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On Surfaces of Constant Negative Curvature and their Deformation.

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Chapter (I)

Definitions and Deductions of Fundamental Formulae.

Let us first define what is meant by curvature. If w represents the angle between the positive directions of two tangents MT and $M'T'$, at two points M and M' which are infinitely near and on a curve c, we then write the radius of curvature of c, at the point M, equal to the limit of the quotient of $\frac{\text{arc } MM'}{w}$ as M approaches M' indefinitely; that is, $R = \frac{ds}{dw}$ which is the reciprocal of the curvature. Therefore the curvature $K = \frac{dw}{ds}$.

The radius of curvature lies in the osculating plane. The radius of curvature of a normal section is measured along the normal to the surface. The radius of curvature of a normal section is obtained when the osculating plane coincides with a normal section of the surface. The principal radii of curvature of a point P on a surface are the radii of curvature of the two principal normal sections, which sections pass through the apes of the indicatrix.

It is evident that a discussion of curvature requires first that we consider the osculating plane. The osculating plane of a

space curve c is determined by three points A, B and C infinitely near each other, and corresponding to the values of the parameters t, t+h, and t+2h respectively, as h approaches zero. Let the coordinates of

A be x, y, z,
B " $x+\Delta x,\ y+\Delta y,\ z+\Delta z$,
C " $x+2\Delta x+\Delta^2 x,\ y+2\Delta y+\Delta^2 y,\ z+2\Delta z+\Delta^2 z$.

The equation of a plane through A is

(1) $a(X-x)+b(Y-y)+c(Z-z)=0$

The same plane, since it passes through B, is represented by the equation

(2) $a\Delta x+b\Delta y+c\Delta z=0$;

and since it also passes through C it is represented by the equation

(3) $a(2\Delta x+\Delta^2 x)+b(2\Delta y+\Delta^2 y)+c(2\Delta z+\Delta^2 z)=0$.

Combining (2) and (3)

(4) $a\Delta^2 x+b\Delta^2 y+c\Delta^2 z=0$

From (2) and (4) we have

$$a:b:c=(\Delta y\Delta^2 z-\Delta z\Delta^2 y):(\Delta z\Delta^2 x-\Delta x\Delta^2 z):(\Delta x\Delta^2 y-\Delta y\Delta^2 x)$$

Therefore the equation of a plane through A, B, C is

(5) $(\Delta y\Delta^2 z-\Delta z\Delta^2 y)(X-x)+(\Delta z\Delta^2 x-\Delta x\Delta^2 z)(Y-y)+(\Delta x\Delta^2 y-\Delta y\Delta^2 x)(Z-z)=0$

Let us divide through by $h\cdot h^2$, and pass to the limit. Noticing that the limit $h=dt$ we have

$$\left(\frac{dy}{dt}\cdot\frac{d^2z}{dt^2}-\frac{dz}{dt}\cdot\frac{d^2y}{dt^2}\right)(X-x)+\left(\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right)(Y-y)+$$
$$\left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)(Z-z)=0;$$

or finally we have

(I) $(y'z''-z'y'')(X-x)+(z'x''-x'z'')(Y-y)+(x'y''-y'x'')(Z-z)=0$

as the equation of the osculating plane.

Now let us proceed to find the expression for the radius of curvature of a space curve. Take three neigboring points $P\,(x,y,z)$; $Q\,(x_1,y_1,z_1)$ $R\,(x_2,y_2,z_2)$ corresponding respectively to the parameters t, $t+k$ and $t-k$. The osculating plane at the point P is the limiting position of the plane determined by the three points P, Q and R, as Q and R approach P. The radius of curvature at the point P is situated on the line of intersection of the two normal planes passing through Q and R as they approach the limiting position; or what is the same by the intersection of the normal planes through P and Q as Q approaches the limiting position P. But the center of curvature also lies in the osculating plane which is the plane of the circle. Therefore the center of curvature is given by the intersection of the normal planes through P and Q and the osculating plane to the curve at P.

The equation of the osculating plane at $P(x,y,z)$ is, as we have seen

$$(I)\ (y'z''-z'y'')(X-x)+(z'x''-x'z'')(Y-y)+(x'y''-y'x'')(Z-z)=0.$$

The normal plane at the point P is, as we know

$$x'(X-x)+y'(Y-y)+z'(Z-z)=0.$$

Let the equation of the space curve be given in the form

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}\right\}$$

Then the normal can be written in the form

$$(\alpha)\ f'(t)[X-x]+g'(t)[Y-y]+h'(t)[Z-z]=0.$$

The equation of the normal plane at Q is

$$(\beta)\ f'(t+k)[X-f(t+k)]+g'(t+k)[Y-g(t+k)]+h'(t+k)[Z-h(t+k)]=0$$

Subtract (α) from (β) and divide by k

$$X\left[\frac{f'(t+k)-f'(t)}{k}\right]+Y\left[\frac{g'(t+k)-g'(t)}{k}\right]+Z\left[\frac{h'(t+k)-h'(t)}{k}\right]=$$

$$\frac{f'(t+k)f(t+k)-f'(t)f(t)}{k}+\frac{g'(t+k)g(t+k)-g'(t)g(t)}{k}+\frac{h'(t+k)h(t+k)-h'(t)h(t)}{k}$$

Pass to the limit.

$$Xf''(t)+Yg''(t)+Zh''(t)$$

$$=\frac{d}{dt}[f'(t)f(t)]+\frac{d}{dt}[g'(t)g(t)]+\frac{d}{dt}[h'(t)h(t)]$$

$$=f''(t)f(t)+[f'(t)]^2+g''(t)g(t)+[g'(t)]^2+h''(t)h(t)+[h'(t)]^2$$

But we have

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \\ z = h(t) \end{array}\right\} \quad \left.\begin{array}{l} x' = f'(t) \\ y' = g'(t) \\ z' = h'(t) \end{array}\right\} \quad \left.\begin{array}{l} x'' = f''(t) \\ y'' = g''(t) \\ z'' = h''(t) \end{array}\right\}$$

Therefore by transposing and substituting we have

$$(\gamma)\ (X-x)x'' + (Y-y)y'' + (Z-z)z'' = (x')^2 + (y')^2 + (z')^2;$$

also we have

$$(I)\ (X-x)(y'z''-z'y'') + (Y-y)(z'x''-x'z'') + (Z-z)(x'y''-y'x'') = 0$$

$$(\alpha)\ (X-x)x' + (Y-y)y' + (Z-z)z' = 0.$$

From these we have

$$X - x = \frac{\begin{vmatrix} (x')^2+(y')^2+(z')^2 & y'' & z'' \\ 0 & z'x''-x'z'', & x'y''-y'x'' \\ 0 & y' & z' \end{vmatrix}}{\begin{vmatrix} x'' & y'' & z'' \\ y'z''-z'y'', & z'x''-x'z'' & x'y''-y'x'' \\ x' & y' & z' \end{vmatrix}}$$

$$= \frac{[x'^2+y'^2+z'^2][(x'^2+y'^2+z'^2)x'' - (x'x''+y'y''+z'z'')x']}{(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2) - (x'x''+y'y''+z'z'')^2}$$

Set $$M = \frac{x'^2+y'^2+z'^2}{(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2) - (x'x''+y'y''+z'z'')^2},$$

then

$$X - x = M[(x'^2+y'^2+z'^2)x'' - (x'x''+y'y''+z'z'')x'];$$ similarly

$$Y - y = M[(x'^2+y'^2+z'^2)y'' - (x'x''+y'y''+z'z'')y']$$

$$Z - z = M[(x'^2+y'^2+z'^2)z'' - (x'x''+y'y''+z'z'')z']$$

But, as we have seen the intersection of these three planes gives the center of the osculating circle; therefore X, Y and Z, are the coordinates of the center. Consequently the radius R, at P, gives

$$R = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2} \quad \text{and}$$

$$R^2 = (X-x)^2 + (Y-y)^2 + (Z-z)^2$$

For the purpose of calculating R let us set

$$\left.\begin{aligned} x'^2 + y'^2 + z'^2 &= a \\ x'x'' + y'y'' + z'z'' &= b \end{aligned}\right\}$$ Making this substitution in $X-x$, $Y-y$ and $Z-z$ we have

$$R^2 = [m^2]\left[(ax'' - bx')^2 + (ay'' - by')^2 + (az'' - bz')^2\right]$$

$$= m^2 \begin{bmatrix} a^2x''^2 - 2abx'x'' + b^2x'^2 \\ a^2y''^2 - 2aby'y'' + b^2y'^2 \\ a^2z''^2 - 2abz'z'' + b^2z'^2 \end{bmatrix} =$$

$$m^2\left[a^2(x''^2 + y''^2 + z''^2) - 2ab(x'x'' + y'y'' + z'z'') + b^2(x'^2 + y'^2 + z'^2)\right]$$

$$= m^2\left[a^2(x''^2 + y''^2 + z''^2) - 2ab^2 + ab^2\right]$$

$$= m^2a\left[a(x''^2 + y''^2 + z''^2 - b^2\right]$$

$$= m^2a^2\left[\frac{1}{m}\right] \qquad \left\{\text{since } a(x''^2 + y''^2 + z''^2) - b^2 = \frac{1}{m}\right\}.$$

$$= ma^2 = m(x'^2 + y'^2 + z'^2).$$

Therefore,

$$R^2 = \frac{(x'^2+y'^2+z'^2)^3}{(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2)-(x'x''+y'y''+z'z'')^2}$$

$$= \frac{(x'^2+y'^2+z'^2)^3}{(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2}$$

$$= \text{(II)} \quad \frac{x'^2+y'^2+z'^2}{A^2+B^2+C^2};$$ where A, B and C are equal respectively to $(y'z''-z'y'')$, $(z'x''-x'z'')$ and $(x'y''-y'x'')$ which are furthermore the coefficients of $(X-x)$, $(Y-y)$ and $(Z-z)$ as found in the osculating plane.

We shall now find the cosine of the angle between the principal normal and the normal to the surface $\begin{cases} x = f(uv) \\ y = g(uv) \\ z = h(uv) \end{cases}$.

Let α, β and γ represent the direction cosines of the principal normal. Then

$\alpha : \beta : \gamma :: (X-x) : (Y-y) : (Z-z)$ since these values are, as we have seen in finding the radius of curvature, porportional to the direction cosines of the principal normal. We also know the equation of the normal to the surface; namely,

$$\frac{X-x}{y_u z_v - z_u y_v} = \frac{Y-y}{z_u x_v - x_u z_v} = \frac{Z-z}{x_u y_v - y_u x_v}$$

Let λ, μ and ν be the direction cosines of the normal to the surface. Then

$\lambda : \mu : \nu :: (y_u z_v - z_u y_v) : (z_u x_v - x_u z_v) : (x_u y_v - y_u x_v)$.

Consequently, $\cos\theta$

$$= \frac{(X-x)(y_u z_v - z_u y_v) + (Y-y)(z_u x_v - x_u z_v) + (Z-z)(x_u y_v - y_u x_v)}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}\sqrt{(y_u z_v - z_u y_v)^2 + (z_u x_v - x_u z_v)^2 + (x_u y_v - y_u x_v)^2}}$$

But we have $R^2 = (X-x)^2 + (Y-y)^2 + (Z-z)^2 =$

$$\frac{R^4}{(x_1'^2 + y_1'^2 + z_1'^2)^4}\left\{\left[(x_1'^2 + y_1'^2 + z_1'^2)x'' - (x'x'' + y'y'' + z'z'')x'\right]^2 + \right.$$

$$\left[(x_1'^2 + y_1'^2 + z_1'^2)y'' - (x'x'' + y'y'' + z'z'')y'\right]^2 +$$

$$\left.\left[(x_1'^2 + y_1'^2 + z_1'^2)z'' - (x'x'' + y'y'' + z'z'')z'\right]^2\right\}$$

Then the sum of squares in brackets must equal $\frac{(x_1'^2 + y_1'^2 + z_1'^2)^4}{R^2}$.

If however we take out of $(X-x)$, $(Y-y)$ and $(Z-z)$ the common factor $\frac{R}{x_1'^2 + y_1'^2 + z_1'^2}$ we also have the direction cosines proportional to

$$\frac{R}{x_1'^2 + y_1'^2 + z_1'^2}\left[(x_1'^2 + y_1'^2 + z_1'^2)x'' - (x'x'' + y'y'' + z'z'')x'\right],$$

$$\frac{R}{x_1'^2 + y_1'^2 + z_1'^2}\left[(\quad '' \quad)y'' - (\quad '' \quad)y'\right],$$

$$\frac{R}{x_1'^2 + y_1'^2 + z_1'^2}\left[(\quad '' \quad)z'' - (\quad '' \quad)z'\right].$$

But we have seen that the sum of the squares in brackets equals $\frac{(x_1'^2 + y_1'^2 + z_1'^2)^4}{R^2}$. Then if we take the sum of the squares of these quantities which

are proportional to the direction cosines of the principal normal, we have their sum

$$= \frac{\rho^2}{(x'^2+y'^2+z'^2)^2}\Big\{\left[(x'^2+y'^2+z'^2)x''-(x'x''+y'y''+z'z'')x'\right]^2 +$$

$$\left[(x'^2+y'^2+z'^2)y''-(x'x''+y'y''+z'z'')y'\right]^2 +$$

$$\left[(x'^2+y'^2+z'^2)z''-(x'x''+y'y''+z'z'')z'\right]^2.$$

$$= \frac{\rho^2}{(x'^2+y'^2+z'^2)^2}\cdot\frac{(x'^2+y'^2+z'^2)^4}{\rho^2} = (x'^2+y'^2+z'^2)^2$$

Then in our formula for $\cos\theta$ we can write

$$\frac{X-x}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} = \rho\frac{\left[(x'^2+y'^2+z'^2)x''-(x'x''+y'y''+z'z'')x'\right]}{(x'^2+y'^2+z'^2)^2}$$

$$\frac{Y-y}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} = \rho\frac{\left[(\quad\text{''}\quad)y''-(\quad\text{''}\quad)y'\right]}{(x'^2+y'^2+z'^2)^2}$$

$$\frac{Z-z}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} = \rho\frac{\left[(\quad\text{''}\quad)z''-(\quad\text{''}\quad)z'\right]}{(x'^2+y'^2+z'^2)^2}$$

Making this substitution we have

$$\cos\theta =$$

$$\frac{\rho\left[x''(y_u z_v - z_u y_v)(x'^2+y'^2+z'^2)+y''(z_u x_v - x_u z_v)(x'^2+y'^2+z'^2)+z''(x_u y_v - y_u x_v)(x'^2+y'^2+z'^2)\right]}{(x'^2+y'^2+z'^2)^2(H)}$$

$$-\frac{\rho\left[x'x''+y'y''+z'z''\right]\left[(y_u z_v - z_u y_v)x'+(z_u x_v - x_u z_v)y'+(x_u y_v - y_u x_v)z'\right]}{(x'^2+y'^2+z'^2)^2\cdot H}$$

where $H^2 = (y_u z_v - y_v z_u)^2 + (z_u x_v - x_u z_v)^2 + (x_u y_v - y_u x_v)^2$.
This can be written in determinant form thus

$$\cos\theta = R\begin{vmatrix} x'' & x_u & x_v \\ y'' & y_u & y_v \\ z'' & z_u & z_v \end{vmatrix} \frac{(x'^2+y'^2+z'^2)}{(x'^2+y'^2+z'^2)^2 \cdot H} - R\begin{vmatrix} x' & x_u & x_v \\ y' & y_u & y_v \\ z' & z_u & z_v \end{vmatrix} \left[\frac{x'x''+y'y''+z'z''}{(x'^2+y'^2+z'^2)^2\, H}\right]$$

But the latter determinant vanishes since

$$x = f(uv)$$
$$x' = x_u + x_v \frac{dv}{du} = x_u + x_v K, \quad \left[K = \frac{dv}{du}\right]$$

This makes the first column a sum of multiples of the other two columns and therefore the determinant vanishes identically. Then we have

$$\cos\theta = \frac{R}{x'^2+y'^2+z'^2}\cdot\frac{1}{H}\begin{vmatrix} x'' & x_u & x_v \\ y'' & y_u & y_v \\ z'' & z_u & z_v \end{vmatrix}.$$

But

$$x = f(uv)$$
$$x' = x_u + x_v K, \quad \text{where } K = \frac{dv}{du}$$
$$x'' = x_{uu} + 2x_{uv}K + x_{vv}K^2$$
$$y'' = y_{uu} + 2y_{uv}K + y_{vv}K^2$$
$$z'' = z_{uu} + 2z_{uv}K + z_{vv}K^2 \text{; therefore,}$$

$$\cos\theta =$$

$$\frac{R}{H(x'^2+y'^2+z'^2)}\begin{vmatrix} (x_{uu}+2Kx_{uv}+K^2x_{vv}), & x_u, & x_v \\ y_{uu}+2Ky_{uv}+K^2y_{vv}), & y_u, & y_v \\ z_{uu}+2Kz_{uv}+K^2z_{vv}), & z_u, & z_v \end{vmatrix} =$$

$$\frac{R}{H(x'^2+y'^2+z'^2)}\left\{\frac{1}{H}\begin{vmatrix} x_{uu}, & x_u, & x_v \\ y_{uu}, & y_u, & y_v \\ z_{uu}, & z_u, & z_v \end{vmatrix} + \frac{2K}{H}\begin{vmatrix} x_{uv}, & x_u, & x_v \\ y_{uv}, & y_u, & y_v \\ z_{uv}, & z_u, & z_v \end{vmatrix} + \frac{K^2}{H}\begin{vmatrix} x_{vv}, & x_u, & x_v \\ y_{vv}, & y_u, & y_v \\ z_{vv}, & z_u, & z_v \end{vmatrix}\right\}$$

In the element of length we have

$$\left(\frac{ds}{du}\right)^2 = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 = x'^2 + y'^2 + z'^2 =$$

$$(x_u + x_v K)^2 + (y_u + y_v K)^2 + (z_u + z_v K)^2 =$$

$$E + 2KF + GK^2 \quad \text{where}$$

$$E = x_u^2 + y_u^2 + z_u^2$$

$$F = x_u x_v + y_u y_v + z_u z_v$$

$$G = x_v^2 + y_v^2 + z_v^2$$

$$K = \frac{dv}{du}.$$

Using this representation we have, at last

$$\cos\theta =$$

$$\text{(III)} \quad R\left[\frac{L + 2MK + NK^2}{E + 2FK + GK^2}\right]; \quad \text{where}$$

$$L = \frac{1}{H}\begin{vmatrix} x_{uu} & x_u & x_v \\ y_{uu} & y_u & y_v \\ z_{uu} & z_u & z_v \end{vmatrix}, \quad M = \frac{1}{H}\begin{vmatrix} x_{uv} & x_u & x_v \\ y_{uv} & x_u & x_v \\ z_{uv} & z_u & z_v \end{vmatrix}$$

, and

$$N = \frac{1}{H}\begin{vmatrix} x_{vv} & x_u & x_v \\ y_{vv} & y_u & y_v \\ z_{vv} & z_u & z_v \end{vmatrix}.$$

Then $\cos\theta$, which is the cosine of the angle between the normal to the surface and the principal normal, equals

$$\rho\left[\frac{L+2MK+NK^2}{E+2FK+GK^2}\right].$$ Consequently, we have

$$\text{(IV)} \quad \rho = \frac{E+2FK+GK^2}{L+2MK+NK^2}\cos\theta.$$

We have now found the radius of curvature of a space curve in terms of E, F, G, K, L, M, N and θ. It is now our purpose to show three things:-

1st. The radius of curvature of all curves whatever on a surface passing through a point P of the surface and having at this point the same osculating plane are equal.

2nd. We wish to show

(α) that the radius of curvature of a normal section is greater than that of any oblique section having the same tangent, or than that of any other curve at the same point and having the same tangent.

(β) That the radius of curvature of an oblique

section equals the projection of the radius of curvature of a normal section having the same tangent.

3d. We wish to show

(α) That there are normal sections that have a maximum and a minimum radius of curvature and that these sections stand at right angles.

(β) That the equation giving the principal radii of curvature can be found.

(γ) That $\frac{1}{R_n} = \frac{\cos^2\psi}{R^1} + \frac{\sin^2\psi}{R^2}$, where ψ is the angle which the tangent K of any normal section makes with K_1 the tangent of a principal normal section, and where R_1 and R_2 denote the radii of curvature of the principal normal sections and R_n that of any other normal section.

1st. E, F, k, L, M and N are constant for any point P and since these curves have the same tangent PT, $K = \frac{dv}{du}$ which determines the direction of the tangent is fixed. But

$$R = \frac{E + 2FK + kK^2}{L + 2MK + NK^2}\cos\theta\ ;$$ then if P and K are both fixed we have

$$R = \text{const. } \cos\theta.$$

Therefore, R is equal for all curves having the same osculating plane.

2nd. Suppose we now let θ vary, that is, let the osculating plane turn about the tangent through P, the point P and the tangent PT being kept fixed. We obtain the normal section when the osculating plane contains the normal to the surface. Then the principal normal and the normal to the surface coincide. This makes $\theta = 0$ and $\cos\theta = 1$; therefore

$$R_n = \frac{E + 2FK + GK^2}{L + 2MK + NK^2}$$

and the radius of curvature of a normal section at P is greater than the radius of curvature of any oblique section at P and having the same tangent; and consequently greater than the radius of curvature of any curve at P having the same tangent. Moreover we have

$$R = \frac{E + 2FK + GK^2}{L + 2MK + NK^2}\cos\theta;$$

therefore

(V) $R = R_n \cos\theta$: This is Munier's law.

3d. (α). Now let us retain our point P and our normal section, but let $\frac{dv}{du} = K$ vary,

that is permit the tangent at the pt. (u,v) to turn. We have

$$R_n = \frac{E+2FK+GK^2}{L+2MK+NK^2}$$

The condition for maximum and minimum is that

$$\frac{dR_n}{dK} = \frac{2\left[(L+2MK+NK^2)(F+GK)-(E+2FK+GK^2)(M+NK)\right]}{(L+2MK+NK^2)^2}=0,$$

that is,

$$\text{(VI)}\quad (FN-GM)K^2+(EN-GK)K+(EM-FL)=0 .$$

Which will give a maximum and a minimum for $K=K_1$ and $K=K_2$ where K_1 and K_2 are roots of equation (VI).

Moreover R_1 and R_2 are real since K_1 and K_2 are real, for the discriminant

$$D=(EN-GL)^2-4(FN-GM)(EM-FL)>0$$

To see this, consider

$$EN-GM=\frac{1}{E}\{G(FL-EM)-F(GL-EN)\};$$

then

$$D=\frac{1}{E^2}\left[4(EG-F^2)(FL-EM)^2+\{2F(FL-EM)-E(GL-EN)\}^2\right]$$

$$=\frac{1}{E^2}\left[4H^2(FL-EM)^2+\{2F(FL-EM)-E(GL-EN)\}^2\right]>0$$

(where $H^2=EG-F^2$ as we know.)

since each term is a square. Therefore the two roots K_1 and K_2 are always real, consequently R_1 and R_2

exist and are real.

We shall now prove that the principal normal sections stand at right angles.

Let C_1' and C_2 designate the curves cut in the surface by these sections; and let the equations of C_1' and C_2' be respectively

$$C_1' \begin{cases} x_1 = f(uv) \\ y_1 = g(uv) \\ z_1 = h(uv) \\ v = \phi(u) \end{cases} \qquad C_2' \begin{cases} x_2 = f(uv) \\ y_2 = g(uv) \\ z_2 = h(uv) \\ v = \psi(u) \end{cases}$$

Let P be their point of intersection. Then K_1 and K_2 will give the directions of the tangents to C_1' and C_2' at P. Let a_1, b_1, c_1 be the direction cosines of the tangents to C_1'; and a_2, b_2, c_2 be the direction cosines to C_2' at the point P in each case. Then

$$\left.\begin{array}{l} (1)\ a_1 : b_1 : c_1 :: x_1' : y_1' : z_1' \\ (2)\ a_2 : b_2 : c_2 :: x_2' : y_2' : z_2' \end{array}\right\} \text{ where } x', y' \text{ and } z' \text{ are different in (1) and (2).}$$

Let $K_1 = \frac{dv}{du}$ in (1)

Let $K_2 = \frac{dv}{du}$ in (2).

For C_1' $\begin{cases} x' = x_u + x_v K_1 \\ y' = y_u + y_v K_1 \\ z' = z_u + z_v K_1 \end{cases}$; for C_2' $\begin{cases} x' = x_u + x_v K_2 \\ y' = y_u + y_v K_2 \\ z' = z_u + z_v K_2 \end{cases}$

Let w denote the angle between the principal

normals. Then

$$\cos w = \frac{(x_u + x_v K_1)(x_u + x_v K_2) + (y_u + y_v K_1)(y_u + y_v K_2) + (z_u + z_v K_1)(z_u + z_v K_2)}{\sqrt{(x_u + x_v K_1)^2 + (y_u + y_v K_1)^2 + (z_u + z_v K_1)^2}\sqrt{(x_u + x_v K_2)^2 + (y_u + y_v K_2)^2 + (z_u + z_v K_2)^2}}$$

$$= \frac{E + F(K_1 + K_2) + G K_1 K_2}{\sqrt{E + 2FK_1 + GK_1^2}\sqrt{E + 2FK_2 + GK_2^2}}$$; But from our equation for maxima and minima radii of curvature we have

$$K_1 K_2 = \frac{EM - FL}{FN - GM}$$

$$K_1 + K_2 = \frac{GL - EN}{FN - GM}$$. Then

$$\cos w = \frac{E + F\left(\frac{GL - EN}{FN - GM}\right) + G\left(\frac{EM - FL}{FN - GM}\right)}{\sqrt{E + 2FK_1 + GK_1^2}\sqrt{E + 2FK_2 + GK_2^2}} =$$

$$\frac{EFN - EGM + FGL - FEN + EGM - FGL}{(FN - GM)\sqrt{E + 2FK_1 + GK_1^2}\sqrt{E + 2FK_2 + GK_2^2}} = 0$$

Therefore, $w = \frac{\pi}{2}$, and the planes stand at right angles.

Let us now find (B) the equation which gives the two principal radii of curvature.

The equation in K which gives the maximum and minimum radius of curvature is as we have seen

$$(L + 2MK + NK^2)(F + GK) - (E + 2FK + GK^2)(M + NK) = 0.$$

Then we have

$$\frac{E+2FK_1+GK_1^2}{L+2MK_1+NK_1^2} = R_1 = \frac{F+GK_1}{M+NK_1};$$

also

$$\frac{E+2FK_2+GK_2^2}{L+2MK_2+NK_2^2} = R_2 = \frac{F+GK_2}{M+NK_2}:$$

therefore,

$$R_1+R_2 = \frac{(F+GK_1)(M+NK_2)+(F+GK_2)(M+NK_1)}{(M+NK_1)(M+NK_2)},$$

and

$$R_1R_2 = \frac{(F+GK_1)(F+GK_2)}{(M+NK_1)(M+NK_2)}.$$

We have here the sum and the product of the two principal radii of curvature. We can accordingly form the equation in R that has R_1 and R_2 for its roots; viz.,

$$(M^2+MN(K_1+K_2)+N^2K_1K_2)R^2 -$$
$$\{2FM+(FN+GM)(K_1+K_2)+GNK_1K_2\}R +$$
$$F^2+FG(K_1+K_2)+G^2K_1K_2 = 0.$$

But

$$K_1+K_2 = \frac{GL-EN}{FN-GM}$$

and

$$K_1K_2 = \frac{EM-FL}{FN-GM}.$$

We thus obtain

$$\text{(VII).}\quad (LN-M^2)R^2-(GL-2FM+EN)R+EG-F^2=0$$

The solution of which gives the two principal radii of curvature. We also have

VIII. $R_1 + R_2 = \frac{GL - 2FM + EN}{LN - M^2}$, and

(IX). $R_1 R_2 = \frac{EG - F^2}{LN - M^2}$

(X). $\frac{1}{R_1 R_2} = K = \frac{LN - M^2}{EG - F^2}$. This last expression is called the Gaussian curvature of a surface.

It may be remarked here that if the equation of a surface is given in the form

$$z = f(xy)$$

our equation for the principal radii of curvature takes the following form:

$$(rt - s^2)R^2 - \left[\sqrt{1+p^2+q^2}\{(1+p^2)t + (1+q^2)r - 2pqs\}\right]R + (1+p^2+q^2)^2 = 0 \quad \text{(XI)}.$$

where

$$p = \frac{\partial z}{\partial x}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}.$$

$$q = \frac{\partial z}{\partial y}, \quad t = \frac{\partial^2 z}{\partial y^2},$$

Let us now show (Y) that

$$\frac{1}{R_n} = \frac{\cos^2\psi}{R'} + \frac{\sin^2\psi}{R_2}.$$

Let K_1 and K_2 determine the principal tangents

at P; that is, the tangents to the two principal normal sections; let K determine the tangents of any other normal section at P. Let ψ denote the angle between K and K_1; ϕ the angle between K_2 and K. Then $\phi + \psi = \omega = \frac{\pi}{2}$.

The direction cosines of PK_1 are proportional to $x_u + x_v K_1$, $y_u + y_v K_1$, $z_u + z_v K_1$; those of PK are proportional to

$$x_u + x_v K, \quad y_u + y_v K, \quad z_u + z_v K .$$

Then

$$\cos\psi = \frac{E + F(K_1 + K) + G K_1 K}{\sqrt{E + 2FK_1 + GK_1^2}\sqrt{E + 2FK + GK^2}} .$$

$$\cos^2\psi = \frac{[E + F(K_1 + K) + G K_1 K]^2}{(E + 2FK_1 + GK_1^2)(E + 2FK + GK^2)};$$

also

$$\sin^2\psi = \cos^2\phi = \frac{E + F(K_2 + K) + G K_2 K}{(E + 2FK_2 + GK_2^2)(E + 2FK + GK^2)} .$$

But $\omega = \frac{\pi}{2}$, therefore,

$$\cos\omega = \frac{E + F(K_1 + K_2) + G K_1 K_2}{\sqrt{E + 2FK_1 + GK_1^2}\sqrt{E + 2FK_2 + GK_2^2}} = 0.$$

Therefore,

$$E + F(K_1 + K_2) + G K_1 K_2 = 0;$$

also

$$E + F(K_1 + K_2) + G K_1 K \Big]_{K = K_2} = 0$$

Therefore $K-K_2$ is a factor; and since K has to be multiplied by $F+GK_1$ to produce the part on the left containing K it follows that $F+GK_1$ is the other factor; consequently we can write

$$E+F(K_1+K)+GK_1K = (K-K_2)(F+GK_1).$$

similarly

$$E+F(K_2+K)+GK_2K = (K-K_1)(F+GK_2).$$

Then

$$\cos^2\psi = \frac{(F+GK_1)^2(K-K_2)^2}{(E+2FK+GK^2)(K_1-K_2)(F+GK_1)};$$

since

$$E+2FK_1+GK_1^2 = E+2F(K+K_1)+GKK_1\Big]_{K=K_1}$$

$$=(F+GK_1)(K-K_2)\Big]_{K=K_1}$$

$= (F+GK_1)(K_1-K_2)$. Then $\cos^2\psi = \dfrac{(F+GK_1)(K-K_2)^2}{(E+2FK+GK^2)(K_1-K_2)}$

But $R_1 = \dfrac{F+GK_1}{M+NK_1}$, therefore,

$$\cos^2\psi = \frac{R_1(M+NK_1)(K-K_2)^2}{(E+2FK+GK^2)(K_1-K_2)}.$$

Similarly

$$\cos^2\phi = \sin^2\psi = \frac{R_2(M+NK_2)(K-K_1)^2}{(E+2FK+GK^2)(K_2-K_1)}$$

Therefore by dividing $\cos^2\psi$ and $\sin^2\psi$ by R_1 and R_2 respectively and adding we have

$$\frac{\cos^2\psi}{R_1} + \frac{\sin^2\psi}{R_2} = \frac{(m+nK_1)(K-K_2)^2}{(E+2FK+GK^2)(K_1-K_2)} + \frac{(m+nK_2)(K-K_1)^2}{(E+2FK+GK^2)(K_2-K_1)}$$

$$= \frac{(m+nK_1)(K-K_2)^2-(m+nK_2)(K-K_1)^2}{(E+2FK+GK^2)(K_1-K_2)} =$$

$$\frac{(K_1-K_2)[nK^2+2mK-m(K_1+K_2)-nK_1K_2]}{(K_1-K_2)(E+2FK+GK^2)}$$

But

$$K_1+K_2 = \frac{GL-En}{Fn-Gm}$$

$$K_1K_2 = \frac{Em-FL}{Fn-Gm}$$. Therefore,

$$\frac{\cos^2\psi}{R_1} + \frac{\sin^2\psi}{R_2} = nK^2+2mK - \left[\frac{m(GL-En)+n(Em-FL)}{Fn-Gm}\right]$$

$$= \frac{L+2mK+nK^2}{E+2FK+GK^2} = R_n$$. Therefore our proposition is established and we have

$$\text{(XII)} \quad \frac{\cos^2\psi}{R_1} + \frac{\sin^2\psi}{R_2} = R_n$$

So in conclusion we have shown: (1) that the study of the curvature of all curves on a surface reduces to that of a plane section; (2) that the study of the curvature of all oblique sections reduces to that of a normal section; (3) that the study of the curvature of all normal sections reduces to that of the principal normal

sections.

Chapter (II).

Rotation Surfaces of Constant, Negative Curvature.

We have seen from our previous discussion that the Gaussian curvature, which is the curvature of a surface gives us

$$\frac{1}{R_1 R_2} = K = \frac{LN - M^2}{EG - F^2}.$$

Now in order to have negative curvature this expression for $\frac{1}{R_1 R_2}$ must be negative; and for constant negative curvature it must be negative and constant and this must hold for any point P whatever of the surface.

We have found that the principal radii of curvature are given by the following equation:

$$(LN - M^2)R^2 - (GL - 2FM + EN)R + EG - F^2 = 0; \quad (VII);$$

or if $z = f(xy)$, we have

$$(rt - s^2)R^2 - \sqrt{1+p^2+q^2}\left[(1+p^2)t + (1+q^2)r - 2pqs\right]R + (1+p^2+q^2)^2 = 0. \quad (XI)$$

Suppose for the surface of revolution we take the form (XI) and see if it will permit of any simplification. The equations of a surface of revolution can be written thus:

$$x = r_1 \cos w$$

$$y = r_1 \sin w$$

$$z = f(r_1) = f(x,y)$$

$$p = \frac{\partial z}{\partial x} = f'(r_1)\frac{\partial r_1}{\partial x}; \qquad r = \frac{\partial^2 z}{\partial x^2} = f''(r_1)\left(\frac{\partial r_1}{\partial x}\right)^2 + f'(r_1)\frac{\partial^2 r_1}{\partial x^2}$$

$$q = \frac{\partial z}{\partial y} = f'(r_1)\frac{\partial r_1}{\partial y}; \qquad t = \frac{\partial^2 z}{\partial y^2} = f''(r_1)\left(\frac{\partial r_1}{\partial y}\right)^2 + f'(r_1)\frac{\partial^2 r_1}{\partial y^2}$$

$$s = f''(r_1)\cdot\frac{\partial r_1}{\partial x}\cdot\frac{\partial r_1}{\partial y} + f'(r_1)\frac{\partial^2 r_1}{\partial x \partial y};$$

also

$$r_1^2 = x^2 + y^2 \quad ; \text{ then}$$

$$r_1\frac{\partial r_1}{\partial x} = x \quad ,$$

$$r_1\frac{\partial r_1}{\partial y} = y$$

and consequently

$$\left(\frac{\partial r_1}{\partial x}\right)^2 + r_1\frac{\partial^2 r_1}{\partial x^2} = 1, \quad \frac{\partial r_1}{\partial x}\cdot\frac{\partial r_1}{\partial y} + r_1\frac{\partial^2 r_1}{\partial x \partial y} = 0, \quad \left(\frac{\partial r_1}{\partial y}\right)^2 + r_1\frac{\partial^2 r_1}{\partial y^2} = 1,$$

therefore

$$p = f'\frac{x}{r_1}; \quad q = f'\cdot\frac{y}{r_1}; \quad r = f''\cdot\frac{x^2}{r_1^2} + f'\frac{r_1^2 - x^2}{r_1^3} = f''\frac{x^2}{r_1^2} + f'\cdot\frac{y^2}{r^3};$$

$$s = f''\cdot\frac{xy}{r_1^2} - f'\cdot\frac{xy}{r_1^3}; \quad t = f''\cdot\frac{y^2}{r_1^2} + f'\cdot\frac{x^2}{r_1^3}$$

Before we substitute these values in our equation for the principal radii of curvature it is of importance to notice that they admit of considerable simplification since a surface of revolution is the same at all points along a parallel circle. Consequently we can always choose our axes so that the principal meridian through the considered point coincides with the ZX plane, and the results obtained for the considered point P hold equally for every point on the parallel circle through P, since r_1 is the same for every point of the circle and $z = f(xy) = f(r_1)$ is the equation of the surface.

Making such a choice of our point P we have

$y=0$ therefore $x=r_1$ and

$p=f', \quad q=0; \quad r=f'', \quad s=0, \quad t=f'\frac{1}{r_1}$.

Accordingly our equation for R becomes

$$\frac{f'f''}{r_1}R^2-\left[f''+(1+f'^2)\frac{f'}{r_1}\right]\sqrt{1+f'^2}\,R+(1+f'^2)^2=0,$$

which can also be written

$$\text{(XIII)}.\quad R^2-\left(\frac{r_1\sqrt{1+f'^2}}{f'}+\frac{(1+f'^2)^{\frac{3}{2}}}{f''}\right)R+\frac{r_1(1+f'^2)^2}{f'f''}=0$$

The two roots for R are

$$\frac{r_1\sqrt{1+f'^2}}{f'} \quad\text{and}\quad \frac{(1+f'^2)^{\frac{3}{2}}}{f''}.$$

The product of the roots gives

$$\text{(XIV)}.\quad R_1R_2=\frac{r_1(1+f'^2)^2}{f'f''},$$

which is the reciprocal of K the Gaussian curvature.

Now let us find the equation of all surfaces of revolution that have constant negative curvature. Let the curvature be represented by $-\frac{1}{a^2}$. Then we have

$$R_1R_2=\frac{r(1+f'^2)^2}{f'f''},$$ (where the subscript of r_1 is dropped)

that is,

$$(1)\quad \frac{r\left(1+\left(\frac{dz}{dr}\right)^2\right)^2}{\frac{dz}{dr}\cdot\frac{d^2z}{dr^2}}=-a^2$$

By means of a simple calculation we hereby

obtain the sought differential equation of all surfaces of revolution that have constant negative curvature.

In equation (1), set

$$\frac{dz}{dr} = t; \quad \frac{d^2z}{dr^2} = \frac{dt}{dr};$$

we then have

$$\frac{r(1+t^2)^2}{t \cdot \frac{dt}{dr}} = -a^2,$$

and

$$\frac{r(1+t^2)^2}{t} = -a^2 \frac{dt}{dr};$$

separate the variables

$$r\,dr = -a^2 \frac{t}{(1+t^2)^2}\, dt;$$

integrating

$$\frac{r^2+K^2}{2} = \frac{a^2}{2} \frac{1}{1+t^2}. \qquad \text{Set } K^2 = a^2 - b^2;$$

then

$$\frac{r^2+a^2-b^2}{2} = \frac{a^2}{2(1+t^2)}; \quad \text{accordingly}$$

$$t^2 = \frac{a^2-(r^2+a^2-b^2)}{r^2+a^2-b^2},$$

$$t = \pm\sqrt{\frac{b^2-r^2}{r^2+a^2-b^2}},$$

that is,

(XV). $$dz = \pm\sqrt{\frac{b^2-r^2}{r^2+a^2-b^2}}\, dr$$

This is our desired diffential equation which will represent the different surfaces that invite our attention in these pages.

Before we notice the different types of sur-

faces represented by equation (XV), let us first find the general expression for the element of length.

We have as the equations of a surface of revolution

$$x = r\cos w$$
$$y = r\sin w$$
$$z = f(r)$$

We have seen that the linear element ds is

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

Let $r = u$, $w = v$ and we have

$$E = (x_u)^2 + (y_u)^2 + (z_u)^2 = \cos^2 w + \sin^2 w + f'(r)^2 = 1 + f'(r)^2$$

$$F = x_u x_v + y_u y_v + z_u z_v = -r\cos w\sin w + r\cos w\sin w + 0 = 0$$

$$G = (x_v)^2 + (y_v)^2 + (z_v)^2 = r^2\sin^2 w + r^2\cos^2 w + 0 = r^2;$$

therefore,

(XVI) $$ds^2 = (1 + f'(r)^2)\,dr^2 + r^2\,dw^2;$$

which is true for all surfaces of revolution.

For our surfaces we have

$$f'(r) = \frac{dz}{dr} = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}.$$ Substituting in XVI. we have

$$ds^2 = \left(1 + \frac{b^2 - r^2}{r^2 + a^2 - b^2}\right)dr^2 + r^2\,dw^2;$$ or simplified

(XVII) $$ds^2 = \frac{a^2 dr^2}{r^2+a^2-b^2} + r^2 d\omega^2.$$

Now let us consider the different surfaces of constant negative curvature that can be obtained from XV by assigning different values to a and b.

Suppose first we let $a=b$. We have then

(1) $$dz = \frac{\sqrt{b^2-r^2}}{r}$$ which gives

(2) $$dz = \frac{\sqrt{b^2-x^2}}{x}$$ as the equation of the generating curve, i.e. the principal meridian. But (2) is the well known form of the equation of the tractrix, therefore (1) represents the surface obtained by revolving a tractrix in the xz plane about the z axis. This surface is called the pseudosphere. When $a=b$, the linear element takes the form, as seen from XVII,

(XVIII) $$ds^2 = \frac{a^2 dr^2}{r^2} + r^2 d\omega^2$$

and this is at once seen to represent the form of the linear element of a surface of revolution whose meridian is the tractrix.

By making proper substitutions the equation for the linear element of the pseudosphere can be reduced to a still simpler form.

Set $r = a \sin\phi$, then $dr = a\cos\phi\, d\phi$, therefore,

$$ds^2 = a^2[\cot^2\phi\, d\phi^2 + \sin^2\phi\, d\omega^2]$$

Now set

$$\cot\phi\, d\phi = du; \qquad w = v;$$

integrating

$$u = \log \sin\phi,$$

taking the exponential

$$e^u = \sin\phi,$$

therefore,

$$\text{(XIX)} \qquad ds^2 = a^2(du^2 + e^{2u}dv^2)$$

2nd. Take $b < a$. We see from our equation,

$$dz = \pm\sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\, dr,$$

that r can take all values $< b$ including zero. If $r = 0$ the meridians must cut the z axis for that point; and by taking the section that coincides with the zx plane, we see that the meridians cut the axis of the surface at an angle whose tangent is

$$\frac{dr}{dz} = \sqrt{\frac{a^2 - b^2}{b^2}}.$$

Furthermore this surface has a cuspidal edge since dz is both positive and negative. In this surface also the linear element given in (XVII) takes a simpler form by placing

$$r = \sqrt{a^2 - b^2}\,\frac{e^u - e^{-u}}{2}; \quad w = \frac{av}{\sqrt{a^2 - b^2}};$$

then

$$dr = \sqrt{a^2 - b^2}\left(\frac{e^u + e^{-u}}{2}\right)du; \quad dr^2 = (a^2 - b^2)\left(\frac{e^u + e^{-u}}{2}\right)^2 du^2,$$

consequently,

$$ds^2 = \frac{a^2(a^2-b^2)\left(\frac{e^u+e^{-u}}{2}\right)^2 du^2}{(a^2-b^2)\left(\frac{e^u-e^{-u}}{2}\right)^2 + a^2-b^2} + \frac{a^2(a^2-b^2)\left(\frac{e^u-e^{-u}}{2}\right)^2 dv^2}{(a^2-b^2)}$$

$$= \frac{a^2(a^2-b^2)\left(\frac{e^u+e^{-u}}{2}\right)^2 du^2}{(a^2-b^2)\left(\frac{e^u+e^{-u}}{2}\right)^2} + \frac{a^2\left(\frac{e^u-e^{-u}}{2}\right)^2 dv^2}{1}$$

$$= \text{(XX)} \qquad a^2\left[du^2 + \left(\frac{e^u-e^{-u}}{2}\right)^2 dv^2\right]$$

3d. Take $b > a$. From our equation

$$dz = \pm\sqrt{\frac{b^2-r^2}{r^2+a^2-b^2}}\,dr$$

we see that

$$\sqrt{b^2-a^2} < r < b.$$

Therefore r, since it has a minimum, cannot cut the axis; it also has a maximum consequently there are two cuspidal edges. In this type of surface the equation for the linear element also admits of simplification. Set

$$r = \sqrt{b^2-a^2}\left(\frac{e^u+e^{-u}}{2}\right); \qquad w = \frac{av}{\sqrt{b^2-a^2}}$$

$$dr = \sqrt{b^2-a^2}\left(\frac{e^u-e^{-u}}{2}\right).$$

we have.

$$ds^2 = \frac{a^2(b^2-a^2)\left(\frac{e^u-e^{-u}}{2}\right)^2 du^2}{(b^2-a^2)\left(\frac{e^u+e^{-u}}{2}\right)^2 + a^2-b^2} + \frac{a^2(b^2-a^2)\left(\frac{e^u+e^{-u}}{2}\right)^2 dv^2}{(b^2-a^2)}$$

$$= \frac{a^2(b^2-a^2)\left(\frac{e^u-e^{-u}}{2}\right)^2 du^2}{(b^2-a^2)\left(\frac{e^u-e^{-u}}{2}\right)^2} + a^2\left(\frac{e^u+e^{-u}}{2}\right)^2 dv^2.$$

Therefore we have

$$\text{(XXI)} \qquad ds^2 = a^2\left[du^2 + \left(\frac{e^u+e^{-u}}{2}\right)^2 dv^2\right].$$

After this short survey of the surfaces of revolution under our consideration we find that we have obtained three different types of such surfaces, all of which have constant negative curvature. We have

the pseudospherical type when $a = b$;
" elliptical " " $a > b$;
" hyperbolical " " $a < b$.

We shall now proceed to plot carefully the generating curves, that is, the meridians, which revolved, produce these surfaces of revolution. We have prepared four carefully constructed models which represent these surfaces together with some of their most important properties. We have made two pseudospheres; one large one suitable for class room demonstrations, and one smaller one which is to show the applicability of these three types of surfaces since it has the same constant a as those of the elliptic and hyperbolic types. For the purpose of shaping the tissues which are used to show the applicablity, we made three moulds from our pseudospherical surfaces.

We have furthermore represented the lines of curvature, the asymptotic lines and the geodetic lines which are situated on these surfaces.

We have to deal first with the pseudosphere. This is produced as we have seen by the revolution of the tractrix. The equation of the pseudosphere is obtained from our general equation (XV) by taking $a = b$; consequently we have as its equation

$$dz = \pm \frac{\sqrt{a^2 - r^2}}{r}\, dr$$

By taking the meridian section which coincides with the XZ plane we have

$$dz = \frac{\sqrt{a^2 - r^2}}{r} \quad \text{where } r = x$$

Integrating we obtain as the equation of the tractrix

$$\text{(XXII).} \quad z = -\sqrt{a^2 - r^2} + a \log \frac{a + \sqrt{a^2 - r^2}}{r}$$

Set $a = 10$; $r = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$ which is a maximum. We shall proceed to calculate the value of z for these values of r together with all necessary intermediary values of r.

$$\log_e(z) = \frac{\log_{10}(z)}{\log_{10}(e)}$$

$$\log_{10}(e) = .43429448\,, \quad \text{therefore,}$$

$$\log_e(z) = \log_{10}(z).(2.302\dot{5}9)$$

Set $2.30259 = Y$. Then $\log_e(z) = Y.\log_{10}(z)$.

Then

$$z = -\sqrt{a^2-r^2} + a \log \frac{a+\sqrt{a^2-r^2}}{r} =$$

$$-\sqrt{99} + a \log_e \frac{a+\sqrt{a^2-r^2}}{r}.$$

But $\log \sqrt{99} = .99782$

$\therefore \sqrt{99} = 9.95$,

also

$$a \log \frac{a+\sqrt{a^2-r^2}}{r} = 10 \log_e \left(\frac{10+\sqrt{99}}{1}\right) = (10)(Y) \log(10+\sqrt{99}),$$

$\log(10+\sqrt{99}) = \log(19.95) = 1.29994$

$\therefore (10)(Y) \log(10+\sqrt{99}) = (23.0259)(1.29994)$

Multiply by logarithms

$\log(23.0259) = 1.36222$

$\log(1.29994) = .11392$

$\therefore \log[(23.0259)(1.2994)] = 1.47614$

$\therefore (23.0259)(1.2994) = 29.9321$

$\therefore z = -9.95 + 29.9321 = 19.9821.$

Proceeding in like manner for successive values of r we obtain after a long calculation the results that are tabulated on the following page. These values of z and r give us, when plotted, a very accurate curve which generates our pseudosphere. It may be remarked that the evolute of the tractrix is the catenary a curve which generates a surface that has its two prin-

cipal radii of curvature equal, at every point but of contrary signs.

Table of values for the tractrix.

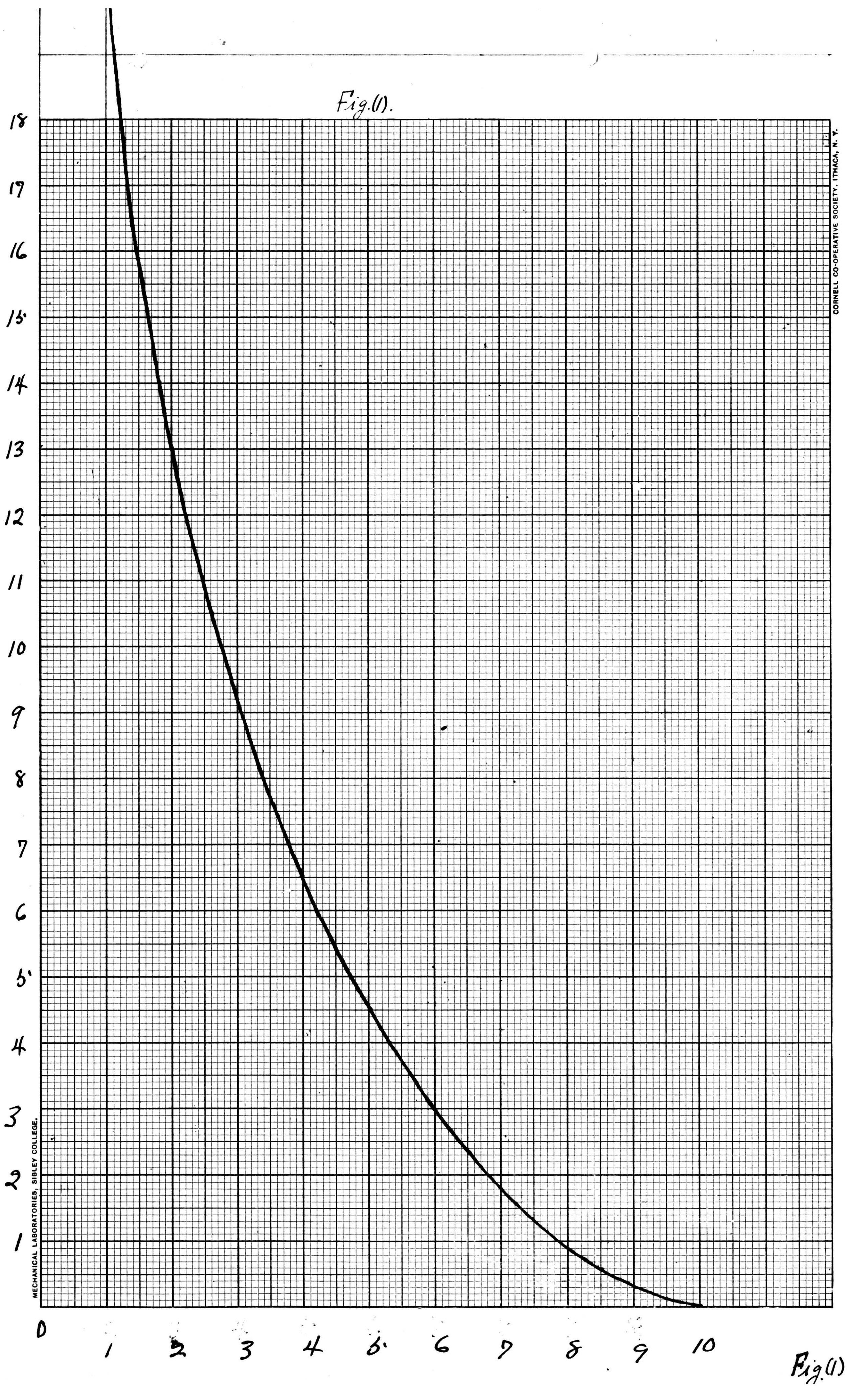
r	z	r	z
.5	26.8962	6	2.9862
1	19.9821	6.5	2.3613
1.2	18.1697	7	1.8147
1.5	15.9589	7.5	1.3392
2	13.0257	8	.9315
2.5	10.9825	8.5	.5891
3	9.1989	9	.3125
3.5	7.7335	9.5	.1079
4	6.5030	10	0
4.5	5.4364		
5	4.5092		
5.5	3.6984		

A plot of the curve is shown on the following page. From this, a templet was made of galvanized tin which served to guide us in making our pseudosphere.

In our second surface of the elliptic type we have

$$dz = \sqrt{\frac{b^2-r^2}{r^2+a^2-b^2}}\,dr$$

Fig.(1).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1
0
1
2
3
4
5
6
7
8
9
10
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Fig.(1)

which is at once recognized to be an elliptic integral and hence cannot be integrated by ordinary methods. Therefore we must have recourse to some special method in order to obtain the values of z and r necessary for plotting our curve. We shall use a special device in that we obtain our curve by the method of areas. For this purpose let us take

$$a^2 = b^2 - K$$

in the expression

$$dz = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\, dr,$$

then

$$b^2 = a^2 - K \quad , K \text{ positive}.$$

Set

$$K = n a^2$$

then

$$dz = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\, dr = \sqrt{\frac{a^2 - (K + r^2)}{r^2 + K}} = \sqrt{\frac{(1-n)a^2 - r^2}{r^2 + n a^2}}\, dr$$

Now let us set

$$y = \sqrt{\frac{(1-n)a^2 - r^2}{r^2 + n a^2}}$$

and find the curve where y is a function of r, then

$$z = \int^{r = 1, 2, \ldots \sqrt{80}} y\, dr$$

which is the area between the curve $y = \phi(r)$, the r axis and any two given ordinates. Then this area for successive values of r will represent the values of z for the same values of r.

We notice also that the value of y for any value of r is $\frac{dz}{dr}$, that is, it is the slope of the tangent at the point corresponding to the same value of r in the curve $z = f(r)$.

Let us take $a = 10$ and $n = \frac{1}{5}$.

We have

$$y = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}} = \sqrt{\frac{(1-n)a^2 - r^2}{r^2 + na^2}} = \sqrt{\frac{80 - r^2}{r^2 + 80}}$$

where $\sqrt{80}$ is a maximum value for r.

Take $r = 0, 1, 2, 3, \text{------} \sqrt{80}$.

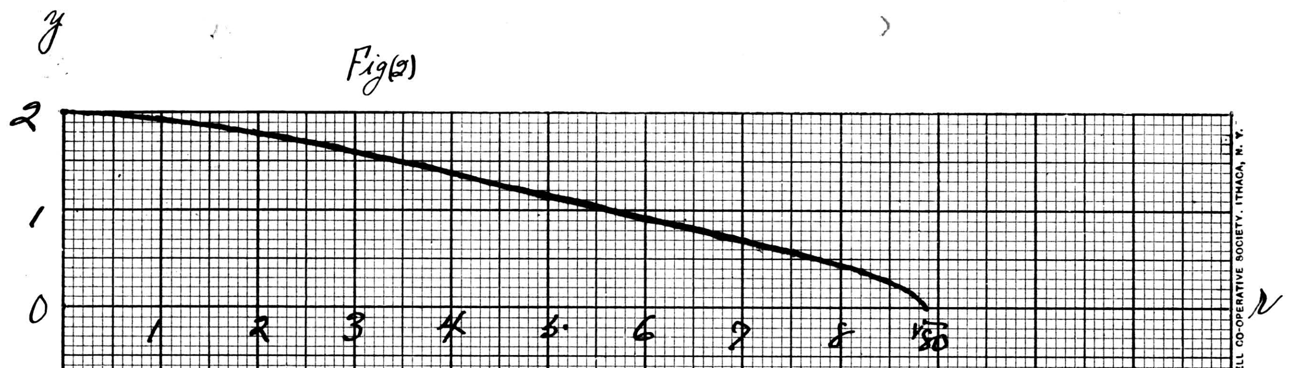
For $r = 0$; $y = \sqrt{\frac{80}{20}} = 2$.

By means of logarithms proceed in like manner for successive values of r. We thus obtain the following table of values for r and y:

r	y	r	y
0	2	6	.8864
1	1.9405	7	.67023
2	1.7804	8	.4344
3	1.5647	8.5	.28946
4	1.3333	$\sqrt{80}$	0
5	1.1055		

This gives the curve $y = \phi(r)$. On the following page Fig.(2) this curve is plotted and from this

Fig(2)


y
2
1
0
1 2 3 4 5 6 7 8 √80
r
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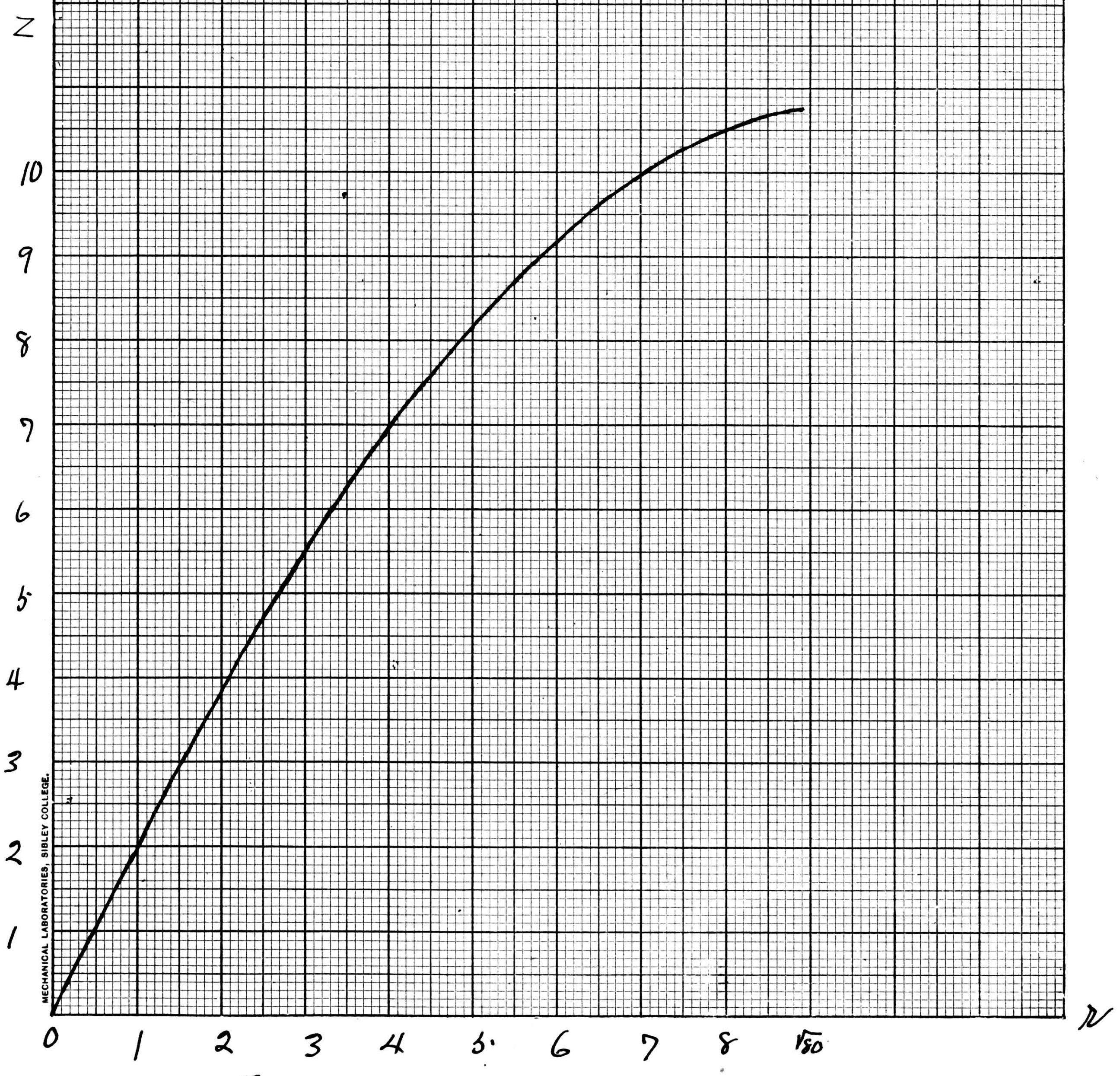
Z
10
9
8
7
6
5
4
3
2
1
0 1 2 3 4 5 6 7 8 √80
r
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Fig (3)

the area is computed for successive values of r; and the area corresponding to a given value of of r in the curve $y = \phi(r)$ is the value of z which corresponds to the same value of r in the curve $z = f(r)$. Thus we have

$$z = \int_0^{r=1,2,\cdots\sqrt{80}} y\,dr$$

Computing z by means of areas, we have below the following table of values for r and z.

r	z	r	z
0	0	6	9.15
1	1.98	7	9.95
2	3.85	8	10.5
3	5.45	8.5	10.68
4	6.95	$\sqrt{80}$	10.75
5	8.15		

This gives us the required curve $z = f(r)\Big]_{r=x}$ which is also plotted on the preceding page Fig.(3)
The revolution of this meridian curve about the z axis gives the second type of our surfaces, that is, the elliptic type.

It now remains for us to consider the surface formed from our general equation when

$$a < b.$$

As before we have to resort to our special

device since we have, again, an elliptic integral. Let us proceed as before to calculate our curve $z = f(r)$ by means of areas. We have

$$dz = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\, dr,$$

where $a < b$.

Let $a^2 = b^2 - K$,

then $b^2 = a^2 + K$; K positive

Let $K = na^2$,

then $dz = \sqrt{\frac{b^2 - r^2}{r^2 + a^2 - b^2}}\, dr = \sqrt{\frac{a^2 + K - r^2}{r^2 - K}}\, dr = \sqrt{\frac{(n+1)a^2 - r^2}{r^2 - na^2}}\, dr$

$\frac{dz}{dr} = \sqrt{\frac{n+1}{-n}}$ at the origin. Hence the slope is imaginary, therefore the curve does not pass through the origin.

Set

$$y = \sqrt{\frac{(n+1)a^2 - r^2}{r^2 - na^2}};$$

take $a = 10$ and $n = \frac{1}{4}$;

y will not assume a real value until $r = 5$, since by making the above substitutions we have

$$y = \sqrt{\frac{125 - r^2}{r^2 - 25}}.$$

For $r = 5$ we have

$$\frac{dz}{dr} = y = \sqrt{\frac{125 - 25}{25 - 25}} = \infty$$

which shows that a tangent to the curve $z = f(r)$ at the point where $r = 5$ is parallel to the z axis. For $r = 5.1$

$$y = \sqrt{\frac{98.99}{1.01}} = 9.9$$

Proceed in like manner for successive values of r we thus obtain the corresponding value of y. As before the area, between the curve $y = \phi(r)$ and the r axis and the ordinates $r=0$, $r=r_0$, will represent the value of z for the same value of r, r_0. We give here a table of values for r and y. where $5 < r < \sqrt{125}$.

r	y	r	y
5	∞	7	1.7795
5.05	198	8	1.2507
5.1	9.9	9	.8863
5.2	6.93	10	.5774
5.3	5.6005	11	.2041
5.5	4.248	$\sqrt{125}$	0
6	2.8455		

The curve $y = \phi(r)$ whose values are given above is shown on the following page. Proceeding, as explained, to calculate the values of z by means of the curve $y = \phi(r)$ we see that we shall have no difficulty in obtaining the areas that give the value of z when $r > 5.2$. But for values of $r \leqq 5.2$ our curve $y = \phi(r)$ becomes, as seen from Fig(4), almost parallel to the y axis and ap-

Line r = 5 parallel to y axis.

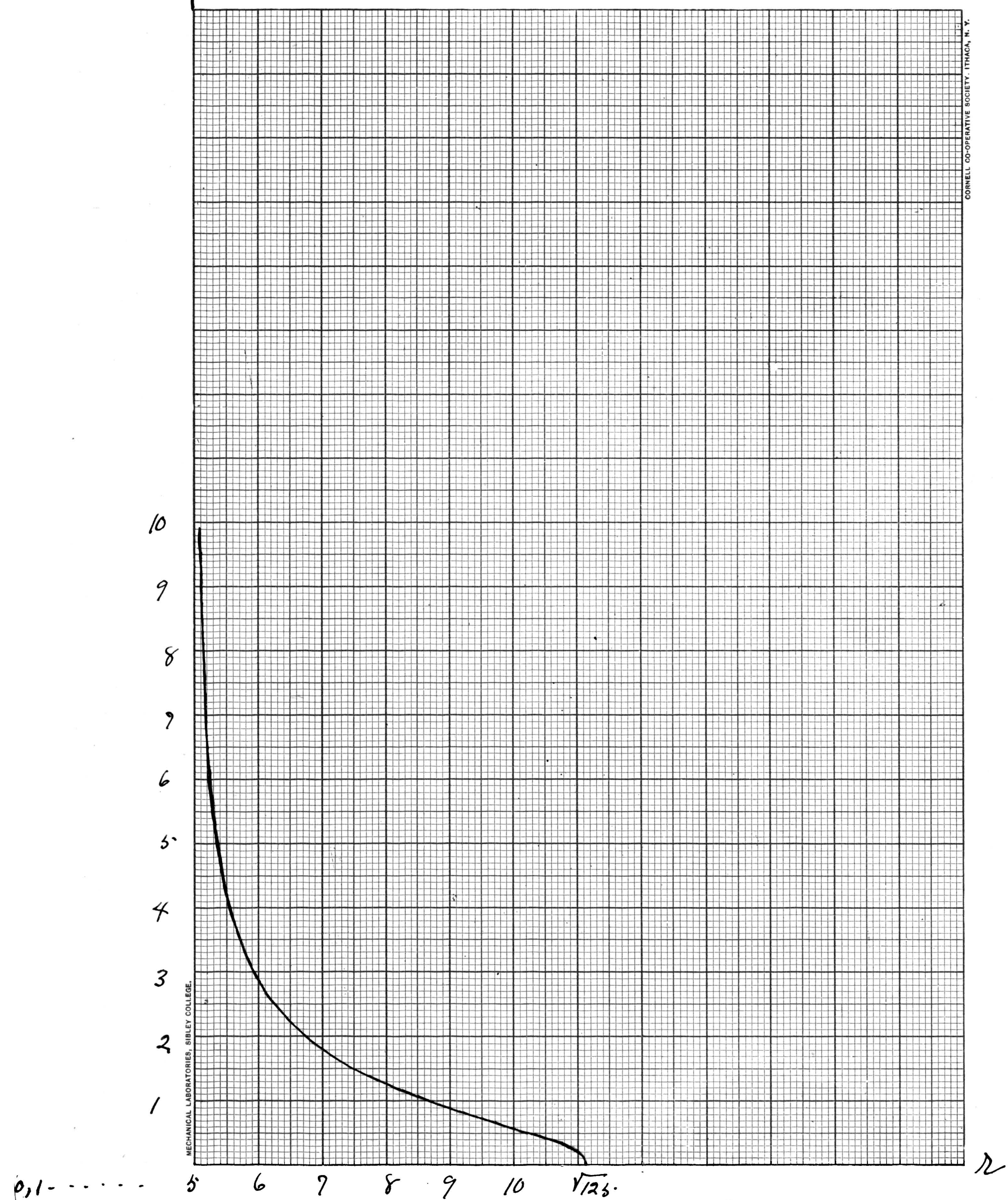


Fig (4)

proaches asymptotically the straight line $r = 5$. It is consequently necessary that we obtain the values of z for $5 \leqq r \leqq 5.2$ by some process other than by the method of areas; using however the method of areas for values of $r > 5.2$.

For the purpose of determining the values of z that correspond to values of $r \leqq 5.2$, let us seek an approximation curve that will give the required area, which corresponds to z, to any required degree of accuracy. We have

$$(1) \quad y = \frac{\sqrt{125 - r^2}}{\sqrt{r^2 - 25}} = \phi(r).$$

Let us take as our approximation curve

$$(2) \quad y_1 = \frac{10}{\sqrt{r^2 - 25}} = \psi(r);$$

let $r \doteq 5$ then

in (1) and (2) $\phi(r)$ and $\psi(r)$ increase and approach infinity, and $\psi(r) \geqq \phi(r)$. also the $\int \psi(r) dr = \int \frac{10}{\sqrt{r^2 - 25}} dr = 10 \log(r + \sqrt{r^2 - 25})$ exists and is finite and determinate for $r = 5$. Then the integral of $\phi(r)$, that is z, also exists. For we have the theorem, that if $\phi(r)$ increases and becomes infininite and $\psi(r) \geqq \phi(r)$ and the integral of $\psi(r)$ exists then the integral of $\phi(r)$ exists also.

Having shown that the integrals of $\psi(r)$ and

and $\phi(r)$ exist, let us consider the difference

$$y-y_1=\left[\sqrt{\frac{125-r^2}{r^2-25}}-\frac{10}{\sqrt{r^2-25}}\right].$$

For $r=5+\epsilon$ we have

$$y-y_1=\frac{\sqrt{100-10\epsilon-\epsilon^2}-10}{\sqrt{\epsilon^2+10\epsilon}};$$

expanding we have

$$y-y_1 = \frac{\frac{1}{2}(\epsilon^2+10\epsilon)\cdot\frac{1}{10}+\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{1}{1000}\,\frac{(\epsilon^2+10\epsilon)^2}{2!}+\cdots}{\sqrt{\epsilon^2+10\epsilon}};$$

but

$$\epsilon=r-5$$
$$d\epsilon=dr,$$

then the area between $y-y_1$ is equal to

$$\int_0^\epsilon\left[\frac{1}{20}\sqrt{\epsilon^2+10\epsilon}-\frac{1}{8000}(\epsilon^2+10\epsilon)^{3/2}+\frac{1}{1600000}(\epsilon^2+10\epsilon)^{5/2}+\cdots\right]d\epsilon,$$

and by the theorem of mean

$$=\frac{1}{20}\sqrt{\xi^2+10\xi}\int_0^\epsilon d\epsilon-\frac{1}{8000}(\xi^2+10\xi)^{3/2}\int_0^\epsilon d\epsilon+\frac{1}{1600000}(\xi^2+10\xi)^{5/2}\int_0^\epsilon d\epsilon+\cdots$$

$$f(\epsilon)=\sqrt{\epsilon^2+10\epsilon},\quad 0<\xi<\epsilon.$$

$$<\frac{1}{20}\epsilon\sqrt{\epsilon^2+10\epsilon}-\frac{1}{8000}\cdot\epsilon(\epsilon^2+10\epsilon)^{3/2}+\frac{1}{1600000}\epsilon(\epsilon^2+10\epsilon)^{5/2}+\cdots$$

and it is at once seen that the expression on the right, which is > the area between the curves, approaches zero for small values of ϵ. If we

designate by A the difference between the areas of the curves y and y_1, we have

$$A\Big]_{\epsilon=0}^{\epsilon=.05} \leqq$$

$$\frac{1}{(20)^2}\sqrt{\frac{1}{400}+\frac{1}{2}} - \frac{1}{8000}\cdot\frac{1}{20}\left(\frac{1}{400}+\frac{1}{2}\right)^{3/2} +$$

$$\frac{1}{1600000}\cdot\frac{1}{20}\left(\frac{1}{400}+\frac{1}{2}\right)^{5/2} + \frac{5}{1280000000}\cdot\frac{1}{20}\left(\frac{1}{400}+\frac{1}{2}\right)^{7/2} + \cdots$$

which converges absolutely since it is a power series in $\left(\frac{1}{400}+\frac{1}{2}\right)$. Then

$$A\Big]_{\epsilon=0}^{\epsilon=.05} < \frac{1}{(20)^2}\cdot\frac{11}{20} = \frac{11}{8000}$$ since all terms after

the first can be dropped and since the first and largest term on the right is dropped our expression on the right is increased and our inequality strengthened since each term is smaller than the preceding. Consequently the error we are liable to make by taking the approximation curve instead of the given curve is, at most, less than $\frac{11}{80}$ of the smallest square shown on our coordinate paper.

By similar calculations we find that

$$A\Big]_{\epsilon=0}^{\epsilon=.1} \leqq \frac{1}{200}\sqrt{1.01} < \frac{1}{200}$$

and

$$A\Big]_{\epsilon=0}^{\epsilon=.2} \leqq \frac{1}{100}\sqrt{2.4} \leqq \frac{16}{1000} = \frac{2}{125}$$

The first of these last two differences is less than $\frac{1}{2}$ of one of the smallest squares; the second at most but a little more than one of these small squares. Therefore we have found that our approximation curve can be used with all necessary accuracy in plotting for $5 < r \leq 6.2$

We shall now proceed with the calculation of z for these three values of r by means of our approximation curve. For the first we have

$$z = 10\int_{5}^{5.05} \frac{dr}{\sqrt{r^2-25}}$$

$$= 10 \log(r + \sqrt{r^2-25})\Big]_{r=5}^{r=5.05}$$

$$= -10 \log 5 + 10 \log(5.05 + \sqrt{.5025})$$

But as we have seen

$$\log_e(z) = \log_{10}(z).(2.30259) = \gamma \log_{10}(z).$$

Hence we have

$$z = 10\gamma(-.69897) + .76034) = 10\gamma(.06137) = 1.4131$$

$$\text{since } \log 10\gamma = 1.36222$$

$$\text{" } \log .06137 = 8.78796 - 10$$

$$\log[(10\gamma).(.06137)] = .15018$$

We proceed in like manner for $r = 5.1$ and $r = 5.2$; finding in the first case

$$z = 1.4131;$$

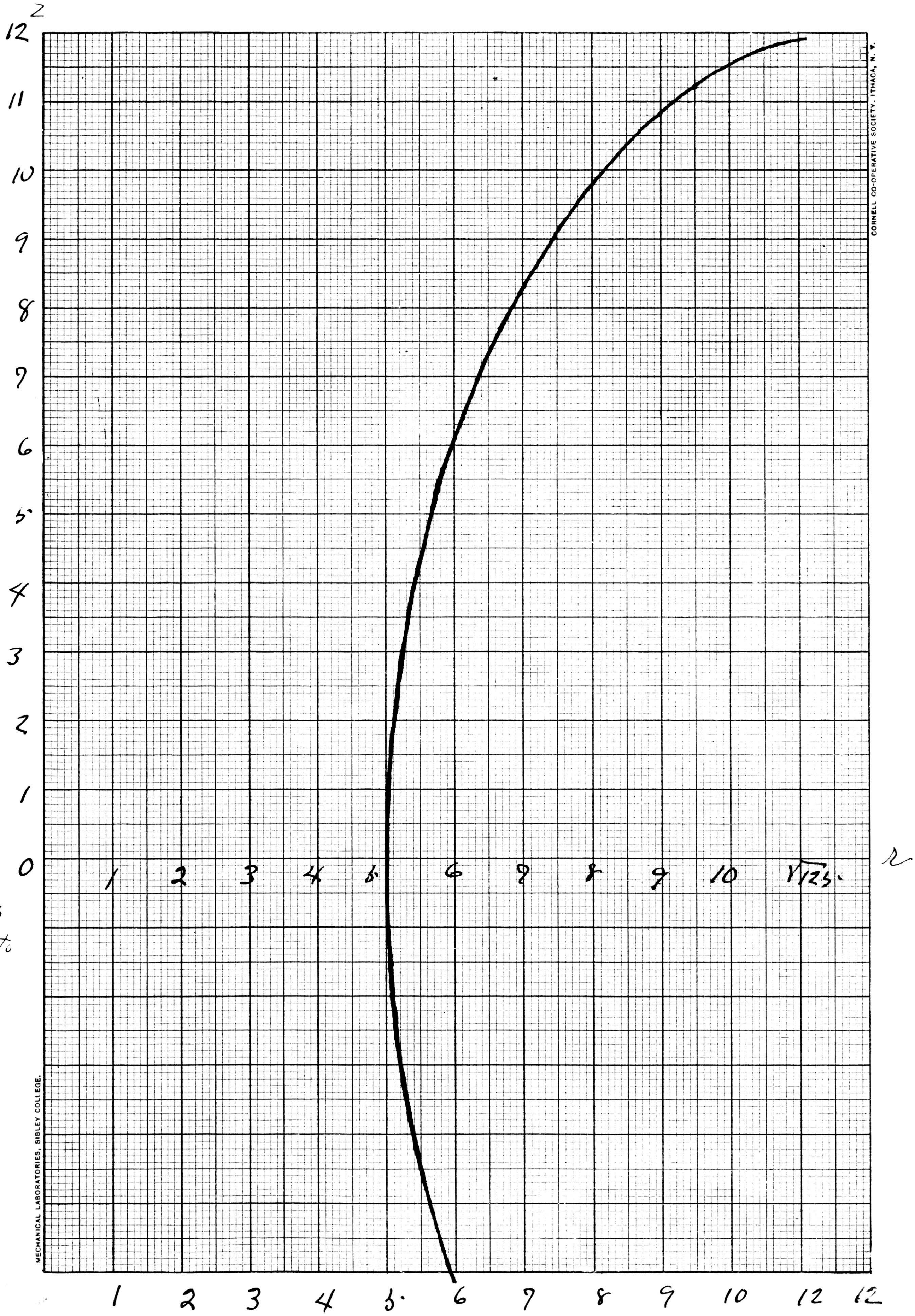
and in the second case

$z = 2.8185$

The remaining values we can, as shown, obtain by the method of areas from our curve $y = \phi(r)$. Writing out the three values already obtained and the remaining values which easily follow from our curve Fig.(5) on the following page, we have the table of values as given below:

r	z	r	z
5	0	6.4	7.15
5.05	1.4131	6.6	7.59
5.1	1.997	6.8	7.99
5.2	2.8185	7	8.35
5.3	3.45	7.5	9.16
5.4	3.96	8	9.83
5.5	4.41	8.5	10.4
5.6	4.81	9	10.86
5.7	5.17	9.5	11.26
5.8	5.51	10	11.56
5.9	5.83	10.5	11.79
6	6.13	11	11.91
6.2	6.67	$\sqrt{125}$	11.93

After making templets of the shape of the generating curves, we began the construction



This curve is symmetrical to both axes.

Fig (5)

of our models. We first made a rude frame work of wood, and covered this with screen wire. Over this we spread plaster of Paris and shaped it as nearly as possible into the desired form. While in the process of hardening we used knives to smooth the surface down. After this we left the plaster to harden, and the remainder of the work in forming the models had to be done with the file. After dressing the models into the correct shape we polished them with sand and emery paper. When this was done we painted the body of the models white, and the ends where they are cut off we painted black. We shall now proceed to calculate the different lines traced on these surfaces.

Chapter (III).

Lines of Curvature, Geodetic and Asymptotic Lines.

Let us consider first the lines of curvature. Lines of curvature are lines such that the normals to the surface at different points of one of them form a developable surface.

Let

$$z = f(x\,y)$$

be the equation of the surface; then the equations of the normal to the surface are

$$\frac{(X-x)}{p} = \frac{Y-y}{q} = -\frac{Z-z}{1},$$

then

$$\begin{cases} X = -pZ + x + pz \\ Y = -qZ + y + qz \end{cases}$$

Now in order that this straight line generate a developable surfaces it is necessary and sufficient (See Goursat (n° 223)) that the two equations obtained by differentiating the two above equations with respect to P should be satisfied by the same value of Z. Differentiating we obtain

$$(1),\ -Z\,dp + d(x+pz) = 0$$

$$(2),\ -Z\,dq + d(y+qz) = 0,$$

consequently,

$$\frac{d(x+pz)}{dp} = \frac{d(y+qz)}{dq}$$

or

$$\frac{dx+p\,dz}{dp} = \frac{dy+q\,dz}{dq}.$$

But

$$\begin{cases} dz = p\,dx + q\,dy \\ dp = r\,dx + s\,dy \\ dq = s\,dx + t\,dy\,; \end{cases}$$

therefore, our equation becomes

$$(XXIII),\quad \frac{(1+p^2)\,dx + pq\,dy}{r\,dx + s\,dy} = \frac{pq\,dx + (1+q^2)\,dy}{s\,dx + t\,dy}$$

This equation can be reduced to a simpler form for surfaces of revolution, and then admits of an easy interpretation. Let us, first, write the above equation in determinant form. It thus becomes

$$\begin{vmatrix} r & s & t \\ 1+p^2 & pq & 1+q^2 \\ dy^2 & -dy\,dx & dx^2 \end{vmatrix} = 0.$$

But in surfaces of revolution this determinant assumes a simpler form, as we shall see by first taking our surface in the form

$$z = f(xy)$$
$$x = x = u$$
$$y = y = v$$

$$L = \frac{1}{H}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} 0 & 0 & z_{xx} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \frac{z_{xx}}{H} = \frac{r}{H}.$$

$$M = \frac{1}{H}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} 0 & 0 & z_{xy} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \frac{z_{xy}}{H} = \frac{s}{H}$$

$$N = \frac{1}{H}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} 0 & 0 & z_{yy} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \frac{z_{yy}}{H} = \frac{t}{H}$$

Hence we find that

$$L : M : N :: r : s : t.$$

We also have

$$E = (x_u)^2 + (y_u)^2 + (z_u)^2 = 1 + z_x^2 = 1 + p^2$$

$$F = x_u x_v + y_u y_v + z_u z_v = z_x z_y = pq$$

$$G = (x_v)^2 + (y_v)^2 + (z_v)^2 = 1 + z_y^2 = 1 + q^2.$$

Therefore, our equation can be written thus

$$(XXIV), \quad \begin{vmatrix} L & M & N \\ E & F & G \\ \left(\frac{dv}{du}\right)^2, & -\frac{dv}{du}, & 1 \end{vmatrix} = 0$$

In a surface of revolution we have

$$\begin{cases} x = r\cos w \\ y = r\sin w \\ z = f(r) \end{cases} \qquad \text{where } \begin{cases} r = u \\ v = w. \end{cases}$$

Then

$$E = (x_u)^2 + (y_u)^2 + (z_u)^2 = \cos^2 w + \sin^2 w + f'(r)^2 = 1 + f'(r)^2$$

$$F = x_u x_v + y_u y_v + z_u z_v = -r\sin w \cos w + r\sin w \cos w + 0 = 0$$

$$G = (x_v)^2 + (y_v)^2 + (z_v)^2 = r^2\sin^2 w + r^2\cos^2 w + 0 = r^2.$$

We also have

$$L = \frac{1}{H}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} 0, & 0, & f''(r) \\ \cos w, & \sin w, & f'(r) \\ -r\sin w, & r\cos w, & 0 \end{vmatrix} = \frac{r f''(r)}{H}$$

$$M = \frac{1}{H}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} -\sin w & \cos w & 0 \\ \cos w & \sin w & f'(r) \\ -r\sin w & r\cos w & 0 \end{vmatrix} = 0$$

$$N = \frac{1}{H}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} -r\cos w, & -r\sin w & 0 \\ \cos w, & \sin w, & f'(r) \\ -r\sin w & r\cos w & 0 \end{vmatrix} = \frac{r^2 f'(r)}{H}$$

Consequently, we can now write the equation of a line of curvature of a surface of revolution in the following form.

$$\begin{vmatrix} 1+f'(r)^2 & 0 & r^2 \\ r f''(r) & 0 & r^2 f'(r) \\ \left(\frac{dw}{dt}\right)^2 & -\frac{dw}{dt}\cdot\frac{dr}{dt} & \left(\frac{dr}{dt}\right)^2 \end{vmatrix} = 0,$$

or dividing through by r we obtain

(XXV)
$$\begin{vmatrix} 1+f'(r)^2 & 0 & r^2 \\ f''(r) & 0 & r\,f'(r) \\ \left(\frac{dw}{dt}\right)^2 & -\frac{dw}{dt}\cdot\frac{dr}{dt}, & \left(\frac{dr}{dt}\right)^2 \end{vmatrix} = 0$$

This gives us
$$\frac{dw}{dt}\cdot\frac{dr}{dt}\left| r\,f'(r)\left[1+f'(r)^2\right] - r^2 f''(r)\right| = 0$$

If the factor in the bars does not vanish we have as the equation of the lines of curvature

(XXVI)
$$\frac{dw}{dt}\cdot\frac{dr}{dt} = 0\ ;$$

that is, w = const, and r = const. Hence they are the lines that represent the meridians and the parallels of a surface of revolution.

Again, we can readily see from our definition of the lines of curvature that the meridians and parallels form such lines. For if we consider a surface of revolution which revolves around the z axis and select, for instance, a normal to the surface in the XZ plane, then as the generating plane revolves the normal NP will describe a circular cone which is a developable surface; therefore, the point P

traces a line of curvature on the surface of revolution. On the other hand the normals along a meridian lie in a plane and therefore constitute a developable surface and the meridian is accordingly the line of curvature. We traced these lines on our surfaces and painted them red.

The geodetic lines next invite our attention. The geodetic line, which is the shortest line between two given points on a surface, has the property that its osculating plane passes through the normal to the surface. (For proof see Joachimsthal's "Anwendung der Differential und Integralrechnung" articles 86 and 87.) Let us take this for the definition of a geodetic line. We have as the equation of the osculating plane

$$\text{(I)}\quad (y'z''-z'y'')(X-x)+(z'x''-x'z'')(Y-y)+(x'y''-y'x'')(Z-z)=0$$

and the equation of the normal to the surface is where the surface is given in the form

$$F(xyz) = 0 \qquad \text{is}$$

$$\frac{X-x}{P} = \frac{Y-y}{Q} = \frac{Z-z}{R}$$

where $\qquad P = F'_{(x)}\,,\ Q = F'_{(y)}\,,\ R = F'_{(z)}.$

Then the condition that the osculating plane

(plane) should contain the normal to the surface, or that the angle between them should be zero, is that,

$$P(y'z''-z'y'')+Q(z'x''-x'z'')+R(x'y''-y'x'')=0$$

This equation can be written in determinant form

$$\text{(XXVII)}, \quad \begin{vmatrix} x' & x'' & P \\ y' & y'' & Q \\ z' & z'' & R \end{vmatrix} = 0$$

For obtaining the geodetic line on a surface of revolution let us write (XXVII) in the following form

$$\begin{vmatrix} d^2x+\lambda dx+\mu P & d^2x & P \\ d^2y+\lambda dy+\mu Q & d^2y & Q \\ d^2z+\lambda dz+\mu R & d^2z & R \end{vmatrix} = 0,$$

where λ and μ are at present unknown but are some functions of (x, y, z)

Now if this determinant vanishes for all values of d^2x, d^2y, d^2z, P, Q and R, then

$$d^2x+\lambda dx+\mu P=0$$
$$d^2y+\lambda dy+\mu Q=0$$
$$d^2z+\lambda dz+\mu R=0.$$

But a surface of revolution is commonly represented by the equations

$$\begin{cases} x = r\cos w \\ y = r\sin w \\ z = f(r) \end{cases}, \quad r^2 = x^2 + y^2 .$$

accordingly

$$P = f'(r)\cdot r_x = f'\cdot\frac{x}{r}$$
$$Q = f'(r)\cdot r_y = f'\cdot\frac{y}{r}$$
$$R = -1$$

Our equations may now be written

(1) $d^2x + \lambda dx + \mu f'\cdot\frac{x}{r} = 0$

(2) $d^2y + \lambda dy + \mu f'\cdot\frac{y}{r} = 0$

(3) $d^2z + \lambda dz - \mu \qquad = 0$

Let us multiply them in order by dx, dy and dz, and add. We then obtain

$$(4) \quad dx\,d^2x + dy\,d^2y + dz\,d^2z$$
$$+\lambda(dx^2 + dy^2 + dz^2)$$
$$+\mu\left(f'\frac{x}{r}dx + f'\frac{y}{r}dy - dz\right)$$
$$= ds\,d^2s + \lambda\,ds^2 + 0\cdot\mu = d^2s + \lambda\,d^2s = 0,$$

since $dz = f'(r)\frac{dr}{dx} + f'(r)\frac{dr}{dy}$
$= f'(r)\frac{x}{r} + f'(r)\frac{y}{r}.$

Considering equations (1) and (2) we see that we can eliminate μ by multiplying (1) by y and (2) by x and subtracting. So we obtain:

$$y\,d^2x - x^2dy + \lambda(y\,dx - x\,dy) = 0.$$

Let us set

$$y\,dx - x\,dy = \eta$$

then

$$d\eta + \lambda\eta = 0$$

also

$$\frac{d\eta}{\eta} = -\lambda\ ;$$

but from (4)

$$\lambda = \frac{d^2s}{ds},$$

therefore

$$\frac{d\eta}{\eta} = \frac{d^2s}{ds}\ ;$$

which can also be written

$$\frac{ds.d\eta - \eta d^2s}{ds^2} = 0.$$

Integrating we obtain

$$\frac{\eta}{ds} = \text{const. } c$$

Substituting the values for η and ds, we have the following equation

$$(5)\qquad y\,dx - x\,dy = c\sqrt{dx^2 + dy^2 + dz^2}.$$

Since

$$\begin{cases} x = r\cos w \\ y = r\sin w \\ z = f(r) \end{cases} \quad \text{and} \quad \begin{cases} dx = -r\sin w\,dw + \cos w\,dr \\ dy = r\cos w\,dw + \sin w\,dr \\ dz = f'(r)\,dr \end{cases}$$

equation (5) can be written as follows:

$$-r^2\sin^2 w\,dw + r\sin w\cos w\,dr - (r^2\cos^2 w\,dw + r\sin w\cos w\,dr)$$
$$= c\sqrt{r^2\sin^2 w\,dw^2 + 2r\sin w\cos w\,dw\,dr + \cos^2 w\,dr^2 + r^2\cos^2 w\,dw^2 + 2r\sin w\cos w\,dw\,dr + \sin^2 w\,dr^2 + f'(r)^2dr^2}$$

$$= -r^2 dw = c\sqrt{dr^2 + r^2 dw^2 + f'(r)^2 dr^2}$$

or $\quad dw^2\left[r^4 - c^2 r^2\right] = c^2 dr^2\left[1 + f'(r)^2\right],$

thus we see the variables can be separated and we obtain the integrable equation

$$dw = \frac{c\,dr}{r}\sqrt{\frac{1+f'^2}{r^2-c^2}}.$$

If we choose a meridian as the meridian of reference $w=0$, and call the radius of the parallel circle where the geodetic line cuts the meridian r_0, and let the radius of the parallel circle where the line cuts the meridian $w=w$ be r, then we have

(XXVIII) $$w = c\int_{r_0}^{r} \frac{dr}{r}\sqrt{\frac{1+f'^2}{r^2-c^2}}$$

Let us apply this formula in obtaining the equation of the geodetic line of the pseudosphere. In the pseudosphere

$$\frac{dz}{dr} = f'(r) = \frac{\sqrt{a^2-r^2}}{r}.$$

Then

$$w = \int_{r_0}^{r} \frac{dr}{r}\sqrt{\frac{1+\frac{a^2-r^2}{r^2}}{r^2-c^2}}$$

$$= ac\int_{r_0}^{r} \frac{dr}{r^2\sqrt{r^2-c^2}} = -\left.\frac{ac\sqrt{r^2-c^2}}{c^2 r}\right]_{r_0}^{r}$$

$$= -\frac{a}{c}\left.\frac{\sqrt{r^2-c^2}}{r}\right]_{r_0}^{r} \quad \text{(XXIX)}.$$

This gives a very simple equation for the geodetic lines and by assigning different values to r the values of w can be readily found. But in the case of the geodetic lines it is unecessary to compute the values of w since a string stretched along the surface will give a geodetic line.

In considering our surfaces of the elliptic and hyperbolic types we have

$$f'(r) = \frac{dz}{dr} = \frac{\sqrt{b^2 - r^2}}{\sqrt{r^2+a^2-b^2}},$$

then

$$w = c\int_{r_0}^{r} \frac{dr}{r}\sqrt{\frac{1+\frac{b^2-r^2}{r^2+a^2-b^2}}{r^2-c^2}}$$

$$= \quad ac\int_{r_0}^{r} \frac{dr}{r\sqrt{(r^2-c^2)(r^2+a^2-b^2)}} \qquad \boxed{XXX}.$$

When $a < b$ this equation gives the geodetic lines on our surface of constant negative curvature of the hyperbolic type; when $a > b$ it gives the geodetic lines on our surface of constant negative curvature of the elliptic type. Here, of course, as in the case of the pseudosphere we do not need to compute the values of w in order to draw our lines on the surface. These lines are painted blue.

Let us now consider the asymptotic lines that are represented on our three types of surface. On surfaces of negative curvature there are at each point two tangents for which the corresponding normal section has an infinite radius of curvature; these tangents are the asymptotes of the indicatrix. The asymptotic lines are the lines situated on the surface which are tangent at each of their points to one of these asymptotes. Then the condition for an asymptotic line is that the radius of curvature R_n should be infinite, i.e.

$$R_n = \frac{E + 2FK + GK^2}{L + 2MK + NK^2} = \infty \quad \Big| \left(K = \frac{dv}{du}\right)$$

for all points of the line; or that

$$L + 2MK + NK^2 = L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$$

since $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ can be chosen finite. We have therefore for the equation of our asymptotic line

$$\text{(XXXI)} \quad L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$$

We may remark here that the asymptotic lines can be defined as the lines of the surface where the osculating plane coincides with the tangent plane and their equation derived from this definition.

For the asymptotic lines, as for the lines of curvature and the geodetic lines, we find that their equation also admits of great simplification in surfaces of revolution where the equations of the surface are in the form

$$x = r\cos w$$

$$y = r\sin w$$

$$z = f(r)$$

Set $u = r$, $v = w$.

Then we have

$$L = \frac{1}{H}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} 0 & 0 & f''(r) \\ \cos w, & \sin w, & f'(r) \\ -r\sin w, & r\cos w, & 0 \end{vmatrix} = \frac{r f''(r)}{H}$$

$$M = \frac{1}{H}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} -\sin w & \cos w & 0 \\ \cos w, & \sin w, & f'(r) \\ -r\sin w, & r\cos w & 0 \end{vmatrix} = 0$$

$$N = \frac{1}{H}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \frac{1}{H}\begin{vmatrix} -r\cos w, & -r\sin w & 0 \\ \cos w, & \sin w & f'r \\ -r\sin w, & r\cos w & 0 \end{vmatrix} = \frac{r^2 f'(r)}{H}$$

Substituting these values in (XXXI) and clearing of fractions we have

$$r f''(r)\,dr^2 + r^2 f'(r)\,dw^2 = 0$$

or

(XXXII) $$dw = \pm\sqrt{-\frac{f''(r)}{r f'(r)}}$$

This gives us two asymptotic lines through each point. The assemblage of such lines forms a double system of lines on the surface.

Let us now find the equation of the asymptotic lines on the pseudosphere and then calculate the values necessary for plotting these lines on the surface. We have

$$x = r\cos w$$

$$y = r\sin w$$

$$z = f(r)$$

$$f'(r) = \frac{dz}{dr} = \frac{\sqrt{a^2-r^2}}{r}$$

$$f''(r) = -\frac{a^2}{r^2\sqrt{a^2-r^2}}$$

$$dw = \pm\sqrt{\frac{\frac{a^2}{r^2\sqrt{a^2-r^2}}}{r\frac{\sqrt{a^2-r^2}}{r}}} = \pm\frac{a}{r\sqrt{a^2-r^2}};$$

therefore, $$w = \pm\frac{1}{2}\log\frac{a-\sqrt{a^2-r^2}}{a+\sqrt{a^2-r^2}}.$$

By means of this equation we have calculated the values of w. After reducing the values of w to angular measure we have the results given in the table on the following page.

r	w	r	w
1	171.6°	7	51.24°
1.5	148.4°	8	39.71°
2	131.6°	9	26.77°
3	107.4°	9.5	18.5°
4	89.91°	9.8	11.6°
5	76.12°	10	0°
6	62.95°		

Having obtained these values we took a large sheet of paper and divided the circumference of a circle, which was drawn on it, into degrees. On the center of this circle we placed the axis of our pseudosphere and by means of the graduated scale on the paper we marked the desired values of w on the edge of the surface. To obtain the radius corresponding to a given angle we used the templet on which we drew a line vertical to the plane of the edge of the pseudosphere; i.e. vertical when the templet rests in its proper place against the surface. Then for radius 10 we marked the distance zero on the templet which gave a point on the edge of the surface and on the zeroth meridian.

For $r = 9$ we marked a distance 1 from the vertical line on the templet to the edge of the generating curve. Then placing our templet vertical at $26°.77^0$ and letting it rest against our surface we marked on the surface the point where the designated point on the templet touched; this gave us for the desired point on the surface; viz, $(9, 26.77^0)$. In like manner we obtained the remaining values and then traced the curve through the points thus located. After tracing the asymptotic lines we painted them black.

We shall now determine the asymptotic lines on the surface of constant negative curvature of the elliptic type.

Our equation (XXXII) is

$$w = \pm \int \sqrt{\frac{-f''(r)}{r\, f'(r)}}$$

We have for the equation of the surface

$$x = r \cos w$$

$$y = r \sin w$$

$$z = f(r)$$

$$f'(r) = \frac{dz}{dr} = \frac{\sqrt{b^2 - r^2}}{\sqrt{r^2 + a^2 - b^2}}$$

$$f''(r) = \frac{d^2z}{dr^2} = -\frac{a^2 r}{\sqrt{b^2 - r^2}\,(r^2 + a^2 - b^2)^{\frac{3}{2}}};$$

then

$$w = \pm \int \sqrt{\frac{\dfrac{a^2 r}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{3/2}}}{\dfrac{(b^2-r^2)^{\frac{1}{2}}}{(r^2+a^2-b^2)^{\frac{1}{2}}}}}\,.$$

$$= a\int \frac{dr}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}}\,.$$

We have again an elliptic integral which we shall treat as before by the method of areas.

In our surface of the elliptic type we have $a > b$, and in constructing our model we took

$$b^2 = a^2 - K = a^2(1-n); \text{ where } K = na^2.$$

We further chose $a = 10$, $n = \frac{1}{5}$.

Let us now set

$$y = \frac{a}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}} =$$

$$\frac{a}{\left\{(1-n)a^2-r^2\right\}^{\frac{1}{2}}(na^2+r^2)^{\frac{1}{2}}}$$

We have seen that the area between the curve $y = \phi(r)$, the r axis and the ordinates corresponding to $r\Big]_0^{r_0}$ will give the value of w for the corresponding value of r. We can thus obtain the values of w after first

obtaining the values of y and then plotting the curve it represents.

Proceeding with the calculation by assigning different values to r we thus obtain the following table of values for r and y.

r	y	r	y
0	.25	7	.2162
1	.2455	8	.2727
2	.23415	8.5	.374
3	.22045	8.75	.5489
4	.20833	8.9	1.126
5	.201	$\sqrt{80}$	∞
6	.2015		

The curve $y = \phi(r)$ is plotted on the next page. And from this the values of w are obtained and reduced to angular measure. It is to be noted that, although $y = \infty$ for $r = \sqrt{80}$, we obtain a value for $r = 8.9$ which is as near the edge as we can possibly draw it; and since the area between the ordinate $r = 8.9$ and $r = \sqrt{80}$ is only added to our w for $r = 8.9$ it will not affect any result

Distances on y axis have been multiplied by 10.

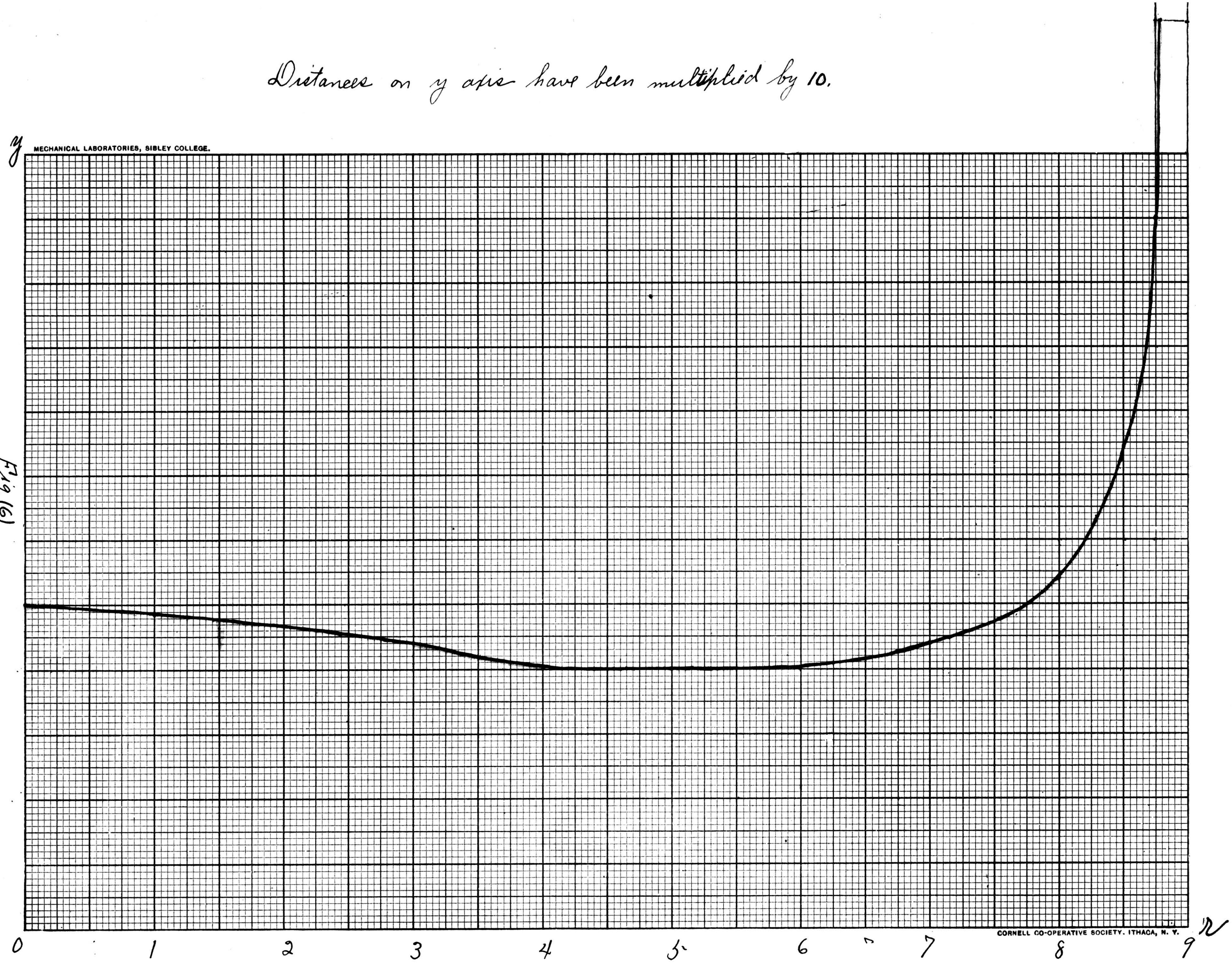


Fig. (6)

we obtain for w previous to value of w corresponding to $r = 8.9$. But this is as near the edge as we can draw consequently whatever the area might be it would not affect our representation on the surface. But this area is easily shown to be finite between $r\Big]_{8.9}^{\sqrt{80}}$; since

$$\int_{8.9}^{\sqrt{80}} \frac{dr}{\left[(1-n)(a^2-r^2)\right]^{\frac{1}{2}}(na^2+r^2)^{\frac{1}{2}}} <$$

$$\int_{8.9}^{\sqrt{80}} \frac{dr}{(80-r^2)^{\frac{1}{2}}\sqrt{20}} = \frac{\sqrt{20}}{2}\int_{8.9}^{\sqrt{80}} \frac{dr}{\sqrt{80-r^2}}$$

$$= \frac{\sqrt{20}}{2} \sin^{-1}\frac{r}{\sqrt{80}}\Big]_{8.9}^{\sqrt{80}} = \frac{\sqrt{20}}{2}\left[\sin^{-1}1 - \sin^{-1}\frac{8.9}{\sqrt{80}}\right] =$$

$$\frac{\sqrt{20}}{2}\left[\frac{\pi}{2} - \frac{129\pi}{270}\right] = \frac{\sqrt{20}}{2}.\pi.\frac{4}{135} = \frac{2\sqrt{20}.\pi}{135} < \frac{\pi}{15}$$

So the area neglected is less than $\frac{\pi}{15}$ of a unit.

We obtain the following table of values for r and w where w is expressed in degrees; ~~firstly~~, of course, w is computed in radians by the method of areas. See, Fig.(6), our area curve.

We have then the afore-named values for r and w.

r	w	r	w
0	0°	6	75.5°
1	14.21°	7	87.3°
2	27.3°	8	101.04°
3	40.5°	8.5	110.3°
4	52.6°	8.75	116.6°
5	64°	8.9	123.8°

Also we know for $r = \sqrt{80}$ $w < 139°$.
We plotted these values on our elliptic surface in the same manner as on the pseudosphere and then drew the asymptotic lines.

This brings us now to our surface of the hyperbolic type. To find the asymptotic lines let us proceed as before by the method of areas.

We have

$$w = a\int \frac{dr}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}}$$

$$a < b \text{, i.e. } b > a$$

$$b^2 = a^2 + K = a^2(1+n) \quad \text{where } K = na^2$$

We took $a = 10$; $y = \frac{1}{4}$

Let

$$y = \frac{a}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}} = \frac{a}{[a^2(1+n)-r^2]^{\frac{1}{2}}(r^2-na^2)^{\frac{1}{2}}}.$$

Assigning r and computing y we have the table of values below.

r	y	r	y
5	∞	8	.205
5.05	1.4145	9	.2015
5.1	.9951	10	.2309
5.3	.5779	10.5	.282
5.5	.4484	11	.5103
6	.3203	11.1	.7542
7	.2341	$\sqrt{125}$	∞

Plotting the curve represented by $y=\phi(r)$ we get that which is represented on the following pages, where y becomes infinite at $r=5$ and $r=\sqrt{125}$. See Fig. (9)

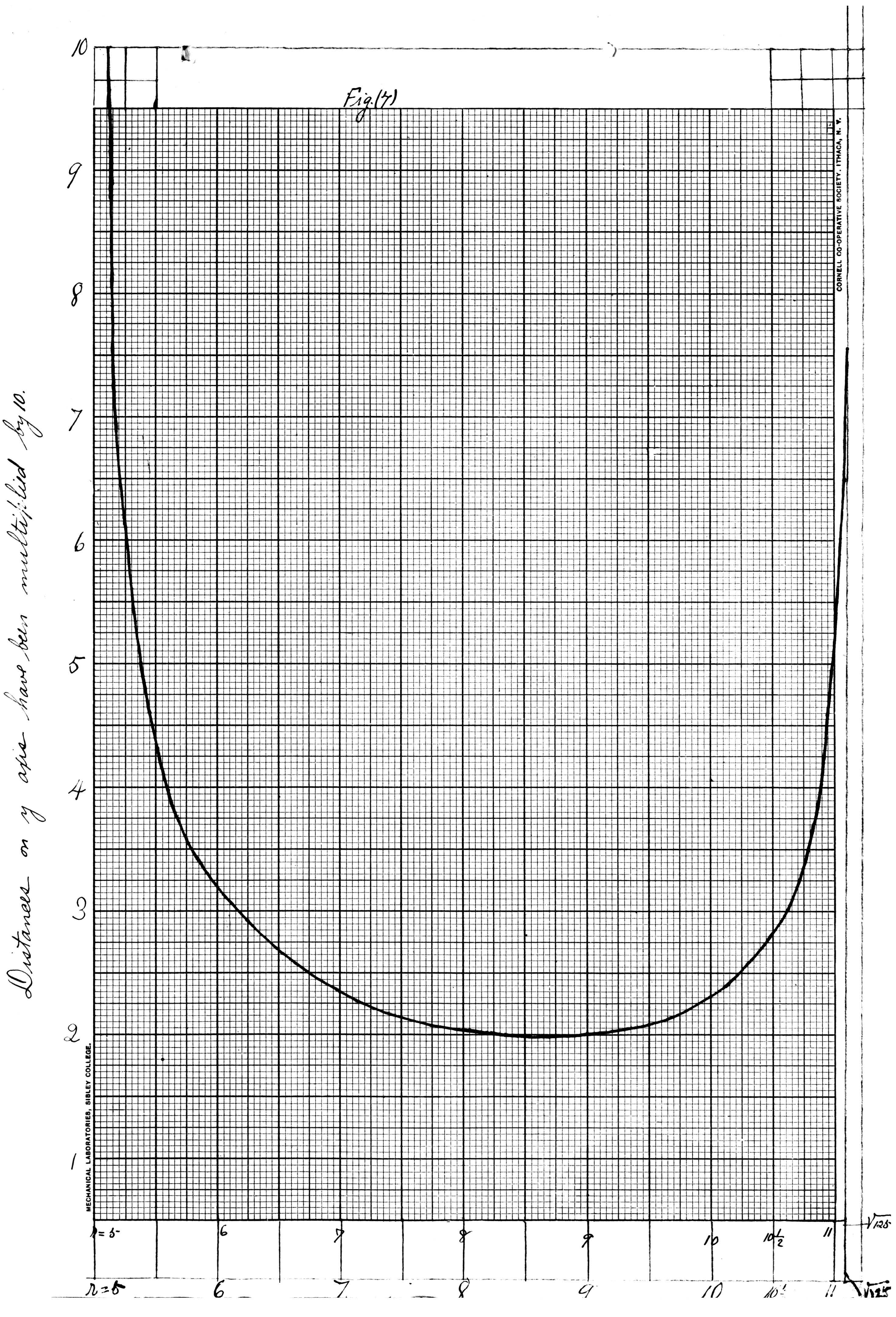
Now before we can obtain the values of w by the method of areas we are obliged first to find the area between $y=\phi(r)$, the r axis and the ordinates $r=5$ and $r=5.1$ say.

$$y = \frac{a}{(r^2-b^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}}$$

$$= \frac{10}{(r^2-25)^{\frac{1}{2}}(125-r^2)^{\frac{1}{2}}}.$$

But $(125-r^2)^{-\frac{1}{2}} =$

Fig.(4)


Distances on y axis have been multiplied by 10.
10
9
8
7
6
5
4
3
2
1
r = 5
6
7
8
9
10
10½
11
√125
MECHANICAL LABORATORIES, SIBLEY COLLEGE.
CORNELL CO-OPERATIVE SOCIETY. ITHACA, N.Y.

$$\frac{1}{\sqrt{125}}\left[1+\frac{1}{2}\left(\frac{r^2}{125}\right)^1+\frac{3}{8}\left(\frac{r^2}{125}\right)^2+\left(\frac{15}{48}\right)\left(\frac{r^2}{125}\right)^3+\cdots\cdots\right]$$

which is a power series in $\left(\frac{r^2}{125}\right)$ and evidently converges for all values of $r^2<125$.

Let us represent the sum of the series by A. Then let us assign to r in the series A the greatest value in question. Consequently, we have

$$w<\left[\frac{10A}{\sqrt{125}}\int_5^{5\cdot 1}\frac{dr}{\sqrt{r^2-5^2}}\right]=\frac{10A}{\sqrt{125}}\log(r+\sqrt{r^2-25})\Big]_5^{5\cdot 1}$$

This last expression is seen to be finite therefore the area in question i.e. w is also finite.

But

$$A=\left[1+\frac{1}{2}\left(\frac{r^2}{125}\right)^1+\frac{3}{8}\left(\frac{r^2}{125}\right)^2+\left(\frac{15}{48}\right)\left(\frac{r^2}{125}\right)^3+\frac{105}{384}\left(\frac{r^2}{125}\right)^4\right.$$
$$\left.+\frac{945}{3840}\left(\frac{r^2}{125}\right)^5+\frac{10395}{46080}\left(\frac{r^2}{125}\right)^6+\cdots\cdots\right]_5^{5\cdot 1}$$

We have

$$\frac{1}{5}\leqq\frac{r^2}{125}<\frac{3}{10},$$

therefore,

$$A<$$
$$\left[1+\frac{1}{2}\cdot\frac{3}{10}+\frac{3}{8}\cdot\frac{9}{100}+\frac{15}{48}\cdot\frac{27}{1000}+\frac{105}{348}\cdot\frac{81}{10000}+\right.$$
$$\left.\frac{945}{3840}\cdot\frac{243}{100000}+\frac{10395}{46080}\cdot\frac{729}{1000000}+\cdots\cdots\right]$$

Call the part to the right of the last inequlality B. Then

$$B < \left[C = \left(1 + \frac{3}{10} + \left(\frac{3}{10}\right)^2 + \left(\frac{3}{10}\right)^3 + \left(\frac{3}{10}\right)^4 + \left(\frac{3}{10}\right)^5 + \left(\frac{3}{10}\right)^6 + \cdots\right]$$

C is also a power series that converges and and each term of C after the first is greater than the corresponding term of B, therefore, the error in stopping with the seventh term of B is less than the error in stopping with the seventh term of C.

But C can be summed since it is an infinite decreasing geometrical series and we have

$$C = \frac{1}{1-\frac{3}{10}} = \frac{10}{7} = 1.42867;$$

and the sum of the first seven terms gives

$$C_7 = \begin{cases} 1. \\ .3 \\ .09 \\ .027 \\ .0081 \\ .00243 \\ .000729 \end{cases}$$

$$\overline{1.428259}$$

Then $C - C_7 = .00032$.

Therefore we know that the error in stopping with the seventh term of B is $< .00032$.

The first seven terms of B give us

$B_7 =$

$$\left\{\begin{array}{l} 1. \\ .15 \\ .03375 \\ .0084375 \\ .0022148 \\ .0005980 \\ .0001644 \end{array}\right.$$
$$\overline{1.1951647}$$

and the error in stopping with the seventh term is $< .00032$. Hence, at most we certainly have

$$B < 1.1956 < 1.2 .$$

Consequently since $A < B$ we have

$$w < \frac{10A}{\sqrt{125}} \log\left(r + \sqrt{r^2 - 25}\right)\Big]_{5}^{5.1} <$$

$$\frac{10}{\sqrt{125}} (1.2) \log\left(r + \sqrt{r^2 - 25}\right)\Big]_{r=5}^{r=5.1} =$$

$$\frac{10}{\sqrt{125}} (.254)(1.2) = .273$$

But from our curve which we have plotted we can easily show that the area in question is $> .143$, hence the error we are liable to make by taking $.273$ as the true value of w is $< .13$ This is sufficiently accurate for careful plotting. For values of $r > 5.1$ we obtain w by the method of areas. Accordingly we have on the following page a table of values for r and w, where w is expressed in degrees.

r	w	r	w
5·	0	8	69°
5·1	15·6°	9	80.2°
5·3	24·8°	10	92.2°
5·5	30·8°	$10\frac{1}{2}$	100.°
6	41.2°	11	109.5°
7	56.6°	$\sqrt{125}=11.18$	113.°

This line was plotted on our surface of the hyperbolic type similarly to the way we plotted the asymptotic lines on our other two surfaces. The asymptotic lines are, in all cases, painted black.

Chapter (IV.)

Deduction of certain properties of the asymptotic lines pertaining to these three types of surfaces.

Let us consider first the torsion of the asymptotic lines of the pseudosphere. The torsion of a space curve is a measure of the rate of turning of the osculating plane. We have as its equation (See Goursat's Cours D'analyse PP 538)

$$T = -\frac{A^2+B^2+C^2}{\Delta};$$

where T means torsion, and

$$A = y'z'' - z'y''$$
$$B = z'x'' - x'z''$$
$$C = x'y'' - y'x''$$

and

$$\Delta = \begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix}$$

For the equations of the pseudosphere we have

$$x = r \cos w$$
$$y = r \sin w$$
$$z = f(r)$$
$$\frac{dz}{dr} = f'(r) = \frac{\sqrt{a^2 - r^2}}{r};$$

for the equation of the asymptotic lines on the pseudosphere we have

$$dw = \pm \frac{a}{r\sqrt{a^2 - r^2}}$$ (we shall choose + sign for this discussion)

Differentiating with respect to r since we may consider r as the parameter t inasmuch as we have a relation connecting r and w; and the values x, y and z on the pseudosphere must satisfy the equation of the asymptotic line on the pseudosphere; we thus obtain:

$$X' = \cos w - r \sin w \frac{dw}{dr}$$

$$X'' = -2 \sin w \frac{dw}{dr} - r \cos w \left(\frac{dw}{dr}\right)^2 - r \sin w \frac{d^2w}{dr^2}$$

$$X''' = -2 \cos w \left(\frac{dw}{dr}\right)^2 - 2 \sin w \frac{d^2w}{dr^2} + r \sin w \left(\frac{dw}{dr}\right)^3 -$$

$$\cos w \left(\frac{dw}{dr}\right)^2 - 2r \cos w \frac{dw}{dr} \frac{d^2w}{dr} - r \cos w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2}$$

$$- \sin w \frac{d^2w}{dr^2} - r \sin w \frac{d^3w}{dr^3},$$

or we may write

$$X''' = -3 \cos w \left(\frac{dw}{dr}\right)^2 - 3 \sin w \frac{d^2w}{dr^2} + r \sin w \left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right]$$

$$- 3r \cos w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2};$$

also

$$y' = \sin w + r \cos w \frac{dw}{dr}$$

$$y'' = 2 \cos w \frac{dw}{dr} - r \sin w \left(\frac{dw}{dr}\right)^2 + r \cos w \frac{d^2w}{dr^2}.$$

$$y''' = 2 \cos w \frac{d^2w}{dr^2} - 2 \sin w \left(\frac{dw}{dr}\right)^2 - r \cos w \left(\frac{dw}{dr}\right)^3$$

$$- \sin w \left(\frac{dw}{dr}\right)^2 - 2r \sin w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2} - r \sin w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2}$$

$$+ \cos w \frac{d^2w}{dr^2} + r \cos w \frac{d^3w}{dr^3},$$

or

$$y''' = 3 \cos w \frac{d^2w}{dr^2} - 3 \sin w \left(\frac{dw}{dr}\right)^2 - r \cos w \left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right]$$

$$- 3r \sin w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2}.$$

We have

$$\frac{dw}{dr} = \frac{a}{r\sqrt{a^2-r^2}},$$

then

$$\frac{d^2w}{dr^2} = \frac{a(2r^2-a^2)}{r^2(a^2-r^2)^{\frac{3}{2}}},$$

also

$$\frac{d^3w}{dr^3} = a\left[\frac{2a^4+6r^4-6a^2r^2}{r^3(a^2-r^2)^{\frac{5}{2}}}\right]$$

Substituting the values of $\frac{dw}{dr}$, $\frac{d^2w}{dr^2}$ and $\frac{d^3w}{dr^3}$, in the expressions on the preceding page, and reducing, we get

$$x' = \cos w - \frac{a\sin w}{\sqrt{a^2-r^2}}$$

$$x'' = -\frac{a^3}{r(a^2-r^2)^{\frac{3}{2}}}\sin w - \frac{a^2\cos w}{r(a^2-r^2)}$$

$$y' = \sin w + \frac{a\cos w}{\sqrt{a^2-r^2}}$$

$$y'' = \frac{a^3}{r(a^2-r^2)^{\frac{3}{2}}}\cos w - \frac{a^2\sin w}{r(a^2-r^2)}.$$

We shall, for the present, leave x''' and y''' in their original form.

We also have

$$z' = \frac{dz}{dr} = \frac{\sqrt{a^2-r^2}}{r}$$

$$z'' = -\frac{a^2}{r^2\sqrt{a^2-r^2}}$$

$$z''' = \frac{a^2(2a^2 - 3r^2)}{r^3(a^2-r^2)^{\frac{3}{2}}}$$

We now find

$$A = y'z'' - z'y''$$

$$= \frac{-a^2}{r^2\sqrt{a^2-r^2}}\sin w - \frac{a^3\cos w}{r^2(a^2-r^2)} - \left[\frac{a^3\cos w}{r^2(a^2-r^2)} - \frac{a^2\sin w}{r^2\sqrt{a^2-r^2}}\right]$$

$$= -\frac{2a^3}{r^2(a^2-r^2)}\cos w \quad \text{or} \quad = -2a\cos w\left(\frac{dw}{dr}\right)^2;$$

Likewise

$$B = z'x'' - x'z''$$

$$= -\frac{a^3}{r^2(a^2-r^2)}\sin w - \frac{a^2\cos w}{r^2(a^2-r^2)^{\frac{1}{2}}} - \left(-\frac{a^2\cos w}{r^2(a^2-r^2)^{\frac{1}{2}}} + \frac{a^3\sin w}{r^2(a^2-r^2)}\right)$$

$$= -\frac{2a^3}{r^2(a^2-r^2)}\sin w \quad \text{or} \quad = -2a\cos w\left(\frac{dw}{dr}\right)^2$$

$$C = x'y'' - y'x''$$

$$= \left(\cos w - \frac{a\sin w}{\sqrt{a^2-r^2}}\right)\left(\frac{a^3\cos w}{r(a^2-r^2)^{\frac{3}{2}}} - \frac{a^2}{r(a^2-r^2)}\sin w\right)$$

$$-\left(\sin w + \frac{a}{\sqrt{a^2-r^2}}\cos w\right)\left(-\frac{a^3\sin w}{r(a^2-r^2)^{\frac{3}{2}}} - \frac{a^2\cos w}{r(a^2-r^2)}\right)$$

$$= \frac{2a^3}{r(a^2-r^2)^{\frac{3}{2}}}.$$

Then

$$A^2 + B^2 + C^2 = (\sin^2 w + \cos^2 w)\frac{4a^6}{r^4(a^2-r^2)^2} + \frac{4a^6}{r^2(a^2-r^2)^3}$$

$$= \frac{4a^8}{r^4(a^2-r^2)^3}$$

$$\Delta = \begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix} =$$

$$x''' \begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix} - y''' \begin{vmatrix} x' & x'' \\ z' & z'' \end{vmatrix} + z''' \begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}$$

$$= A\,x''' + B\,y''' + C\,z'''$$

Now from our values of A and B in the second form we have

$$A = -2a \cos w \left(\frac{dw}{dr}\right)^2$$

$$B = -2a \sin w \left(\frac{dw}{dr}\right)^2;$$

also we have

$$x''' = -3 \cos w \left(\frac{dw}{dr}\right)^2 - 3 \sin w \frac{d^2w}{dr^2}$$

$$+ r \sin w \left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r \cos w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2}$$

$$y''' = -3 \sin w \left(\frac{dw}{dr}\right)^2 + 3 \cos w \frac{d^2w}{dr^2}$$

$$- r \cos w \left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r \sin w \frac{dw}{dr} \cdot \frac{d^2w}{dr^2}.$$

Then

$$A x''' + B y'''$$

$$= 6a \cos^2 w \left(\frac{dw}{dr}\right)^4 + 6a \sin w \cos w \left(\frac{dw}{dr}\right)^2 \frac{d^2w}{dr^2} -$$

(over)

$$2ar\sin w\cos w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] + 6ar\cos^2 w\,\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^3$$

$$+6a\sin^2 w\left(\frac{dw}{dr}\right)^4 - 6a\sin w\cos w\left(\frac{dw}{dr}\right)^2\frac{d^2w}{dr^2}$$

$$+2ar\sin w\cos w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] + 6ar\sin^2 w\,\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^3$$

$$= 6a\left[\left(\frac{dw}{dr}\right)^4 + r\frac{d^2w}{dr^2}\cdot\left(\frac{dw}{dr}\right)^3\right] = \frac{6a^5}{r^2(a^2-r^2)^3}$$

But $Cz''' = \dfrac{2a^3}{r(a^2-r^2)^{\frac{3}{2}}}\cdot\dfrac{a^2(2a^2-3r^2)}{r^3(a^2-r^2)^{\frac{3}{2}}}$,

since $C = \dfrac{2a^3}{r(a^2-r^2)^{\frac{3}{2}}}$, and $z''' = \dfrac{a^2(2a^2-3r^2)}{r^3(a^2-r^2)^{\frac{3}{2}}}$;

Combining we have

$$Cz''' = \frac{2a^5(2a^2-3r^2)}{r^4(a^2-r^2)^3}.$$

Therefore,

$$\Delta = Ax''' + By''' + Cz'''$$

$$= \frac{6a^5}{r^2(a^2-r^2)^{\frac{3}{2}}} + \frac{2a^5(2a^2-3r^2)}{r^4(a^2-r^2)^3} = \frac{4a^7}{r^4(a^2-r^2)^3};$$

therefore,

$$T = -\frac{A^2+B^2+C^2}{\Delta}$$

$$= -\frac{\dfrac{4a^8}{r^4(a^2-r^2)^3}}{\dfrac{4a^7}{r^4(a^2-r^2)^3}} = -a.$$

This shows that the torsion is independent

of the parameter r, in other words the torsion is constant for the asymptotic lines of the pseudosphere.

Now let us see if the same holds true for our surfaces of the elliptic and hyperbolic types.

In these types of surfaces we have as the equation of the asymptotic lines

$$\frac{dw}{dr} = \frac{a}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}}$$

where

$$x = r\cos w$$

$$y = r\sin w$$

$$z = f(r)$$

$$z' = \frac{dz}{dr} = \frac{\sqrt{b^2-r^2}}{\sqrt{r^2+a^2-b^2}}$$

$$z'' = f''(r) = \frac{-a^2r}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}$$

$$z''' = f'''(r) =$$

$$\frac{-a^2(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{3}{2}} + a^2r\left[\frac{(-r)(r^2+a^2-b^2)^{\frac{3}{2}}}{\sqrt{b^2-r^2}} + \frac{3}{2}\sqrt{b^2-r^2}(r^2+a^2-b^2)^{\frac{1}{2}}2r\right]}{(b^2-r^2)(r^2+a^2-b^2)^3}$$

$$= \frac{-a^2(b^2-r^2)(r^2+a^2-b^2) - a^2r(r^2+a^2-b^2) + 3a^2r^2(b^2-r^2)}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{5}{2}}}$$

$$= \frac{-a^2b^2r^2 - a^4b^2 + b^4a^2 + a^2r^4 + a^4r^2 - a^2b^2r^2 - a^2r^4 - a^4r^2 + a^2b^2r^2 + 3a^2b^2r^2 - 3a^2r^4}{(b^2-r^2)^{3/2}\,(r^2+a^2-b^2)^{5/2}}$$

$$= \frac{2a^2b^2r^2 - 3a^2r^4 - a^4b^2 + b^4a^2}{(b^2-r^2)^{3/2}\,(r^2+a^2-b^2)^{5/2}}$$

$$\frac{dw}{dr} = \frac{a}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}}$$

$$\frac{d^2w}{dr^2} = a\left[\frac{\frac{r}{\sqrt{b^2-r^2}}(r^2+a^2-b^2)^{1/2} - \sqrt{b^2-r^2}\cdot\frac{r}{\sqrt{r^2+a^2-b^2}}}{(b^2-r^2)\,(r^2+a^2-b^2)}\right]$$

$$= a\left[\frac{r(r^2+a^2-b^2) - (b^2-r^2)r}{(b^2-r^2)^{3/2}\,(r^2+a^2-b^2)^{3/2}}\right]$$

$$= a\left[\frac{2r^3+a^2r-2b^2r}{(b^2-r^2)^{3/2}\,(r^2+a^2-b^2)^{3/2}}\right]$$

$$\frac{d^3w}{dr^3} = a\left[\frac{(b^2-r^2)^{3/2}(r^2+a^2-b^2)^{3/2}(6r+a^2-2b^2)}{(b^2-r^2)^3\,(r^2+a^2-b^2)^3}\right]$$

$$-a\left[\frac{(2r^3+a^2r-2b^2r)\left\{(-3r)(b^2-r^2)^{1/2}(r^2+a^2-b^2)^{3/2}+(b^2-r^2)^{3/2}\cdot 3r(r^2+a^2-b^2)^{1/2}\right\}}{(b^2-r^2)^3\,(r^2+a^2-b^2)^3}\right]$$

$$= a\left[\frac{(b^2-r^2)(r^2+a^2-b^2)(6r^2+a^2-b^2)}{(b^2-r^2)^{5/2}\,(r^2+a^2-b^2)^{5/2}}\right]$$

$$-a\left[\frac{(2r^3+a^2r-2b^2r)\left\{(-3r)(r^2+a^2-b^2)+(b^2-r^2)3r\right\}}{(b^2-r^2)^{5/2}\,(r^2+a^2-b^2)^{5/2}}\right] =$$

$$\frac{6ar^6+5a^3r^4-10ab^2r^4+2a^5r^2-2a^3b^2r^2+2ab^4r^2+a^5b^2-3a^3b^4+2ab^6}{(b^2-r^2)^{\frac{5}{2}}(r^2+a^2-b^2)^{\frac{5}{2}}}$$

We have already seen from our calculation of the torsion of the asymptotic lines of the pseudosphere that

$$x' = \cos w - r\sin w\frac{dw}{dr}$$

$$x'' = -2\sin w\frac{dw}{dr} - r\cos w\left(\frac{dw}{dr}\right)^2 - r\sin w\frac{d^2w}{dr^2}$$

$$x''' = -3\cos w\left(\frac{dw}{dr}\right)^2 - 3\sin w\frac{d^2w}{dr^2} + r\sin w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r\cos w\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}.$$

$$y' = \sin w + r\cos w\frac{dw}{dr}$$

$$y'' = 2\cos w\frac{dw}{dr} - r\sin w\left(\frac{dw}{dr}\right)^2 + r\cos w\frac{d^2w}{dr^2}$$

$$y''' = 3\cos w\frac{d^2w}{dr^2} - 3\sin w\left(\frac{dw}{dr}\right)^2 - r\cos w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r\sin w\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}$$

Substituting for $\frac{dw}{dr}$, $\frac{d^2w}{dr^2}$ and $\frac{d^3w}{dr^3}$ their values we have

$$x' = \cos w - \frac{ar\sin w}{\sqrt{b^2-r^2}\sqrt{r^2+a^2-b^2}}$$

$$x'' = \frac{-(2ab^2r^2-a^3r^2+2a^3b^2-2ab^4)\sin w}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}} - \frac{a^2r\cos w}{(b^2-r^2)(r^2+a^2-b^2)}.$$

$$y' = \sin w + \frac{a r \cos w}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}}$$

$$y'' = \frac{(2ab^2r^2 - a^3r^2 + 2a^3b^2 - 2ab^4)\cos w}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}} - \frac{a^2 r \sin w}{(b^2-r^2)(r^2+a^2-b^2)}$$

Leave x''' and y''' in their present form for a time. We now have

$$A = y'z'' - z'y'' =$$

$$-\left(\sin w + \frac{a r \cos w}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}}\right)\left(\frac{a^2 r}{(b^2-r^2)^{\frac{1}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}\right)$$

$$-\frac{\sqrt{b^2-r^2}}{\sqrt{r^2+a^2-b^2}}\left[\frac{(2ab^2r^2 - a^3r^2 + 2a^3b^2 - 2ab^4)\cos w}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}} - \frac{a^2 r \sin w}{(b^2-r^2)(r^2+a^2-b^2)}\right]$$

$$= -\left[\frac{(\sin w \cdot \sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2})\,a^2 r}{(b^2-r^2)(r^2+a^2-b^2)} + \frac{a^3 r^2 \cos w}{(b^2-r^2)(r^2+a^2-b^2)} + \right.$$

$$\left.\frac{(2ab^2r^2 - a^3r^2 + 2a^3b^2 - 2ab^4)\cos w}{(b^2-r^2)(r^2+a^2-b^2)^2} - \frac{a^2 r\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}\,\sin w}{(b^2-r^2)(r^2+a^2-b^2)^2}\right]$$

$$= -\frac{2a(b^2r^2 + a^2b^2 - b^4)\cos w}{(b^2-r^2)(r^2+a^2-b^2)^2} = -\frac{2ab^2\cos w}{(b^2-r^2)(r^2+a^2-b^2)}$$

Likewise

$$B = z'x'' - x'z''$$

$$= \frac{2a(b^2r^2 + a^2b^2 - b^4)\sin w}{(b^2-r^2)(r^2+a^2-b^2)^2} = \frac{2ab^2\sin w}{(b^2-r^2)(r^2+a^2-b^2)}$$

$$C = x'y'' - y'x''$$

Let us take x', x'', y' and y'' in terms of $\frac{dw}{dr}$.

$$x' = \cos w - r\sin w \frac{dw}{dr}$$

$$y'' = 2\cos w \frac{dw}{dr} - r\sin w\left(\frac{dw}{dr}\right)^2 + r\cos w \frac{d^2w}{dr^2}$$

$$x'y'' = 2\cos^2 w \frac{dw}{dr} - r\sin w\cos w\left(\frac{dw}{dr}\right)^2 + r\cos^2 w \frac{d^2w}{dr^2}$$

$$+ r^2\sin^2 w\left(\frac{dw}{dr}\right)^3 - 2r\sin w\cos w\left(\frac{dw}{dr}\right)^2 - r^2\sin w\cos w \frac{dw}{dr}\frac{d^2w}{dr^2}$$

$$y' = \sin w + r\cos w \frac{dw}{dr}$$

$$x'' = -2\sin w \frac{dw}{dr} - r\cos w\left(\frac{dw}{dr}\right)^2 - r\sin w \frac{d^2w}{dr^2}$$

$$y'x'' = -2\sin^2 w \frac{dw}{dr} - r\sin w\cos w\left(\frac{dw}{dr}\right)^2 - r\sin^2 w \frac{d^2w}{dr^2}$$

$$- r^2\cos^2 w\left(\frac{dw}{dr}\right)^3 - 2r\sin w\cos w\left(\frac{dw}{dr}\right)^2 - r^2\sin w\cos w \frac{dw}{dr}\frac{d^2w}{dr^2}$$

Then

$$x'y'' - y'x'' =$$

$$2\cos^2 w \frac{dw}{dr} - 3r\sin w\cos w\left(\frac{dw}{dr}\right)^3 + r\cos^2 w \frac{d^2w}{dr^2} + r^2\sin^2 w\left(\frac{dw}{dr}\right)^3 - r^2\sin w\cos w \frac{dw}{dr}\frac{d^2w}{dr^2}$$

$$+2\sin^2 w \frac{dw}{dr} + 3r\sin w\cos w\left(\frac{dw}{dr}\right)^3 + r\sin^2 w \frac{d^2w}{dr^2} + r^2\cos^2 w\left(\frac{dw}{dr}\right)^3 + r^2\sin w\cos w \frac{dw}{dr}\frac{d^2w}{dr^2}$$

$$= 2\frac{dw}{dr} + r^2\left(\frac{dw}{dr}\right)^3 + 2r\frac{d^2w}{dr^2} =$$

$$\frac{2a}{\sqrt{b^2-r^2}\,\sqrt{r^2+a^2-b^2}} + \frac{2r^2a^3}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}} + \frac{ar(2r^3+a^2r-2b^2r)}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}$$

$$= \frac{2a(b^2-r^2)(r^2+a^2-b^2)+2ar^4+2a^3r^2-2ab^2r^2}{(b^2-r^2)^{\frac{3}{2}}\,(r^2+a^2-b^2)^{\frac{3}{2}}}$$

$$= \frac{2ab^2r^2+2a^3b^2-2ab^4-2ar^4-2a^3r^2+2ab^2r^2+2ar^4+2a^3r^2-2ab^2r^2}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}$$

$$= \frac{2ab^2r^2+2a^3b^2-2ab^4}{(b^2-r^2)^{\frac{3}{2}}\,(r^2+a^2-b^2)^{\frac{3}{2}}} = \frac{2ab^2(r^2+a^2-b^2)}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}$$

$$= \frac{2ab^2}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}}.$$

Then

$$A^2+B^2+\mathcal{C}^2 =$$

$$\frac{4a^2b^4}{(b^2-r^2)^2(r^2+a^2-b^2)^2}\left[(\sin^2 w+\cos^2 w) + \frac{r^2+a^2-b^2}{b^2-r^2}\right] =$$

$$\frac{4a^2b^4}{(b^2-r^2)^2(r^2+a^2-b^2)^2}\left[1 + \frac{r^2+a^2-b^2}{b^2-r^2}\right] = \frac{4a^4b^4}{(b^2-r^2)^3(r^2+a^2-b^2)^2}.$$

We have

$$\Delta = \begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix} =$$

$$x'''\begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix} - y'''\begin{vmatrix} x' & x'' \\ z' & z'' \end{vmatrix} + z'''\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}$$

$$= Ax''' + By''' + \mathcal{C}z'''.$$

We have

$$A = -\frac{2ab^2\cos w}{(b^2-r^2)(r^2+a^2-b^2)} = -K\cos w\left(\frac{dw}{dr}\right)^2 \; ; \; K = \frac{2b^2}{a}$$

$$B = \frac{2ab^2\sin w}{(b^2-r^2)(r^2+a^2-b^2)} = -K\sin w\left(\frac{dw}{dr}\right)^2$$

$$C = \frac{2ab^2}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{1}{2}}}.$$

We also have

$$x''' = -3\cos w\left(\frac{dw}{dr}\right)^2 - 3\sin w\,\frac{d^2w}{dr^2}$$

$$+ r\sin w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r\cos w\,\frac{dw}{dr}\,\frac{d^2w}{dr^2}$$

$$y''' = -3\sin w\left(\frac{dw}{dr}\right)^2 + 3\cos w\,\frac{d^2w}{dr^2}$$

$$- r\cos w\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right] - 3r\sin w\,\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}$$

Then

$$Ax''' + By''' =$$

$$3K\cos^2 w\left(\frac{dw}{dr}\right)^4 + 3K\sin w\cancel{\cos w}\,\frac{dw}{dr}\cdot\frac{d^2w}{dr^2} -$$

$$Kr\sin w\cos w\cancel{\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right]} + 3Kr\cos^2 w\,\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^3 +$$

$$3K\sin^2 w\left(\frac{dw}{dr}\right)^4 - 3K\sin w\cancel{\cos w}\,\frac{dw}{dr}\cdot\frac{d^2w}{dr^2} +$$

$$Kr\sin w\cos w\cancel{\left[\left(\frac{dw}{dr}\right)^3 - \frac{d^3w}{dr^3}\right]} + 3Kr\sin^2 w\,\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^3.$$

$$= 3K\left[\left(\frac{dw}{dr}\right)^4 + r\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^3\right] =$$

$$\frac{6b^2}{a}\left[\frac{a^4}{(b^2-r^2)(r^2+a^2-b^2)} + \frac{ar^2(2r^2+a^2-2b^2)}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}} \cdot \frac{a^3}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{3}{2}}}\right]$$

$$= \frac{6b^2}{a}\left[\frac{a^4r^4+a^6b^2-a^4b^4}{(b^2-r^2)^3(r^2+a^2-b^2)^3}\right] = 6a^3b^2\left[\frac{r^4+a^2b^2-b^4}{(b^2-r^2)^3(r^2+a^2-b^2)^3}\right]$$

We also have

$$z''' = \frac{2a^2b^2r^2-3a^2r^4-a^4b^2+a^2b^4}{(b^2-r^2)^{\frac{3}{2}}(r^2+a^2-b^2)^{\frac{5}{2}}}.$$

Therefore,

$$Cz''' =$$

$$\frac{(2ab^2)(2a^2b^2r^2-3a^2r^4-a^4b^2+a^2b^4)}{(b^2-r^2)^3(r^2+a^2-b^2)^3} =$$

$$\frac{2a^3b^2(2b^2r^2-3r^4-a^2b^2+b^4)}{(b^2-r^2)^3(r^2+a^2-b^2)^3}$$

Therefore,

$$Ax'''+By'''+Cz''' =$$

$$\frac{2a^3b^2\cdot 3(r^4+a^2b^2-b^4)}{(b^2-r^2)^3(r^2+a^2-b^2)^3} + \frac{2a^3b^2(2b^2r^2-3r^4-a^2b^2+b^4)}{(b^2-r^2)^3(r^2+a^2-b^2)^3}$$

$$= \frac{4a^3b^4(r^2+a^2-b^2)}{(b^2-r^2)^3(r^2+a^2-b^2)^3} = \frac{4a^3b^4}{(b^2-r^2)(r^2+a^2-b^2)^2}.$$

Therefore,

$$T = -\frac{A^2+B^2+C^2}{\Delta} =$$

$$-\frac{\dfrac{4a^4b^4}{(b^2-r^2)^3(r^2+a^2-b^2)^2}}{\dfrac{4a^3b^4}{(b^2-r^2)^3(r^2+a^2-b^2)^2}} = -a$$

Hence the torsion of the asymptotic lines of our two surfaces of the elliptic and hyperbolic types ~~is~~ also constant. This is then a common property of the three surfaces we have studied and the torsion is the same in each.

We have seen that there are two asymptotic lines passing through each point of our surfaces of constant negative curvature. It is now of interest to see whether these lines cut at right angles as is the case with minima surfaces of revolution, or whether or not they cut at any constant angle.

To find the angle at which the asymptotic lines through the point (x, y, z) cut, it is necessary to find the equations of the tangents to the two curves at the point $(x\, y\, z)$ and determine the angle between the tangents which is the angle at which the asymptotic lines cut. Let us take the asymptotic lines on the pseudosphere.

We have for the equation of the asymptotic lines on the pseudosphere

$$dw = \pm \frac{a}{r\sqrt{a^2-r^2}}\, dr \quad , \quad \text{where}$$

$$x = r\cos w$$
$$y = r\sin w$$
$$z = f(r)$$
$$\frac{dz}{dr} = f'(r) = \frac{\sqrt{a^2-r^2}}{r}$$ where $r =$ parameter t.

We have seen that

$$x' = \cos w - r\sin w\frac{dw}{dr}$$

$$y' = \sin w + r\cos w\frac{dw}{dr}$$

$$z' = \frac{\sqrt{a^2-r^2}}{r}$$

But the equations of a tangent line is

$$\frac{X-x}{\frac{dx}{dt}} = \frac{Y-y}{\frac{dy}{dt}} = \frac{Z-z}{\frac{dz}{dt}}$$

First, taking the positive sign for the equation of our line, we then have as the equation of the tangent line

$$\frac{X-x}{\cos w - \frac{a\sin w}{\sqrt{a^2-r^2}}} = \frac{Y-y}{\sin w + \frac{a\cos w}{\sqrt{a^2-r^2}}} = \frac{Z-z}{\frac{\sqrt{a^2-r^2}}{r}}$$

Using the minus sign we have as the equation of the tangent line to the second asymptotic line at the point (x,y,z)

$$\frac{X-x}{\cos w + \frac{a\sin w}{\sqrt{a^2-r^2}}} = \frac{Y-y}{\sin w - \frac{a\cos w}{\sqrt{a^2-r^2}}} = \frac{Z-z}{\frac{\sqrt{a^2-r^2}}{r}}$$

Let θ be the angle between the two tangents

and consequently the measure of the angle between the two curves. We then have

$$\cos\theta =$$

$$\cos^2 w - \frac{a^2}{a^2-r^2}\sin^2 w + \sin^2 w - \frac{a^2}{a^2-r^2}\cos^2 w + \frac{a^2-r^2}{r^2}$$

divided by

$$\sqrt{\cos^2 w + \frac{2a}{\sqrt{a^2-r^2}}\sin w\cos w + \frac{a^2}{a^2-r^2}\sin^2 w - \frac{2a}{\sqrt{a^2-r^2}}\sin w\cos w + \frac{a^2}{a^2-r^2}\cos^2 w + \frac{a^2-r^2}{r^2} + \sin^2 w}$$

times

$$\sqrt{\cos^2 w - \frac{2a}{\sqrt{a^2-r^2}}\sin w\cos w + \frac{a^2}{a^2-r^2}\sin^2 w + \sin^2 w + \frac{2a}{\sqrt{a^2-r^2}}\sin w\cos w + \frac{a^2}{a^2-r^2}\cos^2 w + \frac{a^2-r^2}{r^2}};$$

simplifying we have

$$\cos\theta =$$

$$\frac{(\sin^2 w + \cos^2 w) - \frac{a^2}{a^2-r^2}(\sin^2 w + \cos^2 w) + \frac{a^2-r^2}{r^2}}{\sqrt{(\sin^2 w + \cos^2 w) + \frac{a^2}{a^2-r^2}(\sin^2 w + \cos^2 w) + \frac{a^2-r^2}{r^2}}\sqrt{(\sin^2 w + \cos^2 w) + \frac{a^2}{a^2-r^2}(\sin^2 w + \cos^2 w) + \frac{a^2-r^2}{r^2}}}$$

$$= \frac{1 - \frac{a^2}{a^2-r^2} + \frac{a^2-r^2}{r^2}}{1 + \frac{a^2}{a^2-r^2} + \frac{a^2-r^2}{r^2}} = \frac{\frac{a^2(a^2-2r^2)}{r^2(a^2-r^2)}}{\frac{a^4}{r^2(a^2-r^2)}} = \frac{a^2-2r^2}{a^2}.$$

This shows that the asymptotic lines do not cut at right angles in the case of the pseudosphere, nor do they cut at a constant angle since the parameter r is found in the value of $\cos\theta$, where θ represents the angle at which the two lines cut.

If we take our point (x, y, z) on the edge of the pseudo sphere, i.e. where $r = a$, then $\cos\theta = -1$ and the lines cut at an angle of $180°$; if we take $r = \frac{a}{2}$, $\cos\theta = 0$ and the

lines cut at right angles.

We shall now pass to the problem of the deformation of surfaces.

Chapter V.

Deformation of Surfaces of constant negative curvature.

We shall first introduce a short discussion concerning the curvature of a surface. The curvature of a surface was defined in Chapter I and its equation

$$\frac{1}{\rho_1\rho_2} = K = \frac{LN - M^2}{EG - F^2}, \quad (X)$$ was obtained.

It is of interest to note that the curvature of a surface may be defined in terms of the surface element of a given surface and the surface element of a sphere of radius unity. It is shown by means of a geometric representation that is striking and also convincing.

Let us consider the curvature of a plane curve different to the way we considered it, in curves in general, in chapter I; and analogous to this we shall represent the curvature of a surface.

Take any plane curve c and at any

point O describe a circle of radius unity. At two neigboring points P and M of c draw two normals and parallel to these normals draw two radii of the circle, OP' and OM'. Let s represent the arch of the curve between the two normals and σ represent the arch of the circle between OP' and OM'. The angle between the two radii equals the angle between the two normals to c. Therefore the angle between the two normals = σ. But the angle between the two normals at P and M equals the angle ω between the two tangents at the same point. Consequently by our previous definition, chapter (I),

$$K = \frac{1}{\rho} = \lim \frac{\omega}{s} = \frac{d\omega}{ds}$$

may be written

$$K = \lim \frac{\sigma}{s}$$

This is true of any space curve.

Now let us take, first, some surface S and then take a sphere Σ of radius unity. On S let us take a closed region, s, and draw the normals to the surface along the boundary of the closed region. From the center of the sphere let us draw radii which are parallel one by one to the normals drawn to the surface. We thus obtain a closed region on the sphere.

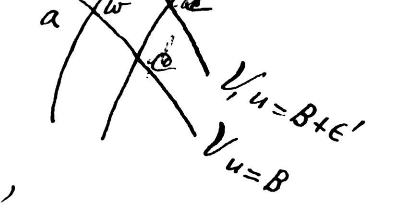
(on the sphere). Let us designate the region on the given surface and on the sphere by S and σ respectively. Now if we imagine that both of these closed regions become infinitely small, which of course σ does provided S does, we shall have in the limit $\frac{\sigma}{S}$, which is called the Gaussian curvature. And the limit $\frac{\sigma}{S}$ can be shown equal to

$$\frac{1}{R_1 R_2} = \frac{LN - M^2}{EG - F^2} = \frac{rt - s^2}{(1+p^2+q^2)^2} \quad \text{when } z = f(x,y).$$

For this purpose we need first the expression for the surface element of any surface. Take the surface

$$x = f(uv)$$
$$y = g(uv)$$
$$z = h(uv)$$

Let us take two infinitely near curves of the system U for which v is constant and two infinitely near curves of the system V for which u is constant.



Let us designate the curves for which v is constant by $v = \alpha$ and $v = \alpha + \epsilon$, and those for which u is constant by $u = B$ and $u = B + \epsilon'$ where ϵ and ϵ' represent quantities

infinitely small. Let us designate the point $(v=\alpha, u=\beta)$ by (a), $(v=\alpha, u=\beta+\epsilon')$ by (b), $(u=\alpha+\epsilon, v=\beta)$ by (c), $(u=\alpha+\epsilon, v=\beta+\epsilon')$ by (d). Now as ϵ and ϵ' are infinitely small we may consider the surface element (a, b, c, d) a parallelogram which is its limiting position as ϵ and ϵ' approach zero. Then the area of (a, b, c, d) equals

$$\overline{ab}\cdot\overline{ac}\sin\omega.$$

But $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$, then for the arc of the curve $v=$ const.

$$ds = \sqrt{E}\,du;$$

for the arc of the curve $u=$ const

$$ds = \sqrt{G}\,dv$$

Then bc may be considered the diagonal of parallelogram which has $\sqrt{E}\,du$ and $\sqrt{G}\,dv$ for its sides. Therefore,

$$\cos\omega = \frac{F}{\sqrt{EG}}\text{; consequently,}$$

$$\sin\omega = \frac{\sqrt{EG-F^2}}{\sqrt{EG}} = \frac{\sqrt{A_1^2+B_1^2+C_1^2}}{\sqrt{EG}},$$

where
$$A_1 = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}$$
$$B_1 = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}$$
$$C_1 = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Therefore, the area of our surface element (a, b, c, d)

$$= \sqrt{E}\,du \cdot \sqrt{G}\cdot dv \frac{\sqrt{A_1^2+B_1^2+C_1^2}}{\sqrt{EG}}$$

$$= \sqrt{A_1^2+B_1^2+C_1^2}\;du\,dv. \quad \text{(XXXIII)}.$$

Now we are prepared to show that

$$\lim \frac{\sigma}{S} = \frac{rt-s^2}{(1+p^2+q^2)^2}$$

Let the equation of our given surface S be given in the form

$$x = f_1(uv)$$
$$y = g_1(uv)$$
$$z = h_1(uv)$$

Let the equation of our determined sphere of radius unity be

$$x^2+y^2+z^2=1, \text{ where}$$
$$x = f_2(uv)$$
$$y = g_2(uv)$$
$$z = h_2(uv)$$

and the origin of coordinate axes is at the center of the sphere.

The equations of the normal to the surface are

$$(1)\quad \frac{X-x_0}{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)} = \frac{Y-y_0}{\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}\frac{\partial z}{\partial z}\right)} = \frac{Z-z_0}{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)} \text{ or}$$

$$(2)\quad \frac{X-x_0}{A_1} = \frac{Y-y_0}{B_1} = \frac{Z-z_0}{C_1}\;;\; A_1 = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} \text{ etc.}$$

accordingly the surface element of S
$$= \sqrt{A_1^2 + B_1^2 + C_1^2}\, du\, dv$$
We have for the equation of our normal to the sphere Σ

(3) $$\frac{x - 0}{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)} = \frac{y - 0}{\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}} = \frac{z - 0}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}}$$

or setting $A_2 = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}$ etc

(4) $$\frac{x}{A_2} = \frac{y}{B_2} = \frac{z}{C_2}$$

since (x, y, z) are different functions of (u, v) to what they are in (1).

Then the surface element of the spere Σ
$$= \sqrt{A_2^2 + B_2^2 + C_2^2}\, du\, dv.$$

Let α, β, γ denote the direction cosines that the normal N of the sphere makes with th x, y and z axes. Then

$$\alpha = \lambda_2\left[\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right] = \lambda_2 A_2$$

$$\beta = \lambda_2\left[\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right] = \lambda_2 B_2$$

$$\gamma = \lambda_2\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right] = \lambda_2 C_2$$

But the radius of the sphere is also the normal to the surface which passes through (x, y, z) and the origin. Therefore,

$$\left.\begin{aligned} \alpha &= \frac{x}{r} = x \\ \beta &= \frac{y}{r} = y \\ \gamma &= \frac{z}{r} = z \end{aligned}\right\}$$ since $r = 1$.

But by construction corresponding

normals to Σ and S have the same direction cosines, therefore:

$$X = \lambda_2 A_2 = \lambda_1 A_1$$
$$Y = \lambda_2 B_2 = \lambda_1 B_1$$
$$Z = \lambda_2 C_2 = \lambda_1 C_1$$

Hence

$$X^2 + Y^2 + Z^2 = \lambda_2^2(A_2^2 + B_2^2 + C_2^2) = \lambda_1(A_1^2 + B_1^2 + C_1^2) = 1,$$

consequently,

$$\lambda_2 = \frac{1}{\sqrt{A_2^2 + B_2^2 + C_2^2}} \; ; \; \lambda_1 = \frac{1}{A_1^2 + B_1^2 + C_1^2}$$

Let us write

$$X.X + Y.Y + Z.Z = 1.$$

Let for one factor in each term their values, then

$$\lambda_2(XA_2 + YB_2 + ZC_2) = 1.$$

Therefore,

$$\lambda_2(XA_2 + YB_2 + ZC_2) = \lambda_2\sqrt{A_2^2 + B_2^2 + C_2^2} \quad \text{or}$$

$$(5.) \quad XA_2 + YB^2 + ZC_2 = \sqrt{A_2^2 + B_2^2 + C_2^2}$$

Multiply both sides by $du\,dv$. We then have

$$(XA_2 + YB_2 + ZC_2)\,du\,dv = \sqrt{A_2^2 + B_2^2 + C_2^2}\,du\,dv,$$

which we recognize as the surface element of Σ. We can also write (5.) in the form of

$$\left[X\left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\cdot\frac{\partial y}{\partial v}\right) + Y\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) + Z\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right] du\,dv. \quad (5.)'$$

which we wish to express in terms of A_1, B_1, C_1 and their partial derivatives. For this purpose

we can take

$$\left.\begin{aligned} x &= \frac{A_1}{\sqrt{A_1^2+B_1^2+C_1^2}} \\ y &= \frac{B_1}{\sqrt{A_1^2+B_1^2+C_1^2}} \\ z &= \frac{C_1}{\sqrt{A_1^2+B_1^2+C_1^2}} \end{aligned}\right\} \quad (6)$$

since these values of x, y and z satisfy the equation of the sphere

$$x^2+y^2+z^2=1.$$

Differentiating the equations in (6) with respect to u and v, we have

$$\frac{\partial x}{\partial u} = \lambda_1\frac{\partial A_1}{\partial u} + A_1\frac{\partial \lambda_1}{\partial u}\ ; \quad \frac{\partial x}{\partial v} = \lambda_1\frac{\partial A_1}{\partial v} + A_1\frac{\partial \lambda_1}{\partial v}$$

$$\frac{\partial y}{\partial u} = \lambda_1\frac{\partial B_1}{\partial u} + B_1\frac{\partial \lambda_1}{\partial u}\ ; \quad \frac{\partial y}{\partial v} = \lambda_1\frac{\partial B_1}{\partial v} + B_1\frac{\partial \lambda_1}{\partial v}$$

$$\frac{\partial z}{\partial u} = \lambda_1\frac{\partial C_1}{\partial u} + C_1\frac{\partial \lambda_1}{\partial u}\ ; \quad \frac{\partial z}{\partial v} = \lambda_1\frac{\partial C_1}{\partial v} + C_1\frac{\partial \lambda_1}{\partial v}.$$

Substituting these values we have

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} =$$

$$\left(\lambda_1\frac{\partial A_1}{\partial u} + A_1\frac{\partial \lambda_1}{\partial u}\right)\left(\lambda_1\frac{\partial B_1}{\partial v} + B_1\frac{\partial \lambda_1}{\partial v}\right) - \left(\lambda_1\frac{\partial B_1}{\partial u} + B_1\frac{\partial \lambda_1}{\partial u}\right)\left(\lambda_1\frac{\partial A_1}{\partial v} + A_1\frac{\partial \lambda_1}{\partial v}\right)$$

$$= \lambda_1^2\left[\frac{\partial A_1}{\partial u}\frac{\partial B_1}{\partial v} - \frac{\partial B_1}{\partial u}\frac{\partial A_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial u}\left[A_1\frac{\partial B_1}{\partial v} - B_1\frac{\partial A_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial v}\left[B_1\frac{\partial A}{\partial u} - A_1\frac{\partial B}{\partial u}\right].$$

From analogy we can write

$$\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} =$$

$$\lambda_1^2\left[\frac{\partial B_1}{\partial u}\cdot\frac{\partial C_1}{\partial v} - \frac{\partial C_1}{\partial u}\cdot\frac{\partial B_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial u}\left[B_1\frac{\partial C_1}{\partial v} - C_1\frac{\partial B_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial v}\left[C_1\frac{\partial B_1}{\partial u} - B_1\frac{\partial C_1}{\partial u}\right]$$

and

$$\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} =$$

$$\lambda_1^2\left[\frac{\partial C_1}{\partial u}\cdot\frac{\partial A_1}{\partial v} - \frac{\partial A_1}{\partial u}\cdot\frac{\partial C_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial u}\left[C_1\frac{\partial A_1}{\partial v} - A_1\frac{\partial C_1}{\partial v}\right] + \lambda_1\frac{\partial \lambda_1}{\partial v}\left[A_1\frac{\partial C_1}{\partial u} - C_1\frac{\partial A_1}{\partial u}\right]$$

Multiplying the members on the left by z, x and y respectively and those on the right by their equals $\lambda_1 C_1$, $\lambda_1 A_1$, $\lambda_1 B_1$ and setting for λ_1 its value $\frac{1}{\sqrt{A_1^2+B_1^2+C_1^2}}$, and adding we obtain for the left member $\frac{(b')'}{du\,dv}$, and in the right member the second and third terms in λ_1^2 obtained from the above three equations cancel. We then have

$$\left[x\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) + y\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) + z\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right] =$$

$$\frac{1}{(A_1^2+B_1^2+C_1^2)^{\frac{3}{2}}}\left[A_1\left(\frac{\partial B_1}{\partial u}\frac{\partial C_1}{\partial v} - \frac{\partial C_1}{\partial u}\frac{\partial B_1}{\partial v}\right) + B_1\left(\frac{\partial C_1}{\partial u}\frac{\partial A_1}{\partial v} - \frac{\partial A_1}{\partial u}\frac{\partial C_1}{\partial v}\right) + C_1\left(\frac{\partial A_1}{\partial u}\frac{\partial B_1}{\partial v} - \frac{\partial B_1}{\partial u}\frac{\partial A_1}{\partial v}\right)\right]$$

So we see that the left member multiplied by $du\,dv$ gives us $(b')'$ i.e. the surface element of the sphere, and it is equal to the expression

on the right multiplied by $du\,dv$. Consequently we have the surface element of the sphere expressed in terms of A_1 B_1 C_1 and their partial derivatives i.e.

$$\sigma = \frac{\left[A_1\left(\frac{\partial B_1}{\partial u}\frac{\partial C_1}{\partial v} - \frac{\partial C_1}{\partial u}\frac{\partial B_1}{\partial v}\right) + B_1\left(\frac{\partial C_1}{\partial u}\frac{\partial A_1}{\partial v} - \frac{\partial A_1}{\partial u}\frac{\partial C_1}{\partial v}\right) + C_1\left(\frac{\partial A_1}{\partial u}\frac{\partial B_1}{\partial v} - \frac{\partial B_1}{\partial u}\frac{\partial A_1}{\partial v}\right)\right]}{\left(A_1^2 + B_1^2 + C_1^2\right)^{3/2}}\,du\,dv$$

also we have

$$S = \sqrt{A_1^2 + B_1^2 + C_1^2}\;du\,dv\,;$$

therefore,

$$\frac{\sigma}{S} = \left[\frac{A_1\left(\frac{\partial B_1}{\partial u}\frac{\partial C_1}{\partial v} - \frac{\partial C_1}{\partial u}\frac{\partial B_1}{\partial v}\right) + B_1\left(\frac{\partial C_1}{\partial u}\frac{\partial A_1}{\partial v} - \frac{\partial A_1}{\partial u}\frac{\partial C_1}{\partial v}\right) + C_1\left(\frac{\partial A_1}{\partial u}\frac{\partial B_1}{\partial v} - \frac{\partial B_1}{\partial u}\frac{\partial A_1}{\partial v}\right)}{\left(A_1^2 + B_1^2 + C_1^2\right)^2}\right], \quad (7)$$

Now if we take our surface in the form

$$z = f(xy) = f(uv)$$
$$x = u$$
$$y = v \,.$$

Then

$$A_1 = \frac{\partial y}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\frac{\partial y}{\partial y} = -p$$

$$B_1 = \frac{\partial z}{\partial x}\frac{\partial x}{\partial y} - \frac{\partial x}{\partial x}\frac{\partial z}{\partial y} = -q$$

$$C_1 = \frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial y}{\partial x}\frac{\partial x}{\partial y} = 1$$

$$\frac{\partial A_1}{\partial u} = \frac{\partial A_1}{\partial x} = -\frac{\partial p}{\partial x} = -r$$

$$\frac{\partial B_1}{\partial u} = \frac{\partial B_1}{\partial x} = -\frac{\partial q}{\partial x} = -s$$

$$\frac{\partial A_1}{\partial v} = \frac{\partial A_1}{\partial y} = -\frac{\partial p}{\partial y} = -s$$

$$\frac{\partial B_1}{\partial v} = \frac{\partial B_1}{\partial y} = -\frac{\partial q}{\partial y} = -t$$

$$\frac{\partial C_1}{\partial u} = \frac{\partial C_1}{\partial x} = \frac{\partial .1}{\partial x} = 0$$

$$\frac{\partial C_1}{\partial v} = \frac{\partial C_1}{\partial y} = \frac{\partial .1}{\partial y} = 0.$$

Substituting these values in (7) the first and second terms fall out and we obtain

$$\frac{\sigma}{S} = \frac{1\left[(-r)(-t)-(-s)(-s)\right]}{(1+p^2+q^2)^2} =$$

$$\frac{rt-s^2}{(1+p^2+q^2)^2} = \frac{1}{R_1 R_2}, \quad (\text{XXXIV}).$$

Also we see that the product of the two principal radii of curvature at a pt. P of a surface S will not change however small the surface element may be therefore

$$\lim \frac{\sigma}{S} = K = \frac{1}{R_1 R_2} \quad (\text{XXXV}).$$

After giving this new definition of the curvature of a surface and having established

its identity to our definition given in Chapter I, we shall now define deformation. A surface Σ is a deformation of a surface S if Σ has passed by a suite of changes from S to its new form Σ without tearing stretching or crumpling the surface. A deformation of a surface is a bending of the surface. Consequently the linear element and the surface element remain invariant during the deformation, hence any surface Σ that is a deformation of S is applicable to S. If we consider a surface Σ composed of a flexible and an inextensible tissue we can then say that it is applicable to a given surface S if Σ can be folded on S without tearing, stretching or crumpling the surface of Σ. In this folding process Σ is seen to pass by a suite of changes from Σ to S and consequently the lengths between corresponding points on the two surfaces are equal.

Let us first see how corresponding points on two surfaces are obtained. Take any two surfaces S and Σ whose equations are given in parametric form

$$S \quad \begin{cases} x = f_1(u,v) \\ y = g_1(u,v) \\ z = h_1(u,v) \end{cases} \qquad \Sigma \quad \begin{cases} x = f_2(s,t) \\ y = g_2(s,t) \\ z = h_2(s,t) \end{cases}$$

To every point P' on surface S corresponds a point P on the uv plane; and to every point Q' on Σ corresponds a point Q on the st plane. Then in order to establish a correspondence between P' and Q' it is only necessary to make P of the uv plane correspond to Q of the st plane. Then for every point P there is a corresponding point Q, and consequently

$$s = \phi(uv)$$
$$t = \psi(uv).$$

Then the equation of Σ may be written

$$x = f_2[\phi(uv), \psi(uv)]$$
$$y = g_2[\phi(uv), \psi(uv)]$$
$$z = h_2[\phi(uv), \psi(uv)].$$

Hence for each pair of values for u and v there is a point P' on S and a corresponding point Q' on S; consequently there is a one to one correspondence established between P' and Q' on S and Σ.

Now if our two surfaces S and Σ are applicable and 'c' represents a line joining P' and P'' on S, and K', terminated by corresponding points Q' and Q'', represents the line containing corresponding points of Σ, then when we apply the two surfaces

the two lines must coincide throughout, and the expression for the linear element

$$ds = \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

is identically equal to

$$d\sigma = \sqrt{E_1\,du^2 + 2F_1\,du\,dv + G_1\,dv^2},$$

where $E = E_1$, $F = F_1$, $G = G_1$

We wish now to show that the necessary condition for the applicability of two surfaces is that the Gaussian curvature be the same in both. Since if the surfaces are applicable $E = E_1$, $F = F_1$, $G = G_1$ we shall only need to show that that the Gaussian curvature depends on E, F and G and their derivatives with respect to u and v.

We have seen from (X) that

$$K = \frac{1}{R_1 R_2} = \frac{LN - M^2}{EG - F^2}.$$

We wish to express $LN - M^2$ in terms of E, F and G, and their derivatives with respect to u and v.

But

$$L = \begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \frac{1}{H}$$

$$M = \begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \frac{1}{H}$$

$$\mathcal{N} = \begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \frac{1}{H}.$$

For brevity let us put

$a = x_u$; $a' = x_v$

$b = y_u$; $b_1 = y_v$

$c = z_u$; $c' = z_v$

$\alpha = \frac{da}{du} = x_{uu}$; $\alpha' = \frac{da'}{du} = \frac{da}{dv} = x_{uv}$

$\beta = \frac{db}{du} = y_{uu}$; $\beta' = \frac{db'}{du} = \frac{db}{dv} = y_{uv}$

$\gamma = \frac{dc}{du} = z_{uu}$; $\gamma' = \frac{dc'}{du} = \frac{dc}{dv} = z_{uv}$

$$\alpha'' = \frac{da'}{dv} = x_{vv}$$

$$\beta'' = \frac{db'}{dv} = y_{vv}$$

$$\gamma'' = \frac{dc'}{dv} = z_{vv}$$

Then

$$H^2(L\mathcal{N} - \mathcal{M}^2) =$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a & b & c \\ a' & b' & c' \end{vmatrix} \cdot \begin{vmatrix} \alpha'' & \beta'' & \gamma'' \\ a & b & c \\ a' & b' & c' \end{vmatrix} - \begin{vmatrix} \alpha' & \beta' & \gamma' \\ a & b & c \\ a' & b' & c' \end{vmatrix}^2 .$$

According to the law for the multiplication of determinants we have

$$H^2(L\mathcal{N} - \mathcal{M}^2) =$$

$$\begin{vmatrix} \alpha\alpha''+\beta\beta''+\gamma\gamma'', & a\alpha''+b\beta''+c\gamma'', & a'\alpha''+b'\beta''+c'\gamma'' \\ a\alpha+b\beta+c\gamma, & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha+b'\beta+c'\gamma, & aa'+bb'+cc', & a_1^2+b_1^2+c_1^2 \end{vmatrix} -$$

$$\begin{vmatrix} \alpha'^2+\beta'^2+\gamma'^2, & a\alpha'+b\beta'+c\gamma', & a'\alpha'+b'\beta'+c'\gamma' \\ a\alpha'+b\beta'+c\gamma', & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha'+b'\beta'+c'\gamma', & aa'+bb'+cc', & a'^2+b'^2+c'^2 \end{vmatrix}.$$

Now let us see what values of E, F and G and their derivatives can be substituted for the different terms in this determinant. We have

$$E = a^2+b^2+c^2$$
$$F = aa_1+bb_1+cc_1$$
$$G = a'^2+b'^2+c'^2$$

$$\frac{1}{2}\frac{dE}{du} = a\alpha+b\beta+c\gamma, \text{ since } \frac{da}{du}=\alpha, \text{ etc}$$

$$\frac{1}{2}\frac{dG}{du} = a'\alpha'+b'\beta'+c'\gamma', \text{ since } \frac{da'}{du}=\alpha', \text{ etc}$$

$$\frac{1}{2}\frac{dE}{dv} = a\alpha'+b\beta'+c\gamma, \text{ since } \frac{da}{dv}=\alpha', \text{ etc}$$

$$\frac{1}{2}\frac{dG}{dv} = a'\alpha''+b'\beta''+c'\gamma'', \text{ since } \frac{da'}{dv}=\alpha'', \text{ etc}$$

also

$$a\alpha''+b\beta''+c\gamma'' = \frac{dF}{dv}-(a'\alpha'+b'\beta'+c'\gamma') = \frac{dF}{dv}-\frac{1}{2}\frac{dG}{du}$$

$$a'\alpha+b'\beta+c'\gamma = \frac{dF}{du}-(a\alpha'+b\beta'+c\gamma') = \frac{dF}{du}-\frac{1}{2}\frac{dE}{dv}$$

We have still if possible to express $\alpha\alpha''+\beta\beta''+\gamma\gamma''$ and $\alpha'^2+\beta'^2+\gamma'^2$ in terms of E, F and G. Let us for this object differentiate

$$a'\alpha+b'\beta+c'\gamma = \frac{dF}{du}-\frac{1}{2}\frac{dE}{dv}$$ with respect to v.

Then

$$(1),\ \alpha\alpha''+\beta\beta''+\gamma\gamma'' = \frac{d^2F}{du\,dv}-\frac{1}{2}\frac{d^2E}{dv^2}-\left(a'\frac{d\alpha}{dv}+b'\frac{d\beta}{dv}+c'\frac{d\gamma}{dv}\right)$$

We also get, by differentiating

$$a'\alpha'+b'\beta'+c'\gamma' = \frac{1}{2}\frac{dG}{du}$$ with respect to u,

$$(2),\ \alpha'^2+\beta'^2+\gamma'^2 = \frac{1}{2}\frac{d^2G}{du^2}-\left(a'\frac{d\alpha'}{du}+b'\frac{d\beta'}{du}+c'\frac{d\gamma'}{du}\right).$$

But $\frac{d\alpha}{dv} = \frac{d\alpha'}{du}$, etc.

Therefore in (1) and (2) the parts in parentheses are equal. Also, we see, that if we develop our two determinants by minors, that $a'\alpha''+b'\beta''+c'\gamma''$ and $\alpha'^2+\beta'^2+\gamma'^2$ are multiplied by the same factor; therefore the product of the two equal factors cancel and the expressions in parentheses may accordingly be omitted without effecting the result. Then we can write for

$\alpha'\alpha'' + \beta\beta'' + \gamma\gamma''$, $\frac{d^2F}{du\,dv} - \frac{1}{2}\frac{d^2E}{dv^2}$;
and for
$\alpha'^2 + \beta_1^2 + \gamma_1^2$, $\frac{1}{2}\frac{d^2G}{du^2}$.
Then

$$H^2(LN - M^2) =$$

$$\begin{vmatrix} \frac{d^2F}{du\,dv} - \frac{1}{2}\frac{d^2E}{dv^2}, & \frac{dF}{dv} - \frac{1}{2}\frac{dG}{du}, & \frac{1}{2}\frac{dG}{dv} \\ \frac{1}{2}\frac{dE}{du}, & E, & F \\ \frac{dF}{du} - \frac{1}{2}\frac{dE}{dv}, & F & G \end{vmatrix} -$$

$$\begin{vmatrix} \frac{1}{2}\frac{d^2G}{du^2}, & \frac{1}{2}\frac{dE}{dv}, & \frac{1}{2}\frac{dG}{du} \\ \frac{1}{2}\frac{dE}{dv}, & E, & F \\ \frac{1}{2}\frac{dG}{du}, & F, & G \end{vmatrix}$$

This gives us $LN - M^2$ in terms of E, F and G and their first and second derivatives; consequently, the Gaussian curvature is now expressed in terms of E, F and G and their first and second derivatives alone.

If we expand these determinants and combine

and substitute in (X),

$$K = \frac{L n - m^2}{E G - F^2},$$

the value of $(Ln - m^2)$ and put for H^2 its value $EG - F^2$ we thus obtain

$$4(EG - F^2)K =$$

$$E\left[\frac{dE}{du}\cdot\frac{dG}{dv} - 2\frac{dF}{du}\cdot\frac{dG}{dv} + \left(\frac{dG}{du}\right)^2\right] +$$

$$F\left[\frac{dE}{du}\cdot\frac{dG}{dv} - \frac{dE}{dv}\cdot\frac{dG}{du} - 2\frac{dE}{dv}\frac{dF}{dv} + 4\frac{dF}{du}\frac{dF}{dv} - 2\frac{dF}{du}\frac{dG}{du}\right] -$$

$$2(EG - F^2)\left[\frac{d^2E}{dv^2} - 2\frac{d^2F}{du\,dv} + \frac{d^2G}{du^2}\right]$$

Thus we have established the proposition that the necessary condition for the applicability of two surfaces is that $K = \frac{1}{R_1 R_2}$ should be the same for both surfaces.

Since our surfaces were determined with the same constant negative curvature we know that this necessary condition for applicability holds. We can however from the equations of the surfaces at once show that the Gaussian curvature is the same in all three cases.

The equations of a pseudosphere are $x = r\cos w$, $y = r\sin w$, $z = f(r)$ where $\frac{dz}{dr} = f'(r) = \frac{\sqrt{a^2 - r^2}}{r}$

and $\frac{d^2z}{dr^2} = f''(r) = -\frac{a^2}{r^2\sqrt{a^2-r^2}}$.

Then by (XIV) for a surface of revolution we have

$$R_1 R_2 = \frac{r(1+f'^2)^2}{f' f''}.$$

Substituting for f' and f'' we have

$$R_1 R_2 = \frac{r\left(1+\frac{a^2-r^2}{r^2}\right)^2}{-\frac{\sqrt{a^2-r^2}}{r}\cdot\frac{a^2}{r^2\sqrt{a^2-r^2}}} = -\frac{\frac{a^4}{r^3}}{\frac{a^2}{r^3}} = -a^2.$$

Therefore $K = \frac{1}{R_1 R_2} = -\frac{1}{a^2}$.

Our surfaces of the elliptic and hyperbolic types are represented by the equations

$$x = r\cos w$$

$$y = r\sin w$$

$$z = f(r) \qquad \text{where}$$

$$\frac{dz}{dr} = f'(r) = \frac{\sqrt{b^2-r^2}}{\sqrt{r^2+a^2-b^2}}$$

$$\frac{d^2z}{dr^2} = f''(r) = -\frac{a^2 r}{\sqrt{b^2-r^2}\,(r^2+a^2-b^2)^{\frac{3}{2}}}.$$

accordingly by (XIV)

$$R_1 R_2 = -\frac{r\left[1+\frac{b^2-r^2}{r^2+a^2-b^2}\right]}{\frac{\sqrt{b^2-r^2}}{\sqrt{r^2+a^2-b^2}}\cdot\frac{a^2 r}{(r^2+a^2-b^2)^{\frac{3}{2}}}} =$$

$$-\frac{\dfrac{a^4}{r(r^2+a^2-b^2)^2}}{\dfrac{a^2 r}{(r^2+a^2-b^2)^2}} = -a^2.$$

Consequently

$$K = \frac{1}{R_1 R_2} = -\frac{1}{a^2}$$

which shows that the Gaussian curvature for our three surfaces is the same and in addition is a constant.

It is our purpose now to determine the equations of surfaces of revolution applicable to each other. Since a surface of revolution is a helicoid whose pitch is zero the same formulae that give the equations of helicoids applicable to each other, will be setting the pitch equal to zero, give the equations of the surfaces of revolution that are applicable to each other.

Then let us take the helicoid defined by the equations

$$x = r\cos w$$

$$y = r\sin w$$

$$z = f(r) + aw$$ where a designates the pitch.

Let us determine the linear element

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

Accordingly

$$E = \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right]_{u=r} = 1 + f'(r)^2$$

$$F = \left(\frac{dx}{du}\right)\left(\frac{dx}{dv}\right) + \left(\frac{dy}{du}\right)\left(\frac{dy}{dv}\right) + \left(\frac{dz}{du}\right)\left(\frac{dz}{dv}\right) = a\,f'(r)$$

$$G = \left[\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2 + \left(\frac{dz}{dv}\right)^2\right]_{v=w} = a^2 + r^2$$

Consequently $ds^2 =$

$$\left[1 + f'(r)\right] dr^2 + 2a\,f'(r)\,dr\,dw + (a^2 + r^2)\,dw^2 \quad \text{(XXXVI)}.$$

This equation can be written in the form:

$$ds^2 = \left(1 + \frac{r^2 f'(r)^2}{a^2 + r^2}\right) dr^2 + (r^2 + a^2)\left[\frac{a\,f'(r)}{a^2 + r^2}\,dr + dw\right]^2 \quad \text{(XXXVII)}.$$

Let us now transform from the rw plane to the uv plane by means of the transformation

$$\left.\begin{aligned} du &= \sqrt{1 + \frac{r^2 f'(r)^2}{a^2 + r^2}}\,dr \\ dv &= dw + \frac{a\,f'(r)\,dr}{a^2 + r^2} \end{aligned}\right\} \quad (1)$$

$$u = \int \sqrt{1 + \frac{r^2 f'(r)^2}{a^2 + r^2}}\,dr = H(r) \text{ alone}.$$

$$v = w + \int \frac{a\,f'(r)}{a^2 + r^2}\,dr.$$

We have

$$u = H(r)$$
$$r = \varphi(u);$$

consequently we see that a^2+r^2 becomes a function of u. Write it

$$a^2+r^2 = U^2(u)$$

Then

$$ds^2 = du^2 + U^2(u)\,dv, \qquad \text{(XXXVIII)}.$$

and since $f(r)$ is arbitrary $U^2(u)$ can be any function we please except a constant for then we could not return to our original $H(r)$. This tells us that if we are given any function $U^2(u)$ that there exists a helicoid that has its linear element in the form (in the form) $ds^2 = du^2 + U(u^2)\,dv^2$ when u and v are properly chosen. Moreover the linear element is the same as that for a surface of revolution which surface is a helicoid whose pitch is zero. We see thus that the helicoid is applicable to a surface of revolution.

Now let us take a second helicoid S' and see when it is applicable to our first helicoid S. S' will be given by the equations

$$x = r\cos w$$

$$y = r\sin w$$

$$z = \phi(r) + hw.$$

Then as before

$$ds'^2 = \left(1 + \frac{r^2\phi'^2(r)}{a^2+r^2}\right)dr^2 + (r^2+h^2)\left[\frac{h\,\phi'(r)}{h^2+r^2}\,dr + dw\right]^2.$$

Take S in its reduced form.

$$ds^2 = du^2 + U^2(u)\,dv^2$$

For these to be applicable we must have

$$\left.\begin{aligned} du &= \sqrt{\left(1 + \frac{r^2\phi'(r)^2}{h^2+r^2}\right)}\,dr = j'(r) \\ \pm\frac{U(u)\,dv}{\sqrt{h^2+r^2}} &= \frac{h\,\phi'(r)}{h^2+r^2}\,dr + dw \end{aligned}\right\} \quad (2)$$

$$dw = -\frac{h\,\phi'(r)\,dr}{h^2+r^2} \mp \frac{U(u)}{\sqrt{h^2+r^2}}\,dv$$

We have here a perfect differential. Therefore

$$\frac{d}{dv}\left(\frac{h\,\phi'(r)}{h^2+r^2}\right) = \pm\frac{d}{dr}\left(\frac{U(u)}{\sqrt{h^2+r^2}}\right).$$

But r is a function of $\underline{u}$ alone, therefore the left member is zero; consequently the right member is zero also. But the second member is a function of r and its derivative with respect to r is zero. Therefore $\dfrac{U(u)}{\sqrt{h^2+r^2}} = \dfrac{1}{m}$ a constant.

We then have

$$\left.\begin{aligned} \sqrt{h^2+r^2} &= m\,U(u) \\ \frac{h\,\phi'(r)}{h^2+r^2}\,dr \pm dw &= \frac{dv}{m} \end{aligned}\right\} \quad (3)$$

We wish now to get r and w in terms of u and v, i.e. pass from the rw to the uv plane.

We have

$$r = \pm \sqrt{m^2 U^2(u) - h^2}$$

$$dr = \frac{m^2 U(u)\, U'(u)\, du}{\sqrt{m^2(U^2(u) - h^2}}$$

$$du^2 = \left(1 + \frac{r^2 \phi'(r)^2}{h^2 + r^2}\right) dr^2$$

$$r^2 \phi'(r)\, dr^2 = du^2(h^2 + r^2) - dr^2(h^2 + r^2)$$

$$\phi'(r)^2 = \frac{(du^2 - dr^2)(h^2 + r^2)}{r^2 dr^2}$$

$$d\phi^2 = \frac{(du^2 - dr^2)(h^2 + r^2)}{r^2}$$

$$= \frac{\left[du^2 - \frac{m^4 U^2(u)\, U'^2(u)}{m^2 U^2(u) - h^2} du^2\right] m^2 U^2(u)}{m^2 U^2(u) - h^2}$$

$$= \frac{du^2 (m^4 U^4(u) - m^2 U^2(u) h^2 - m^6 U^4(u) U'^2(u))}{(m^2 U^2(u) - h^2)^2}$$

$$\begin{cases} d\phi = \dfrac{m^2 U\, du}{m^2 U^2 - h^2} \sqrt{U^2(1 - m^2 U'^2) - \dfrac{h^2}{m^2}} \\ dw = \dfrac{dv}{m} - \dfrac{h\, d\phi}{m^2 U^2} = \dfrac{dv}{m} - \dfrac{h\, du}{U(m^2 U^2 - h^2)} \sqrt{U^2(1 - m^2 U'^2 - \dfrac{h^2}{m}} \\ r = \pm \sqrt{m^2 U^2 - h^2} \end{cases} \qquad (XXXIX)$$

We have thus determined the function ϕ and

and found w and r so that S' is applicable to S; and since m is arbitrary there will be an infinite number of helicoids of the same pitch applicable to S.

Now to find the surfaces of revolution applicable to the given helicoid we need only to take $h=0$. We thus obtain

$$\boxed{XL} \quad \begin{cases} d\phi = \sqrt{1-m^2 U'^2}\, du \\ dw = \dfrac{dv}{m} \\ r = mU \end{cases}$$

Then the equations of surfaces of revolution applicable to each other as well as to the helicoid is given by

$$\boxed{XLI}. \quad \begin{cases} x = r\cos w = mU\cos\frac{v}{m} \\ y = r\sin w = mU\sin\frac{v}{m} \\ z = \phi(r) = \int\sqrt{1-m^2U'^2} = \int\sqrt{1-m^2\frac{dU}{du}} \end{cases}$$

We are now prepared to prove that each of our surfaces is applicable to itself and to the others.

First, let us prove that the pseudosphere

is applicable to itself. We have seen formula (XIX) that the linear element of the pseudosphere is given in the form

$ds^2 = a^2(du^2 + e^{2u}dv^2)$; take $a^2 = 1$, we then have

$$ds^2 = du^2 + e^{2u}dv^2,$$

where $U(u) = e^u$.

By substituting in (XLI) we have

$$\left.\begin{aligned} x &= m e^u \cos\frac{v}{m} \\ y &= m e^u \sin\frac{v}{m} \\ z &= \int\sqrt{1-m^2 e^{2u}}\, du \end{aligned}\right\} (1)$$

The surfaces defined by these formulae are all the same; for, if we place

$$m e^u = \sin\phi$$
$$v = a v'$$

they become

$$(2) \begin{cases} x = \sin\phi \cos v, \\ y = \sin\phi \sin v, \\ z = \cos\phi + \log\tan\frac{\phi}{2}. \end{cases}$$

These formulas are free from m and therefore give the above surfaces as a single surface. In other words we see by means of the transformation $ae^u = e^{u'}$, $v = av'$ that all of these surfaces go over into a single surface and since m is an arbitrary constant we have ∞^1 (such)

such surfaces going over into a single surface therefore this surface which is here the pseudo-sphere is applicable to itself in an ∞ of ways. To obtain real values of x, y and z we must have $a^2 e^{2u} < 1$

Let us now show that our surface of the elliptic type is applicable to the pseudo-sphere. We have seen that the general expression for the linear element in our surfaces of revolution can be written in the form

$$ds^2 = \frac{a^2 dr^2}{r^2+a^2-b^2} + r^2 dw^2 \qquad \boxed{XVII}$$

If we let $a = b$ we have

$$d\sigma^2 = \frac{a^2 dr^2}{r^2} + r^2 dw^2, \qquad \boxed{XVIII}$$

which is the limiting form of $\boxed{XVII}$ as $a \doteq b$

If in $\boxed{XVII}$ we set $r = \sqrt{a^2-b^2}\,\frac{e^u - e^{-u}}{2}$

it takes the form

$$a^2\left[du^2 + \left(\frac{e^u - e^{-u}}{2}\right) dv^2\right], \qquad \boxed{XX}$$

which is the linear element of our surface S of the elliptic type. Equation $\boxed{XX}$ shows us that the linear element of our elliptic surface is independent of b. Then in $\boxed{XVII}$ let us set $\epsilon = \sqrt{a^2-b^2}$ and let $b \doteq a$, then $\epsilon \doteq 0$. As ϵ varies our surface S given by the equation

$$dz = \frac{\sqrt{b^2-r^2}}{\sqrt{r^2+a^2-b^2}}\, dr$$

will change its form and pass through all forms intermediary to its given form and its limiting form, the pseudosphere. Hence it is applicable to all of these intermediary forms, since they are deformations of the surface S. Now, by making $\epsilon \doteq 0$, we can make our surface of the elliptic type approach the pseudosphere Σ as nearly as we please. Let S_1 denote this variable surface and ds' its linear element. Then as $\epsilon \doteq 0$, we see by comparing (XVII) and (XVIII) that the difference of the linear elements $ds_1^2 - d\sigma^2$ also approaches zero, and by choosing ϵ sufficiently small in S_1

$$|ds_1^2 - d\sigma^2|$$

can be made less than any assignable quantity. But the distance between corresponding points on S and Σ is a constant, consequently $ds - d\sigma$ is a constant. Moreover the distance between corresponding points on S and S_1 is the same since S_1 is a deformation of S; therefore $|dS_1 - d\sigma|$ is also a constant. But this difference as we have seen can be made less than any assignable quantity; hence $|dS_1 - d\sigma| = 0$, for the only constant quantity less than any assignable quantity is zero. Therefore $ds_1 = d\sigma$; but $ds_1 = ds$, therefore $ds = d\sigma$, and

consequently S is applicable to Σ. Or we could also say that since $ds_1 = d\sigma$, S_1 is applicable to Σ; but S is applicable to S_1, therefore S is also applicable to Σ.

In a similar way we can show that our surface of the hyperbolic type, whose linear element is given in the form

$$ds^2 = a^2\left(du^2 + \frac{e^u + e^{-u}}{2}\,dv^2\right),$$ is applicable to the pseudosphere. In this case

$$r = \sqrt{b^2-a^2}\,\frac{e^u+e^{-u}}{2},$$ hence we should now set $\epsilon = \sqrt{b^2-a^2}$. The remaining argument is identical to that made for the elliptic type. Accordingly we have that the hyperbolic type of surface is also applicable to the pseudosphere. Then since our surfaces of the elliptic and hyperbolic types are both applicable to the pseudosphere they are consequently applicable to each other.

But we have shown that the pseudosphere is applicable to itself in an infinity of ways. Therefore these surfaces that are applicable to it can, after they are folded on the pseudosphere, be made to pass through the same deformations that the pseudosphere undergoes to reveal its applicability to itself. Hence they are applicable to

the pseudosphere in an infinity of ways; and since in being applied (applied) to the pseudosphere, they are applied to each other and to themselves folded on the pseudosphere, they are therefore applicable to each other and to themselves in an infinity of ways.

To illustrate the applicability of these surfaces we have taken two casts, one from each of our pseudospheres. These casts we formed into molds, and papers pressed in the smaller one which has the same Gaussian curvature as our surfaces of the elliptic and hyperbolic types, can be applied to any portion of these two surfaces or to itself. Papers pressed in the larger mold can be applied to any portion of the large pseudosphere without stretching, tearing or crumpling the paper.

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