The thesis herewith submitted in partial fulfillment of the requirements for the degree of master of arts, and entitled "Vectors in Four Dimensions", by Mr. W. S. Pemberton, is approved.

O. D. Kellogg.

VECTORS IN FOUR DIMENSIONS

by

WALKER S. PEMBERTON, B. S.

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INTRODUCTORY REMARKS

on the Thesis by Mr. W. S. Pemberton

on

VECTORS IN FOUR DIMENSIONS.

The interest attaching to n-dimensional geometry comes chiefly from two sources, first the light thrown upon analysis by a geometric interpretation of its results when more than three variables are involved, and second, the light thrown upon the geometries of a smaller number of dimensions by fitting them into their places in a more general theory, which brings out much more clearly than is otherwise possible their characteristic properties.

The theory of vectors is a special aspect of geometry, and one which takes its inspiration from, and is highly useful to, the science of mechanics. Mr. Pemberton's work, therefore, is an attack upon a problem of considerable interest, yet one apparently only very little developed. Just the extent to which his results are new it is impossible to determine without access to libraries more extensive than he has had at his disposal.

The first troublesome points with which he met were the proper extension of the notion of moment, and the proper convention as to the sense of orientated magnitudes in four space. The generalization of his definitions to n-dimensional space will be immediate. Particular attention should be called to the result that any system of vectors in four space may be reduced to a system of two vectors except when the system is equivalent to two couples in planes intersecting in a point only, and that in this case the reduction is impossible.

This is a striking characteristic of four dimensional space, which opens up a whole series of interesting questions in the mechanics of hyperspace, which it is to be hoped Mr. Pemberton will later have opportunity to investigate.

O. D. Kellogg.

Vectors in Four Dimensions.

Chapter I.

In this discussion, which is based on the ordinary theory of vectors as applied to mechanics, we shall extend the meaning so as to make application to four dimensions, and in doing so we assume the geometric properties of four dimensional space.

1. Definitions:- We shall define a vector geometrically to mean a line segment $A_1 B_1$, having A_1 as the point of application and B_1 as the extremity, its positive sense being from A_1 to B_1.

We shall assume four perpendicular axes and define a vector analytically by the co-ordinates x_1, y_1, z_1, w_1, and x_2, y_2, z_2, w_2 as referred to the four axes.

The vector may also be fixed by x_1, y_1, z_1, w_1, and its projections X, Y, Z, W, on the four axes, $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$, and $W = w_2 - w_1$.

The angle between two lines needs no new definition here since the two lines that determine the angle also determine the plane in which the angle lies and so projection as here used is according to the ordinary definition for projections.

2. Standard Set of Axes:- Given three perpendicular axes through a point in four space, which we take for the X, Y, Z axes, there is one line through this point perpendicular to the three space of the axes. We fix arbitrarily, and once for all, a positive sense on this line and this determines a standard "right hand set" of X, Y, Z, W axes.

If we are given any other set of axes O', X', Y', Z', W'; O'X', O'Y', and O'Z' may be made to coincide with OX, OY, and OZ respectively. The O'W' axis either coincides with OW in a positive or a negative sense. If it coincides in a positive sense the set is to be called a right hand set and if in a negative sense, a left hand set.

3. Moments with Respect to a Point:- The moment of a vector with respect to a point has a magnitude and an orientation.

The magnitude of the moment is equal to two times the area of the triangle formed by the vector and the point of reference.

The orientation is that of the plane in which the point and the vector lie. We shall call the

above plane the plane of the moment.

The moment of a vector with respect to a point is completely determined by the point and the projections of the triangle mentioned above on the six co-ordinate planes. If we take the origin as the point of reference, the projections of the vertices are:

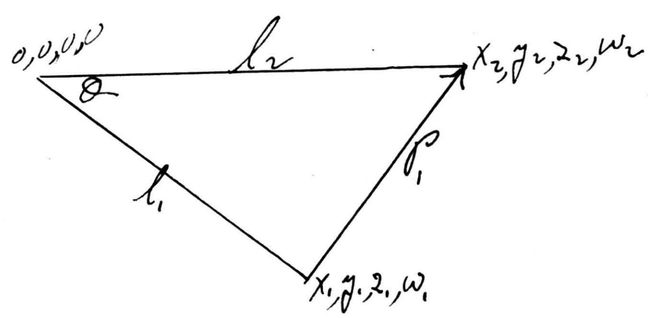
in xy plane, (o,o,o,o), (x_1,y_1,o,o), (x_2,y_2,o,o),
xz " (o,o,o,o), (x_1,o,z_1,o), (x_2,o,z_2,o),
xw " (o,o,o,o), (x_1,o,o,w_1), (x_2,o,o,w_2),
yz " (o,o,o,o), (o,y_1,z_1,o), (o,y_2,z_2,o),
yw " (o,o,o,o), (o,y_1,o,w_1), (o,y_2,o,w_2),
zw " (o,o,o,o), (o,o,z_1,w_1), (o,o,z_2,w_2).

The projections of the triangles in the six coordinate planes, as designated by the subscripts of P, are:

$$P_{xy} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \quad P_{xz} = \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \quad P_{xw} = \begin{vmatrix} x_1 & w_1 \\ x_2 & w_2 \end{vmatrix},$$

$$P_{yw} = \begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix}, \quad P_{zw} = \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix}.$$

The magnitude of the moment with respect to the origin is equal to the square root of the sum of the squares of the six determinants given above. This is a theorem of n-dimensional geometry. It may also be verified as follows:



Take x_1 y_1 z_1 w_1 and x_2 y_2 z_2 w_2 as the two points that determine the vector. Let l_1 equal the distance of $x_1 y_1 z_1 w_1$ from the origin, and l_2 equal the distance $x_2 y_2 z_2 w_2$ from the same point. Let θ equal the angle between the two lines.

Then we have the moment of the vector with respect to the origin equal to $l_1\ l_2 \sin\theta = l_1\ l_2\sqrt{1-\cos^2\theta}$

But $l_1 = \sqrt{x_1^2+y_1^2+z_1^2+w_1^2}$ and $l_2 = \sqrt{x_2^2+y_2^2+z_2^2+w_2^2}$

Then if we let $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$, and $\cos\delta_1$, be the direction cosines of l_1, and let $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$, and $\cos\delta_2$, be the direction cosines of l_2 we

will have :

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 +$$

$\cos\delta_1 \cos\delta_2$. Now if we take the values,

$$\cos\alpha_1 = \frac{x_1}{l_1}, \quad \cos\beta_1 = \frac{y_1}{l_1}, \quad \cos\gamma_1 = \frac{z_1}{l_1}, \quad \cos\delta_1 = \frac{w_1}{l_1},$$

$$\cos\alpha_2 = \frac{x_2}{l_2}, \quad \cos\beta_2 = \frac{y_2}{l_2}, \quad \cos\gamma_2 = \frac{z_2}{l_2}, \quad \cos\delta_2 = \frac{w_2}{l_2},$$

and substitute in the equation above for the moment we get the moment equal to

$$l_1 l_2 \sqrt{1 - \left[\frac{x_1x_2}{l_1l_2} + \frac{y_1y_2}{l_1l_2} + \frac{z_1z_2}{l_1l_2} + \frac{w_1w_2}{l_1l_2}\right]^2} =$$

$$\sqrt{l_1^2 l_2^2 - \left[x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2\right]^2}.$$

By substituting the values of l_1 and l_2 from above we get for the magnitude of the moment

$$\sqrt{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & w_1 \\ x_2 & w_2 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}^2 + \begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix}^2 + \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix}^2}$$

If we use the notation as suggested on page 3 we get for the magnitude of the moment

$$\sqrt{P^2_{xy} + P^2_{xz} + P^2_{xw} + P^2_{yz} + P^2_{yw} + P^2_{zw}}$$

We may extend the definition of the moment of a vector with respect to the origin to that of a vector with respect to any point $(\xi, \eta, \zeta, \omega)$ by the matrix

$$\left\| \begin{matrix} \xi & \eta & \zeta & \omega & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \end{matrix} \right\|$$

The projections of the triangle on the six co-ordinate planes are found by dropping two rows at a time from the above matrix, always retaining the last. In order to verify this we take the vector P with the point of application $x_1 y_1 z_1 w_1$ and its projections X, Y, Z, W. The projections of the moment with respect to the origin are:

$$P_{xy} = \begin{vmatrix} x_1 y_1 \\ X_1 Y_1 \end{vmatrix}, \quad P_{xw} = \begin{vmatrix} x_1 w_1 \\ X \ W \end{vmatrix}, \quad P_{yz} = \begin{vmatrix} y_1 z_1 \\ Y \ Z \end{vmatrix},$$

$$P_{yw} = \begin{vmatrix} y \ w \\ Y \ W \end{vmatrix}, \quad P_{zw} = \begin{vmatrix} z_1 w_1 \\ Z \ W \end{vmatrix}.$$

If we take any other point as $O'(\xi', \eta', \zeta', \omega')$ and

translate the axes so that O' will be the origin, the co-ordinates of the point of application as referred to the new axes are,

Since the projections of the vector remain the same the projections of the moment with respect to O' are ,

$$P'_{xy} = \begin{vmatrix} x_1 - \xi' & y_1 - \eta' \\ X & Y \end{vmatrix}, \quad P'_{xz} = \begin{vmatrix} x_1 - \xi' & z_1 - \zeta' \\ X & Z \end{vmatrix},$$

$$P'_{xw} = \begin{vmatrix} x_1 - \xi' & w_1 - \omega' \\ X & W \end{vmatrix}, \quad P_{yz} = \begin{vmatrix} y_1 - \eta' & z_1 - \zeta' \\ Y & Z \end{vmatrix},$$

$$P_{yw} = \begin{vmatrix} y_1 - \eta' & w_1 - \omega' \\ Y & W \end{vmatrix}, \quad P_{zw} = \begin{vmatrix} z_1 - \zeta' & w_1 - \omega' \\ Z & W \end{vmatrix},$$

$$P'_{xy} = Y(x_1 - \xi') - X(y_1 - y') = x_1y - x'y - y_1X + y'X = P_{xy} - (x'Y - Xy')$$

$$P'_{xz} = Z(x_1 - \xi') - X(z_1 - \zeta) = P_{xz} - (\xi Z - X\eta)$$

$$P'_{xw} = W(x_1 - \xi') - X(\omega - \omega') = P_{xw} - (\xi' W - X\omega')$$

$$P'_{yw} = W(y_1 - \eta') - Y(W - \omega) = P_{yw} - (\eta' W - Y\omega')$$

$$P'_{yw} = Z(y_1 - \eta') - Y(z - \zeta') = P_{yz} - (\eta' Z - Y\zeta)$$

$$P'_{zw} = W(z - \zeta') - Z(w - \omega') = P_{zw} - (\zeta W - Y\omega')$$

The above results are the same as the ones we get from the matrix which defines the moment with respect to any point. They also show the relation between the moment with respect to the origin and the moment with respect to any point in space.

4. Moment with Respect to a Line:- The moment of a vector with respect to a line is defined as a vector whose magnitude is 3! times the volume of the tetrahedron determined by the vector and two points in the line a unit's distance apart, and whose direction is perpendicular to the three-space determined by the line and the vector. The sense on this perpendicular is fixed by taking the X,Y,Z axes of a standard right hand set of axes in the three space of the vector and the line of reference in such a way that the vector tends to rotate about the line in a right hand screw motion. The positive W axis then gives the positive sense on the moment. This vector is in a sense a free vector, inasmuch as its initial point is not fixed. We shall merely make the convention that it intersects the line of reference. Its initial point thus lies in a plane that is determined by the line of reference and the vector.

The magnitude of the moment with respect to a line is also equal to the magnitude of the moment with

respect to a point in the line multiplied by the sine of the angle the line makes with the plane through the vector and the point of reference. For if we take any two points in the given line and the two end points of the vector they will be sufficient to determine a three-space and for such the theorem is well known. See Appell Traite de Mecanique Rationnelle, Vol. I., p 6.

5. The Ten Co-ordinates of a Vector:- The magnitude, direction, and line of action of a vector are determined by its four components and the projections of its moments with respect to a point, that is, by X, Y, Z, W, P_{xy}, P_{xz}, P_{xw}, P_{yz}, P_{yw}, and P_{zw}. These quantities are not arbitrary but satisfy five identical relations which we proceed to obtain. If we write down the matrix

$$\left\|\begin{matrix} X & Y & Z & W \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{matrix}\right\|$$

we are able to produce four determinants, each of which is equal to zero since the difference between the last two rows produce a row that is the same as the first. From the four determinants formed above we get four

equations as follows:

$$
\begin{aligned}
Y P_{zw} - Z P_{yw} + W P_{yz} &= 0 \\
X P_{zw} - Z P_{xw} + W P_{xz} &= 0 \\
X P_{yw} - Y P_{xw} + W P_{xy} &= 0 \\
X P_{yz} - Y P_{xz} + Z P_{xy} &= 0
\end{aligned}
\qquad (1 - 4)
$$

Then from the determinant

$$
\begin{vmatrix}
x_1 & y_1 & z_1 & w_1 \\
x_2 & y_2 & z_2 & w_2 \\
x_1 & y_1 & z_1 & w_1 \\
x_2 & y_2 & z_2 & w_2
\end{vmatrix} = 0
$$

we get the relation $P_{xy} P_{zw} - P_{xz} P_{yw} + P_{xw} P_{yz} = 0$ (5)

It is interesting to note that if (5) is satisfied only two of the relations (1 - 4) are independent. For, if we form the determinant of the coefficients of X, Y, Z, W we get

$$\begin{vmatrix} 0 & P_{zw} & -P_{yw} & P_{yz} \\ P_{zw} & 0 & -P_{xw} & P_{xz} \\ P_{yw} & -P_{xw} & 0 & P_{xy} \\ P_{yz} & -P_{xz} & P_{xy} & 0 \end{vmatrix}$$

By the aid of the relation ~~(1-4) and~~ (5) we are able to show that all the determinants of third order as formed from the above determinant are equal to zero. The minor of 0 in the first row is

$$\begin{vmatrix} 0 & -P_{xw} & P_{xz} \\ -P_{xw} & 0 & P_{xy} \\ -P_{xz} & P_{xy} & 0 \end{vmatrix}$$

which equals $P_{xy} \; (-P_{xw} \; P_{xz} + P_{xz} \; P_{xw}) = 0$

The minor of P_{zw} in the first row is

$$\begin{vmatrix} P_{zw} & -P_{xw} & P_{xz} \\ P_{yw} & 0 & P_{xy} \\ P_{yz} & P_{xy} & 0 \end{vmatrix}$$

which equals, $P_{xy} \; (P_{yw} \; P_{xz} - P_{yz} \; P_{xw} - P_{xy} \; P_{zw}) = 0$

But by (5) the term in the parenthesis is equal to 0 and thus the determinant is equal to 0.

The minor of the third element in the first row is,

$$\begin{vmatrix} P_{zw} & 0 & P_{xz} \\ P_{yw} & -P_{xw} & P_{xy} \\ P_{yz} & -P_{xz} & 0 \end{vmatrix}$$

which equals $P_{xz}(-P_{yw}P_{xz} + P_{zw}P_{xy} + P_{xw}P_{yz})$, and we see that this determinant is equal to 0 for the same reason as just stated.

In the same way we could take the minor of the last element in the first row and also the minors of the elements in the other rows and show that they are all equal to 0.

If now we examine the minors of the second order that we are able to get we find that we may have,

$$\begin{vmatrix} 0 & P_{xy} \\ P_{xy} & 0 \end{vmatrix}, \quad \begin{vmatrix} -P_{xw} & P_{xz} \\ -P_{xz} & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & -P_{xw} \\ -P_{xw} & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & P_{yz} \\ P_{yz} & -P_{xz} \end{vmatrix},$$

$$\begin{vmatrix} 0 & P_{yw} \\ P_{yw} & -P_{xw} \end{vmatrix}, \quad \begin{vmatrix} 0 & P_{zw} \\ P_{zw} & 0 \end{vmatrix}$$

and these determinants are not all equal to 0 unless $P_{xy} = P_{xz} = P_{xw} = P_{yz} = P_{yw} = P_{zw} = 0$, which would be a trivial case and the theorem would be evident. As the rank of the matrix is thus two there are exactly two of the equations (1-4) that are independent.

We proceed now to show that unless $X = Y = Z = W = 0$, the vector is determined by the ten quantities above satisfying the relations (1-4) and (5) and otherwise arbitrary, and thus we shall justify ourselves in calling the ten quantities the co-ordinates of the vector.

Take x, y, z, w, as the co-ordinates of the initial point of the vector and with these and the ten co-ordinates as given above write out the six equations as follows,

$$\begin{aligned} xY - yX &= P_{xy} \\ xZ - zX &= P_{xz} \\ xW - wX &= P_{xw} \\ yZ - zY &= P_{yz} \\ yW - wY &= P_{yw} \\ zW - wZ &= P_{zw} \end{aligned} \qquad (1\text{-}6)$$

By writing out the determinant formed by the coefficients of the above six equations it is easily seen that the minors of the third order do not all vanish

for we can pick out minors equal to $-X^3$, $-Y^3$, $-Z^3$, and $-W^3$, and thus three of the equations are independend. By means of the equations (1-4) and (5) we are able to determine that three of the equations (1-6) are consequences of the other three. The three equations determine a line in four space and that line is the line of the vector as determined by the ten quantities given.

Analytically the magnitude of the moment with respect to a line is given by its projections on the axes, which are equal to the determinants of the matrix

$$\left\| \begin{matrix} \xi_2 & \eta_2 & \rho_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \rho_3 & \omega_3 & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \end{matrix} \right\| \div \sqrt{(\xi_3-\xi_2)^2+(\eta_3-\eta_2)^2+(\rho_3-\rho_2)^2+(\omega_3-\omega_2)^2}$$

found by dropping one column at a time, always retaining the last. A plus or minus sign is attached to each determinant according to whether it requires an even or odd number of transpositions to bring the column dropped to the position of the first column. Here $\xi_2, \eta_2, \rho_2, \omega_2$ and $\xi_3, \eta_3, \rho_3, \omega_3$ represent points on the line and if we take the distance between these points to be equal to one, the projections are

$$P_x = \begin{vmatrix} \eta_2 & \zeta_2 & \omega_2 & 1 \\ \eta_3 & \zeta_3 & \omega_3 & 1 \\ y_1 & z_1 & w_1 & 1 \\ y_2 & z_2 & w_2 & 1 \end{vmatrix}, \qquad P_y = -\begin{vmatrix} \xi_2 & \zeta_2 & \omega_2 & 1 \\ \xi_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & z_1 & w_1 & 1 \\ x_2 & z_2 & w_2 & 1 \end{vmatrix}$$

$$P_z = \begin{vmatrix} \xi_2 & \eta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \omega_3 & 1 \\ x_1 & y_1 & w_1 & 1 \\ x_2 & y_2 & w_2 & 1 \end{vmatrix}, \qquad P_w = -\begin{vmatrix} \xi_2 & \eta_2 & \zeta_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}$$

If now we take $\xi_2\ \eta_2\ \zeta_2\ \omega_2$ as the origin we have as the projections

$$P_x = -\begin{vmatrix} \eta_3 & \zeta_3 & \omega_3 \\ y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \end{vmatrix}, \quad P_y = \begin{vmatrix} \xi_3 & \zeta_3 & \omega_3 \\ x_1 & z_1 & w_1 \\ x_2 & z_2 & w_2 \end{vmatrix}, \quad P_z = -\begin{vmatrix} \xi_3 & \eta_3 & \omega_3 \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{vmatrix}$$

$$P_w = \begin{vmatrix} \xi_3 & \eta_3 & \zeta_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

A relation exists between the given vector and its moment with respect to a line, which shows that the vector lies in a space at right angles to its moment.

For, $XP_x + YP_y + ZP_z + WP_w =$

$$\begin{vmatrix} X & Y & Z & W & 0 \\ \xi_2 & \eta_2 & \zeta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \end{vmatrix}$$

which is seen to vanish because in subtracting the next to the last row from the last, two rows become identical.

It is interesting to add an analytic proof of the relation stated on page 8 for the magnitude of the moment with respect to a line, and with respect to a point. We shall take $\xi_3 \eta_3 \zeta_3 \omega_3$ as the origin, and the XY plane as the plane determined by the vector and the origin. Also take the point $\xi_2 \eta_2 \zeta_2 \omega_2$ so that the line o,o,o,o and $\xi_3 \eta_3 \zeta_3 \omega_3$ is equal to one. Then if we write out the matrix for the moment with respect to the origin we get

$$\left\| \begin{array}{ccccc} o & o & o & o & 1 \\ x_1 & y_1 & o & o & 1 \\ x_2 & y_2 & o & o & 1 \end{array} \right\|$$

From that we see that double the area of the triangle is equal to the absolute value of $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

If we now write down the matrix for the moment with respect to the line and observe the conditions as imposed above we get

$$\left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & y_1 & 0 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 \end{matrix} \right\|$$

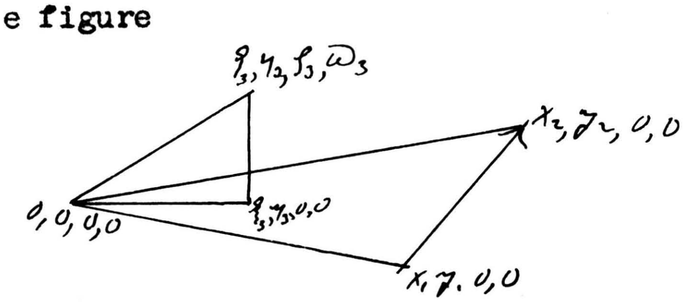
which gives for the projections on the axes

$$-\begin{vmatrix} \eta_3 & \zeta_3 & \omega_3 \\ y_1 & 0 & 0 \\ y_2 & 0 & 0 \end{vmatrix}, \begin{vmatrix} \xi_3 & \zeta_3 & \omega_3 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{vmatrix}, -\begin{vmatrix} \xi_3 & \eta_3 & \omega_3 \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix}, \begin{vmatrix} \xi_3 & \eta_3 & \zeta_3 \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix}$$

This gives for the moment with respect to the line

$$\sqrt{\zeta^2 + \omega^2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

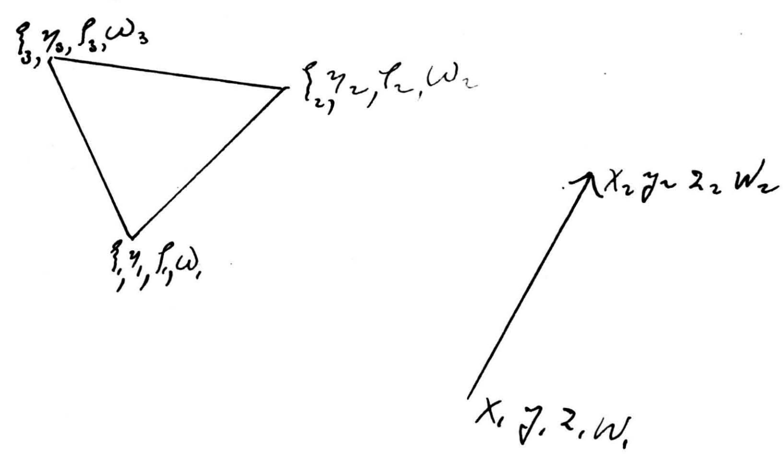
If we construct the figure



and project on the XY plane, the co-ordinates of the projection are . If we call the angle the line makes with the XY plane, sin will equal the distance between and , which is equal to .

Thus we have the result: the magnitude of the moment with respect to a line is equal to the magnitude of the moment with respect to a point in the line multiplied by the sin of the angle the line makes with the plane of the moment.

6. The Moment with Respect to a Plane:- The moment of a vector with respect to a plane is a scalar and is defined as 4! times the content enclosed by the five points as vertices divided by double the area of the triangle formed by the three points which determine the plane of reference.



The positive or negative sign is attached to the moment as follows: First a positive sense of rotation, or a positive side is fixed in the plane. Then taking three points of the plane in the positive order, in the counter clock-wise order as viewed from the positive side of the plane, and adding the initial point of the vector. These four points determine a sense in their space, namely the sense of the vector of the first two points about the vector about the last two. Introducing into this space the X, Y, Z,, axes of our standard set so that the sense of their space is the same, we attach to the moment the plus sign or the minus sign according as the given vector makes an acute or an obtuse angle with the standard positive W axis.

Analytically the moment of a vector with respect to a plane is determined by the determinant

$$\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 & \omega_1 & 1 \\ \xi_2 & \eta_2 & \zeta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \end{vmatrix}$$

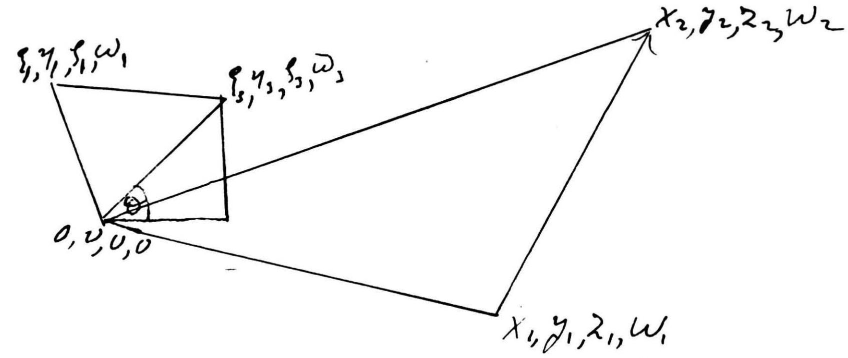
divided by double the area of the triangle formed by the three points ($\xi_1 \eta_1 \zeta_1 \omega_1$), ($\xi_2 \eta_2 \zeta_2 \omega_2$), and

($\xi_3\ \eta_3\ \zeta_3\ \omega_3$). In order to establish a relation between the magnitude of the moment with respect to a plane and with respect to a point in the plane we select the point $\xi_1\eta_1\ \zeta_1\ \omega_1$ as the origin, and take $\xi_2\eta_2\ \zeta_2\ \omega_2$, $\xi_3\ \eta_3\ \zeta_3\ \omega_3$, such that the area of the triangle formed equals $\frac{1}{2}$, and rotate the axes so that the vector may lie in the XY plane. Then we have the magnitude of the moment with respect to the plane equal to

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ \xi_2 & \eta_2 & \zeta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & y_1 & 0 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \zeta_2 & \omega_2 \\ \zeta_3 & \omega_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Thus we see that we have the magnitude of the moment with respect to a plane equal to the magnitude of the moment with respect to a point in the plane multiplied by the projection of the area of a unit parallelogram in the plane of reference on a plane perpendicular to the plane of the moment with respect to a point. In other words, the magnitude of the moment of a vector with respect to a plane equals the magnitude of its moment with respect to a point of the plane times the sine of the angle between the plane of reference and

the plane of the point of reference and the vector.



If we take, as shown in the figure, $\xi_2 \eta_2 \zeta_2 \omega_2$ equal (o,o,o,o) the length of the line between (o,o,o,o), and $\xi_3, \eta_3, \zeta_3 \omega_3$ equal one, and the area of the triangle formed by (o,o,o,o), ($\xi_3 \eta_3 \zeta_3 \omega_3$) and ($\xi_2 \eta_2 \zeta_2 \omega_2$) equal one-half, we will have, according to the relation stated on page 18, the moment of the vector with respect to the line between (o,o,o,o) and ($\xi_3 \eta_3 \zeta_3 \omega_3$) equal

$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ sin θ , where θ is the angle between the line determined by (o,o,o,o) and $\xi_3, \eta_3, \zeta_3, \omega_3$, and the plane determined by the vector and (o,o,o,o).

Also we see that if we add the point ($\xi_1 \eta_1 \zeta_1 \omega_1$) as shown in the figure, and take the moment with respect to the plane determined by (o,o,o,o), ($\xi_3 \eta_3 \zeta_3 \omega_3$) and ($\xi_1 \eta_1 \zeta_1 \omega_1$) it will equal, as stated above,

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \times \begin{vmatrix} \zeta_2 & \omega_2 \\ \zeta_3 & \omega_3 \end{vmatrix}$$

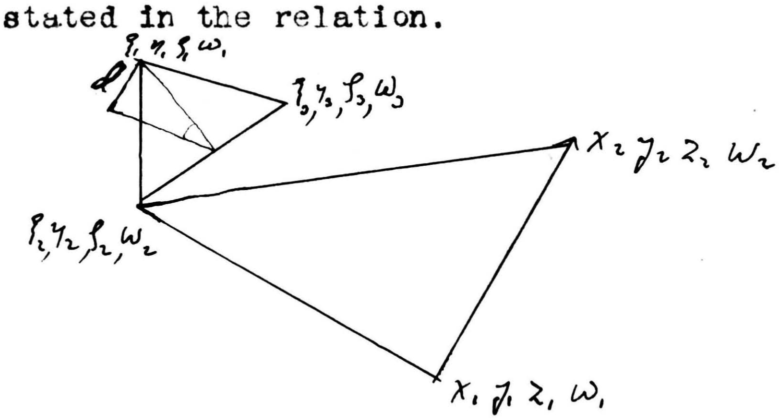
But we see that $\begin{vmatrix} \zeta_2 & \omega_2 \\ \zeta_3 & \omega_3 \end{vmatrix}$ is the projection of the area of the triangle determined by (o,o,o,o), $(\xi_3\ \eta_3\ \zeta_3\ \omega_3)$ and $(\xi_1\ \eta_1\ \zeta_1\ \omega_1)$ on to the plane perpendicular to the plane determined by the vector and (o,o,o,o). If we represent this projection by P_p we see, from the value found above for the moment with respect to a plane and the moment with respect to a line in the plane, that the moment with respect to the plane equals the moment with respect to the line multiplied by P_p and divided by the sin. θ .

A relation between the moment of a vector with respect to a plane and the moment with respect to a line is this. If we take the line of reference in the plane of reference and of unit length and the area of the triangle which determines the plane of reference equal to one half, the moment with respect to the plane is equal to the moment with respect to the line multiplied by the sine of the angle the plane of reference makes with the space determined by the vector and the line of reference.

We assume here that the five points taken determine a four dimensional space.

In order to show that the above relation holds let us take x_1, y_1, z_1, w_1 and $x_2 y_2 z_2 w_2$ as the points that fix the vector, and fix the line of reference with $\xi_2, \eta_2, \zeta_2, \omega_2$ $\xi_3, \eta_3, \zeta_3, \omega_3$ As the third point which is to be taken with the line of reference to determine the plane of reference we shall take $(\xi_1, \eta_1, \zeta_1, \omega_1)$.

The above points we are assuming to satisfy the conditions as stated in the relation.



The moment with respect to the line =

$$\sqrt{\begin{vmatrix} y_2 & \zeta_2 & \omega_2 & 1 \\ y_3 & \zeta_3 & \omega_3 & 1 \\ y_1 & z_1 & w_1 & 1 \\ y_2 & z_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & \zeta_2 & \omega_2 & 1 \\ \xi_3 & \zeta_3 & \omega_3 & 1 \\ x_1 & z_1 & w_1 & 1 \\ x_2 & z_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & y_2 & \omega_2 & 1 \\ \xi_3 & y_3 & \omega_3 & 1 \\ x_1 & y_1 & w_1 & 1 \\ x_2 & y_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & y_2 & \zeta_2 & 1 \\ \xi_3 & y_3 & \zeta_3 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}^2}$$

which we shall designate as M_l. Let d equal the perpendicular distance from $\xi_1\ \eta_1\ \zeta_1\ \omega_1$ to the space determined by $x_1\ y_1\ z_1\ w_1$, $x_2\ y_2\ z_2\ w_2$, $\xi_2\ \eta_2\ \zeta_2\ \omega_2$, and $\xi_3\ \eta_3\ \zeta_3\ \omega_3$. Then d has been found to be equal to

$$\frac{\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 & \omega_1 \\ \xi_2 & \eta_2 & \zeta_2 & \omega_2 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & \zeta_2 & \omega_2 & 1 \\ y_3 & \zeta_3 & \omega_3 & 1 \\ y_1 & z_1 & w_1 & 1 \\ y_2 & z_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & \eta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \omega_3 & 1 \\ x_1 & z_1 & w_1 & 1 \\ x_2 & z_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & \eta_2 & \omega_2 & 1 \\ \xi_3 & \eta_3 & \omega_3 & 1 \\ x_1 & y_1 & w_1 & 1 \\ x_2 & y_2 & w_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} \xi_2 & \eta_2 & \zeta_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}^2}}$$

The numerator of the fraction we shall set equal to D and we see that the denominator is equal to M_l. Thus $d = \frac{D}{M_l}$. The moment with respect to the plane is equal to

$$\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 & \omega_1 \\ \xi_2 & \eta_2 & \zeta_2 & \omega_2 \\ \xi_3 & \eta_3 & \zeta_3 & \omega_3 \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{vmatrix} = D = M_p$$

But from the figure we see that d = sine of the angle between the plane of reference and the space = sine θ . Substituting in the equation above we get

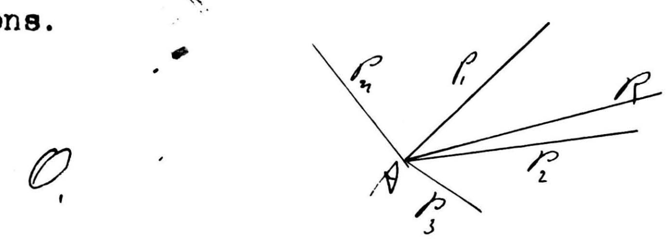
$$\text{Sine}\,\theta = \frac{D}{M\ell}$$

$$M\ell \sin\theta = 4\,D = M_p \;.$$

CHAPTER II.

Systems of Vectors.

1. Geometric sum, or resultant, of concurrent vectors:- Take a system of concurrent vectors P_1, P_2, P_3 ....P_n having for the common point A they may be reduced to a resultant AR as in three dimensions.

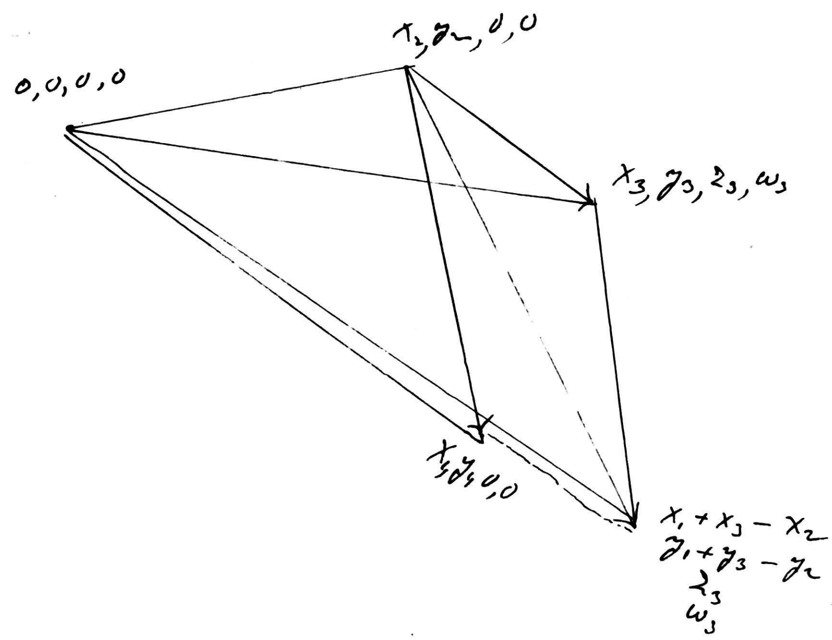


For since two concurrent lines in four dimensions determine a plane we may pass a plane through P_1 and P_2 and find the resultant as in two dimensions. Then by taking that resultant and combining it with another vector, and by continuing the process we may be able to find a final resultant for the system.

The resultant moment of a system of concurrent vectors we shall define as being such that the projections on the six coordinate planes equals the sum of the projections of the moments of the vectors taken separately. The resultant moment of a system of concurrent vectors is equal to the moment of the resultant.

In order to verify this we take a system of vectors

P_1, P_2 .... P_n,



and from this we take P_1 and P_2 and rotate the axes so that P_1 may lie in the XY plane and then translate the axes so that the origin may correspond with the point of reference. Then we have the moment of P_1 equal to

$$\begin{Vmatrix} o & o & o & o & 1 \\ x_1 & y_1 & o & o & 1 \\ x_2 & y_2 & o & o & 1 \end{Vmatrix} .$$

The moment of P_2 is equal to

$$\left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \end{matrix} \right\| .$$

The moment of R is equal to

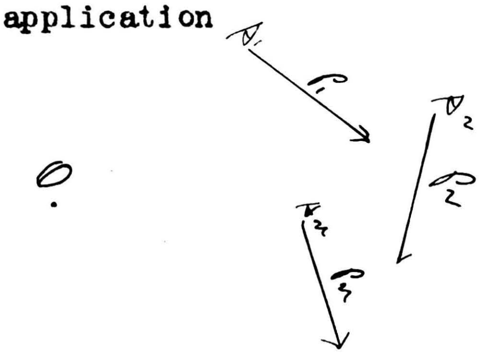
$$\left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ x_1 + x_3 - x_2 & y_1 + y_3 - y_2 & z_3 & w_3 & 1 \\ x_2 & y_2 & 0 & 0 & 1 \end{matrix} \right\| .$$

If we add the third row of the matrix for R to the second to form a new row for the second we readily see that the matrix formed is equal to the matrix for P_1 added to that for P_2 by adding the second row of each. We could take the resultant found and one other vector and the theorem would be true, since we have found it true for two vectors, and by continuing the process we should find it true for a system of any number of vectors.

Since the laws for projections of points onto lines and planes in four dimensions hold in the same way as they do for three, and since in finding the resultant we build up the polygon with the resultant as the closing line, we have the projections of the result-

ant equal to the sum of the projections of the vectors, and thus we see the resultant is independent of the order in which the vectors are taken.

2. Vectors not Concurrent. General Resultant and Resultant Moment.- If we have a system that is not concurrent, as P_1, P_2 ....P_n having A_1, A_2 ..... A_n as points of application



and if we take any arbitrary point in space as O then the general resultant is defined as the resultant of the vectors having O for a common point of application and equal and parallel to the given vectors.

The resultant moment with respect to O is defined as the aggregate of six quantities, P_{xy}, P_{xz}, P_{xw}, P_{yz}, P_{yw}, P_{zw}, which are the sums of the corresponding quantities for the separate vectors. This reduces to the moment of the resultant in the case of concurrent vectors. Otherwise it cannot be considered the moment of any single force, for the relation $P_{xy} P_{zw} - P_{xz} P_{yw} + P_{xw} P_{yz} = 0$ will not in general be satisfied.

It will be useful to notice how the resultant moment varies with the point of reference.

Let P^o_{xy}, P^o_{xz}, P^o_{xw}, P^o_{yz}, P^o_{yw}, P^o_{zw}, denote the components of the moment with respect to the origin, and let P_{xy}, P_{xz}, P_{xw}, P_{yz}, P_{yw}, P_{zw}, denote the components of the resultant moment with respect to any point in space as ξ, η, ζ, ω. Then $P^o_{(xy)k} = \begin{vmatrix} x_k & y_k \\ X_k & Y_k \end{vmatrix}$, $P^o_{(xz)k} =$

$$\begin{vmatrix} x_k & z_k \\ X_k & Z_k \end{vmatrix}, \quad P^o_{(xw)k} = \begin{vmatrix} x_k & w_k \\ X_k & W_k \end{vmatrix}, \quad P^o_{(yz)k} = \begin{vmatrix} y_k & z_k \\ Y_k & Z_k \end{vmatrix}$$

$$P^o_{(yw)k} = \begin{vmatrix} y_k & w_k \\ Y_k & W_k \end{vmatrix}, \quad P^o_{(zw)k} = \begin{vmatrix} z_k & w_k \\ Z_k & W_k \end{vmatrix}.$$

$$P_{(xy)k} = \begin{vmatrix} \xi & y & 1 \\ x_k & y_k & 1 \\ X_k & Y_k & 0 \end{vmatrix} = P^o_{(xy)k} - \xi Y_k + \eta X_k .$$

$$P_{xy} = \sum P_{(xy)k} = P^o_{(xy)} - \xi Y + \eta X .$$

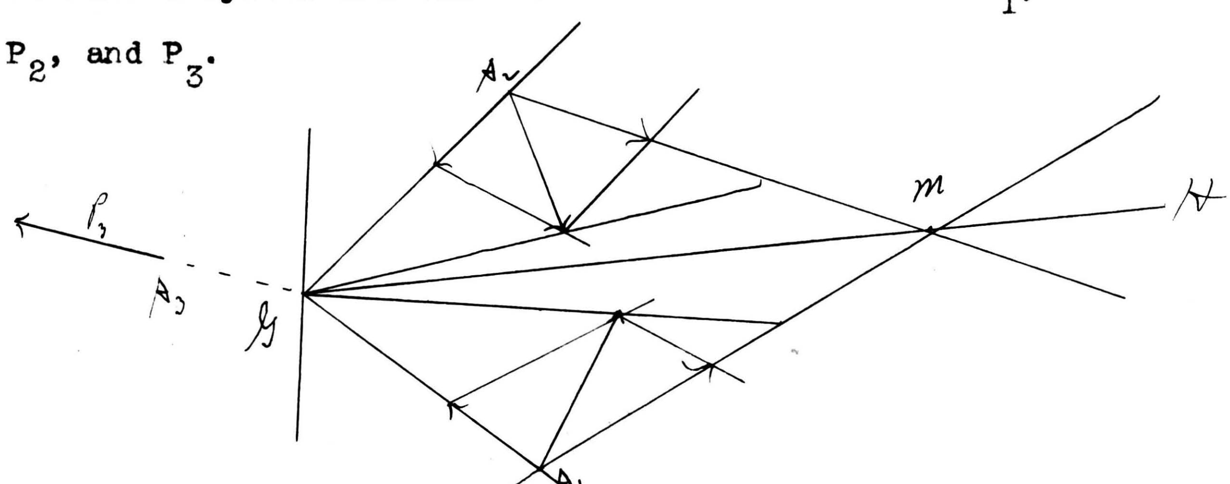
In the same way we could find corresponding values for P_{xz}, P_{xw}, P_{yz}, P_{yw}, and P_{zw}. Thus we see that the resultant moment with respect to any point in space is equal to the resultant moment with respect to the origin decreased by the moment of the general resultant R thought of as applied at the origin and when taken with respect to the point .

3. Elementary Operations:- We define elementary operations as in Appell, Traite de Mechanique Rationnelle, Vol. I., page 19. They have the property of leaving unchanged the general resultant and the resultant moment.

4. Equivalent Systems:- Two systems are defined as being equivalent if one can be reduced to the other by means of elementary operations.

Corollary: Two equivalent systems have the same general resultant and resultant moment.*

5. Reduction of a System of Two:- Any system of vectors may in general be reduced to two. Suppose we have a system and take out of it three vectors P_1, P_2, and P_3.

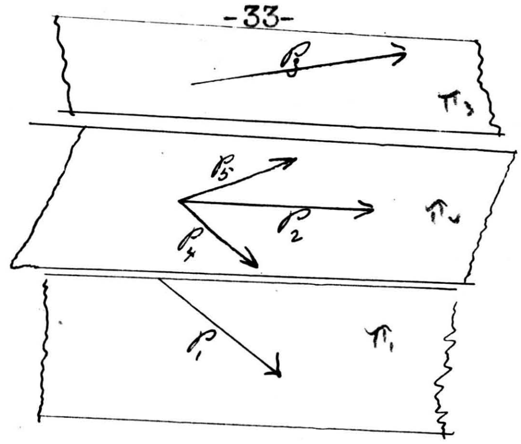


* The converse of this corollary is important, although it is not immediately necessary, namely, two systems are equivalent if they have the same general resultant and resultant moment.

If the three vectors are all in the same space it is known that they may be reduced to two, but suppose P_3 is in a different space from P_1 and P_2 and suppose it meets the space of P_1 and P_2 in some point, say G. We can pass a plane through P_1 and G and one through P_2 and G. Now, since P_1 and P_2 are in the same space the two planes will pass through a common line, say GH. Let us take some arbitrary point on GH and call it M. We are able to resolve P_1 into two vectors that lie in the lines A_1G and A_1M. Also we are able to resolve P_2 into two vectors that lie in the lines A_2G and A_2M. Now let us move P_3 and the vectors formed down to the two points G and M and we will have two systems of concurrent vectors which may each be resolved into one. Thus we will have the system reduced to two.

This process fails only when each vector does not intersect the space of the other two.

Let us suppose we have three vectors P_1, P_2, P_3, where each is parallel to the space determined by the other two. We shall proceed to prove that in general they may be reduced to two.



Since two spaces in four dimensions meet in a plane, we shall designate the plane of intersection of the spaces determined by P_1, P_2 and P_2, P_3 by π_2 . Since P_2 is in each space and they intersect only in π_2 , P_2 must lie in π_2 . In three space a plane may be passed through a line parallel to a plane which the line does not intersect; therefore we are able to pass a plane π_1 through P_1 parallel to π_2 , in the three space of P_1 and P_2, and also one, which we designate as π_3 , through P_3 and parallel to π_2 in the three space of P_2 and P_3. We resolve P_2 into two vectors P_4 and P_5 which are parallel respectively to P_1 and P_3.

We can reduce P_1 and P_4 to a single vector unless they form a couple, and similarly for P_3 and P_5. Thus we have the following cases.

(1) When neither pair forms a couple. Then the reduction of the system to two vectors is accomplished.

(2) One pair forms a couple and the other has a single resultant. As a couple may be moved anywhere in its plane, and its plane may be moved anywhere in space parallel to itself we move the couple so that one of its vectors meets the single resultant, and thus again we may reduce the system to two vectors.

(3) Both pairs form couples. These couples lie in planes which have a point in common, namely, a point of the vector P_2.

There are now two sub-cases.

(3') The planes have a line in common. Then by turning the couples in their planes, one vector of each may be made to lie along this line and we have three vectors in parallel lines, therefore in three space, and they may be reduced to a single resultant or a couple. Thus the reduction is again accomplished.

(3'') If the planes determined by the couples meet in a point only, further reduction is impossible. To show that this case is really irreducible we show

that for a pair of couples in planes intersecting only in a point the general resultant is equal to 0 while the expression

$$P_{xy} P_{zw} - P_{xz} P_{yw} + P_{xw} P_{yz} \neq 0.$$

The above expression we shall designate by $Q_{(p)}$. And for any system of two vectors which has the general resultant equal to 0 the expression

$$P_{xy} P_{zw} - P_{xz} P_{yw} + P_{xw} P_{yz} = 0$$

To prove for the pair of couples that $Q_{(p)} \neq 0$, we take the point of intersection of P_4 P_5 for the origin and designate P_1 by the points x_1 y_1 z_1 w_1 x_2 y_2 z_2 w_2, and P_3 by x_3 y_3 z_3 w_3 and x_4 y_4 z_4 w_4. Since P_4 and P_5 pass through the origin their moment of each with respect to that point is $= 0$. For the moments of P_1 and P_3 with respect to the origin

$$P_{xy} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \qquad \text{etc.}$$

$$\text{Then } Q_{(p)} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} z_3 & w_3 \\ z_4 & w_4 \end{vmatrix} + \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}$$

two similar expressions + two similar ones. For the terms omitted are $Q_{(p_1)}$ and $Q_{(p_2)}$, which are both 0. It follows that

$$Q_{(p)} = \begin{vmatrix} o & o & o & o & 1 \\ x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \end{vmatrix}$$

But the above determinant is the expression for the content of the five point figure as determined by the two vectors P_1 P_3 and the origin, and in order for the content to be 0 the five points would have to lie in the same three space, which contradicts the supposition that the planes through the origin and P_2 and P_3 respectively meet only in a point.

The general resultant for a couple is 0, and hence also for a pair of couples. On the other hand any system of two vectors with general resultant 0 leads to a $Q_{(p)} = 0$. For the resultant vanishing, the system is a couple and one of the vectors of the couple may be made to pass through the origin. The moment thus reduces to the moment of the other vector, and for it $Q_{(p)} = 0$.

This thesis is never to leave this room.
Neither is it to be checked out overnight.