NON-CONFORMAL TRANSFORMATIONS.

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INTRODUCTION.

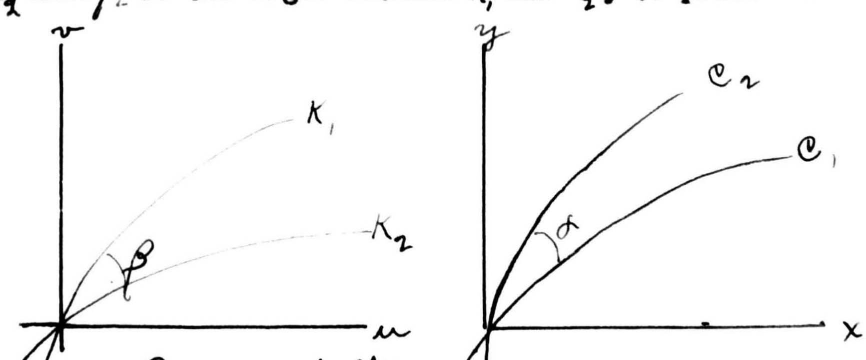
For many years conformal transformations have been studied. Riemann surfaces of many of them have been constructed and their principal properties have been discovered and examined in the study of the theory of functions of a complex variable, for if a transformation is conformal it must satisfy the Cauchy-Riemann equations as will be shown later. Comparatively little, however, has ever been done with conformal transformations. In this paper, we have made a study of some of the properties of such transformations and have constructed Riemann surfaces of a few of them. In Part I, an ordinary conformal transformation has been considered and its Riemann surface constructed. Then by means of non-conformal transformations upon the conformal one, non-conformal transformations between real variables were obtained, and through the properties of the former, the properties of the latter were discovered and the Riemann surfaces were constructed. In Part II, the non-conformal transformations were studied directly.

PART I.

NON-CONFORMAL TRANSFORMATIONS OBTAINED FROM CONFORMAL TRANSFORMATIONS.

Before considering the particular conformal transformation from which the non-conformal transformations were derived, it is necessary to prove that the original transformation was really conformal. This is proved as follows:-

Let $C_1 \begin{cases} x = f_1(t) \\ y = \varphi_1(t) \end{cases}$ and $C_2 \begin{cases} x = f_2(t) \\ y = \varphi_2(t) \end{cases}$ be two curves in the z-plane which intersect at a point, P, and let $K_1 \begin{cases} u = A[f_1(t)\,\varphi_1(t)] \\ v = B[f_1(t)\,\varphi_1(t)] \end{cases}$ and K_2, $\begin{cases} u = A[f_2(t)\,\varphi_2(t)] \\ v = B[f_2(t)\,\varphi_2(t)] \end{cases}$ be their corresponding curves in the w-plane which intersect a some point Q. Let the points P and Q be the origins in the two planes. This can always be made possible by a change of axes. Then $f_1(0) = f_2(0)$ and $\varphi_1(0) = \varphi_2(0)$. Let $\angle\alpha$ be the angle between C_1 and C_2 and $\angle\beta$ be the angle between K_1 and K_2, to prove $\angle\alpha = \angle\beta$.



The slope of $C_1 = \tan^{-1} \dfrac{\varphi_1'(t)}{f_1'(t)}$

The slope of $C_2 = \tan^{-1} \dfrac{\varphi_2'(t)}{f_2'(t)}$,

Then since $\tan^{-1} l - \tan^{-1} m = \tan^{-1} \dfrac{l-m}{1+lm}$,

$$\angle \alpha = \tan^{-1} \frac{\varphi_1'(t) f_2'(t) - \varphi_2'(t) f_1'(t)}{f_1'(t) f_2'(t) + \varphi_1'(t) \varphi_2'(t)}.$$

The slope of $K_1 = \tan^{-1} \dfrac{B'[f_1(t)\,\varphi_1(t)]}{A'[f_1(t)\,\varphi_1(t)]}$

The slope of $K_2 = \tan^{-1} \dfrac{B'[f_2(t)\,\varphi_2(t)]}{A'[f_2(t)\,\varphi_2(t)]}$

$$\therefore \angle \beta = \tan^{-1} \frac{B'[f_1(t)\varphi_1(t)]\,A'[f_2(t)\,\varphi_2(t)] - B'[f_2(t)\varphi_2(t)]\,A'[f_1(t)\varphi_1(t)]}{A'[f_1(t)\varphi_1(t)]\,A'[f_2(t)\varphi_2(t)] + B'[f_1(t)\,\varphi_2(t)]\,B'[f_2(t)\varphi_2(t)]}$$

Then for $\angle\alpha$ to be equal to $\angle\beta$, the following equation must be true.

$$\frac{\varphi_1'(t) f_2'(t) - \varphi_2'(t) f_1'(t)}{f_1'(t) f_2'(t) + \varphi_1'(t) \varphi_2'(t)} = \frac{B'[f_1(t)\,\varphi_1(t)]\,A'[f_2(t)\varphi_2(t)] - B'[f_2(t)\varphi_2(t)]\,A'[f_1(t)\varphi_1(t)]}{A'[f_1(t)\,\varphi_1(t)]\,A'[f_2(t)\varphi_2(t)] + B'[f_1(t)\,\varphi_1(t)]\,B'[f_2(t)\varphi_2(t)]}$$

This equation is only true provided the Cauchy-Riemann equations are satisfied,

i.e. if $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Therefore for a transformation to be conformal, the function must be an analytic function.

The conformal transformation considered was $w^3 - 3w = z$, whose Riemann surface is represented by Planes II and III. Here

there are two planes, a w-plane and a z-plane, which consists of three sheets. If $w^3 - 3w = z$, then the singular points found by setting $\frac{dz}{dw}$ equal to zero are $\begin{cases} w=+1 \\ z=-2 \end{cases}$ and $\begin{cases} w=-1 \\ z=+2 \end{cases}$. Branch lines were drawn in the z-plane from $z=-2$ to $-\infty$ (red line) and $z=+2$ to $+\infty$ (blue line). The branch line is therefore the line $y=0$ or the x-axis.

Substituting $z = x+yi$ and $w = u+vi$ in the original equation,

$$u^3 + 3u^2vi - 3uv^2 - v^3i - 3u - 3vi = x+yi$$

and equating real and imaginary parts,

$$u^3 - 3uv^2 - 3u = x$$
$$3u^2v - v^3 - 3v = y$$

Then substituting this value in the equation $y=0$, we have

$$3u^2 - v^3 - 3v = 0$$

or $v=0$ (the u-axis) and

$$3u^2 - v^2 - 3 = 0 \qquad \text{(an hyperbola).}$$

These two curves in the w-plane correspond to the $y=0$ or x-axis in the z-plane and they are called the images of the x-axis.

If $z=-2$, then $w^3 - 3w + 2 = 0$ or $w = \begin{cases} +1 \\ +1 \\ -2 \end{cases}$ and $z=+2$ then $w^3 - 3w - 2 = 0$ or $w = \begin{cases} -1 \\ -1 \\ +2 \end{cases}$. So the branch points are double.

The portion of the u-axis from $w=-2$ to $w=\infty$ and the right-hand portion of the hyperbola are images of the x-axis from $z=-2$ to $z=-\infty$, while the u-axis from $w=+2$ to $w=+\infty$

and the left-hand portion of the hyperbola correspond to the x-axis from $x = +2$ to $x = +\infty$. The portion of the u-axis between $w = -2$ and $w = +2$ corresponds to the x-axis from $z = -2$ and $z = +2$. Thus the w-plane is divided into three parts, each of which corresponds to one sheet of the z-plane. The images of the origin in the z-plane are the points $(+\sqrt{3}, 0)$, $(0, 0)$, and $(-\sqrt{3}, 0)$ corresponding to the origins in the first, second, and third sheet respectively. Images of the y-axis ($x = 0$) are the v-axis ($u = 0$) and the hyperbola $u^2 - 3v^2 - 3 = 0$ for if $x = 0$, then since $x = u^3 - 3uv^2 - 3u$, $u^3 - 3uv^2 - 3u = 0$ or $u = 0$ (v-axis) and $u^2 - 3v^2 - 3 = 0$ (an hyperbola). This hyperbola has its vertices at $(-\sqrt{3}, 0)$ and $(+\sqrt{3}, 0)$, images of the origin and has asymptotes with the slope $\frac{1}{\sqrt{3}}$ or the 30° lines.

In order to still further subdivide the plane, images of the 45° lines and of the circles $x^2 + y^2 = r^2$ were also found.

The images of the 45° lines were found as follows,

Here $y = x$ and $y = -x$.

Then in polar coordinates, where $w = \rho(\cos\theta + i\sin\theta)$ and $z = r(\cos\varphi + i\sin\varphi)$, $w^3 - 3w = z$ becomes

$\rho^3(\cos 3\theta + i\sin 3\theta) - 3\rho(\cos\theta + i\sin\theta) = r(\cos\varphi + i\sin\varphi)$

from which, equating real and imaginary parts,

$$\rho^3\cos 3\theta - 3\rho\cos\theta = r\cos\theta$$

$$\rho^3\sin 3\theta - 3\rho\sin\theta = r\sin\theta$$

For the 45° line, $r\cos\varphi = r\sin\varphi$

$$\therefore \rho^3\cos 3\theta - 3\rho\cos\theta = \rho^3\sin 3\theta - 3\rho\sin\theta$$

from which $\rho = 0$ and $\rho = \pm\sqrt{\dfrac{3(\cos\theta - \sin\theta)}{\cos 3\theta - \sin 3\theta}}$

The equation $\rho = 0$ is the origin.

To plot the curve $\rho = \pm\sqrt{\dfrac{3(\cos\theta - \sin\theta)}{\cos 3\theta - \sin 3\theta}}$ the following values were used:-

θ	0°	15°	-45°	135°	75°	165°	-105°	-30°	10°	5°
ρ	$\pm\sqrt{3}$	∞	∞	∞	∞	∞	∞	± 2	± 2.6	2

For the -45° line, $r\cos\varphi = -r\sin\varphi$

$\therefore \rho^3\cos 3\theta - 3\rho\cos\theta = -\rho^3\sin 3\theta + 3\rho\sin\theta$

from which $\rho = 0$ and $\pm\sqrt{\dfrac{3(\sin\theta + \cos\theta)}{\cos 3\theta + \sin 3\theta}}$

The equation $\rho = 0$ is again the origin.

To plot the equation $\rho = \pm\sqrt{\dfrac{3(\sin\theta + \cos\theta)}{\cos 3\theta + \sin 3\theta}}$, the following values were used, together with the fact that the curves corresponding to $y = -x$ are the reflection on the x axis of those corresponding to $y = -x$.

θ	-15°	$+45^\circ$	-135°	$+165^\circ$	105°	-75°
ρ	∞	∞	∞	∞	∞	∞

Images of the circle $x^2 + y^2 = 4$ were found as follows. As before

$$\rho^3\cos 3\theta - 3\rho\cos\theta = r\cos\varphi = x$$
$$\rho^3\sin 3\theta - 3\rho\sin\theta = r\sin\varphi = y.$$

Then corresponding to $x^2 + y^2 = 4$, we have

$$(\rho^3\cos 3\theta - 3\rho\cos\theta)^2 + (\rho^3\sin 3\theta - 3\rho\sin\theta)^2 = 4$$

$$\text{or}\quad \cos 2\theta = \frac{4 - 9\rho^2 - \rho^6}{-6\rho^4}$$

From this equation the following values were found:-

$\rho = \pm 1$	$\theta = 0,\ 180°$
$\rho = \pm 2$	$\theta = 0,\ 180°$
$\rho = +2.1$ or greater	θ imaginary
$\rho = 1.8$	$\theta = 11°,\ 169°,\ -11°,\ -169°$
$\rho = 1.5$	$\theta = 14°,\ -14°,\ 166°,\ -166°$
$\rho = 1.1$	$\theta = 5°,\ -5°,\ 175°,\ -175°$
$\rho = 1.9$	$\theta = 10°,\ -10°,\ 170°,\ -170°$
$\rho = .8$	$\theta = 16°,\ -16°,\ 164°,\ -164°$
$\rho = 6.2$	$\theta = 90°,\ 270°$
$\rho = .7$	$\theta = 35°,\ -35°,\ 145°,\ -145°$

By means of these values, remembering that, since the curves in the z-plane are orthogonal, the curves in the w-plane are also orthogonal, and that the angles between the curve and the u-axis at the branch points are $45°$ because the branch points are of the second order, the images of the circle $x^2+y^2=4$ were plotted.

The image of the circle $x^2+y^2=9$ is the curve $\cos 2\theta = \frac{9-9\rho^2-\rho^6}{-6\rho^4}$. By means of the following values and the orthogonal and angle properties, the curve was plotted.

ρ	1	2	2.1	2.2	1.8	1.5	1.1	.8	.806
θ	$\pm 40°$ $\pm 140°$	$\pm 13°$ $\pm 167°$	$\pm 0°$ $\pm 180°$	imag.	$\pm 15°$ $\pm 165°$	$\pm 20°$ $\pm 160°$	$\pm 32°$ $\pm 148°$	imag	$\pm 90°$

The image of $x^2+y^2=1$ is the curve $\cos 2\theta = \frac{1-9\rho^2-\rho^6}{-6\rho^4}$

Remembering the same properties, this curve was platted by means of

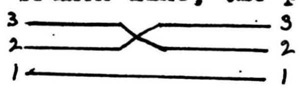
following values.

$\rho = .36$	θ imag.	$\rho = 1.5$	$\theta = 0°$
$\rho = .35$	θ imag.	$\rho = 1.6$	$\theta = 4°$
$\rho = .34$	$31°$	$\rho = 1.7$	$\theta = 5.5°$
$\rho = .33$	$52°$	$\rho = 1.8$	$\theta = 4.5°$
$\rho = .345$	$15°$	$\rho = 1.9$	$\theta = 0°$
$\rho = .335$	$41°$	$\rho = .1$	θ imag.
$\rho = .325$	$72°$	$\rho = .2$	θ imag.

In planes II and III, as in all the figures, corresponding points, lines, and portions of the plane are similarly colored.

The connections between the three sheets of the z-plane are found by following a closed curve which passes through all three portions of the w-plane. If this curve starts in part I, corresponding to sheet I in the z-plane, and passes across a red line it comes into part II, corresponding to sheet II, so the connection across the red cut line is between sheets I and II. If the curve goes on across a blue line, it comes into part II, which corresponds to sheet III in the z-plane, so the connection across the blue cut line is from sheet II to sheet III. If the curve passes on across a red line it still remains in part III, so a passage in sheet III across the red line does not change sheets. From part III, if the curve crosses the blue line, it comes into part II again, so the connection from sheet III across the red cut line is to sheet II. From part II across the red line, the curve passes back again into partI, so the connection across the red line is from sheetII to

sheet I. From part I across the blue line, the curve still remains in part I, so a passage in sheet I across the blue cut line does not change sheets. Hence the connection on the red branch line, the negative end of the x-axis, is denoted by C_1 below. while that on the blue branch line, the positive end of the x axis, is indicated by



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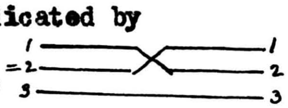
$C_1 =$ 

The connections being established, the Riemann surface is entirely finished, and any point in either plane can be quickly located in the other.

Using this Riemann surface, a Riemann surface of a non-conformal transformation of real variables was constructed. In the original transformation, $w^3 - 3w = z$, let $x = 2\sigma$, $y = \tau$ and $u = 2s$, $v = t$ which gives the transformation,

$$\sigma = 4s^3 - 3st^2 - 3s$$
$$\tau = 12s^2t - t^3 - 3t$$

Instead of constructing the surface by using the correspondences given directly by this transformation, Plane IV was constructed from Plane III by means of the transformation $x = 2\sigma$ and $y = \tau$ and Plane I, from Plane II by means of the transformation $u = 2s$ and $v = t$.

In Planes III and IV, $x = 2\sigma$ and $y = \tau$. Since $y = \tau$, the σ-axis, $\tau = 0$, corresponds to the x-axis, $y = 0$. But since $x = 2\sigma$, that is $\sigma = \frac{1}{2}x$, the branch points ($-2, 0$) and($+2, 0$)

in the z-plane become the branch points ($-1, 0$) and ($+1, 0$) in the σ-τ plane. As x increases from $+2$ along the x-axis, σ increases from $+1$, along the σ-axis and as x decreases from -2 along the x-

axis, σ decreases from -1 along the σ-axis. As x goes from -2 to $+2$, σ goes from -1 to $+1$. Corresponding to the y-axis ($x=0$) is the τ-axis ($\sigma=0$). Corresponding to the 45° lines, $y=x$ and $y=-x$ are the two lines $\tau=2\sigma$ and $\tau=-2\sigma$. The images of the circles $x^2+y^2=r^2$ are ellipses $4\sigma^2+\tau^2=r^2$. By means of these images, the $\sigma-\tau$ plane is divided into portions similar to those of the z-plane.

In Planes I and II, $u=2s$ and $v=t$. By this transformation, the u-axis ($v=0$) in Plane II becomes the s-axis ($t=0$) in Plane I. The branch points $(+1,0)$ and $(-1,0)$ in Plane II, become the branch points $(+\frac{1}{2},0)$ and $(-\frac{1}{2},0)$ respectively in PlaneI,

while the branch points $(+2,0)$ and $(-2,0)$ become respectively the branch points $(+1,0)$ and $(-1,0)$ in Plane I. As u starts from $+2$ and increases towards $+\infty$, s starts from $+1$ and increases towards $+\infty$, and as u starts from -1 and increases towards $-\infty$, s starts from -1 and increases towards $-\infty$. Since $u=2s$, the v-axis ($u=0$) becomes the t-axis ($\sigma=0$). The hyperbola $u^2-\frac{v^2}{3}=1$ becomes the hyperbola $\frac{s^2}{1/4}-\frac{t^2}{3}=1$, which has its vertices at $(\pm 1/2, 0)$ and has asymptotes with the slope $\pm 2\sqrt{3}$. This hyperbola divides the $s-t$ plane into three portions each of which corresponds to a single sheet of the $\sigma-\tau$ plane. The hyperbola $\frac{u^2}{3}-v^2=1$ becomes an hyperbola with the equation $\frac{s^2}{3/4}-t=0$ with vertices at $(\pm\frac{1}{2}\sqrt{3},0)$ and with asymptotes having t the slope $\pm\frac{2}{3}\sqrt{3}$.

The curves corresponding to those in plane II which were

the images of the circles in Plane III have very complicated equations. So these curves were plotted rather freely by means of a few corresponding points.

The images of the lines in Plane II which were the images of the $45°$ line in Plane III were found as follows:-

Since $w = \rho \cos\theta$, $u = 2s$, $s = r\cos\omega$,

$v = \rho\sin\theta$, $v = t$, $t = r\sin\omega$,

then $\rho\cos\theta = 2r\cos\omega$ ------------(1)

$\rho\sin\theta = r\sin\omega$ ----------(2)

In Plane II, $\rho^3\cos 3\theta - 3\rho\cos\theta = \rho^3\sin 3\theta - 3\rho\sin\theta$ -----(3)

$\therefore$ Since $\sin 3x = 3\sin x - 4\sin^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

(3) becomes

$$4\rho^3\cos^3\theta - 3\rho^3\cos\theta - 3\rho\cos\theta = 3\rho^3\sin\theta - 4\rho^3\sin^3\theta$$
$$3\rho\sin\theta \text{ --(4)}$$

From (1) and (2) $\rho^2\cos^2\theta = 4r^2\cos^2\omega$

$\rho^2\sin^2\theta = r^2\sin^2\omega$

By addition, $\rho^2(\cos^2\theta + \sin^2\theta) = 4r^2\cos^2\omega + r^2\sin^2\omega$

$\rho^2 = 4r^2\cos^2\omega + r^2\sin^2\omega$ -------(5)

Using (1), (2) and (5) and substituting in (4)

$$4(8r^3\cos^3\omega) - 3(4r^2\cos^2\omega + r^2\sin^2\omega)(2r\cos\omega)$$
$$-3(2r\cos\omega) =$$
$$3(4r^2\cos^2\omega + r^2\sin^2\omega)(r\sin\omega) - 4r^3\sin^3\omega - 3r\sin\omega$$

or $8r^3\cos^3\omega + r^3\sin^3\omega - 6r^3\sin^2\omega\cos\omega - 12r^3\sin\omega\cos^2\omega$

$$-6r\cos\omega + 3r\sin\omega = 0$$

From which $r = 0$ and

$$r^2(8\cos^3\omega + \sin^3\omega - 6\sin^2\omega\cos\omega - 12\sin\omega\cos^2\omega) = 6\cos\omega - 3\sin\omega$$

or

$$r = \pm\sqrt{\frac{6\cos\omega - 3\sin\omega}{14\cos^3\omega + 13\sin^3\omega - 6\cos\omega - 12\sin\omega}}$$

From this equation the following values were obtained and by means of them, the curves were plotted.

$\omega = 0°, 180°$	$r = .866$	$\omega = .25°$	$r = \pm 2.18$
$\omega = 90°, 270°$	r imag.	$\omega = -25°$	$r = \pm .86$
$\omega = 10°$	$r = \pm .991$	$\omega = 28°$	$r = \pm .876$
$\omega = -10°$	$r = \pm .785$	$\omega = -28°$	$r = \pm .88$
$\omega = -20°$	$r = \pm .882$	$\omega = -45°$	$r = \pm 1.17$
$\omega = 20°$	$r = \pm 1.67$	$\omega = 45°$	r imag.
$\omega = 30°$	r imag.	$\omega = 60°$	r imag.
$\omega = -30°$	$r = \pm .899$	$\omega = -60°$	$r = \pm 2.7$
$\omega = -50°$	$r = \pm 1.39$	$\omega = 75°$	$r = \pm 1.04$
$\omega = \pm 85°$	r imag.	$\omega = 80°$	$r = \pm 2.28$
$\omega = \pm 89°$	r imag.	$\omega = 70°$	$r = \pm .617$

The images of the images of the $-45°$ line were found by the reflection on the s-axis of the line obtained above.

The images of the lines, $\sigma = \pm 1$ and $\tau = \pm 1$ in Plane IV were found in Plane I by means of the real transformation.

$$\sigma = 4s^3 - 3st^2 - 3s$$
$$\tau = 12s^2t - t^3 - 3t$$

If $\sigma = \pm 1$, $t = \pm\sqrt{\dfrac{1 + 3s - 4s^3}{-3s}}$, and if $\tau = \pm 1$, $s = \pm\sqrt{\dfrac{1 + t^3 + 3t}{12t}}$.

By means of values obtained from these equations, the curves were plotted.

The connections between the three sheets of the σ-τ plane were found by following the path of a closed curve in the s-t plane. They are the same as the connections in the x-y plane, because the transformations used were one to one transformations.

In this manner, a complete Riemann surface of a real transformation was constructed and its properties were found to differ but little from the original Riemann surface from which it was derived. In this case, however, the transformation between the two surfaces was very simple and it seems reasonable that the surfaces should differ but little. In constructing the next surface, a more complicated transformation was employed in changing the conformal into nonconformal transformation and the surfaces were found to differ very essentially.

In the original transformation, $w^3 - 3w = z$, let $x = \frac{1}{\sigma}$, $y = \frac{\tau}{\sigma}$ and $u = \frac{1}{s}$, $v = \frac{t}{s}$, then the transformation becomes

$$\sigma = \frac{s^3}{1 - 3t^2 - 3s^2}$$

$$\tau = \frac{3t - t^3 - 3ts^2}{1 - 3t^2 - 3s^2}$$

Here as before most of the surface was not obtained directly from this transformation but each plane was found separately from the planes in the conformal transformation.

Between Planes III and IV (B), the transformation is $y = \frac{\tau}{\sigma}$ and $x = \frac{1}{\sigma}$. So the images of the points $(0,0)$, $(+2,0)$ and $(-2,0)$

in Plane III are the points $(\infty, 0)$, $(\frac{1}{2}, 0)$, and $(-\frac{1}{2}, 0)$ respectively in Plane IV (B). The x- axis ($y=0$) becomes the σ-axis, $\tau=0$ and the y-axis ($x=0$) becomes the line at infinity ($\infty=0$). The images of $x=\pm y$ are $\frac{1}{\sigma}=\pm\frac{\tau}{\sigma}$ or $\tau=\pm 1$. The images of the circles $x^2+y^2=r^2$ are the hyperbolae $\frac{1}{\sigma^2}+\frac{\tau^2}{\sigma^2}=r^2$. The images of these hyperbolae in Plane I (B) were not plotted as the transformation gave a very complicated curve.

The transformation between Planes II and I (B) is $u=\frac{1}{s}$ and $v=\frac{t}{s}$. Then the points $(0,0)$, $(\pm 1,0)$, $(\pm 2,0)$, $(\pm\sqrt{3},0)$, $(\pm\infty,0)$ in Plane II become the points $(\infty, 0)$ $(\pm 1, 0)$ $(\pm\frac{1}{2}, 0)$ $(\pm\frac{1}{3}\sqrt{3}, 0)$, and $(0,0)$ respectively in Plane I (B). The u- axis ($v=0$) becomes the s-axis ($t=0$), The v-axis ($u=0$) becomes the line at infinity ($s=\infty$) while the line at infinity ($u=\infty$) becomes the t-axis ($s=0$). The image of the hyperbola $u^2-\frac{v^2}{3}=1$ is the ellipse, $s^2+\frac{t^2}{3}=1$, and that of the hyperbola $\frac{u^2}{3}-v^2=1$ is the circle $\frac{s^2}{1/3}+\frac{t^2}{1/3}=1$. The images of the lines $\tau=\pm 1$, $\tau=\pm\frac{1}{2}$ and $\sigma=\pm\frac{1}{2}$ were obtained directly from the transformation between I (B) and IV (B), that is

$$\sigma=\frac{s^3}{1-3t^2-3s^2}$$

$$\tau=\frac{3t-t^3-3ts^2}{1-3t^2-3s^2}$$

From this transformation, $\tau=\frac{3t-t^3-3ts^2}{1-3t^2-3s^2}$ and if $\tau=+1$, then $s=\sqrt{\frac{3t-t^3+3t^2-1}{3(t-1)}}$. From this equation, the following values were obtained and by means of them, the curves were plotted.

t	0	.1	-1	.2	-2	+.3	-.3	-.5	-.8	1	1.5	2	3	3.5	4.
s	$\pm\frac{1}{3}\sqrt{3}$	±.5	±.6	±.34	±.62	imag.	±.64	±.6	±.4	0	4.5	1.7	1.1	.6	imag.

The image of $\tau = -1$ was plotted from that of $\tau = +1$ by symmetry.

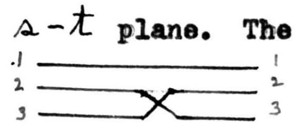
If $\tau = +.5$, then $s = \pm\sqrt{\dfrac{3t - t^3 + 1.5\,t^2 - .5}{3t - 1.5}}$. From this equation, the curve was plotted by means of the following values.

t	0	.1	.15	−.1	−1.	1	1.5	2	2.5	3	.5
s	$\pm\frac{1}{3}\sqrt{3}$	±.39	±.1	±.66	.47	±1.4	±1.1	±.8	.3	imag.	∞

The image of $\tau = -.5$ was plotted by symmetry. If $\sigma = +\frac{1}{2}$, $t = \pm\sqrt{\dfrac{1 - 3s^2 - 3s^3}{3}}$. From this equation, the following values were obtained and the curve plotted.

s	0	.2	.3	.4	.5	−.2	−.4	−.5	−.8	−1.	−1.1	−1.2	−1.5
t	$\pm\frac{1}{3}\sqrt{3}$	±.5	±.4	±.3	.0	±.5	±.46	±.4	±.3	0	±.1	±.21	±.5

The curve $\sigma = -\frac{1}{2}$ was plotted by symmetry. Then in Plane I (B), the portion inside the circle $\frac{s^2}{1/3} + \frac{t^2}{1/3} = 1$, (the purple circle), represents one sheet in the $\sigma - \tau$ plane because this circle represents the line at infinity in the $\sigma - \tau$ plane. The portion above the s-axis outside of the circle represents another sheet of the $\sigma - \tau$ plane while the portion below the s-axis represents the third sheet. If the cuts in the $\sigma - \tau$ plane are taken along the σ-axis from the points ± 2 to ∞, the first sheet joins to itself while the second and third sheets join to each other along each end of the s-axis, as can be easily seen by tracing a curve across the brown lines in the $s - t$ plane. The connections therefore are represented by



.

In this manner, the Riemann surface for the real trans-

formation,

$$\sigma = \frac{s^3}{1-3t^2-3s^2}$$

$$\tau = \frac{3t-t^3-3ts^2}{1-3t^2-3s^2}$$

was constructed. It is represented by Planes IV (B) and I (B). Plane I (B') is a representation of the central portion on Plane I (B) on a larger scale.

This surface differs decidedly from that of the original surface from which it was derived. The non-conformal character of the transformation is also clearly shown, and it is also interesting to note how one sheet of the σ-τ plane, a sheet infinite in extent, has been transformed into a circle of finite extent.

The remaining Riemann surfaces were constructed directly from the non-conformal transformations by means of certain properties the envelope and Jacobian as will be shown in Part II.

PART II.

RIEMANN SURFACES CONSTRUCTED DIRECTLY FROM THE NON-CONFORMAL TRANSFORMATIONS.

Consider a transformation such as $\sigma = f(s,t),\ \tau = g(s,t)$ by means of which for any point, M, in one plane there is a corresponding point, m, in the other plane, and when the point, M, describes a curve then the point, m, likewise describes a curve, and also let the first derivative $\frac{\partial \sigma}{\partial \tau}$ exist for all values of σ and τ. Then, <u>usually</u>, if two curves $C_1 \begin{cases} s = \varphi_1(u) \\ t = \psi_1(u) \end{cases}$ and $C_2 \begin{cases} s = \varphi_2(u) \\ t = \psi_2(u) \end{cases}$, which together with their first derivatives are continuous, are tangent in one plane, two other curves $c_1 \begin{cases} \sigma = f_1(s,t) \\ \tau = g_1(s,t) \end{cases}$ and $c_2 \begin{cases} \sigma = f_2(s,t) \\ \tau = g_2(s,t) \end{cases}$, corresponding to C_1 and C_2 respectively by means of the above transformation and fulfilling the same conditions as C_1 and C_2, will be tangent in the second plane. However, there are exceptions to this under certain circumstances.

Let $C \begin{cases} s = \varphi(u) \\ t = \psi(u) \end{cases}$ be a curve fulfilling the same conditions as the curves above and lying in the first plane and let $c \begin{cases} \sigma = f(s,t) = f[\varphi(u), \psi(u)] \\ \tau = g(s,t) = g[\varphi(u), \psi(u)] \end{cases}$ be the corresponding curve in the second plane. Here the transformation must be so constructed that both $\frac{\partial t}{\partial u}$ and $\frac{\partial s}{\partial u}$ never vanish at the same time. This can always be done by means of some parameter if necessary the length.

Then $\frac{\partial t}{\partial s} = \frac{\partial \psi(u)}{\partial u} \Big/ \frac{\partial \varphi(u)}{\partial u} = t_u / s_u$

and the contact property will in general be retained if

$$\frac{\partial T}{\partial \sigma} = \frac{g_s s_u + g_t t_u}{f_s s_u + f_t t_u} = \frac{g_s + g_t \frac{t_u}{g_u}}{f_s + f_t \frac{t_u}{s_u}} \quad \text{.........} (1)$$

for

$$\frac{\partial T}{\partial \sigma} = \frac{\frac{\partial T}{\partial s}}{\frac{\partial \sigma}{\partial s}} = \frac{\frac{\partial g}{\partial s} + \frac{\partial g}{\partial s} \cdot \frac{\partial t}{\partial s}}{\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial s}} =$$

$$\frac{\partial g}{\partial s} + \frac{\partial g}{\partial t} \frac{\frac{\partial t}{\partial u}}{\frac{\partial s}{\partial u}} \Bigg/ \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \frac{\frac{\partial t}{\partial u}}{\frac{\partial s}{\partial u}} = \frac{g_s s_u + g_t t_u}{f_s s_u + f_t t_u}$$

Hence it is seen that $\frac{\partial T}{\partial \sigma}$ depends only upon s, t and $\frac{\partial t}{\partial s}$, and therefore if the above transformation be applied to two curves C_1 and C_2, which are tangent in the first plane, the corresponding curves c_1 and c_2 will usually be tangent in the second plane since $\frac{\partial t}{\partial s}$ is the same for C_1 and C_2 at their common point.

But if the Jacobian is equal to zero that is if

$$J = \begin{vmatrix} f_s & f_t \\ g_s & g_t \end{vmatrix} = 0, \text{ then } \frac{f_s}{f_t} = \frac{g_t}{g_s} = k$$

Then from (1),

$$\frac{\partial T}{\partial \sigma} = \frac{g_t \left(K + \frac{t_u}{s_u}\right)}{f_t \left(K + \frac{t_u}{s_u}\right)} = \frac{g_t}{f_t},$$

which shows that the slope of c in plane II is independent of the slope of C in Plane I and depends only upon s and t; i. e. any curves through the same points have tangency in the new plane.

Therefore when the Jacobian is zero, the contact law breaks down. Of course the Jacobian is not usually equal to zero, so that this failure of the contact law only occurs along certain curves or at some certain points or point as is clearly shown in the examples which follow.

The only troublesome case appears when in the expression $\frac{\partial \tau}{\partial \sigma} = \frac{g_s s_u + g_t t_u}{f_s s_u + f_t t_u}$ both the numerator and the denominator vanish. This usually only happens at individual points on the envelope and these points may be considered in each case, as will be done in the following examples.

Ex. I. An example of this failure in the contact law is shown in the transformation illustrated by Planes V. and VI. and by Planes V' and VI'. Here the transformation is $\sigma = st,\ \tau = s^2 + t^2$
If the Jacobian is set equal to zero, there results

$$J = \begin{vmatrix} \frac{\partial \sigma}{\partial s}, & \frac{\partial \sigma}{\partial t} \\ \frac{\partial \tau}{\partial s}, & \frac{\partial \tau}{\partial t} \end{vmatrix} = \begin{vmatrix} t, & s \\ 2s, & 2t \end{vmatrix} = 2(t^2 - s^2) = 0$$

If $2(t^2 - s^2) = 0$, $t + s = 0$ and $t - s = 0$, which are the 45° lines in Planes V. and V'.

Corresponding to these lines $t = s$ and $t = -s$, there are in Planes VI and VI', the two lines $\sigma = \frac{\tau}{2}$ and $\sigma = -\frac{\tau}{2}$ which are themselves singular lines as is shown below.

If in the transformation $\begin{cases} \sigma = st \\ \tau = s^2 + t^2 \end{cases}$, we consider the family of lines $t = k$ where $k \neq 0$ and find the envelopes of the images of these lines in Plane VI, these envelopes are found to be the images of the lines which were found by setting

the Jacobian equal to zero, i. e. $t = s$ and $t = -s$ which cut the lines $t = k$ at an angle of $45°$. This is a striking example of the failure of the contact law for curves which are tangent in the $\sigma\tau$ plane have images which cut each other at an angle of $45°$ in the st plane. If $\sigma = st$, $\tau = s^2 + t^2$ corresponding to the family of lines $t = k$ where $k \neq 0$, there is the family of parabolae $\tau = \frac{\sigma^2}{k^2} + k^2$. To find the envelope of these parabolae, the derivative with regard to is set equal to zero. i. e.

$$2k\tau - 4k^3 = 0$$

or $k = 0$ or $k^2 = \tau/2$

Substituting these values in the equation of the parabolae, if

$k^2 = \tau/2$, $\tau^2/4 - \sigma^2 = 0$ or

$$\begin{cases} \sigma - \tau/2 = 0 & (E_1) \\ \sigma + \tau/2 = 0 & (E_2) \end{cases}$$

and if $k = 0$,

$$\sigma = 0 \qquad (E_2)$$

These three curves E_1, E_2, and E_3 are all possible envelopes.

Consider first (E_1). $\sigma - \tau/2 = 0$ and the equation of the parabolae, (P), $\tau = \sigma^2/k^2 + k^2$. Solving E_1 and C, the point of intersection of the two is found to be the point $\begin{cases} \tau = 2k^2 \\ \sigma = k^2 \end{cases}$. The slope of ($P$) is $\frac{\partial\tau}{\partial\sigma} = \frac{2\sigma}{k^2}$ which at the point of intersection is equal to 2. The slope of E_1 is $\frac{\partial\tau}{\partial\sigma}$ which is equal to 2 at all points. Therefore since the slopes are equal at the point of intersection, E_1 is an envelope of the

parabolae. In the same manner it can be proven that E_2 is an envelope of the parabolas. The curve $E_3\ (\sigma = 0)$ is not an envelope, for here the point of intersection is $\tau = k^2,\ \sigma = 0$, and the slope of P, $\partial\tau/\partial\sigma = 2\sigma/k^2$, at this point is zero while the slope of E_3 is always infinite. Therefore since the slopes are not equal at the point of intersection, E_3 is not an envelope.

In Plane V, the lines found by setting the Jacobian equal to zero are shown as the 45° lines. They divide Plane V into four parts each of which represents one sheet in Plane VI. The images of these lines form the envelopes E_1 and E_2 of the four sheets in Plane VI. Beyond these lines, values obtained through transformation do not exist, and as τ is the sum of squares no values exist below the σ-axis. The lines $\sigma = k$ become equilateral hyperbolae in the st plane while the lines $\tau = k$ become circles about the origin. The connections of the four sheets of the $\sigma - \tau$ plane can be found by tracing around one of these circles. Starting in Part I which represents Sheet I and crossing the red line we come into Part II, so at the red line Plane VI folds over and forms Sheet II. Passing on across a blue line we come into Part III, so at the blue line Plane VI folds back and forms Sheet III. Passing on across the red line again, we come into Part IV, so on the red line Plane VI folds back again and forms Sheet IV. Passing on across the blue line we come again into Part I, so at the blue line, Sheet IV joins on to Sheet I. In this way, the four sheets of Plane VI are all connected and if one could look at a cross section of this plane

it would look like this,  . Since all the sheets come to a single point at the origin, the $\sigma\tau$ plane, if opened out, would form a cone with its vertex at the origin, and any curve in the $\sigma\tau$ plane which crossed every sheet in the plane would pass entirely around the cone.

The same transformation $\sigma = st$, $\tau = s^2 + t^2$ is again represented by Planes V' and VI' . Here, however, the property that the images of the lines formed in Plane V' by setting the Jacobian equal to zero become the envelopes in Plane VI' is perhaps more clearly shown as the images of the lines $t=k$ and $s=k$ are found to be the parabolae $\tau = \frac{\sigma^2}{k^2} + k^2$. which have as envelopes the two lines E_1 and E_2 . It also shows these parabolae tangent to E_1 and E_2 while their images in the other plane intersect at an angle of $45°$. Plane VI'(B) is merely Plane VI' on a larger scale.

For the troublesome case where the numerator and denominator both vanish in the expression $\frac{\partial\tau}{\partial\sigma} = \frac{g_s s_u + g_t t_u}{f_s s_u + f_t t_u}$, in this transformation, $\frac{\partial\tau}{\partial\sigma} = \frac{2s}{t}$, and if both the numerator and denominator vanish, then $2s=0$ and $t=0$ which is the origin in the st plane. But this is also the origin in the $\sigma\tau$ plane and is the point where the two envelopes come together and where all four of the sheets in the $\sigma\tau$ plane are joined together.

Ex. II. Another failure of the contact law is even more forcibly illustrated in the transformation represented by Planes VII and VIII.

Here
$$\begin{cases} \sigma = t & \text{------------ (1)} \\ \tau = 2st + s^2 & \text{-------- (2)} \end{cases}$$

If the Jacobian is set equal to zero, we have

$$J = \begin{vmatrix} \frac{\partial \sigma}{\partial s}, & \frac{\partial \sigma}{\partial t} \\ \frac{\partial \tau}{\partial s}, & \frac{\partial \tau}{\partial t} \end{vmatrix} = \begin{vmatrix} 0, & 1 \\ 2t+2s, & 2s \end{vmatrix} = -2t - 2s = 0$$

If $-2t - 2s = 0$, then $s + t = 0$, which is the $-45°$ line. The image of this line in the $\sigma\tau$ plane is the parabola, $\tau = -\sigma^2$.

To find the envelope, from (1) and (2), we have

$$\tau = 2s\sigma + s^2 \text{ -------- (3)}$$

Then $\frac{\partial \tau}{\partial s} = 2\sigma + 2s$

Letting this equal zero, we have $2\sigma + 2s = 0$

or $s = -\sigma$

Substituting this value in (3),

$$\tau = -2\sigma^2 + \sigma^2$$

or $\tau = -\sigma^2$ which is the envelope in the $\sigma\tau$ plane.

In Plane VIII, the envelope is shown and as no value of σ and τ exist inside of this parabola that part of the figure has been cut away and the Riemann surface formed by joining the two sheets at this envelope. The images of the lines $s = k$, which are the straight lines $\tau = 2K\sigma + K^2$, are shown tangent to the envelope, and passing at the envelope from one sheet to the other. In the st plane, the correspponding lines are seen to cut each other at an angle of $45°$. The upper right-hand half of this plane represents one sheet in the $\sigma\tau$ plane while the

lower left-hand half represents the other sheet. The two sheets of the $\sigma\tau$ plane form a ruled surface for which the envelope is the edge of regression. This fact indicates the possibility of further study here since it is clear that any straight line envelope leads to an analogous proof in developable surfaces.

Ex. III. In Planes IX and X, there is another example of the image of the curve obtained by setting the Jacobian equal to zero becoming the envelope in the second plane. However, here both the curve and the envelope have degenerated into the points at the origins. Here the transformation is

$$\sigma = s^2 - t^2, \quad \tau = 2st$$

If the Jacobian vanishes,

$$J = \begin{vmatrix} \frac{\partial \sigma}{\partial s}, & \frac{\partial \sigma}{\partial t} \\ \frac{\partial \tau}{\partial s}, & \frac{\partial \tau}{\partial t} \end{vmatrix} = \begin{vmatrix} 2s, & -2t \\ 2t, & 2s \end{vmatrix} = 4(s^2 + t^2) = 0$$

Therefore $s^2 + t^2 = 0$, which is the origin. Of course, by this transformation the image of the origin is the origin.

To find the envelope, consider the family of lines $t = k$ and the corresponding family of curves $\begin{cases} \sigma = s^2 - k^2 \\ \tau = 2sk \end{cases}$ or $\sigma = \frac{\tau^2}{4k^2} - k^2$.

For these curves

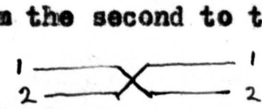
$$4k^4 - 4k^2\sigma - \tau^2 = 0 \quad \text{------------} \quad (1)$$

Taking the derivative with regard to k and setting it equal to zero,

$$16k^2 + 8k\sigma = 0$$

from which $k = 0$ or $k^2 = -\frac{\sigma}{2}$.

Substituting these values in (1) if $k = -\sigma/2$, we have $\sigma^2 + \tau^2 = 0$ which is the origin and if $k = 0$, $\tau = 0$. But $\tau = 0$ is not an envelope because its slope is always zero, and the slope of the curves is changing. Therefore in these planes also, the image of the curve obtained for the vanishing Jacobian is the envelope in the other plane.

The Riemann surface was then constructed by plotting a few corresponding curves. The images of the σ-axis ($\tau = 0$) are the s-axis ($t = 0$) and the t-axis ($s = 0$). The images of the τ-axis ($\sigma = 0$) are the lines $t = \pm s$, the 45° lines. The images of the 45° line ($\sigma = \tau$) are the lines $s = (1 \pm \sqrt{2})t$ and of the -45° line are the lines $s = (-1 \pm \sqrt{2})t$. The images of the lines $\tau = k$ are the hyperbolae $s^2 - t^2 = k$ while those of the lines $\sigma = k$ are the hyperbolae $s^2 - t^2 = k$. These are shown in the surface. The upper and lower half of the st plane represent respectively the first and second sheets of the $\sigma\tau$ plane as can be readily seen by taking values of any point in the upper half of the st plane and then taking the corresponding point in the lower half of it. They will give the same value in the $\sigma\tau$ plane. If the cut is made in the $\sigma\tau$ plane along the σ-axis from the origin to $+\infty$, the connection between the sheets, found by tracing the path of a closed curve about the origin in the st plane, is seen to be from the first to the second sheet and from the second to the first sheet as is indicated in the figure as 

Ex. IV. Turning back to the Planes I and IV, which were formed

indirectly by means of a transformation upon a Riemann surface for a complex variable, it was found that the points found by setting the Jacobian equal to zero were the points formerly denoted as singular points, and that the envelope was also singular points.

For in the transformation,

$$\sigma = 4s^3 - 3st^2 - 3s$$
$$\tau = 12s^2t - t^3 - 3t$$

if the Jacobian vanishes,

$$J = \begin{vmatrix} \frac{\partial\sigma}{\partial s}, & \frac{\partial\sigma}{\partial t} \\ \frac{\partial\tau}{\partial s}, & \frac{\partial\tau}{\partial t} \end{vmatrix} = \begin{vmatrix} 12s^2 - 3t^2 - 3, & -6st \\ 24st, & 12s^2 - 3t^2 - 3 \end{vmatrix} = 0$$

or $16s^4 + 8s^2t^2 - 8s^2 + 2t^2 + t^4 + 1 = 0$

In the solution of this equation, $t = 0$, and therefore $s = \pm \frac{1}{2}$ which are the singular points in the st plane.

(NOTE. In the above equation, that $t = 0$ may be shown by constructing the surface

$$z = 16s^4 + 8s^2t^2 - 8s^2 + 2t^2 + t^4 + 1$$

and examining it for maxima and minima. It will be found that the surface is always positive and that the points $(\pm \frac{1}{2}, 0)$ are minima.)

Or in an easier manner, these points can be found directly from the Jacobian for

$$J = (12s^2 - 3t^2 - 3)^2 + 144s^2t^2 = 0$$

and since this is the sum of two squares and equal to zero then

$$12s^2 - 3t^2 - 3 = 0$$

and $144s^2t^2 = 0$

From which if $t = 0$, $s = \pm \frac{1}{2}$.

The images of these singular points ($\pm \frac{1}{2}, 0$) are of course the points ($\pm 1, 0$), which are singular points in the $\sigma - \tau$ plane. To prove that these points are the envelopes in the $\sigma \tau$ plane gave in this transformation an equation which was too complicated to handle. Therefore it was proved in general that envelope in one plane is equivalent to the Jacobian vanishing in the other. The proof is given below.

To show that the envelope in one plane is equivalent to the vanishing of the Jacobian in the other plane.

Let the transformation be $\sigma = f(s,t)$, $\tau = \varphi(s,t)$.

Let $t = k$, then $\sigma = f(s,k)$, $\tau = \varphi(s,k)$

To find the envelope, s is eliminated, which gives some function of σ, τ and k, as $\mathcal{F}(\sigma, \tau, K$.

Let $\mathcal{F}(\sigma, \tau, K) = 0$.

Then does $\frac{\partial \mathcal{F}}{\partial K} = 0$?

But $\mathcal{F}(\sigma, \tau, K) = \mathcal{F}[f(s,k), \varphi(s,t), k] = 0$

and the total partial derivatives also equal zero.

$$\therefore \frac{\delta \mathcal{F}}{\delta K} = \frac{\partial \mathcal{F}}{\partial \sigma}\frac{\partial \sigma}{\partial K} + \frac{\partial \mathcal{F}}{\partial \tau}\frac{\partial \tau}{\partial K} + \frac{\partial \mathcal{F}}{\partial K} = 0 \quad \text{------} \quad (1)$$

also $$\frac{\delta \mathcal{F}}{\partial s} = \frac{\partial \mathcal{F}}{\partial \sigma}\frac{\partial \sigma}{\partial s} + \frac{\partial \mathcal{F}}{\partial \tau}\cdot\frac{\partial \tau}{\partial s} = 0 \quad \text{------------} \quad (2)$$

For an envelope, both $\mathcal{F}(\sigma, \tau, K)$ and $\frac{\partial \mathcal{F}}{\partial K}(\sigma, \tau, K)$ are set equal to zero, and k eliminated between them, which then gives a function, $$\Phi(\sigma, \tau) = 0.$$

But equations (1) and (2) are always true and on the envelope

$$\frac{\partial \mathcal{F}(\sigma, \tau, K)}{\partial K} = 0 \quad \text{------------} \quad (3)$$

Therefore substituting (3) in (1), on the envelope, it must be that

$$\frac{\partial \mathcal{F}}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial K} + \frac{\partial \mathcal{F}}{\partial \tau} \cdot \frac{\partial \tau}{\partial K} = 0 \quad \text{------------} \quad (4)$$

$$\frac{\partial \mathcal{F}}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial s} + \frac{\partial \mathcal{F}}{\partial \tau} \cdot \frac{\partial \tau}{\partial s} = 0 \quad \text{------------} \quad (5)$$

For equations (4) and (5) to be true, either $\frac{\partial \mathcal{F}}{\partial \sigma}$ and $\frac{\partial \mathcal{F}}{\partial \tau}$ are each equal to zero or else the determinant $\begin{vmatrix} \frac{\partial \sigma}{\partial K}, & \frac{\partial \tau}{\partial K} \\ \frac{\partial \sigma}{\partial s}, & \frac{\partial \tau}{\partial s} \end{vmatrix}$ which is the Jacobian of the transformation, must be equal to zero. But, in general, on the envelope, the derivatives $\frac{\partial \mathcal{F}}{\partial \sigma}$ and $\frac{\partial \mathcal{F}}{\partial \tau}$ cannot both be equal to zero. Therefore the Jacobian must be equal to zero. Therefore, the envelope in one plane is equivalent to the vanishing of the Jacobian in the other plane.

Ex. V. In a similar manner, in Planes I(B) and IV(B) the points found by setting the Jacobian equal to zero were also found to be the singular points.

Here the transformation is

$$\sigma = \frac{s^3}{1 - 3t^2 - 3s^2}$$

$$\tau = \frac{3t - t^3 - 3ts^2}{1 - 3t^2 - 3s^2}$$

and if the Jacobian vanishes

$$J = \begin{vmatrix} \frac{\partial \sigma}{\partial s}, & \frac{\partial \sigma}{\partial t} \\ \frac{\partial \tau}{\partial s}, & \frac{\partial \tau}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{3s^2 - 9s^2t^2 - 3s^4}{(1 - 3t^2 - 3s^2)^2}, & \frac{6s^3t}{(1 - 3t^2 - 3s^2)^2} \\ \frac{12st + 12st^3}{(1 - 3t^2 - 3s^2)^2}, & \frac{3 + 6t^2 - 12s^2 + 3t^4 + 9s^4}{(1 - 3t^2 - 3s^2)^2} \end{vmatrix} = 0$$

This equation, if $t = 0$, which is allowable for the same reason as in the previous example, becomes

$$1 - 5s^2 + 7s^4 - 3s^6 = 0$$

From which $s = \pm 1$, $s = \pm 1$ or $s = \pm\sqrt{\frac{1}{3}}$

The points $(\pm 1, 0)$ are the singular points already found while the points $(\pm\sqrt{\frac{1}{3}}, 0)$ are points where the transformation itself becomes indeterminate.

Ex. VI. The last Riemann surface constructed was that represented by Planes XI and XII. Here the ordinary transformation for polar coordinates was used, i.e.

$$\sigma = \rho \cos \phi,$$
$$\tau = \rho \sin \phi.$$

If the Jacobian is set equal to zero,

$$J = \begin{vmatrix} \frac{\partial\sigma}{\partial\rho}, & \frac{\partial\sigma}{\partial\phi} \\ \frac{\partial\tau}{\partial\rho}, & \frac{\partial\tau}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \cos\phi, & -\rho\sin\phi \\ \sin\phi, & \rho\cos\phi \end{vmatrix} = \rho\left[\cos^2\phi + \sin^2\phi\right] = 0$$

from which, since $\cos^2\phi + \sin^2\phi = 1$, $\rho = 0$. Therefore $\rho = 0$ is the singular line found by letting the Jacobian vanish. The image of this line in the $\sigma\tau$ plane is a single point, the origin.

To find the envelope, let $\rho = k$, then $\begin{cases} \sigma = k\cos\phi \\ \tau = k\sin\phi \end{cases}$

From these $\sin\phi = k/\tau$, and since $\cos\phi = \sqrt{1-\sin^2\phi}$

$$\sigma = k\sqrt{1 - \tau^2/k^2} \quad \text{or} \quad \sigma^2 = k^2 - \tau^2$$, which is the equation of circles about the origin as a center and the images of the lines $\rho = k$.

Taking the derivative with regard to k of this equation and setting it equal to zero, we have

$$2k^2 = 0 \quad \text{or} \quad k = 0$$

If $k = 0$, $\sigma^2 + \tau^2 = 0$ which is the originTherefore in this surface also the envelope is the image of the curve found by setting the Jacobian equal to zero, just as it was known it would be because of the general proof previously given.

In Planes XI and XII, only two sheets out of an infinite number are shown. For since the values of σ and τ are repeated every time ρ increases by 2π, a strip from $\rho = -\pi$ to $\rho = +\pi$ represents two sheets of the $\sigma\tau$ plane, one sheet being represented by that portion with negative values of ρ and the other sheet by that with positive values of ρ. Strips of width 2π above and below that shown in Plane XI would represent other pairs of sheets in the $\sigma\tau$ plane, and it is easily seen that such strips and their corresponding sheets are infinite in number.

In Plane XII, the parabolae $\sigma = \pm\tau^2$ and $\tau = \pm\sigma^2$ are shown . Their images are the curves $\rho = \pm\frac{\cos\phi}{\sin^2\phi}$ and $\rho = \pm\frac{\sin\phi}{\cos^2\phi}$ respectively and are shown in Plane XI. These parabolae, which are tangent to the origin, have images which pass through the line which is the image of the origin at an angle of $45°$. This is another example of the failure of the contact law.

It is also interesting to notice the little portions about the origin which are enclosed by two parabolae. In the

transformation, these become small triangular pieces which have as their bases the line which is the image of the origin. It is also clearly shown by Plane XI, that the only connection between the two sheets is at the origin, and that if the cut in the first sheet is along the σ axis from the origin to $-\infty$ then in the second sheet it is along the same axis from the origin to $+\infty$

Amechanical conception of the manner in which the transformation occurs may be formed as follows. Imagine the two sheets of the $\sigma\tau$ plane, which are joined together at the origin, gradually pulled apart until they become two cones with only their verticew touching. Now imagine these cones surrounded by an infinite cylinder, then if every point on the cones is projected perpendicularly upon this cylinder and the cylinder cut along the element correaponding to the two cut lines, when this cylinder was flattened out it would be exactly the portion of the plane represented in Plane XI. The relation between this failure in the contact law when the Jacobian is zero and the ordinary theory of singular points in complex variables is very easily seen. For if the Jacobian is equal to zero then

$$J = \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 0$$

But in complex variables, the Cauchy--Riemann equations must also be satisfied

i.e. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial v}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \quad \text{or} \quad \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

and since these equations are the sums of squares and also equal to zero, it is necessary in order to satisfy the former that $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$ and in order to satisfy the latter that $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$. It being necessary to satisfy two equations instead of one, points instead of curves will be found by setting the Jacobian equal to zero, and these will be the so-called singular points found in the usual manner, for in finding these singular points, the derivative $\frac{\partial w}{\partial z}$ is set equal to zero. i.e. $\frac{\partial w}{\partial z} = \frac{du}{\partial x} + \frac{\partial u}{\partial y} i = 0$

or

$$\frac{\partial w}{\partial z} = \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} i = 0$$

and equating real and imaginary parts, we have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

or else $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$

which are the same equations as those found before.

The Cauchy-Riemann equations may be considered as special cases of the equations

$$\frac{\partial v}{\partial y} = A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} \quad \text{-------------} \quad (1)$$

$$\frac{\partial v}{\partial x} = C \frac{\partial u}{\partial x} + D \frac{\partial u}{\partial y} \quad \text{-------------} \quad (2)$$

where $A = 1$, $B = 0$, $C = 0$, and $D = -1$.

If the above equations are taken instead of the Cauchy-Riemann equations, the Jacobian becomes

$$J = \frac{\partial u}{\partial x}\left(A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y}\left(C \frac{\partial u}{\partial x} + D \frac{\partial u}{\partial y}\right) = 0$$

or $A\left(\frac{\partial u}{\partial x}\right)^2 + (B - C)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - D \frac{\partial u}{\partial y} = 0$ ---- (3)

A

This equation will have imaginary roots only when $(B-C)^2 + 4AD < 0$ that is when $-AD > (B-C)^2 \geqq 0$ and in that case, when the Jacobian is set equal to zero, singular points will be found, as is shown by Example IV where

$$J = \begin{vmatrix} \frac{\partial\sigma}{\partial s}, & \frac{\partial\sigma}{\partial t} \\ \frac{\partial\tau}{\partial s}, & \frac{\partial\tau}{\partial t} \end{vmatrix} = \begin{vmatrix} 12s^2 - 3t^2 - 3, & -6st \\ 24st, & 12s^2 - 3t^2 - 3 \end{vmatrix}.$$

Here $\frac{\partial\tau}{\partial t} = \frac{\partial\sigma}{\partial s}$ and $\frac{\partial\tau}{\partial s} = -4\frac{\partial\sigma}{\partial t}$ and if equations (1) and(2) are to be satisfied $A = 0$, $B = 0$, $C = 0$ and $D = -4$, hence $(B-C)^2 + 4AD = -4$ which is less than zero.

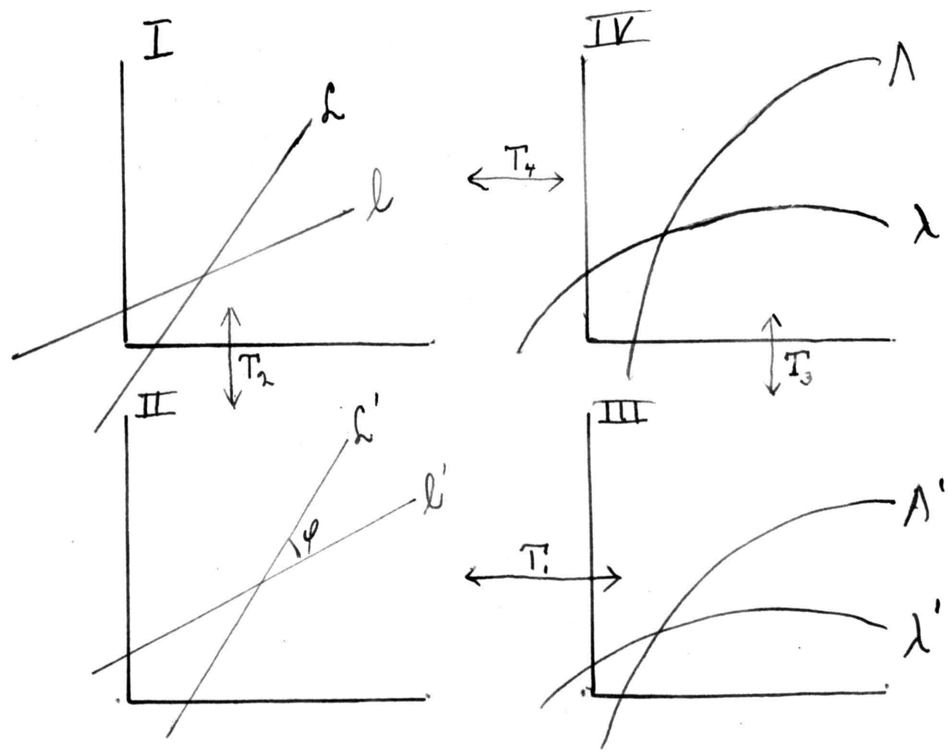
The equation (3) will have equal and real roots if $(B-C)^2 + 4AD = 0$ or $(B-C)^2 = -4AD$ and this is the case when the vanishing Jacobian gives the same curve twice. It will have real and unequal roots when $(B-C)^2 + 4AD > 0$ which is the case when the Jacobian represents two different curves. It would now be possible to study transformations which satisfy such linear differential equations as those given in (1) and (2) and doubtless some very interesting theorems and facts would be discovered.

As described in the preceding paragraphs, some of the properties of non-conformal transformations have been studied and several Riemann surfaces of such transformations have been constructed. There now remains one curious property of some such transformations which will be explained in Part III.

PART III.

A PSEUDO-CONFORMAL PROPERTY OF SOME NON-CONFORMAL TRANSFORMATIONS.

Suppose we start with a conformal transformation and draw the two planes. If we now transform these two planes by means of any fixed transformation, then the transformation between the last two planes will be non-conformal but nevertheless it can be proven that there is one peculiar concept which we have called the alpha angle (written Angle or $\measuredangle$) which remains unchanged. Hence with regard to this concept such a non-conformal transformation may be said to be pseudo-conformal.



Consider four planes as shown above and let the transformation, T_1, between Planes II and III be conformal but let the transformations between Planes I and III, T_2, and between Planes III and IV, T_3, be non-conformal, we will show that the transformation between Planes I and IV, T_4, is nonconformal but that the peculiar angle is unchanged.

In Plane II, let the equations for $\mathcal{L}'$ and l be respectively

$$v = \frac{a}{2}u + b \text{ .... } (\mathcal{L}')$$
$$v = \frac{a'}{2}u + b' \text{ .... } (l')$$

Then the tangent of the angle between $\mathcal{L}'$ and l is tan $\frac{2a-2a'}{4+aa'}$.

If the equations of the lines $\mathcal{L}$ and l' in Plane I are respectively

$$y = ax + b \text{ .... } (\mathcal{L})$$
$$y = a'x + b' \text{ .... } (l)$$

Then we shall define the alpha angle (Angle) between $\mathcal{L}$ and l to be the same as the angle φ between $\mathcal{L}'$ and l', i.e.

$$\tan\left[\text{Angle}(\mathcal{L}, l)\right] = \frac{2a-2a'}{4+aa'}$$

If now in Plane II, we replace the straight lines $\mathcal{L}'$ and l' by the curves C' and c' respectively where C' and c' have the equations

$$v = f(u) \text{ .... } (C')$$
$$v = \varphi(u) \text{ .... } (c')$$

then

$$\tan\left[\text{angle}(C', c)\right] = \frac{f'(u) - \varphi'(u)}{1 + f'(u)\varphi'(u)}$$

and the curves C' and c' become in Plane I the two curves C and c with the equations

$$y = f(2x) \text{ - - - - - - - - - - } (C)$$
$$y = \varphi(2x) \text{ - - - - - - - - - - } (c)$$

Then the tangent, T, to C is

$$y - y_0 = 2f'(2x)(x - x_0) \text{ - - - - - } (T)$$

and the tangent, t, to c is

$$y - y_0 = 2\varphi'(2x)(x - x_0) \text{ - - - - - - - - } (t)$$

and from the definition of Angle,

$$\tan \measuredangle(T,t) = \tan\left[\text{Angle}(T,t)\right] = \frac{f'(2x) - \varphi'(2x)}{1 + f'(2x)\,\varphi'(2x)}$$

which is the same as the tangent of the angle between C' and c.

$$\therefore \tan\left[\text{Angle}(T,t)\right] = \tan\left[\text{angle}(C',c')\right] \text{ - - - - - - - - - - - } (1)$$

If, therefore, we define the Angle between two curves as the Angle between their tangents at their point of intersection, then

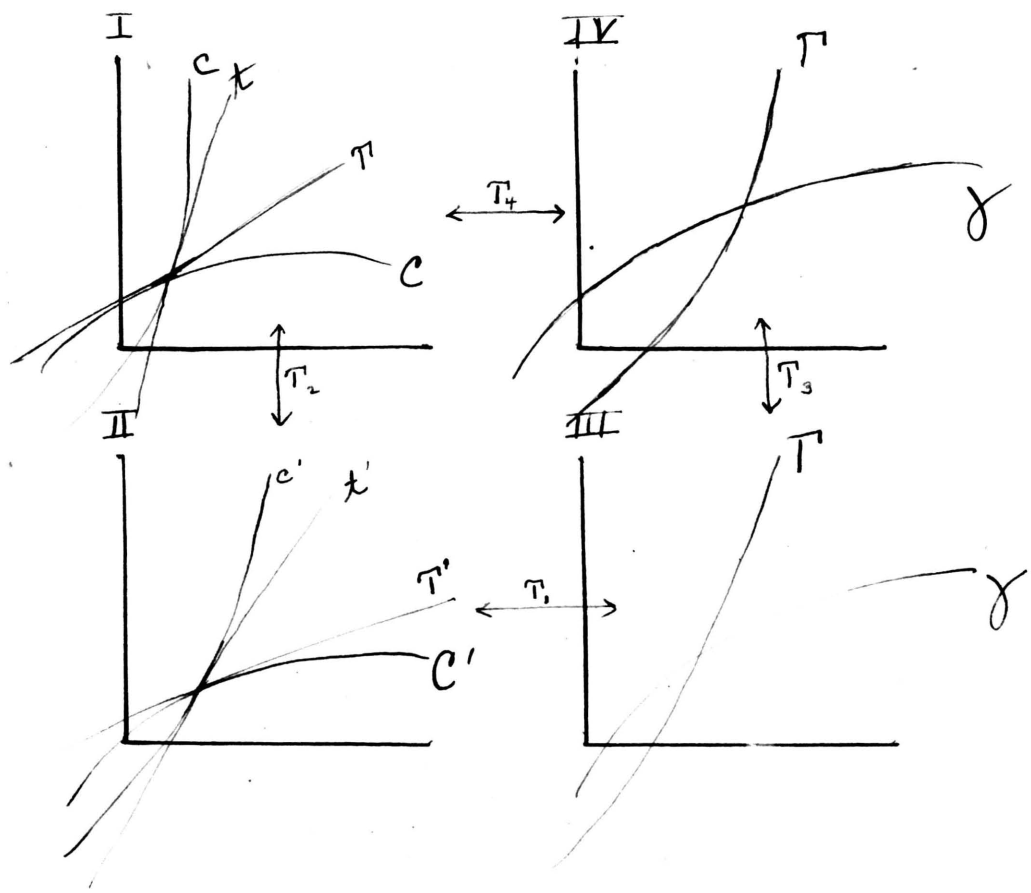
$$\tan\left[\text{Angle}(C,c)\right] = \tan\left[\text{Angle}(T,t)\right] \text{ - - - - - - - - - - - } (2)$$

and from (1) and (2)

$$\tan\left[\text{Angle}(C,c)\right] = \tan\left[\text{angle}(C',c')\right]$$

Therefore the original definition of the Angle can be replaced by

$$\tan \measuredangle(C,c) = \tan \measuredangle(C',c')$$



The transformation, T_1, between Plane II and III is conformal, i. e.

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial t} \quad \text{and} \quad \frac{\partial u}{\partial t} = -\frac{\partial v}{\partial s}$$

If this transformation, T_1, between Plane II and III is $s = s(u,v)$ and $t = t(u,v)$, then the two curves C' and c' in Plane II will become two curves Γ' and γ' in Plane III; for if

$$v = f(u) \ldots\ldots\ldots C'$$

$$v = \varphi(u) \ldots\ldots\ldots c'$$

then by means of

$$\left.\begin{aligned} s &= s[u, f(u)] \\ t &= t[u, f(u)] \end{aligned}\right\} \ldots\ldots \Gamma'$$

$$\left.\begin{aligned} s &= s[u, \varphi(u)] \\ t &= t[u, \varphi(u)] \end{aligned}\right\} \ldots\ldots \gamma'$$

We shall now show that

$$\left.\frac{dt}{ds}\right]_{\Gamma'} = \left.\frac{\frac{dt}{du}}{\frac{ds}{du}}\right]_{\Gamma'} = \frac{\left.\frac{\partial t}{\partial u} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial u}\right]_{C'}}{\left.\frac{\partial s}{\partial u} + \frac{\partial s}{\partial v}\cdot\frac{\partial v}{\partial u}\right]_{C'}} = \frac{-s_v + s_u f'(u)}{s_u + s_v f'(v)}$$

$$\left.\frac{dt}{ds}\right]_{\gamma'} = \left.\frac{\frac{dt}{du}}{\frac{ds}{du}}\right]_{\gamma'} = \frac{\left.\frac{\partial t}{\partial u} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial u}\right]_{C'}}{\left.\frac{\partial s}{\partial u} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial u}\right]_{C'}} = \frac{-s_v + s_u \varphi'(u)}{s_u + s_v \varphi'(u)}$$

Then

$$\tan \measuredangle (\Gamma', \gamma') = \frac{\left.\frac{dt}{ds}\right]_{\Gamma'} - \left.\frac{dt}{ds}\right]_{\gamma'}}{1 + \left.\frac{dt}{ds}\right]_{\Gamma'}\left.\frac{dt}{ds}\right]_{\gamma'}} =$$

$$\frac{[-s_v + s_u f'(u)][s_u + s_v\varphi'(u)] - [s_u + s_v f'(u)][-s_v + s_u \varphi'(u)]}{[s_u + s_v f'(v)][s_u + s_v\varphi'(u)] + [-s_v + s_u f'(u)][-s_u + s_v f'(u)]} =$$

$$\frac{(s_u^2 + s_v^2)[f'(u) - \varphi'(u)]}{(s_u^2 + s_v^2)[1 + f'(u)\varphi'(u)]} = \frac{f'(u) - \varphi'(u)}{1 + f'(u)\varphi'(u)} = \tan \measuredangle (C', c')$$

$\therefore \tan \measuredangle (\Gamma', \gamma') = \tan \measuredangle (C', c')$.

By means of the transformation, T_3, between Planes III and IV, the two curves Γ' and γ' and will become two curves Γ and γ respectively, and if the equations of Γ' and γ' are

$$\left.\begin{array}{l} s = s[u, f(u)] \\ t = t[u, f(u)] \end{array}\right\} \Gamma'$$

$$\left.\begin{array}{l} s = s[u, \varphi(u)] \\ t = t[u, \varphi(u)] \end{array}\right\} \gamma'$$

the corresponding curves will have the equations

$$\left.\begin{array}{l} \sigma = \frac{1}{2} s[u, f(u)] \\ \tau = t[u, f(u)] \end{array}\right\} \Gamma$$

$$\left.\begin{array}{l} \sigma = \frac{1}{2} s[u, \varphi(u)] \\ \tau = t[u, \varphi(u)] \end{array}\right\} \gamma$$

We wish to show that tan $\measuredangle(\Gamma, \gamma) =$ tan $\measuredangle(C, c)$

We have already proved that

tan $\measuredangle(C, c) =$ tan $\measuredangle(C', c')$ - - - - - - - - - - - - - - (1)

and that

tan $\measuredangle(C, c) =$ tan $\measuredangle(\Gamma', \gamma')$ - - - - - - - - - - - - - - (2)

So now it is only necessary to prove that

tan $\measuredangle(\Gamma, \gamma) =$ tan $\measuredangle(\Gamma', \gamma')$

This proof follows.

$$\left.\frac{d\tau}{d\sigma}\right]_{\Gamma} = \left.\frac{\frac{d\tau}{du}}{\frac{d\sigma}{du}}\right]_{\Gamma} = \left.\frac{\frac{dt}{du}}{\frac{1}{2}\frac{ds}{du}}\right]_{\Gamma'} = \left.2\frac{dt}{ds}\right]_{\Gamma'}.$$

$$\left.\frac{d\tau}{d\sigma}\right]_{\gamma} = \left.\frac{\frac{d\tau}{du}}{\frac{d\sigma}{du}}\right]_{\gamma} = \left.\frac{\frac{dt}{du}}{\frac{1}{2}\frac{ds}{du}}\right]_{\gamma'} = \left.2\frac{dt}{ds}\right]_{\gamma'}$$

But, by definition,

$$\tan \measuredangle [\Gamma, \gamma] = \frac{2\frac{d\tau}{d\sigma}\Big]_{\Gamma} - 2\frac{d\tau}{d\sigma}\Big]_{\gamma}}{4 + \left(\frac{d\tau}{d\sigma}\right)_{\Gamma}\left(\frac{d\tau}{d\sigma}\right)_{\gamma}} = \frac{\frac{dt}{ds}\Big]_{\Gamma'} - \frac{dt}{ds}\Big]_{\gamma'}}{1 + \left(\frac{dt}{ds}\right)_{\Gamma'}\left(\frac{dt}{ds}\right)_{\gamma'}} = \tan \measuredangle (\Gamma', \gamma')$$

$$\tan \measuredangle [\Gamma, \gamma] = \tan \measuredangle [\Gamma', \gamma'] \text{ . . . . } (3)$$

Therefore from equations (1), (2) and (3) above

$$\tan \measuredangle [\Gamma', \gamma] = \tan > [C, c]$$

which shows that transformation T_4 is non-conformal but that the Angle is unchanged.

It is now possible to build up whole groups of transformations at will exactly as the one just studied was constructed, and to study the propertied of these in an exactly analogous manner. In fact any fixed non-conformal transformation defines an assemblage of transformations which is in a one-to-one assignment to the conformal transformation and the new transformations will not be conformal if the fixed transformation is not so.

In particular, let the fixed transformation be a linear one. This will give an enormous extension of non-conformal transformations from a conformal one. Suppose fixed transformation is

$$u = \frac{a'x + b'y + c'}{ax + by + c}$$

$$v = \frac{a''x + b''y + c''}{ax + by + c}$$

This transformation will change a line into a line but the angles will not be preserved. Many consequences which come about

through such a transformation can be easily seen, In this case, the concept, alpha angle or Angle, has been studied already. This concept is really very important as it is fundamental in the study of non-euclidean geometry and study of it really means a study of pseudo-conformal transformations in non-euclidean space.

The ordinary theorems of complex variables can be immediately translated into theorems in this study without exception and every equation or formula of complex variables becomes a formula here.

For example a pair of equations analogous to the Cauchy-Riemann equations,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$$
$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$$

can be found as follows. Suppose the fixed transformation is the linear one given above i.e.

$$u = \frac{a'x + b'y + c'}{ax + by + c}$$
$$v = \frac{a''x + b''y + c''}{ax + by + c}$$

Then the non-conformal transformation will have the following form,

$$p = \frac{a's + b't + c'}{as + bt + c}$$
$$q = \frac{a''s + b''t + c''}{as + bt + c}$$

We shall consider the special case where $a = b = 0$ and $c = 1$

Then

$$u = a'x + b'y + c' \text{ ....} \quad (1)$$
$$v = a''x + b''y + c'' \text{ ....} \quad (2)$$

and $$p = a's + b't + c' \quad \text{.............(3)}$$

$$q = a''s + b''t + c'' \quad \text{.............(4)}$$

where $s = f(x,y)$ and $t = \varphi(x,y)$.

and from these we want to get a pair of equations between $\frac{\partial p}{\partial u}$, $\frac{\partial p}{\partial v}$, $\frac{\partial q}{\partial u}$ and $\frac{\partial q}{\partial v}$ which will be analogous to the Cauchy-Riemann equations.

From (3) and (4)

$$\frac{\partial p}{\partial x} = a'\frac{\partial s}{\partial x} + b'\frac{\partial t}{\partial x}$$

$$\frac{\partial p}{\partial y} = a'\frac{\partial s}{\partial y} + b'\frac{\partial t}{\partial y}$$

$$\frac{\partial q}{\partial x} = a''\frac{\partial s}{\partial x} + b''\frac{\partial t}{\partial x}$$

$$\frac{\partial q}{\partial y} = a''\frac{\partial s}{\partial y} + b''\frac{\partial t}{\partial y}$$

From (1) and (2)

$$x = \frac{b''u - b'v - b''c' + b'c''}{a'b'' - a''b'}$$

Then $$y = \frac{a''u - a'v - a''c' - a'c''}{a''b' - a'b''}$$

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial u} = \left[a'\frac{\partial s}{\partial x} + b'\frac{\partial t}{\partial x}\right]\frac{b''}{a'b''-a''b'} + \left[a'\frac{\partial s}{\partial y} + b'\frac{\partial t}{\partial y}\right]\frac{a''}{a''b'-a'b''}$$

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial v} = \left[a'\frac{\partial s}{\partial x} + b'\frac{\partial t}{\partial x}\right]\frac{-b'}{a'b''-a''b'} + \left[a'\frac{\partial s}{\partial y} + b'\frac{\partial t}{\partial y}\right]\frac{-a'}{a''b'-a'b''}$$

$$\frac{\partial q}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial u} = \left[a''\frac{\partial s}{\partial x} + b''\frac{\partial t}{\partial x}\right]\frac{b''}{a'b''-a''b'} + \left[a''\frac{\partial s}{\partial y} + b''\frac{\partial t}{\partial y}\right]\frac{a''}{a''b'-a'b''}$$

$$\frac{\partial q}{\partial v} = \frac{\partial q}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial v} = \left[a''\frac{\partial s}{\partial x} + b''\frac{\partial t}{\partial x}\right]\frac{-b'}{a'b''-a''b'} + \left[a''\frac{\partial s}{\partial y} + b''\frac{\partial t}{\partial y}\right]\frac{-a'}{a''b'-a'b''}$$

But on account of the Cauchy-Riemann equations these equations become

$$\frac{\partial p}{\partial u} = \frac{\partial s}{\partial x} + \left[\frac{-a'a'' - b'b''}{a'b'' - a''b'}\right]\frac{\partial s}{\partial y}$$

$$\frac{\partial p}{\partial u} = 0 + \left[\frac{a'^2 + b'^2}{a'b'' - a''b'}\right]\frac{\partial s}{\partial y}$$

$$\frac{\partial q}{\partial u} = 0 + \left[\frac{-a''^2 - b''^2}{a'b'' - a''b'}\right]\frac{\partial s}{\partial y}$$

$$\frac{\partial q}{\partial v} = \frac{\partial s}{\partial x} + \left[\frac{a'a'' + b'b''}{a'b'' - a''b'}\right]\frac{\partial s}{\partial y}$$

Eliminating $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ between these four equations, we have

$$\frac{\partial q}{\partial v} = \frac{\partial p}{\partial u} + 2\left[\frac{a'a'' + b'b''}{a'^2 + b'^2}\right]\frac{\partial p}{\partial v}$$

$$\frac{\partial q}{\partial u} = -\left[\frac{a''^2 + b''^2}{a'^2 + b'^2}\right]\frac{\partial p}{\partial v}$$

For this particular case of a non-conformal transformation, the two equations just found are analogous to the generalized form of the Cauchy-Riemann equations given on page 31.

Here $A = 1$, $B = 2\left[\frac{a'a'' + b'b''}{a'^2 + b'^2}\right]$, $C = 0$, $D = -\left[\frac{a''^2 + b''^2}{a'^2 + b'^2}\right]$.

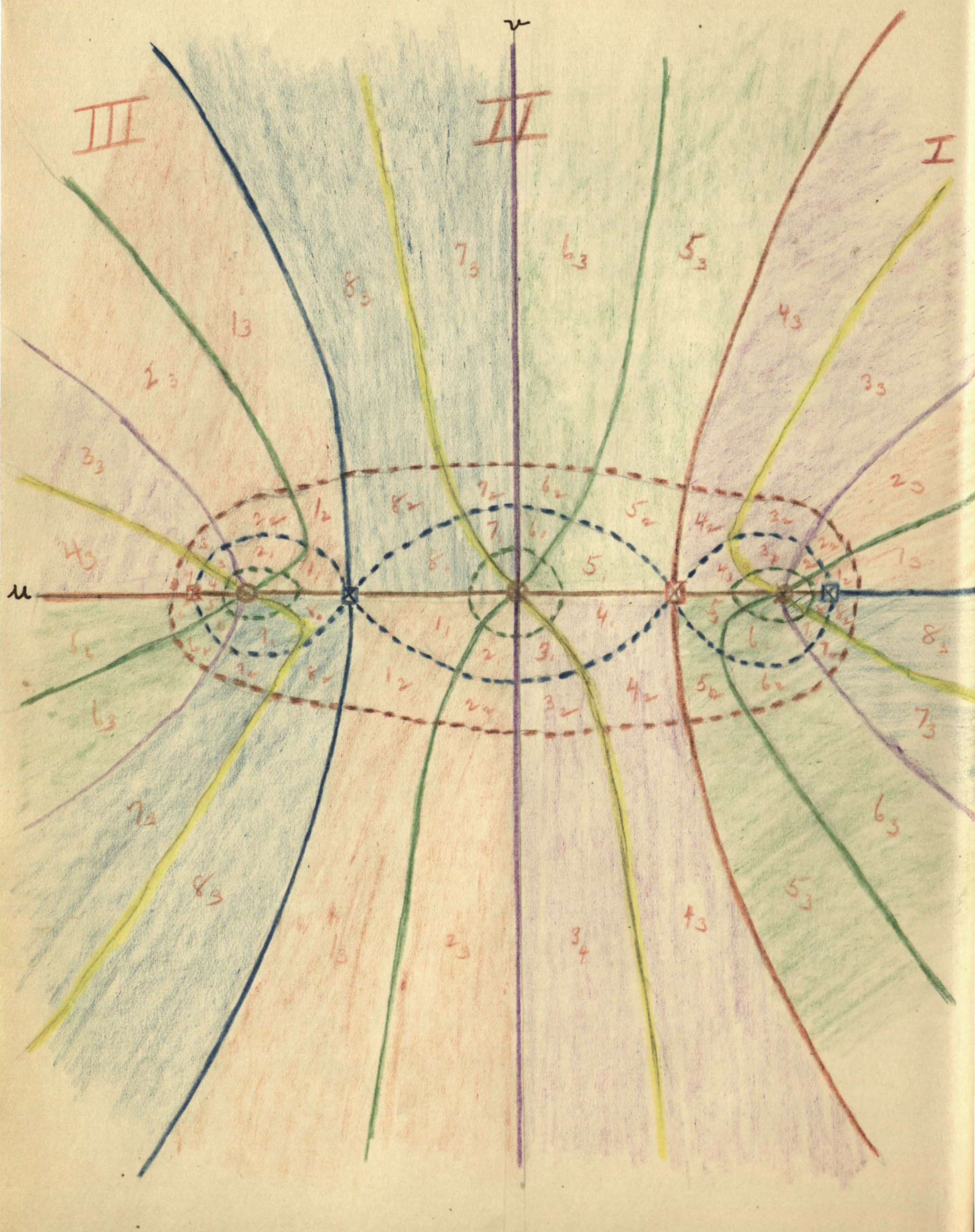
Now just as before if $(B - C)^2 + 4AD < 0$ when the Jacobian is set equal to zero singular points will be found, then since in this particular case $(B - C)^2 + 4AD = 4\left[\frac{a'a'' + b'b''}{a'^2 + b'^2}\right]^2 - 4\left[\frac{a''^2 + b''^2}{a'^2 + b'^2}\right] =$ $-4(a'b'' - a''b')^2$ and this quantity is always less than zero, in any transformation fulfilling the conditions of

this particular case, singular points and not singular lines would be found.

Thus the generalization previously indicated leads to very interesting problems which were perhaps not evident at that time. In a similar manner, every theorem and formula of complex variables could be translated into a theorem or formula for non-conformal transformations and such a study would doubtless reveal many interesting properties of these transformations.

Plane II

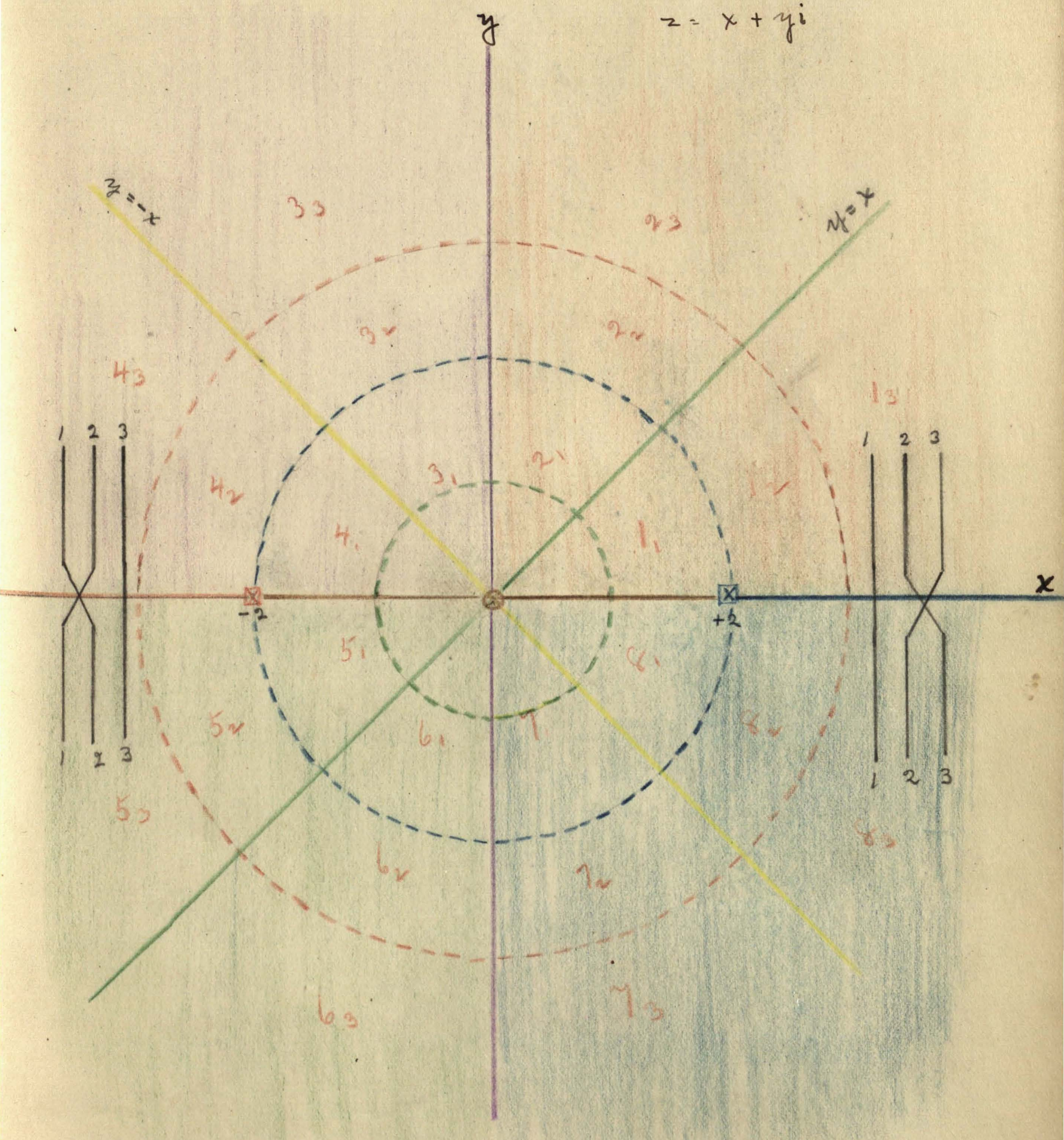
w-plane. 1 sheet.



Plane III.

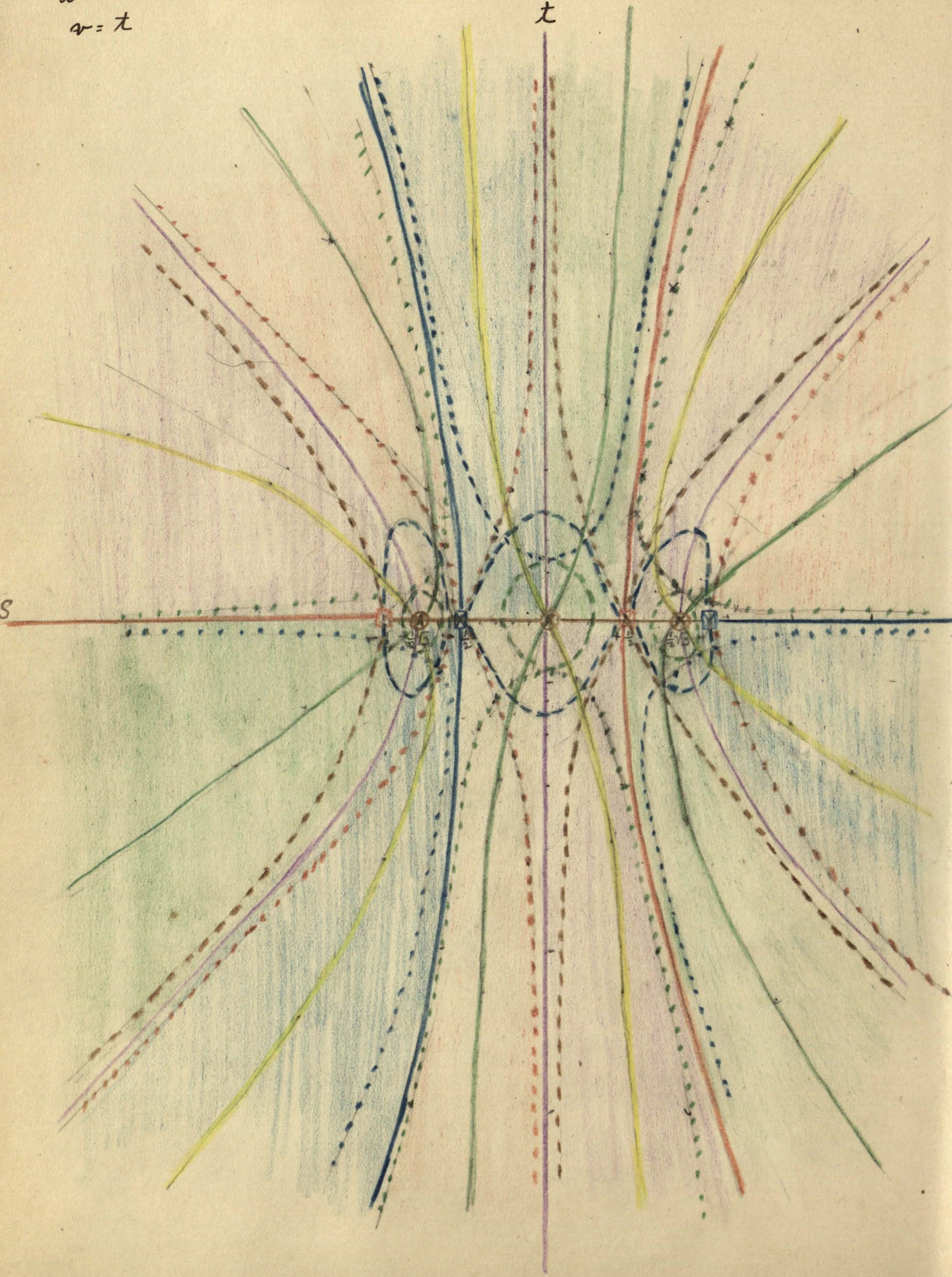
z-plane. 3 sheets.

$w^3 - 3w = z$
$w = u + vi$
$z = x + yi$



Plane I.

$u = 2s$
$v = t$

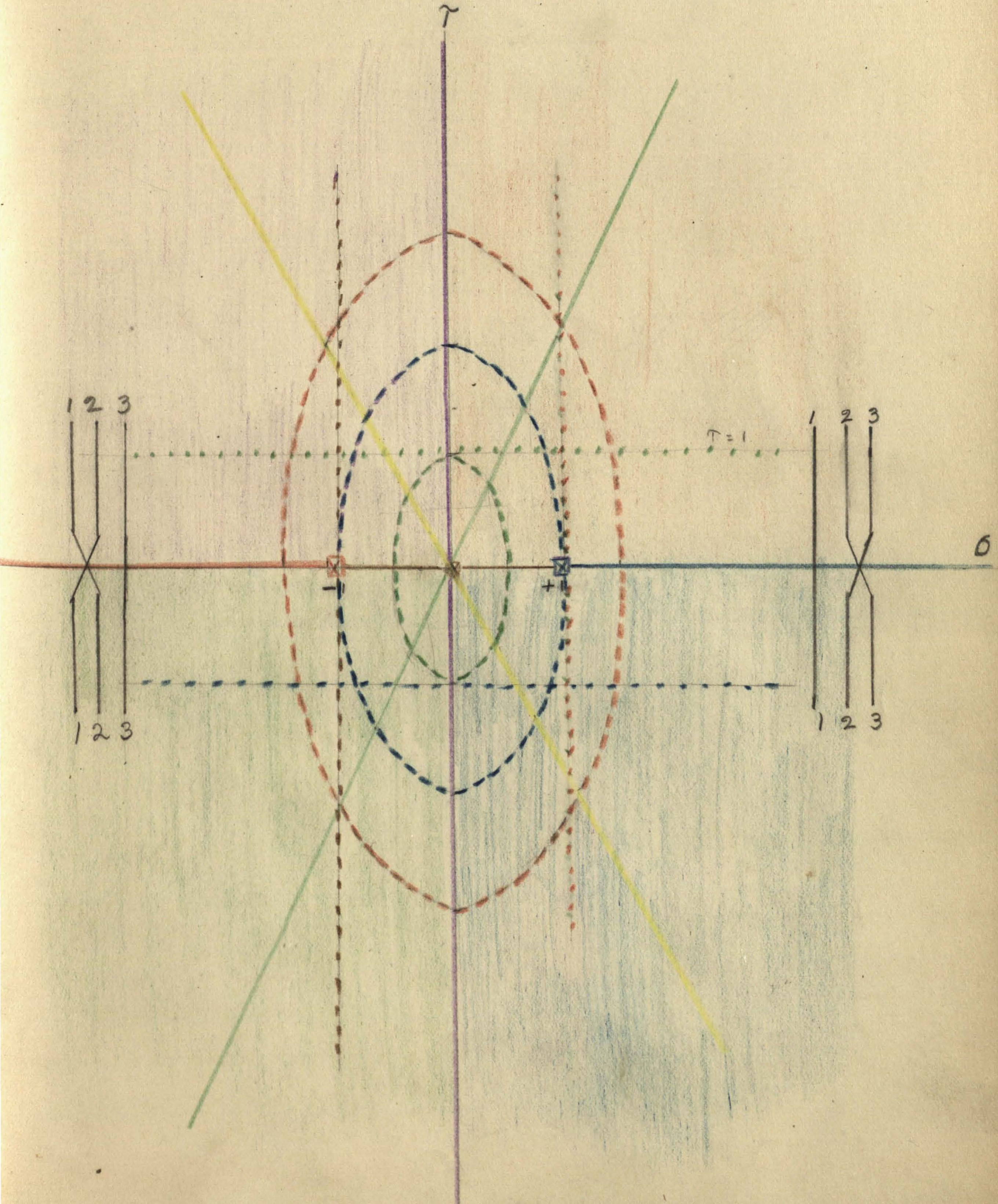


Plane IV.

$$4s^3 - 3st^2 - 3s = \sigma$$
$$12s^2t - t^3 - 3t = \tau$$

$x = 2\sigma$

$y = \tau$



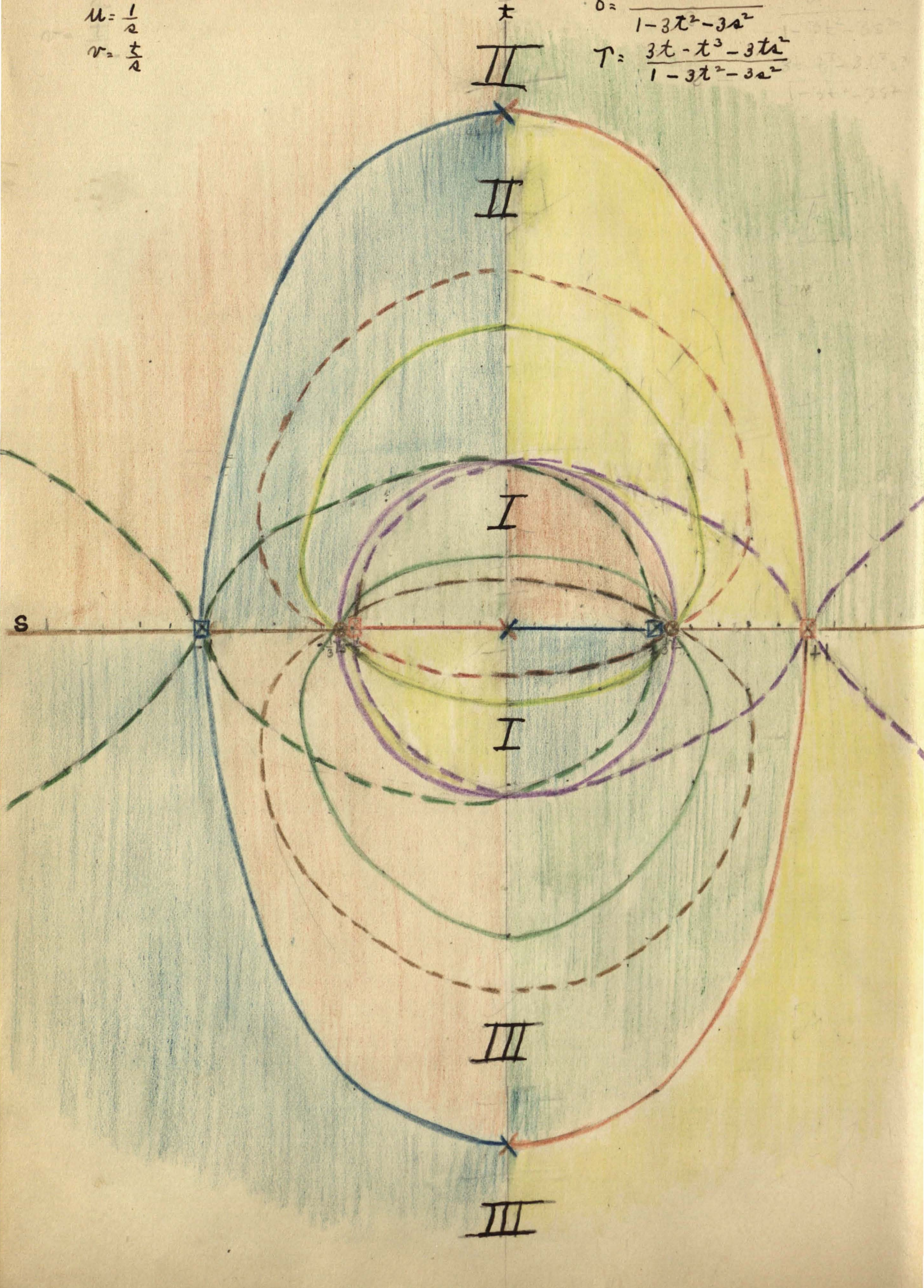
Plane I (B')

$u = \frac{1}{s}$

$v = \frac{t}{s}$

$\sigma = \frac{s^3}{1-3t^2-3s^2}$

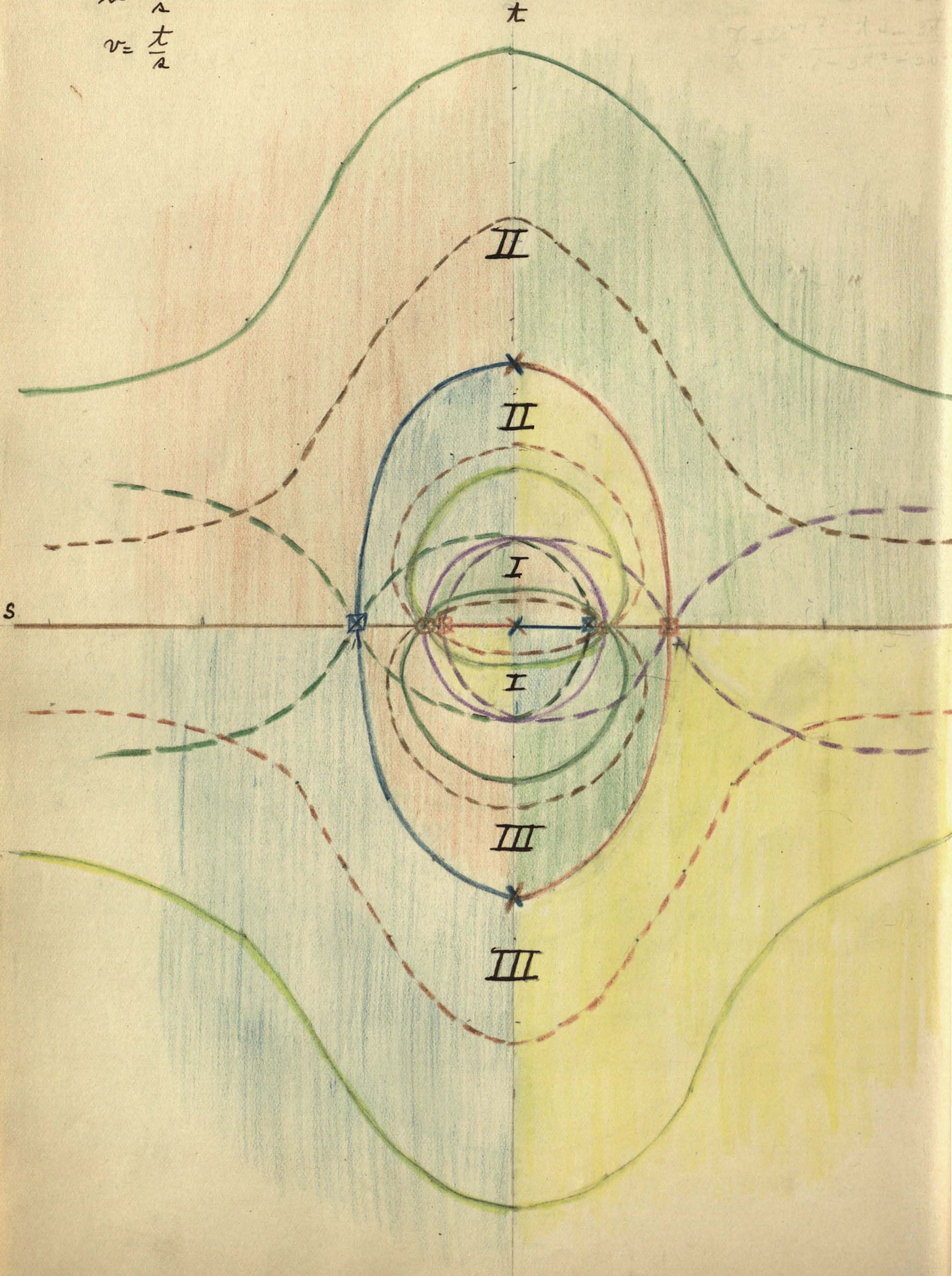
$\tau = \frac{3t - t^3 - 3ts^2}{1-3t^2-3s^2}$



Plane I (B)

$u = \frac{1}{s}$

$v = \frac{t}{s}$

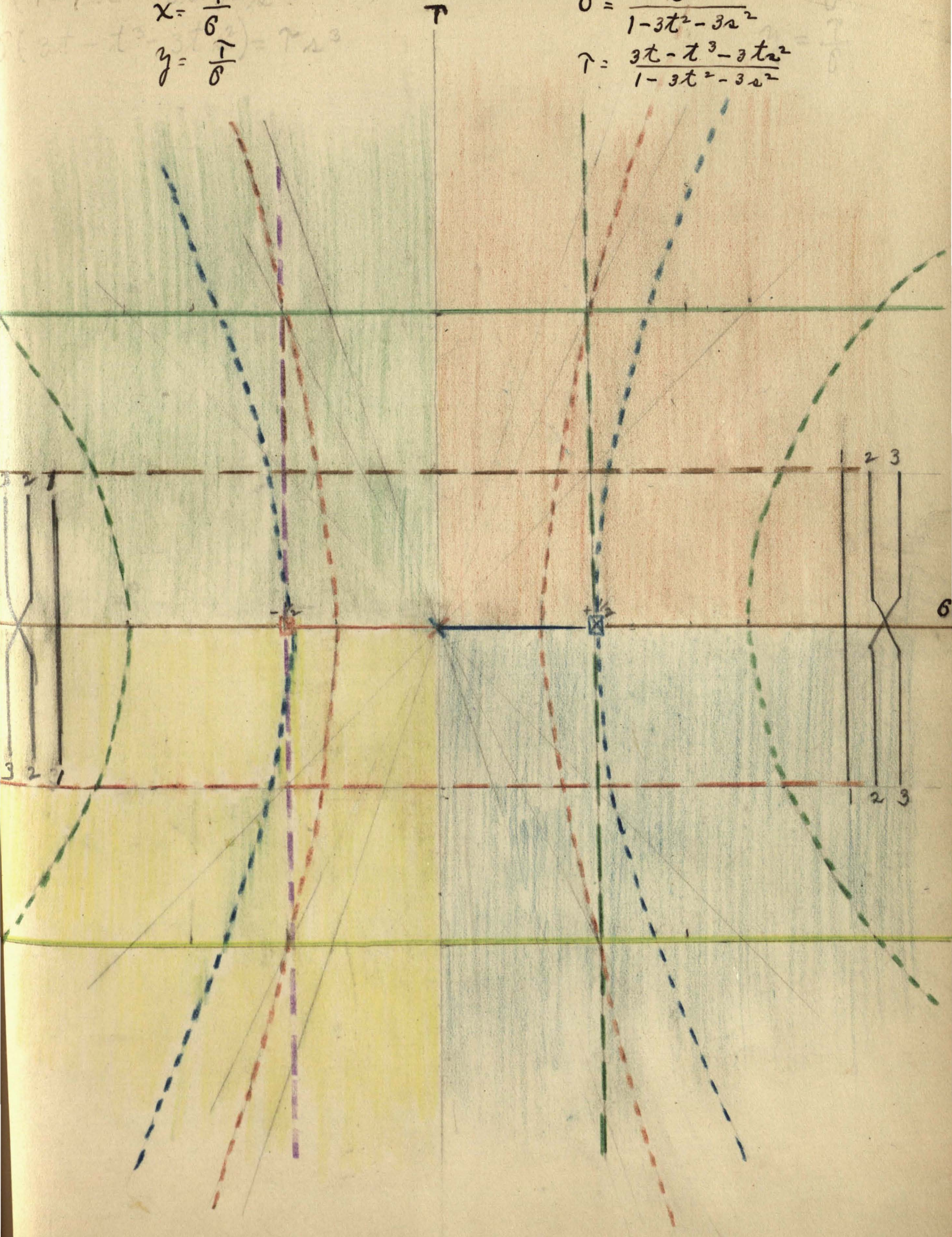


Plane IV (13)

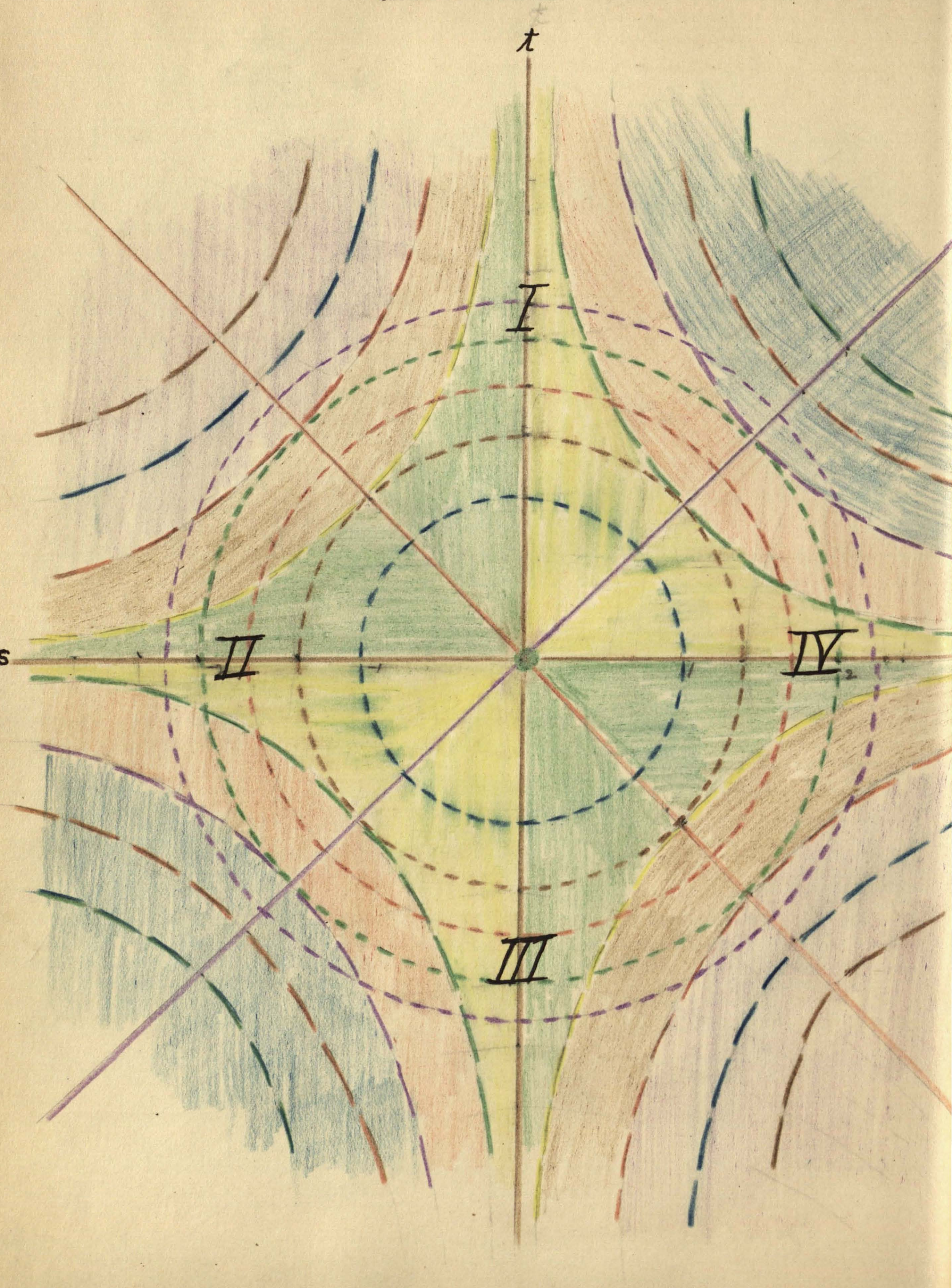
$x = \dfrac{1}{\sigma}$ $y = \dfrac{\tau}{\sigma}$

$\sigma = \dfrac{s^3}{1-3t^2-3s^2}$

$\tau = \dfrac{3t-t^3-3ts^2}{1-3t^2-3s^2}$

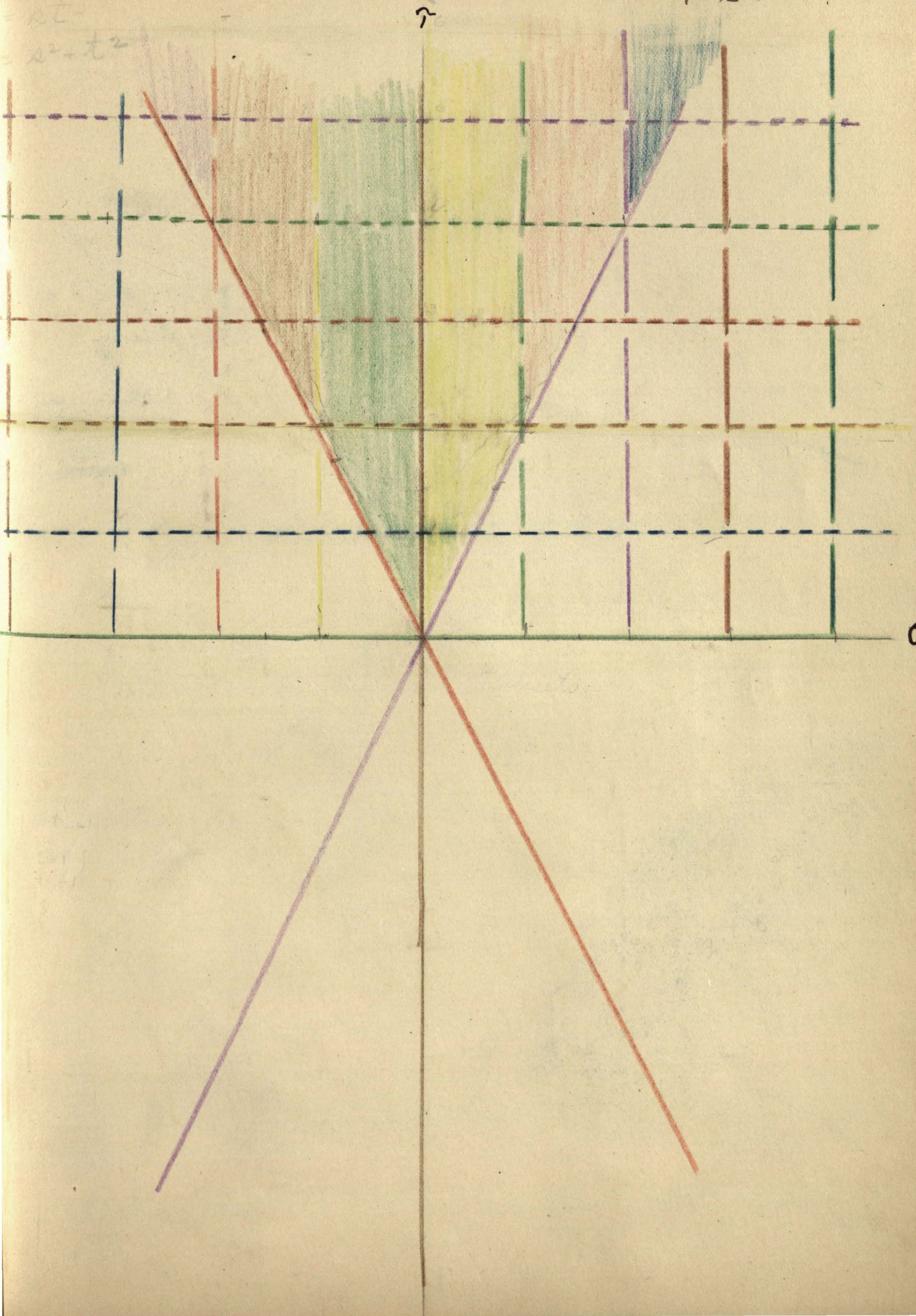


Plane V

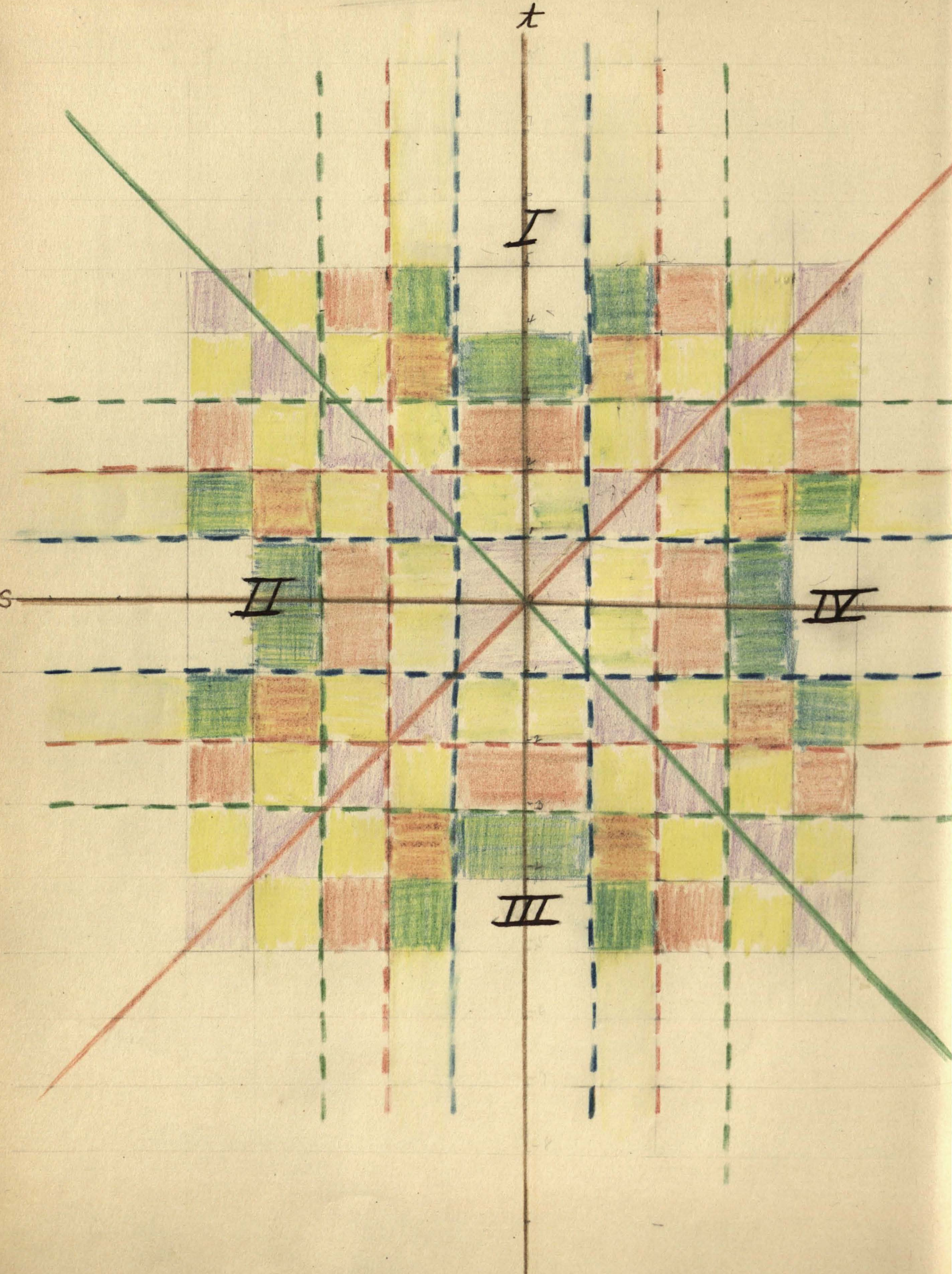


Plane VI

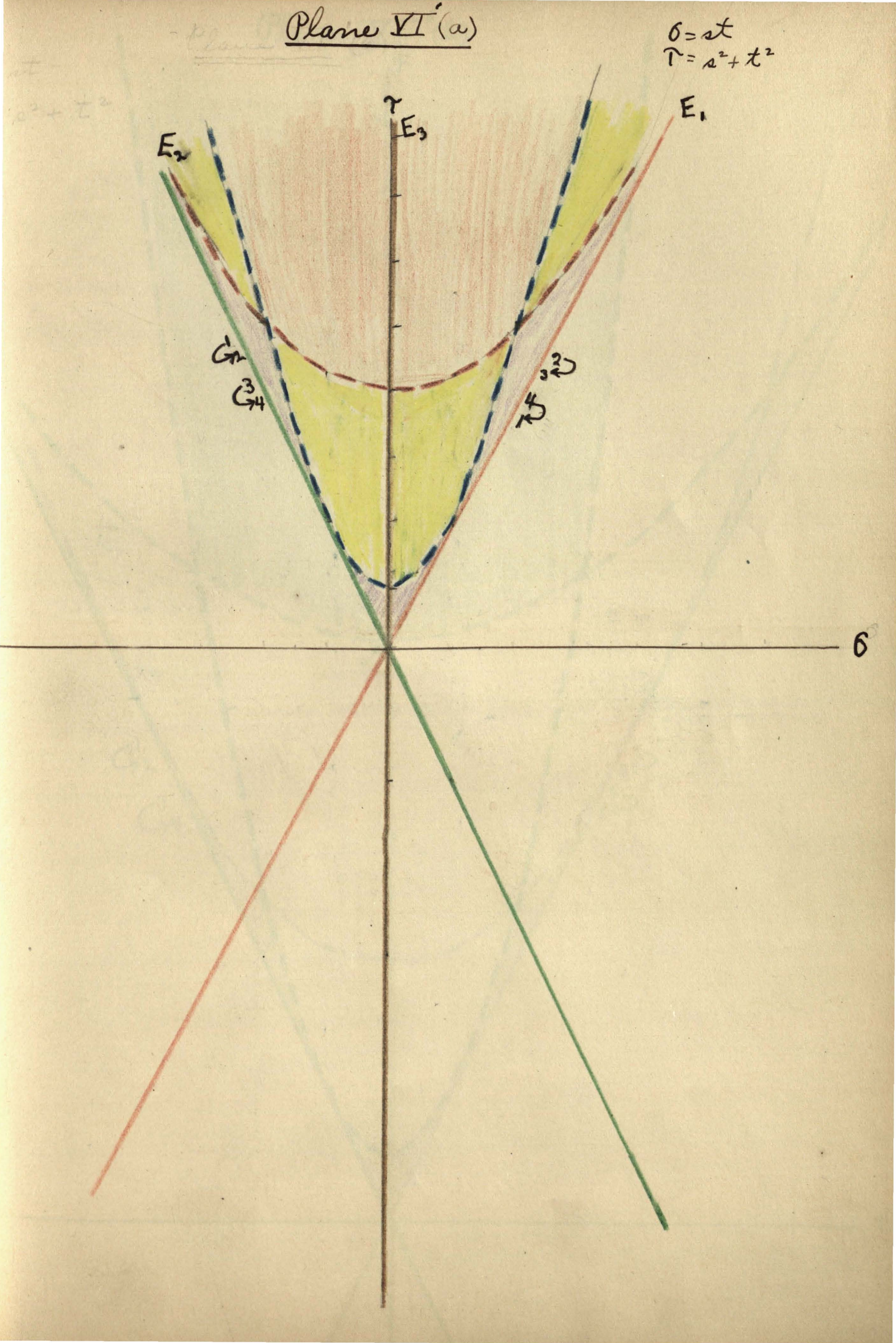
$\sigma = st$
$\tau = s^2 + t^2$



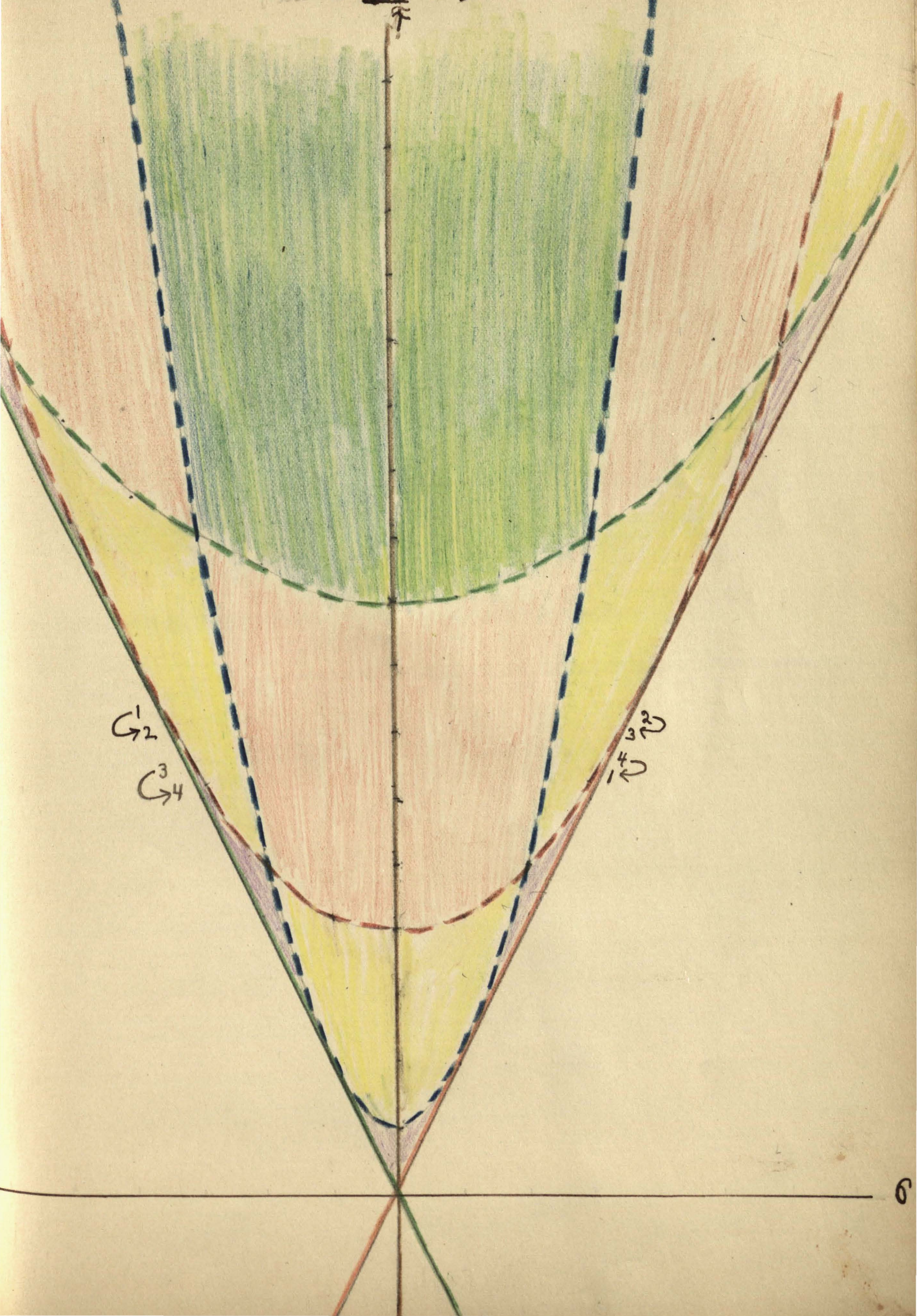
Plane V'



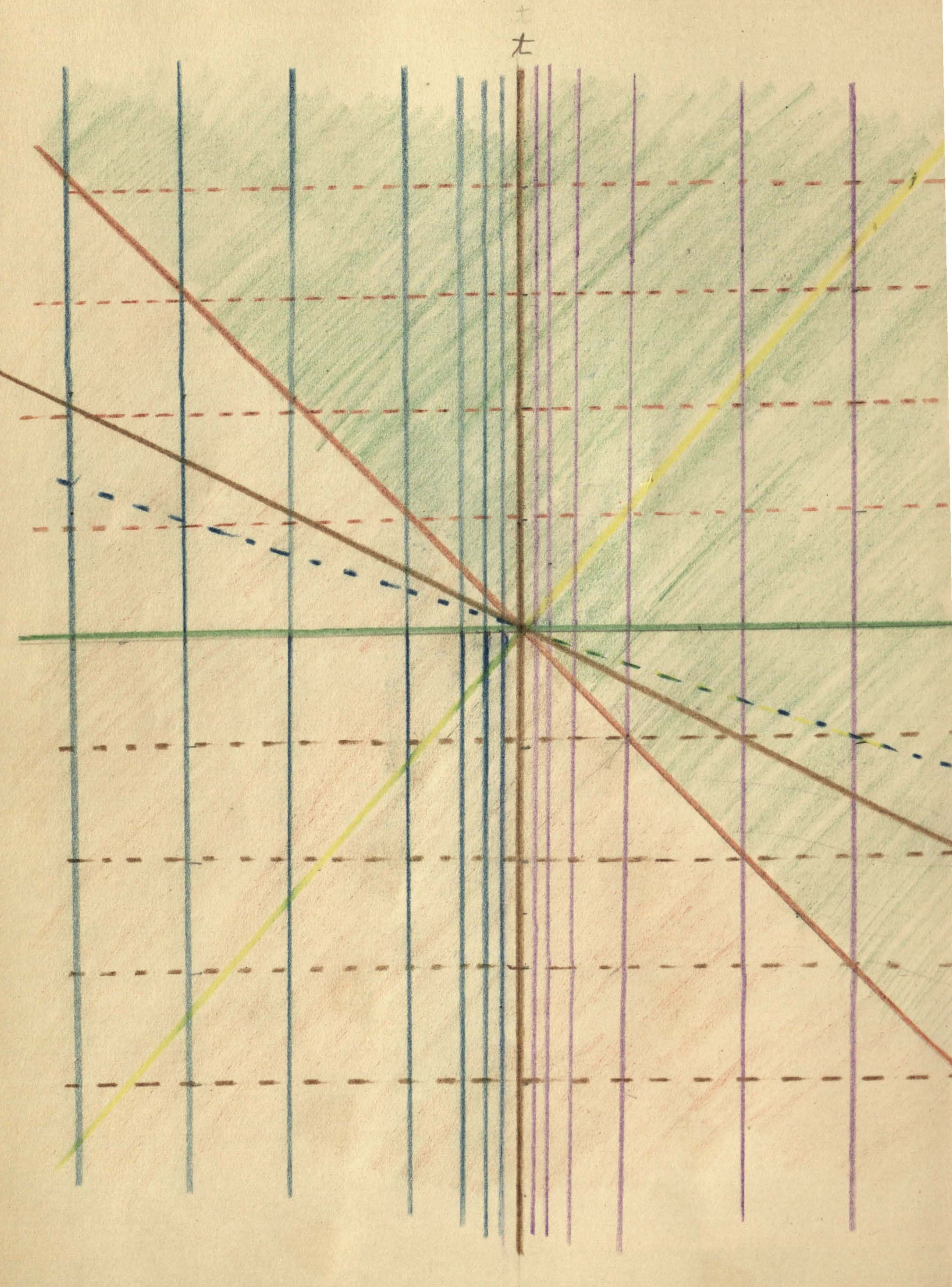
Plane VI′ (a)



Plane VI'(b)

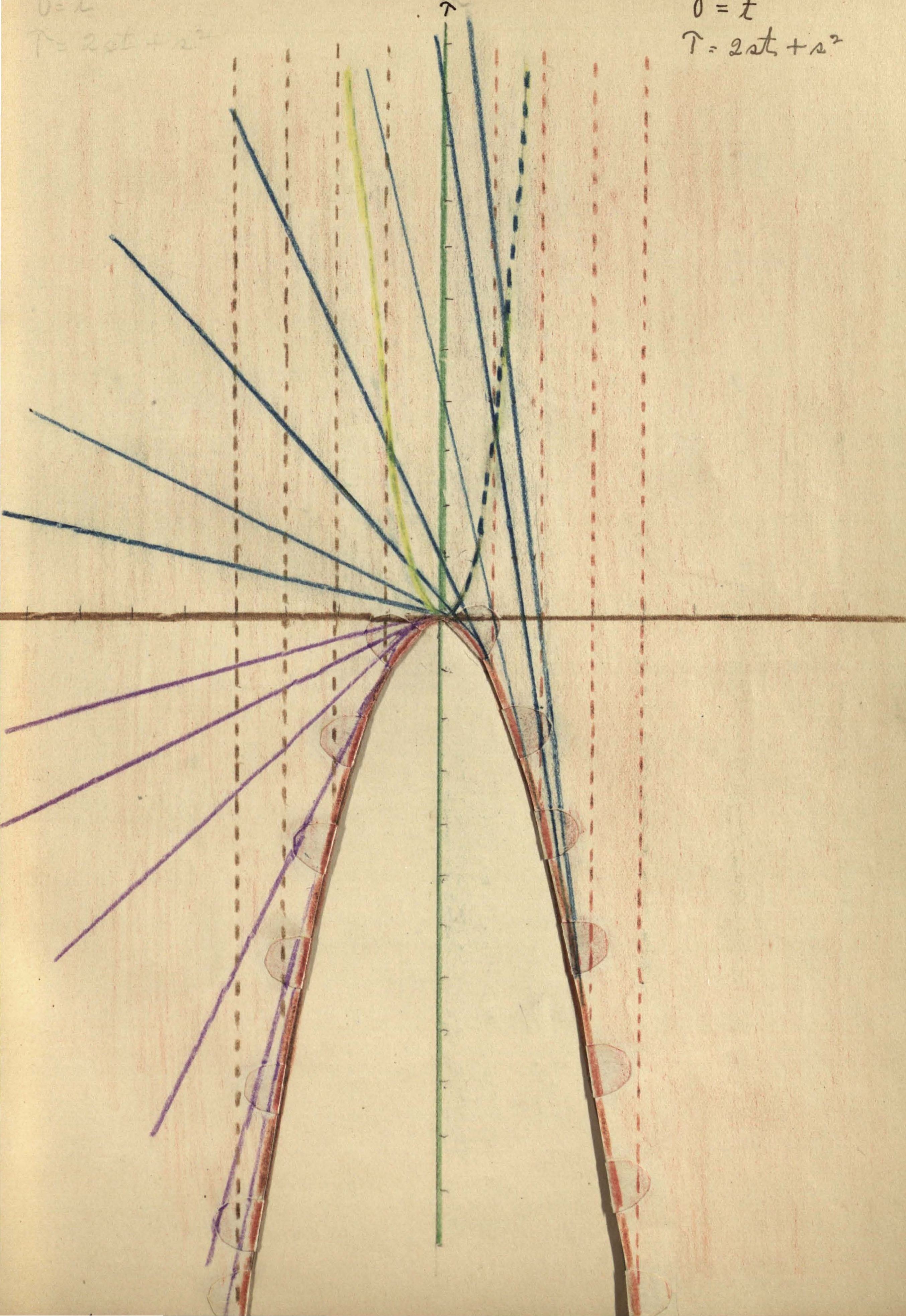


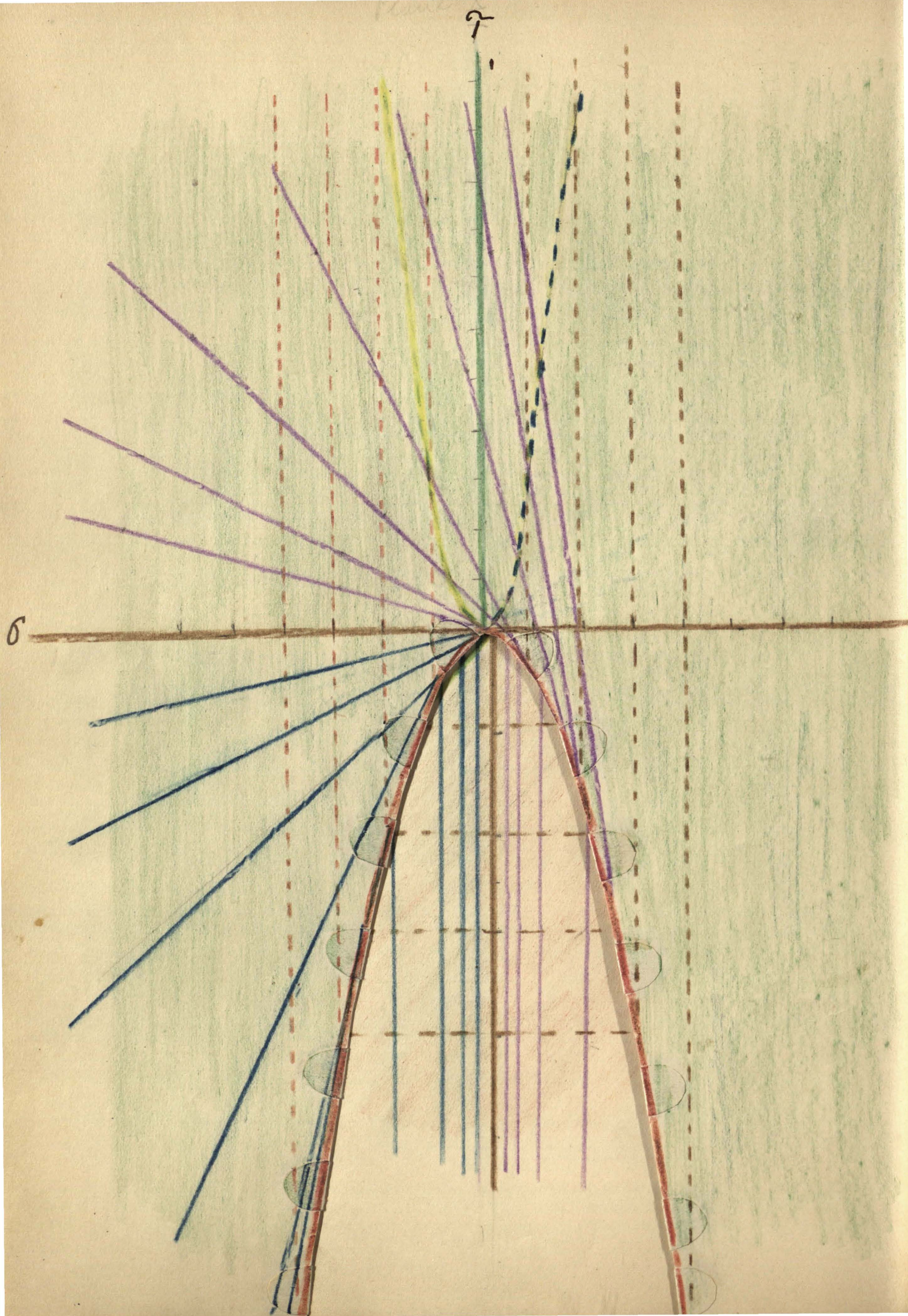
Plane VII



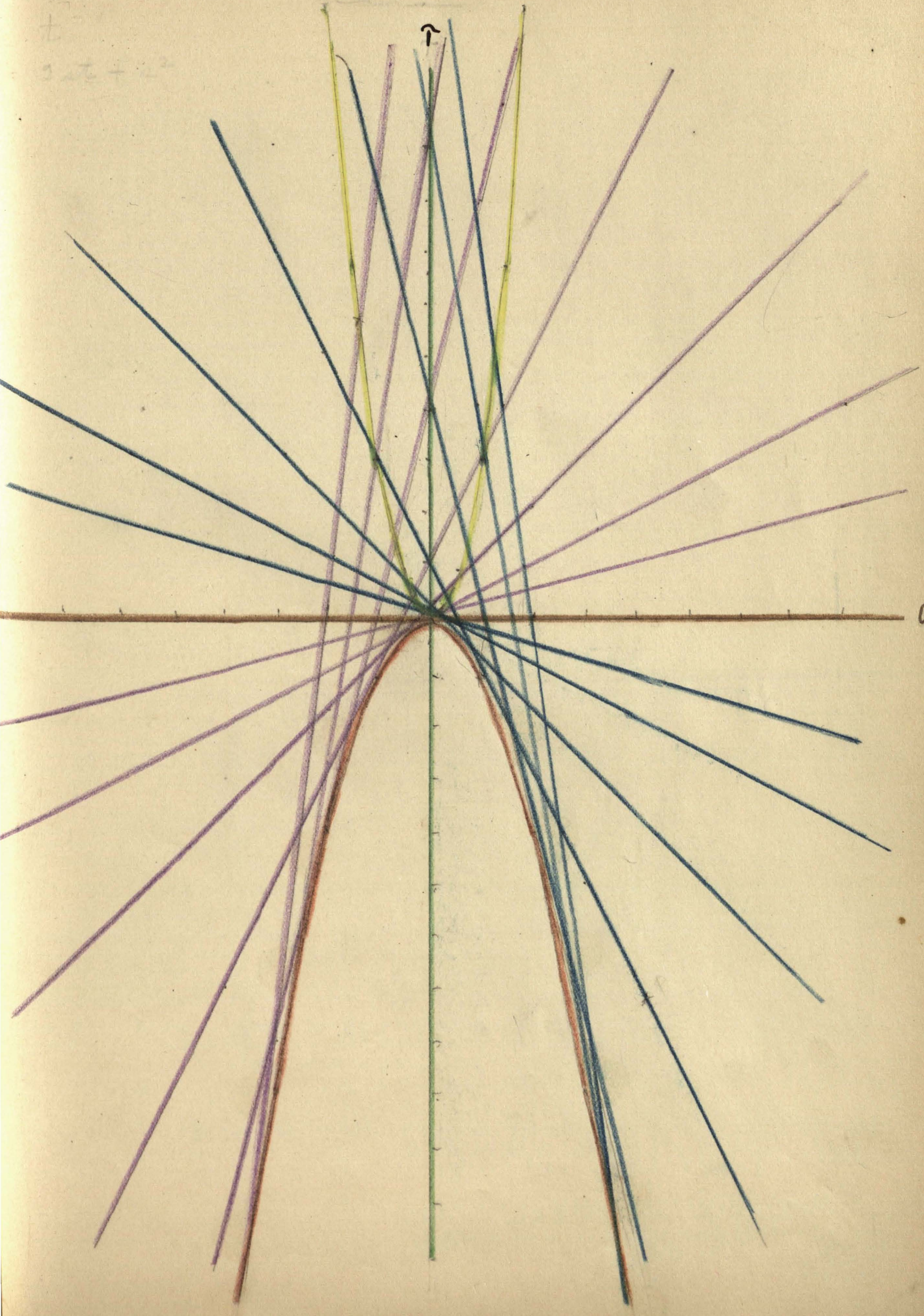
Plane VIII

$\sigma = t$
$\tau = 2st_1 + s^2$

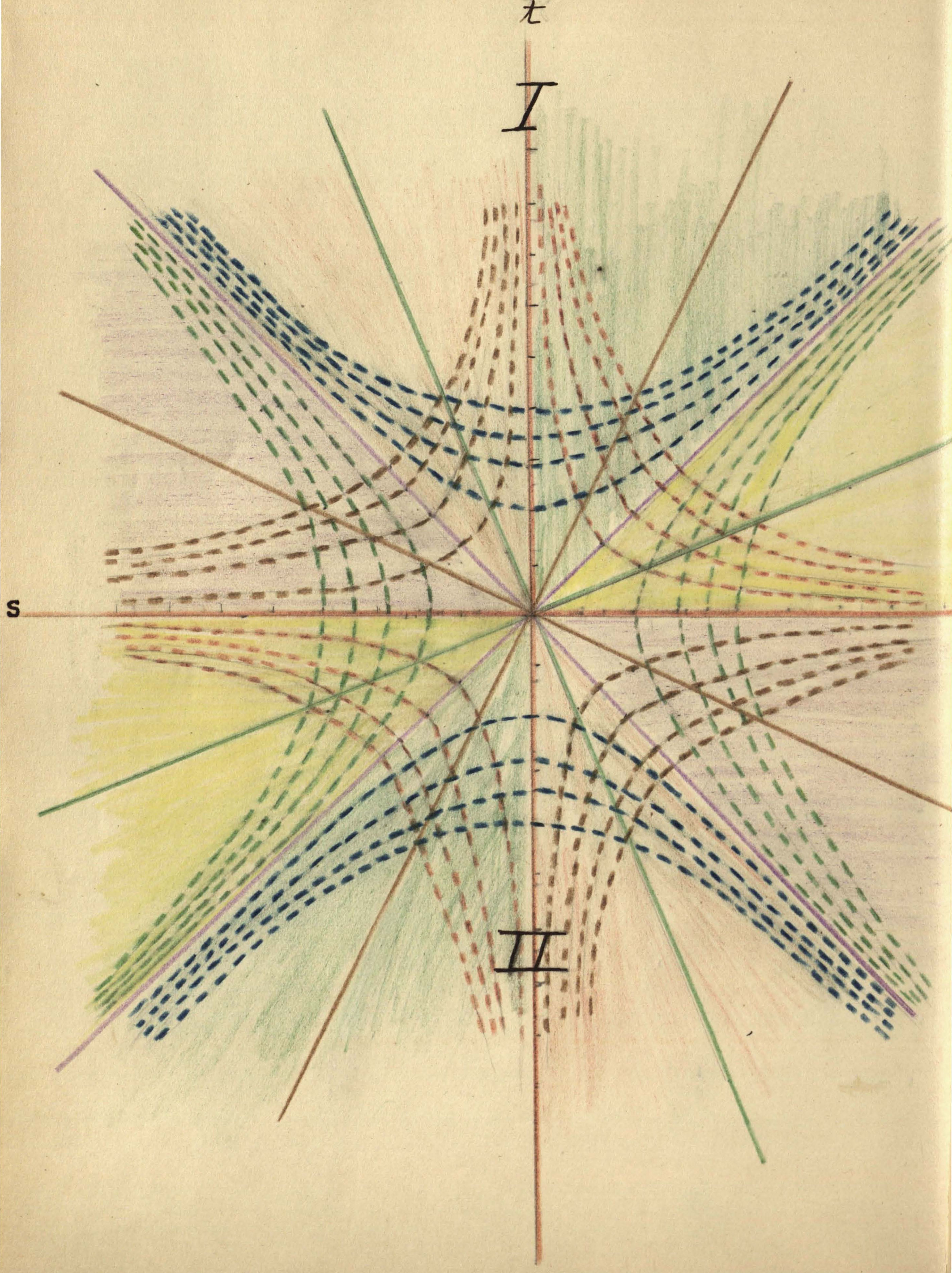



τ
σ

Plane VIII

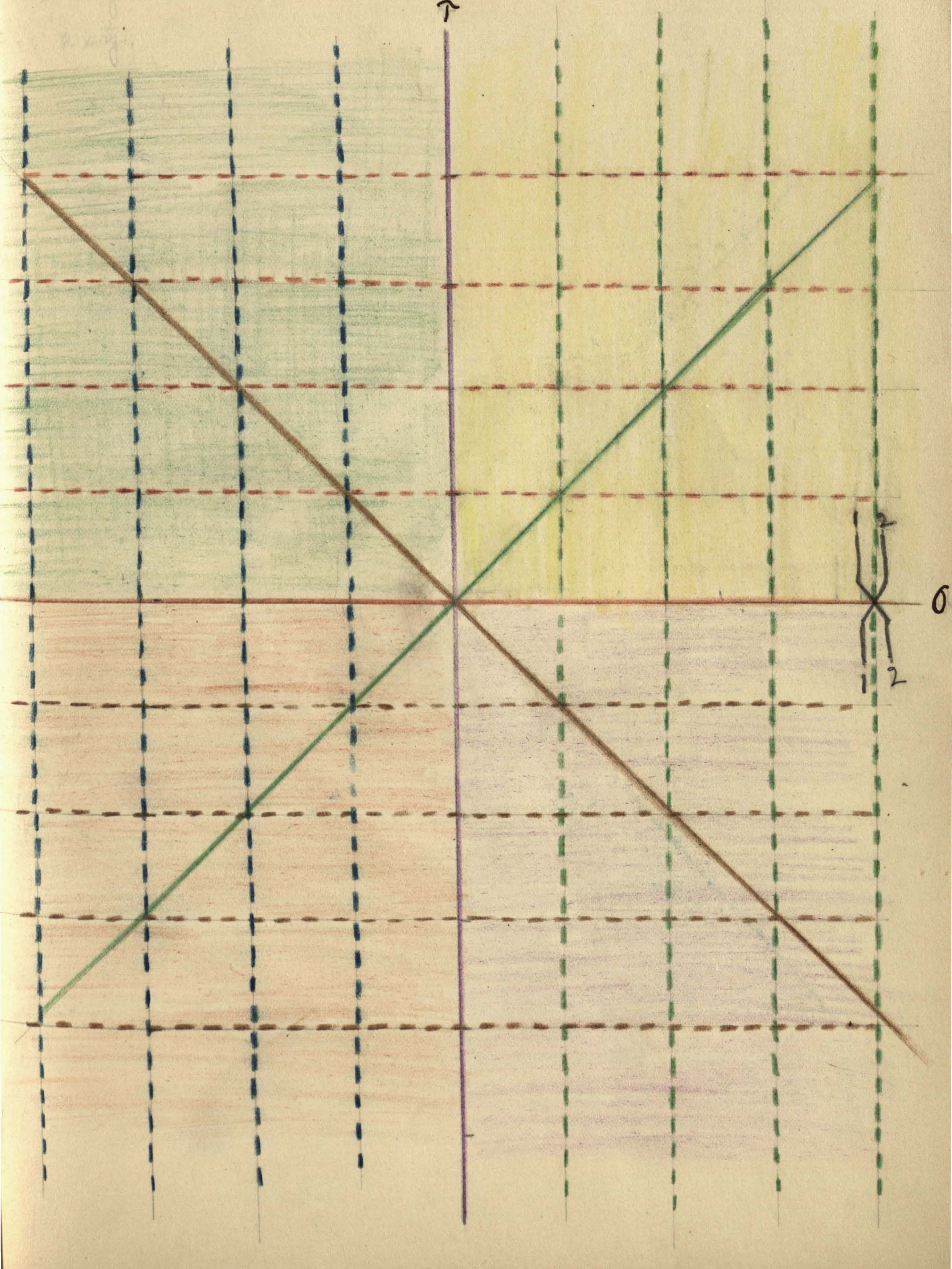


Plane IX

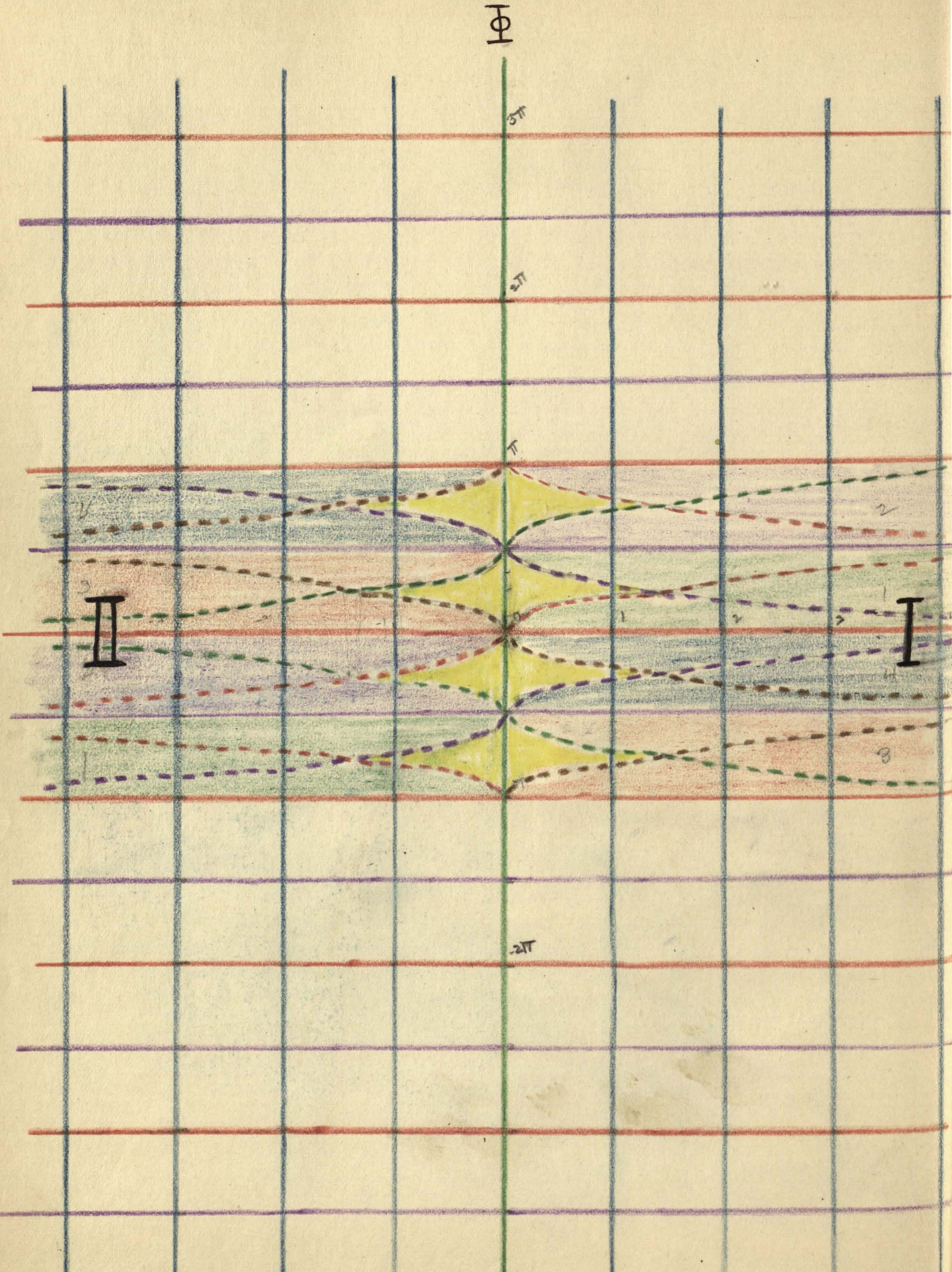


Plane X

$\sigma = s^2 - t^2$
$\tau = 2st$



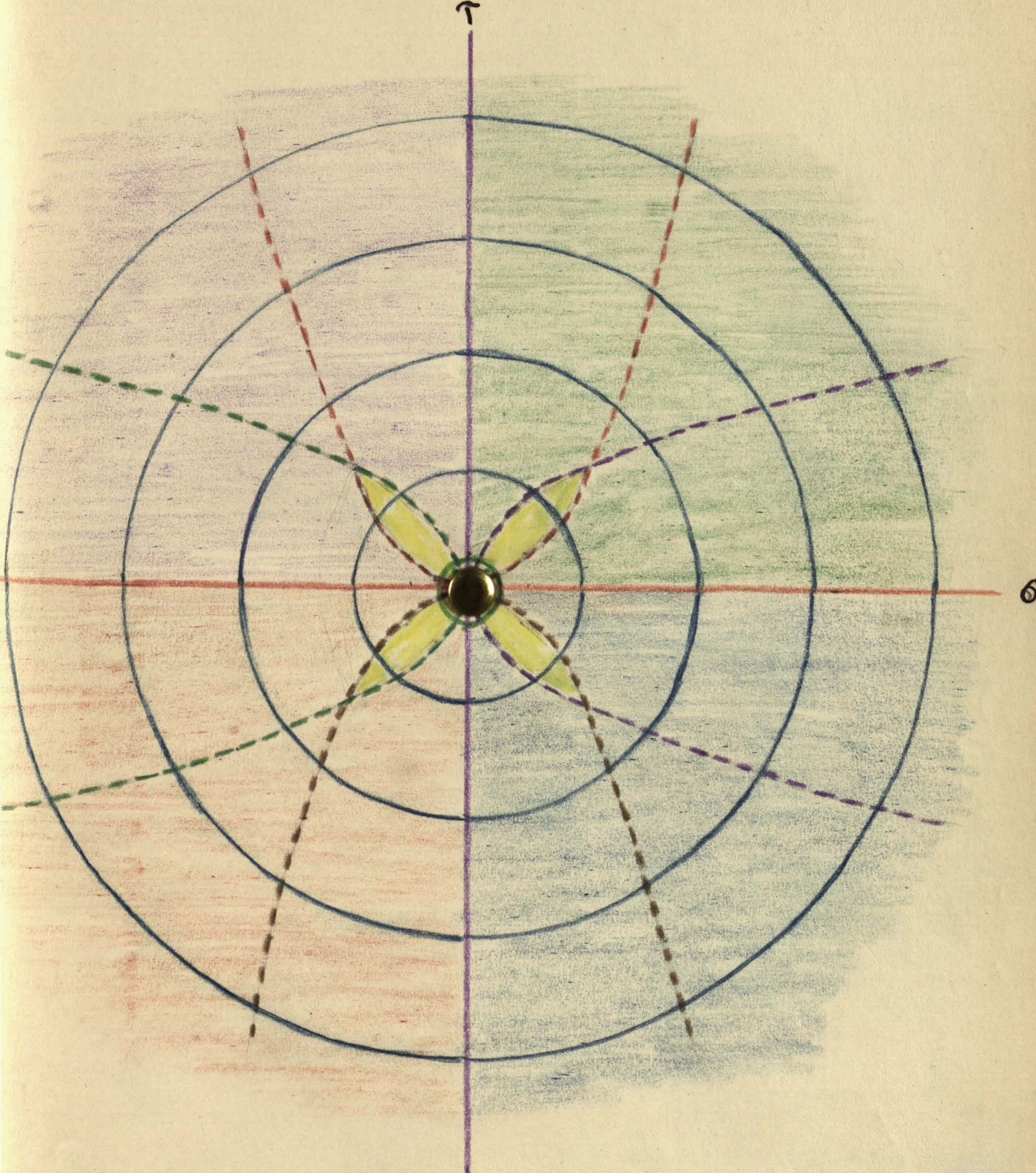
Plane XI

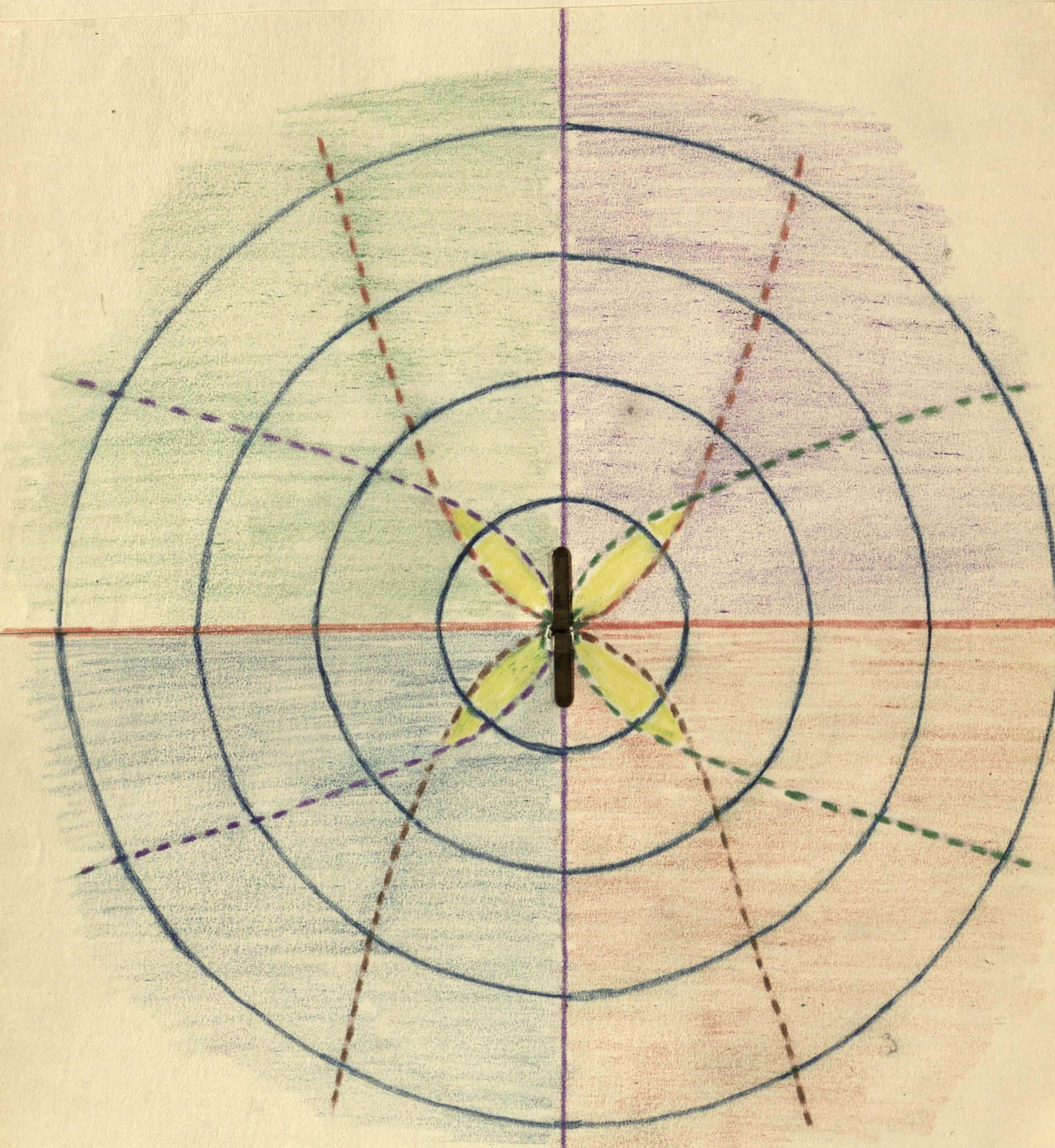


Plane XII

$\sigma = \rho \cos \Phi$

$\tau = \rho \sin \Phi$





This thesis is never to leave this room.
Neither is it to be checked out overnight.