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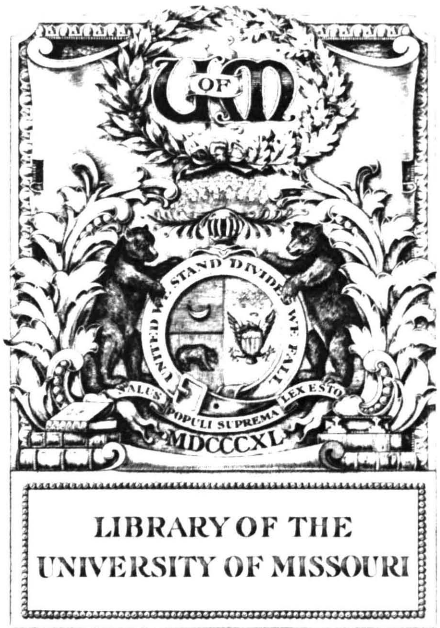

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THESIS

"CONVERGENCE OF INFINITE SERIES."

BY

CARL MANFORD MOORE.

Submitted in partial fulfilment of the requirements for the degree of Master of Arts in the University of the State of Missouri.

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CONVERGENCE OF INFINITE SERIES.

We shall define an infinite series as a succession of terms formed after some definite law. Most generally the terms are actual numbers or are at least regarded as constant, and we are concerned with their sum. There may be either a finite or an infinite number of terms. Thus we have two theories finite and infinite series. But it is only in the latter that the notion of convergency and divergency present themselves.

So long as the number of terms is finite or determinate the sum is found by simply adding together the terms. But when the number of terms is indeterminate or infinite this can no longer be done

Ex. (1) — $1+3+5+7+9+11+ \cdots = n^2$

Ex. (2) — $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+ \cdots = 2-\frac{1}{2^{n-1}}$

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Thus we arrive at the fundamental question in the study of an infinite series, namely, has it a sum or in other words is there a definite value S to which sum constantly approaches as n increases, such that $S - S_n \underset{n=\infty}{=} 0$. There also arises a further question, as to whether this infinite succession of terms can be subjected to laws of algebra laid down for finite number of summands.

Consider the series

$$S_n = U_1 + U_2 + U_3 + U_4 + U_5 + \text{--} \quad U_n$$

where $U_1 \gtreqless U_2 \gtreqless U_3 \gtreqless U_4$ ———, and U_n is a function of n.

$$S_1 = U_1, \quad S_2 = U_1 + U_2, \quad S_3 = U_1 + U_2 + U_3$$

$$S_4 = U_1 + U_2 + U_3 + U_4, \quad S_5 = U_1 + U_2 + U_3 + U_4 + U_5$$

$$S_{10} = U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + U_7 + U_8 + U_9 + U_{10}$$

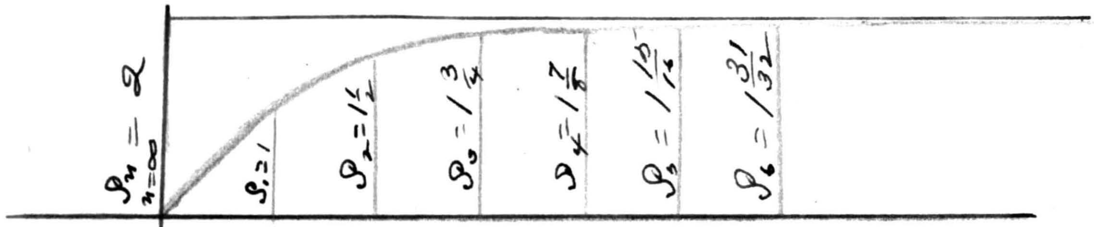
Now when n becomes infinite, [one of] three things must happen.

There will either exist some definite limit S such that $\lim\limits_{n=\infty} S_n = S$.

Ex. $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\cdots \frac{1}{2^{n-1}}$

$S_1 = 1, \quad S_2 = 1 + \frac{1}{2} = 1\frac{1}{2} \quad S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$

$S_4 = 1\frac{7}{8} \quad S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}, \; S_6 = 1\frac{31}{32}$



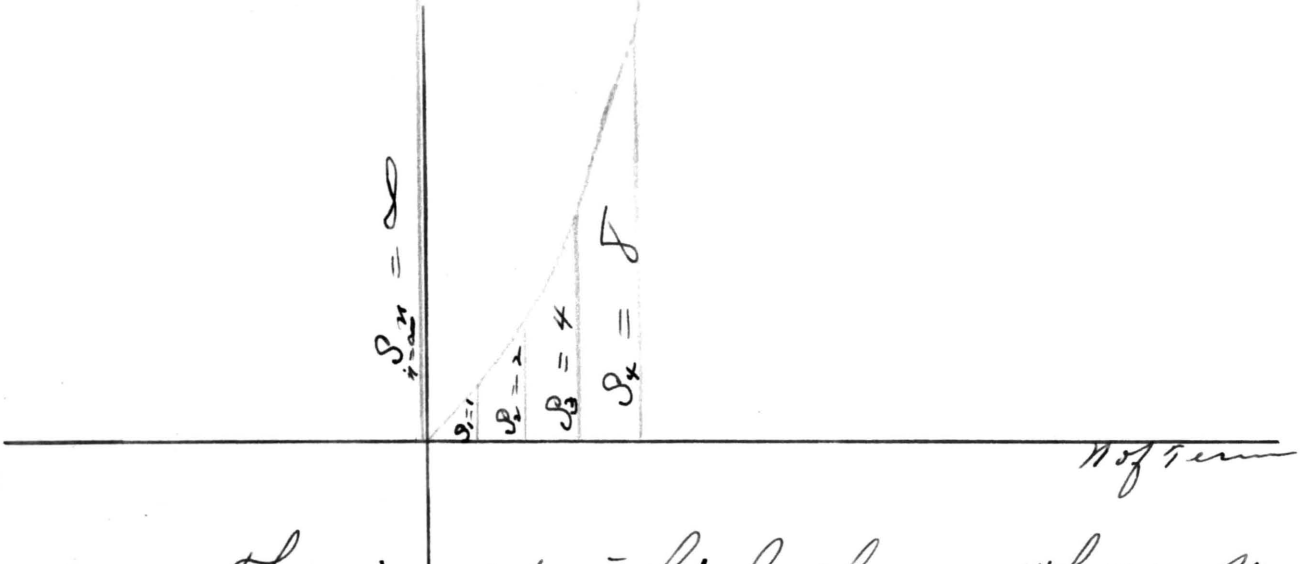
There is a constant approach to the value two, but as each term only reaches over half the distance left at end of preceding no ordinate can exceed two. Hence Sum for $n = \infty$ is 2. In this case series is Convergent

Again S_n may increase as n increases so that by taking a sufficient number of terms we can make S_n greater than any quantity however large; that is

$$\lim_{n=\infty} S_n = S = \infty$$

Ex. – $1 + 2 + 4 + 8 + 16 + \cdots\cdots 2^{n-1}$

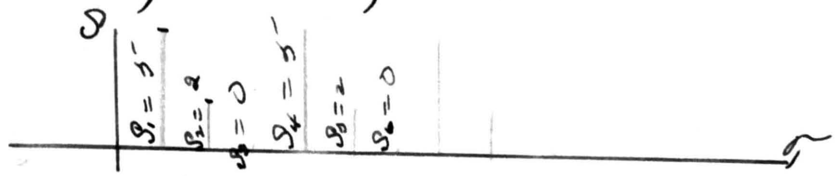
This will become infinite with n, In this case the series is said to be divergent



There is yet a third class, whose sum neither approaches a definite finite quantity nor becomes infinite. This we may term an oscillating series. It may, however, be classed with divergent series.

Ex. $5 - 3 - 2 + 5 - 3 - 2 +$ ————

Lim S_n is 5, 2, or 0 according as we take $3m+1$, $3m+2$, or $3m$



Conditions for the convergency of an infinite series;

It is evident that for a series [to have sum], its terms must decrease indefinitely, and ultimately become equal to zero; but the converse is not true that all such series have sums. In the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+ \cdots \frac{1}{n}$ the terms decrease indefinitely and limit to last term for $n=\infty$ is zero; but the series is divergent.

$$S_n = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+ \cdots\cdots$$
$$> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+ \cdots\cdots \text{ ad inf}$$

Hence $\lim_{n=\infty} S_n = \infty$

We may now state the essential condition for convergency of an infinite series. It must be shown that $n+1$th term ~~term~~ constantly decreases as n increases and that $\lim_{n=\infty} {}_mR_n$ can be made less than any assignable quantity however small.

Ex(1) Take series above. $_mR_n = \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+3}+\cdots$

$$> \frac{n}{n+m} = \frac{1}{1+\frac{m}{n}}.$$ The first condition is satisfied but not second.

Ex(2) - $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+ \cdots\cdots \frac{1}{2^{n-1}}$

$$_mR_n = \frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\frac{1}{2^{n+4}}+ \cdots \text{ ad inf.}$$

$$> \frac{m}{2^{n+m}} = 0.$$ Both conditions are satisfied, and series is convergent.

Tests for Convergence.

Series whose terms are all positive.

I - The most simple test and that from which most others are derived consists in simply comparing series proposed with series known. If series ΣV_n is convergent and series ΣU_n has corresponding terms less than corresponding terms of ΣV_n, then ΣU_n is convergent. If V is divergent and corresponding terms in ΣU are greater than corresponding terms of ΣV, then V is divergent.

A very convenient way to study this is to consider the ratio $\frac{U_n}{V_n}$ as n varies. If after a certain value of n this ratio remains between two positive numbers A and B; that is it neither becomes infinite nor equal to zero, the two series will be convergent or divergent together. The terms of ΣU_n will be comprised between terms of two series; for $A < \frac{U_n}{V_n} < B$

$$AV_1 + AV_2 + AV_3 + AV_4 + AV_5 + \cdots AV_n$$
$$BV_1 + BV_2 + BV_3 + BV_4 + BV_5 + \cdots BV_n$$

convergent or divergent at same time with ΣV. Now if ration remains inferior to B and V is convergent U is convergent; if ratio remains superior to A different from zero and V is divergent U will be divergent.

Ex. $1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+ \cdots \frac{1}{n^m}+\frac{1}{(n+1)^m}$

$\frac{1}{2^m}+\frac{1}{3^m} > \frac{2}{2^m}=\frac{1}{2^{m-1}}, \quad \frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\frac{1}{7^m} < \frac{4}{4^m}=\frac{1}{4^{m-1}}$

$\Sigma \frac{1}{n^m} < 1+\frac{1}{2^{m-1}}+\frac{1}{4^{m-1}}+\frac{1}{8^{m-1}}+ \cdots$

Put ~~x~~ for $\frac{1}{2^{m-1}}$ and consider $m-1>0$, then $x<1$.

$1+\frac{1}{2^{m-1}}+\frac{1}{4^{m-1}}+\frac{1}{8^{m-1}}+\cdots = 1+x+x^2+x^3+x^4+\cdots x^n$

$\frac{x}{1-x} = x+x^2+x^3+x^4+x^5+ \cdots$ ad inf.

no matter how many terms we take the sum will always be less than $\frac{x}{1-x}$. Therefore the series is convergent for $m>1$. It has already been shown to be divergent when $m=1$. From the above criterion it follows immediately that series will be divergent when $m<1$, since terms will be larger that corresponding terms in $\Sigma\frac{1}{n}$.

Ex. $e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\frac{x^5}{\lfloor 5}+\frac{x^6}{\lfloor 6} \cdots \frac{x^n}{\lfloor n}$

$\frac{u_n}{v_n} = \frac{\frac{1}{n^2}}{\frac{x^n}{\lfloor n}} = \frac{1\cdot 2\cdot 3\cdots n-1}{n x^n}$. This remains finite for all values of x. Hence series is convergent

$-\infty \qquad 0 \qquad +\infty$

(a) Suppose we have a series $S_n = u_1 + u_2 + u_3$ --- given which we know to be convergent. Then the series $S' = \frac{u_1}{S_1} + \frac{u_2}{S_2} + \frac{u_3}{S_3} + \frac{u_4}{S_4} +$ ---
where $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, is convergent; for each term is less than the corresponding term in S.

(b) Suppose now the series S is divergent. S' will be divergent. The sum of m terms which follow the nth is $\frac{u_{n+1}}{S_{n+1}} + \frac{u_{n+2}}{S_{n+2}} + \frac{u_{n+3}}{S_{n+3}} + \cdots \frac{u_{n+m}}{S_{n+m}}$

Since $S_{n+1} < S_{n+2} < S_{n+3} < S_{n+4} \cdots < S_{n+m}$ the sum is greater than $\frac{u_{n+1} + u_{n+2} + u_{n+3} \cdots u_{n+m}}{S_n + m}$

$= \frac{1}{1 + \frac{u_1 + u_2 + u_3 + \cdots u_n}{u_{n+1} + u_{n+2} + \cdots u_{n+m}}}$. But as the series is divergent $\frac{\Sigma u_n}{\Sigma u_{n+m}}$ will be a proper fraction. Hence series is not convergent since $m R_n \neq 0$.

(c) If $u_1 + u_2 + u_3 + u_4 + u_5 +$ --- is divergent the series $\frac{u_1}{S_0^\alpha} + \frac{u_2}{S_1^\alpha} + \frac{u_3}{S_3^\alpha} +$ ---- $\frac{u_n}{S_{n-1}^\alpha}$. for $\alpha \ngtr 1$

We have $\operatorname{Log} \frac{S_n}{S_{n-1}} = \operatorname{Log} \frac{u_1 + u_2 + u_3 + \cdots u_n}{S_{n-1}}$
$= \operatorname{Log}\left(1 + \frac{u_n}{S_{n-1}}\right) < \frac{u_n}{S_{n-1}}$

$$S_n' = \frac{u_1}{S_0} + \frac{u_2}{S_1} + \frac{u_3}{S_2} + \frac{u_4}{S_0} + \cdots \frac{u_n}{S_{n-1}} > \operatorname{Log}\frac{S_1}{S_0} + \operatorname{Log}\frac{S_2}{S_1}$$

$$+ \cdots \operatorname{Log}\frac{S_{n-1}}{S_{n-2}} + \operatorname{Log}\frac{S_n}{S_{n-1}}$$

$$= \operatorname{Log} S_1 - \operatorname{Log} S_0 + \operatorname{Log} S_2 - \operatorname{Log} S_1 + \operatorname{Log} S_3 - \operatorname{Log} S_2 + \cdots + \operatorname{Log} S_{n-1}$$

$$- \operatorname{Log} S_{n-2} + \operatorname{Log} S_n - \operatorname{Log} S_{n-1}$$

$= \operatorname{Log} S_n - \operatorname{Log} S_0 < S_n'$. As original series was divergent the last will be divergent, since S_n becomes infinite with n. Therefore

$$S_n = \frac{u_1}{S_0^\alpha} + \frac{u_2}{S_1^\alpha} + \frac{u_3}{S_2^\alpha} + \frac{u_4}{S_3^\alpha} + \cdots$$ is divergent

for $\alpha + 1$

Ex. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots$

$$\frac{1}{2} + \frac{1}{3(1+\frac{1}{2})} + \frac{1}{4(1+\frac{1}{2}+\frac{1}{3}} + \frac{1}{5(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})} + \cdots$$

Since the first series is divergent the last is divergent from theorem just proved.

(d) – If $u_1 + u_2 + u_3 + \cdots$ is divergent the series $\frac{u_1}{S_1^{1+\alpha}} + \frac{u_2}{S_2^{1+\alpha}} + \frac{u_3}{S_3^{1+\alpha}} + \frac{u_4}{S_4^{1+\alpha}}$ is convergent, if α is positive

$$S_{n-1}^{-\alpha} - S_n^{-\alpha} = (S_n - u_n)^{-\alpha} - S_n^{-\alpha} > S_n^{-\alpha} + \alpha S_n^{-\alpha-1} u_n + \cdots - S_n^{-\alpha}$$

$= \alpha u_n S_n^{-\alpha-1} = \frac{\alpha u_n}{S_n^{\alpha+1}}$. Hence the series is convergent

Ex. $1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots$

$(S_1)^{1+\alpha} = (1)^{\alpha+1}, \; S_2^{1+\alpha} = 2^{\alpha+1}, \; S_3^{1+\alpha} = 3^{1+\alpha}, \; S_4^{1+\alpha} = 4^{\alpha+1}$

$1 + \frac{1}{2^{\alpha+1}} + \frac{1}{3^{\alpha+1}} + \frac{1}{4^{\alpha+1}} + \cdots \frac{1}{n^{\alpha+1}}$

If $\alpha = 0$ this series becomes

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \frac{1}{n}$ which we know to be divergent. (a), (b), (c) & (d) are due to Abel.

II Second Test for Convergence.

If $\lim_{n=\infty} \frac{u_{n+1}}{u_n} < 1$ Convergent.

If $\lim_{n=\infty} \frac{u_{n+1}}{u_n} > 1$ Divergent.

If $\lim_{n=\infty} \frac{u_{n+1}}{u_n} = 1$ No Test.

(a) Let us represent ratio by (r) and take K some quantity so near to one that $\frac{u_{n+1}}{u_n}$ shall always be less than K. Then we have

$\frac{u_{m+1}}{u_m} < K, \; \frac{u_{m+2}}{u_{m+1}} < K, \; \frac{u_{m+3}}{u_{m+2}} < K \; \cdots$

$u_{m+1} + u_{m+2} + u_{m+3} + \cdots < u_m(K + K^2 + K^3 + \cdots)$

$< \frac{u_m K}{1-K}$ Hence the

Sum of series beginning with the mth

is finite: and we have simply omitted a finite number of terms. Hence the series is Convergent.

(β)- $\frac{u_{m+1}}{u_m} > K \quad K > 1$. By same reasoning we have $u_{m+1} + u_{m+2} + u_{m+3} + \cdots$

$$> u_m(K + K^2 + K^3 + K^4 + \cdots\cdots)$$

$$> u_m \frac{K}{1-K}$$, But we have already

Shown that geometric series for $K > 1$ is divergent; for its terms increase as n increases.

γ)- It is evident that in the case where $K = 1$ there is no knowledge derived. In some cases the Series may be divergent in some convergent.

It is appropriate here to note that in this test we are concerned with limit to r as n becomes infinite. Consider the divergent Series. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

$\frac{\frac{1}{10}}{\frac{1}{9}} = \frac{9}{10}$, $\frac{\frac{1}{21}}{\frac{1}{20}} = \frac{20}{21}$, ∴ $\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1}$

$\lim_{n=\infty} \frac{1}{1+\frac{1}{n}} = 1$.

In precisely the same manner we may show that when $\sqrt[n]{u_n} < 1$ the series is convergent

$\sqrt[n]{u_n} >$ divergent.

(4) $\sqrt[n]{u_n} < K$, $\sqrt[n+1]{u_{n+1}} < K$, $\sqrt[n+2]{u_{n+2}} < K$, ------

$u_n + u_{n+1} + u_{n+2} + u_{n+3} + ---- < K^n + K^{n+1} + K^{n+2}$

$< \frac{K^n}{1-K}$. This will

approach zero as n increases. Therefore series is convergent.

In a like manner we can establish its divergence when $\sqrt[n]{u_n} > 1$

Applications.

Ex(1) - $1 + e^{-1} + e^{-(1+\frac{1}{2})} + e^{-(1+\frac{1}{2}+\frac{1}{3})} + e^{-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})}$ ------

$\frac{u_{n+1}}{u_n} = \frac{e^{-\Sigma\frac{1}{n+1}}}{e^{-\Sigma\frac{1}{n}}} = e^{-\frac{1}{n+1}} = \frac{1}{e^{\frac{1}{n+1}}}$. This for $n = \infty$ is greater than one. Hence series is divergent.

Ex(2) $\frac{x}{a_1} + \frac{x^2}{a_2^2} + \frac{x^3}{a_3^3} + \frac{x^4}{a_4^4} + ---- \frac{x^n}{a_n^n}$

Where a_1, a_2, a_3, a_4 are numbers which increase indefinitely with n

$\sqrt[n]{\frac{x^n}{a_n^n}} = \frac{x}{a_n}$ $\quad \lim_{n=\infty} \frac{x}{a_n} = 0$

Ex(3) $1 + e^{-1} + \frac{1}{2}e^{-(1+\frac{1}{2})} + \frac{1}{3}e^{-(1+\frac{1}{2}+\frac{1}{3})} + \frac{1}{4}e^{-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})}$ + etc

$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+1}e^{-\Sigma\frac{1}{n+1}}}{\frac{1}{n}e^{-\Sigma\frac{1}{n}}} = \frac{n}{n+1}e^{-\frac{1}{n+1}} = \frac{1}{1+\frac{1}{n}} \times \frac{1}{e^{+\frac{1}{n+1}}} = 1$ for $n = \infty$

Hence our test fails, but the series is convergent

II As a further test ratio the following is sometimes useful.

If $n\left(1-\frac{u_{n+1}}{u_n}\right)$ approaches a limit let this limit be denoted by β. Then the series $u_1+u_2+u_3+u_4+u_5+u_6+$ - - - -

$\beta > 1$ (or $=\infty$) Convergent.

$\beta < 1$ (or $=-\infty$) Divergent.

$\beta = 1$ No Test.

Let us take the series $\Sigma\frac{1}{n^\alpha}$, which we know to be convergent for $\alpha>1$, and divergent, when $\alpha \leqq 1$

$$n\left(1-\frac{u_{n+1}}{u_n}\right)=n\left(1-\frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}}\right)=n\left(\frac{(n+1)^\alpha-n^\alpha}{(n+1)^\alpha}\right)$$

Expanding $=n\left(\frac{n^\alpha+\alpha n^{\alpha-1}+\frac{\alpha(\alpha-1)}{1.2}n^{\alpha-2}+\cdots-n^\alpha}{n^\alpha+\alpha n^{\alpha-1}+\frac{\alpha(\alpha-1)}{1.2}n^{\alpha-2}}\right)=\alpha$

Suppose $\beta>1$, and α between the limit $n\left(1-\frac{u_{n+1}}{u_n}\right)$ and one. Then we must have after some finite limit, that is after some finite value of n

$n\left(1-\frac{u_{n+1}}{u_n}\right)>n\left(1-\frac{v_{n+1}}{v_n}\right)$. Where $v_n=\Sigma\frac{1}{n^\alpha}$

$-\frac{u_{n+1}}{u_n}>-\frac{v_{n+1}}{v_n}$ $\quad\frac{v_{n+1}}{v_n}>\frac{u_{n+1}}{u_n}$

Hence since ΣV_n is convergent ΣU_n is convergent

If β be less than one and α be taken between β and one we can prove in like manner that ΣU_n will be ~~doi~~divergent when ΣV_n is divergent; But the latter series has already been proved divergent for $\alpha \equiv 1$.

The utility of this test is shown by the following series,

$$\frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}x + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}x^2 + \text{------}$$

$$\lim_{n-\infty} \frac{U_{n+1}}{U_n} = \frac{\dfrac{\alpha(\alpha+1)(\alpha+2)(\alpha+3) \text{ --- } \alpha+n}{\beta(\beta+1)(\beta+2) \text{ ---- } (\beta+n)}}{\dfrac{\alpha(\alpha+1) \text{ ------ } (\alpha+n-1)}{\beta(\beta+1)(\beta+2) \text{ --- } (\beta+n-1)}} = \frac{\alpha+n}{\beta+n}x$$

$\lim_{n=\infty} \frac{\alpha+n}{\beta+n}x = \frac{\frac{\alpha}{n}+1}{\frac{\beta}{n}+1}x = x$. Hence the series is convergent when $x<1$, and divergent when $x>1$. When $x=1$ no test. Applying the above test we have

$$n\left(\cancel{1} - \frac{U_{n+1}}{U_n}\right) = n\left(1 - \frac{\alpha+n}{\beta+n}x\right) = \frac{n(\beta-\alpha)}{\beta+n}x = \frac{\beta-\alpha}{\frac{\beta}{n}+1}x$$
$$= (\beta-\alpha)x$$ Therefore convergent for $\beta-\alpha>1$

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Ex (2) $\frac{1}{2^2-a}+\frac{1}{3^2-a}+\frac{1}{4^2-a} --- \frac{1}{n^2-a}$

$$n\left(1-\frac{u_{n+1}}{u_n}\right)=n\left(1-\frac{n^2-a}{(n+1)^2-a}\right)=\frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1-a}{n^2}}.$$

$\lim_{n=\infty} n\left(1-\frac{u_{n+1}}{u_n}\right)=2$. Hence the series is Convergent.

Supplemental Criteria.

Both of the above tests failing when limit for $n=\infty$ of $\frac{u_{n+1}}{u_n}=1$, we preceed to establish Supplemental criteria which may be used in such cases.

We have already shown that the convergence of a series may be established by comparing its terms with the corresponding terms in known convergent series. If Σu_n be convergent and Σv_n has its corresponding terms each less than the corresponding terms in Σu_n it will be convergent. If Σu_n be divergent and corresponding terms in Σv_n be each greater than corresponding terms in Σu_n it will be divergent.

Let us take the series Σu_x, Comparing it with $\Sigma \frac{1}{a^x}$ where ~~x~~ >1, and a being indefinitely increased.

$\frac{1}{u_x} > x^\alpha \qquad \text{Log}\,\frac{1}{u_x} > \alpha\,\text{Log}\,x$

$\alpha < \frac{\text{Log}\,\frac{1}{u_x}}{\text{Log}\,x}$ Since ~~α~~ > 1 was the condition that $\sum \frac{1}{x^\alpha}$ be convergent, the condition for Convergence of u_x is $\frac{\text{Log}\,\frac{1}{u_x}}{\text{Log}\,x} > 1$

(b) If $u_x > \frac{1}{x^\alpha}$, $\frac{1}{u^\alpha} < x^\alpha$

$\text{Log}\,\frac{1}{u^\alpha} < \alpha\,\text{Log}\,x \quad , \quad \alpha > \frac{\text{Log}\,\frac{1}{u_x}}{\text{Log}\,x}$

But $\alpha < 1$ was the condition that $\sum \frac{1}{x^\alpha}$ be convergent. Hence we have

$\frac{\text{Log}\,\frac{1}{u_x}}{\text{Log}\,x} > 1$ Convergent.

$\frac{\text{Log}\,\frac{1}{u_x}}{\text{Log}\,x} < 1$ Divergent.

The limit being unity and the above test failing we pass to next series $\sum \frac{1}{x(\text{Log}\,x)^\alpha}$

$u_x < \frac{1}{x(\text{Log}\,x)^\alpha}, \quad (\text{Log}\,x)^\alpha < \frac{1}{x u_x},$

$\alpha\,\text{Log}\,\text{Log}\,x < \text{Log}\,\frac{1}{x u_x}, \quad \alpha < \frac{\text{Log}\,\frac{1}{x u_x}}{\text{L}\,\text{Log}\,x}$

Therefore $1 < \frac{\text{Log}\,\frac{1}{x u_x}}{\text{Log}\,\text{Log}\,x}$ Convergent.

$1 > \frac{\text{Log}\,\frac{1}{x u_x}}{\text{Log}\,\text{Log}\,x}$ Divergent.

This failing we proceed to the next series $\sum \frac{1}{x \log x (\log \log x)^m}$ and so on ad inf.

Proceeding in this way we establish a system of functions of the form

$$\frac{\operatorname{Log} \frac{1}{x\, u_x}}{\operatorname{Log}\operatorname{Log} x}, \quad \frac{\operatorname{Log} \frac{1}{x \operatorname{Log} x\, u_x}}{\operatorname{Log}\operatorname{Log}\operatorname{Log} x}, \quad \frac{\operatorname{Log} \frac{1}{x \operatorname{Log} x \operatorname{Log}\operatorname{Log} x\, u_x}}{\operatorname{Log}\operatorname{Log}\operatorname{Log}\operatorname{Log} x}, \text{ etc.}$$

> 1 Convergent ,, < 1 Divergent

De Morgan's Criteria.

This criteria was originally presented by De Morgan under a form easily derived from the preceding. Writing $u_x = \frac{1}{f(x)}$ and applying to this form rule for indeterminate functions. If for $x = \infty$ the limit be less than one the series is Divergent. If greater than one the series is convergent.

$$\frac{\operatorname{Log} f x}{\operatorname{Log} x} = \frac{\frac{f'x}{fx}}{\frac{1}{x}} = \frac{x f'x}{fx}$$

$$\frac{\operatorname{Log} \frac{fx}{x}}{\operatorname{Log}\operatorname{Log} x} = \frac{x f'(x) - f(x)}{x f(x)} \div \frac{\frac{1}{x}}{x \operatorname{Log} x} = \operatorname{Log} x \left[\frac{x f'x}{fx} - 1\right]$$

Thus we can replace the system of function by the following. Put $P_0 = \frac{x f'x}{fx}$, $P_1 = (P_0 - 1)\operatorname{Log} x$

$P_2 = (P_1 - 1) \operatorname{Log} \operatorname{Log} x$ ------ ad inf.

These may still be presented in another form which will sometimes possess advantage over other forms. We shall here only show how these forms may be obtained, without attempting to justify our work.

Substitute in the above $\frac{P_0}{x} = \frac{u_x}{u_{x+1}} - 1$, and we have the system of functions

$$x\left(\frac{u_x}{u_{x+1}} - 1\right), \quad x\left(\frac{u_x}{u_{x+1}} - 1\right) - 1\Big) \operatorname{Log} x, \quad x\left[\left(\frac{u_x}{u_{x+1}} - 1\right) - 1\Big) \operatorname{Log} x - 1\right] \operatorname{Log}\operatorname{Log} x$$

III Cauchy Condensation Test

Let $V = u_1 + u_2 + u_3 + u_4 +$ -- be a series of positive terms. Now we can without altering value of series associate them together in groups. $v_1 = u_1$, $v_2 = u_2 + u_3$ $v_3 = u_4 + u_5 + u_6 + u_7$

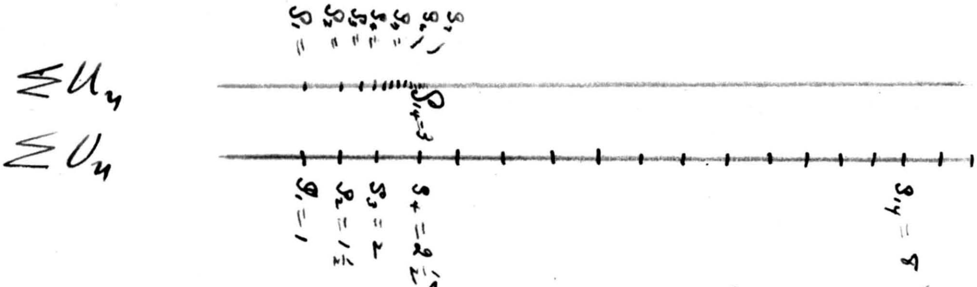
$$V = v_1 + v_2 + v_3 + v_4 + \text{ ---- } v_n$$

$\lim_{n=\infty} \sum u_n = \lim_{m=\infty} \sum v_n$. It is clear that convergency or divergency of $\sum v_n$ will be more apparent than than that of $\sum u_n$.

Consider the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \text{ ---}$$

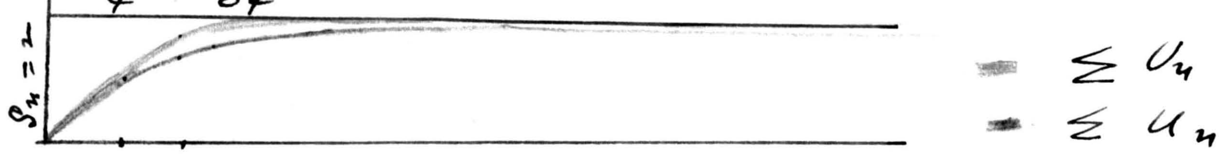
$$\geqq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \text{ --------}$$



Consider the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+ \cdots$
Arranging the terms in groups of the form
$1, 2, 2^2, 2^3, \cdots 2^n$

$1+(\frac{1}{2}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64})+(\frac{1}{128}+ \cdots$

$1+\frac{3}{4}+\frac{15}{64}+ \cdots$



Cauchy's Condensation test is as follows:
If series ΣU_n have all its terms positive, and constantly decrease as n increases, then series will be convergent or divergent according as $\Sigma \alpha^n U_{\alpha^n}$ is convergent or divergent, α being any positive integer $\not< 2$

We have the following relations since

$u_1 > u_2 > u_3 > u_4 > u_5 > u_6 \cdots$

$u_1+u_2+u_3+ \cdots u_\alpha < (\alpha-1)u_1+u_1$

$u_{\alpha+1}+u_{\alpha+2}+ \cdots u_{\alpha^2} < (\alpha^2-\alpha)u_\alpha < (\alpha-1)\alpha u_\alpha$

$u_{\alpha^2+1}+u_{\alpha^2+2}+ \cdots u_{\alpha^3} < (\alpha^3-\alpha^2)u_{\alpha^2} < (\alpha-1)\alpha^2 u_{\alpha^2}$

$$u_{\alpha^n+1} + u_{\alpha^n+2} + \text{------} u_{\alpha^{n+1}} < (\alpha^{n+1} - \alpha^n) u_{\alpha^n} < (\alpha-1)\alpha^n u_{\alpha^n}$$

$$\Sigma u_{\alpha^n+1} < (\alpha-1)[u_1 + (u_1 + \alpha u_\alpha + \alpha^2 u_{\alpha^2} + \alpha^3 u_{\alpha^3} + \text{----})]$$

$$(a) \quad \Sigma u_{\alpha^n+1} < (\alpha-1)\Sigma U_n$$

We may also write

$$\alpha(u_1 + u_2 + u_3 + u_4 + \text{----} u_\alpha) > \alpha u_\alpha$$

$$\alpha(u_{\alpha+1} + u_{\alpha+2} + u_{\alpha+3} + \text{----} u_{\alpha^2}) > \alpha(\alpha^2 - \alpha) u_{\alpha^2} > \alpha^2 u_{\alpha^2}$$

$$\alpha(u_{\alpha^2} + u_{\alpha^2+1} + \text{----} u_{\alpha^3}) > \alpha(\alpha^3 - \alpha^2) u_{\alpha^3} > \alpha^3 u_{\alpha^3}$$

$$\alpha(u_{\alpha^{n-1}+1} + u_{\alpha^{n-1}+2} + \text{-----} u_{\alpha^n} > \alpha^n u_{\alpha^n}$$

adding and we have

$$\alpha(u_1 + u_2 + u_3 + u_4 + \text{-----} u_{\alpha^n} > \alpha(u_\alpha + \alpha^2 u_{\alpha^2} + \alpha^3 u_{\alpha^3}$$

$$(B) \qquad \alpha \Sigma u_{\alpha^n} > U_n - u_1$$

From these two inequalities we see that if ΣU_n is finite if ΣV_n is finite, and infinite when ΣU_n is infinite.

Cauchy's Integration Test

If function x be finite, positive, and decrease in value as x increases continuously from α to ∞ then the series

$$f(\alpha) + f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \cdots$$

will be convergent if $\int_\alpha^\infty f(x)\,dx$ be finite, and divergent when $\int_\alpha^\infty f(x)\,dx$ is infinite

As function x decreases from $x=\alpha$ to $x=\alpha+1$ ---- ad inf., we have

$$\int_\alpha^{\alpha+1} f(x)\,dx < f(\alpha), \quad \int_{\alpha+1}^{\alpha+2} f(x)\,dx < f(\alpha+1)$$

$$\cdots\cdots$$

adding these inequalities we have,

$$\int f(x)\,dx < f(\alpha) + f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \cdots$$

For the same reason we have

$$\int_\alpha^{\alpha+1} f(x)\,dx > f(\alpha+1), \quad \int_{\alpha+1}^{\alpha+2} f(x)\,dx > f(\alpha+2)$$

$$\cdots\cdots$$

adding we have again

$$\int_\alpha^\infty f(x)\,dx > f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \cdots$$

Now from this we see that the value of the $\int_{\alpha}^{\infty} f(x)\,dx$ intermediate in value between

$$f(\alpha) + f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \text{-----}$$

$$f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \text{-----}$$

But these differ by $f(\alpha)$, a finite quantity, and thus $\int_{\alpha}^{\infty} f(x)\,dx$ will differ from original series by finite quantity. Hence the series and $\int_{\alpha}^{\infty} f(x)\,d(x)$ will be finite or infinite together.

By substituting $\alpha - 1$ for α in $\int_{\alpha}^{\infty} f x\,dx > f(\alpha+1) + f(\alpha+2) \text{--}$ We have $\int_{\alpha-1}^{\infty} f(x)\,dx > f(\alpha) + f(\alpha+1) + f(\alpha+2) + \text{------}$ Hence the series will be intermediate in value between $\int_{\alpha}^{\infty} f(x)\,dx$ and $\int_{\alpha-1}^{\infty} f(x)\,dx$

The reader is probably aware that we used a set of series in the preceding, whose convergence we had not yet established. We shall now apply Cauchy's integral test to these series.

Ex (1) — $\dfrac{1}{\alpha(\text{Log}\,\alpha)^m} + \dfrac{1}{(\alpha+1)[\text{Log}(\alpha+1)]^m} + \dfrac{1}{(\alpha+2)[\text{Log}(\alpha+2)]^m} + \text{------}$

$$\int_{\alpha}^{\infty} \frac{dx}{x(\text{Log}\,x)^{\alpha}} = \frac{(\text{Log}\,x)^{1-\alpha} - \text{Log}^{1-\alpha}}{1-\alpha}.$$

If α is less than one ~~which~~ will become infinite with x, and finite for α greater than one. If $\alpha = 1$ the integral assumes the form $\log x - \operatorname{Log} a$, which is infinite when $x = \infty$.

Ex (2) – $\int_a^\infty \frac{dx}{x \operatorname{Log} x (\operatorname{Log} x)^\alpha} = \frac{(\operatorname{Log} x)^{1-\alpha} - (\operatorname{Log} a)^{1-\alpha}}{1-\alpha}$. This will also be infinite for $\alpha < 1$, and finite for $\alpha > 1$. When $\alpha = 1$, we have $\operatorname{Log}\operatorname{Log} x - \operatorname{Log}\operatorname{Log} a$, which is infinite for $x = \infty$.

Thus Series $\Sigma \frac{1}{a^\alpha}$, $\frac{1}{a (\operatorname{Log} a)^\alpha}$, $\frac{1}{a \operatorname{Log} a (\operatorname{Log}\operatorname{Log} a)^m}$; .... are each convergent for $\alpha > 1$ and divergent for $\alpha \leqq 1$.

Ex $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$\int_0^\infty x^n dn = \frac{x^n - 1}{\operatorname{Log} x}$ If $x > 1$ this becomes infinite with n, but when $x < 1$ it is finite and approaches $\frac{-1}{\operatorname{Log} x}$.

When $x = 1$ it is indeterminate.
Treating this expression in the usual way we have $n x^n$ which approaches infinity as n increases indefinitely.

Series with Both Positive and Negative Terms.

It is important here to note that there are two types of convergent series, namely series which converge when all their terms are taken positively, and those whose convergence depends upon presence of recurring negative sign.

Ex $1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{------}$ ad inf.

$1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} + \frac{1}{5}) - (\frac{1}{6} - \frac{1}{7}) + \text{------}$

$1 - \frac{1}{2} + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \text{------}$

Sum is $1 > S > \frac{1}{2}$. Hence series converges

We have already shown that this series is divergent when all its terms are taken positively. Here it is convergent, but its convergence depends upon presence of recurring negative sign. Such a series we shall call semi convergent to distinguish from absolutely convergent or those series which converge when all their terms are taken positively.

(a) Alternating Series.

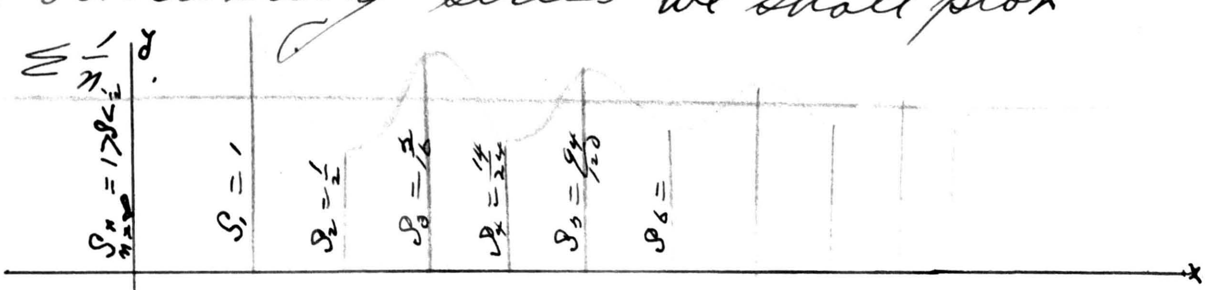
A series is convergent whose terms decrease in absolute value, and are or end with becoming alternately positive and negative

Let $u_0 - u_1 + u_2 - u_3 + u_4 - u_5 + \cdots\cdots u_n$

$$S = u_0 - (u_1 - u_2) - (u_3 - u_4) + \cdots\cdots$$
$$S' = u_0 - u_1 + (u_2 - u_3) + (u_4 - u_5) + \cdots\cdots$$

Observing that the terms in parenthesis are all positive by hypothesis, we see that sum of series lies between u_0 and $u_0 - u_1$. Therefore series is convergent.

To give a better idea of convergence of an alternating series we shall plot series $\Sigma \frac{1}{n}$.



If series be not alternately positive (and negative), but the infinity of negative signs has a periodic arrangement, we can always associate in in groups each succession of negative and positive signs and thus reduce all such series to case where terms are alternately positive and negative.

(b) General Case of Series with positive and negative signs.

Let $S = u_1 + u_2 + u_3 + u_3 + u_4 + \cdots$ be any series with positive and negative terms and denote series of negative terms by

$$-A = -[\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha + \cdots$$

Series of positive terms by B

$$B = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \cdots$$

$$S_n = B_m - A_m$$

1 - First suppose A and B are both convergent $\lim\limits_{n=\infty} \Sigma \alpha_n = -A$, and $\lim\limits_{n=\infty} \Sigma B_n = B$. But limit to S_n for n infinite is S. Hence

$$\lim_{n=\infty} S_n = S = B - A$$ Thus the series is absolutely convergent, as was implied in our definition of absolute convergence

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2- If S is convergent and one of A, B series divergent the other must be

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

$B = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \text{ect}$

$A = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} + \cdots$

$= -\frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\right.$

This last series is convergent as has already been shown. By comparing A and B we see that the corresponding terms in B each greater than the corresponding terms in A. Hence B is divergent.

The above follows as a natural consequence since $S_n = B_n - A_n$. If S be convergent and one of series A, B is divergent, we see clearly that if sum is to remain finite the other must be divergent also.

Test Ratio applied to Series with positive and negative Terms.

By forming series of absolute values of terms of S we can apply the test ratio $\frac{u_{n+1}}{u_n}$, since S will evidently be convergent if S' is convergent

However it is not conversely true that S will be divergent, when S' is divergent.

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = K, \quad -1 < K < 1. \qquad \text{Convergent}$$

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = K, \quad K > 1, \text{ or } K < -1 \qquad \text{Divergent}$$

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = K, \quad K = 1 \qquad \text{no test.}$$

Ex $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \text{-----------}$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{x^{2n+2}}{2n+2}}{\frac{x^{2n+1}}{2n+1}} = \frac{2n+1}{2n+2}x = \frac{2+\frac{1}{n}}{2+\frac{2}{n}}x = x$$

Hence the series of absolute values will be convergent for $x < 1$. But when $x = 1$ Series of absolute values is divergent and the series of original form is convergent.

Convergence of Power Series.

The series of ascending powers of z, $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\cdots$ whose coefficients $a_1, a_2, a_3, a_4, \cdots$ are independent of z is called a power series or integral series in z. A Series of this class may be convergent for all values of z but usually it converges for some values and diverges for others.

In our discussion of this series we shall consider both $a_0, a_1, a_2, a_3, \cdots$ and z complex quantities, that is $a_n = r_n(\cos\theta_n + i\sin\theta_n)$ and $z = \rho(\cos\phi + i\sin\phi$ where r and θ are function of n. This series now is of the form $\sum \rho^n(\cos\phi + i\sin\phi)^n r_n(\cos\theta + i\sin\theta)$. It is now evident that this Series will be convergent when $\sum r_n\rho^n$ is convergent.

Applying to this the test ratio $\frac{u_{n+1}}{u_n}$, we have $\lim_{n=\infty}\frac{u_{n+1}}{u_n} = L\frac{r_{n+1}\rho^{n+1}}{r_n\rho^n} = \rho\lim\frac{r_{n+1}}{r_n} < 1$, series convergent

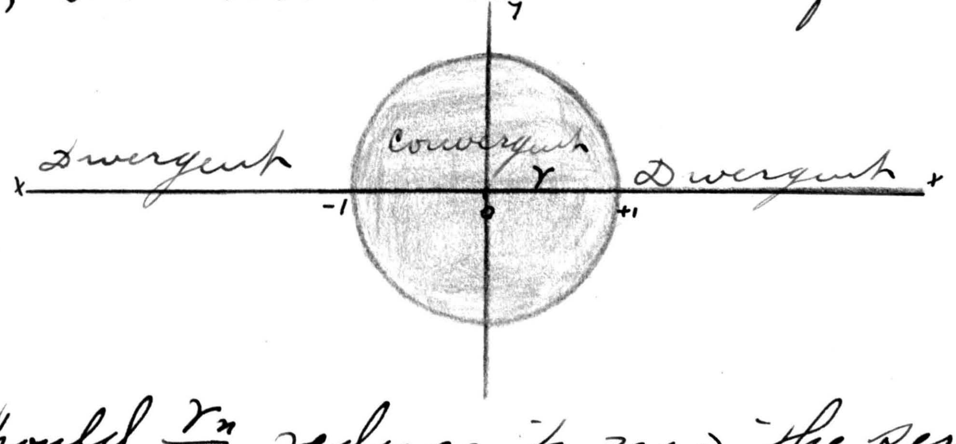
$\rho\lim_{n=\infty}\frac{r_{n+1}}{r_n} > 1$ or $\rho > \frac{r_n}{r_{n+1}}$ Divergent

Now $\frac{r_n}{r_{n+1}}$ must be either finite or infinite

If $\frac{r_n}{r_{n+1}}$ is finite the series will be convergent for every point within region round the origin of radius R. This circle is called the circle of convergence and R is called the radius of convergence.

Ex. $\frac{1}{1-z} = 1+z+z^2+z^3+z^4+ \cdots$ of radius one

If the z be regarded as complex variable the series is convergent for every value of z lying about the origin within the unit circle, but not on the circle of convergence



Should $\frac{r_n}{r_{n+1}}$ reduce to zero the series would be convergent for one point the origin.

In case $\frac{r_n}{r_{n+1}}$ is infinite the series converges for every point in plane.

Ex. $e^z = 1+z+\frac{z^2}{\lfloor 2}+\frac{z^3}{\lfloor 3}+\frac{z^4}{\lfloor 4}+ \cdots$

Convergence of Series with Complex Terms.

A Series of the form $\sum_1^n \alpha_m + i\beta_m$ where α and β are functions of n and i is an imaginary unit will evidently be convergent if $\sum_1^\infty \alpha_m$ $\sum_1^\infty \beta_m$. moreover the series will diverge or oscillate if either $\sum \alpha_n$ or $\sum \beta_n$ diverge or oscillate. Then if $\sum_1^\infty \alpha_m$ and $\sum_1^\infty \beta_m$ tend to a limit as m is indefinitely increased S will be convergent and converge to limit $S = A + iB$

Let $\sum z_n$ be a series with complex terms of the form $\alpha + i\beta$, and represent its modulus by ρ and amplitude by ϕ, then z_n will be of form $z_n = \rho(\cos\phi + i\sin\phi)$. Such a series will be convergent if the series of real terms $\sum \bmod z_n$ be convergent; for if $\sum \rho_n$ is convergent $\sum \rho_n \cos\phi$ and $\sum \rho_n \sin\phi$ will be convergent, since cosine and sine can never exceed unity: that is $\sum \alpha_n$ and $\sum \beta_n$ are both convergent. Hence $\sum z_n$ is convergent.

The condition thus established although

Sufficent is not necessary; for a a Series $\Sigma \alpha_n + i\beta_n$ may be convergent when series of moduli is divergent.

Multiple Series with Real Terms.

In all the series thus far considered the general term depended upon a single integer; but in the multiple series the general term depends upon several integers, which may take independently of each other all values from $-\infty$ to $+\infty$, if not subjected to restrictive conditions.

Before considering this series, however we shall consider a series extending backward and forward to infinity, that is a series of the form $\cdots + u_{-3} + u_{-2} + u_{-1} + u_0 + u_1 + u_2 + u_3 + \cdots$

Ex $\frac{1}{x-1} + \frac{1}{1-x} = \cdots + x^{-2} + x^{-1} + 1 + x + x^2 + x^3 \cdots$

If such a series be absolutely convergent it is independent of the order in which the terms are summed and sum is equal to $\Sigma u_n + \Sigma u_{-n}$. In case it is not absolutely convergent we must say in what order we intend to sum the series, if it is to have a meaning

Consider the series $\cdots -\frac{1}{3}-\frac{1}{2}-\frac{1}{1}+0+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots$

If we take for sum Σu_{-n} and Σu_n we have

$$-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)+0+\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$

$-\Sigma\frac{1}{n}+\Sigma\frac{1}{n}$. This sum can have no meaning since both Σu_n and Σu_{-n} are divergent

Take again for the sum, $0+\left(\frac{1}{1}-\frac{1}{1}\right)+\left(\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{3}\right)+\cdots$

$$=0+0+0+0+0+\cdots$$

The sum of any number of terms will always be zero.

If we consider for sum $\Sigma u_{\rho}+\Sigma u_{\rho'}$ where both ρ and ρ' are infinite this sum will have a definite meaning only when $\frac{\rho}{\rho'}$ is finite.

$$\cdots+x^{-4}+x^{-3}+x^{-2}+x^{-1}+1+x+x^{2}+x^{3}+x^{4}+\cdots$$

Divergent	-1 Convergent 0	$+1$ Divergent	$=\Sigma x^{n}$
Convergent	Divg.	Convergent	$=\Sigma x^{-n}$

$\cdots+x^{-3}+x^{-2}+x^{-1}+1+x+x^{2}+x^{3}+\cdots$ Divergent

$1+\left(x+\frac{1}{x}\right)+\left(x^{2}+\frac{1}{x^{2}}\right)+\left(x^{3}+\frac{1}{x^{3}}\right)+\cdots$ Divergent.

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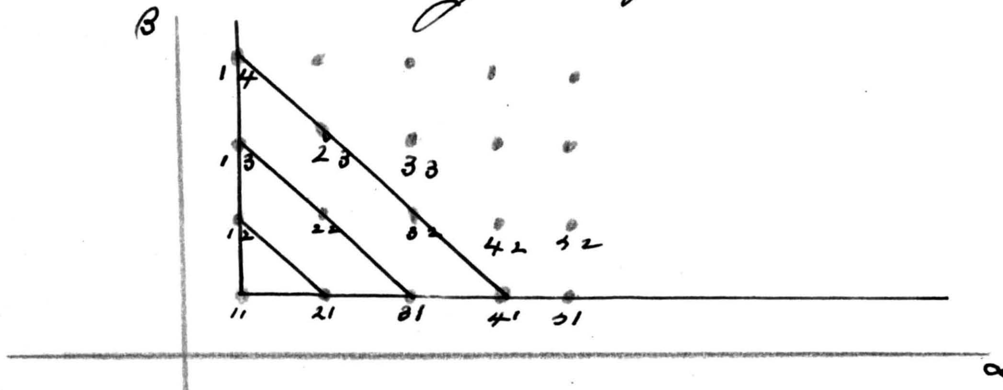
Double Series:-

Let $\sum_1^\infty U_{\alpha\beta}$ be a series where α and β take independently of each other all values from $+\infty$ to 1. We may advantageously display the terms of such a series in the following manner.

$$\begin{array}{l}
U_{11} + U_{12} + U_{13} + U_{14} + U_{15} + U_{16} + \cdots\cdots U_{1n} \\
U_{21} + U_{22} + U_{23} + U_{24} + \cdots\cdots U_{2n-1} + U_{2n} \\
U_{31} + U_{32} + U_{33} + U_{34} + \cdots\cdots \\
U_{41} + U_{42} + U_{43} + U_{44} + \cdots\cdots \\
\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots \\
U_{n1} + U_{n2} + U_{n3}
\end{array}$$

It is evident that a great variety of definitions might be given to the sum of finite number of terms of such a series. Hence in order to give a meaning to the sum of such a series, we might regard α and β as the rectangular coordinates of a point in a plane and the sum as sum series taken along the diagonals.



$$\sum U_n = U_{11} + (U_{12} + U_{21}) + (U_{31} + U_{22} + U_{13}) + \cdots$$

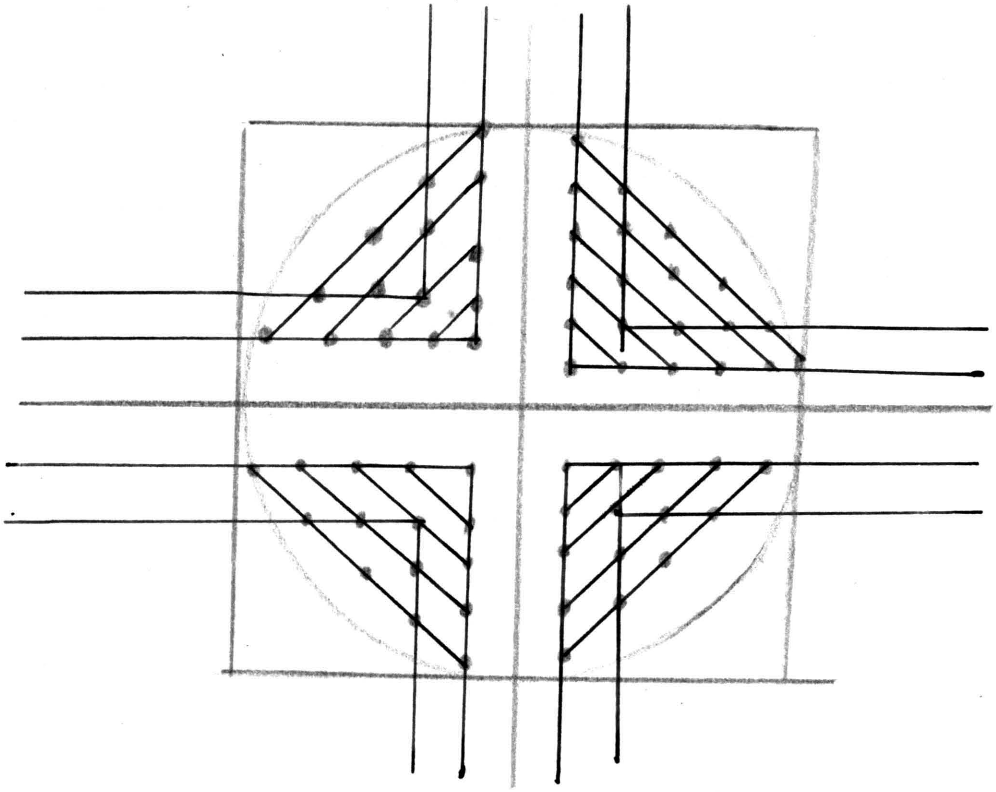
We might also in the same array consider the sum of series in horizontal rows as the sum.

$$\sum_1^\infty U_{\alpha\beta} = \sum_1^\infty U_{\alpha 1} + \sum_1^\infty U_{\alpha 2} + \sum_1^\infty U_{\alpha 3} + \sum_1^\infty U_{\alpha,4} + \cdots$$

Again we might take sum by columns as sum of the series

$$\sum_1^\infty U_{\alpha\beta} = \sum_1^\infty U_{1\beta} + \sum_1^\infty U_{2\beta} + \cdots\cdots \sum_1 U_{\alpha\beta}$$

We may also consider the sum as sum of terms in rectangular array with α and β for sides, that is the sum of all terms in ${}^{\beta}\underset{\alpha}{\boxed{}}$.



If α and β take all values from $-\infty$ to $+\infty$, we may consider the boundary a circle or a four sided figure around the origin as in the above diagram, $\alpha^2+\beta^2=R^2=0$. It is always possible to take the parameter R so large that terms corresponding to points outside of boundary can be made as small as we please.

Double series whose terms are real and ultimately all of the same sign.

As in the single series the important question here is the absolute convergence of the double series. If we have the form of the boundary given and the sum of terms within that boundary approach a definite limit the series is said to be convergent for this particular way of summing; but if the series approaches a definite value for each way of summing it is said to be absolutely convergent. In case all the terms are positive and the series is convergent for any particular way of summing it will be convergent for any other.

Consider the series $\sum U_{pq}$ convergent.

$$\begin{array}{l} U_{11} + U_{12} + U_{13} + U_{14} + U_{15} + U_{16} \\ U_{21} + U_{22} + U_{23} + U_{24} - - - - - - - \\ U_{31} + U_{32} + U_{33} + - - - - - - \\ + \end{array}$$

Now since this series is convergent and its terms are all positive the series formed by taking terms in each vertical column converges $\sum^{\infty} U_{p1}$, $\sum^{\infty} U_{p2}$, $\sum^{\infty} U_{p3}$. Let the sum of these series be denoted by S_1, S_2, S_3, S_4, S_5, S_+ - - - - Considering the first p columns

$$S_1 + S_2 + S_3 + S_4 \cdots = S_{pq} + r_1 + r_2 + r_3 + r_4 \cdots$$

where p is finite. But since each column is convergent $r_1 + r_2 + r_3 + r_4 \cdots$ must vanish for q infinite. Hence $S_1 + S_2 + S_3 + S_4 + \cdots S_p$

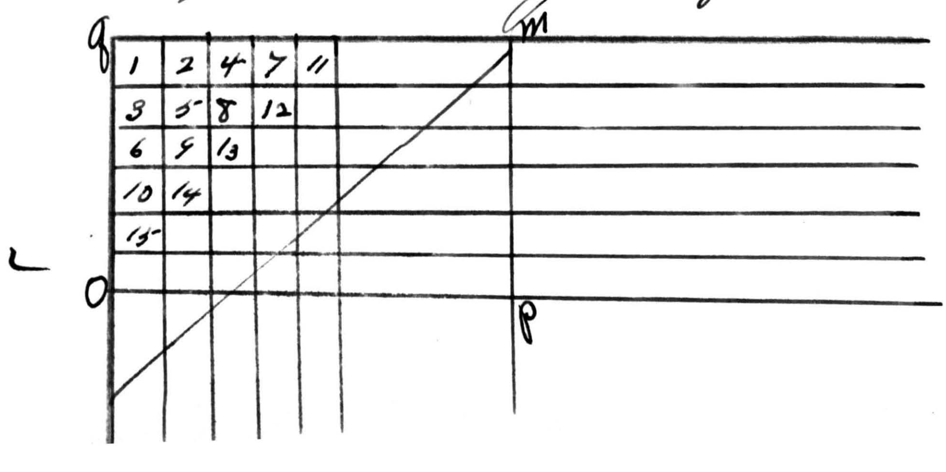
$$= \lim_{q=\infty} S_{pq}$$

$$\lim_{p=\infty} (S_1 + S_2 + S_3 \cdots S_p) = \lim_{p=\infty} \lim_{q=\infty} S_{pq} = S$$

In a like manner if $U_1, U_2, U_3, U_4 - U_q$ be sum of first q rows

$$\lim_{q=\infty} (U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + \cdots U_q) = \lim_{\substack{q=\infty \\ p=\infty}} S_{pq} = S$$

now since this series is convergent if the sum is taken by horizontal rows or by columns the sum of pq terms in the rectangular array is finite.



Hence the series summed by diagonals is convergent, that is

$$\Sigma U_{pq} = U_{11} + (U_{12} + U_{21}) + (U_{13} + U_{22} + U_{31}) + ---$$

the residue outside of PMOQ will always be outside QOM and thus the sum can never exceed the sum of pq terms in OQMP. By sufficently increasing p+q we can make this residue as small as we please. Therefore ΣU_n is convergent and converges to the value S. Thus the double series is unconditionally convergent for the ways of summing above mentioned.

Absolutely Convergent Double with Positive and Negative terms

It is evident that all the theorems proved with regard to convergence of double series with positive terms are true when terms are alternately positive and negative, for restoring negative signs will only render the residue less than before.

Let $U_{pq} - V_{pq}$ be sum of positive and negative terms. Let U' be sum when all terms are all taken positively.

$$V = U_{pq} - U_{pq}$$

$$V' = U_{pq} + U_{pq}$$

Since the series is absolutely convergent U_{pq} and V_{pq} are both finite. Hence $\operatorname{Lim} V_{pq} = V - V'$

Test of Convergence

Cauchy's integral test of convergence of single series may be extended to double series. If terms of $\sum V_{pq}$ be all positive and can be expressed by $f(x\,y)$ which decreases indefinitely as xy increases and limit to the last term is zero, the double series converge according as $\iint f(x\,y)\,dx\,dy$ taken over that part of the plane exterior to the bounding curve has or has not a meaning.

Ex $1 + xy + x^2y^2 + x^3y^3 + x^4y^4 + x^5y^5 + \cdots\cdots$

$$\int_0^\infty\!\!\int_0^\infty x^m y^n\,dm\,dn = \int_0^\infty x^m\,dm \int_0^\infty y^n\,dn = \frac{y^n - 1}{\operatorname{Log} y} \cdot \frac{x^n - 1}{\operatorname{Log} x}$$

Hence the series is convergent when $0 < x < 1$ and $0 < y < 1$.

Algebraic Transformations of Infinite Series.

We shall close this paper with a brief discussion of the application of the fundamental laws of algebra to an infinite series

Law of association.- The sum of a finite number of summands is independent of the order in which we take the sum; but we shall see in what follows that this is not always the case when the number of things become infinite

Ex (1) $1-1+1-1+1-1+1-1+ - - - - -$

This approaches a limit 0 or 1 according as we take an even or an odd number of terms

But the series $(1-1)+(1-1)+(1-1)+(1-1)+ - -$ has alway a sum equal to zero.

Ex (2) $\frac{3^2}{2^2} - \frac{4^2}{3^2} + \frac{5^2}{4^2} - \frac{6^2}{5^2} + \frac{7^2}{6^2} + - - - -$

this is clearly a non convergent oscillating series. But $\left(\frac{3^2}{2^2} - \frac{4^2}{3^2}\right) + \left(\frac{5^2}{4^2} - \frac{6^2}{5^2}\right) + - -$ is convergent

Its n^{th} term will be $\left[\frac{(2n+1)^2}{(2n)^2} - \left(\frac{2n+2}{2n+1}\right)^2\right] = \frac{8n^2+8n+1}{(4n^2+2n)^2}$

$= \frac{8+\frac{8}{n}+\frac{1}{n^2}}{16n^4(1+\frac{1}{n}+\frac{1}{4n^2})} =$ for $n=\infty$ $\frac{8}{16n^2} = \frac{1}{2n^2}$. Comparing with the test Series $\Sigma \frac{1}{n^m}$ where $m = 2$ or $\Sigma \frac{1}{n^2}$ we see it is convergent.

It is evident from these examples that the law of association is not applicable to to an oscillating series where Limit of $U_n \neq 0$. The law of association can however be applied providing the series the series is convergent. Let S'_m denote the sum of m terms of the new series formed by grouping the terms and S_n denote the sum of n terms of the original series. Now it matters not how large we make n and then hold it fast we can make m so large that it will include at least all the terms of S_n. Now we can take n sufficiently large that $_rR_n$ may be made less than any assignable quantity however small

$$\lim_{m=\infty} S'_m - \lim_{n=\infty} S_n = {}_rR_n = 0$$

Commutative Law. Where the number summed is finite the sum is independent of the order in which we take the sum, but when the number of summands become

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infinite this fails to apply. Dirichlet has shown that the commutation of terms may render semi convergent series divergent

Ex $\frac{1}{1^m}+\frac{1}{2^m}+\frac{1}{3^m}-\frac{1}{4^m}+$ ---------- where $m=\frac{1}{2}$

this series is convergent. But the series

$(\frac{1}{1}-\frac{1}{\sqrt2}+\frac{1}{\sqrt3}-\frac{1}{\sqrt4}+\frac{1}{\sqrt5}-\frac{1}{\sqrt6}+$ ------ when transformed

into $(\frac{1}{\sqrt1}+\frac{1}{\sqrt3}-\frac{1}{\sqrt2})+(\frac{1}{\sqrt4}+\frac{1}{\sqrt5}-\frac{1}{\sqrt6})$ ---- $(\frac{1}{\sqrt{4k+1}}+\frac{1}{\sqrt{4k+3}}-\frac{1}{\sqrt{2k+2}})$

is divergent. Comparing this with $\Sigma\frac{1}{\sqrt{k}}$ which we know to be divergent we have

$$\frac{u_m}{v_m}=\frac{\frac{1}{\sqrt{4k+1}}+\frac{1}{\sqrt{4k+3}}-\frac{1}{2k+2}}{\frac{1}{\sqrt{k}}}$$

$$=\frac{\sqrt{k}(\sqrt{4k+3}\sqrt{2k+2})+k\sqrt{4k+1}(2k+2-\sqrt{k}\sqrt{4k+1})\sqrt{4k+3}}{\sqrt{4k+1}\sqrt{4k+3}\sqrt{4k+2}}$$

$$=\frac{1}{\sqrt{4+\frac{1}{k}}}+\frac{1}{\sqrt{4+\frac{3}{k}}}-\frac{1}{\sqrt{2+\frac{2}{k}}}$$ when the

number of terms are increased indefinitely this ratio becomes $1-\frac{1}{2}\sqrt2$. Hence as $\Sigma\frac{1}{\sqrt{k}}$ is divergent Σu_m is also divergent.

The converse of this is often used as a convenient means of deciding upon the convergency of an infinite series. As the grouping of terms in the above produced a divergent series from a semi convergent, so this process might convert a divergent series into a convergent and thus it must be used with precaution

In the above examples we have shown that the law of commutation can not be extended to a Semi Convergent Series. Hence we are led to the last class which converges when all its terms are taken positively. But before considering this class we shall consider one more Semi Convergent Series.

Let us take the Series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+$ writing it in the form

$\left(1+\frac{1}{3}\right)-\frac{1}{2}+\left(\frac{1}{5}+\frac{1}{7}\right)-\frac{1}{4}+\left(\frac{1}{9}+\frac{1}{11}\right)-\frac{1}{6}+\ --$ which is equally a convergent series, but the last converges to value between $\frac{5}{6}$ and $\frac{4}{3}$ while the value of the original series lies between $\frac{1}{2}$ and $\frac{5}{6}$.

Riemann has shown that the same series may by proper commutation be made to converge to any limit whatever. If we assign the limit 500 begin by adding positive terms up to that sum, of the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ Etc When the sum has reached the preassigned value we can add as we please only taking care to keep the sum 500

Let us take now the absolutely convergent series $S_n = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$ and represent by $S_n' = u_0' + u_1' + u_2^2 + u_3^3 + u_4^4 + \cdots$ the series after commuting the terms in which each term has been displaced only a finite number of steps. now it matters not how large we take n we can always take S_m such that it will contain at least all the terms of S_n; that is we can make $S_m - S_n = \rho R_n$ as small as we please $\lim\limits_{m=\infty} \rho R_m = 0$ Hence $\lim\limits_{n=\infty} S_n = \lim\limits_{m=\infty} S_m$ thus the terms of an absolutely convergent series can be rearranged at pleasure without altering the value of the series.

Two infinite series may be added provided both converge. Let S_n and S_n' be two infinite series both of which are convergent and converge to the values S and S'

$U_n = S_n + S_n'$. Hence for n infinite

$\lim U_n = \lim S_n + S_n'$. $U = S + S'$

Let $S_n = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 +$ be a convergent series which converges to the value S. Now if we multiply this series by any quantity B it will converge to the value BS. now

This is evident since $BS = Bu_0 + Bu_1 + \cdots$ is equivalent to $BS = B[u_0 + \cancel{B}u_1 + u_2 + u_3 + \cdots$

But the series in parenthesis converges to value S hence whole series converges to BS.

Ex. $20S = 20(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$= 20 + 10 + 5 + 2\frac{1}{2} + 1\frac{1}{4} + \frac{5}{8} + \cdots$

$= 20 + \frac{20}{2^n} + \frac{20}{2^{n+1}} + \frac{20}{2^{n+2}} + \cdots$

$\frac{u_{n+1}}{u_n} = \frac{20}{2^{n+m+1}} = \frac{2^{n+m}}{2^{n+m+1}} = \frac{1}{2}$ Convergent.

It may also be shown that two infinite series may be multiplied together when at least one of the series be absolutely convergent. In case where both series are absolutely convergent the product will be absolutely convergent.

Let S and S' be two absolutely convergent series.

$$S = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \cdots$$

$$S' = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$$

Then $SS' = \alpha_0 \beta_0 + (\alpha_1 \beta_0 + \alpha_0 \beta_1) x + (\alpha_0 \beta_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0) x^2$

Ex $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{4} + \cdots$

Squaring this series by substituting in the above formula we have

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{8x^3}{3} + \cdots$$

Cauchy has shown that series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ has a definite sum but its square $1 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} \cdots$ ad inf is divergent.