class project in which students had to design a city with buildings that had to preserve stated relationships between perpendicular and parallel lines. Some students drew the town; others created physical models of a town, while one student created the town using gaming software. Although Mr. Walker's whole class instruction often reduced the level of cognitive demand of tasks to *memorization* or *procedures without connections*, I observed that when students worked in groups higher cognitive thinking was evident as well as *analytical proof schemes*. Mr. Walker sought to engage students in doing proofs, by assigning higher cognitive demand proof tasks that he made with his colleague.

Mr. Walker often supplemented the textbook with additional proof tasks, and had students provide justification for given statements. He was mindful that some students had a negative disposition towards proofs, but he would often encourage them to learn the list of reasoning as a means to become proficient at proof propositions. Based on conversations with Mr. Walker, his deviation from the textbook was due to his desire to pose more higher-level cognitive demand tasks. He acknowledged that the textbook had limitations, and he tried to overcome them. Mr. Walker remarked, "I guess, there's just not enough like, if I look in this section in the book there's one, there's two proof of how we want them to be thinking about like" (November 3, 2011- Follow up interview at the end of the lesson). He also noted that sometimes the order in which content is presented in the book might not be logical, so his goal was to ensure the content progressed logically. Mr. Walker commented, "…the book…gives you a bunch of information, but they don't really try to connect it to what you're going to be doing in the future…so when I make my notes I want to connect this …" (November 3, 2011- Follow up interview at the end of the lesson). Mr. Walker used the textbook to assign homework, and structure the lesson. If he deviated from the textbook, the tasks he used aligned with the lesson objective of the textbook, and were meant to emphasize proofs.

Cognitive Demand of Tasks during Mr. Walker's Enacted Lessons

Many of the tasks enacted in Mr. Walkers' lesson were higher cognitive demand tasks. Of the 13 lessons observed of Mr. Walker's teaching of geometry, Table 17 indicates the level of cognitive demand of tasks for the original, planned, and engagement with the task during the enacted lesson as documented on the observation protocol. The original task depicts task as written in the curriculum materials, the planned task is the teachers stated intention of how they intend to use the task during the lesson, and the engagement with the task is how teacher actually use the task during the enacted lesson. In three lessons there existed multiple levels of cognitive demands for the various tasks posed. The shift from the original tasks to engagement with tasks suggest that when enacted the level of cognitive demand was reduced. It further suggests that half of the tasks Mr. Walker posed reflected *procedures with connections.* It is interesting to note that Mr. Walker utilized *McDougal Littell* geometry, and based on the results for question, 50% of the proof tasks were *procedures with connections*.

Table 17. Levels of cognitive demands observed during 13 of Mr. Walker's geometry lessons.

Mathematical Tasks in Relations to the Levels of Cognitive Demands	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
Original Tasks	2	6	8	0
Planned Tasks	2	6	8	0
Engagement with the Tasks during the Enacted Lesson	2	8	6	0

An example of a proof-related task that reflects *memorization* reads, "Which postulate allows you to conclude that there is exactly one plane B that contains X, Y, and Z? Explain your reasoning" [28] (excerpt from Section 2.4 McDougal Littell resource material). Mr. Walker's told the students,

> So the way I would think through this is that we've got three non collinear points, S, Y, and Z, and we're told that three non-collinear points contain exactly one plane. So which postulate says three non-collinear points contain exactly one plane? That should be postulate 8. Right. (September 22, 2011- Enacted lesson)

His *memorization* tasks often required students to restate postulate, theorems, and rules. He believed that, in order for students to prove, they must know a list of reasons.

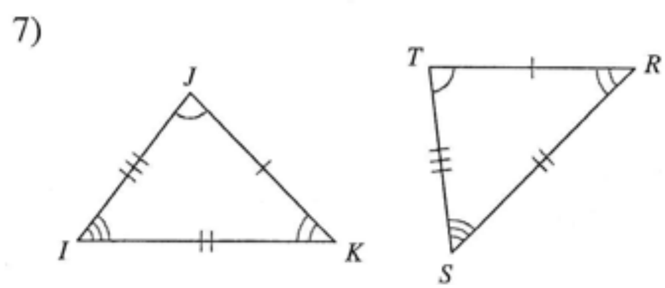
[28] From McDougal Littell Geometry, Teacher's Edition, by Larson, et al.

For example, he reminded students that the definition of angle bisector could be used to prove that, if an angle is bisected, the two angles formed are congruent. Mr. Walker said, “Good, definition of angle bisector. So this is on your list of 28 items. Basically what the definition of an angle bisector just says; it’s a ray, or a line, or a segment that divides and angle into two congruent triangles” (October 18, 2011- Enacted lesson). He readily referenced the list as a tool to identify appropriate reasoning to support claims made.

Writing statements about congruent triangles often were *procedures without connections*. The order in which the second triangle was labeled had to correspond with the congruent sides and angles of the first triangle. For instance, on November 3, 2011, he posed the following task he created (Figure 22), which is similar to practice exercises in Section 4.2.

Figure 22. Mr. Walker’s *procedures without connections* task.

Write a statement that indicates that the triangles in each pair are congruent



During his whole class discussion, he stated,

> Number 7 says write a statement that indicates the two triangles and each pair are congruent. What we are looking for is just to make that congruent statement. First triangle, you can write the letter in any order doesn’t matter. I wrote I, J, K. But the order that the next letters come in is important they

> have to correspond. Which vertex corresponds to I? S, what about J? T? And then for K, as they are. So just match up the corresponding parts that's all I am asking you to do on that section. (Mr. Walker, November 3, 2011- Enacted lesson)

Other tasks posed that I classified as *procedures without connections* required students to place marking on the diagrams to identify corresponding sides and angles, solving equations, and drawing diagrams. For example, Mr. Walker said,

> We've got a lot of problems with segments and whenever we do a proof with segments, and we're going to have to set an equation, there are usually two things that are going to help us set up an equation. With segments, it's either that constant to midpoint or the segmented addition postulate. With angles, it's the exact same thing except instead of, you usually have a midpoint of an angle but we've got angle bisectors so we could use an angle bisector to set up an equation or the angle addition postulate. So you're going to have to look at the given information and kind of decide which of these can I use to set up an equation. Let's keep that in mind. (September 8, 2011- Enacted lesson)

Figure 23 shows an example of a proof he used to illustrate the procedure of using the segment addition postulate on September 8, 2011. This was categorized as a task of *procedures without connections*.

Figure 23. Mr. Walker's proof tasks used to illustrate segment addition postulate.

Given: AC = BD

Prove: AB = CD

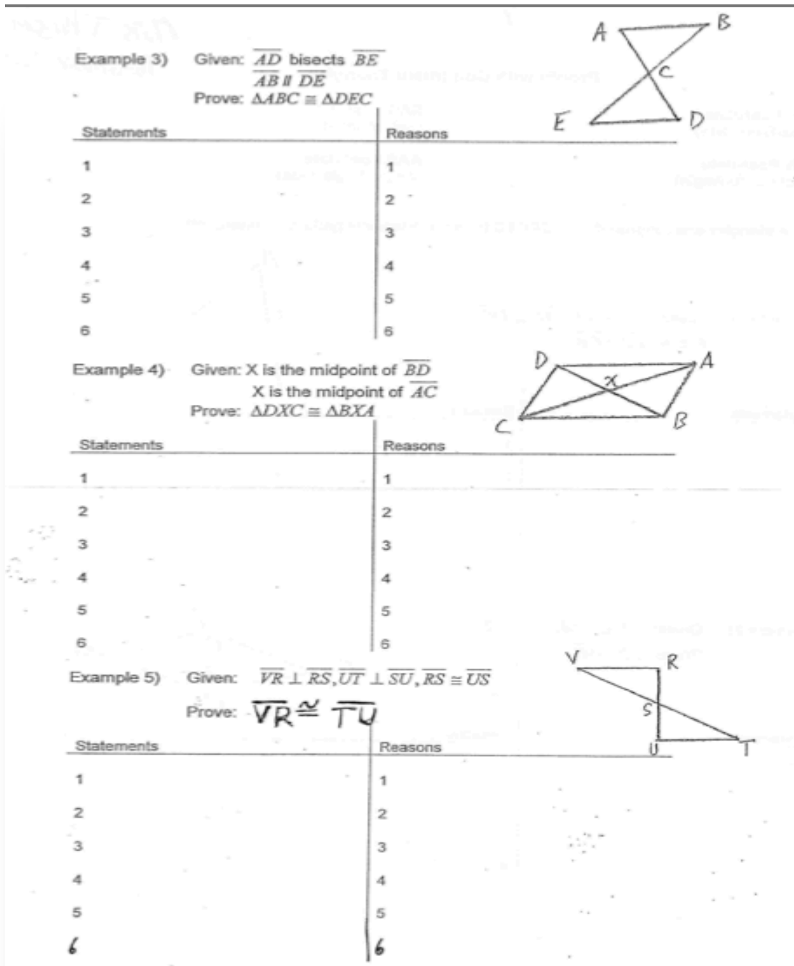
Proof

Statements	Reasons
1. AC = BD	1. Given
2. AB + BC = AC	2. Segment addition postulate
3. BC +CD= BD	3. Segment addition postulate
4. AB + BC = BC +CD	4. Substitution
5. AB = CD	5. Subtraction property

Among the tasks that Mr. Walker posed involving *procedures with connections*, he asked students to organize shuffled proof statements and reasons to make logical proof arguments, and he assigned projects in which students had to construct a town that preserved the placement of buildings in relations to parallel and perpendicular lines, or write a story that logically links 10 conditional statements. He was mindful that thought must be given to sentence structure to ensure the accuracy of proofs is preserved.

Additionally, he posed tasks in which students had to write complete proof arguments. Based on conversations with him at the end of the lessons, he constructed most of these tasks to supplement the textbook. Mr. Walker admitted that he strongly believed, that the textbook did not provide sufficient amount of tasks in which students were required to write complete proofs. It was interesting to observe that most of the proof tasks he used required six or fewer steps and used the two-column proof representation. Figure 24, is an example of a proof task Mr. Walker wrote (November 15, 2011- Teacher artifact) to complement Section 4.6- Use congruent triangles.

Figure 24. Proof tasks Mr. Walker wrote that reflected *procedures with connections.*



Example 3) Given: $\overline{AD}$ bisects $\overline{BE}$

$\overline{AB} \parallel \overline{DE}$

Prove: $\triangle ABC \cong \triangle DEC$

Statements	Reasons
1	1
2	2
3	3
4	4
5	5
6	6

Example 4) Given: X is the midpoint of $\overline{BD}$

X is the midpoint of $\overline{AC}$

Prove: $\triangle DXC \cong \triangle BXA$

Statements	Reasons
1	1
2	2
3	3
4	4
5	5
6	6

Example 5) Given: $\overline{VR} \perp \overline{RS}, \overline{UT} \perp \overline{SU}, \overline{RS} \cong \overline{US}$

Prove: $\overline{VR} \cong \overline{TU}$

Statements	Reasons
1	1
2	2
3	3
4	4
5	5
6	6

Proof Schemes in Mr. Walker's Classroom

In Mr. Walker's classroom, *external conviction proof schemes* and *axiomatic proof schemes* were frequently observed, while *empirical proof schemes* were observed less frequently. *External conviction proof schemes* were observed in nine lessons, *analytical proof schemes* in eight of the lessons, and one lesson of *empirical proof schemes*. Based on my observation, students often viewed the textbook and teacher as the source of truth, and understood the difference between postulates, and theorems. Furthermore, Mr. Walker required students to first prove theorems in order to use them, which appears to have strengthened students' *axiomatic proof schemes*.

External Conviction Proof Schemes

During whole class instruction, the teacher and the textbook were the mathematical authority of the mathematics. According to Harel and Sowder (1998), "when students merely follow formulas to solve problems they learn… prescriptions …and when the teacher is the sole source of knowledge students are unlikely to gain confidence…These learning habits are believed to lead to…*external conviction proof schemes*" (p. 245). The teacher prescribed specific strategies for proof propositions. Mr. Walker led most of the whole class discussions and was the bearer of knowledge. He often gave students rules to use in their homework assignments. Students often preferred that he give them the proofs rather than do them themselves. Hence, *authoritarian*, *ritual* and *symbolic* proof schemes were observed.

Authoritarian Proof Scheme.

The *Authoritarian* proof scheme was evident in many lessons. Harel and Sowder (1998) wrote, "students…are not concerned with the burden of proof; their main source for conviction is a statement appearing in a textbook or uttered by a teacher. Such a conception of proof we call *authoritarian*" (p. 247). In 9 of 13 lessons authoritarian proof schemes were evident. For instance on November 10, 2011, Mr. Walker posed a proof task that required students to write a proof. He said, "all right, I'll get you started" and proceeded to complete the proof in its entirety. Similarly, for a *McDougal Littell* Resource proof-related activity for Section 2.4, he simply told students the answer to the problem, rather than requiring students to provide some insight about the task. The task asked, "Which postulate guarantees that point *Y* is not on line *n*? Explain your reasoning"[29]. To answer the task, Mr. Walker responded,

> …There's really in my opinion, two answers that work. Either postulate 7 or postulate 5. I'll go through the reasoning for postulate 7 first. ….The other way will be to use postulate 5. And that one says, "There is exactly one line between two points", ….. So that's how you could use postulate 5 to reason through proving that fault. Is there any questions on that? All right. Speak up if you do have a question or concern. (Mr. Walker, November 10, 2011- Enacted lesson)

Hence, his solution to the task was the primary source of truth, and the students never challenged it. Students simply recorded his remarks without considering other possibilities, and erased their answer if it did not align with the teacher's answer. Furthermore, it was more common during whole class discussion for students to be told a proof than do the proof themselves. For instance on November 15, 2011 Mr. Walker posed the following task (Figure 25).

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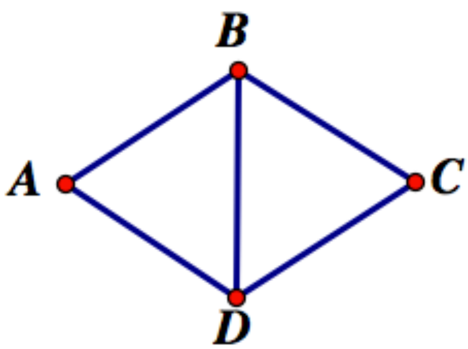
[29] From McDougal Littell Geometry, Teacher's Edition, by Larson, et al.

Figure 25. Proof task posed by Mr. Walker.



Given: $AB \cong BC$, $AD \cong DC$,
Prove: $\angle A \cong \angle C$

The original task that was considered a *procedures with connections* proof task, however, when enacted the level of cognitive demand was reduced because the teacher did the proof for the students. While recording his solution in a two-column proof representation, he stated,

> All right. So here's what I'm looking at when I look at our problem. We're trying to prove A is congruent to C, okay? Notice our end result is not trying to get two triangles congruent. It's trying to get a pair of individual parts congruent, A congruent to C. So what I'm going to be thinking about is, is there a quick way to get angle A congruent to C. If not, then I know if I can get the two triangles congruent to each other, then I can prove these two angles congruent to each other by CPCTC. So I'm going to go through that process, trying to get the triangles congruent to each other. So write down our given. So we're just getting down what we already know. And then I was hoping people last time, I realized that people that are having the most trouble identifying the way to prove a triangle congruent

are the ones who are not doing anything to their pictures. You have got to mark what you know is congruent. AB and BC, AD and CD, make sure you are giving yourself a visual reference; mark them congruent in your picture. All right. What else could we do to try and get these triangles congruent?… So if we knew of an angle bisector that would help us get some angles congruent... BD is congruent to BD by the reflective property of congruence. So now we've got three sides of one in terms of three sides of the other. And if we look at our list of ways that I left on the board, that's one of our six ways to prove a triangle is congruent. So triangle ABD is congruent to triangle CBD by side, side, side congruence. So what I know is if these triangles as a whole are congruent to each other, all of their individual parts, we got the three sides, so we've got enough information to prove those two angles are congruent, those two angles are congruent, and those two angles are congruent. And obviously the ones we are interested in are just angle A and angle C. So angle A is congruent to angle C by CPCTC. (Mr. Walker, November 15, 2011- Enacted lesson)

Students observed the finalized proof and often sought to mirror the teacher's steps when constructing proofs working in their groups. Although some students could construct the proof independently, they would always call the teacher to confirm that their proof was correct, as desired by the teacher.

Ritual Proof Scheme.

Harel and Sowder (1998) stated, "Accepting false proof verifications on the basis of their appearance is a deficiency in one's mathematical education, which is possibly attributable to the over-emphasis in school on proof writing…" (p.246). Mr. Walker's class used *ritual* proof scheme in constructing proofs, especially for congruent triangles. For instance on the right side of the board, Mr. Walker listed the following to prove congruent triangles: Definition of congruent triangles, SSS congruent postulate, SAS congruent postulate, ASA congruent postulate, AAS congruent theorem, and Hypotenuse Leg congruent theorem. Students were expected to use those terms in their proofs. Mr. Walker enforced this ritualistic behavior by drawing reference to this list while writing proofs. For example, the following dialogue between Mr. Walker and his student illustrates how students simply provided short responses, and stated reasons for congruent without considering the validity of the congruence statement.

> Mr. Walker: …Now, do we have enough information to prove the triangle is congruent? Yes or no.
>
> Student: Yes.
>
> Mr. Walker: We sure do. So, triangle A, C, S is congruent to triangle A, R, S, which of these five methods we've learned? Okay? There are two angles and there's one side. Is this side the included side between the two angles?
>
> Student: No.
>
> Mr. Walker: No. So, which method?
>
> Student: That's A, S, A.
>
> Mr. Walker: A, A, S, angle, angle, side. But we have one side and two angles. But it's not in the included side, so it can't be in between them. If

> they were going to be angle side angle, we would have to know these sides were congruent, but we don't know that…. (November 10, 2011- Enacted lesson)

Although Mr. Walker corrected the students it appeared that the student felt the appearance of one of the rules on the board would have been sufficient to make the proof correct, without considering which reason of congruence fits the context of what needs to prove. Mr. Walker was aware that students learning how to write proofs were ritualistic at that times, and that he may have facilitated it. He said,

> …You can learn like one rule that sums up everything. That's more complex than that. It's deeper than that. So I think that they- I think from the little, little kids rely on that ritualistic behavior… Do you remember when I gave you that initial sheet; there are all the proof reasons? If you look on the back, it has little two-step proofs just like I did up here for the warm up. That would be something that would promote them to use a ritualistic approach because it's like when I see this, then I see this. But for some of them, it works do you know what I mean? Like for some of them the only time I'm going to use alternate interior converse is when I've got alternate interior congruent and I'm proving the line is parallel. That is the only time that idea would be used. For some things it works, but I can see how that would hurt them as well. (November 8, 2011- Follow up interview at the end of the lesson)

Mr. Walker's remark suggested that he was aware that students might construct proofs based on appearance and use reasons from the list provided, rather than the accuracy of the proof argument.

Symbolic Proof Scheme.

Harel and Sowder (1998) wrote, "Thinking of symbols as though they possess a life of their own without reference to their possible functional or quantitative reference we call symbolic reasoning… symbolic proof scheme is a scheme by which mathematical observations are proven by… symbolic reasoning" (Harel & Sowder, 1998, p. 250). I observed the *symbolic* proof scheme in Mr. Walker's classroom, in few instances. *Symbolic* proof scheme was used to guide students in constructing proof or writing equations. For instance, Mr. Walker told students "So whenever we're given information, we're going to use that information somehow, okay" (October 6, 2011- Enacted lesson). This remark suggests that the use of symbols and provided information will always be used without regard for the reasoning behind it, the practicality of the usage, and the placement of the information in relation to the logical flow of the argument. He justified such action by saying, "I honestly don't know if that's going to help you get to the end, but you're writing down something that you guys know is true. So, that exactly what we need to do" (Mr. Walker, October 6, 2011- Enacted lesson). Students were attempting to move toward a solution using known facts, without understanding the extent the ideas contributed to the problem; nevertheless, they used it anyway.

Empirical Proof Schemes

Students in Mr. Walker's class used *empirical proof schemes* scarcely. According to Harel and Sowder (1998), "In an empirical proof scheme conjectures are validated impugned or subverted by appeal to physical fact or sensory experiences" (p.252). There was a single incidence of *perceptual proof schemes* and no observation of *inductive* proof schemes.

Perceptual Proof Scheme.

"Perceptual observations are made by means of rudimentary mental images – images that consist of perceptions and a coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation" (Harel & Sowder, 1998, p. 255). Evidence of a *perceptual* scheme was observed in a male student during his presentation of his project, in which he placed building within a community, using guidelines of angle relationship between buildings in relation to perpendicular and parallel lines. He chose to work by himself, rather than in a group. The young man displayed his presentation using SIMS software. The SIMS Software allows students to simulate real world situations using 3D graphics. Users of the software can design neighborhoods, and create storyline, thereby creating life simulation models. The student noted, "it's a life simulation though; it is some what good for building things like that" (October 18, 2011- Enacted lesson). Although his usage of the technological tool was very creative and innovative, it appeared that he had elementary mental images of the underlying mathematical relationships of angles formed by parallel lines and transversals, and intersecting lines. He was unable to explain the implications of what would happen to the relationships between buildings if he moved one building. For example if two buildings originally formed corresponding angles on parallel streets, what will happen if one building is moved on the other side of the street. When asked how did he conceptualize the program, and his presentation, he responded, " I drew that first. I guess that was my plan". The student was unable to describe changes to the angle relationship of buildings if changes were made to his original design of the town. Therefore, although he envisioned a 3-D presentation that can rotate 360 degrees, he was unable to visualize completely the geometrical relationships between the buildings, or draw adequate conclusions about the relationships of the angles.

Analytical Proof Schemes

According to Harel and Sowder (1998), "Simply stated an analytical proof scheme is one that validates conjectures by means of logical deductions" (1998, p. 259). Some students in Mr. Walker's classroom exhibited *analytical proof schemes*, which reflected *axiomatic* and *transformational* proof schemes. Other sub-schemes were not evident in the coded data of the enacted lessons. *Analytical proof schemes* were observed when students worked in groups or independently to write proof arguments, or construct proof arguments by arranging statements and reasons in a logical manner. On multiple occasions students commented that the number of steps could be reduced if you use a theorem that embodied several steps. A few students acknowledge in their group that although Mr. Walker did the proof in a particular order, changing some of the order of the steps would not affect the correctness of the proof argument. *Analytical proof schemes* were observed in 8 out of the 13 observed lessons. The *analytical proof schemes* were more prevalent whenever higher-level cognitive demand tasks were posed.

Axiomatic Proof Scheme.

"When a person understands that at least in principal a mathematical justification must have started originally from undefined terms and axioms (facts or statements accepted without proof) we say that person possesses an *axiomatic proof scheme*" (Harel & Sowder, 1998, p. 273) *Axiomatic* proof scheme was the most common of the *analytical proof schemes*. All of the students assumed that a proof must begin with a given statement and axioms. The students accepted that some statements are accepted in a proof without further justification, and can be used to validate a claim made, and some students incorporated known theorems and axioms in an effort to reduce the length of their proofs. When a student was questioned if the theorem would change the meaning of the proof, he responded “No, because you’re linking the axioms” (November 10, 2011- Enacted lesson). His remark suggested that he was aware of the relationship between axioms, and theorems, and understood that the merit of the proof is based on a set of axioms. This may be due to textbook authors distinction of the terms at the beginning of the book. Section 1.2 of *McDougal Little Geometry* states, “ In Geometry, a rule that is accepted without proof is called a postulate or axiom. A rule that can be proved is called a theorem” (Larson et al., 2007, p. 9). The definition of theorem was restated in Section 2.6 and the textbook authors added that “Once you have proven a theorem, you can use the theorem as a reason in other proofs” (Larson et al., 2007, p. 113). Mr. Walker was enthused that some students recognized that using theorems rather than multiple axioms could reduce the length of a proof. He said,

> …I am happy that some of them are getting that they can be done shorter. A lot of times, they don’t realize that they’re cutting corners. But sometimes they really do come up with ways to do things shorter, which is to me is really doing math…. In this chapter where I have them prove something that takes a bunch of steps, but there’s actually like a theorem we learned that can do it in like two steps. (November 15, 2011- Follow up interview at the end of the lesson)

Therefore, Mr. Walker was mindful that students were aware that proofs are underpinned with axioms, and that theorems can be used to minimize the amount of axioms needed to complete a proof argument. Furthermore, Mr. Walker facilitated the development of *axiomatic* scheme because he reinforced the notion that statements must be supported with appropriate reasoning, and often modeled this behavior during his whole class discussion of teaching proof. For example, Mr. Walker stated,

> That's what we're doing every time we do these proofs. We're basically gathering new information based on an all the information, right? We need to know this phrase right here, CPCTC. It stands for…Corresponding parts of congruent triangle are congruent…so what that's saying is exactly what our definition of congruent triangle says, is that every pair of congruent parts between congruent figures are congruent. (November 15, 2011-Enacted lesson)

His remarks highlights that known information accepted as truth can be used to create new arguments, and support claims made in constructing a proof. Furthermore, his explanation of corresponding parts of congruent triangles are congruent (CPCTC) connects the theorem to a definition; hence he was building on the mathematical language that was previously communicated. Furthermore, the list of reasons that he provided to students fostered students development of axiomatic schemes because it reinforced that proof arguments must begin with axioms, and possibly some undefined terms.

Transformational Proof Scheme.

According to Harel and Sowder (1998) "Transformational observations involve operation on objects and anticipations of the operation results"(1998, p. 259). During the lessons I observed, there was only one instance of a *transformational* proof scheme. A student commented on how the teacher wrote congruent triangles on the board, such that the naming of the second triangle aligned with the corresponding sides of the first triangle. She asked, "Can't you write it backward?" (November 3, 2011- Enacted lesson). She realized that although it was customary that the naming of the second triangle corresponds with the congruent sides of the first triangle, in actuality the naming of the triangle did not matter because it was the same triangle. She acknowledged that as long as the angles and sides of both triangles are congruent it could be deduced the triangles were congruent, and the naming of the triangle did not change that. Her remarks were transformative because she operated on naming the object, and anticipated the implication of how naming the triangles had on the fact that the two triangles were indeed congruent. The student's remarks were indeed a surprise, considering that Mr. Walker emphasized that the order in which congruent triangles are named matters. On November 15, 2011 (during the enacted lesson), Mr. Walker said

> Mr. Walker: …So does the order of these letter matter?
>
> Student: No
>
> Mr. Walker: No, it does not, because if I name that KL or LK, I'm still on the same object. But now if we talk about the triangles being congruent, triangles JKL this order does matter because I have to match up the corresponding parts. J should go with M. K should go with N, and L goes with L. So in this statement the order does matter.

Hence, the female student's observations of the customary practice impact on the overall triangle, was indeed transformative in respect to the established classroom norm.

Relationship between Cognitive Demand and Proof Schemes

Table 18 summarizes the relationship between the codes for the level of cognitive demand, and proof schemes observed. Codes were drawn from the enacted lessons of tasks. In some instances, multiple codes came from a single lesson. For example, the 10 *perceptual* code came from a conversation with a student who used technology to display his project. The sole *transformational* code came from a single student's remarks. Nevertheless, the table suggests that there exists a possible relationship between the level of cognitive demands and the proof schemes observed. These data suggest that when lower-level demand tasks were posed *external conviction proof schemes* were more likely to be observed, while when higher cognitive demand tasks were posed, *analytical proof schemes* were more evident. More particularly, the table highlights that the greatest frequency of codes (18) occurred for the relationship between *authoritarian* and *procedures without connections*. Whereas, the second largest frequency of codes (13) occurred for the relationship between *axiomatic* and *procedures with connections*. Although *empirical proof schemes* did not occur often, it was more evident for *procedures with connections* task. Mr. Walker posed tasks that required various levels of cognitive and reflected multiple proof schemes. The frequency of axiomatic and authoritarian proof schemes suggests that they were the dominant proof schemes in his teaching of geometry.

Table 18. Relationship between cognitive demand of engagement with tasks, and proof schemes observed in Mr. Walker's classroom.

Proof Schemes	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
External conviction				
Authoritarian	5	18	10	0
Ritual	2	12	1	0
Symbolic	1	3	0	0
Empirical				
Perceptual	0	0	10	0
Inductive	0	0	0	0
Analytical				
Axiomatic	4	5	13	0
Transformational	0	1	0	0

Mrs. Lillie Davis

Mrs. Davis's tertiary training focused primarily on mathematical pedagogies. She obtained her Bachelor's Degree in Mathematics Education with an emphasis in Mathematics, and subsequently a Master's Degree in Curriculum and Instruction. Mrs. Davis is passionate about teaching mathematics and often said that she wanted her students to have fun while learning mathematics. She proudly proclaims, "Yes, I love math. Hell, I love math" (September 23, 2011- Enacted lesson). She has taught geometry for 6 years and since *Prentice Hall Geometry* is the sole curriculum that she ever used to teach geometry, she had nothing to compare the curriculum to. She explained that the curriculum team plans the lessons, but there is flexibility to alter lessons if needed. At the beginning of each chapter, students are provided with a chapter outline for the number of days for a particular objective and the assignments from the textbook they will be required to complete. The planned chapter outlines that were distributed by Mrs. Davis skipped one lesson per chapter. The lessons skipped were: Section 2.3 -Deductive reasoning, Section 3.7- Constructing parallel and perpendicular lines, and Section 4.4 – Using congruent triangles: CPCTC. She never explained the reasons for skipping those lessons, and I did not follow up as to why. Furthermore, although the pacing for each lesson generally aligned with the textbook recommendation, in Chapter 3 an extra day was allocated for Section 3.2- Proving lines parallel and Section 3.4- the polygon angle-sum theorems. The handout informed students whether a worksheet would be provided for figures, and whether they will receive a test review packet. Since the class was taught on a block schedule (88minutes per lesson), Chapter 2 was allocated 5 instructional days; Chapter 3 was allocated 10 instructional days, and Chapter 4 was allocated 9 instructional days.

Mrs. Davis admitted that initially she was not great at teaching proof. However, due to advice and guidance from her colleagues, her confidence in teaching proof had significantly improved. Mrs. Davis said, "…when I first starting teaching [proof] I wasn't very good at it. But I would say that I have improved. Just having seen other people and observed other teachers and teaching of proofs and their examples of how they teach…" (September 23, 2011- Follow up interview during lunch break). During the initial interview, Mrs. Davis noted that the proof tasks taught would be low-level, because students were not familiar with proofs. She said,

> … We teach two-column proof, paragraph proof and flow proof. And we would like them to be able to construct it ideally, construct proof on their own from beginning to end and sometimes they can do that with the two-column proof with say segment addition postulate. In the real world a lot of students can't construct a proof from beginning to end on their own, so we will have fill in the blank, two-column where maybe a step is provided on the left and they have to provide the reason on the right, or the reasons on the right with the matching step on the left. (Mrs. Davis, September 2, 2011- Initial interview)

The homework proof tasks assigned were generally fill in the blanks of skeletal proofs, or required students to use the word bank to complete proofs. However, when she provided instructions on proofs, she would often provide proof tasks that required writing proofs. Such tasks were used to model how to fill in the blanks for similar proof tasks. Additionally, the skeletal flow proof she used in the September 23, 2011 for a Do I Remember This (D.I.R.T) activity (a task that requires students to prove the transitive property of parallel lines) had only blank lines and boxes, and required students to consider how to progress logically from the given to the conclusion. It hinted at the amount of steps needed and the relationship between the steps. For such tasks, she reminded students the given goes first, what needs to be prove goes last, and gave hints about how to complete the middle part of the proof. Most of the lessons observed focused on proof-related activities, rather than the writing of proofs, which mirrors the little opportunity for doing proofs provided in the textbook. Thus, most of the tasks observed simply required *procedures without connections*.

Furthermore, she was aware of students' negative disposition to proofs. She noted that students were challenged to support claims. Mrs. Davis said, "Well they have a hard time explaining why they did something. A lot of times students don't like to show their work how did you get to that answer" (September 2, 2011- Initial interview). She also acknowledged that in many instances students do not complete the proof tasks on homework assignments, which suggests that students were generally not intrinsically motivated to engage in doing proofs.

The structure of most lessons observed began with students working on D.I.R.T. assignments. Subsequently, students graded their homework assignment, the teacher taught the lesson, and students practiced the activity for that lesson. If time permitted, students got a head start on their homework. The textbook was used as a source for the homework assignment and to structure the lesson. According to Mrs. Davis, "…the way that the book lays out proofs is how model proof on the test and on our homework" (September 23, 2011-Follow up interview during lunch break). Additionally the geometry team created packets that readily aligned and reflected the content in the textbook such that students can write their responses on rather than writing in the textbook. Overall, Mrs. Davis offloaded the curriculum most of the time (Remillard, 2005), and adapted it based on decisions of the geometry team, or to accommodate students learning. Mrs. Davis willingness to follow the textbook as is, without any objections, may be due in part to the fact that this is the only textbook that she have used to teach geometry. According to Mrs. Davis, "...this is the only geometry textbook that I have seen because it's the only one that we've had while I've been here. We haven't change textbook since I've been here" (September 23, 2011- Follow up interview during lunch break); hence, "I don't know how other textbook teaches it" (September 23, 2011- Follow up interview during lunch break). Therefore, considering this was the sole textbook curriculum she ever used, she had nothing to compare it to. Hence, she chose to follow the textbook during her instructional practices, without any objections or concerns.

In Mrs. Davis class, students often abbreviated definitions and postulates to reduce the amount of writing required. For instance, she said, "I told students that they could abbreviate their classification for polygons even though that was not included on the tasks itself. EL for equilateral, EA for equiangular, CC for concave, etc" (Mrs. Davis, September 24, 2011- Task reflection sheet after implementation-sent via email).

Cognitive Demand of Tasks during Mrs. Davis's Enacted Lessons

Data were collected via classroom observation protocol, audio, and video of six classroom observations. I used audio-recorded data from a lesson I did not observe and supplementary materials and teacher's reflections about lessons. Based on the data sources, the cognitive demands of the tasks were examined, engagement with the tasks, and proof schemes used by the classroom community.

Table 19 summarizes data collected from the observation protocol of the six observed lessons. It suggests that in most lessons, students' engagement with the tasks were primarily low-level as suggested by the teacher; and that it was highly likely that the tasks posed required *procedures without connections*. Furthermore, the table suggests that although original tasks may require higher-levels of cognitive demand the enacted lesson is reduced. It is interesting to note that Mrs. Davis used *Prentice Hall Geometry* which had more than the proof tasks requiring lower-levels of cognitive demands. Similarly, more than half of Mrs. Davis's original tasks reflected lower levels-of cognitive demands. Furthermore, it appears that Mrs. Davis shifted the level of cognitive demand for tasks during her planning and subsequent enactment.

Table 19. Levels of cognitive demands observed during six of Mrs. Davis geometry lessons.

Mathematical Tasks in Relations to the Levels of Cognitive Demands	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
Original Tasks	1	4	3	0
Planned Tasks	1	6	1	0
Engagement with the Tasks during the Enacted Lesson	1	6	1	0

In tasks classified as *memorization*, students had to recall definitions, or fill in the blank (as observed on the homework assignments). An example of a *memorization* task (Figure 26) was posed on September 8, 2011 in the D.I.R.T activity. The task read:

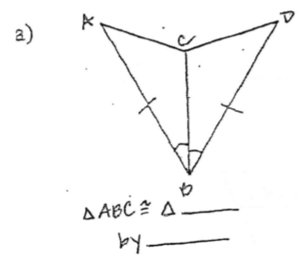
Figure 26. Do I Remember This (D.I.R.T) task posed by Mrs. Davis.

> Name the property that justifies each statement.
>
> If $\overline{JK} \cong \overline{MN}$ and $\overline{MN} \cong \overline{PQ}$, Then $\overline{JK} \cong \overline{PQ}$

Additionally, there were instances in which the teacher reduced the level of the original proof-related task when enacted to reflect *memorization*. For example, on October 11, 2011, a test review task (Figure 27), read:

Figure 27. *Memorization* task posed by Mrs. Davis.

Code and name the postulate or theorem that proves the congruence. If not congruent, write none.



Although, the coding of the task is *procedures without connections* because students have to identify (if possible) congruent sides, congruent angles, and congruent triangles, when assisting a student Mrs. Davis engaged with the task as merely *memorization*. Mrs. Davis said,

> Mrs. Davis: You see those 5 things up on the board
>
> Student: Yeah
>
> Mrs. Davis: Vertical, reflexive…those are the only things you can code.
>
> Student: Oh.
>
> Mrs. Davis: and then depending on your coding you're going to choose is it congruent by SSS, SAS. (October 11, 2011 – Enacted lesson)

Thus, the level of cognitive demand of *procedures without connections* task was reduced to *memorization* because the teacher posted a list of what the codes ought to be.

Sometimes the handouts for proof tasks assigned as homework, modified rich tasks (*procedures with connections*) in the book to tasks that simply required *memorization*. For example, in Section 4.6- Congruence in right triangles, students had to write a brief paragraph proof to explain the congruent relationship between triangles. However, when the teacher assigned the task, she required students to complete skeletal proofs by filling in the blank. Examples of the tasks in the textbook, and the task posed by the teacher are below in Figure 28 and Figure 29 respectively:

Figure 28. Section 4.6- Questions 1-2 in *Prentice Hall Geometry* (Bass, 2004, p.219).[30]

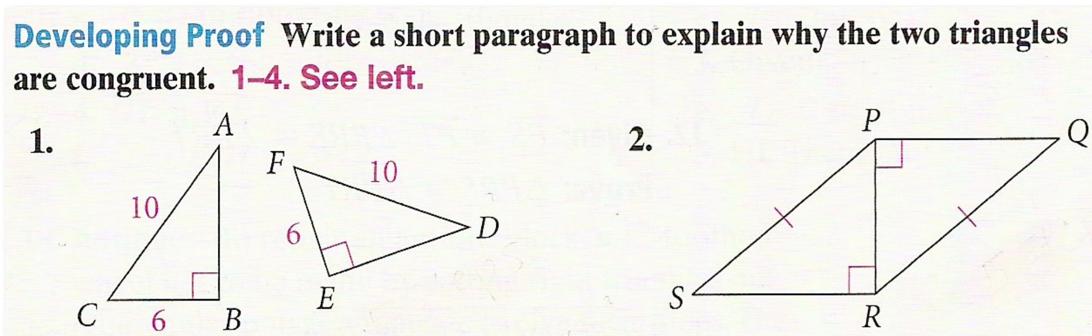
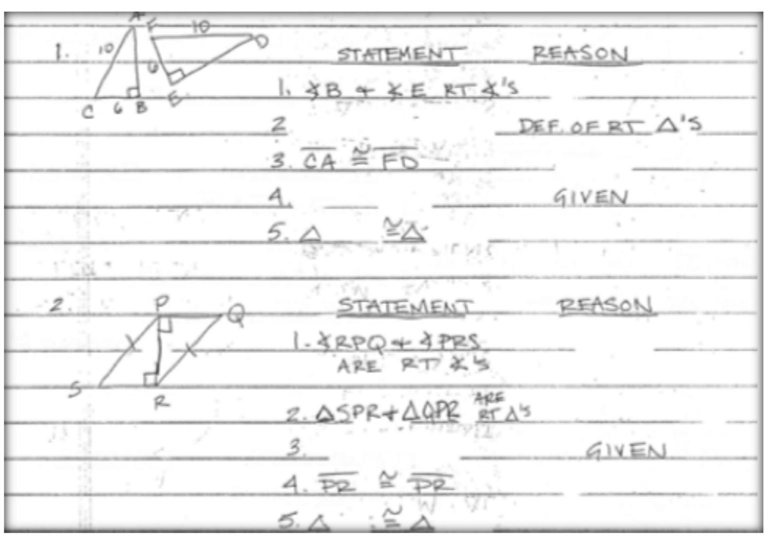


Figure 29. Mrs. Davis's adaption of the *Prentice Hall Geometry* tasks for Section 4.6 Questions 1-2 (Bass, 2004, p.219).



Admittedly the homework tasks were not the primary focus of the observation

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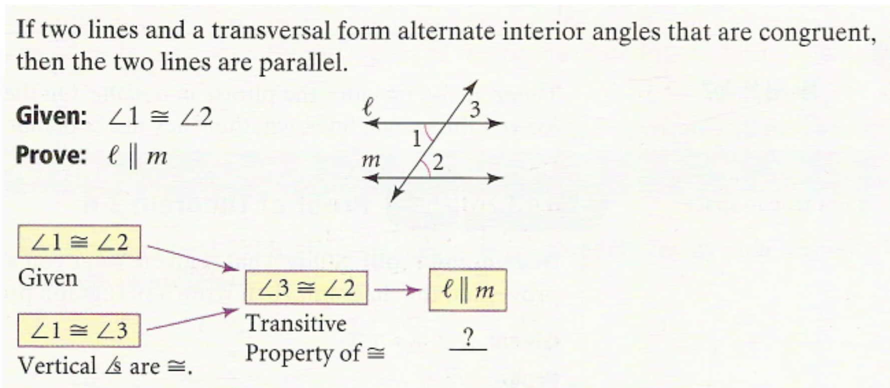
protocol, but it was interesting to note that the reinforcement of low-level tasks were encouraged when students had more time to practice the writing of proofs. Hence, it was deduced that although tasks may originally require higher-level cognitive demand, when enacted the level of difficulty of tasks was reduced.

Tasks that required *procedures without connections* involved finding counterexamples, writing the converse, and conditional statements, solving equations, using the distance formula, coding triangles (when list is not on the board), bisecting angles, classifying polygons, finding the perimeter and area, or using a two-column proof to complete a flow proof. Students were able to attain success on such tasks if they were able to memorize the rules and procedures.

The two-column proof representation was used as the primary proof representation; and was often used as the root while the other representations were considered the branches. Mrs. Davis told students, "So if you ever have to construct a flow proof on your own, I suggest doing a two-column for it first and then cutting it for a flow proof" (September 20, 2011- Enacted lesson). For example, the task depicted in Figure 30 (which is Example 1 in Section 3.2-Parallel lines), required students to construct a flow proof.

Figure 30. Section 3.2 Example 1 in *Prentice Hall Geometry* (Bass et al., 2004, p. 123) - Flow proof task.[31]



To enact the task, Mrs. Davis gave the following instruction to students,

> We're going to take our two-column proof and we are going to make a flow proof. So in the same blank area where I told you to leave room for flow proof, you need to draw this out, we're going to fill it in. This will be your first flow proof. (Mrs. Davis, September 20, 2011- Enacted lesson)

She subsequently discussed the proof. She said,

> …What do you think one of these boxes needs to be? In your given let's just do the top of your given. What was our given? Angle 1 is congruent to angle 2 and then your reason goes below it. What was our reason, angle one is congruent to angle 2? So this box is for what we saw in our picture. What can be concluded from our picture? Angle 1 was congruent to angle three, why? Vertical angles, why do we have two arrows to one box here? It takes the information from these two and combined it into one using one property,

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the transitive…So we have angle 2 is congruent to angle 3 and what's their relationship, corresponding angles which means are two lines are parallel, so converse corresponding. (September 20, 2011- Enacted lesson)

Thus, Mrs. Davis enacted the writing of proofs using particular proof representations as a procedure. She answered her own questions while illustrating how to write the flow proof. The subsequent flow proof posed was rather similar to the examples.

The only lesson observed, where she enacted tasks as *procedures with connections* was the lesson entitled "Classifying polygons". Mrs. Davis gave her students 20 figures and shapes and asked them to select particular characteristics that represent a number of shapes. Mrs. Davis informed students, "In your group, you're going to look at this paper of shapes. You are going to pick one characteristic, and you are going to circle the number by the shape of the shapes that fit your group's characteristic" (September 23, 2011- Enacted lesson). Students chose characteristics such as open figures, logos, symmetrical, stars, hexagon etc. The discussion had students develop their understanding of concave and convex polygons. Therefore, although the classification appeared superficial it was not done mindlessly. Students connected their knowledge of shapes to real world examples; however, the task did not contribute to students developing an understanding of doing proofs.

During the lessons observed, Mrs. Davis did not enact any tasks that could be classified as *doing mathematics.*

Proof Schemes in Mrs. Davis's Classroom

Most of Mrs. Davis' classes facilitated the development of *external conviction proof schemes*. Mrs. Davis was the authority of the mathematics, and students often called her to verify the accuracy of their responses. Students seems more interested in learning the procedures that would help them complete their homework and test. They seem more interested in being told proofs, rather than write proofs by themselves. Hence, Mrs. Davis usually assigned proofs that required students to complete a skeletal proof rather than write proofs in its entirety. *Analytical proof schemes* were observed when students knew that proofs must start with postulates, or theorems. I observed only superficial instances of *analytical proof schemes.* Since students were expected to follow the actions modeled by Mrs. Davis, they rarely engaged in analytical thinking. *Inductive proof schemes*, in which students try to 'prove by examples,' were not observed. It is likely that the teachers' emphasis on the procedures of writing a proof, rather than on the role of proof as validation of a statement, discouraged students from using their intuition or non-standard methods of proof. As a consequence, students did not try to prove any proposition by examining different cases. It is also to be expected that *inductive proof schemes* would be more likely when students are facing statements about numbers or algebraic propositions, rather than geometric theorems, where generating different examples in a timely fashion can only be done through technology (such as Geometer's Sketchpad, or GeoGebra), which was not used by teachers in this study. Overall, the low-level cognitive demand tasks posed by Mrs. Davis generally promoted students development of *external conviction proof schemes*.

External Conviction Proof Schemes

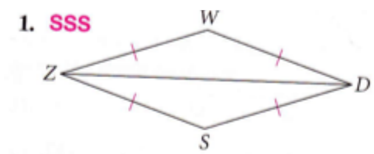
External conviction proof schemes were dominant in Mrs. Davis' instructional practices. The tasks posed, usually drawn from the textbook, seldom encouraged creative thinking, or students discovering new ideas. Students were taught rituals for argumentations, defer to the teacher as the authority of the mathematics, and sometimes treated symbols as if they were independent of a mathematical context.

Authoritarian Proof Scheme.

Mrs. Davis and the textbook were the authority of mathematics; students often called on her or referred to the textbook as a source of guidance. Her instructions always aligned with the instructions published in the textbook, and she determined the correctness of an answer rather than providing an argument to support the answer. For instance on September 6, 2011students were required to write conditional statements and their converse. Although some converse of statements may not necessarily be true, she said, " No I don't want true or false. I just want the statement" (Mrs. Davis, September 6, 2011- Enacted lesson). She continued, "You don't want to do that in this section, we are not there yet". There will be a portion [of the book] where you will change the order" (Mrs. Davis, September 6, 2011- Enacted lesson). Her remarks suggested that the textbook dictates when and how students are exposed to content. Although this was an opportunity to extend students' thinking by considering the validity of the converse, the teacher restricted critical thinking because it was outside the boundaries of the planned lesson, and it would be addressed in a future chapter. A similar incident occurred on October 6, 2011. The task posed was Exercise 1 in Section 4.2- Triangles congruence by SSS and SAS, and is depicted in Figure 31.

Figure 31. Section 4.2 Question 1 in *Prentice Hall Geometry* (Bass et al., 2004, p. 189).[32]

Developing Proof Which postulate, if any, could you use to prove that the two triangles are congruent?



To complete the task, a student wrote the following two-column proof.

Statements	Reasons
1. $\overline{WS} \cong \overline{SZ}$	Given
2. $\overline{WD} \cong \overline{SD}$	Given
3. $\overline{SD} \cong \overline{SD}$	Reflexive
4. $\triangle ZWD \cong \triangle ZSD$	SSS

Instead of encouraging the writing of proofs, Mrs. Davis told the student, "No, you're doing, like, way more than you need to do. But you did it correctly". Based on her remark, the student ceased writing proof and said to his peer and to me, "I am doing too much" (October 6, 2011- Enacted lesson). Therefore, instead of writing proofs he simply stated a congruence theorem. Because she was the authority of the mathematics, her comment guided him on what ought to be done, rather than encouraged him on considering possibilities of what could be done. In describing *authoritarian proof schemes,* Harel and Sowder (1998) wrote, "instrumental understanding rather than

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relational understanding is emphasized throughout the curriculum. As a consequence students build the view of mathematics as a subject that does not require intrinsic justification" (p. 247). Hence, the student did not continue his natural inclination to write proofs, and merely followed the directions of the prescriptive tasks.

The adherence to the textbook, encouraged by the teacher was evident in the lesson. Students placed their textbooks on the desk during each lesson, and would often refer to it when attempting to complete tasks. Since the tasks in the textbook were not rich enough to provide opportunities to engage in actual proofs, students seem to develop misconceptions about proofs. For example, Mrs. Davis noted, "…So, its one thing you're forgetting because right now you only have an angle and the side. We've only written down two things congruent in our proof" (October 6, 2011- Enacted lesson). So in this example, the student could not remember theorems about congruence, and simply relied on the teacher for guidance in constructing the proof. Even when students worked in groups, the correctness of their responses was confirmed by the teacher rather than by their peers. Therefore, *authoritarian* proof scheme was promoted in Mrs. Davis class.

Ritual Proof Scheme.

In Mrs. Davis' classroom, students proved propositions about triangles in a very ritualistic way. Mrs. Davis encouraged students to place marking on the congruent triangles, code the triangles for congruent sides and angles, and conclude with a congruent statement. Whenever the students and Mrs. Davis talked about the proofs, "tick" and "swoosh" were used to refer to marking on the diagrams. She said,

> You can never swoosh or tick what you're trying to prove…if you tick or swoosh what you're trying to prove, this reason is going to be wrong.

> ...Prove the triangles are congruent first. So you code what you're given, then you code anything else you know, vertical, reflexive, alternate interior with parallel lines, the word mid-point or the word bisect. (Mrs. Davis, October 14, 2011-Enacted lesson)

The marking on the diagram was emphasized more than the correctness of the proof argument. Mrs. Davis stated, "If you see a coding like this, we can conclude those two triangles are congruent" (Mrs. Davis, October 6, 2011- Enacted lesson). As a result, students often ensured they placed markings on the diagram, rather than focusing their attention on writing logical ideas. Seldom were students challenged to write a proof in its entirety. Mrs. Davis made clear to her students that the given and what needs to proven should always be included in the proof. She said,

> There are two things that you should never miss in a flow proof. One is the given and one is what you're trying to prove. What you're trying to prove will always go where? At the end, so you should never miss this last box of the flow proof, even if you don't know how to do the proof you should be able to get yourself a point of a test. (Mrs. Davis, September 20, 2011-Enacted lesson)

Her comments suggested the ritual of writing the given and concluding statements without giving students the opportunity to think about the complete proof. The emphasis on the use of the markings and on simple procedures such as writing the given and conclusion, without focus on the development and understanding of the proof, characterized the approach to proof in the textbook and Mrs. Davis's instructional practices when facilitating students learning to prove.

Symbolic Proof Scheme.

In Ms. Davis's classroom, the marking on the congruent triangles varied in usage from being used meaninglessly to demonstrating a deep understanding of why triangles are congruent. In proving triangles are congruent by SSS or CPCTC, Mrs. Davis said, "Make sure they correspond…you have to understand the symbols" (October 14, 2011). Her comment suggested that symbols might be used without reflecting an understanding of the content matter. Admittedly, such practices may have been encouraged by the importance she gave to the use of tick marks. For example, Mrs. Davis said,

> So whenever you see a tick mark, that's going to represent an S for a side for a congruent side. So if they're in 4.1, it says you have three sides of one triangle, bam, bam, bam are congruent to three sides of another triangle, bam, bam, bam, and then the two triangles are congruent. (Mrs. Davis, October 6, 2011- Enacted lesson)

So students may perceive markings are symbolic of congruency or is simply required when doing proofs.

Analytical Proof Schemes

In Mrs. Davis's class, the *analytical proof schemes* observed were generally *axiomatic*. Students were aware that some statements could be used in a proof without being proved (such as definitions, and theorems). Definitions, and theorems used included: complementary angles sum to 90^{o}, reflexive property, definition of midpoint, vertical angles etc. Students knew that proofs had to be supported by definitions and theorems, which is represented in the column for reasoning. For example during a lesson about proving congruence by SSS and SAS, Mrs. Davis said, "The only thing that you're going to know is congruent are the reflexive property or vertical angles. You have either of those in there" (October 6, 2011-Enacted lesson). Additionally, she noted that information given is used and is never questioned. For instance, she said, "…Reason is given. So you need one of those in your given first and then you want to figure out if there's anything else you know about your picture, anything else you think you can include in that". She pointed out that some reasons can be accepted as truth without proof, while other facts can be concluded based on observable information (for example congruent triangles). Overall, *analytical proof schemes* were observed in 3 of the 6 observed lessons, and when it was observed it was rather rudimentary in nature, since most task did not require students to engage in critical thinking to arrive at the correct solution.

Relationship between Cognitive Demand and Proof Schemes

In most lessons, Mrs. Davis posed low-level cognitive demand tasks, which often encouraged *external conviction proof schemes*. Table 20 summarizes the relationship between the level of cognitive demand of tasks and proof schemes. The greatest frequency of codes (43) occurred for the relationship between *ritual* and *procedures and connections*, and *authoritarian* and *procedures and connections* (38) respectively.

Table 20. Relationship between cognitive demand of engagement with tasks, and proof schemes observed in Mrs. Davis' classroom.

Proof Schemes	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
External conviction				
Authoritarian	5	38	1	0
Ritual	7	43	0	0
Symbolic	1	6	0	0
Empirical				
Perceptual	0	0	0	0
Inductive	0	0	0	0
Analytical				
Axiomatic	8	10	4	0
Transformational	0	0	0	0

The table illustrates that *perceptual*, *inductive,* and *transformational* proof schemes were never observed when the class engaged with tasks. Additionally, it suggests that *external conviction proof schemes* occurred most often when tasks posed required *procedures without connections*. Conversely, the frequency of *analytical proof schemes* exceeded the *external conviction proof schemes* when the task(s) posed required a higher-level cognitive demand.

Mrs. Barbara Bethel

Mrs. Bethel has 18 years experience teaching mathematics, 15 years of which were devoted to teaching geometry. She often reflected on her practice and considered what she could do to improve the lesson in the future. Mrs. Bethel believed that proof is vital to mathematics, and sought to promote students development of logical thinking. She acknowledged that proofs taught using two-column representation are very easy because these proofs involve content that students know to be true or false for the most part. In addition to the two-column proof representation, she also exposes students to flow proof and paragraph proof as presented by her geometry textbook. During my observations, when Mrs. Bethel deviated from this textbook, she often used "The Boss/Secretary" activities courtesy of Becky Bride- Geometry- Kagan Cooperative learning series, which are rather similar to the pedagogical emphasis and structure of the *Prentice Hall Geometry*.

Like Mrs. Davis, Mrs. Bethel collaborated with her fellow geometry team members to plan lessons, and to choose tasks for students to complete as homework for each chapter. She normally modeled behaviors that she expected from her students. She provided students with tips that would help them in constructing proofs, and emphasized that the order matters in progressing from one step to another in doing proofs.

Mrs. Bethel class usually began with a warm –up or Do I Remember This (D.I.R.T.) activity, followed by homework review. After this, she taught the lesson of the day and gave students time to do practice problems. If students completed the class assignment before the end of the class, they were allowed to begin their homework assignment.

Cognitive Demand of Tasks during Mrs. Bethel's Enacted Lessons

I observed Mrs. Bethel's classroom eight times during Fall 2011. Writing proofs was the primary focus of only two of these lessons. Other times, most of the lesson focused on either proof-related tasks or non-proof tasks. Despite the small attentions placed explicitly on proof, which reflected the curriculum utilized, I was still able to gain insight into how proof or proof-related content were taught. In these lessons, Mrs. Bethel posed warm up tasks where students were required to complete skeletal proofs (tasks reflecting *memorization*), as a means to review proofs when observed, since most of the lesson focused on proof-related or non-proof tasks. Data collected during these observations (Table 21) suggested that most of the tasks assigned by Mrs. Bethel required *procedures without connections,* or *memorization*. The level of cognitive demand of proof tasks that were designed as requiring higher-level cognitive demand was diminished when these tasks were enacted, since Mrs. Bethel wrote most of the proofs during lessons, and emphasized procedures for completing these proofs. Flow proofs were generally taught as fill in the blank. Furthermore, fill in the blank proof tasks were assigned as homework and practice (just before the end of the lesson). When Mrs. Bethel wrote proofs, she would often give direct indications about what words and phrases should be used to fill in the blank on the practice tasks. Hence, students had little (if any) opportunity to write proofs by themselves during class time. Their engagement with proof tasks was generally limited to providing missing information for skeletal proofs when they worked independently. The textbook was used as the primary source of mathematical tasks, and was used to structure the mathematical lessons. Table 21 summarizes the cognitive demand of the mathematical tasks observed during Mrs.

Bethel's instruction.

Table 21. Levels of cognitive demand observed during 8 of Mrs. Bethel's geometry lessons.

Mathematical Tasks in Relations to the Levels of Cognitive Demands	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
Original Tasks	1	5	3	0
Planned Tasks	1	6	2	0
Engagement with the Tasks during the Enacted Lesson	1	8	0	0

The table suggests that although some of the original tasks, and planned tasks might have demanded higher-level cognitive demand, when enacted the tasks required *procedures without connections*. The shift in numbers, illustrates that a task had the potential to be a *procedures with connections*, when enacted it reflected *procedures without connections*. A task reflecting *procedures with connections* but enacted as *procedures without connections* is drawn from the exercise of Section 3.4 question 66 (Figure 32). The task reads:

Figure 32. Section 3.4 Question 66 in *Prentice Hall Geometry* (Bass et al., 2004, p. 150).[33]

66. The car at each vertex of a Ferris wheel holds a maximum of 5 people. The sum of the measures of the angles of the Ferris wheel is 7740. What is the maximum number of people that Ferris wheel can hold?

The task linked a realistic setting, to the sum of the interior angle of the polygon and required some degree of cognitive thinking; however, when enacted the teacher reminded students of the formula for finding the sum of the interior angles and solved the problem for the students. Mrs. Bethel enacted the task as follow,

> Mrs. Bethel: All right so here's our Ferris wheel. I don't know how many folks are on this thing but that's what I've got to figure out, right? I've got to figure out how many angles we have on that thing before I can do anything else. So here's what I know. Based on formula that we've been working with, we know that the total degrees is N minus 2 times 180. We've been told what the total is so I'm going to set this equal to 7740, okay? So we start by dividing by 180 since you have the calculator out, [Student name], if you would be so kind to say it once more.
>
> Student 1: 43
>
> Mrs. Bethel: 43
>
> Student 1: Plus 2

[33] From *Prentice Hall Mathematics Geometry* Teacher's Edition by Bass, et al.

Mrs. Bethel: Plus 2

Student 2: 45

Student 1: 45

Mrs. Bethel: That's big huh? That's big. Okay sounded as how many folks we're going to have out there, right? Can you visualize that? That's big. Now 45 at the end of each one of those we've got 5 people. [Student name], would be so kind as to continue 45 times 5 is?

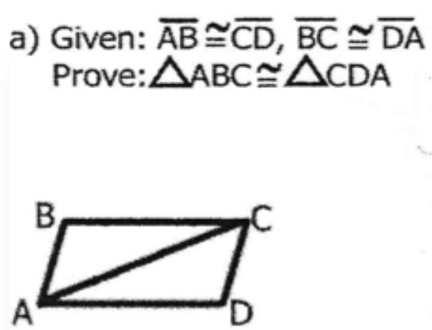
Student 1: 225

Mrs. Bethel: It is 225 people that can ride that Ferris wheel once....

(September 29, 2011- Enacted lesson)

Although the task could have been classified as a *procedures with connections*, the way it was presented by Mrs. Bethel's made the task simply a *procedures without connections*. Similarly, proof tasks that could have been deemed *procedures with connections* were enacted as *procedures without connections.* For example on October 13, 2011 Mrs. Bethel posed a task that required students to prove triangles congruent by SSS. For the most part, she did the proof rather than the students. The task is shown in Figure 33.

Figure 33. Proof tasks assigned by Mrs. Bethel's on October 13, 2011.



Mrs. Bethel: Okay, so I'm going to re-write it, we've got segment AB is congruent to segment CD, and I choose to put the other one in here as well, so segment BC is congruent to segment DA. I don't think there is any reason to put those into separate line, one line is good. I just ran out of space. Okay, that's given, now before I go on, I want to code my picture. So, if AB is congruent to CD. I want to mark it that way, and if BC is congruent to DA, I want to mark that as well.

Student 1: Do you have a reason?

Mrs. Bethel: Yes, we do. Okay, so now Angel is telling us that we've got a reflexive property and you see it. AC is congruent to AC; it's congruent to itself, right. So, we mark it 1, 2, 3, here we go, and it tells the reason again?

Student 2: Reverse

Mrs. Bethel: Absolutely. Okay, so my question is, do we have enough information to prove that these triangles are congruent?

Student 1: Yes.

Mrs. Bethel: How so?

Student 1: Because all sides are….

Student 2: S-S-S.

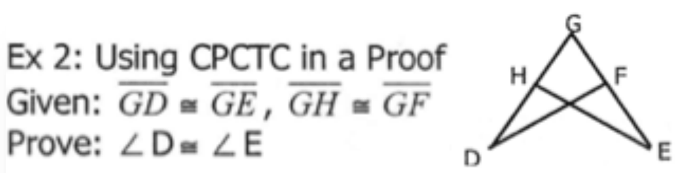
Mrs. Bethel: S-S-S that's right. So, we say triangle ABC is congruent to triangle CDA by side-side-side. Then look at that, pretty cool, huh.

(October 13, 2011- Enacted lesson)

Mrs. Bethel's decision to guide her students throughout the writing of the proof was motivated by her goal that students would recognize shortcuts when writing proofs for these theorems. In fact, as a precursor to proving the task she said, “Yeah, we are looking for some shortcuts, okay. Now, we know how to start our proofs, so we’ll go through this process once more, the first statement, 9 times out of 10 it’s going to be what we have for the given” (Mrs. Bethel, October 13, 2011- Enacted lesson). Mrs. Bethel desired that students recognize patterns in proving and sought to ensure the proving tasks were not difficult to students. This diminished the level of cognitive demand of the tasks that she assigned to her students.

Another example of a task originally deemed *procedures with connections* but enacted as a *procedures without connection*s, was evident on October 25, 2011. For the task, the teacher proved triangles within an overlapping triangle were congruent. The task (Figure 34) was an extension of the example 1 in Section 4.7- Using corresponding parts of congruent triangles (Bass et al., 2004, p. 224).

Figure 34. Proof task assigned by Mrs. Bethel’s on October 25, 2011.



Mrs. Bethel: I'm going to mark both picture so that you can understand sometimes how difficult it can be to look at the picture with them still together. Notice what we have. We had GF congruent to GH. And then we had GD compared to GE. And those were going to be congruent. So it's very hard to show those things when we're talking about that single picture. But if we can kind of pull those things apart, then it makes it a little bit easier for us to follow. Okay, so here's what we got. We knew that GD was congruent to GE, and GH congruent to GF…As we are looking at overlapping triangles then, what we're trying to do is utilize any pieces that are in both the triangles at a given time. So if you notice this, of course, that's the one from straight above here. But we have angle G in this particular triangle already taken care of. So as we look at this, angle G is going to be congruent to angle G. And that's going to be where we use our reflexive. It doesn't always have to be a segment. It can be an angle. So angle G is used in both of those. And then if I take the time to mark that, and I'll do that in a different color, here's what it looks like in the jumbled picture if I pull it out and put it in the little picture, a little easier to follow. And now, take a look at what we know. Do we have congruent triangles? Yeah, we do. So now, and depending on how you want to start this out, I'm going to do GFD, that's my choice. GFD is congruent to – and of course, we've got to get that other one named the same way. So notice what we did. We started in G; we went through the side that was marked with the two-strokes, so this one's going to be GHE. And how are those two triangles congruent? SAS, very good, Side-Angle-Side, okay? So my triangles are now congruent. And the last thing to do is to use a little CPCTC. If the triangles are already congruent, then isn't it true that all corresponding pieces are congruent? Absolutely, so step 4, angle D is congruent to angle E by the soon to be favorite CPCTC. Okay? So it didn't change the difficulty level too much. We just had to deal with that

> picture that was a little bit more complicated. Now on the second one, we won't have overlapping but we'll use our CPCTC again just so we can kind of get the practice with it. (October 25, 2011- Enacted lesson)

While completing the task, Mrs. Bethel told students that the level of difficulty of the task did not change too much; rather it was simply the picture that compounded the situation. Color-coding, and marking on the diagram was a method that she customarily used. The follow up task was rather similar and simply reiterated the skill learned. In all cases, the teacher wrote the proofs, and students simply recorded the proof in their notes. Hence, the level of difficulty of proofs was generally expected to be low-level when enacted by the students.

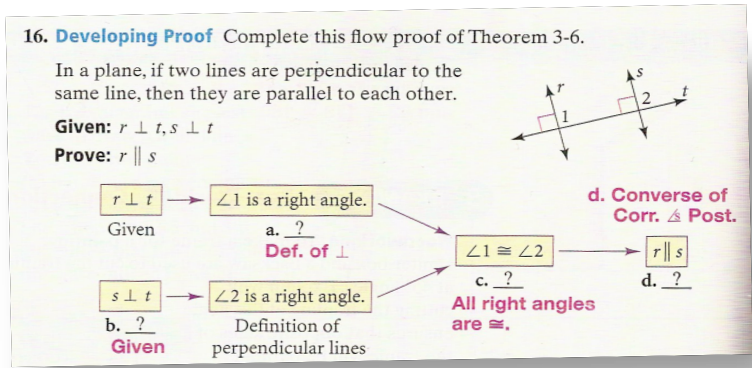
Mrs. Bethel admitted during the initial interview that the proofs taught are elementary and had less than 10 steps. Mrs. Bethel said,

> [We teach] Very basic proofs. Very obvious proofs, and I would say consistently less than 10 steps, never anything complicated but yet trying to still get the main idea across several. The proofs that we do in our geometry are three to six steps and we guide them an awful lot at this stage. (Mrs. Bethel, September 6, 2011- Initial interview)

Tasks deemed *procedures without connections* that were enacted as such included: identifying angle pairs, solving equations, finding measures of unknown angles, coding triangles, using and finding slopes of parallel and perpendicular lines.

A sample flow proof task that required filling in the blank and deemed a *memorization* task was posed on September 22, 2011. The task is depicted in Figure 35 and is drawn from Section 3.2 Question 16 (Bass et al., 2004, p. 126).

Figure 35. Section 3.2 Question 16 in *Prentice Hall Geometry* (Bass et al., 2004, p. 126) - flow proof task reflecting *memorization.*[34]



To enact the task, Mrs. Bethel encouraged students to look for clues in solving the task, and alluded to the fact that the skeletal proof provides guidance on what the missing information can be.

Mrs. Bethel discussion for the task was as follow:

> Mrs. Bethel: …When we're looking at these kinds of things and they're started for us, this is a skeleton. We want to look for clues along the way, so I'm going try to guide you through a couple of these things, maybe you've missed somewhere along the line. If you have a given statement and a proof statement these things need to be inside the proof. So, if I've got two pieces of information that are given, there better be two locations of it down here in my proof. So, you notice this first box right here, R is perpendicular to T, there it is, that was given, but there's one more piece.

[34] From *Prentice Hall Mathematics Geometry* Teacher's Edition by Bass, et al. Copyright 2004 © Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved

So, if it's me, I might go ahead and scan all the way through until I find that last piece, so I can easily complete that part, okay?

Student 1: Right.

Mrs. Bethel: If I move on through this notes that it says, R is perpendicular to T, and then all of a sudden it says angle 1 is a right angle, but down here S is perpendicular to T, and then it says, angle 2 is a right angle, and it tells me the definition of perpendicular that's probably going to give me a big hint … If I have a perpendicular in the step before, and now all of a sudden I'm talking about a right angle that's because I'm using the definition of perpendicular. If two lines are perpendicular, and we know they're going to form a right angle. Now, angle 1 is a right angle, angle 2 is a right angle all of a sudden angle 1 is congruent to angle 2, we did one like this yesterday. What do we know about all right angles? Nothing, we know nothing.

Student 1: On the right angle.

Student 2: They are 90°

Ms. Bethel: They're all 90°, right. All right angles are congruent, you remember that one.

Student 1: Yes.

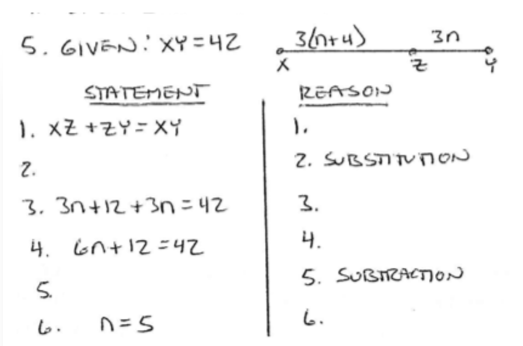
Mrs. Bethel: Because we talked about that yesterday, and there was a little theorem that we learned a little way back that said, all right angles are congruent, and we talked about how easy that would be to prove, because we know that right angle is 90°, and if this one is 90° and so is this one,

> and then they've got to be congruent by definition of congruence. The last thing then, now R is parallel to S, notice what we're looking at here, angle 1 and angle 2 are both congruent. What kind of angles are those? What do we call those?
>
> Student 3: Corresponding.
>
> Mrs. Bethel: Corresponding, very good. So, that's giving us a hint. If R is parallel to S, then that's because we're using the corresponding angles, what's the keyword I have to put here when I'm proving lines parallel?
>
> Student 3: Converse.
>
> Mrs. Bethel: The converse, very good. All right, so when you're going through a proof that's already been skeleton for you, take a look and see if there're any clues in there that you can work with, and try to get with those things first, all right. Fill in the bits and pieces that you know, and maybe spend a couple of minutes thinking about those other things there, all right. (September 22, 2011- Enacted lesson)

It is apparent that in this episode Mrs. Bethel spoke for the bulk of the discussion. When students provided responses, it was usually less than five words, which required the repetition of memorized facts. In all cases, the teacher led the discussions, even when the task posed required simply *memorization* to be completed.

Mrs. Bethel also did reviews for tests, which were generally not related to proofs. Nevertheless, proof-related tasks in these reviews were generally *procedures without connections*, requiring students to fill in the blanks. Tasks that could have been *procedures without connections* were thus reduced to *memorization* for the review activity in which students had to provide missing information (as depicted in Figure 36).

Figure 36. Mrs. Bethel's chapter review proof task posed September 15, 2011.



Overall, the level of engagement of tasks required low-level cognitive demand. Hence, the observed lessons readily aligned with Mrs. Bethel initial interview comments that proof tasks are "very basic" and require limited thinking.

Proof Schemes in Mrs. Bethel's Classroom

Mrs. Bethel primarily facilitated *external conviction proof schemes*, in which *authoritarian* proof scheme was dominant, although *ritual* and *symbolic* schemes were also observed. Mrs. Bethel was the authority of mathematics in the classroom, and students followed the procedures that she modeled in her instruction. When *analytical proof schemes* (particularly *axiomatic* proof scheme) were facilitated, it was rather rudimentary in the sense the teacher emphasized that to progress from given to concluding statements proofs must logically link theorems and definitions together. *Empirical proof schemes* were not visible in any of the observed lesson.

External Conviction Proof Schemes

Mrs. Bethel generally influenced students developing *external conviction proof schemes*. Since she wrote most of the proofs, students were not required to think independently and simply recorded her notes. Additionally, she provided specific procedures for writing proofs, which hindered students from considering a logical relationship between the statements, and encouraged them simply to recall facts previously taught to them.

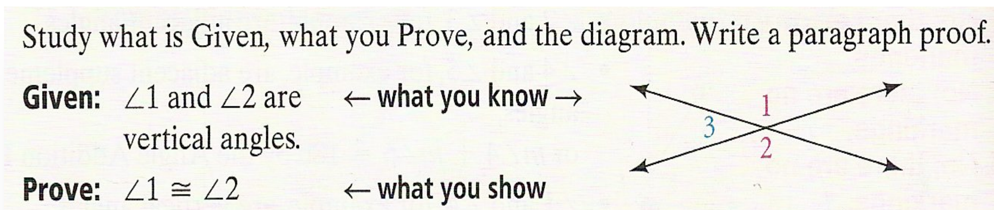
Authoritarian Proof Scheme.

Students were not provided the opportunity to engage in the thinking practice of the discipline and write complete proofs. Tasks that had the potential to challenge students to think critically were reduced significantly when enacted. Hence, students seldom demonstrated justifications that were intrinsic, and relied on the teacher, the word bank and the textbook as the sources of mathematical certainty. Students did not challenge Mrs. Bethel's comments and accepted her statements as true. Mrs. Bethel encouraged such behavior, because she acknowledged during the initial interview,

> I tell my students to pretend there is a little me standing on their shoulder, "what am I going to ask you next, what am I going to ask you next?…" to try to guide them to write something down...in our geometry classes we use our banks to help them choose the correct reason. We do what I like to call skeleton proofs where we have a step and maybe reason blank and the kid have to fill in one or the other" (September 6, 2011- Initial interview).

By encouraging students to follow the model set by her and draw on the word bank, she conveyed to her students that the teacher is the authority of the mathematics. The students got used to being given finished proofs, rather than engaging in writing a proof by themselves. Furthermore, it was customary for Mrs. Bethel to tell students how to solve problems on the homework, and if they view a homework task as difficult, she would do it for them the next day in class. For instance in going over a proof task for a student, she said “ Okay. We have been given the following information…Let’s see what we got here” (October 25, 2011-Enacted lesson), and proceeded to solve the task. Similarly, on September 13, 2011 she proceeded to prove the task rather than have students attempt to write proofs themselves (which is Example 3 in Section 2.5) (Figure 37). She said,

Figure 37. Section 2.5 Example 3 in *Prentice Hall Geometry* (Bass et al., 2004, p. 98).


Study what is Given, what you Prove, and the diagram. Write a paragraph proof.

Given: ∠1 and ∠2 are vertical angles. ← **what you know** →

Prove: ∠1 ≅ ∠2 ← **what you show**

[35]

> Mrs. Bethel: So what I can do is I can take that sentence and I can put it together in this proof. We see two things that are going on. The measure of Angle 1 plus the measure of Angle 3 is going to equal a 180 degrees. And the same thing is happening here. The measure of Angle 2 plus the measure of Angle 3 is equal to 180 degrees. As my reason stated in the paragraph here we’re going to use the angle addition postulate. There are a couple of other things that we could consider some of which we haven't discussed yet. We have a linear pair postulate that says that if two angles form a straight line then they are going to be supplementary. We also know that by definition of supplementary angles that are going to add up

to be 180 degrees. So we can approach this a little bit differently pending on what our mood happens to be, all right. Now what you think about something else here, if we look at what we are supposed to prove. We are supposed to prove that Angle 1 and Angle 2 are congruent. Now look at my equation right now. I've got an Angle 1 and an Angle 2 on both sides of the equal sign. We also see measure of Angle 3 into it. By the subtraction property of equality we could say that the measure of Angle 1 equals the measure of Angle 2, isn't it true that if I subtract the measure of Angle 3 from both sides I am going to get the measure of Angle 1 equals the measure of Angle 2 is that true by subtraction property and then finally when we go out to equals and is congruent we're going to say that Angle 1 is concurrent to Angle 2 by the definition of congruent angles. This one needs the steps so it's depending on which way we trap. (September 13, 2011- Enacted lesson)

Hence, Mrs. Bethel exercised the authority regarding the mathematics, and determined what constitutes a proof. She readily provided formulas and tricks that students were expected to follow in order to be successful in mathematics.

Symbolic Proof Scheme.

Mrs. Bethel encouraged students to mark on diagrams, and remember theorems for congruent triangles. Students knew that to prove that two triangles were congruent one of the theorems must be stated. They rarely considered the merit of the theorem in relation to what needed to be proven. For example, Mrs. Bethel said, " …I heard an SAS and I heard an angle side angle. Look at your picture. There's only one side mark, right? So we've got to go angle side angle" (October 25, 2011- Enacted lesson). Students knew that they had to use symbols, but the extent to which they used them varied in meaning and understanding. Mrs. Bethel said, "So we'll use some symbols, and we'll use some abbreviation in there as well. Okay" (September 13, 2011- Enacted lesson). Mrs. Bethel remarks suggested to students that symbols are needed in doing proofs, without making clear that the use of symbols is only one way to state a proof argument.

Ritual Proof Scheme.

Mrs. Bethel frequently used the two-column proof representation and emphasized that the given goes first and what needs to be proven be written last. When students see a two-column proof or see that the given and the concluding statements are in the standard positions, they may not focus on the body of the argument and are likely to accept the proof as correct without examining its merits. Mrs. Bethel cautioned students,

> Now, we've talked about the fact that nine times out of ten, our first statement are going to be given, however, we've also seen a couple of proofs where it didn't actually start with that for your first statement, so you have to be paying attention to that. (September 15, 2011- Enacted lesson)

Additionally the use of the reflexive property in doing proofs was rather ritual. In proving students would often suggest it be used in a proof. Mrs. Bethel said, "That's my reflexive piece, okay. Now, we all understand that that's automatically going to be true because SK is the same in both pictures" (October 13, 2011- Enacted lesson). Mrs. Bethel's remarks could potentially influence students to focus on the appearance of the proof rather than the logical argument.

Analytical Proof Schemes

Admittedly, Mrs. Bethel modeled proofs and emphasized that a proof utilizes logical deduction, and links theorems and definition to confirm concluding statements. Hence, the few instances in which *analytical proof schemes* were observed, it was generally *axiomatic*.

Axiomatic Proof Scheme.

Mrs. Bethel was aware that proofs were ultimately based on axioms. Arguments for proofs used postulates, theorems, and definitions. She told students, "Now if it's a theorem that means that we can prove it" (September 13, 2011- Enacted lesson). In her reflection for the lesson in which students classify angles, she wrote, "The lesson continues the idea that we must be given certain information for a diagram. We cannot assume things like congruence without the diagram being marked or the fact stated" (Mrs. Bethel, September 13, 2011-Task reflection sheet- after implementation). Her statement suggested that every proof must have some undefined terms, and statements that are accepted because they have been proven before, while other information must be supported. Mrs. Bethel's remarks about not trusting diagram is perceived as a common practice; however from a pedagogical point of view, not using the diagram may restrict students from using their intuitions about the proof. Nevertheless, the observation of *axiomatic* proof scheme appeared sparingly.

Relationship between Cognitive Demand and Proof Schemes

Generally, Mrs. Bethel posed tasks of lower-level cognitive demand, which more readily facilitated *external conviction proof schemes*. *Empirical proof schemes* and *transformational* scheme were never observed. Table 22 displays the relationship between codes for cognitive demand of engagement with tasks and proof schemes observed.

Table 22. Relationship between cognitive demand of engagement with tasks, and proof schemes observed in Mrs. Bethel's classroom.

Proof Schemes	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
External conviction				
Authoritarian	14	56	0	0
Ritual	10	20	0	0
Symbolic	1	5	0	0
Empirical				
Perceptual	0	0	0	0
Inductive	0	0	0	0
Analytical				
Axiomatic	5	4	0	0
Transformational	0	0	0	0

The data highlight that *authoritarian* proof schemes were observed most frequently during Mrs. Bethel's instruction. Furthermore, *authoritarian* proof scheme occurred most often when *procedures without connections* tasks were posed.

Summary across the Three Observed Teachers

The three geometry teachers observed often enacted tasks as *procedures without connections*, which resulted in students developing an *external conviction proof schemes*. *Analytical proof schemes* appeared more often than *external conviction proof schemes* whenever higher-level cognitive demands were posed. Classes' engagement with tasks that reflected high-level cognitive demands happened when students were given more autonomy in their learning. Students working in groups and teachers reducing the level of guidance provided, increased the opportunity for students to demonstrate critical thinking skills, and engage in tasks that required high-levels of cognitive demands.

The one instance (in Mr. Walker's class) of a *transformational* proof schemes occurred when a single student questioned the customary practice of how congruent triangles are named. *Empirical proof schemes* appeared infrequently in all of the teachers' lessons. This is not surprising since most of the proof tasks required students to prove things that were known to be true. Hence, the only opportunity for students to develop *empirical proof schemes* occurred during the lesson in which students had to design a town based on specific guidelines.

Textbooks were used as a tool to structure the order in which teachers presented mathematical content to students, and were the main source for homework assignments. The geometry textbooks contributed to the number of low-level tasks that students were given, since most of the tasks in these textbooks were of low-level cognitive demand. In order to pose higher-level tasks on a regular basis, one of these teachers, Mr. Walker, had to create tasks himself, or seek alternative sources. Although Mrs. Bethel posed alternative tasks in some lessons, the level of difficulty reflected the tasks posed in the textbook. Nevertheless, there was a greater likelihood that if these textbooks were used as the primary source of teachers' instructional practices, then the tasks posed during instruction and for homework will generally require *procedures without connections*.

Teachers' Decision to Deviate from Their Textbooks

In this section, I discuss findings related to the third research question, namely, what influences teachers' decisions to deviate or not from the *McDougal Littell Geometry* and *Prentice Hall Geometry* Teachers' Edition implied or explicit instructions and lesson plans?

During my observations, it was evident that geometry teachers use their district-adopted textbook as a basis for their instructional practices. The textbook provided a structure for teachers throughout the semester, and was used as a resource for instructional content, practice problems, and homework assignment. Even when teachers chose to deviate from the curriculum materials and supplement instruction with alternative materials, the tasks assigned were relatively similar to the chapter content within the textbook. Nevertheless, geometry teachers' decisions to deviate or not from the district-adopted textbook were influenced by factors such as their beliefs, experience, their desire to make mathematics easy, assessment, and their professional community. Teachers' believed that proof was important, that students struggled with proof and that two-column proof was a practical means to facilitate students learning to prove. Furthermore, teachers' experience of teaching proof influenced proof representations used, and how proof tasks were enacted during the lesson. The professional community in which teachers' work was a source of support, which provided resources and guidance as to how proof ought to be taught. Additionally, teachers desire for students to achieve success in mathematics, and on assessment, also contributed to the amount of guidance they provided to students learning to prove. Initially factors were coded based on identified factors in MTF (Henningsen & Stein, 1997), however, additional factors emerged during the examination of the data. All three teachers desired that their students achieve success in mathematics, and their decisions to deviate or not from the textbook were aimed at their goal of facilitating students learning. I will describe each of these factors, in light of evidence provided by each of the three teachers.

Teachers' Belief

Teachers' beliefs contributed to teachers' decisions to follow or divert from their district-adopted textbook. Considering that less than half of the tasks in their geometry textbooks were proof tasks, teachers must first deem proof valuable so that they chose to assign these proof tasks. Fortunately, all of the teachers deemed proof important enough to expose their students to proofs, as stated during their initial interviews. Furthermore, teachers' beliefs about the strengths and limitations of the textbook can influence their decision as to whether or not they choose to offload or divert from the assigned curriculum. When a teacher offload a curriculum, he or she follows it rather closely, and generally adheres to the instructional guidance provided (Brown, 2009). Teachers' preferences for particular forms of proof representation determined they type of proofs they would be more likely to assign.

Importance of Proof

According to the teachers' initial interview, they believed that teaching proof facilitates logical thinking and foster skills students can use in other disciplines. Although most teachers acknowledged that they preferred the two-column proofs, they agreed that the form of the proof was not as important as the logical structure of the argument. They valued students being able to reason and communicate their thoughts effectively. Hence, teachers made a conscious decision to follow the written curriculum pedagogical emphasis related to doing proofs.

Mr. Walker sought for his students to appreciate mathematics and value the proofs that are published in textbooks. He acknowledged that mathematicians placed great thought into the proofs represented in the textbook hence students should appreciate the rationale behind accepted theorems and postulates that are used. Mr. Walker stated,

> I think it [proof] helps you gain an appreciation for mathematics because you know, its not like there is everything that we know in math right now; its just sitting there in the textbook for people hundreds of years ago; you know, people had to come up with these ideas, so I think it helps you gain an appreciation for math and everything that it can do and I also think it helps, kind of, people become a little more logical in their areas of thought not just math. (Mr. Walker, August 23, 2011- Initial interview)

Like Mr. Walker, Mrs. Davis deemed proof important and desired her students to learn how to prove. She believed that the teaching of proof would foster students' development of reasoning and logic and could be used beyond the realms of the classroom. Mrs. Davis believed that learning to prove definitely helps students to function effectively within their daily lives. She knew that students may not use the mathematical symbols or the proof representations in their everyday life, but the skills involved in doing proofs can be used. Mrs. Davis stated,

> I believe that proof helps them. Some kids will say why are we doing this. But it's not just to teach proof, it's to teach reasoning and logic

> skills that they will need for life. They may not have to write down why this segment plus the segment equals a big segment in real life but they need to develop the skills of reasoning to help them in life. (Mrs. Davis, September 2, 2011- Initial interview)

Mrs. Davis reiterated that proof was important to secondary mathematics, because it fosters the development of skills that can be used in real world situations. She believed that developing proofs facilitate students' critical thinking skills, and strengthen their ability to solve problems. In response to an initial interview question, which asked, "Do you think that teaching of proof is important at the secondary level?" Mrs. Davis said,

> I very much do… I will kind of reiterate myself, not necessarily will they have to do mathematical proof in life, but I really do think that it helps them how to think through a problem, how to problem solve, why am I doing this, kind of it's reasoning and reflection upon what they are doing. And I think that they can take those skills through mathematical proof that they are developing and apply them to life. (Mrs. Davis September 2, 2011- Initial interview)

In addition to developing logical reasoning, Mrs. Bethel believed teaching proof enhanced students ability to communicate their ideas. Mrs. Bethel said, "Well, I think it's a growth in learning and communication" (September 6, 2011- Initial interview). During her initial interview, Mrs. Bethel acknowledged that she was aware that proof in geometry might be frightful for students. She said that she tries to encourage students to prove by considering the practicality of doing proofs in business and everyday life; although it was not observed during my classroom visits. According to Mrs. Bethel,

> Proof to me is being able to justify what you doing and I said that very broad and basic because that's where I introduced it to the students. Instead of scaring them and thinking that they have to worry about a proof that's geometry related I explain to them that as you are going through like whatever you choose to do if you need to let's say come up with a business proposal to get money for your particular organization or some aspect of that you have to be able to build up why you need this and I try to explain that it's a way of thinking and justifying why we think in a certain ways, so it's not just geometry… (Mrs. Bethel, September 6, 2011-Initial interview)

Therefore, teachers' decision to follow their textbook's approach to proof was very likely due to the importance that they thought proof had beyond the realms of geometry. If proof was not deemed valuable, the teachers might have chosen to ignore sections, or tasks that explicitly required students to prove.

Strength of the Textbook

Teachers' decisions to follow the organization structure and pedagogical emphasis of a geometry textbook may be due in part to inherent strengths of the textbook. Teachers experiences with other textbooks may have helped them develop an opinion on the practicality of a particular textbook, however, if teachers used only one textbook for their career, they were more inclined to accept their geometry textbook as is.

Mr. Walker valued reasoning as emphasized in the textbook, and used the textbook as a primary source of all homework assignments. He often encouraged his students to provide justifications for their responses, and believed that the textbook also requires students to explain their responses. Mr. Walker commented,

> …The book probably at least half the problem will say, what's the answer, explain... When I look through their homework, their explanations aren't that great so the homework usually isn't enough to hold them accountable for their explanations. So I got to kind of press them to do that in class and have them speak up because like I tell them all the time that the reasoning and the explanation is more important than that final numerical answer. (Mr. Walker, November 3, 2011- Follow up interview at the end of the lesson).

Since Mr. Walker wanted students to provide rich explanations, he was appreciative that many tasks in the textbook require students to explain their responses, and often would assign tasks from the textbook that required students to provide their reasoning. He acknowledged that students providing reasoning is of greater importance than providing the solution only.

Mrs. Davis had no complaints about the district-adopted textbook; nevertheless, hers was the only geometry textbook she had ever used. She did not consider any changes, since this had been her primary resource since she started teaching six years ago.

Mrs. Davis accepted the textbook as it was, and chose to follow it throughout the semester, both the content and the order in which it was presented. For instance, when a student exhibited a skill from a future section in the book early in the semester Mrs. Davis said "And you don't want to do that in this section, we are not there yet. There will be a portion where you will change the order" (September 6, 2011- Enacted lesson). Hence, the skills encouraged in the classroom clearly aligned with the goals of each unit within a particular chapter. Mrs. Davis stated, "how the book lays out proof is how we model proofs on our test and on our homework" (September 23, 2011- Follow up interview during lunch break). Mrs. Davis dutifully integrated the textbook into the class without adaptations, and regularly referenced mathematical ideas in terms of chapter and unit location, rather than as explicit mathematical topics. Hence, Mrs. Davis saw no limitations in the textbook, and in her lessons, she regularly taught the content as presented in the textbook.

Unlike Mrs. Davis, Mrs. Bethel had experience with other curriculum materials, and was aware of potential differences among them. Nevertheless, she chose to follow the textbook closely. If she improvised and posed alternative activities, the content aligned closely with the goals of the chapter or unit as outlined for that day by the geometry team. As indicated earlier, Mrs. Bethel used the table of contents of *Prentice Hall Geometry* as a point of reference: she used the chapter and section as reference, instead of the mathematical content. For example, on October 4, 2011 (during the enacted lesson) she noted "…quiz is going to cover 3.5 and 3.6", rather than lines in a coordinate plane, and slope of perpendicular and parallel lines. Mrs. Bethel, being aware that the textbook was used as an organizational tool, cautioned students that the reference point might vary in other books. For example, she said,

> We've got that theorem 2.2 now remember what we talked about with those numbers. The numbers don't mean anything more than telling us where in our book we're going to find them. If you go to another school

> and you take geometry and they have a different book. This particular theorem that we are getting ready to talk about might not be in the same location and it might not be the second theorem if you introduce chapter two, all right. So number is not always important as what they are saying. (Mrs. Bethel, September 13, 2011- Enacted lesson).

While this quote shows that she wanted her students to think about the content and not about the chapter and section numbers, it does reveal the extent to which she and her students depended on the organization of the textbook.

Although Mrs. Bethel, acknowledged that the language in the geometry textbook may differ from other textbooks, she would follow her textbook, the *Prentice Hall Geometry* without deviation. For example on September 27, 2011 (during the enacted lesson) she acknowledged that other books may refer to a seven sided figure as a septagon, however "we say heptagon… because this is the way that our book represent it and this is the way that we will enforce it". Mrs. Bethel never acknowledged any limitations of her textbook, and so she chose to follow it. The textbook provided students an opportunity to learn core geometry ideas and provided an organizational structure for her about how to proceed. Her decision to follow the textbook as is, suggests that she perceived strengths of the textbook within her particular environment, considering that in her school the teaching of geometry was a team effort. Mrs. Bethel acknowledged that "the geometry textbooks actually set you up with something that you know so that you feel more comfortable when you are actually doing those proofs" (September, 6, 2011- Initial interview).

Therefore, it seems teachers chose to follow their curriculum materials based, among other things, on the perceived strengths that their textbooks have.

Limitation of the Textbook

Of the three participants, Mr. Walker was the only teacher who voiced any concerns about potential limitations of the district adopted geometry textbook. He acknowledged that the *McDougal Littell Geometry* textbook had multiple limitations, namely, that it does not provide sufficient problems to practice proofs, that it did not address the need to connect mathematical ideas, and that the order in which content was presented could be made more logical. Due to these concerns, Mr. Walker supplemented his textbook with additional proof tasks.

Mr. Walker strongly believed that the textbook needed to have more tasks that are proof. In reference to Section 4.2 ("Apply congruence and triangles") Mr. Walker remarked, " … If I look in this section in the book there's one, there's two proofs of how we want them to be thinking about [it]…. I need more than that, it's not good enough" (November 3, 2011- Follow up interview at the end of the lesson). Convinced that the geometry textbook could be improved, Mr. Walker said,

> Well I told you the one thing [I would change about the book] would be that it has more problems of each type because I feel like they have like one problem of every type. So if there's something I mean there still only one of two of those, but there's a lot of them [non-proof tasks]. So, I don't know that's my biggest thing. I just think there needs to be more problems in general, you know of like the same concept. (Mr. Walker, November 8, 2011- Follow up interview at the end of the lesson)

Considering that Mr. Walker had a greater autonomy in his practice than Mrs. Davis or Mrs. Bethel who planned lessons with their geometry team, Mr. Walker chose to supplement the textbook. Mr. Walker commented,

> … One thing about the book is I feel like there's never enough of each type of problem. And I know that it would be the longest book in the world but like for example there's basically numbers three and four that corresponds to everything on this worksheet. Do you know what I mean? Everything else involves these concepts, but they don't have the repetition that I think is necessary, the practice that's necessary to get it, so … most of the times you have to find more practice than just problems out of the book because the book just doesn't have enough of them. (Mr. Walker, November 3, 2011)

In addition to his concern that the textbook needed more proof tasks, Mr. Walker also thought that the order in which the textbook introduced content may not necessarily be ideal. He believed that there were gaps in the connections of ideas as presented in the textbook, and thought that the mathematical ideas be connected rather than taught as isolated units. According to Mr. Walker,

> … That's one thing about the book, they think that the book will a lot of times just give you a bunch of information but they don't really try to connect it to what you're doing in the future. Because I think when you go through textbook, the textbook just kind of assume hey we're here right this section 4.2 so we know everything that happened before that. I think the textbook makes that kind of assumption but so when I make my notes I

> want to connect this to what we are doing, going for, which we're looking for ways to get as many of these corresponding pairs congruent to each other to prove. (Mr. Walker, November 3, 2011)

Mr. Walker was discontent with the departmentalized structure of the textbook. He preferred to teach the content as a whole, rather than as individualized units. He felt that, once students are provided with the relevant postulates and theorems, they could engage in doing proofs for the chapter. Mr. Walker admitted,

> The textbook will spend like a section on each one or maybe a section or two …I'll usually just kind of give them say now these are all of our shortcuts and then we want to see if we can use any of these. So I kind of almost teach it as one big picture with relevant individual smaller pieces. That way because I guess I feel like I'm trying to teach the idea of that we're taking a shortcut to prove this triangle congruent. So, I feel like I can teach that with all of them together rather as individual days. (Mr. Walker, November 3, 2011)

Although Mr. Walker credited the geometry textbook as functional, he highlighted potential inconsistencies in it. He felt that sometimes the way in which proof arguments and definitions were presented might not necessarily be a logical progression. Mr. Walker, stated,

> Yeah the book is inconsistent based on what they're trying to teach. So like for example, the way I teach it and this is how the book starts teaching it is, after you're given a midpoint, you need to first state that like two segments are congruent and then you could go to say that their

> measurement are equal because based on the definition that point just says that it divides the segments into particular segments. It doesn't say it creates two segments of equal measure. So, if we're going by the definition, you have to first say they're congruent and then you could say the measures are equal before you can substitute it. … That's really the logical step. (Mr. Walker, September 8, 2011- Follow up interview at the end of the lesson)

Mr. Walker's observation about the limitations of the textbook was a catalyst for him to choose to deviate from the curriculum. His decision to deviate was prompted by his desire to increase the number of proofs that his students would experience.

Group Work is Important

Although neither *Prentice Hall Geometry* nor *McDougal Littell Geometry* include tasks that explicitly required students to work in groups, the three teachers encouraged group collaboration and the sharing of ideas. They expressed the belief that group work presented opportunities to obtain different perspectives on the matter. Additionally, one of the teachers, Mr. Walker, acknowledged that when students work in their group they would be more productive than during a whole class discussion. Nevertheless, teachers' decisions to promote group work represented a pedagogical persuasion that was not explicitly identified in geometry textbooks.

Mr. Walker viewed group work as a support system for learning. Mr. Walker claimed, "One thing about math, is you just sit there and kind of do it by yourself and write it on paper. But when you talk about it and you ask questions, and you help each other out that's even better" (Mr. Walker, September 22, 2011- Enacted lesson). He felt that the sharing of the ideas and questioning each other thinking was a good practice within a mathematics classroom.

Furthermore, Mr. Walker suggested students were more engaged during group work than whole class discussion, due to students' limited attention span. According to Mr. Walker,

> Well as far as bringing them together as a whole, like if I go through a proof as a whole, I definitely can't do more than about one at a time because then they will be their attention span is probably like five minutes tops. (November 8, 2011- Follow up interview at the end of the lesson)

Recognizing that students would engage more actively during group work, Mr. Walker often gave students cards with proof tasks in which students had to construct proofs at their table, or gave students a shuffled deck of statements and reason, and asked students to logically present a mathematical argument.

Like Mr. Walker, Mrs. Davis encouraged group work in her lessons, although it was not an explicit requirement of any of the tasks included within the geometry textbook. Mrs. Davis told students, "Work together, four brains are better than one brain" (September 20, 2011-Enacted lesson), and encouraged students to "Talk with your groups, throw out some ideas, if you are having trouble figuring it out, talk with each other…" (September 6, 2011- Enacted lesson). Mrs. Davis believed that group members could help one another make sense of the mathematics when they were confused or uncertain about a particular idea. Therefore, she would often encourage her students to collaborate with each other.

Likewise, Mrs. Bethel often encouraged her students to work with their groups or shoulder partner to solve problems. Students who were shoulder partners sat adjacent to each other. Mrs. Bethel acknowledged, "Students are developing new understanding which is noticeable through discussion at tables…" (Mrs. Bethel September 13, 2011 – Task reflection sheet- after implementation). Mrs. Bethel encouraged her students to work collaboratively because she thought that group work was a productive mechanism to facilitate students learning geometrical content.

Therefore, despite the neutrality of textbooks towards group work, teachers encouraged students to work within groups because they believed students could gain a lot from sharing ideas with their peers. Collaboration provided opportunities for multiple perspectives and a source of support to help with challenging tasks.

Teachers' Favored Two-Column Proofs

Although all teachers used at one time or another other representations of proof, they expressed a preference towards teaching proof using two-column proof representation. Unlike the paragraph proof representation, in which attention must be paid to grammar, the two-column proof is a perceived as a precise means to logically link mathematical ideas.

Mr. Walker acknowledged that of the time he allocated to teaching proof, he spent more time working on two-column proofs. He believed the structure of the two-column proof is an adequate means to progress between steps. According to Mr. Walker,

> …I spend a lot more time on two-column proof than I do the paragraph proofs. I – I kind of introduce it and just tell them this is something you can use, but I don't prefer it, so the—I guess I feel like I don't want my

mistake in grammar in English to disrupt a proof, so I like just kind of the traditional two-column proof and so, I – think that kind of structure helps see a sequence better, a sequence of thoughts. (Mr. Walker, August 23, 2011- Initial interview)

Mr. Walker was aware that word choices and sentence structure can potentially change the meaning of a proof. He felt two-column proofs provided a clear and precise structure to progress from the given facts to what needs to proved.

Mrs. Davis explained that although the geometry team at her school teaches multiple representations of proofs (two-column, paragraph and flow), the team often prefers the two-column proofs to foster students' development of proofs. They believe that two-column proofs are a practical way for students to learn how to prove, and were confident that over time students would be able to construct complete proofs. With two-column proofs, students can be introduced to proofs by completing skeletal proof arguments. Mrs. Davis said,

> Well, we in our geometry classes, we teach three different types of proofs. We teach two-column proof, paragraph proof and flow proof. And we would like them to be able to construct it ideally, construct proof on their own from beginning to end and sometimes they can do that with the two-column proof with say segment addition postulate. In the real world a lot of the students can't construct a proof from beginning to end on their own, so we will have fill in the blank, two-column where maybe a step is provided on the left and they have to provide the reason on the

right or the reasons on the right what's the matching step on the left.

(Mrs. Davis, September 2, 2011- Initial interview)

So, although Mrs. Davis exposes students to other forms of proofs, the two-column proof representation is used more frequently, since Mrs. Davis is convinced that it fosters students' ability to provide reasons for statements. She considers that students are not necessarily experts on proofs, so she sees the two-column proof representation as a practical means to introduce students to proofs.

Mrs. Bethel also acknowledged her preference for proofs using a two-column proof, although her textbook provided students more opportunities to utilize paragraph and flow proof. Mrs. Bethel stated, "… we see two-column proof a little bit more often than the other two forms but you will find that in your homework assignment you will get plenty of practice with the paragraph proof or a flow proof" (Mrs. Bethel, September 13, 2011- Enacted lesson). Mrs. Bethel considered that two-column proof was more natural for her. Mrs. Bethel stated, "… So what I am thinking in geometry my brain thinks in two-column proof because I can build the sentence on the statement and the reason…" (September 6, 2011- Initial interview). She believed that in other setting, like algebra two-column proof may not be ideal, and that professors may use paragraph proofs more, but in geometry, two-column proof was prime. Mrs. Bethel stated, "The paragraph indicates that you got sentence construction, and with sentence construction you are going to have statement and you're going to have reason built into it... If you remember I never liked that one" (September 6, 2011- Initial interview). Her preference for a particular proof representation contributed to her emphasis of utilizing two-column proofs in teaching geometrical proof.

Despite teachers exposing students to multiple representations of proof, the two-column proof representation was seen the most in the enacted lessons. In fact, when teachers exposed students to paragraph proof, they would often begin with a two-column proof and convert the argument into a paragraph. Teachers' decision to utilize two-column proof more often, even in chapters that had more paragraph proofs than two-column proofs was due in part to their belief that two-column proof was an effective means to teach proof.

Low-level Proofs are Taught Due to Students' Limited Ability to Prove

It is not a coincidence that when teachers lectured, and exposed students to proofs, most of the tasks required lower-levels of cognitive demands. All three teachers admitted, that most students had little, if any, experience doing proofs prior to this geometry class, so they normally posed low-level tasks so that students were comfortable, while developing an understanding of the notion of what it means to prove.

Mr. Walker suggested that the procedural nature of proof in geometry reduces the potential value of the proof. He acknowledged that students do not necessarily learn new things by doing proofs as much as he would like that to happen. Mr. Walker said,

> [Proof] its just kind of, you know, building on what we know to learn something new, but in practical use, it doesn't' quite work out that way like it – it can become a little more procedural especially in geometry where there's you know freshmen, sophomores that haven't quite developed yet I don't think, so they don't really think about it as okay, I'm trying to learn something new; they just want this process to go through,

> so I think it loses some of its value. (Mr. Walker, August 23 2011- Initial interview)

Since high school geometry students are normally not adept to doing proofs, Mr. Walker often utilized a ritual procedure when teaching proof, rather encouraging creative ideas.

Similarly, Mrs. Davis said, "It's not a high-level proof that we are teaching" (September 2, 201- Initial interview). The reason for assigning tasks of low-level cognitive demand, or reducing the level of cognitive demand of a rich task was because students had limited experience doing proofs. Mrs. Davis stated,

> I don't think they had enough experience with proof. So by 10 or 11th or 12th graders they don't have enough experience with proof prior to this. Mini- M-I-N-I like Mini Proof … because this is, lot of the times, this is the first time they've ever seen proofs… When they get to my geometry class and it's hard for them, if they've never had to show why they did something. You know when you are solving 2x + 4 = 5x and you are standing, you are solving for x, they've never had to show their steps of subtraction, dividing. They never had to list their reasons. Yes they've done it they know what they are doing but they've never had to tell why until now. (September 2, 2011- Initial interview)

Mrs. Davis was aware that students were new to the notion of doing proofs. So she sought to introduce proof in an elementary fashion so that students could provide appropriate reasoning for each mathematical step involved in the proof.

Like Mrs. Davis, Mrs. Bethel admitted, “we teach the basics of proof and do so with the idea that students can ‘master’ proofs if given a ‘skeleton’ and a word bank to fill in the blanks” (September 22, 2011- Task reflection sheet-after implementation). She described the proofs taught as “Very basic proof. Very obvious proofs and I would say consistently less than 10 steps, never anything that’s complicated…and we guide them an awful lot at this stage” (September 6, 2011- Initial interview). Mrs. Bethel acknowledged that she wanted her students to be successful in doing proofs, but because of their lack of readiness, the tasks she posed usually had a low-level cognitive demand. She chose proofs that were short, and provided students with a word bank in order to increase students’ success in writing proofs or completing skeletal proofs.

Therefore, these three teachers opted for assigning proof tasks of low-level of cognitive demand with the expectation that these tasks would be more accessible for students. Since students were inexperienced with proofs, the low-level tasks provided students the opportunity to learn various proof representations, in a context that was simple and supported.

Proof is Challenging for Students

Teachers expressed concerns that students viewed proofs as challenging. In many instances, students did not complete their homework assignments in its entirety, if the assignments included proofs. Students’ negative disposition towards doing proofs potentially influenced tasks teachers selected from the textbooks.

Mr. Walker suggested that students’ peers tell them that proofs are difficult, and therefore students are biased against proofs before entering the class. He also believed that students' negative disposition towards proofs were due in part to a lack of motivation to state their ideas with appropriate reasoning. He asserted,

> A lot of them decide right away that they don't like it. It's got that stigma that gets passed down from grade to grade or they're like we hate this. It's kind of the laziness like they don't want to have to write it down. They don't want to write it down all their thoughts. So that's kind of a laziness type of thing. (Mr. Walker, November 15, 2011- Follow up interview at the end of the lesson)

Due to students' preconceived notion about proofs, it is a challenge to have students engaged with proofs. Since they view proofs as overwhelming, they do not bother to attempt it.

Mrs. Davis knew that students found it difficult to explain their actions when doing proofs. She said,

> Well they have a hard time explaining why they did something. A lot of time students don't like to show their work and how did they get to that answer. …They are not use to thinking and showing why they did something and that's a problem that we encountered when we are doing proofs. They are not use to telling why they do something. They just do it…. (Mrs. Davis, September 2, 2011- Initial interview)

Since students have difficulty supporting their reasons, the use of a word bank made proofs more manageable for students during classroom instruction. Mrs. Davis stated, "I was getting incomplete homework and students who refused to do the proof portion" [September 13, 2011- Task reflection sheet-after implementation- sent via email]. Mrs.

Davis was aware that students struggled in class while doing proofs and chose not to complete proof tasks on homework assignments. To curb this problem, she posed tasks that required either filling in the blanks, or tasks that could be completed if students used memorized theorems and postulates.

Likewise, Mrs. Bethel was aware that students struggled with proofs and often sought to ensure students were comfortable with proofs. For instance on September 15, 2011 (during the enacted lesson), she reminded students, "We can get through a couple more of these properties that we don't always feel comfortable with". She encouraged students to learn particular postulates and theorems that were often used in proofs. Having students comfortable with proofs increased the likelihood that students proved.

Since teachers knew that students view proofs as difficult, many times they sought to make proofs more comfortable for students which reduced the tasks' level of cognitive demand. Students' negative disposition toward proofs may be a result of lack of effort, or because, for them, it is a novel terrain. All three teachers were nevertheless aware that students have difficulty doing proofs.

Making Mathematics Easy

Teachers wanted students to be comfortable with mathematical tasks posed. They ensured that their instructional practices aligned with textbook examples and that students were able to complete homework assignments very easily. Additionally, teachers valued the formulas and rules provided on the back of the textbook. The teachers in the study frequently assured students that the tasks posed were easy and accessible.

Mr. Walker wanted students to have little difficulty completing homework assignments. He provided examples that were similar to tasks posed on the homework.

For instance, Mr. Walker said, “I didn’t get to as many examples as I would have liked. Their homework was probably harder because of that fact” (September 14, 2011- Follow up interview at the end of the lesson). His remarks suggested that tasks in the geometry textbook influenced his instructional practice because he sought to minimize the challenges students had on homework assignments. Mr. Walker frequently encouraged students to use their textbook as a resource in geometry. Mr. Walker told students, “If you get stuck, you have the back page. That page is a guide to give you tips and ways to find corresponding parts. So, it has a section on angles with alternate interior, with vertical angles…reflexive property” (Mr. Walker, November 10, 2011- Enacted lesson). Students used the textbook as a reference for key words and phrases.

Mrs. Davis informed students that the tasks were easy. In most lessons, she used phrases with the word “easy” or “comfortable”. For example, Mrs. Davis said, “this is so easy” (October 11, 2011- Enacted lesson) in reviewing tasks for the geometry tests. Additionally, she perused the tasks in advance and considered means to make them easier for students. An example of this was when Mrs. Davis provided instruction on overlapping triangles in Section 4.7, she said “ okay so there is a way to make it easier. If you rip apart the triangle okay, it doesn’t mean you literally get to rip your paper. Okay. Rip apart means you’re going to separate and redraw the triangles....” (October 14, 2011- Enacted lesson). Mrs. Davis evidently made a conscious decision not to deviate from the textbook. Her instructional practice ensured that tasks posed were not only deemed easy, but that students were able to achieve success while completing them.

Similarly, Mrs. Bethel ensured that tasks were manageable when enacted. For instance, in her discussion on proofs of congruent triangles, she told students, “So it didn't change the difficulty level too much. We just had to deal with that picture that was a little bit more complicated " (October 25, 2011-Enacted lesson). She carefully illustrated how tasks could be made easier and reduced their difficulty level to one that students were accustomed. Additionally, when she illustrated how to prove theorem about angles, she announced to the students, "we can prove that pretty easily" (Mrs. Bethel, September 13, 2011-Enacted lesson).

Hence, teachers' desire to make mathematics easy for students can influence their decision on whether to deviate or not from the textbook. The teachers sought to facilitate students' learning to prove and the likelihood that students would complete homework assignments successfully.

Teachers' Experience

Teachers experience can contribute to their decision to deviate or not from geometry textbooks during their teaching of proof. Based on their experience, they may choose to emphasize a particular proof representation, select preferred tasks, etc. Conversely, teachers' lack of experience may influence their practices as well.

Mr. Walker suggested that a teacher's experience could influence how proof is taught. Mr. Walker has taught for six years and often searched the Internet for new examples/ resources. Based on his experience, he believed two-column proof is a preferred representation to introduce students to proofs. According to Mr. Walker,

> A lot of factors influence how proof is taught… A lot of it has to do with teachers' experience with proofs. For example, I feel like the two-column proofs are the easiest to see logical steps, so that is what I spend the most time teaching. Also, I may be more rigid in the steps that the students must

show me, than other teachers. I don't like to find missing steps in logic according to our geometric postulates or theorems. (September 29, 2011- Follow up interview- sent via email)

Mr. Walker's experience suggests that two-column proof is a practical way for students to develop proof arguments, because the form of the two-column proof provides a mean to logically progress between steps. Furthermore, based on his experience, he valued logical statements and that the steps in the two-column proof minimized students skipping steps in their arguments.

Mrs. Davis stated that she only had experience teaching from the *Prentice Hall Geometry* textbook and her teaching of proof was initially weak. However, due to her interaction with colleagues and observation of their instructional practices, Mrs. Davis believed that her teaching of proof improved over the past six years. According to Mrs. Davis,

> Okay. I would say that since my six years of teaching when I first started teaching I wasn't very good at it. But I would say that I have improved. Just having seen other people and observe other teachers and teaching of proofs and their examples of how they teach, I guess I should say. (Mrs. Davis, September 23, 2011- Follow up interview during lunch break)

Therefore, Mrs. Davis's lack of experience of teaching proof could have contributed to her decision to follow the textbook during her instructional practice. Fortunately, she had supportive colleagues that allowed her to observe their practice so that she can improve her practice as well.

Mrs. Bethel taught mathematics for 18 years, 15 of which were devoted to the teaching of geometry. Mrs. Bethel knew that "Proof is taught differently in different schools" (September 22, 2011-Task reflection sheet -after implementation). Due to her experience, she was able to locate resources (such as the Becky Bride Geometry- Kagan Cooperative Learning Series) that can supplement the goals of the lesson, while fostering group work.

The teachers' experience contributed to their decision to deviate or not from the district-adopted textbooks. Based on their teaching experience, interaction with peers, and exposure to other curriculum, the teachers chose to offload, adapt, or improvise the curriculum (Remillard, 2005).

Professional Community

Teachers' decision to follow or deviate from the curriculum could also be influenced by the school's professional community. If teachers have autonomy in their practice, there existed greater flexibility to deviate from the curriculum. If teachers planned lessons as a team, teachers were more likely to follow the curriculum to ensure quality control of content across sections. All of the teachers acknowledged that their instructional practices were due in part to a collaborative effort.

Mr. Walker had autonomy to decide the extent he would follow the textbook during his instructional practice. His department consisted of two teachers: himself and a male counterpart. They shared tasks and discussed content matter in relations to state and national goals. Nevertheless, Mr. Walker had freedom to decide what tasks will be posed within his geometry class. Mr. Walker noted that some of his tasks were not from the geometry textbook, and were created by either himself or his fellow colleague. For instance, with respect to the tasks assigned to review proofs relevant to congruent line segments, Mr. Walker stated that some of the tasks might have come from the textbook, "but the teacher next door made them up actually. So we just kind of share a lot of problems. So they may have, but from my standpoint, I got them from the teacher next door." (Mr. Walker, October 6, 2011- Follow up interview at the end of the lesson). Based on my conversations with him, I deduced that within his academic community, he chose how to proceed and structure the mathematical content, despite the sharing of ideas.

Mrs. Davis and Mrs. Bethel's instructional practices aligned with the geometry team decisions of how to structure lessons. For each chapter, students in Mrs. Davis and Mrs. Bethel class were given an outline of topics, the number of days assigned for each topic, as well as the tasks they had to complete for homework. Mrs. Davis noted, "The homework assignments were problems from the book turned into a worksheet" (September 8, 2011- Task cover sheet- before implementation-sent via email). Although teachers had the flexibility to adjust the plan based on students learning, both teachers mostly sought to ensure that the content aligned with the planned lessons. According to Mrs. Bethel, **"**This section [Section 2.5 – Proving angles congruent] is part of the geometry curriculum and was in place, as agreed upon by the geometry team, before I came ... Teachers can adjust the assignment as they see fit for their specific students" (September 13, 2011- Task cover sheet- before implementation). Mrs. Bethel suggested that the teaching of geometry is a team effort. She said, "School level – (Geometry Team) has dictated how I check for understanding in the classroom." (Mrs. Bethel September 22, 2011-Task reflection sheet-after implementation). Therefore, the geometry team significantly influenced the extent teachers chose to deviate from the curriculum materials during their instructional practice.

Assessment

All of the teachers desired that their students achieve success in mathematics. A means to measure students' success in geometry is by end of chapter exams, and state mandated end of course exams. Teachers were mindful that most external assessment of geometry seldom required students to construct complete proofs. If proofs were assessed externally, the questions required students to match statements with reasons, or to complete a proof by filling in the blanks of missing statements and reasons. Hence, teachers wanted to expose students to the type of tasks they would be assessed on.

During invigilation of state mandated End of Course exams, Mr. Walker glanced at tasks and noticed that there were few, if any, proof tasks. Mr. Walker stated,

> …You can't really see the End of Course [state exam] problems but if you walk around you can kind of glance and see a couple [of questions] but you're not suppose to sit there and like read the book but from what I can tell just like seeing the problem because it doesn't look like there's a whole lot of proof and reasoning that they need on the end of course testing. (Mr. Walker, November 3, 2011 – Follow up interview at the end of the lesson)

The lack of assessment of proofs can be a challenge for teachers, because what is graded is what gets valued (Wilson, 1994). Fortunately, Mr. Walker valued proofs and felt that proofs developed logical thinking, which can enhance students' performance on testing. So, he continued to teach it. Mr. Walker stated, "My problem with that is, I feel like if we're explaining our reasoning, then we're going to build our understanding. So to me, doing the proofs in class will indirectly help us on testing" (Mr. Walker November 3, 2011- Follow up interview at the end of the lesson). Therefore, because Mr. Walker believed proofs could be beneficial to students, he chose to assign tasks from the textbook pertinent to proofs, despite the fact that proofs are rarely assessed.

Similarly, Mrs. Davis was mindful that tests seldom assessed students on proofs. If proofs are assessed it often required low-level cognitive demand; be it fill in the blanks or matching reasons with statements. Mrs. Davis said,

> Okay our EOCT [end of course test]...the state test, the math test and the EOCT test don't have a performance event so the students wouldn't have to create a proof completely on their own. It would be something, what's step that missing or what step is incorrect in the proof. It will have the entire proof listed on the … test and then students have to come up with either what's missing or what's wrong with it. So that's probably influences the fact that we don't have them create entire proof on their own because we want them to be familiar with how… they would be assessed on a state test. (Mrs. Davis, September 23, 2011- Follow up interview during lunch break)

The structure of assessment questions influenced how teachers used the textbook to teach proof in geometry. The low-level proof tasks assigned by teachers during enacted lessons, reflected external assessment of such tasks. Mrs. Davis, affirmed, "they are to teach as to how the students will probably be successful… I think assessment do influence the way things are taught" (September 2, 2011- Initial interview).

Similarly, Mrs. Bethel stated that proofs are seldom assessed on state exams, and if they are assessed, they generally require low-level cognitive demand. She said "… State assessments do not require students to write the proof from scratch, instead they present the proof as a set of statements and reasons, and have the students "assemble" them in the correct order" (Mrs. Bethel September 22, 2011 – Task reflection sheet- after implementation).

Considering that teachers want students to be successful on assessments and that assessments seldom require students to construct proofs, this may influence teachers decision to deviate or not from the textbook on tasks pertinent to proofs. The teaching of low-level proof tasks may therefore be a result of teachers desires to expose students to formats similar to those on which they will be assessed.

Summary

There are factors that influences teachers' decisions to deviate or not from the subject specific geometry curriculum: teachers' belief, teachers' experience, the desire to make mathematics "easy", professional community, and assessment. Teachers believed that students experience difficulty while doing proofs, hence teachers often chose to enact proof tasks, which required low levels of cognitive demands such that students can obtain a degree of success. Furthermore, teachers experience teaching proofs can influence the proof representation used, and other pedagogical strategies employed to facilitate students learning to prove. The professional community in which the teachers work often shared resources, and in some instances collectively planned lessons. The extent of autonomy teachers' had in planning lessons significantly contributed to the likelihood as to whether or not they would deviate from the classroom. Additionally, assessment contributed to the emphasis placed on proof in geometry lessons. The sad reality is that proofs are seldom assessed, and when proofs are assessed the cognitive demands of the tasks are low. Considering what is evaluated, tend to become what is valued, assessment plays an important role in how proof is taught. Therefore, the identified factors influences tasks teachers pose, and how teachers enact such tasks within the classroom. Nevertheless, teachers' act with good intentions, that is, to facilitate students learning mathematics.

CHAPTER V – CONCLUSIONS AND IMPLICATION

Summary of the Problem

Researchers in mathematics education need to gain a deeper understanding about how proof is taught. Students perform poorly on proof tasks (Healy & Hoyles, 2000; Senk, 1985) and researchers have examined teachers' conceptions about proof (Knuth, 2002a). However, there is a scarcity of studies addressing how geometry teachers use curriculum materials to teach proof. Therefore, the goal of this study was to address these gaps in the literature by examining how geometry teachers use their curriculum materials to teach proof. I addressed the following research questions:

1. How do *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions present proof for segments and angles, parallel and perpendicular lines and congruent triangles to facilitate students learning to prove?
2. To what extent do geometry teachers use *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions to teach proof for segments and angles, parallel and perpendicular lines and congruent triangles to facilitate students learning to prove?
3. What influence teachers' decisions to deviate or not from the *McDougal Littell Geometry* and *Prentice Hall Geometry* Teacher's Editions implied or explicit instructions and lesson plans?

Method

I chose a case study research design to investigate how three geometry teachers use curriculum materials to teach proof. To analyze the data, I used a conceptual analytical framework. My conceptual analytical framework consisted of three dimensions: the first two dimensions are drawn from the mathematical task framework (MTF) (Henningsen & Stein, 1997) and the last dimension is based on Harel and Sowder's (Harel & Sowder, 1998) proof schemes. The first dimension represents the task features; the second dimension, the cognitive demands of the tasks; and the third dimension, the proof schemes apparent in the completion of the tasks. I collected data via video and audio- recorded classroom observations, teacher interviews, teacher artifacts and classroom observation protocols.

I aligned the coding of the data with the conceptual analytical framework that consisted of three dimensions. Among the tasks features that I coded, I considered whether the tasks were proof or proof-related, whether answers were provided with reasons or as answers only; if the tasks were labeled "challenge;" if the tasks required one solution strategy or multiple solution strategies; whether the setting of the tasks were abstract, or in a realistic context; if the tasks required students to fill in the blanks; and if they were multiple choice or composed of multiple parts. The cognitive demand of the tasks ranged from *memorization* to *doing mathematics* (Henningsen & Stein, 1997). Proof tasks deemed as *memorization* generally required students to complete skeletal proofs in which they had to fill in the blanks, while proof tasks deemed as *procedures without connections* included matching statements with appropriate reasoning to complete a proof. Proof tasks that I classified as *procedures with connections* included items that required students to prove congruence of triangles in a Cartesian plane using a particular theorem, or required students to write proof plans. I classified proof tasks as *doing mathematics* when tasks generally required students to write complete proofs. More particularly, a proof task reflecting *doing mathematics* required writing a complete proof, that was not similar to previous tasks and examples, and may change the context, utilize a different representation, and is not algorithmic. Proof schemes were coded based on the interactions between the teachers and students during the enacted lessons. Five researchers participated in the analysis of the textbooks and the classroom observations to increase the reliability of findings. For the mathematical tasks coded we achieve an inter-rater reliability of 89%.

Findings

In examining my first research question, I observed similarities in Chapter 2-4 of the textbooks. In both textbooks, most of the tasks were situated in abstract context, required *procedures without connections*, and could be solved using one solution strategy. Few (less than 4%) tasks were multiple-choice and slightly less than one fifth of tasks were composed of multiple parts.

Despite these similarities in task features and the low-level of cognitive demand of most tasks, a further analysis of proof tasks revealed differences in the textbooks on the task features and cognitive demand dimensions of proof tasks. For instance, of the 1066 tasks analyzed in *Prentice Hall Geometry* only 79 of them were proof tasks. Almost half (46.8%) of the 79 proof tasks in this textbook required students to fill in the blanks, thus requiring only *memorization* of facts. Slightly more than half (58.2%) of the 79 proof tasks in *Prentice Hall Geometry* required low-level cognitive demand (*memorization* or *procedures without connections*). These results suggest that *Prentice Hall Geometry* facilitated students engaging with lower-levels of cognitive demand tasks. On the other hand, *McDougal Littell Geometry* had 977 tasks, of which 128 were proof tasks. Of the proof tasks posed in *McDougal Littell Geometry*, only 10.2% required students to complete skeletal proofs; while over half (64.8%) of *McDougal Littell Geometry* proof tasks was higher-level cognitive demand tasks which required students to write complete proofs, 14.8% of which were considered *doing mathematics*. Therefore, for the chapters studied, *McDougal Littell Geometry* provided more opportunities for students to write complete proofs than *Prentice Hall Geometry*. Furthermore, two-column proof representations were far more frequent than other forms of representation in *McDougal Littell Geometry*, while in *Prentice Hall Geometry* two-column proofs and paragraph proofs were used just as often. The results suggest that although the organization structure of the textbooks were the same, the attention given to proofs varied for tasks features and levels of cognitive demands. *McDougal Littell Geometry* was more likely to pose higher-level proof tasks, whereas *Prentice Hall Geometry* was more likely to pose lower-levels of cognitive demands. All taken into account, it seems that *McDougal Littell* presented richer opportunities for students to write proofs.

Regarding my second research question, I found that Mrs. Davis and Mrs. Bethel generally offloaded (Brown, 2009) or adapted the textbook to align with the geometry team planning efforts; however Mr. Walker often improvised by posing additional proof tasks. Mr. Walker's action increased opportunities for students to engage with tasks that required a higher-level of cognitive demand by supplementing the textbook. In these three classrooms, proof and proof-related tasks were enacted generally as *procedures without connections*. In the few instances in which enacted tasks required higher-levels of cognitive demands, teachers usually switched roles from being instructors to facilitators. More explicitly, for such tasks, teachers talked less and provided more time for students to work on tasks independently as well as within their groups, without providing excessive guidance. During whole class discussions, the teachers generally completed the proofs for students and posed questions that required recollection of facts rather than the elaboration of arguments.

These results suggest that there is a relationship between the levels of cognitive demand of enacted tasks and the proof schemes utilized. *External conviction proof schemes* were more visible when *memorization* and *procedures without connections* tasks were posed. In fact the frequency of codes relative to relationship of cognitive demands and proof schemes were greater than *external conviction proof schemes* in relations to lower-levels of cognitive demand than any other relations. Teachers and textbooks were viewed as the authority of the mathematics. Students were encouraged to mirror the teacher's actions in constructing proofs and recall the appropriate reason to fill in the blanks from the list of reasons. Teachers sought to make the learning of proof more "comfortable" or "easy" such that students attain some form of success. The emphasis on rules and memorized reasons from a prepared list could have discouraged any original attempts at proofs. Moreover, sometimes students randomly selected reasons without evaluating the appropriateness of their choices. The few instances in which higher-level cognitive demand proof tasks were posed, *analytical proof schemes* were more evident than *external conviction proof schemes*. Whenever higher cognitive demand proof tasks were posed, students were challenged to logically link statements and reasons to create an acceptable proof, rather than merely focus on how many steps were needed to construct the proof. There were scant instances in which *empirical proof schemes* were observed. Based on the number of instances I observed, if *empirical proof schemes* existed, they were more likely for a proof-related task that required *procedures with connections*. Mr.

Walker was the only teacher that posed proof tasks with a higher-level of cognitive demand (namely, *procedures with connections*).

For my third research question, my analysis indicates that teachers' beliefs, experience, and desire to make mathematics easy, as well as the demand of community of practice and assessment, contributed to the three participants teachers decision regarding textbook use when teaching proof.

Teachers believed that proof is important, that two-column proof is an effective means to teach proof, that proof is challenging to students and that students' limited ability in proofs demands that low level proofs are taught. Furthermore, teachers had opinions about the potential strengths and limitations of their textbooks, and these opinions appeared to influence their decisions. For example, Mrs. Davis used her textbook for the past six years and had no objections to it, so she used the textbook without adaptation during her instructional practices. On the other hand, Mr. Walker objected to the small number of proof tasks in his book and chose to provide additional proof tasks during his lessons.

Teachers' experience influenced whether they deviate from their district-adopted textbook. For example, Mr. Walker noted that based on his experience he believed that two-column proof representation was the most practical means to introduce students to proofs. Although it may appear obvious that teachers' experience influences how they teach proof and how they use textbook to do so, it is important to document it, since some teachers may trust the textbook more than others, and the adaptations they make depends on previous experiences.

The participant teachers' desire to make mathematics "easy" significantly influenced enacted lessons and how students engaged with proof tasks. In many instances, the teachers completed the proof on the board and the students recorded the proof in their notes. The number of steps for a proof was a generally five or less. Proof tasks assigned as homework frequently required students to fill in the blanks, using words from the word bank or list of reasons.

The academic community in a specific school can influence a teacher's instructional practice. In this study, I document important differences among the participant teachers. For instance, the geometry team significantly contributed to Mrs. Davis's and Mrs. Bethel's instructional practices since this team plan collectively tasks to assign. However, Mr. Walker had more autonomy, since his department consisted of only two teachers. Mrs. Davis stated that she improved her instructional strategies by observing her colleagues teach proof and later following their actions as a model for her own teaching. Therefore, the professional community can influence how proofs are taught and what proofs are taught.

Since proof is not the primary focus of the end of course assessment in geometry or in-class end of chapter exams, the value teachers place on proof is not as high as other topics (such as finding angles measurements, etc). Assessment influences in this way the priorities that teachers set when making decisions about teaching proof in geometry.

Discussion

Despite proof tasks being visible in subject specific geometry textbooks (Bass et al., 2004; Larson et al., 2007), more proof tasks should be included in geometry textbooks to facilitate students learning to prove. Of the three chapters examined, 7.41%

of the tasks in *Prentice Hall Geometry* and 13.1% of the tasks in *McDougal Littell Geometry* were proof tasks. The few instances of proofs, and the noticeable difference of level cognitive demands of proof tasks between the book supports the claim that textbooks needs to devote more attention to proofs, and should seek to increase the percentage of *doing mathematics* proof tasks.

Chávez (2003) and Grouws and Smith (2000) have reported teachers use textbooks as primary resources for mathematical tasks, and my study confirms their conclusion. Even when teachers chose to deviate from the textbook during enacted lessons, most of the homework assignments came from the textbook. Additionally, the textbook structured the means in which teachers progressed from one mathematical idea to another. Teachers did not sporadically assign lessons, but carefully followed the sequence indicated by their textbooks. The decision to ignore particular lessons or use extra instructional days for certain lessons, are common examples of how teachers at not completely faithful to their textbooks and deviates from the authors' recommendation (McNaught et al., 2010). Even considering Mr. Walker's decision to pose additional proof tasks, the textbook was still a primary source for his homework assignments. In all three cases, it is clear that the textbook was a primary source of mathematics problems and a guide for content and sequence of their lessons.

The overarching theoretical framework (Remillard, 2009) used for this study adequately captured teachers' interaction with their curriculum resources as they taught proof in geometry. Remillard (2009) suggested that there exist a bidirectional relationship between teacher resources and curriculum resources, which can influence students learning. Based on data sources, the three teachers in the study readily utilized the tasks and their instructional practices aligned with emphasis of the geometry textbooks, as well as the agency and social capital within the school community. The teachers acknowledged that they collaborate with their fellow colleagues in sharing tasks. Unlike Mrs. Bethel and Mrs. Davis, who planned their instructional lesson with the geometry team, Mr. Walker had a greater autonomy in his practice. Given this particular freedom, Mr. Walker more readily deviated from the textbook when teaching proof. This interaction between curriculum resources and teacher resources greatly influenced the content covered and how it was covered.

Bieda (2010) reported teachers enacted proof tasks in the written curriculum as such. Extending the works of Bieda, my study found that even if the tasks require students to write complete proofs, teachers may reduce the level of cognitive demand of tasks in an effort to make the task more accessible to students. This finding is not isolated to teaching of proofs, considering that other researchers have documented that teachers have a tendency to reduce the levels of cognitive demand of tasks when enacted (Henningsen & Stein, 1997; Stigler & Hiebert, 1999).

The teaching of proof, as I observed in these three classrooms, is consistent with previous findings (Martin & McCrone, 2001; McCrone et al., 2002): teachers were the mathematical authority in the classroom, proofs could be solved rather quickly, the proof tasks posed did not necessarily need to be proved and few opportunities were provided for students to make sense of the mathematics. Unfortunately, proof as it is currently taught, appears mundane and students can hardly see the importance of doing proofs in mathematics. This further compounds the complexity of teaching proofs, considering that Battista and Clements (1995) found creative ways of teaching proofs are ineffective to facilitate students learning to prove. Efforts must be made by textbook publishers and educators to pose richer proof tasks that require students to engage in the process of creating arguments and making sense of what needs to be proved. Furthermore, whenever teachers led whole class discussions the teachers primarily wrote the proofs for students. This practice diminished opportunities for students to think about proofs or associate hypotheses and theses. Even if the teaching of geometry is not restricted to proofs, (Jones, 2002), it is usually the context in which high school students traditionally learn about proofs. It is therefore important to place a greater emphasis as to how proofs are taught and what experience students have with proofs in geometry.

Although various forms of proof representations appears in textbooks (Cirillo & Herbst, 2010), the teachers in this study used the two–column proof representation as the primary proof representation. The use of two-column proof representation reaffirmed claims made by researchers (Lakatos, 1976; Schoenfeld, 1986) that students view proofs as a linear process, which obscures students engaging with proofs.

Factors influencing the teaching of proof such as teachers' belief, although it cannot necessarily be controlled, should not be ignored. Knuth (2002) found that that teachers believed that proof was a means to communicate mathematics and construct new knowledge. He also found that teachers considered the characteristics of a convincing proof argument to include concrete features and amount of details. Similar to Knuth, my results indicate that teachers' beliefs is a factor that can contribute to their instructional practices of teaching proofs. The teachers believed that proof was important, students experience difficulty writing proofs and that two-column proofs are ideal when writing proofs in geometry. In fact, teachers encouraged students to write two-column proof before constructing proofs using other proof representations.

Additionally, the lack of emphasis on proofs in assessment measures may influence teachers' decision to reduce the attention given to proof during their instructional practices. Teachers seek for their students to do well on state-mandated exams. Hence they seek to ensure students are exposed to the content that will be primarily assessed. With that said, if proofs are not assessed frequently, and if assessed seldom requires at higher-levels of cognitive demands, the emphasis on doing proofs may be devalued.

Finally, the results suggest that despite the three primary proof schemes identified by Harel and Sowder (1998), teachers who used the kind of subject specific geometry curriculum that these teachers used generally encourages students to develop *external conviction proof schemes*. When doing proofs, students were generally encouraged to memorize teachers' actions and subsequently follow the model when doing similar proofs. Such practices limited students opportunities to engage in the *doing of mathematics* and suggested that teachers and textbooks are the sole bearers of mathematical knowledge. Not requiring students to think independently hinders students from creating their own arguments, as well as curtails the prospect of students considering how to logically connect ideas. Most of the proof tasks that the students of these three teachers had to do were known facts and did not require students to consider individual cases or new possibilities. Proof tasks of this nature further contributed to the development of *authoritarian* and *ritual* proof schemes, because the concluding statements were generally known to be true, so the attention was not placed on evaluating the validity of statements but rather on the number of steps and key phrases that will generally prove the theorem. Thus, there were minute opportunities for students to strengthen their *empirical proof schemes* or *analytical proof schemes*. There were few lessons that fostered students developing *empirical proof schemes.* Students were seldom required to consider individual cases to make generalizable claims, or find contradictory arguments. To some extent, this is due to the number of tasks included in the textbooks. Moreover, the proof tasks posed in the exercises seldom varied from the examples provided in the lesson notes. If students had to provide a contradictory argument or counterexample, the context was generally known to be false (thus little to no thinking required). When *analytical proof schemes* were observed, generally the teachers allowed students to work independently. However, for most of the whole class discussion, the teachers in this study did the talking, and most of the thinking. Instructions about proofs would be significantly strengthened if teachers posed richer proof tasks and provided more opportunities for students to share their ideas about proofs as a class, rather than simply telling students.

Implications

My results make clear that subject-specific geometry textbooks used by the teachers in this study provided limited opportunities for students to write proofs, or visualize complete proof arguments. I also examined cases of how geometry teachers use curriculum materials to teach proof. Teachers often enacted proof tasks as *procedures without connections* and *authoritarian proof schemes* were more evident when low-level cognitive demand tasks were posed. As I have discussed above, the results of this study have implications on textbook development and changing teachers' instructional practices.

Considering the major role geometry textbooks plays in the teaching and learning of proof, developers should revise the tasks they include in their textbooks so that students have more opportunities to create original proofs and experience tasks of a higher-level of cognitive demand. Including tasks that encourage students to explore and discover mathematical ideas could enhance the geometry textbooks I studied. The textbooks need to reduce the excessive amounts of proof tasks, which only require students to fill in blanks or identify missing links in proof tasks.

Teachers' decision to make proof "easy" for students might have hindered students conceptualizing how to construct rich proofs. Emphasizing the memorization of a list of reasons and the structure of proof is a strategy that results from the decision and is counterproductive. Teachers should seek opportunities, either adapting or choosing carefully tasks, to challenge their students and let them explore different ways of writing proofs so they move beyond two-column proofs and learn that the format of the proof is less important than the arguments used in a proof.

The potential relationship between the levels of cognitive demands of tasks and proof schemes suggest that teachers should consider higher cognitive demand tasks in order to encourage the development of *empirical proof schemes* or *analytical proof schemes*. If textbook developers include these higher-level cognitive demand proofs and proof-related tasks, students would have more opportunities to develop these schemes.

Factors such as teachers' beliefs and their intention to make proof easier significantly influenced the level of the cognitive demand of the enacted tasks. Consequently, teachers gave excessive amounts of guidance when working on proof tasks. While these factors are related to the goal to have students obtain success in mathematics, such factors are subtracting opportunities for students to engage in the type of reasoning that proof entails. Although factors influencing the teaching of proof may be guided by good intentions, they may be a catapult for catastrophic results in students developing a feeble conception of writing proofs. Considering that students' experience with proofs generally suggest that proof can be solved quickly, are less than 5 steps and require filling in the blanks, the notion that proofs embodies the essence of *doing mathematics* may never be realized.

Implications for Future Research

In examining how geometry teachers use subject-specific curriculum materials for teaching proof, I found some areas of concern. Specifically the fact that there are few proof tasks in geometry textbooks and when enacting these tasks teachers primarily complete the proofs for students during whole class instruction. Since this study included only three cases, future studies can seek to validate these results by utilizing a larger sample of teachers, across multiple locations to increase the generalizability of claims made. Additionally, since this study was restricted to subject-specific textbooks in geometry, future studies can examine these questions using integrated curriculum materials, across multiple content strands or across grade levels. Given my finding that there exist a relationship between the levels of cognitive demands and proof schemes, future studies can seek to document the impact curriculum materials have on the proof schemes students develop. Considering the literature is limited relative to the teaching of proof in relation to curriculum materials, future studies are needed for the field of mathematics education.

Another interesting observation was that teachers always required students to mark on the diagrams before writing proofs. Hence, future studies can seek to explore the role diagrams play in teaching proofs. Reflecting on the fact that teachers generally provided the diagrams, researchers could examine the effectiveness of teaching proofs in which students are required to construct the diagrams themselves.

Furthermore, teachers should have opportunities to reflect and learn from their practice. Changing ones practices is not always a flick of a switch. Therefore, we need to document teachers' reflections of their instructional practices if interventions and models of "good" practices are provided. Hence, researchers can provide professional development on teaching proofs, and subsequently study teachers who participated in the program to document the influence of professional development on teachers' instructional practices pertinent to proof.

Implications for Teacher Education

Considering the different factors that contribute to how proof is taught and that teachers resort to tasks of low levels of cognitive demand when they teach proof, teacher education programs need to strengthen teachers understanding of how to develop rich proof arguments and promote the enactment of tasks that require higher-level thinking skills. For instance, they could provide samples of tasks representing both higher and lower levels of cognitive demands, and encourage preservice teachers to write their own tasks that require higher-level demands. There is no doubt that teachers' knowledge can potentially affect their instructional practices of how they unpack proofs in the classroom. Teachers who are unable to prove may avoid teaching proofs due to their frail understanding of proof, or choose to stay on the periphery of facilitating rich proof tasks.

Mrs. Davis acknowledged that initially she was not good at proof and that her instructional practice improved over time after observing her fellow colleagues. Her desire to improve her practice is admirable, but it is unlikely that she observed effective practices related to the teaching of proof. Teachers should seek to have students engage in writing proofs, rather than doing the proofs for the students. For the same reason, it is not enough that teachers see how proofs are done. During their preparation, teachers should have abundant experiences with proofs in different areas of mathematics, using different formats and representations, and should experience themselves the struggles and successes of putting together convincing arguments. By deepening preservice teachers understanding of proofs and enriching their experiences with doing proofs teacher educators will positively influence how teachers teach proof to future students.

As it relates to assessment of proof, teacher education programs ought to encourage proof being viewed as a process that can be used in all aspects of mathematics, rather than simply a topic that is represented in few lessons within the textbook. Teachers ought to be encouraged to integrate proof in their assessment practices.

Finally, considering that not all of the tasks included diagrams in the book, however tasks enacted generally did, teacher educators needs to be more explicit about the use of diagrams in doing proofs. For example, teacher educators can recommend to preservice teachers to allow students to construct the diagrams rather than doing it for them. Hence, they can suggests that when teaching proofs not to use similar diagrams or aids to exercise tasks. This will seek to ensure that students are provided the opportunity to engage in the thinking practice of the discipline.

Implications for Professional Development

Teachers probably lack the pedagogical content knowledge needed to develop convincing arguments. If professional development can be a means enhance teachers' ability to prove and teach proof, then it needs to provide opportunities for teachers to write proofs and to examine samples of students proof. These experiences may help teachers to reflect on students' conceptions and misconceptions of proof.

Furthermore, considering that all of the proof tasks observed were situated within an abstract setting, professional developers could discuss with teachers how could realistic problems help students provide conjectures that could be proved later.

Professional development experiences could help teachers become knowledgeable about factors that contribute to the maintaining or decline of the level of cognitive demand of mathematical tasks and help teachers see how this can affect the way their students experience proofs in the classroom. Mrs. Davis and Mrs. Bethel acknowledged during the initial interview that they pose low-level proof tasks, and I confirmed this during observed lessons. Both teachers were knowledgeable of the fact that the proof tasks posed required limited thinking, however they chose not to increase the levels of difficulty of the tasks. Professional development should seek to encourage teachers to pose higher-level tasks, and to avoid doing anything that contributes to the decline of the level of cognitive demand. While all these recommendations can be applied to other circumstances, my findings suggest that it is particularly important for teachers who need to enrich their strategies for teaching proof.

Limitations of the Study

This study focused on only three geometry teachers within the midwest region of the United States. It is not generalizable to all teachers, or across multiple mathematical content strands. Nevertheless, this study provides insight into how curriculum materials are used to teach proof in geometry. Considering that research is scarce in documenting how proof is taught, this study extends the literature with the transformative view that there exist a relationship between the levels of cognitive demand of tasks and proof schemes utilized during the enacted lessons.

Another limitation of the study is that it focused on only three chapters of district-adopted textbooks during a portion of the semester, rather than the entire textbook or for a full semester. Nonetheless, the patterns in teachers' instructional practices were repeated in all three chapters and during multiple observations. The results emphasized that generally geometry teachers' engagement with tasks during whole class instruction encouraged *procedures without connections* and that teachers were the authority of the mathematics.

Finally, another limitation of the study was that the research design did not include an analysis of students learning of proof. However, previous literature acknowledged that students have difficulty constructing proof. My study did not seek to explore this notion further; rather it sought to gain an understanding about how proof is taught. By understanding how proof is taught future studies may seek to bridge effective teaching practices to students being able to write complete proof arguments.

Summary

In closing, my study found that district adopted curriculum materials provide minimum opportunities for students to construct proofs. I found teachers engagement with proof tasks required lower levels of cognitive demands. Furthermore, the results suggest that there exist a possible relationship between the level of cognitive demands of proof tasks and proof schemes used, and that various factors can contribute to how proofs are taught. The findings of this study have implications on teacher education programs and professional development initiatives. Moreover, the results of this study brings to light the need to improve how proof is taught in geometry. The results raise some concerns and underscore the need for teachers to provide greater opportunities for students to write proofs in its entirety by themselves. The questions teachers pose to students should not be merely to recall facts, but should develop students' critical thinking skills and engagement in meaningful tasks. Furthermore, textbook developers should seek to reduce the number of low-level tasks, such as fill in the blank exercises, and increase the number of tasks in which students are required to write proofs. It was particularly notable that almost half of the proof tasks (for the chapters studied) in *Prentice Hall Geometry* required students to complete skeletal proofs by filling in the blanks. It is hoped that future subject-specific curriculum materials significantly increase students' opportunity to prove and that a large portion of the proof tasks are situated within real world contexts. The process of learning to prove needs to be presented in a meaningful way such that students can value the practicality of proofs beyond the borders of their geometry classrooms.

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APPENDIX A: CLASSROOM OBSERVATION PROTOCOL

This protocol is adapted from Horizon Research Inc [36], and is used to examine how geometry teachers use curriculum materials to teach proof.

PART I A. Background Information

1. Observer: ____

2. Date(s) of Site Visit: ____________________________________

3. School Name: ____

4. Teacher: ____

5. Class Period ____

B. Context and Nature of the Lesson

1. Instructional Material used for this class of students (include text, chapter, section, page numbers if applicable):

Mark which of the following math strands was the focus of this lesson:

Geometry and Measurement	___	Reasoning-and-Proving	___
Patterns, Functions and Algebra	___	Number Concepts	___
Data and Probability	___	Other	___

If other, please provide details: ________________________________

____

____

C. Students:

1. Grade: ____________________ 2. Total number: ___________________

D. Outline of the Lesson:

1. Please list the stated goals for the lesson as described in the instructional materials and/or provided by the teacher.

[36] This is adapted from Horizon Research, Inc. classroom observation protocol developed for the Cases of Reasoning and Proving in Secondary Mathematics Project, with funding from the National Science Foundation (Award No. DRL-0732798)

2. Briefly (2-3 paragraphs) describe the structure and flow of the lesson you observed [non- evaluative].

3. Describe how reasoning-and-proving was incorporated into the lesson. For example, are students intended to learn about aspects of reasoning-and-proving or use reasoning-and-proving in order to learn a different mathematical concept?

PART II A. Classroom Culture

1. To what extent did the classroom culture/learning environment facilitate students' opportunity to learn the targeted mathematical ideas in this lesson? For example, were students respectful to the teacher and each other? Did there appear to be high expectations for learning of all students? Were students willing and motivated to learn?

1	2	3	4
Greatly inhibited	Somewhat inhibited	Somewhat facilitated	Greatly facilitated

Please provide a rationale to support your rating. Use the questions above to guide your response:

2. To what extent did the mathematical norms of the classroom facilitate students' opportunity to learn the targeted mathematical ideas in this lesson? For example, do the students understand what counts as an acceptable mathematical explanation and justification? Do students challenge others thinking? Does the teacher press students to explain their thinking? Are the students doing most of the intellectual work?

1	2	3	4
Greatly inhibited	Somewhat inhibited	Somewhat facilitated	Greatly facilitated

Please provide a rationale to support your rating. Use the questions above to guide your response:

B. Use of Instructional Tools

1. Teacher Tools
To what extent did the teacher's use of tools facilitate students' opportunity to learn the targeted mathematical ideas in this lesson? Did the teacher use tools to attend to and/or keep track of student ideas (e.g., monitoring, selecting, or sequencing work)? Did the teacher's use of tools distract students or did the absence of tools limit the teacher's ability to aid students in learning?

1	2	3	4
Greatly inhibited	Somewhat inhibited	Somewhat facilitated	Greatly facilitated

Please provide a rationale to support your rating. Use the questions above to guide your response:

C. Facilitation of the Task(s)

To what extent did the teachers' facilitation of the proof task and/or proof-related task[37] support students' in learning the targeted mathematical ideas in this lesson? For example, did the teacher appear to have anticipated likely student responses to the task(s)? Did the teacher monitor students' responses while they worked? Did the teacher select and purposefully sequence student work to be displayed? Did the teacher help the class make mathematical connections between different student responses and between student responses and the targeted mathematical ideas?

1	2	3	4
Greatly inhibited	Somewhat inhibited	Somewhat facilitated	Greatly facilitated

Please provide a rationale to support your rating. Use the questions above to guide your response:

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[37] **Proof Tasks** – are tasks designed to engage students in proving, constructing mathematical conjectures and pattern generalization, as well as task that requires students to develop a proof argument. For this study, proofs and proof tasks may be used interchangeably.
Proof-related tasks are tasks that are related to a proof in the sense that are meant to provide students with an opportunity to perform a step that may be used in later proofs and are not necessarily proof tasks by themselves. For example, completing the steps of an unfinished proof, identifying corresponding sides in a triangle, identifying the congruence criterion that must be used in a given proof, etc.

D. Cognitive Demand of the Task(s)

A teacher may choose mathematically simpler task(s) in order to focus on higher-levels of reasoning-and-proving. Focus on the cognitive demand with respect to the purpose of the task(s) when providing your rating and rationale.

1. Intellectual Potential of the Original Task(s)
Identify the primary kind of cognitive demand of the task(s) as it appeared in the resource materials.

0	1	2	3	4
No academic thinking required	Memorization	Use of procedures without connection to meaning, concepts or understanding.	Use of procedures with connection to meaning, concepts or understanding.	Engaging in the thinking practices of the discipline

Please provide a rationale to support your rating:

2. Intellectual Potential of the Planned Task(s)
Identify the primary kind of cognitive demand of the task(s) as set up by the teacher.

0	1	2	3	4
No academic thinking required	Memorization	Use of procedures without connection to meaning, concepts or understanding.	Use of procedures with connection to meaning, concepts or understanding.	Engaging in the thinking practices of the discipline

Please provide a rationale to support your rating:

3. Engagement with the Task(s)
Identify the manner in which most students and the teacher actually engaged with the task(s) for the majority of the time.

0	1	2	3	4
No academic thinking required	Memorization	Use of procedures without connection to meaning, concepts or understanding.	Use of procedures with connection to meaning, concepts or understanding.	Engaging in the thinking practices of the discipline

Please provide a rationale to support your rating:

E. Proof Schemes
A teacher may "apply different methods to remove doubts in the processes of ascertaining and persuading accordingly: "A person proof scheme consists of what constitutes ascertaining and persuading for that person. …It is important to note that these schemes are not mutually exclusive; people can simultaneously hold more than one kind of scheme" (Harel and Sowder, 1998, p. 244).

Please identify the proof scheme(s) encouraged by the teacher during the enacted lesson.

Task	**External conviction Proof scheme**			**Empirical Proof schemes**		**Analytical proof schemes**	
	Authoritarian	**Ritual**	**Symbolic**	**Inductive**	**Perceptual**	**Transformational**	**Axiomatic**

Please provide a rationale to support your rating:

APPENDIX B: TASK COVER SHEET – BEFORE IMPLEMENTATION

This instrument is adapted from Horizon Research Inc [38], and is used to examine how geometry teachers use curriculum materials to teach proof.

Teacher Name: ________________

Date(s) the task will be implemented: ________________

1. Where did you find the task assigned during the class, and for the homework assignment (e.g. textbook, website, etc)?

2. What modifications, if any, did you make to the original task and why?

3. What specific mathematics idea (s) will be targeted in the lesson(s) with this task?

4. What do you think students already understand about these idea(s), either from previous instruction this year or in previous years or from their experiences outside of school?

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[38] This is adapted from Horizon Research, Inc. classroom observation protocol developed for the Cases of Reasoning and Proving in Secondary Mathematics Project, with funding from the National Science Foundation (Award No. DRL-0732798)

APPENDIX C: TASK REFLECTION SHEET- AFTER IMPLEMENTATION

This instrument is adapted from Horizon Research Inc [39], and is used to examine how geometry teachers use curriculum materials to teach proof.

Teacher Name: ________________

Date(s) the task was implemented: _____________________________________

1. Describe any directions oral or written you gave to students that are not included on the task itself. Include what you communicated to students about expectations for their work.

2. Did you implement the task differently than you had planned? If so, what changed did you make and why?

3 Did students develop new understanding about the mathematical idea(s) targeted in this task? How do you know?

4. Did students make mathematical arguments, either proofs or non-proofs, during the task? if so, what argument did they make? How did their engagement compare with what you had envisioned before implementing the task?

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[39] This is adapted from Horizon Research, Inc. classroom observation protocol developed for the Cases of Reasoning and Proving in Secondary Mathematics Project, with funding from the National Science Foundation (Award No. DRL-0732798)

VITA

Ruthmae Sears was a student and teacher at C.R. Walker Senior High School in Nassau Bahamas. She obtained her undergraduate degrees from The College of The Bahamas (Associate of Arts in Mathematics and Statistics, and a Bachelors of Education in Secondary Mathematics). In 2006 she completed her Masters degree at Indiana University in Mathematics Education, and in 2008 she enrolled in the doctoral program at the University of Missouri (Columbia). Four years later, she successfully defended her doctoral dissertation on April 9, 2012.

During her doctoral studies, she worked as a graduate research assistant, and teaching assistant at the University of Missouri (Columbia). She was assigned to two National Science Foundation research projects: namely (i) Researching Science and Mathematics Teacher Learning in Alternative Certification Models (ReSMAR2T project) and (ii) Comparing Options in Secondary Mathematics Investigation Curriculum (COSMIC). Additionally, she taught Intermediate Algebra and College Algebra in the Mathematics Department.

Ruthmae is indeed a community minded individual. She is a member of the Bahamas National Science Council, Kappa Delta Pi Honor Society in Education, Golden Key International Honour Society, and Griffiths Leadership Society for Women. Additionally, she served as an Advisor for the Bahamas' National Teacher Cadet Program, volunteers for P.A.C.E. School for teen mothers and Ronald McDonald House Charities, and was the founder of C.R. Walker Senior High School Maidens club.

As a scholar, Ruthmae was a recipient of the Organization of American States Fellowship Award, the Lyford Cay Foundation Graduate Scholarship Award and Kappa Delta Pi - Hollis L Caswell Laureate Scholarship. Additionally, she has presented at national and international conferences on topics pertinent to reasoning and proof, teaching of geometry, and curriculum materials.

Ruthmae is an Assistant Professor of Mathematics Education in the Department of Secondary Education at the University of South Florida.