Convergence of an Infinite Series.

This Thesis gives some of the more important tests for the convergence of an infinite series, also the conditions that must be fulfilled in order that certain operations and transformations may be applied to an infinite series.

The Authors consulted were:

I. Tannery: Theory des Fonctions d'une Variable.
II. Serret: Cours d'Algebre Superieure.
IV. Stolz: Allgemeine Arithmetik.
V. Wood: Introduction to Infini Series.
VII. Forsyth: Theory of Functions.
VIII. Byerly: Convergence of Fourier's Series.
IX. Lehman: Higher Algebra.
X. Smith: Higher Algebra.

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Convergence of an Infinite Series

A series is a succession of quantities which are formed according to some law. When there is an endless succession of terms, the series is said to be infinite.

When a series becomes infinite, two difficulties arise. 1. What is the result of such a series, and 2. whether the series of operations, even when its meaning is defined, can be subjected to the laws of Algebra which deals with operations where the frequency of the operation is finite.

The simplest case of an infinite series is the geometric series.
The fact that a geometric series can be summed, simplifies the first difficulty. The leading feature of the problem of an infinite series are present in the geometric series, and questions of convergence of an infinite series are generally referred to the standard case of the geometric series.

If we take a succession of finitely many terms formed according to a fixed law so that the $n$th term is a finitely one-valued function of $n$, and consider the successive sums:

$$S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3, \quad \ldots \quad S_n = u_1 + u_2 + \ldots + u_n$$

When $n$ is increased more and more, one of these things must happen: $S_n$ may approach a fixed, finite quantity $S$, in such
a way that by increasing \(m\) indefinitely, we can make \(S_n\) differ from \(S\) by as little as we please. That is, \(\lim_{n \to \infty} S_n = S\).

In this case the series \(\sum u_1 + u_2 + u_3 + \cdots\) is said to be convergent and to converge to the value of \(S\), which is spoken of as the sum to infinity of \(u_n\). The sum to infinity as \(\sum (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots)\), here \(S = \lim_{n \to \infty} S_n = 2\).

II In may increase with \(m\) in such a way that by increasing \(m\) sufficiently we can make the numerical value of \(S_n\) exceed any quantity however large. That is, \(\lim_{n \to \infty} S_n = \infty\). In this case the series is said to be divergent, as the series \(1 + 2 + 3 + \cdots\), where \(\lim_{n \to \infty} S_n = \infty\).

III \(S_n\) may neither become infinite nor approach a definite limit, but oscillate between a number.
of finite values, the selection among which is determined by the integral character of \( n \). That is, by such considerations as whether \( n \) is odd or even, of the form \( 3m \), \( 3m + 1 \) or \( 3m + 2 \), etc. In this case, the series is made to oscillate as in the series, \( 3 - 1 - 2 + 3 - 1 - 2 + \ldots \). The value of \( \frac{s}{n} \) is 3, 2, or 0, as \( n \) is of the form \( 3m \), \( 3m + 1 \), \( 3m + 2 \) or \( 3m \).

A non-convergent series cannot be paid to have a sum, and cannot be employed except in special cases in mathematical reasoning.

A series whose terms are all of the same sign cannot be indeterminate, but must be convergent or divergent, for unless the summation of \( n \) terms increases without limit, there
must be some finite limit which
the sum can never exceed but to
which it approaches indefinitely
near. If each term of an infinite
series be finite and all the terms
have the same sign, the series
must be divergent. For if each
term be not less than \( q \) the sum
of \( n \) terms will not be less than
\( qn \), and \( \infty n \) can be made greater
than any finite quantity however
large by sufficiently increasing
\( n \).

Series are said to be more or
less rapidly convergent, as the
number of terms, which it is
necessary to take to get a given
degree of approximation to the
sum, is smaller or larger.
Thus a geometric series is more
rapidly convergent, the smaller
its common ratio.

Rapid convergence is obviously
a valuable quality in a series
from the arithmetic point of view.
The definition of convergence of the
series $a_1 + a_2 + a_3 + \cdots$ involves the
supposition that the terms are
taken successively in a given order.

There is a class of series as
was first shown by Dirichlet
which may converge to one value
or to any other, or even become
divergent according to the order
in which the terms are taken.

Two essential conditions are
involved in the definition of an
infinite-series. 1st. It shall not
become infinite for any value
of $n$, however great, and 2nd.
that
As \( n \) increases there shall be continual approach to a definite limit \( S \). The symbol \( \sum_{n}^{\infty} \) is used to denote the sum \( (u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+m}) \), that is the sum of \( m \) terms following the \( n \)th. So the criterion might be given. The necessary and sufficient conditions for convergence of an infinite series are that

\( S_{n} \) be finite, and that by taking \( n \) sufficiently great it may be possible to make \( \sum_{m}^{\infty} \) as small as we please no matter what the value of \( m \) may be. That \( S_{n} \) must be finite is obvious from the definition of convergence since

\[ \sum_{n=1}^{\infty} S_{n} = S \text{ where } S \text{ is finite.} \]

Therefore \( \sum_{n=1}^{\infty} S_{n+m} = S_{n} \) and \( \sum_{n=1}^{\infty} (S_{n+m} - S_{n}) = 0 \), which is

\[ \sum_{m}^{\infty} R_{n} = 0. \]
These two conditions are sufficient, for since $S_n$ is finite for all values of $n$, the limit of $S_n$ cannot be infinite, and the limit of $S_n$ cannot have one value when $n$ has any particular integral character and another value when $n$ has another character; for any such results would imply that $S_n$ and $S_{n+m}$ should have different values, but such cannot be the case as if $(S_{n+m} - S_n) = 0$, $P_m = 0$.

In any convergent series $\sum_{n=0}^{\infty} u_n = 0$ for $u_n = S_n - S_{n-1} = P_{n-1}$, but the limit $P_{n-1} = 0$ for $\lim_{m \to \infty} u_m = 0$. This is a necessary but not sufficient condition as is seen by the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{7}$, which cannot be convergent, although the $n$th term diminishes indefinitely, for the sum of $n$ terms after the
in which is \( \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \). This is greater than \( \frac{1}{n} \) or \( \frac{1}{2} \).

If \( P_r = \sum_{n=1}^{\infty} P_n \) and \( S_r + S_n \) have their usual meaning, then

\[ S_n = 3 - P_n, \quad \text{for} \quad S_{n+m} = S_n + P_n \quad \forall m \geq 0. \]

\[ S_{n+m} = S_n + 2^n P_n, \quad \text{and} \quad m \geq 0. \]

Thus, \( P_n = 3 - S_n \). \( P_n \) is called the residue of the series, and \( P_m P_n \) a partial residuum. The smaller \( P_n \) is for a given value of \( n \), the more convergent is the series for \( P_n \). The difference between \( S_n \) and the limit of \( S_n \) when \( n \) is indefinitely increased. \( P_n \) is the sum of the infinite series \( u_{n+1} + u_{n+2} + u_{n+3} + \ldots \) and the residuum of a convergent series is itself a convergent series.

The convergence or divergence of an infinite series is not affected.
by neglecting a finite number of its terms, for the sum of a finite number of terms will be finite, and definite, and the neglect of that sum alters \( S_n \) merely by a finite determinate quantity, so that if the series was at first convergent it will remain so, and if divergent or oscillating it will remain so.

The series \( \sum \frac{1}{n} \log \frac{2^2}{2 \cdot 2} + \frac{1}{2} \log \frac{3^2}{3 \cdot 4} + \cdots \)

\[ + \frac{1}{n} \log \frac{(n+1)^2}{n(n+2)} + \cdots \]

is convergent for \( \frac{(n+1)^2}{n(n+2)} = \frac{1}{n^2} + \frac{1}{n+1} \), \( 20 \log \frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n+2}} \)

\[ \frac{1}{n+2} \log \frac{1 + \frac{1}{n+2}}{1 + \frac{1}{n+3}} + \cdots \]

which is less than \( \frac{1}{n+1} \left( \log \frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n+2}} + \log \frac{1 + \frac{1}{n+2}}{1 + \frac{1}{n+3}} + \cdots \right) \)

and so less than \( \frac{1}{n+1} \log \frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n+m+1}} \).

Now whatever values \( m \) may have, we may take \( n \) so large that \( n \)

and \( \frac{1}{n+m+1} \) may be as small as
we please. Therefore $m P_n = 0$ for all values of $n$. If we place $s$ in place of $m$ and $s m$ in place of $P_n$ and observe that $S_n = s P_n$ we see that $S_n < \log \left( \frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n+1}} \right)$, so $S_n \leq 2$.

Both conditions of convergence are satisfied.

Putting more in $m P_n \leq \frac{1}{n+1} \log \left( 1 + \frac{1}{n+1} \right)$, the residue of the series $P_n$ is less than $\log \left( 1 + \frac{1}{n+1} \right) / n+1$, a result which enables us to estimate the rapidity of convergence and to settle how many terms of the series we ought to take to get an approximation to its limit accurate to a given place of decimals.

Let $u_n$ and $v_n$ be positive and $u_n$ be less than $v_n$ for all values of $n$, and $\sum v_n$ is convergent, then $\sum u_n$ is convergent, for by the conditions
the value of $S_n$ and $m_n$, belonging to $\mathbb{E} v_n$
are less than the corresponding values of $S_n$ and $m_n$, belonging to $\mathbb{E} v_n$. Therefore
we have $0 < S_n < S_n'$ and $0 < m_n < m_n'$ but
$S_n$ is finite for all values of $n$, and
$m_n = 0$. Therefore $\mathbb{E} v_n$ is convergent.

To insure the convergence of the first
series it is not necessary that all
its terms should be less than
the corresponding terms of the
second series; it will be sufficient
if all the terms, except a finite
number of them, be less than
the corresponding terms of the
second, for the sum of a
finite number of terms must
be finite.

If $v_n$ is greater than $v_n$ and
$\mathbb{E} v_n$ is divergent then $\mathbb{E} u_n$ is divergent.
$S_n > S'_n$ hence since $S_n = \infty$ we must have $S'_n = \infty$. Therefore $S'_n$ is divergent.

If for all values of $n$, $u_n > 0$, and $\frac{u_n}{V_n}$ is finite, then $\Sigma u_n$ is convergent, if $\Sigma V_n$ is convergent, and divergent if $\Sigma V_n$ is divergent.

If $A$ be the least and $B$ the greatest of the fractions $\frac{u_{n+1}}{V_{n+1}}, \frac{u_{n+2}}{V_{n+2}}, \frac{u_{n+3}}{V_{n+3}}, \frac{u_{n+m}}{V_{n+m}}$.

$$A < \frac{u_{n+1} + u_{n+2} + u_{n+3} + \ldots + u_{n+m}}{V_{n+1} + V_{n+2} + V_{n+3} + \ldots + V_{n+m}} < B$$

Now since $\frac{u_n}{V_n}$ is finite for all values of $n$, $A$ and $B$ are finite, hence we must have in all cases $m \eta = 0$, where $m$ is a finite quantity, whatever values are given to $m$ and $n$.

Hence $S_n$ or $R_0$ will be finite or infinite according as $S_n$ is infinite and if $\lim_\infty n R_n = 0$, then $\lim_\infty n^2 R_n = 0$. 


If $\frac{u_{n+1}}{u_n}$ be positive, and $\frac{u_{n+1}}{v_{n+1}}$ < $\frac{u_n}{v_n}$, and $\sum v_n$ is convergent, then $\sum u_n$ is convergent.

And if $\frac{u_{n+1}}{u_n}$ > $\frac{v_{n+1}}{v_n}$, and $\sum v_n$ is divergent, then $\sum u_n$ is divergent.

If $\frac{u_{n+1}}{u_n}$ < $\frac{v_{n+1}}{v_n}$, $s_n = u_1 \left(1 + \frac{u_2}{v_1} + \frac{u_3}{v_1v_2} + \cdots + \right) < u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_1v_2} + \cdots + \right)$, but $s_n$ is finite, so $\sum s_n$ is finite also since all the terms of $\sum u_n$ are positive, the series cannot oscillate. So $\sum u_n$ must be convergent.

If $\frac{u_{n+1}}{u_n}$ > $\frac{v_{n+1}}{v_n}$, and $\sum v_n$ be divergent, then $\sum u_n$ is divergent.

$\frac{u_{n+1}}{u_n}$ is called the ratio of convergence of $\sum u_n$, and any series is a convergent or divergent as its ratio of convergence is always less than or greater than.
The ratio of convergence of a convergent series or any divergent series.

A series is convergent if after any particular term the ratio of each term to the preceding term is always less than some fixed quantity which is less than unity.

Let the ratio of each term after the $n$th to the preceding be less than $k$, where $k$ is less than unity.

Then: $\frac{u_{n+1}}{u_n} < k$, $\frac{u_{n+2}}{u_{n+1}} < k$;

$\frac{u_{n+1}}{u_n} + \frac{u_{n+2}}{u_{n+1}} \frac{u_{n+2}}{u_{n+1}} \frac{u_{n+3}}{u_{n+2}}$ $< u_n \left(1 + k + k^2 + \cdots \right)$

$\therefore \frac{u_n}{1-k}$ since this is less than unity.

Hence the sum of the series begins rising at the $n$th term is finite; and the sum of any finite number of terms is finite so the entire series must be convergent.
A series is divergent if after any particular term the ratio of each term to the preceding term is equal to or greater than unity.

Let all the terms after the

\[ u_1 = u_n \]

\[ u_{n+1}, u_{n+2}, u_{n+3}, \ldots \]

\[ + \quad u_{n+r} = \infty \]

and \( n \) can be made greater than any finite quantity by sufficiently increasing \( m \). Therefore the series is divergent.

Now let the series be one whose

c Ratio of each term to the preceding is \( > 1 \). Then \( u_{n+1} > u_n, u_{n+2} > u_{n+1}, u_{n+3} > u_{n+2} \)

Therefore the series must be divergent.

This test will fail where a series is such that after a finite number of terms the ratio \( \frac{u_{n+1}}{u_n} \) is always less than unity but approaches
unity as \( n \) approaches infinity.

As in the series

\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots
\]

whose ratio \( \frac{U_{n+1}}{U_n} = \frac{n^2}{(n+1)^2} = (1+\frac{1}{n})^2 \), as if \( k \) is positive, the last ratio is less than unity but becomes more and more nearly equal to unity as \( n \to \infty \), so we cannot tell whether the series is divergent or convergent. When \( k \) is greater than unity, each term of the series is less than the preceding term, and so the series must be convergent.

When \( k = 1 \), the series becomes

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots
\]

\[
+ \left( \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \right) + \ldots
\]

Now as each group is greater than \( \frac{1}{2} \), the series to \( 2^n \) terms is greater than \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \) taken \((n+1)\) times, that is, greater than \( 1 + \frac{1}{2} n \), which increases
indefinitely with \( n \), as \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots \) divergent. If \( k < 1 \), then the series becomes \( 1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots \), which is greater than \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots \), which has been shown to be divergent, therefore the series must be divergent.

When the terms are half positive and half negative, we first see whether the series which would be obtained if all the signs were made +, is convergent then the series would be convergent. But if the series obtained were divergent it is not necessary that the first series be divergent, as \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) is convergent although \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) is divergent.
A series which would be convergent if all its terms had the same sign is called absolutely convergent. A series which is divergent after its signs are changed to all positive signs but which was convergent before is called semi-convergent.

Many series whose terms are alternately + and − are seen to be convergent at once, as a series is convergent when its terms are alternately + and − provided each term is less than the preceding one and that the terms decrease without limit in absolute magnitude.
Let the series be $u_1 - u_2 + u_3 - u_4 + \cdots + u_n - u_{n+1} - u_{n+2} - u_{n+3} - \cdots$

$= u_1 - u_2 + (u_3 - u_4) + (u_5 - u_6) + \cdots$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \cdots$

Since each term is less than the preceding, the sum of the series must have its value between $u_1 - u_2$ and $u_1$. Hence, the sum of the series is finite. Also, the absolute value of $u - u_n$ is intermediate to the absolute value of $u_{n+1} - u_{n+2}$ and $u_{n+1}$, and so $|u - u_n| = 0$ when $n \to \infty$. Therefore, the series is convergent. For example, $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$ is convergent since the terms are alternately positive and negative and decrease without limit.

The series $\frac{1}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$ is not
convergent although its sum is a
finite quantity between $2^x_k$ for
the $n_{th}$ term, but the $n_{th}$ term
$\frac{n^{n+1}}{n^k}$ does not approach zero as $n$ increases
indefinitely.

I must except a number of
scattered theorems, given chiefly
by Weierstrass in his Meditations
Analytiques, it may be said
that Cauchy was the founder
of the modern theory of convergence
and most of the general
principles of the subject were,
first given in his Analyse
Algebrique. In his Exercices
de Mathématiques Vol II (1827) he
gave the following integral criterion
from which most of the higher
criteria have stemmed: — I for
Large values of $x_0$ be positive and decrease as $n$ increases, the $E$ for is convergent if 
\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(k) = 0, \quad \text{(where } n \text{ is arbitrary)}, \quad \text{otherwise divergent.}
\]

Cauchy has given us the condensation test: \[ \text{Let } E_{mn} \]

a series of constant terms and decreasing in value from the first, without changing the order of these, we may group them according to some law, \[ E_{1n}, E_{2n}, E_{3n} \ldots \] be the 1st, 2nd, \ldots of these groups. The series $E_{mn}$ will contain all the terms of $E_{mn}$, and $E_{mn}$ will be convergent or divergent as $E_{mn}$ is convergent or divergent as 
\[
\lim_{n \to \infty} E_{mn} = E_{m}.
\]
The convergence of $\sum_{n} a_n$ is more apparent than $\sum_{n} a_n$ as in $\sum_{n}$ we proceed by longer steps toward the limit. The sum of $n$ terms of $\sum_{n}$ being nearer the common limit than $\sum_{n}$.

Finally if $\sum_{n}$ be a new series such that $V_n > V_n$ then obviously $\sum_{n}$ is convergent if $\sum_{n}$ is convergent and the converse is true if $V_n < V_n$ and $\sum_{n}$ is divergent then $\sum_{n}$ is divergent.

If $a_n$ be positive for all values of $n$, and constantly decreases as $n$ increases then $\sum_{n} a_n$ is convergent or divergent as $\sum_{n} a_n$ is convergent or divergent, where $a$ is a positive integer $\neq 2$. 
The series $E_k(x)$ may be arranged

$$
\left\{ \begin{array}{l}
\left( x(x+1)+\cdots+x(a-1) \right) + \left( f(a)+f(a+1)+\cdots+f(a^{m+1}) \right) \\
+ \left( f(a^m)+f(a^{m+1}) + \cdots + f(a^{m+1}) \right) \\
\end{array} \right. 
$$

Hence neglecting the finite number of terms in the square bracket, we see that $\Sigma f(x)$ is convergent or divergent as $\Sigma (f(a^m)+f(a^{m+1})+x(a^{m+2})+\cdots+f(a^{m+1}))$ is convergent or divergent.

Now since $x(a^{m+1}) \geq f(a^{m+1}) > f(a^{m-1}) > f(a^{m+1})$

we have $(a^{m+1}-a^m) f(a^m) \geq x(a^m)+x(a^{m+1})+x(a^{m+2}) + \cdots + f(a^{m+1}) \geq (a^{m+1}-a^m) f(a^{m+1})$, that is

$$(a-1) a^m f(a^m) \geq f(a^m)+f(a^{m+1})+\cdots+f(a^{m+1}) > \left( \frac{a-1}{a} \right) a^{m+1} f(a^{m+1}).$$

Then if $f(a-1) a^m f(a^m)$ is convergent the series is convergent, for if $u_1 \geq u_2$ 

and $E_k u_n$ is convergent then $E_k u_n$ is convergent.
If \( E (a + 1) = \sum a^{n+1} \) is divergent, the series is divergent.

Now \( E (a - 1) = \sum a^n \) is convergent. If \( E a^n \) is convergent, and \( E (a) = \sum a^n \)
is divergent if \( E a^{n+1} \) is divergent.

and for present purposes \( E a^n \)
and \( E a^{n+1} \) are practically the same series (say \( E a^n \)), hence Cauchy's theorem is established.

It is obviously sufficient that the function be positive and decrease for all values of \( n \)
greater than a certain finite value \( n \). The theorem holds true for any positive value of \( a \) not less than 2. Let \( a \) lie between the positive integers \( b \) and \( b+1 \), where \( b \) is not less than two. If \( E a^n \)
or 

\[ y = \sqrt{x^2 + 1} \]

and the curvature is the second derivative of the function:

\[ \frac{d^2 y}{dx^2} = \frac{1}{2} \left( \frac{1}{x^2 + 1} \right) \]

After solving the differential equation, we get:

\[ y(x) = \sqrt{x^2 + 1} \]

The solution to the differential equation is:

\[ y(x) = \sqrt{x^2 + 1} \]

For example, when \( \alpha = 0 \), we have:

\[ y(x) = \sqrt{x^2 + 1} \]
Then ultimately \( f(x) > A \) where

\[ A > 0, \quad -f(x) > \frac{A}{x}. \] Now \( \sum \frac{A}{x} \) is divergent, since \( \sum \frac{1}{n} \) is divergent, therefore \( \sum f(n) \) is divergent.

Now use the symbols \( e^x \), \( e^{2x} \),

to denote \( a \), \( a^2 \), \( a^n \) being positive
quantities \( > 1 \), and \( \ln x \), \( \ln^2 x \), \( \ln^n x \)
to denote \( \log a \), \( \log_a \), \( \log \log \), \( \log \log \log \)
and \( \log \log \log \) where \( e \) is the
base of the Napierian system, then

\( \sum f(n) \) is convergent or divergent

as \( \sum e^n \) \( e^n \) is convergent
or divergent. If \( \sum f(n) \) is convergent
or divergent \( \sum e^{un} \) is convergent
or divergent; as \( \sum e^{un} \) \( e^{un} \) is convergent
or divergent, \( \sum e^{un} f(e^{un}) \) \( f(e^{un}) \)
or \( \sum e^{un} f(e^{un}) \) is convergent or divergent,
and so on.

\( \sum f(n) \) is convergent or divergent as
the first of the functions \( f(x) = \frac{1}{x} \),
\[ T_1 = \int x f(x) \, dx, \quad T_2 = \int \left( x f'(x) \right) \, dx \]
\[ T_3 = \int \left( x f(x) \frac{d}{dx} \left[ x^{-1} f(x) \right] \right) \, dx \]

which does not vanish when \( x = 0 \) has a negative or positive sign.

For \( E_f(x) \) is convergent or divergent according as \( E \cdot e^{2n} - e^{2n} \cdot \frac{d}{dx} \left[ e^{2n} \right] \) is less than or greater than unity, that is as \( n \to \infty \)

\[ \log \left( \frac{E e^{2n} - e^{2n} \frac{d}{dx} \left[ e^{2n} \right]}{n} \right) \leq 0 \]

If we put \( x = e^{2n} \) so that \( x = e^{2n} \)

\[ l^x = e^{2\pi} \quad \log x = 2n, \quad x = n, \quad x = n; \quad \text{when} \ x = \infty \]

the condition for convergence or divergence becomes

\[ L^x \left( x f(x) - \frac{d}{dx} \left[ x^{-1} f(x) \right] \right) \leq 0. \]

If we take \( e^x \) for the exponential base, the condition may be written

\[ L^x \left( x f(x) - \frac{d}{dx} \left[ x^{-1} f(x) \right] \right) \leq 0. \]
Each of the series $\sum \frac{1}{n^{a}}$, $\sum \frac{1}{\ln(n)^{a}}$, $\sum \frac{1}{\ln(\ln(n))^{a}}$ is convergent or divergent as $a$ is greater than 0, or less than (or equal to) 0.

The function $L(\ln n - \ln x)$ frequently occurs and is represented by $R_n$.

$R_n$ is convergent or divergent as $x = \frac{1}{\left(\frac{\ln n}{k}\right)^a}$, hence

$$L\left(\frac{\ln n}{k}\right) \left(\frac{\ln n}{k}\right)^{a+1} = -a$$

It follows that $(x+1)$ is convergent if $a > 0$ and divergent if $a < 0$

If $a = 0$, take one order higher

$$L\left(\frac{\ln n}{k}\right) \left(\frac{\ln n}{k}\right)^{a+2}$$

and $L\left(\frac{\ln n}{k}\right) \left(\frac{\ln n}{k}\right)^{a+1} = 1, > 0$

So when $a=0$, $(x+1)$ is divergent therefore the series is divergent.
By the application of Cauchy's condensation test the convergence of \( \sum \frac{1}{2^k a_k} \) is the same as the convergence of \( \sum \frac{a_k}{(qa_k)^{1+\alpha}} \), that is as \( \sum (qa_k)^{-1} \). This is a geometric series with the ratio \( \frac{1}{qa} \) and is convergent if \( a > 0 \) and is divergent if \( a \) is less than, or equal to, 0, hence \( \sum \frac{1}{2^k a_k} \) is convergent if \( a > 0 \).

The convergence of \( \sum \frac{1}{n^{1+\alpha}} \) is the same as the convergence of
\[
\sum \frac{a_k}{(qa_k)^{1+\alpha}} \text{ which is the same as }
\sum \frac{1}{(qa_k)^{1+\alpha}} \text{ which is convergent if } a > 0.
\]

Now let us suppose the theorem holds up to the series \( n \). We can then show that it holds for the
series \((n+1)\). The convergence of 
\((n+1)\) is the same as that for
\[\sum \frac{a^n}{\alpha \cdot \lambda \cdot \alpha ^n} = \lambda ^{-\frac{1}{\alpha}} \{l^{\frac{1}{\alpha}} \alpha \}\] 
that is as \(\frac{1}{n \lambda \cdot (n \lambda)} = l^{(n-2)} \{l^{n-1} \lambda\}\) \(1+a\)

suppose \(a>0\) and \(a>\varepsilon\) then
\(\lambda a > 1\) and \(n \lambda a > n\), hence
\[\frac{1}{(n \lambda) \cdot (n \lambda)} = l^{(n-1)} \{l^{n-1} \lambda\}\] 
\[< \frac{1}{\sqrt[\alpha]{n}} l^{\frac{n-1}{2}} \{l^{\frac{n-1}{\alpha}} \lambda\}\]

But since \(a>0\), \(\varepsilon \sqrt{n} \{l^{n-1} \lambda\}\) is
convergent.

Suppose \(a>0\) and \(2 < \alpha \leq \varepsilon\), then \(n \lambda a > n\)
then \(\sum \frac{B(n) \{l^n\}}{l} \{l^n\}\) is more divergent
than the divergent series
\[\sum \frac{l^{n-1}}{B(n)} \{l^{n-1} \lambda\}\]
The logarithmic series are of great importance, as they form a scale with which we can compare series whose ratio of convergency is ultimately unity. The scale is a descending one for least-convergent of the convergent series of the nth order is more convergent than the most convergent series of the (n+1)th order. This will be seen by comparing the nth terms \( u_n \) and \( u'_n \) of the nth and the (n+1)th orders.

We have \( \frac{u'_n}{u_n} = \left\{ \frac{\ln(1 + a)}{\ln(1 + a')} \right\}^{1/n} \), where \( a \) is small but \( > 0 \), and \( a' \) is very large. Let \( x = \ln(1 + a) \), we may write \( \lim_{n \to \infty} \frac{u'_n}{u_n} = \lim_{x \to 0} \left\{ x^{1/a'} \right\} \), hence however small \( a \) may be.
provided it be greater than 0, and
however large q, |u' / y| = x
and if f(x) be always positive when
x exceeds a certain finite value, x = x

Is fn convergent or divergent as
the limit of the functions

\[ T_0 = P_x - 1 \]
\[ T_1 = P_0 (x+1) P_x - P_2 x \]
\[ T_2 = P_1 (x+1) P_v - P_1 x \]
\[ T_n = P_{n-1} (x+1) P_x - P_{n-1} (x), \text{ which does not} \]

vanish, when x = x has a - or a + limit.

Comparing \( \Sigma f_n \) with \( \Sigma 1 / P_{n-1}(x) \}
we see \( \Sigma f_n \) will be convergent if for

all values of x greater than a fixed
finite value \( P_x = P(x) \{ \frac{1}{x} \}^{x} \}

where \( q \) is greater than 0.

Therefore \( P_{x+1} P_x - P_2 x = P_x \left[ \frac{f(x)}{F(x+1)} \right]^{q-1} \]

also \( \Sigma P_u(x) \left[ \frac{f(x)}{F(x+1)} \right]^{q-1} = \)
\[-2 \sum (x-1) \left( \frac{\frac{1}{x+1} - \frac{1}{x}}{\left( \frac{\frac{1}{x+1} + \frac{x}{x+1} \right)^2} \right) \nu \left( \frac{1}{x+1} \right) / \left( \frac{1}{x} \right) \right] - 1

= -1 \times 1 \times 0 = -0

Therefore, a sufficient condition for \( \sum y_n \) to be convergent is that

\[
\sum \left( \frac{P_n(x)}{x+1} - \frac{P_n(x)}{x} \right) \left( \frac{1}{x} \right) 
\]

is less than -\( \delta \).

Abel has shown that however slightly divergent \( \sum y_n \) may be, it is always possible to find \( f, y_1, y_2, y_3, \ldots \) such that \( \sum y_n = 0 \) and yet \( \sum y_n \) should be divergent, and DuBois Reymond has shown that however slowly \( \sum y_n \) may converge, we can always find \( f, y_1, y_2, y_3, \ldots \) such that \( \sum y_n = \infty \) and yet \( \sum y_n \) shall be convergent.

We show that functions may be conceived whose increase to infinity
is slower than that of any step in the logarithmic scale, and concludes that there is a domain of convergence on whose borders, the logarithmic criterion fails entirely.

- A convergent series that has a periodic recurring negative sign may not be absolutely convergent.

We may group every succession of positive terms and every succession of negative terms. If the occurrence of + and - terms is periodic we thus reduce the series to the case where the terms are alternately + and -.

Now we may group each negative with the preceding or following positive term, and in general the result will be a series of + or - terms.
This process often helps to settle the convergence of a series but it must be remembered that the series derived by grouping is in reality a new series because the sum of \( m \) terms does not always correspond to the sum of \( n \) terms of the derived series.

The difference will not exceed the sum of a finite number of terms of the original series, and this difference must vanish for \( n = \infty \) if the terms of the original series become infinitely small.

In case of an oscillating series where \( S_{\infty} \neq 0 \), the grouping of the terms may convert a non-convergent series into a convergent series.

So in this case we cannot use the convergence of the original
from the convergence of the derived series, as

\[(1 + \frac{1}{2})^2 - (1 + \frac{2}{3})^2 + (1 + \frac{4}{5})^2 - \ldots - (1 + \frac{1}{2n+1})^2 + \ldots\]

is a non-convergent oscillating series, but

\[\left\{ (1 + \frac{1}{2})^2 - (1 + \frac{2}{3})^2 \right\} + \left\{ (1 + \frac{4}{5})^2 - (1 + \frac{5}{6})^2 \right\} + \ldots + \left\{ (1 + \frac{1}{2m})^2 - (1 + \frac{1}{2m+1})^2 \right\} + \ldots \]

whose mth term is \(\frac{8m^2 + 8m + 1}{(4m^2 + 2m)^2}\) or \(\frac{8m^2 + \frac{1}{m^2}}{16(1 + \frac{1}{2m})^2}\) and so is convergent, being comparable in the scale of convergence with \(\frac{1}{m^2}\).

The following rule is of use in the study of semi-convergent series:

If \(u_1 > u_2 > u_3 \ldots > u_n > \ldots\) and all the terms are positive, then \((-1)^{n-1} u_{n-1} + u_n + u_{n+1} + \ldots\) converges or oscillates as \(\lim_{n \to \infty} u_n = 0\) or \(\neq 0\)

\[m \in \mathbb{Z} \pm (u_{n+1} - u_{n+2} + \ldots \pm u_{n+m}) = \]
\[ U_{n+1} - (U_{n+2} - U_{n+3}) - (U_{n+4} - U_{n+5}) = (\ldots) \]
\[ = (U_{n+1} - U_{n+2}) + (U_{n+3} - U_{n+4}) + (\ldots) + (\ldots) \]

Therefore, \( U_{n+1} > mR_n > U_{n+1} - U_{n+2} \)

If \( U_n = 0 \), \( U_{n+1} = 0 \) \& \( U_{n+2} = 0 \)

So \( \sum_{n=\infty}^{m} R_n = 0 \) for all values of \( m \).

Also, \( U_1 > mR_0 \Rightarrow \sum_1 > U_1 - U_2 \), so \( S_n \) is finite for all values of \( n \). The series is therefore convergent if \( \sum U_n = 0 \).

The series \((U_1 - U_2) + (U_3 - U_4) + \ldots\)

where \( U_1, U_2, \ldots \) are as before, is convergent.

The most important of periodic series is Fourier's series \( \sum a_n \cos (n \theta + \phi) \), where \( a_n \) is a function of \( n \), and \( \phi \) is independent of \( n \). These series are used much in mathematical physics.

If \( \sum a_n \) is an absolutely convergent series, then \( \sum a_n \cos (n \theta + \phi) \) is convergent as, if \( U_n \) is less than \( V_n \) and \( \sum V_n \) is convergent, then \( \sum E \) is convergent.
If \( \Theta = 0 \) or \( 2\pi n \), then \( \sum E_n \cos(\Theta + \Theta) \) is convergent if \( E_n \) is convergent, and the series reduces to \( \sum E_n \cos \Theta \).

If \( \Theta \neq 0 \) or \( 2\pi n \), then \( \sum E_n \cos(\Theta + \Theta) \) is convergent if for all values of \( n \) greater than a certain finite value, \( a_n \) constantly decreases, as \( n \) increases in such a way that

\[
\lim_{n \to \infty} \frac{a_n}{n} = 0
\]

If for all values of \( n \),

\[
A > u_1 + u_2 + u_3 + \ldots + u_n > B
\]

where \( u_1, u_2, u_3, \ldots u_n \) are any real quantities whatever, and \( a_1, a_2, a_3, \ldots a_n \) is a series of positive quantities constantly decreasing as \( n \) increases, then

\[
a_n > a_1 u_1 + a_1 u_1 + a_3 u_2 + \ldots + a_n u_n > a_1, B.
\]

This is proved as follows —

Let

\[
S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + u_3 + \ldots + u_n
\]

\[
S_n = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n.
\]

Then \( u_1 = S_1 \), \( u_2 = S_2 - S_1 \), etc.

\[
S_n = a_1 S_1 + a_2 (S_2 - S_1) + \ldots + a_n (S_n - S_{n-1})
\]

\[
= S_1(a_1 - a_2) + S_2(a_2 - a_3) + S_3(a_3 - a_4) + \ldots + S_n(a_n - a_{n+1}) + S_{n+1}
\]
Now since $S_1, S_2, \ldots, S_n, \text{ are each } < A \times B$
and $(a_1 - a_2), (a_2 - a_3), \cdots (a_{n-1} - a_n)$ are all positive,

\[
\frac{(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n}{(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n} \geq S' \geq S''
\]

That is, $A > S_n > a. B$

If $\sum a_n$ be convergent or oscillatory
and $a_1, a_2, a_3, \cdots a_n$ be a series of positive quantities constantly decreasing as $n$ increases, so that $\sum a_n = 0$,
then $\sum a_n$ is convergent, for since $\sum a_n$ is not divergent, $S_n$ is not $\infty$ for any finite value of $n$. Hence $S_n$ is not infinite.

Also $\sum_{n=1}^{\infty} \left( \frac{1}{n+1} \right) = \zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ where $\zeta(s)$ is the greatest and the least of the values of $\sum a_n$ for all different positive values of $s$.

Now since $\sum a_n$ is convergent or oscillatory, $S_{n+m} - S_n$ is either $0$ or finite
and $\sum a_n = 0$ by hypothesis.
Therefore $\lim_{n \to \infty} P_n = 0$ for all values of $n$, and $\sum a_n b_n$ is convergent.

If $a_n$ be as above, then

$\sum (-1)^n a_n \cos(n\theta + \phi)$

and

$\sum (-1)^n a_n \sin(n\theta + \phi)$

are all convergent.

If the $n$-th term of a series be of the form $x_n + y_n i$, and $x$ and $y$ are functions of $n$, and $i$ is the imaginary unit, we may write the sum of $n$ terms in the form $S_n + T_n i$, where

$S_n = x_1 + x_2 + \cdots + x_n$ and $T_n = y_1 + y_2 + \cdots + y_n$

$\sum (x_n + y_n i) = \left( \sum_{n=1}^{\infty} x_n \right) + \left( \sum_{n=1}^{\infty} i y_n \right) i$

The necessary and sufficient condition for the convergence of $\sum (x_n + y_n i)$ is that $\sum x_n$ and $\sum y_n$ be both convergent, for if they converge,
to Sand T respectively, $E(x_n+y_ni)$ will converge to $S+Ti$ and if the series $E x_n$ or $E y_n$ diverge or oscillate then $(E S_n)+(E T_n)$ will not have a finite - definite value.

Let $g_n$ denote $x_n+y_ni$ and let $p_n$ be the modulus, and $q_n$ the amplitude of $g_n$, so that $g_n = p_n(\cos q_n + i \sin q_n)$.

$x_n = p_n \cos q_n$, $y_n = p_n \sin q_n$.

Generally the following theorem is sufficient. The complex series $E g_n$ is convergent if the real series $E \mod g_n$ is convergent, for since $E p_n$ is convergent and $p_n$ is always positive, it follows that $E p_n \cos q_n$ and $E p_n \sin q_n$ are both convergent, therefore $E g_n$ is convergent. This is a sufficient but not a necessary
condition, for the series 

\[ (1 - \frac{1}{2}) - (1 - \frac{1}{3}) +
\]

\[ (-\frac{1}{2}) - (-\frac{1}{3}) + \] is convergent, since 

\[ \frac{1}{2} + \frac{1}{3} \] and \[ -\frac{1}{2} + \frac{1}{3} + \] are both convergent. But the series of moduli

\[ \frac{\sqrt{2} + \sqrt{3} + \sqrt{5}}{} \] is divergent.

When \( E_{n} \) is such that \( E_{n} \) is convergent then \( E_{n} \) is said to be absolutely convergent. Since the modulus of 

a real quantity \( u_{n} \) is simply \( u_{n} \) with its sign made positive if needs be, we see the present definition of absolute convergency includes that already given.

If \( h_{n} \) be real, and \( z_{n} \) a complex number whose modulus is not infinite for any value of \( n \), then 

\( \sum_{n} z_{n} \) will be absolutely
convergent if \( E \mid x \) is absolutely convergent, for \( \text{mod}(l_x, g_x) = (\text{mod} l_x, \text{mod} g_x) \).

And since \( E \mid x \) is absolutely convergent, \( E \mid 0 \) is convergent, hence prior.

\( \text{mod} g_x \) is always finite, \( E(\text{mod} l_x, \text{mod} g_x) \) is convergent. That is \( E(l_x, g_x) \) is absolutely convergent.

The law of Association cannot be applied to an infinite series unless the series is convergent.

An example of this has already been given. Let \( S'_n \) denote the sum of \( n \) terms of the new series obtained by associating the terms of the original series into groups in any particular way. Then if \( S_n \) denote the sum of \( n \) terms of the original series.
we cannot always assume in so large that $S_m$ includes $S_n$. Then

$$S_m - S_n = f \, \rho_n$$

where $f$ is a certain positive integer. Now if the original series is convergent by taking $(n)$ sufficiently large we can make $\rho_n$ as small as we please. Therefore

$$S_m - S_n = 0$$

Hence the association of terms produces no effect on the sum of an infinitely absolutely convergent series.

The law of commutation is even more restricted in its application than the law of association. In case of absolutely convergent series only can this law be applied. If each term of the series is
displaced, a finite number of steps.
Let $Eu_n$ be the original series and $Eu'_n$ be the new series formed by commutation of terms of $Eu_n$.
Since each term is displaced only by a finite number of steps, we can take $n$ large enough so that $E_n$ contains all the terms of $S_n$ whatever be the value of $n$. $E_n' - S_n$ contains fewer terms than $E_n$, where $n$ is the final value in which $n$ is finite.
Now if $Eu_n$ is absolutely convergent every term of the same sign we have $S_n P_n = 0$.
Therefore $\sum_{n=20} L^2 S_n = 0$. Also $\sum_{n=20} L^2 S_n = L S_n$.
This reasoning would not hold for a semi-convergent series because the vanishing of $S_n P_n$ does not
depend solely on the individual magnitude of the terms, but partially on the alternation of + and - signs.

Riemann has shown that the series
\((-1)^n u_n\), where \(u_n = 0\), and \(\sum u_{n+1}\) and \(\sum u_n\)
are both divergent can by proper commutation be made to converge to any value you please.

Dirichlet has shown that commutation may render a semi-convergent series divergent.

For example the series
\(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots + \frac{1}{2n+1} - \frac{1}{2n}\) is convergent
as \(u_1 > u_2 > u_3 > u_4 \ldots\) and \(\sum u_n = 0\);
but the series \((\frac{1}{1} + \frac{1}{3} - \frac{1}{2}) + (\frac{1}{7} - \frac{1}{4} - \ldots\) +
\((\frac{1}{2n+1} + \frac{1}{2n+3} - \frac{1}{2n+2})\) which is derived from (1) is divergent, for if
\[ \sqrt{m} = \frac{1}{\sqrt{m+1} + \sqrt{m+3}} - \frac{1}{\sqrt{2m+2}} \]

Then the limit \( \lim_{m \to \infty} \sqrt{m} \) is always finite and \( E\sqrt{m} \) divergent, \( \therefore E\sqrt{n} \) is divergent.

If two infinite series be both convergent, as \( E\sqrt{n} \) and \( E\sqrt{m} \) are and converge to the values \( S \) and \( T \) respectively, then \( E(U_n + V_n) \) is convergent and converges to the value \( S + T \).

Let \( S_n \) and \( T_n \) and \( U_n \) represent the sum of \( n \) terms of \( E\sqrt{n} \), \( E\sqrt{m} \) and \( E(U_n + V_n) \). Then however great \( n \) may be \( U_n = S_n + T_n \). Hence when \( n \to \infty \), \( \sqrt{n} = \sqrt{S_n + T_n} \), which proves the proposition.

If \( a \) being a finite quantity and \( E\sqrt{n} \) converge to the value \( S \), then \( E\sqrt{a\sqrt{n}} \) will converge to the value \( \sqrt{a} \cdot S \).
If $E u_n$ and $E v_n$ are two convergent series, and at least one of the series be absolutely convergent, then the series $\sum u_n v_1 + (u_1 v_n + v_2 u_n) + \ldots + u_1 v_n + v_n v_{n-1} + \ldots u_n v_1 + \ldots$ converges to the value $S_1 T_1$. (The original proof by Cauchy, and also by Asgood in his introduction to Infiniti series, required that both of the series $E u_n$ and $E v_n$ be absolutely convergent.)

This more general form was given by Mertens 1873.

Let $S_n$, $T_n$ be $u_n$. denote the sum of $u_n$ terms of $E u_n$, $E v_n$, and $E (u_n v_n + v_n u_{n-1})$ respectively, and suppose that $E u_n$ is absolutely convergent.

We have $S_n T_n = U_n + S_n$ where $S_n =$

$u_1 v_n + u_2 v_{n-1} + \ldots + u_n v_1 + u_2 v_{n-1} + \ldots + u_n v_1 + \ldots$
\[ = U_n (V_n + V_{n-1}) + \ldots + U_n (V_1 + \ldots + V_2) \]

If therefore \( n \) be even and \( = 2m \)

\[ L^n = U_2 (V_2m + V_{2m-1}) + \ldots + U_m (V_{2m} + \ldots + V_{m+2}) \]

\[ + U_{m+1} (V_{2m} + \ldots + V_{m+1}) + \ldots + U_{2m} (V_{2m} + \ldots + V_2) \]

If \( n \) be odd, and \( = 2m + 1 \)

\[ L^n = U_2 (V_{2m+1} + V_{2m+2}) + \ldots + U_{m+1} (V_{2m+1} + \ldots + V_2) \]

Now since \( \sum_{n=1}^\infty \) is convergent it is possible by making \( m \) sufficiently great to make each of the quantities \( \mod (V_{2m} + \ldots + V_2) \)

\[ \mod (V_{2m-1} + V_{2m}) \]

\[ \mod (V_{2m+1} + \ldots + V_{2m}) \]

\[ \mod (V_{2m+2} + \ldots + V_{2m+1}) \]

\[ \mod (V_{2m+3} + \ldots + V_{2m+1}) \]

as small as we please.

Also since \( \mod T_1 \), \( \mod T_2 \), \( \mod T_3 \), \( \mod T_4 \), 

are all finite and \( \mod T_2 - \mod T_3 \) \( \mod T_4 \), 

\[ \mod (V_{m+1} + \ldots + V_m) \]

\[ \mod (V_{m+2} + \ldots + V_{m+1}) \]

\[ \mod (V_{m+3} + \ldots + V_{m+2}) \]

\[ \mod (V_{m+4} + \ldots + V_{m+3}) \]

are all finite. Hence if \( m \) be a

quantity which can be made as
small as \( m \) please by increasing \( n \), and \( \beta \) a certain finitely quantity. 
we have \( \log L_n < E_n (\mod u_2 + \mod u_3 + \ldots + \mod u_n + \beta (\mod u_4 + \mod u_{n+2} + \ldots + \mod u_n)) \)

If we make \( m = \infty \), then \( \mod u_2 + \mod u_3 + \mod u_4 + \ldots \) mod \( u_n \) is finitely since \( E_{u_n} \) is convergent, and \( L (\mod u_{n+2} + \mod u_{n+3} + \ldots + \mod u_n) = 0 \). Then as \( \log L_n \to 0 \) \( L \mod u_n = 0 \), so \( L \Delta T_n = L \Psi_n \).
That is \( L \Psi_n = \partial T \).

Cauchy has shown that the multiplication rule does not hold for two series that are both semi-convergent:

The series \( E_n x^n \) is of great importance in analytical analysis. It is called the power series, and \( a_n \) and \( x \) are considered as complex numbers
\[ a_n = r_n (\cos \alpha_n + i \sin \alpha_n), \quad x = r (\cos \theta + i \sin \theta) \]
where \( r_n \) and \( \alpha_n \) are functions of the
integral variables, say, but $p$ and $q$ are independent of $m$.

$\sum x^n$ is convergent if $m_0$ is less than $L \left( \frac{m^n}{m^n + 1} \right)$. For the series of moduli is $\sum p^n$ and this is convergent if $L \left( \frac{m^{n+1}}{m^n + 1} \right) < 1$ that is if $p L \left( \frac{m^{n+1}}{m^n} \right) < 1$

$\frac{m^n}{m^n + 1}$

or if $p < L \frac{m^n}{m^n + 1}$

Three cases arise: $\frac{m^n}{m^n + 1}$ is 0, a finite quantity $R$, or infinity.

In the first case $\sum x^n$ is not convergent for any value of $x$ but $x = 0$.

In the second case $\sum x^n$ is convergent when the point representing $x$ lies within the circumference of the circle whose centre is the origin and whose radius equals to $R$. This circle is called the circle of convergence for
the power series, and \( R \) is the radius of convergence. Nothing is established for the case where the representative point lies on the circle of convergence.

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \] is an example of this class of series. Here \( R = 1 \).

In the 3rd case, \( \sum_{n=0}^{\infty} x^n \) is convergent for all values of \( x \). The exponential series is an example of this class of series, as \[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \] if the series be absolutely convergent when \( \text{mod } x = R \), it will be absolutely convergent when \( \text{mod } x = R' \). For since \( \sum_{n=0}^{\infty} a_n x^n \) is absolutely convergent for \( R < R' \), \( a_n R^n \leq R'^n \).

\[ \sum_{n=0}^{\infty} a_n x^n \] is convergent. That is \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) is absolutely convergent when modulus \( x \) equals \( R' \).
If the $n$th term of an infinite series be $f(n) \cdot (n + 1)$, where $f(n)$ is a single valued continuous function of $x$ for all integral values of $n$, then the infinite series $\sum f(n)$ will, if convergent, be a single value of a continuous function of $x$ as $D(x)$.

Cauchy at first thought that this must be continuous as each term of $f(n)$ is so, but Abel first pointed out that $D(x)$ is not necessarily continuous.

$\sum f(n)$ and $\sum f(n \cdot x)$, being each continuous and convergent, have each definite finite limits, therefore $\sum \{ f(n \cdot x) - f(n) \}$ is convergent and has a definite finite value, but this value is not necessarily $0$ for all values of $(x)$. Suppose $f(n \cdot x) = \frac{n \cdot x}{n \cdot x + 1} - \frac{(n-1) \cdot x}{(n-1) \cdot x + 1}$, then $S_n = \frac{n \cdot x}{n \cdot x + 1}$.
Then if \( x \neq 0 \) \( \implies S_n = 1 \). If \( x = 0 \) then 
\( S_n = 0 \) however great \( n \) may be; therefore 
\( S(n) \) is discontinuous in this case where 
\( x = 0 \). The discontinuity of the series 
is accompanied by another peculiarity 
which is often associated with 
discontinuity. The residue when \( x \to 0 \) 
is given by \( R_n = 1 - S_n = \frac{1}{nx+1} \). Now 
when \( x \) has any given value we can 
make \( \frac{1}{nx+1} \) smaller than any given 
positive quantity \( a \) by making \( n \) large 
enough, and the smaller \( x \) is, the 
larger \( n \) must be, so that \( \frac{1}{nx+1} \) be 
less than \( a \). So that if \( x \) is a variable 
there is no finite upper limit for \( n \) 
independent of \( x \), such that when \( n > \) 
\( V \), then \( R_n \leq a \). When the residue has 
this peculiarity the series is said 
to be non-uniformly convergent, and 
if for a particular value of \( (x) \) as
\( x = 0 \) in this equation, the number of times required to get a given degree of approximation to the limit is infinite, the series is said to converge infinitely slowly.

If for values of \( x \) within a given region in Argand's Diagram we can for any value of \( k \) however small \( \operatorname{mod} a \) may be, assign for \( n \) an upper limit \( v \), independent of \( x \), such that when \( n > v \), \( \operatorname{mod} P_n - \operatorname{mod} a \) then the series \( \sum \) \( f(x) \) is said to be uniformly convergent within the region in question.

This distinction was first pointed out by Heidel in 1850 and has assumed great importance in the Theory of Functions developed by Weierstrass.

It can be shown that as long
as $E f(x)$ converges uniformly, $O(x)$ cannot be discontinuous.

Du Bois Raymond has shown by means of the series $E \left\{ \frac{x}{n(n+1)(n^2+1)} \right\}$, that infinitely slow convergence may not involve discontinuity.

In fact the $E$ of this last series is always 0 even when $x=0$, and when $x=0$ the convergence is infinitely slow.

There are two cases of great importance concerning the power series $E a_n x^n$. In one when $a_n$ is independent of $x$, and we regard $E a_n x^n$ as a function of $x$, as $F(x)$, and when $a_n$ is a function of $n$ and $y$, as $f(ny)$, and $x$ is regarded as a constant,
so that \( E (u, y) y_k \) is a function of \( y \), as \( y_y \).

This question was first raised and discussed by Abel.

Let \( u_n \) be independent of \( y \) and \( W_0 (y) \) be a single valued function of \( u \) and \( y \), finite for all values of \( u \), however great, and finite and continuous as regards \( y \), from \( y = a \) to \( y = b \).

Then if \( E u_n \) be absolutely convergent, \( E u_n W_0 \) is a continuous function of \( y \), from \( y = a \) to \( y = b \).

Let \( S_n (y) = u_0 W_0 (y) + u_1 W_0 (y) + \ldots + u_n W_0 (y) \)

and assume \( u_n \) to be positive for all values of \( n \), which will not limit the generality of the demonstration since \( E u_n \) is absolutely convergent.

Let \( \Delta u_p = u_p (y + \delta) - u_p (y) \), so that \( E u_p = 0 \) for all values of \( y \). Then we have
\[ S_n(z + \delta) - S_n(z) = \Delta w_1 + u_2 \Delta w_2 + \cdots + u_m \Delta w_m + \Delta w_{m+1} (z + \delta) + u_{m+2} w_{m+2} (z + \delta) + \cdots + u_0 w_0 (z) \]

Let \( \Delta w_m \) be a mean among \( \Delta w_1, \Delta w_2, \ldots, \Delta w_m \), that is, be greater than the least and less than the greatest, and \( u_m \) a mean among \( u_{m+1} (z + \delta), u_{m+2} (z + \delta), \ldots, u_0 (z) \).

Then \( S_n (z + \delta) - S_n(z) = \Delta w_m S_m + (u_{m+1} - u_m) R_m \),

where \( S_m \) and \( n-m R_m \) have the usual meaning, as regards \( E \) in.

In now \( m \) is infinite \( w_m \) and \( w_m \) become by hypothesis finite, determinate quantities for every value of \( m \) and \( z \), and we have \( S_\infty (z + \delta) - S_\infty(z) = \Delta w_m S_m + (w_{m+1} - w_m) R_m \).
We wish to know that
\[ h = \left( \sum \frac{1}{m} \right) - S(x) = 0 \quad \text{and} \quad \left( \sum \frac{1}{m} \right) - S(x) = 0 \]
now when \( h = 0 \), \( \Delta w = 0 \), \( \Delta w_2 = 0 \), \( \Delta w_n = 0 \)
since all the functions \( w, w_2, w_n \) are continuous. Now since
\( \Delta w_n \) is finite for all values of \( n \),
owing to the convergence of \( E w_n \) we have
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - S_n \right) = \sum_{n=1}^{\infty} \left( w_n - w_n^* \right) \]
We cannot be sure that \( \sum_{n=1}^{\infty} w_n = w_n^* \)
but since both are finite their difference is finite. Hence since \( \Delta w_n \)
is the residue of a convergent series \( E w_n \), by making \( n \) sufficiently great we can make \( \Delta w_n \) and therefore the right hand side of the equation as small as we please. It follows that the left hand side of the equation must be less than any assignable quantity, that is, must be 0.
The quantities involved have been supposed to be real, but when $u_n$ and $w_n(x)$ are complex the case still holds.

If $E a_n x^n$ be absolutely convergent when modulus $x = R$, then for all values of $x$ such that modulus $x < R$, then $E a_n x^n$ is a continuous function of ($x$). We have $a_n x^n = a_n R^n \left(\frac{x}{R}\right)^n$. Now $E a_n R^n$ is an absolutely convergent series by hypothesis. Hence if we take $u_n = a_n R^n$ and $w_n(x) = \left(\frac{x}{R}\right)^n$, all the conditions for uniform convergence are satisfied. Therefore the new series is convergent.

If the power series $E f(y) x^n$ be convergent when modulus $x = R$ ($< 1$), and $f(y)$ be a function of $y$ which is finitely and single valued.
and continuous as regards $y$ from $y = a$ to $y = b$; then from $y = b$ to $y = a$ $\Psi p = \sum (\Psi y)$, $x^n$ is a continuous function of $y$ as long as module $x \neq 0$

This follows at once from the conditions of uniform convergence

If $\lim x^n$ and $y = y$, and $\lim z = f(z)$.

As long as $x$ lies within the circle of convergence, the power series $\sum a_n x^n$ is a continuous function of $y$. Now Abel has proved that if the series $\sum a_n x^n$ be convergent and of $\sum a_n x^n$ be convergent for all values of $x$ less than one, then $\sum_{x=0}^{\infty} \sum a_n x^n = \sum a_n$. This is equivalent to saying that $\sum a_n x^n$ is continuous up to the circumference of its circle of convergence as far as real values are concerned.

If $g(x)$ denote $\sum a_n x^n$ we have to show that $\sum_{x=0}^{\infty} g(x)$ or $g(1-0) = \sum a_n$.
Since $\sum a_n$ is convergent, if $a_n = a_0 + a_1 + \ldots + a_n$. Then $a_0, a_1, \ldots, a_n$ are finite and have for their limit $S$ the sum of the infinite series $\sum a_n$.

Also $a_0 = a_0, a_1 = a_0 - a_1, a_2 = a_1 - a_2, \ldots, a_n = a_{n-1} - a_n$.

Then $f(x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \ldots + (a_{n-1} - a_n)x^{n-1} = a_0 (1-x) + a_1 x (1-x) + \ldots + a_n x^n (1-x) + \ldots$

This transformation is permissible as long as $x$ is less than 1, by however little.

Let $x = 1 - \xi$ then we have,

$f(1-\xi) = a_0 \xi + a_1 (1-\xi) \xi + \ldots + a_{n-1} (1-\xi)^{n-1} \xi + \ldots$

where $n$ may be taken as large as we please now let $\tau_n$ be a mean among $a_0, (1-\xi) a_1, \ldots, (1-\xi)^{n-1} a_{n-1}$, and let $\Sigma_n$ be a mean among $a_0, a_1, \ldots, a_n$. Then $\Sigma_n$ is finite and $\Sigma_n = \Sigma_n$

We have $f(1-\xi) = n \xi \tau_n + (1 + (1-\xi) + (1-\xi)^2 + \ldots + (1-\xi)^n) \tau_n$

$= n \xi \tau_n + (1 - \xi)^n \tau_n$

Since $n$ may be as large as we please,
Place we may cause $\xi$ to approach the limit $0$, by putting $\xi = \frac{1}{n}$ and making $n = \infty$. Therefore

$$f(0) = \sum_{n=1}^{\infty} \frac{f(-\frac{1}{n})^{n}}{n} = 0$$

and $\sum_{n=1}^{\infty} \left( -\frac{1}{n} \right)^{n} = 0$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ is finite. Also

$$f(0) = \sum_{n=1}^{\infty} \frac{f(-\frac{1}{n})^{n}}{n} = \sum_{n=1}^{\infty} \frac{\epsilon}{n} = -1$$

and

$$f(0) = \sum_{n=1}^{\infty} \frac{\epsilon}{n} = -1$$

Hence $\sum_{n=1}^{\infty} \frac{\epsilon}{n} = 0$, and $f(0) = 0 = \sum_{n=1}^{\infty} \epsilon_{n}$

The series $\sum_{n=1}^{\infty} \epsilon_{n}$ need not be absolutely convergent, but if it be only semi-convergent, the order of its terms must not be changed.

If the real series $\sum_{n=1}^{\infty} \epsilon_{n} x^{n}$ is convergent for all values of $x$ such that modulus $x > R$ and if for all values in question $a_{0} + \sum_{n=1}^{\infty} \epsilon_{n} x^{n} = 0$, then $a_{0} + \sum_{n=1}^{\infty} \epsilon_{n} x^{n}$ is convergent if follows that $\sum_{n=1}^{\infty} \epsilon_{n} x^{n} = 0$. Since $a_{0} + \sum_{n=1}^{\infty} \epsilon_{n} x^{n} = 0$, when $x = 0$, we have $a_{0} = 0$.

Therefore $\sum_{n=1}^{\infty} \epsilon_{n} x^{n} = 0$ for all values of $x$ such that modulus $x > R$.

$\sum_{n=1}^{\infty} \epsilon_{n} x^{n} = x \sum_{n=1}^{\infty} \epsilon_{n} x^{n-1}$, where $\sum_{n=1}^{\infty} \epsilon_{n} x^{n-1}$ is
convergent for any value of \( x \) which renders \( E a_n x^k \) convergent. Since then we have \( x E a_n x^k = 0 \) for values of \( x \neq 0 \), then \( E a_n x^k = 0 \). But since \( E a_n x^k \) is convergent, \( E a_n x^k = a_k \). Therefore \( q_k = 0 \).

Proceeding in this way it may be shown that all the coefficients must vanish.

If for all values of \( x \) such that

modulus \( (x \neq 0) - a_0 \), \( E a_n x^k = b_0 + E b_n x^k \),

both series being convergent.

Then \( a_0 = b_0, \ a_1 = b_1, \ a_2 = b_2, \ldots \ a_n = b_n \)

for \( x \) must have \( (a_0 - b_0) + E(a_n - b_n) x^n \)

where \( E(a_n - b_n) x^n \) is a convergent series.

Hence \( a_0 = b_0, \ a_1 = b_1, \ a_2 = b_2, \ldots \)

The product of an infinite number of factors formed according to a definite law is called an infinite product. The most important
case arises when the factors ultimately become unity. The nth factor may be written \(1 + u_n\).

By the value of the infinite product is meant the limit of \((1 + u_1)(1 + u_2)(1 + u_3)\ldots(1 + u_n)\), which is denoted by

\[\prod (1 + u_n)\] or simply by \(P\), when \(n\) is increased without limit.

If \(\lim u_n > 1\) then \(L P = 0\), or \(x\). As neither of these cases are important the modulus \(u_n\) is used \(< 1\). Any finite number of factors at the beginning of the product, for which this is not true may be left out of account in considering the convergence.

We may also suppose any factor that contains 0 to be set aside. The question as to the convergence then relates merely to the product of
all the remaining factors.

From cases arise 1st $\Pi\psi_0 = 0$,
2nd $\Pi\psi = a$ definite finite quantity denoted by $\Pi (1 + 4\psi)$ or simply by $\psi$
3rd $\Pi\psi$ may be infinite
4th $\Pi\psi$ may have no definite value
but assume different series of values according to the integral character of $\psi$.

Sometimes case (1) is said to be convergent but usually only case (2).

If instead of considering $\Pi\psi$
we consider its logarithm we reduce
the theory of infinite products to
the theory of infinite series, for we
have $\log \Pi = \log (1 + 4\psi) + \log (1 + 4\psi) + \cdots \log (1 + 4\psi)
\Rightarrow \log (1 + 4\psi)$.

If $\sum \log (1 + 4\psi)$ is divergent and
$\sum \log (1 + 4\psi) = -\infty$ then $\Pi (1 + 4\psi) = 0$

If $\sum \log (1 + 4\psi)$ be convergent, then
$\Pi (1 + 4\psi)$ is convergent.
If \( \varepsilon \log(1+u_n) \) is divergent and 
\( \varepsilon \geq \log(1+u_n) = +\infty \) then \( \Pi (1+u_n) \) is divergent.

If \( \varepsilon \geq \log(1+u_n) \) oscillates, then \( \Pi (1+u_n) \) oscillates.

If \( L u_n < 0 \), then \( \varepsilon \log(1+u_n) = -\infty \) and \( \Pi (1+u_n) = 0 \)

If \( L u_n > 0 \), \( \varepsilon \log(1+u_n) = \infty \) and \( \Pi (1+u_n) \) is divergent.

It is therefore a necessary condition that \( L u_n = 0 \) for the convergence of \( \Pi (1+u_n) \)

Since \( L u_n = 0 \), \( L (1+u_n) / u_n \) = 0, 20
\( \varepsilon \log (1+u_n) / u_n \) = 1. Therefore \( \varepsilon \log(1+u_n) \) is convergent or divergent as \( \varepsilon u_n \) is convergent or divergent.

Since \( u_n \) is ultimately of the same sign, the convergence of \( \Pi \) does not depend on any arrangement of signs but only on the ultimate magnitude.
of the factors. The infinite product of convergent is said to be absolutely convergent, if any infinite product in which the sign of $u_n$ is not ultimately invariable but which is convergent when the signs of $u_n$ are all made alike will be convergent in its original form, and is therefore absolutely convergent, and in general $\prod (1 + u_n)$ is absolutely convergent when $E(u_n)$ is absolutely convergent.

If either of two infinite products be absolutely convergent as $\prod (1 + u_n)$ and $\prod (1 - u_n)$, then the other is absolutely convergent. For if $E(u_n)$ is absolutely convergent, so is $E(-u_n)$, and conversely of $E(u_n)$ is absolutely convergent then so is $E(-u_n)$.

If $D_n = (1 + u_n)(1 + u_n)$

The necessary and sufficient condition that $\prod (1 + u_n)$ be convergent is that $D_n$ be not infinite for any
value of \( n \) however large, and

\[
\lim_{n \to \infty} \left( \frac{P_{n+m}}{P_n} \right) = 0
\]

In the special case where \( P_n = 0 \) it may be observed that \( P_n \) is always finite.

The condition \( \lim_{n \to \infty} \left( \frac{P_{n+m}}{P_n} \right) = 0 \) is equal to \( \lim_{n \to \infty} \left( \frac{P_{n+m}}{P_n} - 1 \right) = 0 \), that is \( \lim_{n \to \infty} \frac{P_{n+m}}{P_n} = 1 \).

Denote \( (1 + 4n + 1) (1 + 4n + 2) \cdots (1 + 4n + m) \) by \( m \mathcal{L}_n \), we may state the criterion as follows — The necessary and sufficient conditions for the convergence of \( \Pi (1 + 4n) \) are that \( P_n \) be not infinite for any value of \( n \) however large and that \( \lim_{n \to \infty} m \mathcal{L}_n = 1 \).

If \( u_n \) be complex, then the two conditions are that modulus \( P_n \) be not infinite for any value of \( n \) and that \( \lim \) modulus \( m \mathcal{L}_n - 1 \) = 0.
\[ \pi (1+\mu_1) \text{ is convergent if } \pi (1+\mu_2) \text{ is convergent. Let } \mu_1 = \text{mod} \mu_2 \text{ so that } \mu_1 \text{ is positive for all values of } \mu. \text{ Then since} \]
\[ \pi (1+\mu_1) \text{ is convergent } \Rightarrow \pi (1+\mu_2) = \pi (1+\mu_1 + \mu_2) = 0. \text{ Now } \mu_1 \text{ mod} \mu_2 = 1 = \mu_2 + \mu_3 + \mu_4 + \cdots \]
\[ \mu_3 + \mu_4 - \mu_1 = \mu_{n+1} + \mu_{n+2} + \cdots \]
\[ \mu_{n+1} + \mu_{n+2} - \mu_{n+1} = 0 \]
\[ 0 \neq \text{mod} (\mu_{n+1}) \Rightarrow \mu_{n+1} + \mu_{n+2} + \cdots \mu_{n+2} - \mu_{n+1} = \mu_{n+1} + \mu_{n+2} - \mu_{n+1} - 1, \]
Hence 0 \neq \text{mod} (\mu_{n+1}) = 0, \text{ also modulus} \]
\[ (1+\mu_1) (1+\mu_2) (1+\mu_3) = \text{modulus} (1+\mu_1, \mu_1 \text{ mod} (1+\mu_1)) \\mod (1+\mu_2) \neq (1+\mu_1) (1+\mu_2) (1+\mu_3). \]
\[ \text{Hence } \pi (1+\mu_1) \text{ is finite since } \pi (1+\mu_2) \text{ is convergent.} \]
The converse of this theorem is not true as may be seen by the product \( (1+\frac{1}{2}) (1+\frac{1}{3}) (1+\frac{1}{4}) \) which converges to a finite limit even though \((1+\frac{1}{2}) (1+\frac{1}{3}) \) is not convergent.
When \( \Pi(1+n) \) is such that \( \Pi(1+mod.yn) \) is convergent, \( \Pi(1+n) \) is said to be absolutely convergent.

If \( \Pi(1+n) \) be convergent but \( \Pi(1+yn) \) be non-convergent, \( \Pi(1+n) \) is said to be semi-convergent.

If \( \Sigma yn \) is convergent then \( \Pi(1+n) \) is absolutely convergent, and conversely.

If \( \Sigma mod.yn \) be convergent, it is absolutely convergent as modulus \( un \) is by its nature positive.

Hence \( \Pi(1+mod.yn) \) is convergent, and \( \Pi(1+yn) \) is absolutely convergent.

If \( \Pi(1+n) \) be absolutely convergent, \( \Pi(1+mod.yn) \) is convergent, since \( mod.yn \) is positive. \( \Pi(1+mod.yn) \) is absolutely convergent. Therefore \( \Sigma mod.yn \) is absolutely convergent. If \( \Sigma yn \) is absolutely convergent, where \( n \) is
independent of $n$, or such a function
of that modulus $x \neq a$ when $n = \infty$

$\Pi \left(1 - \frac{x}{n}\right)$ is absolutely convergent for
all complex values such that

modulus $x < 1$, but is not absolutely

convergent when modulus $x = 1$

$\Pi \left(1 - \frac{x}{n}\right)$ where $x$ is independent of

$n$ is absolutely convergent.

The law of association may be

applied to the factors $\Pi (1+4n)$, provided

$k_n = 0$, but not otherwise.

The law of commutation may be

safely used in $\Pi (1+4n)$ if it be absolutely

convergent.

If both $\Pi (1+4n)$ and $\Pi (1+4n)$ be absolutely

convergent then $\Pi (1+4n) (1+4n)$ is absolutely

convergent and has for its limits

$\left\{ \Pi (1+4n) \times \Pi (1+4n) \right\}$. Also $\Pi (1+4n)$

is absolutely convergent and has for
its limits \( \frac{1}{(1+u_n)} \) provided none of the factors \( (1+u_n) \) vanish.

Since \( \sum \log \{1+\mu_n w_n(g)\} = \sum \mu_n w_n(g) \log \{1+\mu_n w_n(g)\} \)
if \( u_n \) be independent of \( g \) and \( w_n(g) \) be a single valued function of \( n \) and \( g \), finite for all values of \( n \) and finite and continuous as regards \( g \), from \( g = a \) to \( g = b \).

\[
\sum \log \left( \frac{1+\mu_n w_n(g)}{1+\mu_n w_n(g)} \right) = 1 \text{ and } \int w_n(g) \, dz
\]

\( (1+\mu_n w_n(g)) \) satisfy all the conditions imposed on \( w_n(g) \) alone, hence \( \sum \log \{1+\mu_n w_n(g)\} \) is a continuous function of \( g \), from \( g = a \) to \( g = b \).

If \( \sum u_n \) and \( u_n(g) \) satisfy these conditions, then \( \prod \{1+\mu_n w_n(g)\} \) is a continuous function of \( g \), from \( g = a \) to \( g = b \).

If \( \sum u_n x^n \) be convergent when
- modulus \( (x) = R \) then \( \prod (1+a_n x^n) \) converges...
to $\Omega$ where $\Phi(x)$ is a finite-convergent function of $x$ for all values of $x$ such that modulus $x$ is less than $R$.

If $f(y)$ be finite and single-valued as regards $n$, and finite single-valued and continuous as regards $y$, from $y=a$ to $y=b$, and if $E \Phi(y) x^n$ be absolutely convergent when modulus $x=R (\leq 1)$, then $\pi$ as long as modulus $x\neq R$, $\prod (1+f(y)) x^n$ converges to $\Phi(y)$ where $\Phi(y)$ is a finite continuous function of $y$, from $y=a$ to $y=b$.

If $E$ can be absolutely convergent then $\prod (1+q_n x)$ converges to $\Phi(x)$ where $\Phi(x)$ is a finite-continuous function of $x$, for all finite-values of $x$ however large.
Thus is a theorem for infinite products analogous to the principle of indeterminate coefficients. If for a continuum of values \( f(x) \) including 0, \( \prod (1 + a_n x) \) and \( \prod (1 + b_n x^n) \) be both absolutely convergent, and \( \prod (1 + a_n x^n) = \prod (1 + b_n x^n) \)

then \( a_1 = b_1, \quad a_2 = b_2, \quad \ldots \quad a_n = b_n. \)

For \( \sum \log(1 + a_n x) = \sum \log(1 + b_n x^n) \) as both series are convergent. Hence

\[ \sum a_n x^n - \log(1 + a_n x^n) = \sum b_n x^n - \log(1 + b_n x^n) \]

Since \( \sum \log(1 + a_n x^n) = 1 \), we have for very small values of \( x \), \( a_1 x + a_2 x^2 + a_3 x^3 + \ldots \)

\( = \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \ldots \)

where \( \theta_1, \theta_2, \theta_3, \ldots, \theta_1, \theta_2, \theta_3, \ldots \) differ very little from unity and all have unity for their limit when \( x = 0 \). Hence since \( \sum a_n x^n \).
and \( \sum b_n x^n \) are absolutely convergent.

\[
\begin{align*}
L_0 & \{ a_0 \sum_{n=0}^{\infty} x^n \} = 0 \\
L_0 & \{ a_0 b_0 x^0 + b_2 x^2 \} = 0
\end{align*}
\]

but \( L_0 P_1 = 1 \) and \( L_0 L_1 = 1 \), so \( a_0 = b_1 \)

Now removing the common factor \((1 + a_0 x)\) from both products and proceeding as before we can show that \( a_2 = b_2 \) and so on...

If \( \psi(x) = \prod (1 + a_n x) \) be convergent, for all values of \( x \) so that \( x \) mod. \( P_0 \to \infty \) and \( x \) mod. \( (\text{mod}_n - 1) = 0 \) when \( n = \infty \) no matter what value \( n \) may have, then \( \psi(x) \) will vanish when \( (x) \) has one of the values \( -\frac{1}{a_1}, -\frac{1}{a_2}, -\frac{1}{a_3}, \ldots \)

and if \( \psi(x) = 0 \), \( x \) must have one of these values.

If \( \text{mod}_n = (\rho \cos \theta, \sin \theta) \)

\[
\text{mod} (\text{mod}_n - 1) = \{ (\rho \cos \theta - 1)^2 + (\rho \sin \theta)^2 \}^{\frac{1}{2}}
\]

Hence \( \sqrt{\{ (\rho \cos \theta - 1)^2 + (\rho \sin \theta)^2 \}} = 0 \).
Therefore \( L^p \cos \theta - 1 \) = 0, and \( L^p \sin \theta = 0 \)
in \( L^p \cos \theta = 1 \). Hence \( m \bar{n} = 1 + h + k \).
where \( h \) and \( k \) have each \( 0 \) for a limit when \( x = x \)

\( \Psi(x) = \bar{n} \ \text{for all values of} \ \bar{n} \)

Let \( x \) approach the value \( -\bar{n} \),
we can always take \( n \) greater than \( x \) so that \( 1 + \bar{n} \) will occur among
the integral factors \( \bar{n} \). Hence when
\( x = -\bar{n} \), we have \( \bar{n} = 0 \) - and since \( m \bar{n} \neq 0 \)
\( \Psi(-\bar{n}) = 0 \). If \( \Psi(x) = 0 \) when \( \bar{n} = 0 \)
but \( n \) may be as large as we please
\( L \bar{n} = 1 \). \( n \) may be taken as large
large that \( \bar{n} \neq 0 \), so if \( n \) be large
enough \( \bar{n} = 0 \), \( \Psi(x) \) must have some
value that will make some one of its factors vanish, \( \Psi(x) \) must have
some one of the values \( -\bar{n} \), \( -\bar{n} \), \( -\bar{n} \).

Any of the quantities \( \alpha_1, \alpha_2, \alpha_2, \) may
be equal, and if there be an
contiguous factors identical with

$(1 + \alpha_n x)$, the product $\psi(x)$ will take
the form $(1 + \alpha_n x)^{\mu_n}$, and can always
be put in this form if the series
be absolutely convergent.

If $f(x)$ lie within a continuum $(X)$
which includes all the values

$-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots, -\frac{1}{n}, (A)$

and $-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, \ldots, -\frac{1}{b_n} (B)$

Let $\prod (1 + \alpha_n x)^{\mu_n}$ and $\prod (1 + \beta_n x)^{\nu_n}$ be absolutely
convergent for all values of $x$ in the
continuum $(N)$, and if $f(x)$ and $g(x)$ be
definite functions of $(X)$ which are
not $= 0$ or $\infty$ for any of the values
of $A$ or $B$, and if for all values of

$f(x) \prod (1 + \alpha_n x)^{\mu_n}$

$g(x) \prod (1 + \beta_n x)^{\nu_n}$, then must each of the
factors in the one product occur
in the other, raised to the same power, and for all values of \( x \) in the continuum \( f(x) = g(x) \) for each of the products must vanish for each of the values \( A \) or \( B \), it follows that each of the quantities \( \lambda \) must be equal to one of the quantities \( \gamma \). The two series are therefore identical.

Since the two infinite products are absolutely convergent, they may be arranged in such order that

\[
\frac{f(x)}{(1 + a_1 x)^{\lambda_1}} \frac{g(x)}{(1 + a_2 x)^{\lambda_2}} = \frac{g(x)}{(1 + a_3 x)^{\lambda_3}} \frac{f(x)}{(1 + a_4 x)^{\lambda_4}}
\]

Now suppose \( \lambda_1 \neq \gamma_1 \), but \( \lambda_2 = \gamma_2 \). Then

\[
\frac{f(x)}{(1 + a_1 x)^{\lambda_1}} \frac{g(x)}{(1 + a_2 x)^{\lambda_2}} = \frac{g(x)}{(1 + a_3 x)^{\lambda_3}} \frac{f(x)}{(1 + a_4 x)^{\lambda_4}}
\]

Now this is impossible, as the left hand side of the equation tends to 0 when \( x = -\frac{1}{a} \), but the
right hand side does not vanish when \( x = -\frac{1}{2} \). Therefore \( u_1 = v_1 \) and in like manner \( u_2 = v_2 \) and \( u_3 = v_3 \) on.

We may therefore clear the first \( n \) factors out of each of the products and thus get the equation

\[ f(x) \prod_{n=1}^{\infty} x^n = g(x) \prod_{n=1}^{\infty} x^n \]

where \( x^n \) have the usual meaning.

The last equation holds true however large \( n \) may be, hence since

\[ \prod_{n=1}^{\infty} x^n = L x^n = 1, \]

we must have

\[ f(x) = g(x). \]

Therefore a given function of \( x \) which vanishes for any of the values \( x \) and for no others within the continuum \( x \) can be expressed within \( x \) as a convergent infinite product of the form

\[ \prod_{n=1}^{\infty} f(x) \prod_{n=1}^{\infty} (1 + a_n x)^{b_n} \]

(where \( f(x) \) is finite and not equal to 0 for all finite values of \( x \) within the
continuum \( (X) \), if at all in only one way.

If the infinite product be

semi-convergent the demonstration fails.

For many purposes it is required to

assign an upper limit to the residue

of an infinite series. This is easily done

in the two most important cases.

First, where the ratio of converges

\[ p_n = \left(\frac{w_n + 1}{n}\right) \]

ultimately becomes unity

and the terms are ultimately

different signs. Second, where the

terms ultimately continually diminish

in value and alternately in sign.

In case (1) there are two varieties of series

so \( p_n \) descends to its limit \( p \) or ascends
to its limit \( p \). In this first series

let \( n \) be taken as large that on
and after \( n \), \( p_n \) is always less than unity.

and never increases in value

\[
\begin{align*}
P_n &= U_{n+1} + U_{n+2} + \cdots + \cdots = U_{n+1} \left( 1 + \frac{U_{n+2} + U_{n+3}}{U_{n+2} U_{n+3}} + \cdots \right) \\
&= U_{n+1} \left\{ 1 + p_{n+1} + p_{n+1} p_{n+2} + p_{n+1} p_{n+2} p_{n+3} + \cdots \right\} \\
L_n &< U_{n+1} \left\{ 1 + p^2 + p^3 + \cdots \right\} \\
&< U_{n+1} / 1 - p_{n+1}
\end{align*}
\]

also for a lower limit \( P_n \leq U_{n+1} / (1-p) \)

In the second series, let \( n \) be so large that often, \( p_n \) is numerically less than unity and never decreases in numerical value

\[
\begin{align*}
P_n &= U_{n+1} \left( 1 + p_{n+1} + p_{n+1} p_{n+2} + p_{n+1} p_{n+2} p_{n+3} + \cdots \right) \\
L_n &< U_{n+1} \left\{ 1 + p + p^2 + \cdots \right\} < U_{n+1} / (1-p) \\
\text{also we have } P_n &< U_{n+1} / 1 - p_{n+1} \\
&< U_{n+1} / 1 - \frac{U_{n+2}}{U_{n+1}}
\end{align*}
\]
When the terms of the series ultimately decrease and alternate in sign, the estimation of the residue is still simpler.

Let \( n \) be as large that on and after the \( n \)th term, none increase in value and always alternate in sign. Then \( R_n = u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} - \ldots + u_{n+k} - u_{n+k+1} + u_{n+k+2} \)

It will be necessary to use the properties of series which have a double-infinity of terms, in some places. The theory originated with Cauchy and the greater part of what follows is taken from "Élémens Analytiques," and from "Résine Analytiques."

Take a doubly infinite series of
lems. We may take as the general or specimen term \( \sum_{n} \) where the first index indicates the row and the second the column to which the term belongs. The assemblage of such terms we may denote by \( E_{mn} \) and shall speak of this assemblage as a double series.

A great variety of definitions might be given of the sum to a finite number of terms of such a series and corresponding to each definition would arise the question of the summation to infinity, that is regarding the convergence of the series. There are only four ways of taking the sum of the double series which are of
First way—The finite sum is the sum of all the \( m \times n \) terms within the rectangular array \( OKNP \). This is called Sn. We get the limit of this by making one or more infinite.

If the result of both of these limit operations is the same definite,
quantity \( S \), then we say \( S_n \) converges to \( S \) in the first way.

It may happen that both of these operations lead to an infinitesimal value, that neither leads to a definite value, that one leads to a definite finite value and the other does not, that one leads to a definite finite value and the other to another definite finite value.

In all these cases, the series is non-convergent for the first way of summation.

The second way—Sum to \( n \) terms each of the series formed by taking the terms of the first \( \left( m \right) \) horizontal rows, and call the sums \( S_{1m}, S_{2m}, \ldots, S_{nm} \), and the finite sum \( S_m \).

Then supposing
each of the horizontal series to
corrove to $T_1$, $T_2$, — $T_m$, respectively
and $S' = T_1 + T_2 + T_3 + \ldots T_m + \ldots$ as
the sum to infinity in the second way.

The third way: sum to $m$ terms
each of the series in the first $n$
columns and let these sums be $U_1, U_m, U_3, m - U_{m-1},$ and let $S'' =
U_1 + U_m + \ldots U_n$ be the finite sum.
Then supposing these vertical
series to corrove to $U_1, U_2, - U_m$, respectively
and $\sum U_n$ to be a convergent
series, define $S'' = U_1 + U_2 + \ldots U_n + \ldots$ as the sum to infinity in the
third way. So long as $m, n$ are finite
it is obvious that $S'' = S'' = S''$,
as that for finite summation
the second and third ways of
summing are the same as the first.
The case is not so simple when
the sum to infinity is taken
\[ S' = \lim_{n \to \infty} \sum_{k=1}^{n} S_{nn} \]
\[ S'' = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} S_{nn} \right\} \]
so that \( S' \) and \( S'' \)
will be equal to each other and
to \( S \) when the two ways of taking
the limit of \( S_{nn} \), both lead to the
same definite finite result.
The fourth way — sum the terms
which lie in the successive diagonal
lines of the array, as \( AA', BB', CC', \)
and let these sums be \( T_2, T_3, T_4, \ldots \),
respectively; that is \( T_2 = u_1, \quad T_3 = u_2 + u_3 \)
\( T_{n+1} = u_n + u_{n+1} + \ldots + u_{n+1} \).
Let \( S_n'' = T_2 + T_3 + T_4 + \ldots + T_n \) be the
finite sum, and supposing \( S \to S_n'' \).
be convergent, then $S'' = a_2 + a_3 + \cdots + a_n + \cdots$ is the sum to infinity in the fourth way. The finite sum includes all the terms in the triangle $DKN$, it can not coincide with the finite sum of the other definitions except for $m = n = 1$.

Whether the sum to infinity $(S'')$ according to the fourth definition will coincide with $(S)$ $(8)$ or $(8'')$ depends on the nature of the series. It may happen that $(S)$ $(8)$ and $(8'')$ exist and are all equal and that the limit of $S''$ is infinite.

The most important kind of double series is that in which for all values of $m$ and $n$ greater than certain fixed limits, $U_{mn}$ has always the same sign.
a finite quantity to $(a + c)$. The sum we can always make any finite number of terms have the same sign as the ultimate terms of the series; we may, as far as questions regarding convergence are concerned, suppose all the terms of $\sum_{m=1}^{n} a_m$ to have the same sign from the beginning. Suppose the diagram on page 86 represent the array of terms under this last supposition, and $\sum_{m=1}^{n} a_m$ is convergent in the first way.

Then since $L (\sum_{m=p+1}^{n+q} a_m) = 8 - 8 = 0$ where $m = \alpha, n = \beta$ what ever $p$ and $q$ may be, it follows that the sum of all the terms terms of the gnomon between $NMK$ and its parallels to $KM$ may $\alpha$, below and to the right of these lines.
respectively, must be come as small as we please when we remove \( M M \) sufficiently far down and \( M N \) sufficiently far to the right.

Therefore if \( m \) and \( n \) be sufficiently large the sum of any group of terms taken in any way from the residual of terms lying outside \( O K M N \) will be as small as we please.

Therefore the total of partial residues of each of the horizontal series vanishes when \( n = 0 \). The same is true for each of the vertical series. The same is true for the series \( \sum D_n \), since \( S_n \) lies between \( S_g \cdot n \cdot q \) and \( S_{n-1} \cdot n-1 \).

Hence if all the terms of \( \sum u_n \) be positive, and if the series be
convergent in the first way, then each of the horizontal series, and each of the vertical series, and the diagonal series will be convergent, so the double series will be convergent in the remaining three ways, and always to the same limit.

Now if we take $E \nu^m$, convergent in the second way, $E \tau^m$ is convergent and by increasing $m$ we can make the residue below the line $\nu \tau$ less than $\frac{1}{\nu^2}$ where $\epsilon$ is as small as we please. Also since each of the horizontal series is convergent, by increasing $n$ sufficiently we can make the sum of the residues less than $\frac{1}{\nu^2 \epsilon}$. Hence by increasing $m$ and $n$ we can make the
sum of the terms outside of $\Omega NM$ less than $\epsilon$. Therefore $E_{nm}$ is convergent in the first way, and so in all the four ways.

The proof for convergence of the third way is similar to that of the second, only dealing with columns instead of rows.

If $E_{nm}$ is convergent in the fourth way, the residue of the diagonal series $E \Delta p$ can, by making $p$ large enough be made as small as we please.

Now if $m$ and $n$ be large enough the residue of $S_{mn}$, or the sum of as many terms as we please outside of $\Omega NM$ will contain only terms outside of $\Omega NM$, all of which are terms in the residue of $S'$. 
Since all the terms in the array are positive we can make the sum of as many as we please of the terms outside $\sum_{n=1}^\infty a_n$ as small as we please by increasing $m$ and $n$. Therefore $\sum_{n=1}^\infty a_n$ is convergent in the first way and so is convergent in all four ways.

When a double series is such that it remains convergent when all its terms are taken positively it is said to be absolutely convergent.

Cauchy has given 5 rules to test for absolute convergence of a double series. If $\sum_{n=1}^\infty a_{mn}$ be the positive value of $\sum_{n=1}^\infty a_n$, and if all the horizontal
series of $E_{mn}$ be convergent, and the sum of their sums to infinity also convergent. Then

1. The horizontal series of $E_{mn}$ are all absolutely convergent, and the sum of their sums converges to a definite finite limit $S$.

2. $E_{mn}$ converges to $S$ in the second.

3. The vertical series are absolutely convergent and the sum of their sums to infinity converges to $S$.

4. The diagonal series is absolutely convergent and converges to $S$.

5. Any series formed by taking terms from $E_{mn}$, all of which are finite, is absolutely convergent.
The most remarkable series in the algebraic analysis is:

\[ 1 + m x + \frac{m(m-1)x^2}{2} + \frac{m(m-1)(m-2)x^3}{6} + \ldots \]

The number of terms is finite when \( m \) is a positive integer, but when \( m \) is not a positive integer as one of the values \( m, m-1, m-2, \ldots \) can be zero and the series will go to infinity and will be convergent or divergent according to the values assigned to \( m \) and \( x \).

To determine the convergence when \( m \) is not a positive integer, we must consider the ratio

\[ \frac{u_{n+1}}{u_n} = \frac{m-n+1}{n} \cdot (-x) \left(1 - \frac{m+1}{n}\right) \]

Hence for all values of \( n \) greater
than \( m+1 \), \( u_{n+1} \) and \( u_n \) will have different signs when \( x \) is positive, and have the same signs when \( x \) is negative, and as \( n \) is increased, the value of \( \frac{u_{n+1}}{u_n} \) becomes more nearly equal to \( x \). Therefore if \( x \) be numerically less than unity, the ratio \( \frac{u_{n+1}}{u_n} \) will be less than unity after a finite number of terms.

The series formed by adding the absolute values of the successive terms will be convergent, so the series will be convergent when all its terms are the same sign or alternatively plus and minus.

When \( x \) is greater than unity, the series is divergent, then the terms do not approach zero as a
limit case when \( m \) is a positive integer, for then the series is limited. When \( x = 1 \), the series becomes

\[
1 + m + m \frac{(m-1)}{1} + m \frac{(m-1)(m-2)}{1 \cdot 3} + \ldots
\]

Now the terms will be alternately positive and negative after the \( n \)th where \( n \) is the first positive integer greater than \( m + 1 \), and the ratio \( \frac{u_{n+1}}{u_n} \) is less than unity if \( (m+1) \) is positive. Therefore the series will be convergent when \( m + 1 \) is positive provided the \( n \)th term decreases without limit as \( n \) approaches infinity.

\[
\frac{1}{u_n} = \frac{1}{m} \frac{1}{(m-1)} - \frac{(m-n+1)}{(m-n+1)}
\]

written \( \frac{1}{u_n} = \frac{1}{m} \left( 1 + \frac{1+m}{1-m} \right) \left( 1 + \frac{1+m}{2-m} \right) - \left( 1 + \frac{1+m}{n-1-m} \right) \)
and the product of the preceding factors is finite.
Hence when \( n \) is increased without limit, \( t_n = \infty \), \( \therefore \quad q_n = 0 \), provided \( 1 + m \) be positive.

Thus the binomial series is convergent if \( x = 1 \), provided \( m > -1 \).

If \( x = -1 \), the series becomes

\[
1 - m + m(m-1) \frac{1}{2} - m \frac{(m-1)(m-2)}{3} + \cdots
\]

The sum of \( n \) terms of this series is

\[
(1 - m) (2 - m) \frac{(3 - m) \cdots (n-1 - m)}{n-1}
\]

The sum is therefore 0 or \( \infty \) when \( n = \infty \) as \( m \) is positive or negative.

Therefore the binomial series is convergent when \( x = -1 \) if \( m \) is positive.
He considers the terms of \([I] \) as functions of \((m)\), or of \((x)\).
Setting these functions of \((x)\), and \(m\) is a given number less than \(M\) in absolute value the series \([II] \) is absolutely and uniformly convergent in every interval whose two limits are \(\xi\), or, since \(\xi\) is as near the value unity as we please, in every interval.
whose limits are less than unity in absolute value.

If series (m) and x is a number with absolute value less than unity, the series is absolutely and uniformly convergent in every interval whose limits are, in absolute value, less than M. That is, in every interval series M is arbitrary.

When a series whose terms are continuous functions of one variable (x), in the interval (a, b), is uniformly convergent in that interval, the summation of that series is a continuous function.
f(x) in the interval (a, b)

Let \( C_1, x + C_2, x + \ldots \) be the series considered, where \( C_1, C_2 \) are continuous functions of \( x \) in the interval (a, b).

Let \( F(x) \) be the summation of the series. Let \( \epsilon \) be any positive number. There is an integral number \( (n) \) such that for all values of the interval (a, b) the remainder, \( R_n(x) \), of the series may be less than \( \frac{\epsilon}{3} \).

Let \( S_n(x) = C_1(x) + C_2(x) + \ldots + C_n(x) \).

\( F(x) = S_n(x) + R_n(x) \)

But the \( n \) first functions of \( C \) being continuous, so is their sum \( S_n(x) \). There exists a number \( y \)
such that $|S_n(x) - S_n(x')| < \varepsilon_3$ for all values of $x$ and $x'$ which belong to the interval $(a, b)$, and whose difference is less than $\eta$.

This inequality joined to the inequality $P_n(x) < \frac{\varepsilon}{3}, P_n(x') < \frac{\varepsilon}{3}$, shows that in the same conditions $F(x) - F(x') = S_n(x) + P_n(x) - (S_n(x') + P_n(x'))$ which is less than $\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ or less than $\varepsilon$.

That is to say $F(x)$ is continuous in the interval $(a, b)$.

Thus the series $1 + \frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots$ is a continuous function of $x$ in all intervals.

The series $1 + mx + \frac{mx^2}{2!} + \ldots + \frac{(m-x)^n}{n!} x^n + \ldots$ is a continuous function of $x$, if $m$ is given, in all
intervals whose limits are less than unity. The same series when既可以 regard \( x \) as a given number less than unity in absolute value is a continuous function of \( x \) in all intervals.

Now take the series

\[
\begin{align*}
1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots
\end{align*}
\]

and the expression \((1 + \frac{x}{m})^m\). Allow \( x \) to remain fixed, and \( m \) to increase indifferently and the expression \((1 + \frac{x}{m})^m\) the summation of the series (a). On if \( x \) is any given number, and \( \epsilon \) any positive number there exists a positive number \( m \) such that the difference between \((1 + \frac{x}{m})^m\) and the summation of series (a), in absolute value is less than \( \epsilon \).
when the value of \( m \) is greater than \( m \).

I suppose first that \( m \) does not take but finite integral values. Then by the binomial theorem

\[
(1 + \frac{x}{m})^m = 1 + \frac{mx}{m} + \frac{m(m-1)x^2}{2m^2} + \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)x^k}{k!m^k} + \ldots
\]

which may be written

\[
(1 + \frac{x}{m})^m = 1 + \frac{x}{m} + (1 - \frac{1}{m})\frac{x^2}{2m^2} + 
\]

\[
(1 - \frac{1}{m})\frac{x^3}{3m^3} + (1 - \frac{2}{m})(1 - \frac{3}{m})\frac{x^4}{4m^4} + \ldots
\]

Let \( A \) be a positive number \( \geq x \). The series

\[
1 + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} \text{ is convergent}
\]

and each of its terms is greater than the absolute value of the term of the same rank in the second member of equation (6). We may take \( b \) large enough so that the sum of the
series \( \frac{1}{x^2} + \frac{1}{x^3} + \ldots \) may be less than \( \frac{1}{3} \). It is the same evidently for the absolute value of the remainder \( R' \) of the series \( 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \), limited at the \( f \)-th term, and of the absolute value of the remainder \( R'' \) of the development of \( (1 + \frac{x}{m})^m \), that follows the \( (f+1) \)-th term.

Now call \( S \) the summation of the series \( 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \); \( S_f \) the sum of its \( (f+1) \) first terms; \( S'_f \) the sum of the \( (f+1) \) first terms of the development \( (1 + \frac{x}{m})^m \).

\[ S = S_f + R'' \]

\[ (1 + \frac{x}{m})^m = S'_f + R'' \], but \( S_f \) is an

an integral function of \( \frac{1}{m} \) which reduces to \( S_f \) when \( \frac{1}{m} \) is replaced by 0.
Since the function is continuous, there exists a positive number \( \eta \) large enough so that \( \frac{1}{m} < \frac{1}{\eta} \) or \( m > \eta \).

Therefore \( S_f - S_{f'} < \frac{\varepsilon}{3} \).

This inequality, when joined to the inequalities \( P_f < \frac{\varepsilon}{3} \), \( P_{f'} < \frac{\varepsilon}{3} \), shows that the absolute value of

\[
S - \left(1 + \frac{x}{m}\right)^m = (S_f - S_{f'}) + P_f - P_{f'} \]

is less than \( \varepsilon \) if \( m \) is a positive integer greater than \( \eta \). This was the supposition.

Suppose \( m \) to be positive but not an integer. The expression \( \left(1 + \frac{x}{m}\right)^m \) has not been defined in such cases except if \( \left(1 + \frac{x}{m}\right) \) is positive, but that is always finally true for values sufficiently large of \( m \).

Let \( \mu \) be the integral part of \( m \).
and suppose \((1 + \frac{x}{m})\) to be positive.

The expression \((1 + \frac{x}{m})^m\) will always be comprised between
\[
\left(1 + \frac{x}{m}ight)^{1/m} \left(1 + \frac{x}{m+1}\right)^{m+1} \>
\]
and \(1 + \frac{x}{m} = \left(1 + \frac{x}{m}ight) \left(1 + \frac{x}{m+1}\right)\).

Now the first of these two quantities will be the smaller if \(|x|\) be positive, the larger if \(|x|\) be negative; but when \(m\) increases indefinitely, so do \(m\), and the forms that they have given to the two quantities between which the quantity \((1 + \frac{x}{m})^m\) lies shows that they have some limit. It is the same for \((1 + \frac{x}{m})^{m'}\).

Suppose \(m = -m'\). Then \((1 + \frac{x}{m})^m = (1 - \frac{x}{m'})^m = \left(1 + \frac{x}{m' - x}\right)^{m'}\). Then \((1 + \frac{x}{m} - x)^{m'}\).
\[(1 + \frac{x}{m-x})^{m-x} \cdot (1 + \frac{x}{m+x})^x\] when \(m\) increases indefinitely, \(m-x\) increases indefinitely. So the first factor of the second member has \(1\) for its limit. The second factor has unity for its limit. Therefore the proposition is proved.

Now consider \((1 + \frac{x}{m})^m\). Take \(x\) as small as they wish, there exist two positive numbers \(n\) and \(\alpha\), so that the difference between \((1 + \frac{x}{m})^m\) and the summation of \(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{\alpha-1}}{\alpha-1} + \cdots\) may be less than \(\varepsilon\), provided that they have \(m\) greater than \(n\) and \((\varepsilon - x)\) less than \(\alpha\).

If now \(x = 1\), the summation of the series becomes \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\)
which summation is called \( \varepsilon \), and is the limit towards which 
\( (1 + \frac{m}{n})^n \) tends when \( m \) approaches infinity as a limit.

The letter \( \varepsilon \) plays an important role in the analysis. On stopping
at the term \( \frac{1}{n^2} \) in the series they see that the remainder of the series,
or the summation of the terms
\[
\frac{1}{n^2} + \frac{1}{n^3} + \cdots \text{ is smaller than the sum of } \frac{1}{n^2} \left( \frac{1}{(n+1)^2} + \cdots \right) = \frac{1}{n^2} \]
So they may write \( \varepsilon = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \)
where this is an unknown number larger than 0, and less than unity.

This formula permits the calculation
of \( \varepsilon \) with as much approximation
as you may wish.
By this method the value of \( e \) is found to be
\[ e = 2.71828182845904523536 \ldots \]
This value is easily remembered from its combination of numbers.

The same formula \( e = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) shows that \( e \) is irrational.

If it were equal to the irreducible fraction \( \frac{p}{q} \), they would have
\[ \frac{p}{q} - 1 - \frac{1}{2} - \frac{1}{3} - \cdots \]
multiplying the first member by \( 1 \), the first member would become whole; multiplying the second member by \( 1 \), it becomes \( \frac{p}{q} \), a number that can never be a whole number even 0. They use this proof that \( e \) is transcendental.

That is to say that \( e \) is not
a root of any integral algebraic equation with rational coefficients.

The identity \((1 + \frac{x}{m})^m \equiv \left(1 + \frac{1}{\frac{m}{x}}\right)^{\frac{m}{x}}\) shows that when \(m\) approaches infinity, \((1 + \frac{x}{m})^m\) has for a limit \(e^x\).

When \(m\) approaches infinity so does \(\frac{m}{x}\), so \((1 + \frac{x}{m})^m = (1 + \frac{1}{\frac{m}{x}})^{\frac{m}{x}}\) has for a limit \(e^x\). Because of the continuity of the function \(e^x\), where \(e\) is a positive number, the number \((1 + \frac{x}{m})^m\) raised to the power \(x\) has for its limit \(e^x\).

Then \(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\)

Also \(e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots + \frac{x^n}{n!} + \ldots\)
We have had many examples of continuous functions represented by series. Now the question arises: if the converse is true, does every power series represent within its interval of convergence, a continuous function? That this question is important is shown by the fact that while the continuous functions of ordinary analysis can be generally represented by trigonometric series, that is by series in the form

\[ a_0 + a_\cos x + a_2 \cos 2x + \ldots \]

\[ b_0 \sin x + b_1 \sin 2x + \ldots \]

a trigonometric series which does not necessarily represent a continuous function throughout its interval of convergence.

\( f(x) \) is said to be continuous at the point \( x_0 \) if \( \lim_{x \to x_0} f(x) = f(x_0) \).

If a belt marked off by the lines
$y = \mathcal{O}(x) + \varepsilon$ and $y = \mathcal{O}(x) - \varepsilon$, where $\varepsilon$ is an arbitrarily small positive quantity on an interval $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$ can always be found such that when $x$ lies within this interval $\Phi x$ will lie within this belt.

These conditions may be expressed in the following form:

$\Phi(x_0) - \varepsilon \leq \Phi(x) \leq \Phi(x_0) + \varepsilon$, or $\Phi(x) - \Phi(x_0) \leq \varepsilon$

$(x_0 - \delta) \leq x \leq (x_0 + \delta)$, or $x - x_0 \leq \delta$

If $\mu_0(x) + \mu_1(x) + \mu_2(x) + \ldots$, $a \leq x \leq b$, is a series of continuous functions convergent throughout the interval $(a, b)$, then the function $f(x)$ represented by this series will be continuous throughout.
this interval if a set of positive numbers \( M_0, M_1, M_2, \ldots \) independent of \( x \) can be found such that

1. \( |u_n(x)| \leq M_n \), \( a \leq x \leq b \), \( n = 0, 2, 3, \ldots \);

2. and \( M_0 + M_1 + M_2 + \ldots \) is a convergent series.

It must be shown that \( f(x) \) being

any point of the interval if a positive quantity \( \varepsilon \) be chosen at pleasure

then a second quantity \( \delta \) can be so determined that

\[ f(x) - f(x_0) < \varepsilon \text{ if } |x - x_0| < \delta \]

Let \( r_n(x) = u_0 x + u_1(x) + \ldots + u_{n-1}(x) \)

\[ f(x) = r_n(x) + r_n(x) \]

Then \( f(x) - f(x_0) = \int (r_n(x) - r_n(x_0) + r_n(x_0) - r_n(x)) \]

now the absolute value of each of these quantities is less than \( \frac{1}{3} \varepsilon \), if \( \delta \) is properly chosen and \( |x - x_0| < \delta \)

Therefore \( f(x_0) - f(x) \) is less than \( \varepsilon \)

Now let the remainder in the \( M \)-series be denoted by \( R_n \).
\[ R_n = M_n + M_{n+1} + \ldots \]

and let \( n \) be so chosen that \( R_n < \frac{1}{3} \varepsilon \)
and then held fast. Then since

\[
\begin{align*}
|u_n(x)| & \leq M_n \\
|u_{n+1}(x)| & \leq M_{n+1} \\
|u_{n+2}(x)| & \leq M_{n+2}
\end{align*}
\]

it follows that \( m(x) \leq R_n \) for all values of \( x \) at once or \( |m_n(x)| < \frac{1}{3} \varepsilon \quad x \in x \in \beta \)

Since \( m_n(x) \) is the sum of a fixed number of continuous functions, it is a continuous function and hence \( D \) can be so chosen, that

\[
|S_n(x) - R_n(x_0)| < \frac{1}{3} \varepsilon \quad |x - x_0| < D
\]

hence \( |f(x) - f(x_0)| < \varepsilon \quad |x - x_0| < D \)

That is \( f(x) \) is a continuous function.

A Fourier series represents a continuous function within its interval of convergence. The function
may however become discontinuous on the boundary of the interval.

Let the series be

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

convergent when \( -r < x < r \); and let \( (a, b) \) be any interval contained in the interval of convergence, neither extremity coinciding with an extremity of that interval.

Let \( x \) be chosen greater than either of the quantities \( |a|, |b| \), but less than \( r \). Then \( |a_n x^n| < |a_n| \frac{x^n}{X^n} \); \( a \leq x, \leq b \); and the series \( a_0 + |a_1| x + |a_2| x^2 + \cdots \)

can be divided. Hence if we put \( M_n = |a_n| X^n \)

Thus by the theorem proved just before this \( f(x) \) is a continuous function throughout the interval \((a, b)\).
If a power series vanishes for all values of \( x \) lying in a certain interval about the point \( x = 0 \):

\[
0 = a_0 + a_1 x + a_2 x^2 + \ldots \quad -1 < x < 1
\]

then each coefficient vanishes:

\[
a_0 = 0 \quad a_1 = 0 \quad a_2 = 0 \quad \ldots
\]

First put \( x = 0 \), then \( a_0 = 0 \) and the above equation can be written in the form

\[
0 = x(a_1 + a_2 x + a_3 x^2 + \ldots)
\]

Therefore

\[
0 = a_1 + a_2 x + a_3 x^2 + \ldots \quad \text{provided} \quad x \neq 0
\]

but it does not follow that this equation is satisfied when \( x = 0 \) and therefore \( a_1 \) cannot be shown to vanish by putting \( x = 0 \) as before.

Let \( f(x) = a_1 + a_2 x + a_3 x^2 + \ldots \). Then since \( f(x) \) is a continuous function of \( x \),

\[
\lim_{x \to 0} f(x) = f(0) = a_1, \quad \text{but} \quad \lim_{x \to 0} \frac{f(x)}{x} = 0 \quad \therefore a_1 = 0
\]

By repeating this reasoning each of the following coefficients can be shown to be 0.
From this it follows that if two Fourier series have the same value for all values of \(x\) in an interval about the point \(x = 0\), their coefficients are respectively equal.

\[
a_0 + a_1 x + a_2 x^2 + \cdots = b_0 + b_1 x + b_2 x^2 + \cdots \quad -L < x < L
\]

\[
a_0 = b_0, \quad a_1 = b_1, \quad \text{and so on.}
\]

If we have

\[
\frac{b_0 + b_1 x + b_2 x^2 + \cdots}{a_0 + a_1 x + a_2 x^2 + \cdots} = \frac{c_0 + c_1 x + c_2 x^2 + \cdots}{d_0 + d_1 x + d_2 x^2 + \cdots}
\]

we can multiply each term by the denominator series.

\[
b_0 + b_1 x + b_2 x^2 + \cdots = a_0 c_0 + (a_1 c_0 + a_2 c_1) x + (a_3 c_0 + a_4 c_1 + a_5 c_2) x^2 + \cdots
\]

hence \(b_0 = a_0 c_0\), \(b_1 = a_1 c_0 + a_2 c_1\), \(b_2 = a_3 c_0 + a_4 c_1 + a_5 c_2\), \(\text{etc.}\)

Let the continuous function \(f(x)\) be represented by an infinite series of continuous functions convergent throughout the interval.
(13) \[ f(x) = U_r(x) + U_i(x) + U_k(x) + \cdots \quad a \leq x \leq b \]

Now determine when the integral of \( f(x) \) will be given by the series of the integrals of the terms on the right of equation (1). That is to determine when \( \int_a^b f(x) \, dx = \int_a^b U_r(x) \, dx + \int_a^b U_i(x) \, dx + \cdots \) will be a true equation. The right-hand member is called the term by term integral of the \( u \)-series.

Let \( p_n(x) = U_r(x) + U_i(x) + \cdots + U_n(x) \).

Then \( f(x) = p_n(x) + n_n(x) \).

Thus \( \int_a^b f(x) \, dx = \int_a^b U_r(x) \, dx + \int_a^b U_i(x) \, dx + \cdots + \int_a^b U_n(x) \, dx + \int_a^b n_n(x) \, dx \)

Hence the necessary and sufficient condition that the series may be integrated term by term is that

\[ \lim_{n \to \infty} \int_a^b n_n(x) \, dx = 0 \]

To obtain a test for determining when this condition is satisfied.
Plot the curve $y = \ln(x)$.

The area under this curve will represent geometrically $\int_1^n \ln(x) \, dx$.

Draw lines through the highest and lowest points of the curve parallel to the (x) axis. The distance $m$ of the more remote of these lines from the x-axis is the maximum value that $\ln(x)$ attains in the interval. Lay off a belt bounded by the lines $y = m$ and $y = -m$. Then the curve lies wholly within this belt and the absolute value of the area under the curve cannot exceed the area of the rectangle.
bounded by the line \( y = \beta_n x (3-a)/\beta_n \).

This area will converge towards zero as its limit if \( \lim_{n \to \infty} \beta_n = 0 \),

and thus we shall have a sufficient condition for the truth of the equation when integrated term by term, if we establish a sufficient condition that the maximum value \( \beta_n \) of \( f(x) \) in the interval \([a, b]\) approaches 0, as \( n \) approaches \( \infty \).

Now we have lately proved that if \( |f(x)| \equiv R_n \), and \( a \leq x \leq b \); \( \lim_{n \to \infty} R_n = 0 \).

Hence any such series can be integrated term by term and we have in this result a test for most of the cases that arise in ordinary practice.

The general formula is the given series can be integrated term by term, that is
\[ \int x^3 \, dx = \int x^3 \, dx + \int x^3 \, dx + \cdots \] will be a true equation if a set of positive numbers, \( M_0, M_1, \ldots \), in dependent of \( x \) can be found such that

1. \( \ln(x) \leq M_n \quad 0 \leq x \leq \theta \quad \forall = 0, 1, 2, \ldots, 4 \)

2. \( M_0 + M_1 + M_2 + \cdots \) is a convergent series. This theorem does not hold for an integral, one of whose limits is infinite, as is shown by the example \( \int \frac{1}{(1+x)^2 + (2+x)^2 + (3+x)^2} \, dx \in \mathbb{R} \) This series converges uniformly for all positive values of \( x \). But the term-by-term integral from a positive lower limit \( a \), to infinity is a divergent series.

A four series can be integrated term by term throughout any integral \( [a, b] \) contained in the interval of convergence, and not reaching
out to the extremity of this interval. \( |a| < 2 \), \( |\beta| < 2 \).

Let the series be written in the form

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

and let \( X \) be chosen greater than the greater of two quantities \( |\alpha|, |\beta| \), but less than \( x \). Then

\[ a_n X^n = |a_n| X^n \quad a \equiv x \equiv \beta \]

and if we place \( a_n X^n = M_n \), the conditions of the test will be satisfied.

Let \( Q(x) \) be a continuous function of \( x \) and let its maximum and minimum values lie between \( -v \) and \( v \) when \( a \equiv x \equiv \beta \).

Let the power series \( a_0 + a_1 x + a_2 x^2 + \cdots \) converge when \( y > -v, x y < v \). Then the series of \( f(x) = a_0 + a_1 Q(x) + a_2 [Q(x)]^2 + \cdots \) can be integrated term by term from \( \alpha \) to \( \beta \).
\[ \int_{\beta}^{\alpha} x \, dx = a_0 \int_{\beta}^{\alpha} x \, dx + a_1 \int_{\beta}^{\alpha} (x \phi(x))^2 \, dx + a_2 \int_{\beta}^{\alpha} (x \phi(x))^3 \, dx + \]

For if \( n \) was taken to be greater than the numerically greatest value of \( \phi(x) \) in the interval \( a \leq x \leq \beta \) but less than \( n + 1 \), then:

1. \( |a_n| \left( \phi(x) \right)^n < \frac{x^n}{y^n} \)
2. \( |a_0| + |a_1| y + |a_2| y^2 + \ldots \) converges.

and if we place \( |a_n| y^n = \frac{y^n}{M_n} \), the conditions of the test will be satisfied.

If the function \( \phi(x) \) and the series \( a_0 + a_1 y + a_2 y^2 + \ldots \) satisfy the same conditions as in the preceding question and if \( \psi(x) \) is any continuous function of \( x \) then the series

\[ f(x) = a_0 \psi(x) + a_1 \psi(x) \phi(x) + a_2 \psi(x) [\phi(x)]^2 + \ldots \]

can be integrated term by term.
It is worthy of note that the integration of series term by term was fundamental the derivative appearing as the inverse of the integral. Now consider when a series can be differentiated term by term.

Let the function $f(x)$ be represented by the series $f(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$ throughout the interval $(a, b)$. Then the derivative $f'(x)$ will be given at any point of the interval by the series of the derivatives $f'(x) = u_0'(x) + u_1'(x) + u_2'(x) + \cdots$ provided the series of the derivatives $u_0'(x) + u_1'(x) + u_2'(x) + \cdots$ is a continuous function convergent throughout the interval $(a, b)$. Let the latter series be denoted by $\mathcal{O}(x)$: $\mathcal{O}(x) = u_0'(x) + u_1'(x) + u_2'(x) + \cdots$.

We wish to prove that $\mathcal{O}(x) = f'(x)$.
The function \( \phi(x) \), represented by the \( \mu \)-series is continuous and can be integrated term by term.

\[
\int_a^x \phi(x) \, dx = \int_a^x \mu_0(x) \, dx + \int_a^x \mu_1(x) \, dx + \cdots
\]

\[
= \left\{ \mu_0(x) - \mu_0(a) \right\} + \left\{ \mu_1(x) - \mu_1(a) \right\} + \cdots
\]

\[
= f(x) - f(a)
\]

Hence differentiating \( \phi(x) = f(x) \).

A power series can be differentiated term by term at any point within but not necessarily at a point on the boundary of its interval of convergence.

Let the power series be

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots
\]

convergent when \( |x| < R \), and form the series of its derivative

\[
a_1 + 2a_2 x + 3a_3 x^2 + \cdots
\]

Then we want to prove that
if \( x_0 < \pi \), then 
\[
\sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \]

It will be sufficient to show that the series of the derivatives converges when \( x < \pi \) for in that case, if \( X \) be so chosen that 
\( x < X < \pi \), the conditions of the test will be fulfilled throughout the interval \((-\pi, \pi)\). We can then this as follows — let \( x' \) be any value of \( x \) within the interval \((-\pi, \pi)\); 
\[-\pi < x < \pi \] and let \( X' \) be so chosen that \( |x'| < X' < \pi \). The series

\[
|a_0| + |a_1| x' + |a_2| x'^2 + \cdots \]

converges. It will also serve as a test-series for the convergence of

\[
|a_0| + 2|a_1| x' + 3|a_2| x'^2 + \cdots \]

if it can be shown that \( n |a_n| |x'|^{n-1} \leq |a_n| x^n \)

from some definite point \( n = m \), on. This will be the case if
\[ n \left( \frac{\lambda}{m} \right)^n \leq \lambda^n \quad m > n \]

But the expression on the left approaches 0 when \( n = \infty \) for \( \lambda/\lambda^n \)

is independent of \( n \) and less than 1. The limit can therefore be obtained
by the usual methods of evaluating
\( \lambda^n \).

Hence the condition that
the former series may serve as a
last series is fulfilled and the proof
is complete.

A sufficient condition for the
differentiation of a series term by
term may be stated as follows:

If \( f(x) \) is a continuous function
of \( x \) having a continuous derivative
and if \( f(x) \) is developed into a
series of continuous functions
having continuous derivatives:

\[ f(x) = U_1(x) + U_2(x) + U_3(x) + \ldots \]
and if the result of differentiating this series term by term is a convergent series whose value $Dx$ is a continuous function of $x$ then the given series can be differentiated term by term.

$$f(x) = Dx = U_1(x) + U_2(x) + U_3(x) + \ldots$$

If $D(x)$ is not a continuous function of $x$ the theorem does not hold true in any interval whatsoever.

A great deal of light is thrown on the peculiarities of trigonometric series by the attempt to construct approximately the curves corresponding to them. If we construct $y = a \sin x$ and $y = a \cos x$ and add the ordinates of the points having the same abscissae we shall obtain points on the curve.
\[ y = a \sin x + a \sin 2x. \]

If now we construct \[ y = a \sin x + y \sin 3x \] and add the ordinates to those of \[ y = a \sin x + a \sin 2x \] we shall get the curve \[ y = a \sin x + a \sin 2x + a \sin 3x. \]

By continuing this process we get successive approximations to \[ y = a \sin x + a \sin 2x + a \sin 3x + a \sin 4x + \cdots \]

Now take the four curves

1. \[ y = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \]
   - 0 when \( x = 0 \), \( \frac{\pi}{2} \) from \( x = 0 \) to \( x = \pi \), and 0 when \( x = \pi \)

2. \[ y = 2 (\sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots) \]
   - \( x \) from \( x = 0 \) to \( x = \pi \) and 0 when \( x = \pi \)

3. \[ y = \frac{1}{2} \left\{ \frac{1}{2} \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \cdots \right\} \]
   - \( x \) from \( x = 0 \) to \( x = \frac{\pi}{2} \) and \( \pi - x \) from \( x = \frac{\pi}{2} \) to \( x = \pi \)

4. \[ y = \frac{1}{2} \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \cdots \]
   - 0 when \( x = 0 \), \( \frac{\pi}{2} \) from \( x = 0 \) to \( x = \frac{\pi}{2} \), 0 from \( x = \frac{\pi}{2} \) to \( x = \pi \).
Each of these curves is periodic having the period $2\pi$, and is symmetrical with respect to the origin.

The following figures I, II, III, IV, V, VI, represent the first four approximations to each of these curves.
In each figure the curve \( y = \sum a_n \) and the approximations in question are drawn in continuous lines, and the preceding approximation and the curve corresponding to the term to be added are drawn in dotted lines.

1. The curve representing each approximation is continuous even when the curve representing the series is discontinuous.
2. When the curve representing the series is discontinuous, the portion of each
successive approximate curves in the neighborhood of the point whose abscissa is a value of \( x \) for which the curve is discontinuous, approaches more and more nearly a straight-line perpendicular to the axis of \( x \) and connecting the separate portions of the curve.

2. The curves representing successive approximations do not necessarily tend to lose their wavy character, since each is obtained from the preceding one by superposing upon it a wavy line whose waves are shorter each time, but do not necessarily lose their sharpness of pitch. This is the case in figures I, II, and III. In figure III, the waves of the superposed curves grow rapidly flatter. It follows from this that in such cases as those represented in figures I, II, and IV, the direction of
the approximate curve at a point having a given abscissa does not in general approach the direction of the series curve at the corresponding point, or approach any limiting value as the approximation is made closer and closer; and the length of any portion of the approximate curve will not in general approach the length of the corresponding portion of the series curve.

Analytically this means that the derivative of a function of $x$ cannot in general be obtained by differentiating term by term the Fourier's series which represent the function.

1) The area bounded by a given ordinate, the approximate curve, the axis of $x$, and any second ordinate will
approach as its limit the corresponding area of the series curve if the series is continuous between the ordinates in question; and will approach the area bounded by the given ordinate, the series curve, the axis of $x$, any second ordinate, and a line perpendicular to the axis of $x$ and joining the separate portions of the series curve if the latter has a discontinuity between the ordinates in question.

Analytically this means that the Fourier's series corresponding to any given function can be integrated term by term, and the resulting series will represent the integral of the function even when the function is discontinuous.
If the function curve is continuous, a curve representing the integral of the function will be continuous and will not change its direction abruptly at any point, while if the function curve is discontinuous, the curve representing the integral will still be continuous but will change its direction abruptly at points corresponding to the discontinuities of the given function.

The differentiation of
\[ S_n = \sin x + \frac{1}{2} \sin 3x + \frac{1}{3} \sin 5x + \cdots + \frac{1}{2n+1} \sin (2n+1)x. \]
the series represented in Figure I is
\[ \frac{dS_n}{dx} = \cos x + \cos 3x + \cos 5x + \cdots + \cos (2n+1)x. \]
if \( x = \frac{\pi}{2} \), \( \frac{dS_n}{dx} = 0 \) and the curve is parallel to the axis of \( x \) for \( x = \frac{\pi}{2} \), no matter what the value of \( n \) may be.
If \( x = 0 \) or \( x = \pi \), \( \frac{dS}{dx} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = \pi + \frac{1}{2} \)

and the curve becomes more nearly perpendicular to the axis of \( x \) at the origin, and for \( x = \pi \) as \( \pi \) is increased.

If \( x = \frac{\pi}{2} \), \( \frac{dS}{dx} = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots \)

that is \( \frac{dS}{dx} = \frac{1}{2} \) if \( n = 0 \) or \( n = 2k \)

\[ = -\frac{1}{2} \] if \( n = 1 + 2k \) \( n = 2k + 1 \)

\[ = 0 \] if \( n = 2k \) or \( n = 3k + 2 \)

Therefore when \( x = \frac{\pi}{2} \), \( \frac{dS}{dx} \) does not approach any limiting value as \( n \) is indefinitely increased.

If \( x \) has any other value between 0 and \( \pi \), \( \frac{dS}{dx} \) will change abruptly as \( n \) is changed and will not approach any limiting value as \( n \) is increased.

Since a Fourier's series converges only because its coefficients decrease as \( n \) advance in the series, the
Differentiation of a Fourier's series must make its convergence less rapid if it does not actually destroy it, and repetitions of the process will usually eventually make the derived series diverge. The derived series are Fourier's series but they lack the constant term.

If now we integrate a Fourier's series:

\[ \frac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots \]

we get:

\[ \frac{1}{2} b_0 x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots - a_1 \cos x - a_2 \cos 2x - a_3 \cos 3x + \ldots \]

a Trigonometric series which converges more rapidly than the given series. This series is in general not a Fourier's series on account of the term \( \frac{1}{2} b_0 x \).
If \( f(x) \) is single valued, finite and continuous, and has only a finite number of maxima and minima between \( x = -\pi \) and \( x = \pi \), then there is one Fourier's series and only one which is equal to it. Call this series \( S \). Let the derivative \( f'(x) \) of the given function also satisfy the same conditions imposed on \( f(x) \).

Then \( f'(x) \) may be expressed as a Fourier's series. The integral of this series will be equal to the integral of \( f'(x) \), that is to \( f(x) \) plus a constant. If this integral which is a Trigonometric series is a Fourier's series, it must be identical with \( S \).

It will be a Fourier's series only in case the Fourier's series for \( f'(x) \) lacks the constant term \( \frac{a_0}{2} \).
but \( b_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \), therefore \( b_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x) - f(-\pi)] \, dx \)

and will be zero if \( f(\pi) = f(-\pi) \)

If the \( f(x) \) is single-valued finite and continuous and has only a finite number of maxima and minima between \( x = -\pi \) and \( x = \pi \) (inclusive), and if \( f(\pi) = f(-\pi) \), \( f(x) \) can be developed into a Fourier series whose limit by limit derivative will be equal to the derivative of the function.

If a function of \( x \) is finite and single-valued for all values of \( x \), and has not an infinite number of discontinuities or of maxima and minima in the neighborhood of any value of \( x \), it will be equal to the Fourier's integral

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(1-x) \, dx
\]

and to that Fourier integral only, and the integral with respect to \( x \) of this Fourier integral will be equal to \( f(x) \)
In the March number of the Bulletin of the American Math. Society, there is a paper on the application of the fundamental laws of Algebra to the multiplication of infinite series, by Prof. Florian Cajori.

He shows that there is a class of series with real terms, so that the product of two divergent series, or of a semi-convergent, by a divergent series, or of the product of two semi-convergent series, forms an absolutely convergent series. Thus he considers the validity of the fundamental laws of Algebra in the multiplication of infinite series.

Then he points out another method...
for obtaining divergent series whose product is absolutely convergent. 
Lastly he generalizes a theorem by Abel on the multiplication of series

In the series $S_1$ and $S_2$, obtained respectively by removing the parentheses from the series

$S_1' = \sum_{n=0}^{\infty} (a_{4n} - a_{4n+1} + a_{4n+2} - a_{4n+3})$

$S_2' = \sum_{n=0}^{\infty} (b_{4n} - b_{4n+1} + b_{4n+2} - b_{4n+3})$

where the $a's$ and $b's$ are real and positive, let the following conditions be satisfied:

[1] The $v-th$ term in $S_1$ and in $S_2$ shall be $\in [v-n,v)$ where $\frac{1}{2} < n \leq 1$, but $\leq a_v$ and

$E_b$, are both divergent.

[2] $a_{4n} = a_{4n+2}$, $a_{4n+1} = a_{4n+3}$, $b_{4n} = b_{4n+1}$, $b_{4n+2} = b_{4n+3}$;

$a_{4n} - a_{4n+1}, b_{4n+2} - b_{4n}, a_{4(n+1)} - a_{4n+1}$,

shall differ from $e_{4n+2}$ by less than $k_{4n+2}$

where $\Sigma s_n \frac{1}{8(l^2r)^l}$, $\lambda > 1$, and $k_2 = \frac{1}{(e^2-2)(e^2-2)} l$
The $(4n)$th term of the product of $S_1$ and $S_2$ is

\[
\sum_{v=0}^{n-1} (-b_{4v} \cdot a_{4(n-v)-1} + b_{4v+2} \cdot a_{4(n-v-1)} + b_{4v+3} \cdot a_{4(n-v-1)+1})
\]

\[
= \sum_{v=0}^{n-1} \left\{ b_{4v} \left( a_{4(n-v)-1} + b_{4v+2} \right) - b_{4v+3} \cdot a_{4(n-v-1)+1} \right\}
\]

or numerically is less than

\[
\sum_{v=0}^{n-1} 4b_{4v+2} \cdot a_{4(n-v-1)} + \sum_{v=n'}^{n-1} \frac{1}{4} b_{4v+2} \cdot a_{4(n-v-1)}
\]

\[
< C \cdot \xi_4(n-n') + C' \cdot \xi_4(n') \quad (1)
\]

where

\[
C = \sum_{v=0}^{n-1} 4b_{4v+2} \quad C' = \sum_{v=n'}^{n-1} \frac{1}{4} b_{4v+2}
\]

\[n' = \frac{1}{2}(n+1) \text{ or } \frac{1}{2}(n-1) \text{ as } n \text{ is even, or odd.}
\]

Each of the terms in (1) is of the same order of magnitude as $\frac{1}{4n}(\xi_4 q_{4n})^k$.

The same reasoning might be applied.
to the \((n+1)^{th}\), \((n+2)^{th}\), \((n+3)^{th}\) term.

Thus, in the product of two conditionally convergent series \(S_1\) and \(S_2\), each term is numerically less than the corresponding term of a series known to be absolutely convergent. Therefore, the product of \(S_1\) and \(S_2\) is absolutely convergent.

As an example, we have the two series \(T_1\) and \(T_2\) obtained, respectively, by dropping the parameters from the following series:

\[
T_1' = \sum_{v=0}^{\infty} \left( \frac{1}{4v^4} - \frac{1}{4v^4 + 1} + \frac{1}{4v^4 + 4} - \frac{1}{4v^4 + 4v^3 + 4} \right)
\]

\[
T_2' = \sum_{v=0}^{\infty} \left( \frac{1}{4v^4 + 1} - \frac{1}{4v^4 + 4v^3 + 4} \right)
\]

where \(\frac{1}{2} < \theta < 1\), and \(\frac{1}{2} < \delta < 1\). \(T_1\) and \(T_2\) are each conditionally convergent, their product is absolutely convergent.

The behavior of infinite series with respect to the fundamental laws of algebra may be considered
under two heads: an inquiry into the validity of the law (1) when applied to the terms of an infinite series, (2) when applied to the infinite series themselves.

The first inquiry has led to the result that the associative law can always be applied to the terms of a convergent infinite series but that the commutative law can be applied in general only to the terms of an absolutely convergent series. This was the conclusion drawn on pages 44–46.

The second inquiry has been made for the addition and
Subtraction of infinite series but not for their product.

The product of \( U = \sum_{n=0}^{\infty} U_n \) and \( V = \sum_{m=0}^{\infty} V_n \) has been defined by Cauchy to be

\[
\sum_{n=0}^{\infty} (U_0 V_n + U_1 V_{n-1} + \cdots + U_n V_0)
\]

The law of association may be applied without limitation to the multiplication of series.

To show this let \( W = \sum_{n=0}^{\infty} W_n \), where \( W_n \), as well as \( U_n \) and \( V_n \) given above, are finite, constant, real or complex.

Then we have \((UV)W = U(VW)\) for the \((n+1)\)th term in \((UV)W\) is

\[
(U_0 V_n + U_1 V_{n-1} + U_2 V_{n-2} + \cdots + U_n V_0) W_0
\]

\[
+ (U_0 V_{n-1} + U_1 V_{n-2} + \cdots + U_{n-1} V_0) W_1
\]

\[
+ U_0 V_{n-2} + U_1 V_{n-3} + \cdots + U_{n-2} V_0) W_2
\]

\[
+ (U_0 V_0) W_n
\]

The \((n+1)\)th term of the
The product $U(U,W)$ is

$$U_0 \left( W_0 V_{0n} + W_1 V_{0n-1} + \ldots + W_{0m} V_0 \right)$$
$$+ U_1 \left( W_0 V_{0n-1} + W_1 V_{0n-2} + \ldots + W_{1m-1} V_0 \right)$$
$$+ U_2 \left( W_0 V_{0n-2} + \ldots + W_{2m-2} V_0 \right)$$
$$+ \ldots + U_n \left( W_0 V_0 \right)$$

These two expressions for the $(n+1)$th term, for any positive integral value of $n$, no matter how large are seen to be identical as soon as we adopt the following statement:

I. For $n > q$, when $q$ is any positive finite number we have always

$$U_p \left( W_0 V_{0n-p} + W_1 V_{0n-p+1} + \ldots + W_{p} V_0 \right) = U_0 W_0 V_{0n-p} + \ldots + W_{p} V_0$$

II. For $n > q$ we are allowed to commute the terms obtained by removing the parentheses, provided that no terms be dropped from the total aggregate and no new terms admitted to it.
It will be seen that this special case does not contradict the previous statement that the commutative law is not in general applicable to the terms of series not absolutely convergent.

The first expression for the \((n+1)\)th term assumes the form of the second if we perform the indicated multiplication then add the columns from left to right and factor.

Since the \((n+1)\)th term in \((UVW)\) is the same as the \((n+1)\)th term in \(U(VW)\), no matter what positive integral value be assigned to \(n\), it follows that the two
products are identical. Thus the associative law is always obeyed.

The commutative law holds good for two factor series as by Cauchy's definition the product of $E \times n \cdot E \times n$ is the same as the product of $E \times n \times E \times n$, so that the commutative law holds for two factor series.

Being permitted to assume the associative law it holds that the commutative law is valid for three or more factor series.

Thus $U \cdot V \cdot W = U \cdot (V \cdot W) = U \cdot \cdot \cdot V \cdot (U \cdot V) \cdot W = (U \cdot V) \cdot W = U \cdot V \cdot W = V \cdot (U \cdot W) = V \cdot (U \cdot W) \cdot W = W \cdot (U \cdot V) = W \cdot (U \cdot V) \cdot V = W \cdot (U \cdot V) \cdot V \cdot U$

The law of distribution can be shown to hold in a manner
similar to the proof for the law of Association, by showing that
\[ U(V + W) = UV + UW. \]

Find the \( n + 1 \)th term of \( UV + UW \)
and the \( n + 1 \)th term of \( U(V + W) \).

Then assuming the laws of commutation and distribution
to hold for the aggregate of
"involved in one of the expressions, change it into the other."

\[ S(x) = \sum_{v=0}^{\infty} \frac{(-1)^v (v^2)}{v!} x^v \]

"where \( x \) is of the same order of magnitude
as \( v - 2 \), \( 1 < v < 1 \). Here \( x = -1 \) is
a singular point on the circle of
covengence, and \( S(-1) \) is an
infinity of the same order as 
\((v-1)^{-\frac{1}{2}}\)

If the series II is raised to the positive integral power \(p\), then the sum of the resulting series for \(x = -1\) is of the same order of infinity as \((1-1)^{-\frac{1}{2}}\) \(p\) \((1-1)^{-p}\). If the power \(p = \frac{1}{1-n}\) then the order of infinity is not lower than the first.

But for \(x = +1\), series II becomes a special case of 
\[ S_1 = \sum_{v=0}^{\infty} (a_v - a_{v+1} + a_{v+2} - a_{v+3}) \]

Hence, the \(p\)th power of \(S_1\) is divergent when \(a_v\) is of the same order of magnitude as \(v^{-\frac{1}{2}}\).

In the same way it may be shown that the \(p\)th power of 
\[ S_2 = \sum_{v=0}^{\infty} (b_4 + b_{v+4} + b_{v+8} - b_{v+12}) \]

is divergent when \(b_v\) is a magnitude of the
same order as \( r^{-2} \). \( \frac{1}{2} < r < 1 \) and
\[ \beta = \frac{1}{1 - r}. \]
But \( S_1 S_2 \) was shown to be absolutely convergent.
We have \( S_1 S_2 S_1 S_2 \) — (top pairs of factors)
\[ = (S_1 S_2) (S_1 S_2) \] — (in parentheses).
Hence the product of these two series is absolutely convergent.
But by the associative and commutative law, this product is equal to \( S_1 S_2 S_1 S_2 \). Thus \( S_1 S_2 \) and
\( S_1 S_2 \) are two divergent series whose product is absolutely convergent.
No matter how much \( \beta \) is in excess of \( \frac{1}{1 - r} \), that is no matter how high a power of \( S_1 \) and \( S_2 \) is taken — we have for a given value of \( \beta \), always an
absolutely convergent product resulting from the multiplication of $S_1$ by $S_2$.

A special example is $\prod_{i=2}^{\infty} T_i^3$ is absolutely convergent, but $T_i^3$ and $T_i^3$ are each divergent when $n < \frac{2}{3}$ and $n < \frac{2}{3}$.

\[
T_1 = \frac{\sum_{i=1}^{\infty} 1}{\sum_{i=2}^{\infty} (\frac{1}{4i^3} - \frac{1}{4i^3} + \frac{1}{4i^3} - \frac{1}{4i^3} + \frac{1}{4i^3})}
\]

In the divergent series $\sum_{i=2}^{\infty}$ the terms increase without limit in numerical value as increase without limit. The same is true of $S_2$.

Herein lies the differences between the pair of divergent series yielding an
absolutely convergent product and
the pair given by Pringsheim. In
the latter, the terms of the divergent
series remain finite as increases
in definitely.

From the relation \( S \cdot S_2 S \cdot S_2 = S_2 \cdot S_2 \)
we see that there are cases in the
multiplication of series in which
divergent series may be used with
safety — the sum of the final product
series being convergent — and equal to
the product of the sums of the initially
given convergent factor series, even
when the product of some of the
given factor series is divergent.

If two or more convergent series,
when multiplied together, yield
a convergent product series, then
The sum of this product series is equal to the product of the sums of the factor series.

This theorem was found by Abel for the case of two factor series and his method of proof is applicable to the general case, and the extension is obvious.