HEDGES — Irrational Numbers THESIS 1905

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THE TREATMENT OF IRRATIONAL NUMBERS IN THE SECONDARY SCHOOLS.

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INTRODUCTION

The sub ject matter of this paper was suggested by the belief that a treatment of irrational numbers, from the stand point of the "cut" number, has certain points of superiority over the common treatment from the standpoint of limits. From the point of view of the high school and lower college courses, it is desirable to present a treatment of these topics at once as free as possible from all abstract and intricate logical considerations, and at the same time in harmony with a strictly logical treatment. It is desirable if possible that the student should have nothing to unlearn when he goes into higher work along these lines. It might be stated in passing that this has never practically been done by the limit process.

A thorough canvass of this subject might well embrace, first, a treatment of all the topics related to limits or irrational numbers commonly met with in elementary work, based upon fundamental postulates and carried out in all logical detail from the point of view of the "cut"; second, a comparison of the logical difficulties of this treatment with the treatment based upon the idea of limits; third, a psychological study of the secondary student in his approach to these topics; fourth, a study of the traditional treatment from a historical, a logical and a psychological point of view; fifth, the actual construction of a course adopted to the capacities of secondary students which, at the same time shall be free from actual logical errors and shall fit the student for passing naturally to more rigorous treatments. This is too long a

program for the present paper and we shall confine ourselves to a few phases of it.

The first step in such a study, that of making a rigorous treatment based on fundamental postulates, has generally been passed over by the mathematician with the feeling that he could supply it if necessary. Already familiar with the traditional method of limits, which has been worked down to a concise statement, the method of the "cut" in its early crude form seems cumbersome. Hence, even the mathematician cannot easily judge of the relative difficulties of the two treatments in an unprejudiced way. It is not desirable that the method of the "cut" be passed down to the secondary schools until it has been thoroughly worked over by mathematicians and made a part of their way of thinking. If carried prematurely into the secondary school it is liable to be adopted in a crude form as much open to objection as the traditional method of limits. For this reason the main part of this paper is devoted to an effort to put into concise and convenient and at the same time logical form, certain of the most elementary notions arising in arithmetic and geometry.

That the methods of limits is not adapted to secondary teaching, is quite widely believed at the present time. Secondary teachers are divided largely between the old method of limits, on the one hand, and not teaching these topics at all, on the other. It is not our purpose to enter into this discussion formally. It is our aim to contribute towards making the method of the "cut" a little more familiar, resting its adoption in its own intrinsic simplicity, if in the outcome it shall

succeed in showing itself more simple than the method of limits.

It should be borne in mind that a method which may be most useful to a trained mathematician, may not be most easily adapted to the young student. And even should the method of limit hold its present place in higher mathematics, it does not follow from this that it is best adapted to the secondary s student.

While it is quite probable that the two methods may ultimately be fused together into a method superior to either alone, it seems best in this paper to keep as far as possible from the peculiar devices of the method of limits. Probably the most characteristic feature of the traditional treatment of infinite processes, is the so called "epsilon proof". The essence of this idea of course cannot be avoided. It is however open to question whether it is not possible without any sacrifice of logical rigor, to avoid the complicated phraseology of the epsilon proof. In this paper an effort has been made to avoid the fo form of the epsilon proof even at a slight sacrifice, in the hope that the effort might lead to an equally simple form of statement in other respects without the obvious disadvantages of the epsilon methods. That this is possible in some cases is clear from a comparison of the "cut" definition of an irrational number as presented in this paper with the postulate by which Dr. Huntington provides for the existence of irrationals in his article "Complete Sets of Postulates for the Theory of Real Quantities" (Transactions of the American Mathematical Society, Vol. 4, No. 3, p. 360, July 1903) The assumption preliminary to this definition and to Dr. Huntington's postulates are essentially

the same, the results are equivalent. The answer to the question as to which is the simpler will appear only as we use them in further work.

Chapter I.

A New Treatment of Limits.

During the earlier period of mathematical work only positive numbers are needed to represent the facts with which we wish to deal - later positive fractions and negative numbers are defined and used. But still later in order to carry on mathematical processes we must define a new kind of number.

To show the need of such new number, let us try to find a number which squared will give 3.

We first prove that no rational number squared will produce 3.

Rational numbers are either integral or fractional.

We can easily try the integers $1^2=1$; $2^2=4$. Any number greater than 2 squared will give a result greater than 4. Therefore no integer squared will give 3.

Suppose some fraction squared equals 3. We shall show this to be an absurdity.

Let a/b be the fraction in its lowest terms, which squared equals 3(If there is any). Then $a/b \times a/b = a^2/b^2 = 3$. But $a/b \times a/b$ is still a fraction, since there are no factors in the numerator and denominator which are common. But this gives us a fraction in its lowest terms equal to an integer; which is absurd.

Therefore no fraction squared will give 3.

We shall proceed to define a new number.

We assume all rational numbers and the principles governing them.

Let us put all positive rational numbers into two classes according to a definite law which we now state. (In this work we deal with positive numbers only).

No rational number squared equals 3. If a number squared is less than 3, put it in class A; if greater than 3 put it in class B.

It follows that every A $<$ every B.

As there is no number which squared will give 3, we establish a number K, which is defined by the relation: Every A $<$ K $<$ every B(See general process below).

This number K divides all positive rational numbers into two distinct classes. For this reason it is called a "cut number."

We proceed to lay down a general process by which cut numbers can be established.

Suppose we have given any law by which all positive rational numbers* can be classified into two classes A and B, such that: 1. No rational number is in two classes; 2. Every A $<$ Every B.

Then we define a number K which by definition shall have the following property:

Every A $<$ K $<$ every B(In all our work in this article, K may possibly turn out to be rational. In such case we shall say it is the smallest of the B's. It may be stated also that there is no largest A.

*While we here define "cuts" in rational numbers only, however, when irrationals have been defined under suitable restrictions, we could extend our "cut" to assemblages of irrationals, and we such cuts in later proofs.

In order that our new number may be useful, we must define some of the fundamental operations.

(I) Definition of Equality, $(=)$. If two cut numbers, K and K^{1} are defined by A and B, A^{1} and B^{1} respectively - as described above-a necessary and sufficient condition that $K = K^{1}$ is that every A is an A^{1} and every A^{1} is an A.

(2) Definition of Addition $(+)$. If K and K^{1} are two cut numbers, established by the classes A and B; A^{1} and B^{1} respectively, we shall define K^{11} as follows: Given any rational number, C to be classified, if there exists an A and an A^{1} such that $C < A + A^{1}$; put C in class A^{11}. If there exists no such sum, put C in Class B^{11}.

By this law all rational numbers may be classified into two distinct classes.

(a) No rational number is in two classes, for numbers in class A^{11} are less than some $(A + A^{1})$ and all other numbers are in class B^{11}.

(b) Every $A^{11} <$ Every B^{11}. Therefore according to our general process, we establish a number K^{11} such that every $A^{11} < K^{11} <$ every B^{11}.

We then define $K + K^{1} = K^{11}$ where K, K^{1} and K^{11} are determined as above.

The necessary and sufficient condition in order that $K^{1} + K = K^{11}$ is that every $A + A^{1} = A^{11}$ and every A^{11} is an $A + A^{1}$.

We omit the proof.

(3) Definition of subtraction (–). We define subtraction so that the relation $K^{11} - K^{1} = K$ shall mean the same as $K + K^{1} = K^{11}$ (K, K^{1} and K^{11} defined as above.)

(4) Definition of Multiplication (x). If K is defined by Classes A and B and K^{1} is defined by A^{1} and B^{1}, we shall define K^{11} as follows: Given any rational number C to be classified, if there exists an $A \times A^{1} > C$ put C in class A^{11}; but if not, put C in class B^{11}.

By this law all rational numbers may be put into two distinct classes;

(a) No rational number is in both classes, for all numbers in class A^{11} are less than some product $A \times A^{1}$, and all other numbers are in class B^{11}.

(b) Every $A^{11} <$ every B^{11}.

Therefore according to our general process, we establish a number K^{11}, such that every $A^{11} < K'' <$ every B^{11}.

We then define $K \times K^{1} = K^{11}$ when K, K^{1} and K^{11} are determined as above.

A necessary and sufficient condition that $K \times K^{1} = K^{11}$ is that $A \times A^{1} = A^{11}$ and every A'' is an $A \times$ an A'. This proof we shall omit also.

(5) Definition of Division (÷). $K^{11}/K^{1} = K$ shall mean $K \times K^{1} = K^{11}$ or $K^{11}/K = K^{1}$ where K, K^{1} and K^{11} are defined in multiplication above.

K^{11}/K^{1} shall also be defined as the ratio of K^{11} to K^{1}.

Proportion shall mean an equality of two such ratios.

We assume without proof that the associative, commutative and distributive laws of rational numbers may be extended to cut numbers.

(6) Definition of K^b where K is irrational .

I. Suppose b integral. We define $K^b = K \times K$ --------$\times K$----- b factors, where $K \times K$ has been defined in multiplication.

II. Suppose b a positive fraction c/d, when c and d are both integral.

We define $K^{c/d} = L$ to mean the same as $K^c = L^d$. We may write this $K^{c/d} = \sqrt[d]{K^c}$.

(7) Definition of K^{k^1} when K and K^1 are both irrational.

Suppose K defined by A and B and K^1 by A^1 and B^1 according to our general law laid down above. Then $K^{k'}$ shall be defined as follows: Given any rational number C to be classified, we write the law of classification as follows:

If there exists an $K^{A'}$ such that $K^{A'} > C$, put C in class A^{11}- but if not put C in class B^{11}.

By this law all rational numbers are completely classified into two distinct classes.

(a) No rational number is in two classes, for all numbers in A^{11} are less than some number $K^{A'}$ and all other numbers are in class B^{11}.

(b) Every $A^{11} <$ Every B^{11}.

Therefore according to our general process, we shall establish a number K^{11} such that $A^{11} < K^{11} < B^{11}$.

We then define $K^{K'} = K^{11}$ when K, K^1 and K^{11} are determined as above.

Supplement to Chapter I.

Since writing this chapter, it appears evident that there are a variety of ways to define the elementary operations and the theorems connected with them.

In this supplement we think we have defined our operations so that we get greater symmetry and we have made it more complete by adding a number of simple and at the same time useful theorems connected with irrationals. All the processes defining area, length, etc. will go equally as well by this m method. We proceed to lay down a general process by which cut numbers can be established. Suppose we have given any law by which all positive rational numbers can be classified into two distinct classes A and B, such that : (1) No rational number is in two classes; (2) Every A $<$ every B.

Then we define a number K which by definition shall have the following property.

Every A $<$ K $<$ every B. In this article we shall not have any largest A or smallest B. If the cut number turns out rational, we will let it be the K(K then may sometimes be rational and sometimes irrational).

Definition of Equality. ($=$).

Given K and K' defined by classes A and B, A' and B' respectively.

Given also that every A $=$ an A' and every A' $=$ an A, we shall say by definition K $=$ K'.

Theorem: A necessary and sufficient condition that K $=$ K' is that:

(1) Every B $=$ a B' and every B' $=$ a B.

(2) Every $A <$ some A' and every $A' <$ some A.

(3) Every $B >$ some B' and every $B' >$ some B.

(4) Every $A <$ every B' and every $A' <$ every B.

(5) Every $A =$ an A' and every $B =$ a B'.

(1) Every $B =$ a B' and every B' is a B.

(a) Given: every $A =$ an A' and every $A' =$ an A.
To prove that every $B =$ a B' and every $B' =$ a B.
Suppose a $B \neq$ a B', then $B =$ an A' by definition of cut.

But by definition of equality every $A' =$ an A.

$\therefore$ $B =$ an A which contradicts our definition of a cut.

$\therefore$ Every $B =$ a B'.

Similarly every $B' =$ a B.

(b) Given every $B =$ a B' and every $B' =$ a B.

To prove that every $A' =$ an A and every $A =$ an A'.

Suppose $A' \neq$ an A, then $A' =$ a B and by hypothesis $=$ a B' but this contradicts our definition of cut numbers.

$\therefore$ Every $A' =$ an A and conversely every $A =$ an A'.

(2) Every $A <$ some A' and every $A' <$ some A.

(a) Given: every $A =$ some A' and every $A' =$ an A.

To prove that every $A <$ some A' and every $A' <$ some A.

Proof: Every $A <$ some other $A = A'$

$\therefore$ Every $A <$ some A'.

Similarly every $A' <$ some A.

(b) Given every $A <$ some A' and every $A' <$ some A.

To prove that every $A =$ an A' and every $A' =$ an A.

Proof: Consider any A.

By hypothesis it is less than some A'.

But by definition all numbers $<$ A's are A's.

$\therefore$ As A $<$ A' it is an A'. $\therefore$ Every A = an A'

Similarly every A' = an A'.

(3) Every B $>$ some B' and every B' $>$ some B.

(a) Given every A = an A' and every A' = an A.

To prove that every B $>$ some B' and every B' $>$ some B.

Proof: Every B $>$ some other B.

But by (1) every B = a B'.

$\therefore$ Every B $>$ some B'

Similarly every B' $>$ some B.

(b) Given every B $>$ some B' and every B' $>$ some B.

To prove that every A = an A' and every A' an A.

Proof: Consider any B.

By hypothesis B $>$ some B'.

But all numbers $>$ B' are B's and by hypothesis B $>$ B'.

$\therefore$ Every B = a B'.

Similarly every B' = a B.

But by (1) if every B = a B' and every B' = a B.

Every A = an A' and every A' = an A.

(4) Every A $<$ every B' and every A' $<$ every B.

(a) Given every A = an A' and every A' = an A.

To prove that every A $<$ every B' and every A' $<$ every B.

Proof: Every A = an A' by hypothesis.

Every A' $<$ every B' by definition of cut.

$\therefore$ Every A $<$ every B'

Similarly every A' $<$ every B.

(b) Given every A $<$ every B' and every A' $<$ every B.

To prove that every A = an A' and every A' = an A.

Proof: Consider any A.

By hypothesis every A $<$ every B'.

$\therefore$ Every A must $=$ an A'.

Similarly every B $=$ a B'

(5) Every A $=$ an A' and every B $=$ a B'.

(a) Given every A $=$ an A' and every B $=$ a B'.

To prove that every A' $=$ an A and every B' $=$ a B.

Suppose $A'_n \neq$ an A. Then $A'_n =$ a B.

But by hypothesis every B $=$ a B'.

$\therefore$ $A'_n =$ a B' which contradicts our definition of a cut number.

$\therefore$ Every A' $=$ an A.

Similarly every B' $=$ a B.

It can be easily proved that if $K = K'$ and $K' = K''$, that $K = K''$

Definition of Inequalities ($\gtrless$).

Given $A < K < B$.

Also $A' < K' < B'$.

If an A' exists such that every A' $>$ some B, we shall say by definition that $K' > K$ or $K < K'$.

A necessary and sufficient condition that $K' > K$ is that:

(1) An A' $=$ a B.

(a) Given that every A' $>$ some B.

To prove that some A' $=$ some B.

Proof: Every number $>$ a B is a B by definition of cut.

A' $>$ some B.

$\therefore$ A' is a B.

(b) Given that some $A' =$ some B.

To prove that some $A' >$ some B.

Proof: Some $A' =$ a $B >$ another B.

$\therefore$ Some $A' >$ a B.

If $K > K'$ and $K' > K''$.

To prove that $K > K''$.

Given $A < K < B$
$A' < K' < B'$ Defining K, K' and K''.
$A'' < K'' < B''$

By hypothesis $K > K'$

By theorem (1) some $A =$ a B'.

$\therefore$ Some $A >$ every A'

Also by hypothesis, $K' > K''$.

By theorem (1) some $A' =$ a B''.

$\therefore$ Some $A' >$ every A''.

$\therefore$ Some $A >$ every $A'' =$ a B''.

But by (1) if some $A =$ a B''

$K > K''$.

Given $A < K < B$ and $A' < K' < B'$.

To prove that one and only one of the relation $K = K'$ or $K < K'$ or

$K > K'$ holds .
Let us assume $K \not> K'$ nor $K \not< K'$
To prove $K = K'$.

By theorem of unequality.

$K \not> K'$ means that no $A' =$ a B and

$K \not< K'$ " " " $A =$ a B'.

By definition of cut numbers, if any $A' \neq$ a B it must equal an A.

Similarly every A must equal an A'.

$\therefore K = K'$.

(2) To prove that no two of the relations above hold at the same time.

(a) Given $K = K'$.

To prove $K \not> K'$ nor $K \not< K'$.

By hypothesis $K = K'$.

$\therefore$ Every A = an A' and every A' = an A.

But $A' \neq$ a B and $A \neq$ a B'.

$\therefore$ $K \not> K'$ nor $K \not< K'$.

1. Suppose $K > K'$. To prove $K \neq K'$ nor $K \not< K'$.
2. By (a) $K \neq K'$. If $K > K'$ some A = a B'.
3. Suppose $K < K'$ then some A' = a B.
4. We shall prove this to be absurd.
5. Every $A <$ every B.
6. Every $A' <$ Every B'

$\left.\begin{array}{l}5.\\6.\end{array}\right\}$ By definition of cut.

7. By hypothesis some A = a B'.
8. By supposition some A' = a B.

Substitute (7) in (5) Some $B' <$ every B.
Substitute (7) in (6) some $B <$ Every B'

$\left.\right\}$ Contradiction.

Chapter II_1

In this chapter, we shall apply our principles of Chapter I in defining Area, Lenght, etc. For this purpose, we shall make the following assumptions.

(1) Assume all rational numbers.

(2) Assume that to every rational number corresponds a point on a straight line, which we shall call a rational point.

(This may be done in a variety of ways. See Young's "Theory of Sets of Points. The patticular method which we adopt is defined by our third assumption.

We shall designate points on a straight line by $\alpha_1, ---- \alpha_K$ . ... and the corresponding numbers by a_1, a_2, a_3.. ......a_n, where a_k corresponds to α_K.

(3) Assume that any line or line segment may be placed upon any other line or line segment, and when so placed, if α_j' corresponding to a_j of one line l shall fall upon α_i' corresponding to a_k' may be made to fall upon α_s of l' corresponding to a_s whén and only when $|a_j - a_k| = |a_i - a_s|$.

If $a_i' \angle a_j$ we shall say α_i' (L) α_j' which we may read α_i' lies to the left of α_j'.

Definition of Lenght.

Given any two rational numbers a_k and a_L corresponding to α_K and α_L, we define the length from α_K to α_L to be the number $l = |a_L - a_k| = \overline{\alpha_K \alpha_L}$. It follows from our definition of length, that a necessary and sufficient condition that two lines or line segments are equal is, they may be made to coincide,

II$_2$

Irrationals

Let an irrational number K be defined by the classes A and B according to our definition laid down in Chapter I.

Let the rational points corresponding to the A's and B's fine K, be marked down upon a line.

$$A_1 \ldots\ldots\ldots A_i \ldots\ldots\ldots A_n \,|\, K \,|\, B_n \ldots\ldots\ldots B_1 \ldots\ldots\ldots B$$

Now $A_s < B_r$, hence, $\alpha_s \bigcirc\!\!\!\!< \beta_r$.

But $A_n < K < B_n$ and there is no point corresponding to K.

Let us establish a point κ, corresponding to K . Then by extending our definition of the symbol $\bigcirc\!\!\!\!<$ to these new points, it follwos from the above that $\alpha_n \bigcirc\!\!\!\!< \kappa \bigcirc\!\!\!\!< \beta_n$.

By a similar process we can establish as many irrational numbers and their corresponding points as we desire.

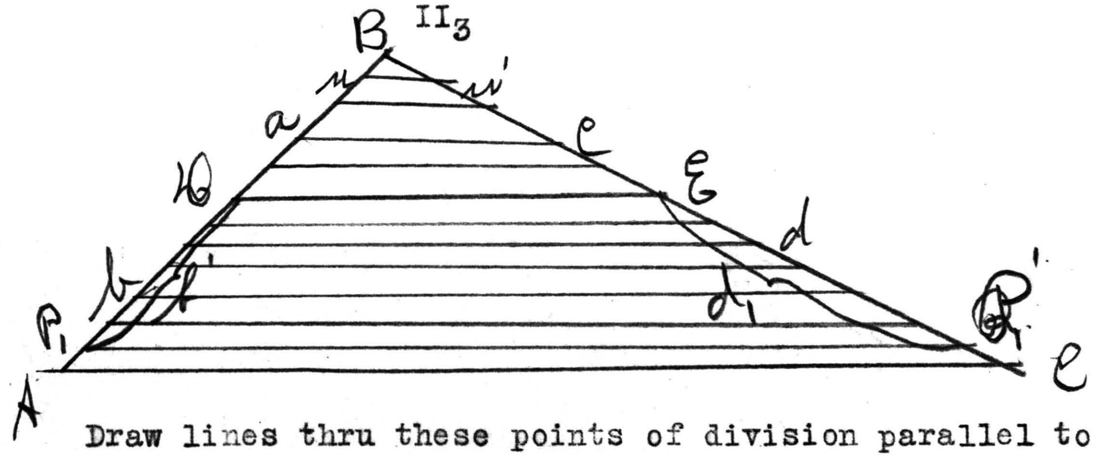
We extend our definition of length to irrationals, $\ell = |K-K^1|$, where classes A and B define K and classes A^1 and B^1 define K^1.

<u>Theorem of proportionality of sides of triangle</u>.

Given any triangle ABC with any line DE drawn parallel to any base, say AC and parts marked as below.

To prove $a/b = c/d$.

Divide a into any suitable number of equal divisions. Call thelength of any one of these divisions u.



Draw lines thru these points of division parallel to ED. These lines will cut C into equal parts (By theoren of plane geometry) Call the length of one of these divisions u'.

Now $a/u = c/u'$.

Next apply this unit u to b and mark points of division and draw parallel lines as before.

It will go a number of times with possibly a remainder, The remainder, however, is less than the unit; if it were not we should apply it again.

Mark the last point of division P_1. There will be a point on EC corresponding to P_1, which we shall call P_1'. Call the length of the line segments DP_1', b, and EP_1', d, respectively.

Now divide u in any way you choose into equal divisions, u' will be divided into the same number of equal divisions by our parallel lines.

Apply these new units to b and d respectively, beginning at D and E.

Mark points of last application on b, P_r $(r = 1 \ldots\ldots n)$ and on d, Q_r. Mark lengths of segments DP_r and EQ_r, b_r and d_r respectively.

b and d are defined by the cuts $\begin{cases} A < b < B \\ A' < d < B' \end{cases}$

$A/u < b/u < B/u$ and $A'/u' < d/u' < B'/u'$ by division. (This is a cut in irrationals)

By definition of equality of two cut numbers it will follow that $b/u = d/u'$ if we can show that:

(1) Any $a/u <$ some A'/u'.

(2) Any $A'/u' <$ some A/u.

Any $A <$ some b_n.

$\therefore$ Any $A/u <$ some b_n/u (1)

But any $b_n/u =$ some d_n/u' (2)

Also any $d_n <$ some A'.

$\therefore$ Any $d_n/u' <$ some A'/u' (3)

Hence from (1), (2), (3), any $A/u <$ some A'/u'.

Similarly any $A'/u' <$ some A/u.

Hence $b/u = d/u'$ by definition of equality.

Then by alternation $b/d = u/u'$ which was to be proved.

But $a/c = u/u'$. $\therefore$ $a/c = b/d$. By multiplying by c/b we get $a/b = c/d$.

This is readily seen to include commensurable cases as well as incommensurable, for b_n for some value of n will equal b, also d_n for the same value of n will equal d, if commensurable.

We now assume the following propositions from plane geometry.

(1) In the same circle or in equal circles, equal central angles intercept equal arcs and conversely.

(2) In the same circle of in equal circles, equal chords subtend equal arcs and are equally distant from the center.

Cor. 1. If the chords are unequal, the greater is closer to the center and subtends the greater arc.

(3) We assume superposition of figures and elementary notions connected therewith.

We are now ready to define <u>Area</u>.

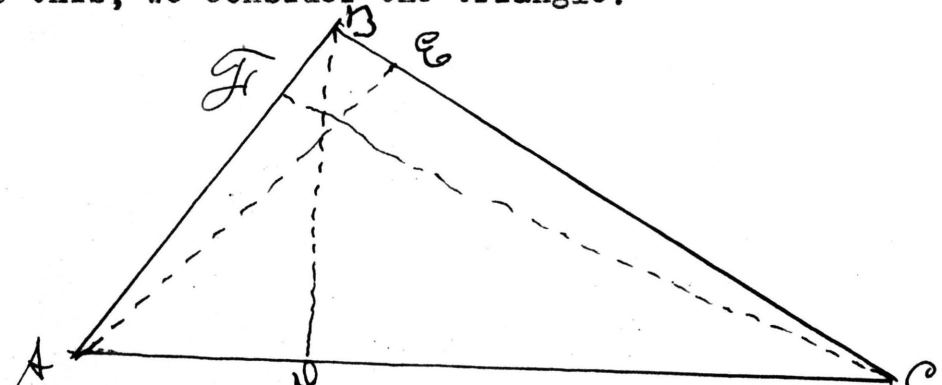
Another simple way of defining area, would be to define the area of a rectangle as the base multiplied by the altitude. It would then be necessary to define the area of a triangle and to prove that the area of a triangle equals the base, multiplied by one half the altitude.

In ordinary treatments the area of a triangle is not proved unique.

The two methods of treatment are essentially of equal simplicity. The triangle is used more. It is easier to divide most figures into triangles than rectangles.

We define the area of a triangle to be the base times one half of the altitude.

It will be necessary to show that the base multiplied by the altitude is independent of the base and altitude used. To do this, we consider the triangle:



We desire to prove that base AC x Altitude BD = base AB x Altitude FC = base BC x Altitude AE.

~~In another place~~ It follows from our proof II_3 we proved the proportionality of sides of triangles which are similar(commensurable and incommensurable)

In the similar rt. triangles, ACF and ABD, AC/CF = AB/BD. Therefore ACBD = ABFC.

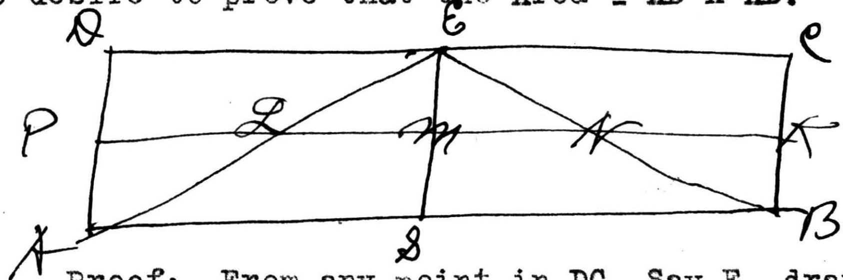
Similarly, BCAB = ACBD = ABFC.

We shall now prove that the area of a rectangle equals the base times the altitude.

We assume that any figure may be cut into any number of parts and rearranged in any way desirable without changing its area, and that the area of the whole equals the sum of the areas of the parts.

Given the rectangle ABCD with base AB and altitude AD.

We desire to prove that the Area = AB x AD.



Proof: From any point in DC, Say E, draw line s to A and B thus forming the triangle AEB, with base AB and altitude AD.

By definition of Area of a triangle, the area of AEB equals AB.AD/2. We shall show that the area of the rectangle is twice the area of the triangle with the same base AB and the same altitude AD.

Draw PK parallel to AB cutting AD so that DP = PA.

By geometric proof of transversals cut by parallel lines, EL = LA, EM = MS, EN = NB and CK = KB.

Triangle PLA = triangle LEM for:

∠PLA = ∠ELM

∠EML = ∠LPA

∠LEM = ∠LAP

Side EL = side LA.

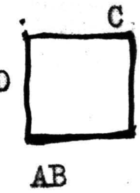
In like manner we can prove that triangle EMN = triangle NBK. But triangles PLA and NBK added to the trapezoids LMAS and MNBS just fill out the triangle EAB and at the same time uses up the redtangle ABCD.

Therefore the area of a redtangle is equal to two times the area of a triangle with an equal base and an equal altitude.

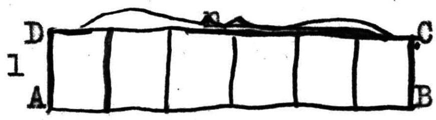
We shall now show that our definition and proof corresponds to the general notions of area.

If AB = 1 and AD = 1, then the area AB.AD =1.

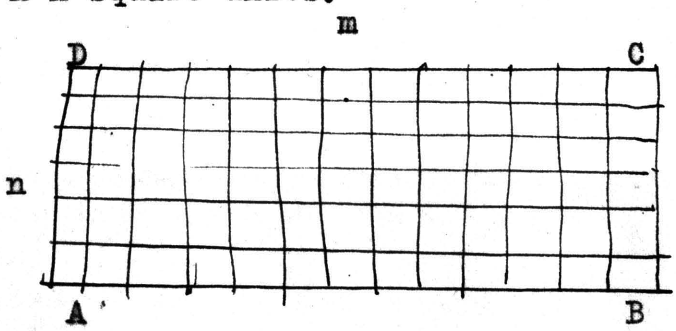
We designate this as one square unit.



If AB = 1 and AD = n units then we see that AB x AD = n square units:



If AB = m units and AD = n units, then we see that AB AD equals m x n square units.



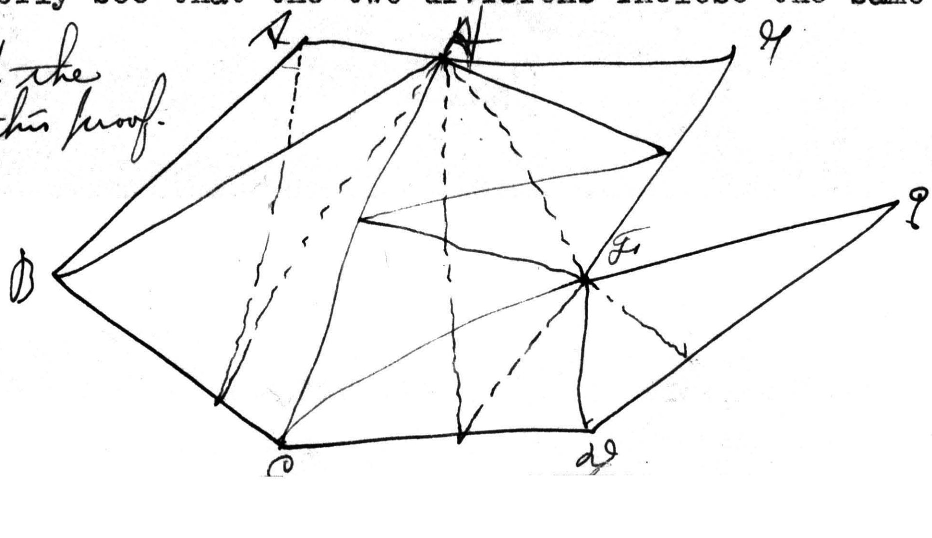
From our definition of the Area of triangles, we can compute the area of trapeziods, trapeziums, parallelograms, and all figures which can be divided into triangles.

We define the area of any plane figure to mean the sum of the areas of the parts into which the figure may be divided.

We now show that this area will be the same, however the figure is divided.

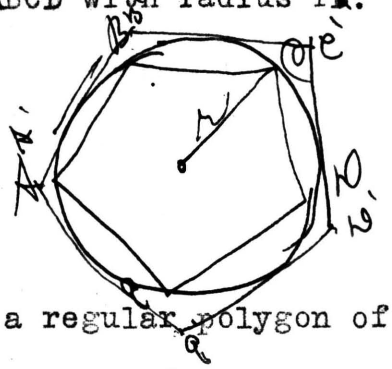
Let us choose any plane, figure ABCDEFGH. and divide it by passing red lines thru it any way desired. Then devide by blue lines in any other way. Now by counting common areas we c can easily see that the two divisions inclose the same area.

We omit the details of this proof.



Definition of the circumference of a circle.

Given a circle ABCD with radius r.



Let us inscribe a regular polygon of n sides. We shall also circumscribe a polygon of n sides. Next increase the number of sides of the inscribed and circumscribed polygons in any way you choose.

We now state the law for determining the circumference of the circle.

Divide all rational numbers into two classes A and B.

Put all rational numbers equal to or less than the perimeter of any inscribed polygon into class A. Put all other numbers into Class B.

All rational numbers are classified in such a manner that ,
1. No rational number is in two classes, for every A is less than the perimeter of some inscribed polygon, and all other numbers are in class B. 2. Every A $<$ every B

According to our general process laid down in Chapter I, classes A and B establish a cut number K which is defined by the relation. Every A $<$ K $<$ every B. We shall call the length of the circumference so defined $2\pi r$, where r is the radius of the inscribed circle.

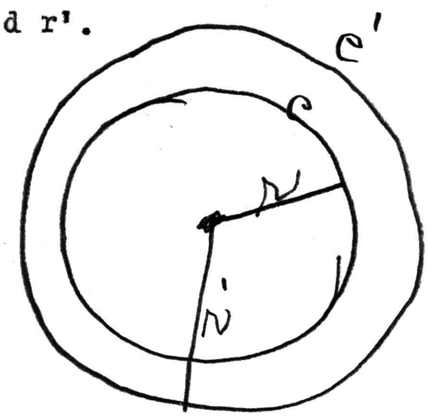
Later we shall show how to compute π to any required decimal place. It still remains to be proved that $C:C^1 = r:r^1$, where c and c^1, r and r^1 are the circumferences and radii respectively of two circles. It then follows that π is a constant.

This we shall proceed to prove.

Given any two circumberences C and c' with radii r and r' respectively.

To prove $C:C' = r:r'$.

Proof: Place the circles concentric at O as in figure. Draw radii r and r'.



The inequalities $A < C < B$ and $A' < C' < B'$ define C and C'. $A/r < C/r < B/r$ and $A'/r' < C/r' < B'/r'$.

By definition of equlity of two cut numbers it will follow that $C/r = C'/r'$, if we can show that:

(1) Any $A/r <$ some A'/r'.

(2) Any $A'/r' <$ some A/r.

Let P_k and P'_k represent the perimeter of the corresponding inscribed polygons respectively.

Any $A <$ some P_n. $\therefore$ Any $A/r < P_n/r$ (1)

But any $P_n/r =$ some P'_n/r' (2)

Also any $P'_n <$ some A'. $\therefore$ $P'_n/r' <$ some A'/r' (3)

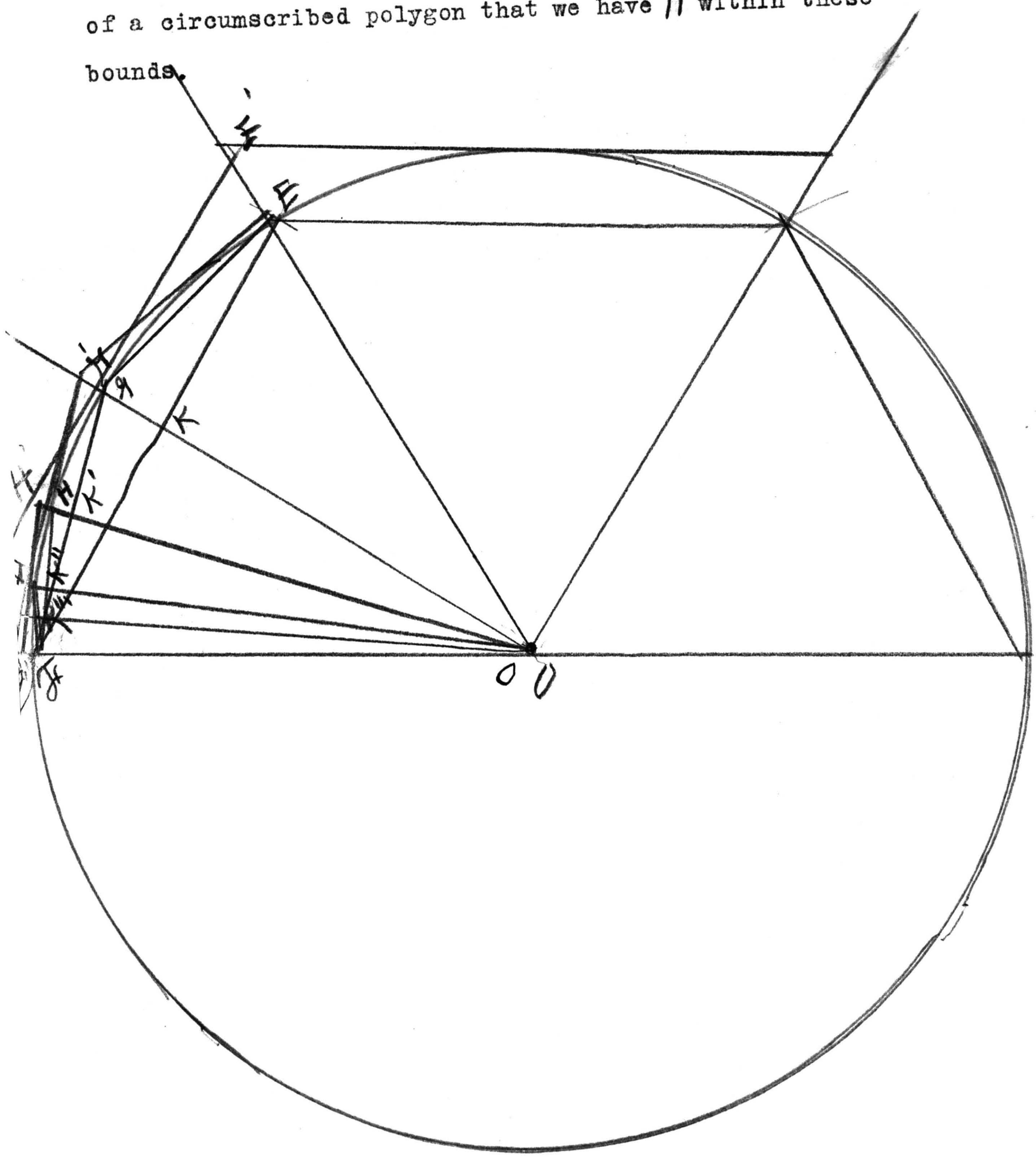
Hence from (1), (2), (3), Any $A/r <$ some A'/r'.

Similarly Any $A'/r' <$ some A/r.

Hence $C/r = C'/r'$.

II_{10}.

We shall now compute π correct to two decimal places by the aid of our inscribed polygons and then show by the aid of a circumscribed polygon that we have π within these bounds.



Inscribe a regular hexagon in the circle (O C).

Call radius of circle r (r = 1)

The perimeter of the inscribed hexagon = 6r = 6.

Draw OK to centre of line EF and extend to the circumference. It will bisect the chord and the arc (EF)

$OK = \sqrt{1^2 - (1/2)^2} = 1/2\sqrt{3}.$

$KG = 1 - 1/2\sqrt{3}.$

Double the number of sides of the inscribed polygon as indicated in the accompaning figure. FGK is a rt Δ with base FK, altitude KG and hypotenuse FG.

$FG = \sqrt{\overline{FK}^2 + \overline{KG}^2} = \sqrt{(1/2)^2 - (1 - 1/2\sqrt{3})^2} = \sqrt{2 - \sqrt{3}}.$

Draw from O, OK' perpendicular to FG as before.

OK' will bisect chord and arc FG.

$FK' = \frac{\sqrt{2 - 3}}{2} = \frac{.5176}{2} = .2588$

$OK' = \sqrt{1^2 - \left(\frac{\sqrt{2 - \sqrt{3}}}{2}\right)^2} = .966$

$K'H = 1 - .966 = .034$

$FH = \sqrt{\overline{FK}^2 + \overline{HK'}^2} = .2612$

Double the number of sides as before and draw OK''.

$FK'' = \frac{.2612}{2} = .1306$

$OK'' = \sqrt{1^2 - (.1306)^2} = .9915$

$K''I = 1 - .9915 = .0085$

$FI = \sqrt{\overline{FK''}^2 \quad K''I^2} = .13087$

Our perimeter has now 48 sides.

$\frac{48 \times .13087}{2} = 3.1408$ for the value of π to two decimal places.

$\therefore \pi > 3.1408$

This process may be continued as often as desired.

We shall now show that our result is correct to two decimal places.

Take a circumscribed polygon of the same number of sides constructed so that the sides of the inscribed and circumscribed polygons will be parallel.

We now have two similar rt. $\triangle$s, OFK''' and OF"'J.

F"'J:FK"'= 1:.9979 (See figure)

$\therefore$ F"'J = .06543 $\div$ .9979 = .06547.

But F"'J is one of the equal sides of a 96 sided polygon.

$\therefore \pi < \frac{.06547 \times 96}{2} = 3.1425.$ Correct to two decimal places.

$\pi = \frac{3.1408 \quad 3.1425}{2} \pm$ remainder.

$= 3.14165 \pm$ remainder. Remainder $< .001$.

Chapter III_1

In this chapter we shall critically examine some of the most widely known textbooks in plane geometry and algebra and inquire into the present teaching of limits in the high schools.

If we carefully examine present text books on limits, we readily conclude that pupils studying them would have the wrong conception of infinity, infinitesimal and limit. They would think of infinity as a large number, of an infinitesimal as a small number, and of a limit as something mysterious of which they cannot form a conception.

These are a few definitions and implications from some of our leading high school texts on geometry and algebra.

Jocelyn of Ann Arvor, in his High School and College Algebra, defines a limit as follows: "The limit of a variable is a constant, the value of which the variable can never reach but to which it may approach so nearly by successive steps, that the difference shall be less than any assignable quantity".

Professor Wells of Massachusetts Institute of Technology says,(Plane and Solid Geometry, Page 81 Art. 186)"The difference between a variable and its limit can be made as small as you please but cannot be made equal to zero" To show the "sum" of an infinite series, which according to any definition he had given, did not exist, he takes the series $1+1/2+1/4. \ldots\ldots\ldots\ldots 1/2n$

He asserts that the last term can be made as small as we please, but not zero. From this statement, he draws the conclusion that the limit of the sum exists. Suppose he had tried some other simple series, say $\div\ 1+1/2+1/3+1/4 \ldots\ldots\ldots\ldots 1/n$

The n th term can be made as small as we please by making n large enough. Yet it is easily proved that there is no limit to the sum, for by taking n large enough we can make the sum greater than any quantity previously assigned.

Schultz and Sevenoak say, (Plain Geometry, page 90 Art.211) "A variable can never reach its limit.

Milne leads us to believe (Academic Algebra p.56) that one can add an infinite number of terms to get a sum. He says (Academic Algebra ; p 380, Art. 314) "If a finite number be divided by an infinitesimal number, the quotient will be an infinite number." Now it is well established usage that an infinitesimal is not a number at all. It is a variable which approaches zero as a limit. Neither is infinity a number (according to any definition of number which had been defined at this stage of mathematical work) but rather a condition.

Schultze in his elementary algebra defines a root as one of the equal factors of a power. He then gives such expressions as $\sqrt{17}$ in his exercises without so much as mentioning incommensurables or even approximations.

These persons who claim that a variable never reaches its limit surely have forgotten such function as :

$\lim_{x \doteq 0} \sin x$, $\lim_{x \doteq a} x^2$, $\lim_{x \doteq 45^\circ} \sin x$, $\lim_{x \doteq a} \frac{1}{x}$, in each of these cases exists and the function reaches it ~~but~~ $(a \neq 0)$ but the limit $\lim_{x \doteq 0} \frac{1}{x}$ does not exist

$\lim_{x \doteq \infty} \frac{1}{x}$ exists but the function does not reach it.

To be sure some variables do not reach their limits, but these are the exceptions and not the rule.

Many variables also pass their limits.

For definitions of the various terms used I refer you to some good calculus. (Campbell or Osgood or any good text)