THESIS 1902

Geometry of Four Dimensions

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Geometry of Four Dimensions

In this thesis a brief outline of Four Dimensional Geometry, as far as the classification of quadrics, is attempted.
The following works have been consulted:

C. Smith's Solid Geometry.
Salmon's Geom. of Three Dimensions,
Salmon's Mod. Higher Algebra.
Veronese's Grundzüge der Geometria, von Mehreren Dimensionen.

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Columbia, Mo.,
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I approve this thesis for the Master degree.

J. N. Fellows.

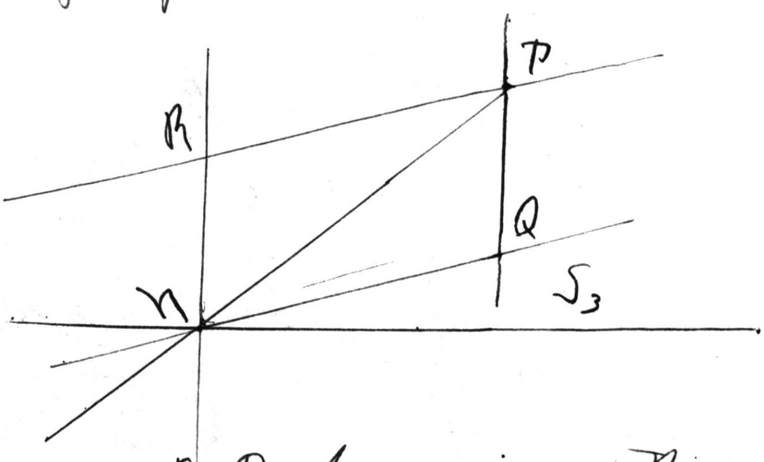
May 30, 1902.

1. S_n will be used throughout this essay to denote space of n dimensions S_1 denoting a line S_2 a plane, S_0 a point and so on

2. Def. As we may generate S_3 by moving each point of a plane perpendicular to each line of the plane passing thro that point, so we may generate S_n by moving each point, P, of S_{n-1} in a direction perpendicular to each line of S_{n-1} passing through P.

3. When two spaces that do not coincide throughout, have certain points in common, they are said to intersect. When two spaces in S_n have no finite points of intersection they are said to be parallel, Providing, spaces of the same order, in general, do intersect in one or more finite points for example, two lines (S_1) in general intersect if they lie in the same plane (S_2) If they do not intersect in any finite point they are parallel, but two lines in S_3 not lying in the same plane, while they have no finite point of intersection, yet are not said to be parallel,

7. The provision in Art. 2, that the motion take place in a direction perpendicular to the generating space is convenient but is not necessary; take for instance any point P in S_4 and connect it by a straight line with any point N of the generating space (S_3) which does not lie perpendicularly under it. draw also a perpendicular PQ to S_3 and at N erect a perpendicular NR; draw NQ



and RP forming the rectangle NRPQ. If the line NQ move in a direction perpendicular to S_3 it will generate the plane NRPQ and must pass over the diagonal NP. therefore the region generated by moving S_3 in a direction parallel to NP contains

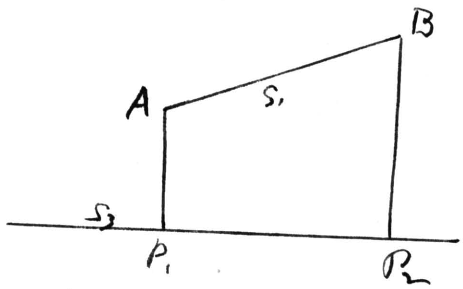
no points that do not lie in S_4.
But does this region include all of the points of S_4? In other words, Is it possible to draw from every point of S_4 a line parallel to PN which will meet S_3? This was assumed when PN was drawn. It may be seen to be always possible as follows:
Through each point of S_3 may be drawn a line parallel to NQ. This line will generate a plane parallel to NRPQ. Hence, through any point of S_4 may be passed a plane parallel to NRPQ and through each point of this plane it is possible to draw a line parallel to any line of NRPQ.

5: Through any point of S_3 an infinite number of straight lines can be drawn which have nothing in common with S_3 except that point.

Let p be the point and consider a straight line passing through p and lying in S_3. When S_3 generates S_4 the straight

line (S_1) generates a plane S_2.
S_1 contains all the points common to S_2 and S_3 and an infinite number of lines can be drawn in S_2 through p. Considering, therefore, only the lines of S_2 an infinite number can be found which meet S_3 in no other point than p.

6. A line which does not lie entirely within a space S_3 has only one point in common with that space

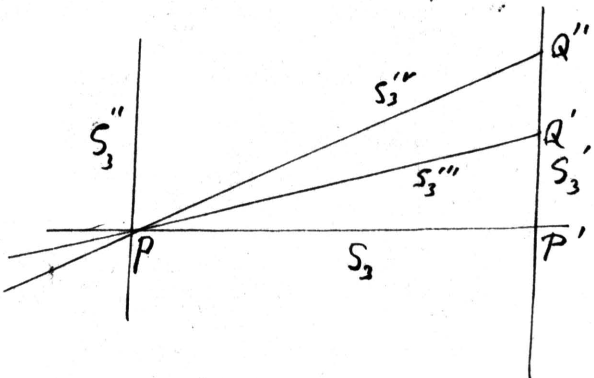


Consider the line S_1 in S_4 and suppose S_4 generated by S_3. From any two points of S_1 as A & B drop perpendiculars AP_1, BP_2 into S_3, P_1P_2 is a line in S_3 which generates the plane AP_1P_2B The point of intersection of S_1 with P_1P_2 is the only point common to S_1 & S_3

7. Through any line of S_3 an infinite number of planes can be passed which have nothing in common with S_3 except that line.
When S_3 generates S_4 any plane S_2 of S_3 generates a three dimensional space S_3'. S_2 contains all the points common to S_3 and S_3'.
An infinite number of planes in S_3' may be passed thro any line of S_2.

8. A plane and a space (S_3) intersect in a right line if the plane does not lie wholly within S_3.
If from three different points in the given plane perpendiculars be dropped into the given space, a plane S_2 will be determined there which can be considered as the generator of a space S_3' containing the given plane.
S_3 and S_3' have only S_2 common.
S_2 and the given plane since they both lie in S_3' intersect in a line which will contain all of the points common to S_3 and the given plane.

9. An infinite number of Spaces (S_3) may be passed thro any plane of a given S_3 which will contain nothing in common except that that plane.



In the figure let S_3 represent the given space and p the given plane. Let p' be any other plane of S_3 parallel to p. Let p and p' generate S_3'' and S_3' when S_3 generates S_4. Let Q' be any position of p' as it generates S_3' let the points of Q' be joined by parallel lines with the points of p. Then since p and Q' are planes the sum total of all such lines will be a space S_3'''. If Q'' is another position of p' we may in a similar manner construct S_3^{iv}. Now Q'' does not lie in S_3''', hence, S_3''' and S_3^{iv} intersect only in p by art. 6.

10. It is evident now that three points are not sufficient to determine a space (S_3) but it can be shown that four will suffice providing they do not all lie in the same plane

It is already known that three points determine a plane P say.

If a fourth be given which does not lie in P, let a plane Q be passed through it parallel to P. Now if all the points of Q be joined with the points of P by parallel lines, the sum total of these lines will be a space S_3 containing the fourth point. If any other plane passing through this point and lying in the above determined space be joined in a similar manner to P the same space will evidently result. Hence, if there exists another space containing the fourth point, it must contain a plane Q, passing thro' the given point which does not lie altogether in S_3. If possible

let S_3' be a space containing both P and Q'. Any line drawn from the fourth point to P lies in both S_3 and S_3'. (Art. 6).

Any plane passing through the fourth point, meeting P in a right line lies in both S_3 and S_3'. Q' is such a plane hence Q' is common to S_3 and S_3' contrary to supposition. Hence, only one space (S_3) can be drawn through the four given points.

11. We have seen that two spaces (S_3) intersect in a plane and that a line and a space in general meet in only one point. It follows that a line and a plane will not in general intersect. If, however, they both lie in the same space (S_3) they will have a common point.

Suppose, now, we have a line and a plane which do not meet. Each point of the line determines with the plane a space (S_3)

and no two points determine the same space for in that case the line and the plane would meet. Thus it can be conceived that we could revolve a space about a plane making it include successively the points of a given right line. In this way we could make the space include any given point of S_4 for we could draw any line through the point. This line would meet the space in one point. The space could then be revolved until it included any given point of the line.

12. Three spaces of three dimensions if they have no plane common intersect in a line and four will have a single point common because any two will intersect in a plane, and the intersection of this plane with a third space is a line which will meet any fourth space in a point. This point is common to the four spaces

13. At any point in a plane which is the intersection of two spaces (S_3) let two perpendiculars be drawn, one into each space. The angle between these two lines will be taken as the measure of the angle between the two spaces. It ought to be shown that this angle will be the same whatever point in the plane be selected. This is evidently true because all perpendiculars to a plane in S_3 are parallel

14. If two spaces are perpendicular to each other, a line in one perpendicular to the plane of intersection is perpendicular to the other

Let S_3 and S_3' be perpendicular and let π be their plane of intersection. In S_3 draw p perpendicular to π and through the foot of p draw p' in S_3' also perpendicular to π. Then

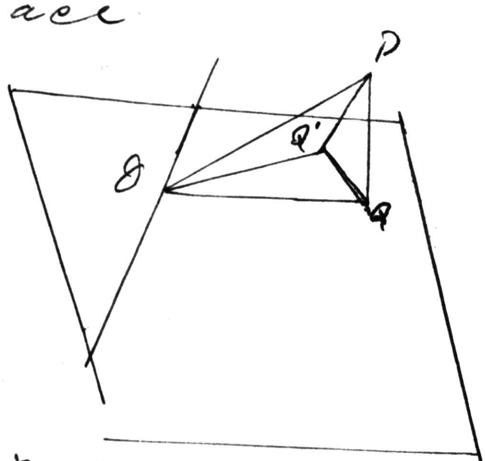
P is perpendicular to S_3' and P' is perpendicular to S_3

Through P draw any plane Q in S_3. It will be perpendicular to Π. Similarly through P' draw Q' which will also be perpendicular to Π.

Let L be the line of intersection of Q and Q'. P' and L both lie in Q'. P is perpendicular to P' because S_3 and S_3' are perpendicular. Also P is perpendicular to L because L lies in Π and passes through the foot of P. ~~Hence~~, P is perpendicular to all the lines of Q' passing through the foot of P. But Q' is any plane through P' lying in S_3'. Therefore P is perpendicular to all the lines of S_3' which pass through its foot

In the same way P' is perpendicular to Q and hence, to S_3

15 The angle between a plane and a space S, is the smallest of the angles which the given plane makes with the planes that pass through the line of intersection of the given plane and space. This is the same as the angle between the given plane and its projection into the given space



Let P be any point in the given plane. draw P O perpendicular to the line of intersection. Let Q be the projection (foot of perpendicular from P) into the given space. Draw O Q in the given space perpendicular to the line of intersection of the given space and plane but in a plane different from the projection

of the given plane in the given space. make $OQ' = OQ$
POQ is the angle between the given plane and its projection. POQ' the angle between the given plane and any other which passes thro the line of intersection

PQQ' is a triangle right angled at Q PQ' is greater than PQ. In the triangles PQO and $PQ'O$, $OQ = OQ'$ OP is common hence the angle POQ' is greater than POQ

16. When two parallel spaces (S_3) are met by a third, their intersections are parallel

Let S_3 and S_3' be met by S_3'' and let their intersections with S_3'' be m and m', m and m' lie in S_3'', hence, they have finite points of intersection or they are parallel. If S_3 and S_3' are parallel, m and m' cannot meet in a finite part of S_3'' because m lies in S_3 and m' in S_3'. hence they are parallel

17. If two straight lines are met by three parallel spaces (S_3) the corresponding segments are proportional

The two lines, if they are not parallel determine a space (S_3) which meets the given spaces in three parallel planes

The segments of the lines by these planes are the same as those made by the given spaces and are proportional

18. In measuring the contents of figures in four dimensional space, a four dimensional unit must be employed. The natural unit is a right four dimensional figure having a unit cube for base and extending one unit perpendicular to this base. This is the four space analogue of the cube and has been called a Tesseract. It is bounded by eight cubes twenty four faces twenty four edges and sixteen vertices. The figure having for lower base a right parallelapiped and extending any given distance perpendicular to the space of the base and having for upper base a right parallelopiped equal to the lower base may be called a right parallelopipedonal prism. We shall see presently that its contents in four dimension-

-al units is equal to the product of its four dimensions and that the contents of any right polyhedronal prism is equal to the volume of the base multiplied by the altitude

19. Two right parallelepipedonal prisms having equal bases are proportional to their altitudes

If the altitudes are commensurable and in the ratio m : n each may be divided into equal prisms by spaces parallel to the base, the first into m the second into n

If the altitudes are incommensurable, we may arrive at the same result by the method of limits

20. Two prisms having equal altitudes, are to each other as their bases

Let the dimensions of the two prisms be a b c – d and x y z – d and denote the first by A and the second by B. Construct a third C, having dimensions y z d – c and a fourth D with dimensions a z d – c.

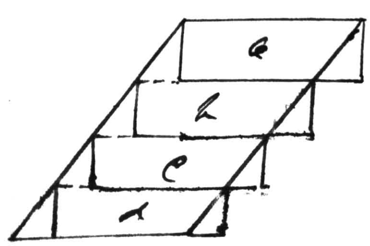
$$\frac{D}{A} = \frac{z}{b}, \quad \frac{C}{D} = \frac{y}{a}, \quad \frac{B}{C} = \frac{x}{c}$$

multiplying we obtain $\frac{B}{A} = \frac{xyz}{abc}$

It easily follows that two parallel apipedonal prisms are to each other as the product of their four dimensions

20. An oblique polyedronal prism is equivalent to to a right polyedronal prism having an equivalent base and an equal altitude

Lay off on the altitude a number of equal parts



and construct the prisms a b c d &c. The sum of these as their number is indefinitely increased is the oblique prism and they are equal to a right prism having an equivalent base and an equal altitude

Similar propositions are true of pyramids,

21a The orthogonal projection of any volume in S_3 upon S_3' is equal to the volume multiplied by the cosine of the angle between S_3 & S_3'. for the volume may be considered to be built up of an indefinite number of parallel plane sections, and each of these, multiplied by the cosine of the angle between S_3 & S_3' will give its projection in S_3'

22. By our definition of four dimensional space it is possible to construct at any point, four lines mutually perpendicular. These four lines may be taken as the axes of a coordinate system. In a coordinate geometry of four dimensions we would naturally expect many of the formulae to be analogous to the corresponding formulae in three dimensional geometry. This is actually the case, the demonstrations in many cases being identical.

(a) The coordinates of the point which divides the line joining the two points $(x_1\ y_1\ z_1\ w_1)$, $(x_2\ y_2\ z_2\ w_2)$ in the ratio $m_1 : m_2$ are

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \qquad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \qquad w = \frac{m_1 w_2 + m_2 w_1}{m_1 + m_2}$$

(C. Smith Solid Geom Art. 4)

(b) the distance between the points $x_1\, y_1\, z_1\, w_1$ and $x_2\, y_2\, z_2\, w_2$ is $\left[\overline{x_1-x_2}^2 + \overline{y_1-y_2}^2 + \overline{z_1-z_2}^2 + \overline{w_1-w_2}^2\right]^{\frac{1}{2}}$

(Same, art. 5.)

(c) If $\alpha, \beta, \gamma, \delta$ are the angles which any line makes with the coordinate axes

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 1$$

(Same art. 6.)

(d) an equation of the first degree represents a space of three dimensions

(Art. 6. and C. Smith, art. 13)

22. Let l m n and r be the direction cosines, and p the length, of the perpendicular from the origin upon a space (S_3)

The equation of the space will be $lx + my + nz + rw = p$

This also is proved as in three space geometry. The equation of any three dimensional space can be written in the above form

It easily follows that the

perpendicular distance from any point $x'y'z'w'$ upon the space whose equation is

$$lx + my + nz + rw = p$$

is $lx' + my' + nz' + rw' - p$.

24. To find the equation of the space (S_3) through four points.

Let the four points be

$x_1\ y_1\ z_1\ w_1 \quad x_2\ y_2\ z_2\ w_2$

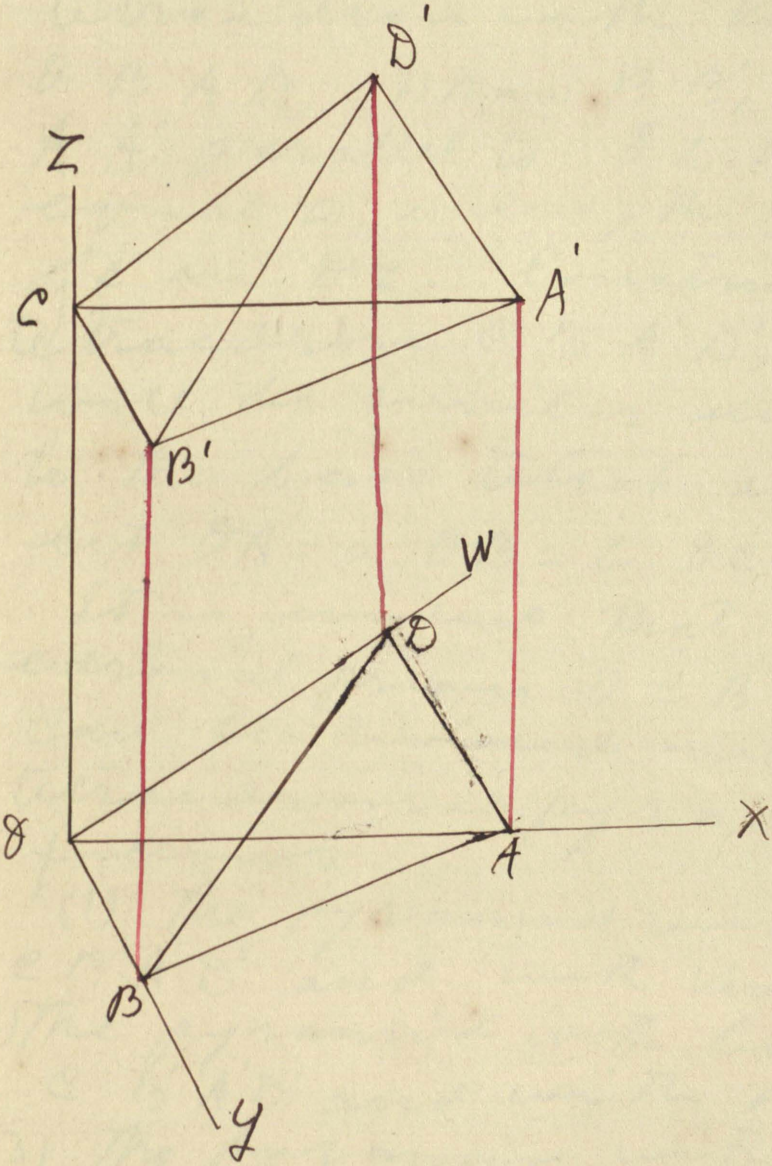
$x_3\ y_3\ z_3\ w_3 \quad x_4\ y_4\ z_4\ w_4$

$$Ax + By + Cz + Dw + E = 0$$

represents any three dimensional space. If each of the points above satisfies this equation we have four relations connecting the five constants, from these relations and the equation itself we may eliminate A B C D and E giving

$$\begin{vmatrix} x & y & z & w & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \end{vmatrix} = 0$$

25. A Right tetraedronal prism is equivalent to four equal tetraedronal pyramids



Let OX, OY, OZ, OW be drawn mutually at right angles and in the space determined by the three lines OX OY and OW construct the tetraedron with vertices O B A D, Draw B B', D D' and A A' parallel to OZ and each equal to a length OC laid off on OZ. Construct the tetraedron C B' A' D', which will be similar and equal to the base tetrahedron OBAD let OA = a OB = b OC = c OD = d

It is evident that the tetraedronal prism O C B' B A A' D D' can be divided into four tetraedronal pyramids as follows

(1) the pyramid with base C B' A' D' and with vertex A

(2) the pyramid with base C B' A D' and with vertex D

(3) the pyramid with base C B A D and with vertex B'

(4) the remaining figure C O B A D

(1) and (4) are equal having equal bases and equal altitudes

(1) and (2) we may see to be equal in the following manner:—

They may be considered as standing on the common base $C B' D' A$. The equation of the space (S_3) determined by these four points is

$$\begin{vmatrix} x & y & z & w & 1 \\ 0 & 0 & c & 0 & 1 \\ 0 & 0 & c & d & 1 \\ 0 & b & c & 0 & 1 \\ a & 0 & 0 & 0 & 1 \end{vmatrix} = 0$$

which reduces to $\frac{x}{a} - \frac{z}{c} - 1 = 0$

The perpendicular distance from A' (the vertex of (1)) into this space is found by substituting the coordinates of A' $(a\ 0\ c\ 0)$ in the expression $\dfrac{\frac{x}{a} + \frac{z}{c} - 1}{\sqrt{1/a^2 + \frac{1}{c^2}}}$ to be $\dfrac{1}{\sqrt{1/a^2 + 1/c^2}}$,

substituting the coordinates of D the vertex of (2) $[0\,0\,0\,d]$, in the same expression for its distance into the same space — $\dfrac{1}{\sqrt{1/a^2 + \frac{1}{c^2}}}$

thus they stand on a common base and have equal altitudes and are therefore, equal

(3) and (4) have the common base C B A D. The equation of the space determined by these four points is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} = 1$
the distance from the origin into this space is $-\frac{1}{(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2})^{\frac{1}{2}}}$
the from B' into the same space is found by substituting the coordinates of B' (0 b c 0) in the expression $\frac{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} - 1}{(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2})^{\frac{1}{2}}}$
to be. $\frac{1}{(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2})^{\frac{1}{2}}}$

(3) and (4) are therefore equal and consequently all four of the pyramids are equal

It follows that the content in four dimensional units, of a polyedronal pyramid is equal to one fourth the product of the volume of its base into its altitude

26 To find the volume of the tetracaronal pyramid in terms of the coordinates of its vertices

Let the vertices be $(x_1\, y_1\, z_1\, w_1)$ $(x_2\, y_2\, z_2\, w_2)$, $(x_3\, y_3\, z_3\, w_3)$, $(x_4\, y_4\, z_4\, w_4)$ $(x_5\, y_5\, z_5\, w_5)$. The equation of the space containing the last four points is

$$\begin{vmatrix} x & y & z & w & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \\ x_5 & y_5 & z_5 & w_5 & 1 \end{vmatrix} = 0$$

The perpendicular from $x_1\, y_1\, z_1\, w_1$ unto this space is

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \\ x_5 & y_5 & z_5 & w_5 & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & w_2 & 1 \\ y_3 & z_3 & w_3 & 1 \\ y_4 & z_4 & w_4 & 1 \\ y_5 & z_5 & w_5 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & z_2 & w_2 & 1 \\ x_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & w_4 & 1 \\ x_5 & z_5 & w_5 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & w_2 & 1 \\ x_3 & y_3 & w_3 & 1 \\ x_4 & y_4 & w_4 & 1 \\ x_5 & y_5 & w_5 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \\ x_5 & y_5 & z_5 & 1 \end{vmatrix}^2}}$$

the denominator of this expression is $\lfloor 3$ times the volume of the tetraedron which determines the space whose equation was given above. Denoting the numerator by D, the denominator by $\lfloor 3 . B$ and the required contents by C we have $C = \frac{1}{4} \cdot \frac{D}{\lfloor 3 B} \cdot B = \frac{1}{\lfloor 4} D = \frac{1}{\lfloor 4} | x_1 y_2 z_3 w_4 1 |$

27. If θ is the angle between two lines passing through the origin, we have, denoting by $x' y' z' w'$ and $x'' y'' z'' w''$, any two points on these lines, and by ρ', ρ'' distances from the origin to these two points

$$(x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2 + (w'-w'')^2 = \rho'^2 + \rho''^2 - 2\rho'\rho''\cos\theta$$

$$\cos\theta = \frac{x'}{\rho'}\frac{x''}{\rho''} + \frac{y'}{\rho'}\frac{y''}{\rho''} + \frac{z'}{\rho'}\frac{z''}{\rho''} + \frac{w'}{\rho'}\frac{w''}{\rho''}$$

or if $a\,b\,c\,d$, and $a'\,b'\,c'\,d'$ are the direction cosines of the two lines, $\cos\theta = aa' + bb' + cc' + dd'$

$$\sin^2\theta = (a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2) - (aa'+bb'+cc'+dd')^2$$

$$= (ab'-a'b)^2 + (ac'-a'c)^2 + (ad'-a'd)^2 + (bc'-b'c)^2 + (bd'-b'd)^2 + (cd'-c'd)^2$$

28. To find the direction cosines of a line perpendicular to three given lines which meet in the Origin

Let the given lines be OP', OP'', OP'''; their direction cosines, $a'b'c'd'$, $a''b''c''d''$, $a'''b'''c'''d'''$, their lengths, ρ', ρ'', ρ''', and the angles which they make with each other, λ, μ, ν. Let a, b, c, d be the required direction cosines

The volume of the tetrahedron $OP'P''P'''$ is $\frac{1}{6}\rho'\rho''\rho'''(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}$

Its projection onto the XYZ-space is $\frac{d}{6}\rho'\rho''\rho'''(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}$

This projection may be denoted also by

$$\frac{1}{6}\rho'\rho''\rho'''\begin{vmatrix} a' & b' & c' & 1 \\ a'' & b'' & c'' & 1 \\ a''' & b''' & c''' & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

This gives $d = \dfrac{|a'\,b''\,c'''|}{(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}$

$$a = -\frac{|b'\,c''\,d'''|}{(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}$$

$$b = \frac{|a'\,c''\,d'''|}{(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}$$

$$c = -\frac{|a'\,b''\,d'''|}{(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}$$

29. If $\lambda = \mu = \nu = \frac{\pi}{2}$ the denominators of the preceding expressions for a b c and d become unity, giving

$a = -|b' c'' d'''|$

$b = |c' d'' a'''|$

$c = -|d' a'' b'''|$

$d = |a' b'' c'''|$

multiplying in order by a' b' c' d' and adding we obtain

$a a' + b b' + c c' + d d' = 0$

similarly

$a' a'' + b' b'' + c' c'' + d' d'' = 0$

$a'' a''' + b'' b''' + c'' c''' + d'' d''' = 0$

$a''' a' + b''' b' + c''' c' + d''' d' = 0$

we have also the relations

$a^2 + b^2 + c^2 + d^2 = 1$

$a'^2 + b'^2 + c'^2 + d'^2 = 1$

$a''^2 + b''^2 + c''^2 + d''^2 = 1$

$a'''^2 + b'''^2 + c'''^2 + d'''^2 = 1$

30. The formulae of the preceding article give the relations connecting the direction cosines of a new set of axes referred to the old when both systems are rectangular. It is easy to see that by such a transformation $x^2 + y^2 + z^2 + w^2$ becomes $X^2 + Y^2 + Z^2 + W^2$. The eight relations are equivalent to the following:—

$$a^2 + a'^2 + a''^2 + a'''^2 = 1$$
$$b^2 + b'^2 + b''^2 + b'''^2 = 1$$
$$c^2 + c'^2 + c''^2 + c'''^2 = 1$$
$$d^2 + d'^2 + d''^2 + d'''^2 = 1$$

$$ab + a'b' + a''b'' + a'''b''' = 0$$
$$bc + b'c' + b''c'' + b'''c''' = 0$$
$$cd + c'd' + c''d'' + c'''d''' = 0$$
$$da + d'a' + d''a'' + d'''a''' = 0$$

31. The projections on the axes of the length from $x\,y\,z\,w$, to $x'\,y'\,z'\,w'$ are $x-x'$, $y-y'$, $z-z'$, and $w-w'$. If these are divided in turn by the direction cosines, the results will each equal the length itself. $\frac{x-x'}{\cos\alpha} = \frac{y-y'}{\cos\beta} = \frac{z-z'}{\cos\gamma} = \frac{w-w'}{\cos\delta}$ may be taken as the equations of the line passing through x', y', z', w', and making the angles $\alpha, \beta, \gamma, \delta$, with the axes

Since three spaces (S_3) intersect in a right line, any three equations of the first degree will represent a line. They may be written in the form $x = lw + a$ $y = mw + b$, $z = nw + c$ and contain six arbitrary constants

Since two spaces (S_3) intersect in a plane two equations of the first degree will represent a plane

They may be written in the forms

$$x = lz + mw + n$$
$$y = rz + sw + t$$

These equations also, contain six arbitrary constants. This result might have been anticipated in both cases, — because in case of the line, we know that four conditions are necessary and sufficient to determine its projection in any one of the coordinate spaces and two perpendiculars from points in the line upon its projection are sufficient to fix its position making in all six conditions.

Similarly for the plane, three conditions are necessary and sufficient to determine its projection in any one of the coordinate spaces and the plane will then be completely determined when perpendiculars from three of its points upon its projection are known

32. We may now find the condition that a line or a plane may lie altogether in a given space (S_3)

Take the space $Ax+By+Cz+Dw+E=0$ (1)

and the line

$$x = lw+a \quad y = mw+b \quad z = nw+c \quad (2)$$

substituting for x, y, z in (1)

we obtain

$$(Al+Bm+Cn+D)w = -(Aa+Bb+Cc+E)$$

If $Al+Bm+Dn+C = Aa+Bb+Cc+E = 0$

the equation will be satisfied for all values of w and the line will lie altogether in (1)

Consider the same space and the plane

$$\left.\begin{aligned} x &= lz+mw+n \\ y &= rz+sw+t \end{aligned}\right| (3)$$

substituting for x and y from (3) in (1) we obtain

$$(Al+Br+C)z+(Am+Bs+D)w+(An+Bt+E)=0$$

If $Al+Br+C = Am+Bs+D = An+Bt+E = 0$, this equation will be satisfied by all values of z and w and the plane will lie altogether in the space

33 By the method of the preceding article we may find the condition that a line or a plane may lie in any four dimensional quantic. we have for either the plane or the line, six arbitrary constants for determining the constants in the equations of the line we can obtain six equations from the quintic. for the plane we can obtain six from the quadric. hence all quantics below the quintic contain an infinite number of right lines all below the quadric an infinite number of planes. All higher than the quintic contain no right lines, the quintic contains a finite number. all higher than the quadric contain no planes, ~~the~~ quadric contains a finite number. Of course in all these cases exception must be made of cylindric, conical, and degenerate forms

34 The general quaternary quadric contains fifteen terms and fourteen arbitrary constants. Written at full length it is

$$U = ax^2 + by^2 + cz^2 + dw^2 + e$$
$$+ 2fyz + 2gzx + 2hxy + 2lwx + 2mwy + 2nwz$$
$$+ 2px + 2ry + 2sz + 2tw = 0$$

This equation may be handled in the same manner as the ternary. If place $x'+x$, for x, $y+y'$, for y, &c, the coefficients of the second degree terms will not be changed, the new absolute term will be U' the result of substituting $x'y'z'w'$ for $x y z w$ in U. The coefficients of the first degree terms will be $\frac{\partial U'}{\partial x'}$ $\frac{\partial U'}{\partial y'}$ $\frac{\partial U'}{\partial z'}$ $\frac{\partial U'}{\partial w'}$

The abbreviations U_1 U_2 U_3 U_4 may be used for

$$\frac{1}{2}\frac{\partial U}{\partial x} \quad \frac{1}{2}\frac{\partial U}{\partial y} \quad \frac{1}{2}\frac{\partial U}{\partial z} \quad \frac{1}{2}\frac{\partial U}{\partial w}$$

35: If we put $\lambda\rho, \mu\rho, \nu\rho, \omega\rho$, for x, y, z, w, we obtain a quadratic in ρ. Here $\lambda\ \mu\ \nu\ \omega$ give the direction of the line from the origin to the point, x, y, z, w. We see, then, that this line meets the quadric in two points. If e equals zero, or in other words, if the origin is on the quadric, one root of the equation in ρ becomes zero. If, in addition, the coefficient of ρ is zero, both roots are zero. The coefficient of ρ is $p\lambda + r\mu + s\nu + t\omega$

Replacing $x\ y\ z\ w$ we have

$px + ry + sz + tw = 0$. (1).

which is the relation that $x\ y\ z\ w$ must satisfy when the line from the origin to $x\ y\ z\ w$ meets the quadric in coincident points at the origin

(1) is the equation of a space (S_3) and is tangent to the quadric at the origin

36. By transferring the origin to the point $x'y'z'w'$, and back again, we see that the tangent space at that point is

$$(x-x')U_1' + (y-y')U_2' + (z-z')U_3' + (w-w')U_4' = 0$$

37. If in U, $p = r = s = t = e = 0$
$px + ry + sz + tw$ will equal zero for all values of $x\ y\ z\ w$, that is, any space (S_3) passing through the origin will meet the quadric in coincident points. The equation then denotes a conical quadric. Such a quadric is represented by the general equation if by transformation of coordinates it is possible to make the first degree coefficients and the absolute term vanish we will have then:

$$U' = 0 \quad U_1' = 0 \quad U_2' = 0 \quad U_3' = 0 \quad U_4' = 0$$

these are equivalent to

$$\begin{aligned} ax' + hy' + gz' + lw' + p &= 0 \\ hx' + by' + fz' + mw' + r &= 0 \\ gx' + fy' + cz' + nw' + s &= 0 \\ lx' + my' + nz' + dw' + t &= 0 \\ px' + ry' + sz' + tw' + e &= 0 \end{aligned}$$

Eliminating $x'y'z'w'$ we have

$$\begin{vmatrix} a & h & g & l & p \\ h & b & f & m & r \\ g & f & c & n & s \\ l & m & n & d & t \\ p & r & s & t & \varepsilon \end{vmatrix} = 0$$

When the above condition holds, the tangent space at any point touches the quadric in a line, and in special cases in a plane. The Point $x'y'z'w'$ is the Vertex of the Conical

38. If c is not zero, and the first degree terms vanish the roots of the equation in ρ will be equal but opposite in sign and every line passing through the origin will be bisected at that point. The Equations for determining the centre are

$$ax' + hy' + gz' + lw' + p = 0$$
$$hx' + by' + fz' + mw' + r = 0$$
$$gx' + fy' + cz' + nw' + s = 0$$
$$lx' + my' + nz' + dw' + t = 0$$

If the determinant of the coefficients vanishes there will be no finite centre

39. The most general transformation which does not change the origin is of the form

$$x = \lambda x_1 + \mu y_1 + \nu z_1 + \omega w_1$$
$$y = \lambda' x_1 + \mu' y_1 + \nu' z_1 + \omega' w_1$$
$$z = \lambda'' x_1 + \mu'' y_1 + \nu'' z_1 + \omega'' w_1$$
$$w = \lambda''' x_1 + \mu''' y_1 + \nu''' z_1 + \omega''' w_1$$

Let $u = ax^2 + by^2 + cz^2 + dw^2 + 2fyz$
$+ 2gzx + 2hxy + 2lwx + 2mwy + 2nwz$

u_1 = the result of writing x_1 &c for λ &c. We know that $x^2+y^2+z^2+w^2$ becomes $x_1^2 + y_1^2 + z_1^2 + w_1^2$

Let $S = x^2 + y^2 + z^2 + w^2$

$KS - u$ becomes $KS_1 - u_1$

The discriminants of the two expressions must be equal

$$\begin{vmatrix} a-K & h & g & l \\ h & b-K & f & m \\ g & f & c-K & n \\ l & m & n & d-K \end{vmatrix} = \begin{vmatrix} a'-K & h' & g' & l' \\ h' & b'-K & f' & m' \\ g' & f' & c'-K & n' \\ l' & m' & n' & d'-K \end{vmatrix}$$

Equating coefficients of K we obtain

$$a+b+c+d = a'+b'+c'+d'$$

$$ab+ac+ad+bc+bd+cd-l^2-m^2-n^2-f^2-g^2-h^2 =$$
$$a'b'+a'c'+a'd'+b'c'+b'd'+c'd'-l'^2-m'^2-n'^2-f'^2-g'^2-h'^2$$

$$abc+abd+acd+bcd+2fmn+2gnl+2hlm+2fgh =$$
$$a'b'c'+a'b'd'+a'c'd'+b'c'd'+2f'm'n'+2g'n'l'+2h'l'm'+2f'g'h'$$

$$abcd+2afmn+2bgnl+2chlm+2dfgh$$
$$-bcl^2-cam^2-abn^2-adf^2-bdg^2-cdh^2$$
$$+f^2l^2+g^2m^2+h^2n^2-2ghmn-2hfnl-2fglm$$
$$= a'b'c'd' + (\&c)'$$

Now if we select such a transformation as will make a', b', c', d' equal to the roots of the biquadratic in K

$$\begin{vmatrix} a-K & h & g & l \\ h & b-K & f & m \\ g & f & c-K & n \\ l & m & n & d-K \end{vmatrix} = 0$$

l', m', n', f', g', h' must all vanish and the products of the variables will disappear from the equations. The roots of the equation in K are all real (Salmons Higher Alg. P. 48.)

40. From art. 35. we see that the line $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = \frac{w}{\omega}$ is bisected at the origin when $p\lambda + r\mu + s\nu + t\omega = 0$. This condition will still hold when the origin

is changed to any point in the space $\lambda u_1 + \mu u_2 + \nu u_3 + \omega u_4 = 0$ hence this space bisects all chords of the quadric parallel to the line given above.

$$ax + hy + gz + lw + p = 0$$

bisects chords parallel to the x-axis

$$hx + by + fz + mw + r = 0$$

bisects all chords parallel to the y-axis when $h = 0$ the first is parallel to the y-axis, the second to the x-axis. Such spaces are said to be conjugate to the direction of the chords which they bisect, and we see that if a space conjugate to one line be parallel to a second, the space conjugate to the second will be parallel to the first

41. If, $\lambda\ \mu\ \nu\ \omega$, are the direction cosines of a chord, its conjugate diametral space is given by

$$\lambda(ax + hy + gz + lw + p)$$
$$+ \mu(hx + by + fz + mw + r)$$
$$+ \nu(gx + fy + cz + nw + s)$$
$$+ \omega(lx + my + nz + dw + t) = 0$$

The conditions that the chord be perpendicular to this space are the following:

$$\lambda a + \mu h + \nu g + \omega l = K\lambda$$
$$\lambda h + \mu b + \nu f + \omega m = K\mu$$
$$\lambda g + \mu f + \nu c + \omega n = K\nu$$
$$\lambda l + \mu m + \nu n + \omega d = K\omega$$

eliminating $\lambda\ \mu\ \nu\ \omega$ we have

$$\begin{vmatrix} a-K & h & g & l \\ h & b-K & f & m \\ g & f & c-K & n \\ l & m & n & d-K \end{vmatrix} = 0$$

a biquadratic to determine K
The four corresponding spaces are called principal Diametral spaces

42. From art. 39. it follows that by transformation of axes we may reduce the equation of the second degree to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{d^2} = 1$$

providing that the quadric which it represents has a finite centre. When written in this form the principal spaces are the coordinate spaces. The signs of the quantities $a^2\ b^2\ c^2\ d^2$ depend on the signs of the roots of the discriminating biquadratic
(1) suppose all the terms positive. The locus is limited in every direction. Sections perpendicular to the axes are Ellipsoids

(2) Suppose one term negative. Sections perpendicular to three of the axes are Hyperboloids of one sheet, - perpendicular to the other - Ellipsoids

(3) Let two terms be negative: Sections perpendicular to two of the axes are hyperboloids of one sheet, those perpendicular to each of the others are Hyperboloids of two sheets

(4) Suppose three terms negative. Sections perpendicular to three of the axes are Hyperboloids of two sheets. Those perpendicular to the other axis are Ellipsoids when the positive term is greater than unity

(5) If all the terms are negative no real values of the variables will satisfy the equation

(6) If the absolute term is zero the following forms are possible

(α) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{d^2} = 0$

(β) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{w^2}{d^2} = 0$

(γ) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{w^2}{d^2} = 0$

(α) is satisfied by no real values of the variables except each equal zero.

Sections of (β) perpendicular to the w-axis are ellipsoids, — perpendicular to either one of the other axes — Hyperboloids of one sheet. W = 0 gives a point — the origin, — x y or z = 0 will give a cone with the vertex at the origin.

Sections of (γ) perpendicular to any one of the axes are Hyperboloids of one sheet. When any one of the coordinates is zero the section is a cone with the vertex at the origin

43. If the quadric contains no finite centre the absolute term in the discriminating biquadratic will be zero and of course one of its roots is zero. Let it be the new coefficient of w. We may then, by changing the direction of the axis reduce the equation

to the form, $a'x^2 + b'y^2 + c'z^2 + 2p'x + 2r'y + 2s'z + 2t'w + e = 0$ and obviously, we may determine a new origin so as to make all the first degree terms, except the one containing w, vanish. The equation may now be written

$a'x^2 + b'y^2 + c'z^2 + 2t'w + e' = 0$

44. Let us now examine the forms of the quadric represented by this last equation

(1) Let $t' = 0$. The equation represents a cylinder, generated by moving the conicoid $a'x^2 + b'y^2 + c'z^2 + e' = 0$, parallel to the w-axis. If e' also equal zero, the generating conicoid becomes a cone. Each element of the cone generates a plane and we have a figure made up of planes. These figures may be called conicoidal and conal cylinders resp'y. If one of the second degree terms vanishes, the following figures are possible

(a) Cylindric cylinders including as special case, two intersecting spaces
(b) Elliptic paraboloidal cylinders.
(c) Hyperbolic paraboloidal cylinders. This last being another figure composed entirely of planes
(2) If t' is not equal to zero the origin may be changed to a point on the quadric when the absolute term will disappear. The equation will then take the form
$a'x^2 + b'y^2 + c'z^2 + 2t'w = 0$
(a) Let all the coefficients be positive. No part of the locus lies in the positive direction of the w-axis.
Sections perpendicular to the w-axis are Ellipsoids. Those perpendicular to the other axes are Elliptic paraboloids
(b) If one of the squared terms is negative, sections across the w-axis will be Hyperboloids of one sheet. Those across the other will be Elliptic Paraboloids.

(c) If two of the coefficients of squared terms are negative, sections across the w-axis will be Hyperboloids of two sheets. Sections across two of the remaining axes will be Hyperbolic paraboloids, - those across the remaining ax-is will be Elliptic paraboloids

(3) If $c' = 0$ the equation may be reduced to the form

$$a'x^2 + b'y^2 + 2s'z + e' = 0$$

This equation represents a cylinder with an Elliptic or Hyperbolic paraboloidal base according as a' and b' have the same or opposite signs

(4) If b' and c' each equal zero the equation can be reduced to the form

$$a'x^2 + 2r'y + e' = 0$$

which represents a parabolic cylindric cylinder