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AN INVESTIGATION IN THE TEACHING OF ELEMENTARY MATHEMATICS FROM THE STANDPOINT OF CHOICE AND ARRANGEMENT OF SUBJECT MATTER

by

WALTER SCOTT MONROE, A. B., B.S. 1882 –

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Preface

This investigation was begun because the writer feels that the results of teaching mathematics in the high school are not commensurate with the time devoted to it and that there is a lack of appreciation of mathematics shown by the pupils. The problem thus raised, How to Improve the Teaching of Elementary Mathematics, was attacked from the standpoint of motive. From this humble beginning the scope of the investigation grew to its present dimensions.

The investigation is confined, for the most part, to the teaching of mathematics in the first year of the high school altho portions of the subject matter have been drawn from the work which is traditionally given at a later period. Some geometry and even trigonometry is included in the course but no attempt has been made to correlate the subjects of elementary mathematics.

The following course is not presented as a final solution of the problem, but it is rather an attempt to base a course upon pedagogical principles which are widely accepted. The success of this or any proposed course depends largely upon the teacher, and any one attempting to follow the plan should first be sure that they understand and are in sympathy with it.

I am especially indebted to Dr. L. D. Ames, Assistant Professor of Mathematics, for many suggestions and valuable criticism; to Dr. E. R. Hedrick, Professor of Mathematics, for his assistance in developing the mathematical point of view herein presented; and to Dr. W. W. Charters, Professor of the Theory of Teaching, under whose direction this work was originally begun, for his constant encouragment and guidance as well as for valuable suggestions and criticism.

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Chapter I.

Evidences of dissatis-
faction with the results
of the present teaching
of elementary mathematics,
and definition of the problem.

Chapter I

Introduction

To one engaged in the teaching profession, or even to a laymen who has come into contact with the educational system of the present time, it is scarcely necessary to present evidences of dissatisfaction regarding the results of the instruction in our schools. In fact, this element of dissatisfaction is characteristic of the present educational situation. It shows itself in many forms, many, if not all, recent innovations being traced to this source.

Although this dissatisfaction extends to all subjects of the curriculum, and perhaps is justified in all of them, it is nowhere more acute than in mathematics. Centered as it now is largely around arithmetic and secondary mathematics it is beginning to extend to the college work.

It is proposed in this paper to investigate the cause or causes of this dissatisfaction with respect to elementary mathematics* and to offer a solution of the difficulty based upon certain well accepted educational principles.

As was stated in the opening paragraph, it probably is not necessary to present argument to convince one who is in touch with our schools that a dissatisfaction exists in regard

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* The term "elementary mathematics" when used in this discussion refers to the mathematics taught in the secondary school; more specifically, to the work of the first years of that school. The course we finally formulate may not be considered elementary in the usual sense, but we shall call it elementary mathematics because it is to be taught to pupils in the first year of the secondary school in the place of the traditional first year of algebra usually given. Unless stated explicitly to the contrary, the term "elementary mathematics" will always carry with it this meaning.

to the results of present instruction in elementary mathematics. But for the purpose of locating the causes and sources of this dissatisfaction and thus defining our problem, it may be well to examine the evidences.

A consideration of the available data may not give us an absolute result since it is, in general, based upon opinion, either directly or indirectly. But these opinions are from men who are looking at the matter from somewhat different angles and, if found to agree fairly closely, they should be sufficiently near the truth to be accepted for, at least, a tentative working basis.

Section I: General evidences.

A general unrest and agitation is shown by the already great and constantly increasing variety of text books each of which is intended to correct evils of our teaching, by the organizations of secondary teachers for the study and improvement of the teaching of mathematics, by the "real applied problem" movement, etc.

In this country new text books were never more numerous. It is scarcely a decade since the graph, (one of the first radical innovations), the omission of the Euclidean process for finding the highest common divisor, etc. made their appearance in our texts. Since then texts representing radical departures from both the traditional subject matter and the traditional order of arrangement have been presented. More radical than the texts are the experiments which are being carried on in some of our secondary schools. Of these that should be especially mentioned are : (1) An experiment at Lincoln, Nebraska, carried on by Miss Edith Long during the past ten years in which she has worked out a first year's work in elementary science which forms the basis for the three following years of correlated mathematics; (2) the courses in first and second year mathematics which have been worked out by the School of Education of the University of Chicago, which is an attempt to formulate a correlated course in mathematics which shall not separate algebra, geometry, etc. The course of the second year has appeared within the last few months.*

In many of the states we find organizations of the teachers of mathematics in the secondary schools.# In connection with these associations there are committees working on various phases of the instruction in mathematics, such as, "Real Applied Problems", "Fundamentals", etc.**

Section 2: Attitude of Mathematicians.

Mathematicians, who have given the matter of secondary instruction serious thought, insist upon the unsatisfactory condition of our instruction. The editor of "The Teaching of Elementary Mathematics" begins the preface with the statement that "Perhaps no single subject of elementary instruction has suffered so much from lack of scholarship on the part of those who teach it as mathematics. ***** The true significance and the symbolism of the processes employed are concealed from the pupil and teacher alike." ##

In his presidential address before the American Mathema-

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* For the latest report of this work see School Science and Mathematics, Jan. 1911, p. 64.

\# For a list of many of these associations see School Science and Mathematics, February, 1911, p.186, which is the official publication of practically all such organizations in this section of the country.

** For reports of these Committees see School Science and Mathematics. Since Spring 1909.

\#\# Smith: Teaching of Elementary Mathematics, preface IX.

tical Society,* Professor E.H. Moore says, "Engineers tell us that in schools algebra is taught in one water tight compartment, geometry in another and physics in another, and that the student learns to appreciate (if ever) only very late the absolutely close connection between these subjects, and then, if he credits the fraternity of teachers with knowing the closeness of this relation, he blames them most heartily for their unaccountably stupid way of teaching him."

In a recent report we find the statement that, "The present period is one of protest against methods and matter; a period of dissatisfaction and retesting the curriculum."#

In the introduction to his book, Der Mathematische Unterricht an der höheren Schulen, Felix Klein of Germany laments the condition existing in the teaching of mathematics in German secondary schools and sums up the present evils in the expression, "A system of a double forgettery, (System des doppelten Vergessens)" By this he means that what the student is taught in the secondary school is of little or no value to him in the University, and also the University student who has been trained in pure mathematics finds, when he goes out to teach in the secondary schools, that he can make little use of his knowledge of higher mathematics; neither the secondary schools nor the University efficiently prepare for the other.

Professor Forsyth, of England, states his position by saying that "As regards algebra, many of the text-books used to be certainly very repellent."** The great movement in England, known

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* Bulletin of the A. M. S. 1903, Vol. 9, p. 415.

\# Report of the Committee on Fundamentals of the Central Association of Science and Mathematics; School Science and Mathematics, December 1910, p. 805.

** British Association Meeting, Glasgow, 1901, p. 37.

as the "Perry Movement" is an agitation for the bettering of instruction in elementary mathematics.

Statements corroborating the above evidence could be multiplied almost indefinitely by quotation from almost any one who has spoken upon the subject.

Section 3: Poor preparation of freshmen in the University.

Another source of evidence of the inefficiency of the instruction in elementary mathematics is the quality of the product, i. e. the abilities of the graduates of the secondary schools. In the University of Missouri, during the present year 1910-1911, 20% of the freshmen taking the first year work in mathematics (algebra and trigonometry) were forced to drop the course because of insufficient preparation.* At the University of Wisconsin a study# was made recently of the effective preparation of freshmen entering from accredited schools. It was shown that good work in mathematics in the secondary school was less indicative of good work in mathematics in the University than in the case of history or english.

Section 4: Problem Stated.

The evidences which we have just presented, coming as it does from various sources and various view points, agrees to a surprising degree. While a careful study of the abilities of the pupil during their secondary school course and at the completion of that

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* The per cent given does not include those who continued the work and failed at the end of the semester. Those who were asked to drop the course made a grade of less than 30% (upon a scale of 100) upon a test given during the third week of the semester. The test covered only the topics of elementary algebra which had been reviewed by the classes.

W.F.Dearborn, The relative Standing of Pupils in the High School and in the University, Bulletin of the University of Wisconsin, High School Series, No. 6, 1909.

course, would be exceedingly valuable here, the data at hand suffices to define our problem on general lines.

One point upon which the authorities mentioned agree and which is emphasized to a greater extent in the complete articles from which the quotations were made, is that we err in what we teach. A second point upon which there is an equal unaminous agreement is that we use faulty methods of teaching.

Thus our problem divides naturally into two phases; (1) subject matter, (2) method. At first thought it may appear that the first is exclusively the problem of the mathematician and the second a problem for the educator. That the present conditions exist and that we have been somewhat slow in reaching a solution of the problem may be because of this very opinion. Although certainly the mathematician must be consulted and must, to a great extent, finally pass upon the subject matter, it is equally certain that it is folly to attempt to teach things to the child at an age when he is not prepared for them. The determination of these factors is within the field of psychology and sociology, things which the mathematician as such is not able to do. On the other hand, it is equally as dangerous for the educational thinker to determine upon a method of teaching without having it passed upon by a person who has a comprehensive knowledge of mathematics.

In the remainder of this paper we shall attempt, from the standpoint of both the educator and the mathematician, to solve the first phase of the problem and shall introduce as much of the second as is necessary to make the first clear. Our problem, in brief, is to work out the organization of a course in elementary mathematics* for pupils in the first year of the high school.

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* See footnote on page 6.

Chapter II

In which we state the aim of education and define mathematics in order to have a basis for the selection of subject matter.

Chapter II

Basis for Selection of Subject Matter.

There are three factors to be considered in the selection of subject matter in mathematics suitable for the child in the first year of the high school.

First, we must consider the aim of education in order that we may have spread before us the whole array of subject matter which the child should be taught in order to realize that aim. Second, from this whole array the mathematician should select that subject matter which lies within his field. Finally, the instincts and acquired tendencies resident and potential in the child of fifteen must be considered in order that we may select from the realm of mathematics the subject matter which may be taught during the first year of the high school.

Section 1: Aim of education.

Because of the scope of this investigation, a thorough going consideration of the aim of education will not be attempted. We shall accept as our aim of education this statement, "Education is to make youth willing and able to realize the ideal purposes".*

In order that this aim may be of use to us in the choice of subject matter, it will be necessary to analyse the concept "realizing ideal values". Professor Münsterberg, whose aim of education we have quoted, goes into this analysis in a quite thoro

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* Münsterberg; Psychology and the Teacher, p. 70.

going manner and we shall epitomize what he says. "Take the material of our experience in its crude state without the meanings which we read into it because of our education." In our experience "There are an infinite number of impulses, impressions, and suggestions, of demands and ideas and things." It is chaos, or to quote a well known expression, "a blooming, buzzing confusion." This is the world in which we live. In this chaotic and disconnected succession of experiences, it is unreal, meaningless. To make this world real, to give it meaning, is the human ideal or end of life.

How is this world to be made real? How are we to give meaning to this chaotic experience? It is by noting that certain bits of experience repeat themselves, that they occur in certain connections, that they in turn give rise to other bits of experience. Thus all need not be chaos; single experiences may be connected and related to each other. To determine and understand these connections, these relations existing between the factors of our experience is and ever will be the human ideal. For, to use a mathematical expression, our experiences do not form a convergent series. In the words of Münsterberg, "If we go to the deeper meaning of human knowledge, it shows itself everywhere as the endlessly complex effort of mankind to understand the chaos of experience in such a way that single experiences may assert themselves."

This is not done because of the chance desires of individuals or races; it is not because mankind or individuals will derive pleasure from it; it is not because their wealth will be increased; it is not because their leisure or span of years will be augmented; it is rather because such is the inborn natuse of of the human being. Why have we inherited from the past masterpieces of literature and art? Why have these come down to us inviolate? Why are they treasured by us today? Why have men drawn and continue to draw inspiration from that which was written two thousand years ago? Is it not because these are records of successful attempts to understand the chaos of experience? It is because the writers were able to grasp clearly the factors of experience, to disregard that which was superfluous, to give meaning to the relations of man to man, of the individual to society, of the state to the individual. We today recognize these same relations as valuable because they help to make our world real.

To summarize, to make our world real, to realize the ideal values of life, we must learn to appreciate and control those relations which exist between the bits of our experience.

Section 2: The field of mathematics.

Our aim of education having been stated, it is necessary to define the part that mathematics plays in helping to realize it.

When we consider this question we find that there is a great diversity of opinion regarding the aim or field of mathematics. We find sponsors for a great variety of aims ranging from the scientific enthusiast who claims the aesthetic pleasure derived from the highly developed processes of mathematics is paramount, to the other extreme of the educational theorist who advocates that mathematics be taught "incidentally". Another equally unjustifiable position is to urge as the reasons for teaching mathematics or the basis for the selection of topics the manifold indirect values which are supposed to result therefrom. I refer to such values as "good judgment", "constructive imagination", "rational memory", "logical and accurate reasoning power", etc.* Thatthese values constitute a sufficient argument for the study of mathematics or for the selection of topics is doubtful, especially when we recognize that many if not all of these same values are claimed by other subjects of the curriculum. Furthermore, in the light of recent investigation it seems doubtful if such <u>general</u> powers as these can result from the study of any particular subject. That these values may, to some extent, result from a study of mathematics is not disputed; indeed, if mathematics is properly taught it is our opinion that many of these powers will be developed in the pupil in the realm of mathematics, but they do not constitute a real, direct motive for the study of mathematics.

The consensus of opinion among the leading mathematicians seems to be tending toward the point of view that mathematics is to aid one to understand the experiences of life. What sort of experience does mathematics help one to understand? What are the relations of experience which we call mathematics? Take very simple cases which certainly fall within the realm of mathematics; volume and pressure of a gas, distance and velocity, the weight of a body and its distance from the center of the earth, the time of day and the temperature, the cost of any commodity and the amount purchased, and so on. What is the relation with which we are concerned in each of these cases from the standpoint of the mathematician? What is it in the case of the volume and pressure of a gas?

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* W. H. Metzler; The Educational Value of Mathematics, Report of Association of Teachers of Mathematics in the Middle States and Maryland, July 1905, p. 39. Mr. Metzler, in this article, enumerates some fourteen values which he recognizes as resulting from a study of mathematics. He arrives at this conclusion by taking as his thesis "every individual possesses certain qualities, abilities and powers of mind and heart" and by showing what of these"qualities, abilities and powers of mind and heart" mathematics develops.

What is it, in the words of Professor Münsterberg, that "asserts itself?" Is it not the quantitative relation that the volume varies inversely as the pressure? What is the relation between distance and velocity in the case of uniform motion that "asserts itself?" Is it not the quantitative relation that the distance is the product of the time and velocity, or the distance varies directly as the velocity? and so on, we might show in each case the factor that "asserts itself", that "persists" is some form of a quantitative relation between varying quantities.

The study of these quantitative relations between varying quantities or factors and the means of controling these relations we call mathematics.*

This definition is essentially the well known mathematical "function" notion. When we speak of one variable quantity as being a function of another variable quantity, we mean that they are so related that when one is known, the other can be determined.

This is exactly the case in the illustrations given above. In the case of a gas, when we know the exact relation between the volume and pressure, we can determine the volume for any given pressure. For instance, if the relation is $pv = 100$, the volume is 10 when the pressure is 10; the volume is 25 when the pressure is 4; and so on.

Besides these specific illustrations, we find a common example of the "function" notion in the general "cause and effect" relation. This relation we employ in the ordinary conversation and by it we mean that given certain known causes, certain effects necessarily follow. The farmer knows that neglect of his crops necessitate a failure, that his stock demand a certain amount of

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* It is not intended to include higher mathematics in this definition. Also there is no attempt to bring under this definition demonstrative geometry.

food, etc.

Lest we may be accused of having stated an entirely new view of mathematics and one which is radical, we may investigate the aims of mathematics as stated by some mathematicians who have given the matter serious thought.

This statement of the aim is supported by Professor E. H. Moore, in the address, to which reference has been made. It also received the support of the Committee on Fundamentals in its report to the Central Association of Science and Mathematics.*

Professor D. E. Smith says, "The teacher who fails to emphasize the idea of algebraic function fails to reach the pith of the science."#

In another place he quotes Comte** as saying "Algebra is the science of functions," and endorses it by saying, "Taking Comte's definition as a point of departure, it is evident that one of the first steps in the scientific teaching of algebra is the fixing of the idea of <u>function</u>".##

So well does Professor Smith explain his position regarding the "function" that we quote at some length.

"Happily this is not only pedagogically one of the first steps, but practically it is a very easy one because of the abundance of familiar illustrations. 'Two general circumstances strike the mind; one, that all that we see is subjected to continual transformation, and the other that these changes are mutually interdependent.' Among the best elementary illustrations are those involving time; a stone falls, and the distance varies

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* School Science and Mathematics, December, 1910, p. 807-8 .
\# D. E. Smith; Teaching of Elementary Mathematics, p. 168.
** The Philosophy of Mathematics, translated from Cours de Philosophy positive, by W. M. Gillespie, New York, p.55.
\## D. E. Smith, loc. cit. p. 163.

as the time, and vice versa; we call the distance a function of the time, and the time a function of the distance. We take a railway journey; the distance again varies as the time, and again time and distance are functions of each other. Similarly, the interest on a note is a function of the time, and also of the rate and the principal.

"This notion of function is not necessarily foreign to the common way of presenting algebra, except that here the idea is emphasized and the name is made prominent. Teachers always give to beginners problems of this nature: Evaluate $x^2 + 2x + 1$ for $x = 2, 3$, etc., which is nothing else than finding the value of a function for various values of the variable. Similarly, to find the value of $a^3 + 3a^2b + 3ab^2 + b^3$ for $a = 1$, $b = 2$, is merely to evaluate a certain function of a and b, or, as the mathematician would say, f(a,b), for special values of the variables. It is thus seen that the emphasizing of the nature of the function and the introduction of the name and symbol are not at all difficult for beginners, and they constitute a natural point of departure. The introduction of algebra should therefore include the giving of values to the quantities which enter into a function, and thus the evaluation of the function itself."

"Having now defined algebra as the study of certain functions, which includes as a large portion the solution of equations, the question arises as to its value in the curriculum."*

Professor J. W. A. Young states his position in these words. "The most salient feature of natural phenomena is change, variation; the most important single branch of mathematics ---- the Calculus --- is a study of variation, and may be called, in an

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* D. E. Smith; Teaching of Elementary Mathematics, p.164-5.

important sense, the mathematics of nature. Geometry is an outgrowth of field measurements, as its name implies, and in fact there is little of secondary mathematics, at least, that might not have come into existence as direct or indirect consequence of mathematical formulation of the quantitative relations which exist in nature."*

"To these may be added", says Professor Young, "the following from the important report of the commission appointed in 1904 by the society of German Natural Scientists and Physicians to investigate questions relative to instruction in mathematics and the natural sciences.

"' With full recognition of the formal culture value of mathematics, one-sided and practically meaningless special topics may be omitted, but on the other hand the power of viewing mathematically the world of phenomena surrounding us should be developed as highly as possible. From this there arises two special problems: the strengthening of the power of space intuition and the training to # habit of functional thinking. The task of logical training, from time immemorial alloted to mathematics, is not hampered thereby, but we can say that this task only gains through the more pronounced fostering of the aspects mentioned, for thereby mathematics is brought into closer relation with the other domains of interest of the pupil, in which he is to set his logical powers to work.'"**

Professor Felix Klein in Germany is emphatic in his sup-

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* Young, J.W.A. The Teaching of Mathematics, p. 15.
\# The italics are mine.
** Quoted by Young, loc. cit. p. 48. For an extended treatment of the value of mathematics, see chapter II of Young.

port of the habit of functional thinking as the aim of Mathematics.*
In one of the most recent German texts#, which he enthusiastically endorses, the equation is introduced as the equality of two functions and the first work is the evaluation of functions for special values of the variable.

In a recent article Professor Hedrick says, "The chief direct value of algebra, in fact the real subject-matter of algebra, aside from the rather insignificant chapter of shorthand which I have mentioned, consists in the study of variable quantities, the relations between such variable quantities and the acquisition of the ability to control and to interpret such relations."-------"With this motive, then, firmly established, algebra receives for the first time a thoroughly firm foundation in modern pedagogy; for we have seen that neither the shorthand of symbolism, nor the search for problems affords a satisfactory basis. Moreover, algebra itself emerges strengthened and beautified, no longer needing any apologist, but manifesting itself as a true need of the modern world, which is, both in its manifold scientific enterprises and in its every day affairs, vitally interested in controlling and interpreting the relations between varying quantities, and in assuring itself that even its humblest citizens have some appreciation of the possibilities of such relations. As a means for selection of topics, this modern view of algebra is therefore absolutely satisfactory."**

Although as we have just shown, this goal or aim, i.e. the habit of functional thinking, has been advocated by the authorities cited and by others, this point of view has not been fully appreciated by most mathematicians and has not been generally

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* Klein; Der Mathematische Unterricht an der höheren Schulen.
\# Müller, H. Die mathematik auf den Gymnasium und Realschulen, Erster Teil, p. 120 et sequa.
** Hedrick, E.R. School Science and Mathematics, Jan. 1911.pp.54-

accepted by teachers; neither has it been worked into our texts, embodied in treatises on the teaching of mathematics, nor become influential in our teaching of the subject.

Chapter III

In which we discuss the interests and acquired tendencies of the child, his method of thinking, his attitude toward scientific processes, and his standard of excellence. (The third factor in determining the course of study.)

Chapter III

Psychology of the Child.

Section 1: His instincts and acquired tendencies.

Having indicated on rather broad lines the sort of subject-matter that mathematics provides for the realization of the aim of education, we may now consider what portions of this subject matter can best be acquired by pupils in the first year of the high school, i.e. by pupils during their fifteenth year. This question can be answered only by considering the nature of the child.

Unfortunately, there is a scarcity of accurate information about the child of fifteen. But in the training of children in general, the following classes of activities are chief importance: "imitation, play, construction, curiosity, or investigation, sociability, collecting, ownership, expression, love, sympathy, manipulation, ambition, emulation, rivalry, love of approbation, pride, independence, defiance, courage, aesthetic and ethical appreciations (as approval and disapproval), tendencies to avoid inactivity and pain (whether mental or physical), pugnacity and fear."*

In addition to this general information, we know that the child of fifteen has just entered upon his adolescent years, and that during this period of his life certain activities become very prominent, more so than either before or after.

His curiosity is never stronger. There is nothing in the child's physical or phenomenal environment to which he does

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* Rowe; Habit Formation and the Science of Teaching, p. 74.

not respond and respond enthusiastically. He wants to understand it; he wants to know the why; his whole attitude towards nature suffers a radical change. He begins to appreciate the extent of time and space; he is coming to realize the significance of cause and effect.

President Hall says: "The normal boy in the teens is essentially in the popular science age. He wants and needs great wholes, facts in profusion, but few formulae. He would go far to see scores and hundreds of demonstrative experiments made in physics, and would like to repeat them in his own imperfect and perhaps even clumsy way without being bothered by equations. He is often a walking interrogation point about ether, atoms, X-ray, nature of electricity, motors of many kinds, with a native gravity of his mind towards those frontier questions where even the great masters know as little as he. He is the questioning age, ******".

"Last, and perhaps most important of all for our purpose today, the high school boy is in the stage of beginning to be a utilitarian. The age of pure science has not come for him, but applications, though not logically first, precede in order of growth and interest the knowledge of the laws, forms, and abstractions." *

Section 2: His method of thinking and his learning process.

Not only should we consider the sorts of things the high school freshmen normally wishes to do but also his method of thinking and his learning process.

According to Professor Dewey intellectual thinking is divided into "Five logically distinct steps". (1) A felt dif-

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* Hall; Adolescence, Vol. II,p. 156.

ficulty; (2) its location and definition; (3) suggestion of possible solution; (4) development by reasoning of the bearings of the suggestion, (tentative selection); (5) further observation and experiment leading to its acceptance or rejection.*

(1) "A felt difficulty" means a break in our accustomed activity, a situation in which our usual way of acting does not give satisfaction; a puzzling situation. This is called a "problem".#

Consider the case of a boy solving a quadratic equation, who has not studied square root. As long as he deals with equations whose roots are rational he has no need for square root, provided of course that he is dealing with fairly small numbers. But as soon as he attempts an equation whose roots are irrational, he finds his accustomed action blocked; he feels a difficulty; he does not completely understand the situation; he has a "problem".

For any pupil the "problems" lie in the path of that pupil's accustomed activity or tendencies to action, that is in the path of his instincts and acquired tendencies. The interruption of these activities furnishes the "felt difficulty". The feeling of value attached to the "problem" thus raised varies directly as the value of the activity interrupted.

What is a "problem" for one may not be a "problem" for another; what is a "problem" for an adult teacher does not necessarily constitute a "problem" for the student. For their

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* Dewey; How We Think, p. 72 et sequa. Compare McMurry's five formal steps; McMurry, Method of the Recitation.

\# "Problem" is used here in a technical sense to indicate a psychological problem and not a problem in mathematics. Hereafter in this paper when a psychological problem is meant the quotation marks will be used except in the case of quotations from authorities.

tendencies and interests are not the same and the break or difficulty in their action may not occur at the same place. This suggests that "problems" cannot be given to the students ready made nor to students collectively any more than knowledge can be poured into them.

"It is, indeed, a stupid error to suppose that arbitrary tasks must be imposed from without in order to furnish the factor of perplexity and difficulty which is the necessary cue to thought. Every vital activity of any depth and range inevitably meets obstacles in the course of its efforts to realize itself ---- a fact that renders the search for artificial or external problems quite superfluous. The difficulties that present themselves within the development of an experience are, however, to be cherished by the educator, not minimized, for they are the natural stimuli to reflective inquiry."*

"A problem is a mental thing, a psychical thing; it involves a certain mental attitude and process on the part of the one to whom it presents itself. Nothing is made really a problem by being labeled such, ----- or even because it is "hard" and repulsive. To appreciate a problem as such, the child must feel it as his own difficulty, which has arisen within and out of his own experience, as an obstacle which he has to overcome, in order to secure his own end, the integrity and fullness of his own experience. But this means that problems shall arise in and grow out of the child's own impulses, ideas, habits, out of his attempts to express and fulfill them --- out of his efforts to realize his interests, in a word."**

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* Dewey; How We Think, pp. 64-65.
** Dewey; Interest as Related to the Will, pp. 32-33.

In case the student has no "problem", thought is practically impossible. There is no steadying or guiding factor in the process of reflection. Suggestions flow at random; the mind wanders; nothing is aimed at; a conclusion is never reached. But on the other hand, if there is a "problem" the student is aiming at something. Not only does the "problem" direct inferences but also, to some extent, the very suggestions themselves.

The effect of no "problem" is illustrated in laboratory work in which the student is simply asked to make observations. Because it is laboratory work and material apparatus is used the work is supposed to be concrete and real to the student. But unless the observations and instruments are tools for solving a "problem", the whole process is abstract. Readings and observations are made at random; they have no meaning; a conclusion is not formed and the student has acquired no new knowledge, at least of the sort we wished him to acquire.*

(2) We have refered to a "problem" as a felt difficulty in the childs activities. This implies the location of the difficulty between certain limits and its definition. This is the second step.

The location of the "problem" is very important. Without the exact location of the difficulty, any attempts at a solution must be random and uncontrolled. For instance, in attacking a problem in interest, if the pupil begins multiplying and dividing before he has decided what quantities are given and what quantity is to be determined, his obtaining the correct result is a mere matter of chance. There is nothing to guide his efforts until he decides whether he is to find the rate, interest, principal, or time,

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* See Dewey, How We Think, p. 191 et sequa.

i.e. until he defines his "problem".

A large part of the pupil's technique consists of his ability to define accurately his "problem". The importance of the definite location of the problem suggests that one of the characteristics of good thinking is the suspension of judgment and not the jumping at a conclusion before the "problem" is well defined

(3) This brings us to the third step, suggestion of possible solutions. The situation in which the difficulty occurs will call up certain meanings and past experiennes. The number of things called up, of course, depends largely upon the individual. The ability to call up a large number is a mark of good thinking. Suspended judgment is also an essential factor in this phase of the process of thinking. But mere observation or the calling up of past experiences is valueless in itself. To be of value, they must be in connection with a "problem", i.e. they must be for a purpose.

To illustrate, if a pupil is solving a problem dealing with levers, calling up what he knows about falling bodies, interest problems, etc. willbe of no value to him. Only facts connected with levers are valuable.

(4) The next step is to make a tentative selection from the suggestions which have been called up. We usually refer to this as reasoning. In it all obviously impossible and irrelevant suggestions are rejected.

(5) The fifth step is the verification or experimental corroboration of the tentatively accepted solution. If this solution efficiently controls the situation; if the method solves the "problem" it is accepted as correct. Im mathematics this means substituting the result in the problem and if it satisfied the conditions of the problem the result is correct. In addition to this

the verification often includes the consideration of the implication of the result.

Besides the five steps which we have considered there is a sixth, i.e. the application or use of the result. To simply arrive at a correct result is to leave a task half finished.

The importance of the application cannot be over estimated . Without this the result will probably have no meaning for the individual, that is to say it is valueless for him, and he will probably not be able to make use of it in a future situation. For example, the statement of the Pythagorean theorem and even its logical proof may be familiar to a pupil before he has necessarily grasped its meaning.

In this connection Professor Dewey says,* "A true conception is a moving idea, and it seeks outlet, or application to the interpretation of particulars and the guidance of action, as naturally as water runs down hill. ******** Application is as much an intrinsic part of genuine reflective inquiry as is alert observation or reasoning itself. Truly general principles tend to apply themselves. The teacher needs, indeed, to supply conditions favorable to use and exercise; but something is wrong when artificial tasks have arbitrarily to be invented in order to secure application for principles."

Moreover, it is only through application and use that we acquire meanings. Take, for example, the case of some simple meaning. When we hear the word chair spoken or see a chair, a meaning is at once present in our minds. This meaning may not be the same to all persons or to the same person under different circumstances. To a person who is tired, a chair means a place to rest. Under other conditions or to another person, a chair may signify an

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* Dewey; How We Think, p. 213.

article of furniture which will assist in beautifying a room. A chair may mean something to stand on in order to hang a picture; to a child it may mean something with which to form a train of cars. This suggests, first that the meaning is not dependent upon the mere object, its material, construction or physical qualities; second, that the meaning is connected with or grows out of the use we may make of the object.*

Professor Dewey says, "The acquisition of definiteness and of coherency (or constancy) of meanings is derived primarily from practical activities. By rolling an object, the child makes its roundness appreciable; by bouncing it, he singles out its elasticity; by throwing it, he makes weight its conspicuous distinctive factor. Not through the senses, but by means of the reaction, the responsive adjustment, is the impression made distinctive, and given a character marked off from other qualities that call out unlike reactions. ******* Variations in form, size, color, and arrangement of parts have much less to do, and the uses, purposes and functions of things and of their parts much more to do, with distinctness of character and meaning than we should be likely to think."#

The same notion may be illustrated by the futility of teaching a child to skate away from ice; of teaching a child to swim without going into water. It is only by participation in the real activity that the child learns how ; learns the meaning of skating , of swimming.

Section 3: His attitude toward scientific processes.

In the preceding pages we have placed much emphasis upon activity, the doing of things, the acquiring of meanings through use. However, as we have already pointed out, this activity

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* See in this connection Charters, Method of Teaching, p. 26.
\# Dewey; How We Think, pp. 122 - 3.

must be in connection with a "problem"; it must be the means of controling and end which appeals to the child as valuable.

It is a common observation that when the child is interested in a thing, he applies himself diligently, he turns aside because of no difficulty. Saying that he is interested, is saying that he has an end in view. The tools he uses, the processes he employs, his activity are a means to that end. Aside from the use he can make of them, he has little appreciation of scientific processes.

In mathematics the processes, operations, and symbols are, from the point of view of the child, means to an end. He exerts himself that he may control that end and not because he appreciates mathematics as a science. For him the equation is a means for controling a problem; clearing of fractions is a means for solving an equation; completing the square is a way of controling a quadratic equation; exponents are ways of writing products; etc.

What is at one time a means may become an end and the value which was attached to the original end is transfered to the means as an end. This is the case in clearing of fractions as a means of solving an equation. The equation originally was a means but now is thought of as an end; the feeling of value which was attached to the problem is now attached to its means, i.e. the equation.

Section 4: His standard of excellence.

The preceding discussion has implied a psychological order of procedure and organization of subject-matter as opposed to a logical order. The acceptance of the psychological order will mean, in actual teaching, that often the student must be allowed to use facts and processes which he may not understand.

In mathematics he may, and often does, make use of the theorems regarding the similarity of triangles long before he is acquainted with their logical proof or even understands the full significance of the theorems.

"Our progress in genuine knowledge always consists in part in the discovery of something not understood in what had previously been taken for granted as plain, obvious, matter-of-course, and in part in the use of meanings that are directly grasped without question, as instruments for getting hold of obscure, doubtful, and perplexing meanings. No object is so familiar, so obvious, so commonplace that it may not unexpectedly present, in a novel situation, some problem, and thus arouse reflection in order to understand it ."*

But however seriously this non-rigorous treatment of subject-matter may be attacked by the man of pure science, just such a procedure is in full accord with the nature of the adolescent

Speaking of the adolescent youth, President Hall says, "Never is the power to appreciate so far ahead of the power to express, and never does the understanding so outstrip ability to explain. Over accuracy is atrophy. Both mental and moral acquistion sink at once too deep to be reproduced by examination without injury to both the intellect and will. **** With pedagogic tact we can teach about everything we know that is really worth knowing; but if we amplify and morselize instead of giving great wholes, if we let the hammer that strikes the bell rest too long against it and deaden the sound, and if we wait before each methodic step till the pupil has reproduced all the last we starve and retard the soul which is not all insight and receptivity."**

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* Dewey; How We Think, p. 120.

** Hall; Adolescence Vol. II, p. 453.

In the next chapter we shall refer to this non- rigorous method in connection with the problems in the course we shall propose. There we shall show that mathematicians also favor this attitude toward the work in elementary mathematics.

Chapter IV

In which we apply the principles of Chapters III and IV in discussing the details of organizing a course in elementary mathematics for the first year of the high school.

Chapter IV

Details of organozation.

A. Problems

Section 1: Problems the core of the course.

The nature of the child as presented in Chapter III is very significant when applied to elementary mathematics. A course in elementary mathematics must be based upon valuable problems, in fact they must form the core of the whole course, and the processes and the tools of mathematics are to come as a means to an end,* i.e. the solving of valuable problems. The child normally exerts himself only in response to a "problem" and hence to motivate his activities a "Problem" is a prerequisite.

The traditional organization of the subject-matter of elementary mathematics, i.e. placing the theory and processes first and making little of application, is opposed to this point of view.

Section 2: Non-rigorous solution of problems.

In chapter III, it was stated that the non-rigorousness of solution of problems which is necessitated by psychological facts is in keeping with the nature of the child in the first year of the high school.

At first sight, it may be felt that such an attitude with respect to elementary mathematics is extremely radical and would not have authoritative support from mathematicians. As a matter of fact it is not necessarily radical and has strong support.

Refering to the objections which may be raised to a less formal and systematic procedure in mathematics, Professor Moore

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* Chapter III, Section 3.

says, "That the boy will be learning to make practical use in his scientific investigations, to be sure in a naïve and elementary way, of the finest mathematical tools which the centuries have forged, that under skillful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions."*

"As a pure mathematician, I hold as the most important suggestion of the English movement *****, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods generally, in continous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is to be accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy ******, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics."#

Professor John Perry of England states his position so specifically that we quote at length.

"As soon as we give up the idea of absolute correctness we see that a perfectly new departure may be made in the study of mathematics. The ancients devoted a lifetime to the study of arithmetic; it required days to extract the square root or to

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* E. H. Moore, On the Foundations of Mathematics, Bulletin American Mathematical Society, Vol. 9, 1903, p. 411-412.

\# E. H. Moore, loc. cit. p.411.

multiply two numbers together. Is there any great harm in skipping all that, in letting a schoolboy learn multiplication sums, and in starting his more abstract reasoning at a more advanced point? Where would be the harm in letting a boy assume the truth of manypropositions of the first four books of Euclid, letting him accept their truth partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra? Letting him assume the sixth book to be axiomatic? Letting him, in fact, begin his severer studies where he is now in the habit of leaving off? We do much less orthodox things. Every here and there in one's mathematical studies one makes exceedingly large assumptions, because the methodical study would be ridiculous even in the eyes of the most pedantic of teachers. I can imagine a whole year devoted to the philosophical study of many things that a student now takes in his stride without trouble. The present method of training the mind of a mathematical teacher causes it to strain at gnats and to swallow camels. Such gnats are most of the propositions of the sixth book of Euclid; propositions generally about incommensurables; the use of arithmetic in geometry; the parallelogram of forces; etc.; decimals. The camels I do not care to mention, because I am in favor of their being swallowed, and indeed I should like to see them greatly increased in number; they exist in the simplest arithmetic, and geometry, and algebra. Why not put aside ever so much more, so as to let a young boy get quickly to the solution of partial differential equations and useful parts of mathematics that only a few men now ever reach? I have no right to dictate in these matters to the pure mathematicians.* They may see more clearly than I do the necessity for a great mathematician going through

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* See citation from Professor Moore given on p. 37

the whole grind in the orthodox way; but if so, I hardly see their position in regard to arithmetic and other things in the study of which they allow skipping. I should have thought that the advantage of knowing how to use spherical harmonics or Bessel functions at the age of seventeen, so as to be able to start in mathematics at Cambridge just about the place where some of the best mathematical men now end their studies for ever, of starting at this high level with youthful enthusiasm, and individuality, and inventiveness, would more than compensate for the evils of skipping."#

This very emphatic stand for a non-rigorous course,** using theorems of geometry early without proof,etc., which is taken by Professor Perry is also taken by the Committee on Fundamentals of the Central Association of Science and Mathematics in its report last December.

It would be possible to cite other authorities who have taken a stand for what we have a non-rigorous course, but these these are sufficient to show that such a procedure has support among mathematicians.

It should be mentioned in this connection while we are pleading for this non-rigorous course that by this it is not meant that the course shall not give training which is accurate or that slip-shod methods and work will be accepted. Our position is rather that we will not insist on the pupil going through each of the logical steps before he is allowed to use a fact. For instance, the pupil may use the Pythagorean theorem before he goes through each logical step necessary for its proof. But when a student solves a problem he should be able to satisfy himself as to the reliability of his result.

John Perry; The Teaching of Mathematics, British Assoc. Meeting at Glasgow, 1901., p. 12-3.

** See in this connection Dewey, How We Think, p. 120.

Section 3: Types of problems.

Our next step is to consider in detail what sort of problems we should use as the basis of a course in elementary mathematics. As has been pointed out in Chapter III, a "problem" is a mental thing and because a situation is presented to the pupil by the statement in the text, it does not necessarily follow that this situation will become a "problem" for any particular pupil. The probability that it will do so varies as the closeness of its connection with the life of the pupil.

In the course of a supplementary investigation we have examined and classified the problems found in three algebra texts now widely used . The principal criteria of the classification was the degree of probability that they would occur in the life of an average individual.

We find the following classes of problems.

(1) The first class consists of those problems whose probability of arising in the life of an individual is quite large. Typical of this class are simple problems in mensuration, percentage problems, problems involving uniform motion, the lever, simple machines, etc.

(2) Following this class with no well defined line of distinction is a second class of problems whose probability of arising in the life of an individual is, in general, much less than that of the first class. I refer to the problems which are encountered in special activities. Typical of this class are the more difficult problems of specific gravity, mensuration of unusual solids or areas, problems from concave mirrors and concave and convex lenses, etc.

(3) A third class exists, probably the most important of all classes, which may never arise in actual life but which we may

define as"analogous" to those which do arise in actual life. By "analogous" problems we mean problems which deal with the variation (functional) relations which actually exist between varying quantities but which present them usually in a simplified form so that they are within the comprehension of the pupil. Examples of this type of problem are mechanics problems in which friction is neglected, problems to which approximate laws are applied, the well known clock problems illustrating relative motion, Et

The hypothesis or conditions given inall three of these classes are those which normally and in the natural order of events would present themselves to an individual. Problems belonging to any one of these first three classes, we shall speak of as practical.

(4) In contrastwith the three classes of problems just mentioned, we find problems such that the hypothesis or conditions given do not exist in the natural order of events. Some parts of the hypothesis must of necessity be determined by the solution of the converse problem. This type of problems may be illustrated by the following example.

"A rectangle is four feet longer and three feet narrower than the side of the equivalent square. Find the side of the square."

In the natural order of events, the hypothesis that the length of the rectangle was four feet longer and three feet narrower than the side of the equivalent square, could not be known until the side of the equivalent square was known. Thus the hypothesis could be known only by solving the converse problem; what is the side of a square which is equivalent to a rectangle nine by sixteen? For this reason the first problem would never arise in the life of any individual. Nor is it valuable in that it illustrates especially any functional relation.

(5) We find problems of the following type in the algebra texts: "The area of New Hampshire is 3/11 of that of Maine. 12 times the area of New Hampshire diminished by 3 times the area of maine is 7000 square miles. Find the area of each."

This problem presumably attempts to justify its presence on the ground that since it deals with actual objects and objects of which the pupil has probably heard before, it is concrete and hence interesting to the pupil because of this "sugar-coating", but as has been pointed out on page 26 the mere fact that a problem has to do with material objects does not make it concrete. It is needless to say that such a problem or analogous one would not arise in the life of an individual and further more that the hypothesis could only be known by first knowing the result or conclusion.

(6) Somewhat in contrast to these we find a class of problems of which the following is an example: "Said one boy to another: 'If you give me one-half of your money and 50 cents, I shall have 4 times as much as you; but I give you 50 cents you will have $2.50 more than B.' How much has each?" Problems of this type should be classed as puzzles. To this class belong the problems propounded by the philosophers and logicians of the past.

(7) There is a last class of problems which deal explicitly with number relations. We might call them denatured problems since they do not pretend to be concrete or to possess qualities which should excite interest.

In the same sense as we called the problem of the first three classes practical, we may call those of the remaining four classes artificial. In the light of our discussion in chapter III, these artificial problems would never become "problems" for a child except possibly those of the sixth and seventh classes if they are presented for what they are.

A count and classification of the problems in three of our present day texts is given below.

Table A

Types / Text	1	2	3	4	5	6	7	Total	Drill
A(1908)	2	134	12	21	0	13	69	236	1950
*B(1907)	0	64	26	7	11	7	7	122	1286
C(1907)	29	165	24	121	135	1	113	589	2174
A text of thirty years ago								398#	4000

Table B; Percentages.

	1	2	3	4	5	6	7
A	.8	56.8	5.0	9	0	5.5	29.
B	0	52.5	21.3	5.7	9	5.7	5.7
C	5	28	4.2	20.5	23.	.02	19.2

Table C

Ratio of problems of application to drill problems.

A	1 to 8
B	1 to 10.5
C	1 to 3.7
Text of thirty years ago	1 to 10

These tables, while not including a large number of texts, are representative and are sufficient to give us some idea of the problem situation. Three conclusions may be pointed out. First, the trend of the texts is toward a proportionately larger number of problems of application; second, desirable problems of application seem to be scarce since the only text

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* This text is not strictly an algebra but combines some geometry with the work of the first year. With the exception of text A, each text is supposed to cover only one year's work.
# These were not classified but with very few exceptions they come under classes 6 and 7.

which furnishes a large number of problems of application shows a large percentage of artificial problems; third, proper care is not always taken in the selection of problems for texts. These is also a decrease in the amount of work expected of the pupils. In the text of thirty years ago (it is an early edition of a text which became widely used) it is stated explicitly in the preface that the 4000 problems should all be solved in the course of the first year. The texts of today reduse this number by at least half.

During the past few years there have been many attempts made to collect new problems suitable for use in the teaching of elementary algebra and geometry. The most serious and thorough of these is the investigation now in progress by a Committee of the Mathematical Section of the Central Association of Science and Mathematics. This Committee was created by the Association at its meeting in November 1908, and is still pursuing its work.

The "problem" of their investigation is; "(1) the determination of the extent of the direct applications of the elementary algebra and geometry and the collection of a fund of real applied problems," (2) the determination of their adaptability for the teaching of algebra and geometry to secondary school pupils."*

The bulk of the work done and reported upon at this date has to do with the first phase of their problem. An ex-

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* The preliminary report of the Committee on the first phase of their problem is found in the School Science and Mathematics for November, 1909. Problems collected by the Committee are found inall numbers of School Science and Mathematics since March 1909 until sometime in the fall of 1910. The essential conclusions presented in the report seem to be that algebra and geometry are to be taught as tools for controling certain situations. Together with a knowledge of the processes and facts of the subjects, a true and adequate conception of their uses in practical life is to be taught.

amination of the reports of the Committee suggests to one that the number of problems reported is surprisingly small if a large number of "real applied problems" exist.

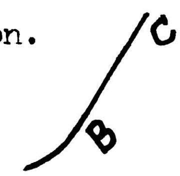
The classes are given in the report* and the following problems are given by the editors as typical of these classes.

1. Problems involving a geometrical demonstration.

In laying out a "turnout" or switch, on a railroad track a "frog" is used at the intersection of the two rails to allow the flanges of the wheels moving onone rail to cross to the other rail. The angle of the frog which must be selected for any place depends upon the central angle of the curves of the two tracks. If one track is straight and the other curved, prove that the angle of the frog equal the central angle between the radii of the curved track drawn to the points of its intersection with the straight track.#

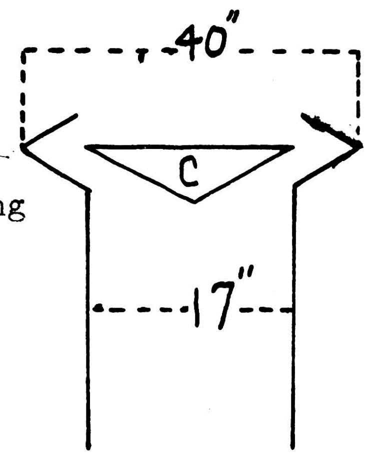
2. Problems involving a geometrical construction.

To draw easement of cornice tangent to rake cornice BC at B



3. Problems involving computation.

In a smokestack for a locomotive, find the diameter of the base of the cone C so that the annular space may be equivalent to the opening of the cylindrical part of the stack S. The diameter of the section of the stack containing the base of C is 40 inches. The diameter of the cylindrical part of the stack is 17 inches.



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* For this report see School Science and Mathematics, November, 1909, p. 673. In addition to this the Committee has issued a complete classified list of the problems collected. This may be obtained from the Committee.

\# The wording of the problem has been changed slightly so as to avoid drawing a figure.

4. Problems involving the solution of equations.

In accurate tool work where holes are to be bored close together in a metal plate by means of a lathe, the centers are first marked carefully to thousandths of an inch. This may be done by first turning the discs on the lathe such that when placed tahgeht to each other their centers mark the positions of the centers of the required holes. These circular discs are then fastened on the metal plate in tangent positions and holes are bored at their centers.

Three holes are to be bored, the distance between whose centers shall be 0.650 in., 0.790 in., and 0. 865 in., respectively. Find the radii of the required discs.

5. Problems involving the evaluation of formulae.

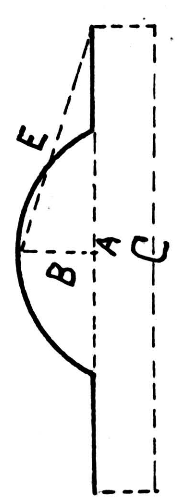
The following formulae is used for computing the horse power of automobile engines:

$$H.P. = KND(D - 1)(R + 2)$$

where K is 0.33 for racing cars, and 0.197 for commercial touring cars: N is the number of cylinders; D is the diameter of the cylinders; and R is the ratio of the stroke(in inches) to the diameter (in inches). What will be the horse power of a four cylinder engine of a racing car in which the diameter of the cylinder is 4 in. and the stroke 5 in.?

6. Problems involving both algebra and geometry.

A flat circular disc of metal is to be stamped in the form of a spherical segment with a flange. The figure shows a cross-section of the resulting piece of metal, A being the width of the spherical part, B the depth or altitude of the segment, and C the outer diameter of the flange. The problem is to determine the size to cut the sheet of metal



in order that when stamped the piece may have these dimensions. Prove that the radius of the required flat circular sheet is equal to E.

Although the Committee has probably done all that they could do, they have not done what they claim. For while problem 1 which deals with the relation between the angle of a frog and the central angle of the curve of the track, is expressed in concepts which may lie within the experience of the average pupil, itis expressed in concepts for which he probably feels but little interest. Moreover, it seems that as the frog is already made and that the track is to be laid so as to suit the frog, the problem should be stated in converse order; i.e. to prove that the central angle of the curve of the track is equal to the angle of the frog. Another criticism is that it is not true to facts since tha curve of the track is not circular but has a variable radius of curvature.

The second problem, to draw easement of cornice tangent to rake cornice, is stated in terms which the pupil will not comprehend and in which he will therefore have no interest.

The third problem is unreal in that locomotives do not have a smokestack of the type represented and also no reason is apparent to the inverted cone being placed in the opening. These conditions are apparently introduced to make the problem seem concrete but they really tend to make the hypothesis more obscure without adding interest.

An analysis of the problem involving the solution of equations reveal about the same condition.

Without stating an analysis of the other types, these conclusions may be drawn:

1. They are not in all cases real in the sense that they present a problem as it actually occurs.

(2) Where they do represent problems that actually occur the concepts of the problems are not, in all cases, closely connected with the life of the child.

(3) For these reasons the "real applied problems" as presented in the report of the Committee do not seem to be suitable for pupils in the first year of the high school.

Section 4: The type of problem we shall use.

We have found, in the case of both our current texts and the report of the Committee on "Real Applied Problems", the number of desirable problems of application to be small, especially if we keep in mind that problems are to be the core of elementary mathematics. Also in both cases, problems were found which had no connection with the life of the child and problems which presented unreal hypotheses

It is therefore appropiate that we consider, in the light of our previous discussion, what sort of problems we should base a course in elementary mathematics upon.

From the standpoint of motive, the ideal method would be to take those problems which the pupil actually has from day to day. But we would find that the average pupil will have few problems which afford simple direct applications of algebra and geometry; that class work would be impossible since probably no two pupils would have the same type of problem at the same time and not in the order which is demanded by the plan of the course; the work would be very irregular since many days might elapse without a problem. Such conditions, while from the ideal standpoint they may not present unsurmountable difficulties, considered in their relation to the schools and the curriculum of the present day are undesirable. We are, then, forced to admit the necessity of introducing problems which are to some extent extraneous to the life of the pupil.

To decide upon the sort of problems we should introduce we will return to the aim of elementary mathematics which we stated on page 17. Viewing the matter in this light we find that the life of every individual is filled with situations involving quantitative relations: the relation of his income to his expenditures; the choice of the best route from a possible two or two or more; the time necessary to complete a task; the most profitable method of cultivating a field; the most profitable selection of feed for stock; etc. In fact, his whole life is an inquiry of how best to do things to gain a certain end; of what is the relation between cause and effect.

To be sure, few if any of these "problems" could be said to be a problem to be solved by simultaneous equation, or a quadratic equation, or a theorem in geometry. In fact, very few of them would be recognized as problems in the usual sense. But in each there are certain "variation relations" between the quantities, probably quite complex, but nevertheless the relations exist and it is only but recognizing the existence of these relations and possessing a knowledge of them that a dependable solution of the puzzling situation can be reached.

The complex relations which exist are often combinations of simpler relations or, in many instances, they differ only so slightly from a simple variation relation that a very approximate result may be obtained by the use of a simple relation.

The problems of the course should be such as to illustrate the simple variation (functional) relations which exist between naturally varying quantities. They should also be expressed in concepts which are suitable to the child. These problems are not necessarily "real applied problems" or problems which may arise in the life of the individual. So much the better if they are but the important thing is that they offer a means for the study of the variation relations existing between the factors of the child's environment.

The type of problem just indicated is not only desirable from the standpoint of the aim of mathematics but also from the point of view of the instincts and acquired tendencies of the child. The high school freshman is especially interested in understanding his physical environment; a knowledge of "variation relations" is a means to this end.

B. The Functional Idea Carried Out.

With problems of the type just described as the core, and following the aim of mathematics we have presented, we have organized a course which follows. Many of the problems(probably about half) have been taken from current texts, care being taken to avoid objectionable types. The great majority of the problems would fall under the class of analogous problems (see page 41).

In the study of the variation relations presented in the problems, the equation* is used where the relation may be easily stated in the form of an equation. When this is not practicable, other means of solution are employed.

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* Equation is used here in the sense of an equation of condition rather than as an identity. This distinction is not usually made in elementary mathematics and should not be forced upon the pupil. However the teacher should be familiar with the significance of the equation of both types and the distinction should be kept in mind. Doing this the teacher's attitude toward the equation will be decidedly different to what it is when the equation is thought of only as an identity.

In solving equations the processes necessary for their control are introduced. Only after these processes have forced themselves upon the child, so to speak, are they studied as such, and then only as a means to an end, i.e. the solving of equations which come from valuable problems. For an example of how this has been carried out, see Chapter III, p.37 et sequa of the following course. Short drill lists have been inserted for use when the need has arisen but the bulk of the child's knowledge of the processes of algebra is acquired in the solution of valuable problems.

Such an attitude toward the processes of algebra has necessitated the entire omission of some processes found in the traditional course and the slighting of others. For example; long division, radicals, and exponents as topics have been omitted factoring, multiplication, fractions, etc. have been given much less space than in standard texts.

In the earlier part of the course the notion of "variation relation" has not been made as prominent as in Chapters IV and V, but even in the first chapters of the text emphasis is placed upon the relation existing between the quantities in any given problem.

When it has been desirable to study a variation relation which is impracticable to express by means of an equation, other means, graphical illustration, has been employed. This is the case with the relation between the time of day and the temperature some problems from trigonometry, maximum and minimum problems, etc. (See Chapter V and the first half of Chapter IV .)

In the last pages of Chapter V, an attempt is made to sum up and systematize the pupil's knowledge of the various types of variation relations which have been taken up in the course.

Results: Tentative Report.

This course, although the notions upon which it is based are largely theoretical, has been partially tried out. Last September the writer was given a first year class of thirteen boys (which number was later reduced to eleven) in the University High School. The class was about the average of first year classes in the high school, although the average age of the class was slightly above the average age.

At the date of this writing, May 1, 1911, the course has been completed by the class. The general interest of the class and the attitude of the pupils in particular have been especially good. The writer has not found it equalled in any of his classes in his former experience as a teacher. The ability of the pupils to attack the problems, which must be admitted to be difficult, has surpassed the expectations of even the writer.

While it is impossible to tell accurately whether the pupils have grasped the meaning of the relations represented by the problems, it seems that they have.

In regard to the processes of algebra, not all of the drill exercises printed in the text were used. An informal test upon the processes studied was given the class after they finished Chapter III. This test was given also to the other first year class and to the second year classes, all of which had used the traditional type of text. Owing to the fact that there was little common ground upon which to compare the two first year classes and that the second year classes had had far more preparation, no dependable conclusions could be drawn. But there appeared to be little difference between the two first year classes in the solving of simple equations; in the case of quadratic equations (which the other first year class could not solve at all) the second year classes were superior although individuals in my class did as well as the average of the second year classes. On the whole, it would seem that, so far as the processes of algebra which are necessary for solving equations, my class compares favorably with other first year classes. Besides this they have gained other valuable knowledge of which the other classes know nothing.

A FIRST COURSE IN ALGEBRA

Chapter I.

In arithmetic, we are given in the chapter on percentage the following rules:

To find the percentage, profit, or loss, multiply the base, or, principal, by the rate expressed as hundredths.

To find the rate divide the percentage, profit, or loss, by the principal, or base.

To find the base, or principal, divide the percentage, profit, or loss, by the rate expressed as hundredths.

Solve the following problems by the above rules.

1. If you purchase a horse for $140.00 and wish to sell him so as to make 15% what price must you ask?

2. Suppose you are feeding a bunch of cattle and they, together with the feed, have cost you $1280.00. If they sell for $1425.00 what is your per cent of profit?

3. Suppose an agent who is selling goods on a commission of 3% makes during a certain year, a gross profit of $3784.20. What volume of business did he do?

4. If you purchase a house and lot for $5800.00 and sell it for $6500.00, what per cent have you made on your money if you pay an agent 1 1/2% for making the sale?

5. In a certain town the population according to the last census is 6784, of which 2967 are colored. What is the per cent of colored population in the town?

What is your greatest difficulty in solving the above problems? Do you have difficulty in deciding which rule you need to use for a particular problem? Would it not simplify the work involved, if, instead of having to make a choice from three rules, we could have one rule which might be used to solve all such problems? We shall see that this is possible if only we know how to use this one rule when it is given us. And it may be encouraging to know that the processes we learn in the use of this rule will be very useful on other occasions.

Take the first rule which we stated on page 1. If we write it briefly, it says; base multiplied by rate is equal to percentage. This statement is still too lengthy and somewhat cumbersome for convenient use. As we shall use it for all problems of the kind which we have been solving, it is advisable to shorten the statement still more and hence put it in a more workable form. If abbreviations are used for the words, the statement will be very much shortened. You have already used the sign = for "is equal to". A very convenient plan for choosing abbreviations for words such as base, rate, etc., is to take the first letter of the word for the abbreviation. Thus we would represent "base" by b, "rate" by r, and "percentage" by p. Suppose we represent "multiplied by" by agreeing that when two or more letters, or a number and a letter are written together with no symbol between them as b r or 7 b, they are to be understood as being multiplied together.

This agreement is just the same as we use when we write 3 bu., 15 min., 20 A., etc. The meaning in these instances is obvious: 3 bu., meaning 3 times 1 bu.; 15 min., meaning 15 times 1 min.; etc. This same meaning holds in the case of 7 b.

Our rule now becomes when the abbreviations just suggested are used,

$$b\ r = p$$

which is read, base times the rate equals percentage or more briefly, b r equals p.

Applied to problem 1, we have; $140 \times .15 = p$

Applied to problem 2, we have $1280\ r = 145$

Applied to problem 3, we have $b.03 = 3784.20$

140 x .15 is read 140 times .15. Two numbers, for example 2 and 3, cannot be written together as we have agreed to write letters because numbers written in that manner have another meaning. Thus 23 means not 2 times 3, but 20 plus 3. The expression 1280 r is read 1280 times r or simply 1280 r. b.03 is read b times .03. It is better written .03 b.

You should note that each of these groups of symbols is a definate statement. For instance, the second group says that 1280 times the rate is equal to 145. Such a statement as this, in which two quantitative expressions are declared to be equal, is spoken of as an equation. You will also note that each of these equations contains one letter, which of course represents some quantity, whose value or amount is not known; so one may look upon these equations not only as a statement of a fact, but also as asking a question, i.e. 1280 r = 145 may be thought of as asking the question, what number multiplied by 1280 will give the product of 145. The process of answering this question is spoken of as solving the equation.

What question does the first equation ask? Would p = 140 x .15 ask the same question? Can we then, if we wish to do so, write p = 140 x .15 in the place of 140 x .15 = p? We speak of this as reversing the order of the equation and it is seen that this may be done without destroying the truth of the statement.

To solve the first equation it is only necessary to multiply 140 by .15. We then have the value of p or our answer.

If you will condier the following questions, they will probably suggest a method for solving the other equations. How do you find the number which multiplied by 5 gives 35? The number which multiplied by 13 gives 51? The number which mutiplied by 37 gives 851? These questions might have been asked by means of the following equations, if we will agree to represent the word number by n:

$$5n = 35$$
$$13n = 51$$
$$37n = 851$$

In the simpler problems the result is so well known that you will be able to say it off at once and the process by which it was obtained may not be thought of. But in the third a method will probably be necessary. If you think of the questions asked by the equation, it is evident that we must divide 851 by 37. If we say this with respect to the equation, we would say that we divide both sides of the equation by 37.

To solve our second problem on page 3, we will divide both sides of the equation by 1280. Doing this we have

$$r = \frac{145}{1280} \quad \text{or } .113\ldots\ldots \quad \text{or } 11.3\%.$$

The answer to the question, what number multiplied by 1280 gives 145, is then 11.3%.

In order to determine whether this answer is correct or not, we try the result in the problem. If it fits the conditions of the problem, we say that it is correct; if it does not fit the conditions of the problem, it is not the correct result. This process is called checking our work.

If we refer to the original problem on page 1, it is clear that to check our work we should take 11.3% of 1280. That is .113 times 1280 which is 144.64. This is nearly 145. Since 11.3%

was not exact, we should not expect 11.3% of 1280 to be exactly 145. We may then say that this result checks and is sufficiently correct.

The solution of the third problem is effected in the same manner. Solve it and check the result.

You will observe that we have thus been able to solve by one rule, problems which ordinarily would have required three rules. This is a saving of time and energy in lerning rules.

In studying the problems of the following list, which you are to solve by very much the same methods, always read the problem over carefully, picking out the number which is the base, the number which is the rate, etc. Keep in mind what we have been refering to as "our rule", i.e. base multiplied by rate is equal to percentage or br = p. This means of course that the percentage is the product of two numbers, (one of which is usually a decimal fraction) and hence must depend upon them for its value. Our problem always is to find one of the three, base, rate, or percentage, when the other two are given.

1. If you purchase property for $2360 and wish to sell it so as to make a profit of 15%, what amount must you ask for it?

2. A merchant buys butter for 30¢ a pound and sells it for 36¢ a pound. What is his per cent of profit on the cost?

3. What must be the per cent of profit on an investment if $525 is to produce $600?

4. The valuation of a school district is $2,900,000. The citizens of the district vote to raise $50,000 for a new school building by direct tax. What will be the tax on each $100 valuation, if they wish to raise the amount in 5 years?

5. A dealer buys 100 piars of shoes at $2.00 each and sells 75 pairs at $2.50 each. He wishes to mark down the price of the remaining 25 pairs and yet make a profit of 20% on the whole. What price per pair will give this result?

6. The weight of bread is 133 1/3% of the weight of the flour. If wheat loses 18% of its weight in being made into flour, how many pounds of bread can be madd from 20 bushels of wheat? (There is 60 pounds in one bushel of wheat.)

7. An agent received 3% commission for buying and 3 1/2% commission for selling some property. He paid $5750 for it and sold it for $7200. What was his total commission?

8. I paid a collector $24.64 for collecting outstanding accounts. If I paid him 7% for collecting, how much did he collect? How much did I receive?

9. If coal is purchased for $2.83 per ton and sold for $3.25 what is the per cent of profit?

10. A dealer pays $1235 for a bill of goods upon which he was allowed a discount of 20% and 5% off. What was the list price of the goods?

11. Express with appropiate symbols the following statements:
7 pounds plus 15 ounces.

12. 5 feet and 3 inches; 9 feet and 2 inches.

13. 8 yards and 2 feet; 20 yards and 2 1/2 feet.

14. 3 heurs and 20 minutes; one hour and 45 minutes.

15. What question does 5n = 90 ask if n is an abbreviation for the word number? 7n = 84? 12n = 216?

16. What question does 137n = 1507 ask? 2n ≁ 5 = 43?

17. Ask these questions by means of equations: What number multiplied by 9 gives 377? What number multiplied by 15 gives 210?

18. What number multiplied by 6 and added to 30 gives 54?

19. What number is there such that 1/2 of it added to 1/3 of it equals 30?

In our future work we shall often wish to put other rules into a similar compact and definite form by stating them in terms of symbols and abbreviations. In general we shall use the first letter of a word as an abbreviation for the word. For a few words and phrases, we have adopted special symbols by common consent and they are universally used. Some of these are =, $, %, -, etc.

State in compact and definite form the following rules, most of which you have heard before.

1. The area of a rectangle is equal to the product of its length by its width.

2. The area of a triangle is equal to one half of the product of its base and altitude.

3. The amount is equal to the base, or principal, plus the product of the base, or principal, by the rate.

4. The interest is equal to the product of the principal, rate and time (in days) divided by 360.

Solve the problem: If you know that a merchant's profit is 30% and you purchase a suit of clothes for $22, what did the suit cost him?

You will observe that this problem is one which comes under the third rule just stated. You have already stated the rule in terms of symbols as,

$$a = b + br.$$

Proceeding as usual, we have

$$22 = b + .30b$$

or

$$b + .30b = 22.$$

We have here b added to $.30b$. The sum is evidently 1 times b + .30 times b which may be expressed by writing $(1 + .30)b$ or $1.30b = 22$. The remainder of the solution is as in previous problems. Complete the solution and check your result.

Solve the following problems by finding the value of the letter in the equation and check the results. Use decimal fractions in expressing results where necessary.

1. $5x + 2x = 30$
2. $2b + 5b = 28$
3. $6b + b - 3b = 12$
4. $7b + 9b - 11b + 3b = 32$

5. $16r + 13r - 9r - 11r = 24$

6. $\frac{1}{2}r + \frac{2}{3}r - \frac{1}{4}r = 2$

7. $4a + 3a + .25a - 5a = 10$

8. $3x + .7x + 3.25x - 6x = 27$

9. $x - .4x + 7.2x + 5.6x = 29.7$

10. $7b + \frac{1}{2}b + \frac{3}{4}b - \frac{1}{8}b = 48$

11. $2r + 7r + 4r - 3r - 2r = 17$

12. $3k + k + 7k - 11k + 2k = 14$

Solve the following list of problems which deal with percentage and interest. In each case form an equation by applying the symbolic rule which you have previously stated. Solve the equation thus formed and check the result by trying it in the problem.

1. What sum of money must you invest in an enterprise yielding 25% profit in order that the amount may be $1000?

2. Water in freezing expands 10% of its volume. How much water when frozen will just fill a five gallon freezer?

3. How much must you ask for a house, if you wish to receive $4000 after the agent's commission of 3 1/2% is deducted?

4. An agent received $945 with which to buy lumber after deducting his commission of 5% on the price of the lumber. How much was his commission?

5. A merchant received $918 with which to buy corn after deducting his commission of 2% on the price of the corn. What was his commission and how much was used to purchase corn?

6. What must be the sale price of a piece of land in order that, after deducting an agent's commission of 4%, the seller may receive $1200?

7. A broker sold stocks for $1728 and remitted $1693.44 to his principal. What was the rate of his commission?

8. How much must I remit to my broker in order that he may buy $600 worth of bonds for me and reserve 5% for his commission?

9. Wheat loses 18% of its weight in being ground into flour. There 60 pounds of wheat in a bushel and 196 pounds of flour in a barrel. How many pounds of wheat are required for 75 barrels of flour?

10. A father invests $1000 at 6 1/2% interest. When the principal and interest amounted to $2235, he gave it to his son for his college education. How long had the money been invested?

11. Six years ago a man invested $470 at simple interest. The amount at present is $611. What is the rate per cent?

To sum up the work of the previous pages; we have learned that rules which we are compelled to use in solving problems may be more convenient for use if they are expressed by using letters and other symbols for abbreviations. We are able to not only express the rules more briefly and definitely but we are often able to combine in one, two or more rules as was done in the case of percentage.

The statement which we obtain when we apply a rule stated in symbolic form to a problem, we call an equation. The equation is thought of not only as stating a fact but also as asking a question. The process by means of which we obtain the required result to the equation, or answer to the question, is called solving the equation. The result is sometimes spoken of as the root of the equation or the value of the unknown.

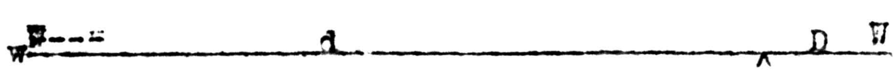
In expressions such as 7b, 7 which is written immediately before the b is called the coefficient of b. It tells how many b's are being considered.

Practice reading the problems on page 7 and tell what numbers are coefficients.

How great a weight can you balance with a lever by bearing down with your weight of 120 pounds, if you use a lever 10 feet long and the fulcrum or turning point is 1 foot from the weight?

The teeter-board is a very simple form of a lever and one with which you are probably acquainted. You will recall or can easily find out by trying it, that the farther you are from the fulcrum or turning,point, the greater the weight you can raise. Also, two boys on one end of a teeter can raise more than one is able to raise. Evidently there are two quantities here which determine the turning effort (moment) and hence directly concern us, i.e. the weight and the distance of the weight from the fulcrum. As to just how these two quantities determine the turning effect or moment, we shall have to take the statement from mechanics in which we are told that when a force, such as your weight, tends to cause a rigid bar, such as a lever or teeter, to turn about a fixed point and the bar just balances, the product of the forces tending to cause the bar to rotate in one direction by its distance from the fulcrum is equal to the product of the force tending to cause the bar to rotate in the opposite direction by its distance from the fulcrum. This means in the case of a teeter-board, that your weight multiplied by your distance from the fulcrum is equal to the product of the other boy's weight times his distance from the fulcrum.

How might you express the above statement in terms of symbols? We may choose our abbreviations as represented by the following diagram:



Using these symbols, our rule becomes

$$wd = WD$$

Comparing our abbreviations with the problem proposed, we
f
find that W is to be found,

D is given as 1 foot,

w is given as 120 pounds,

d is given as 10 feet minus 1 foot, or 9 feet.

Hence our equation is $1\ W = 9 \times 120$

or $W = 1080$ pounds.

Checking, 1 times 1080 = 9 times 120

1080 = 1080.

Therefore we may say that our result is correct.

Solve the following list of problems which come under the rule which we have just expressed in symbolic form. Check all results by trying them in the problem.

1. What weight can a man who weighs 160 pounds, raise with a lever 5 feet long if he can place his fulcrum 6 inches from the weight?

2. One boy, who weighs 75 pounds, sits 7 feet from the fulcrum, and a second boy, who weighs 105 pounds, sits on the other side. At what distance from the fulcrum should the second boy sit in order to make the teeter just balance?

3. A weight rests on a crowbar 6 feet in length, one end of which acts as a fulcrum and the other is supported by a spring balance which reads 42 pounds. If the weight is 6 inches from the fulcrum, how great is it? (The arrangement is illustrated by the following figure.)

4. After having solved the above problems can you suggest how you might determine the weight of a steel beam, which weighs over 500 pounds, with a spring balance which does not read to over 50 pounds.

Can you suggest more than one method? Illustrate your plans by drawings.

5. A suction pump is a device for raising water from wells. The handle works against a pin so that the longer end of the handle is pushed down, the shorter end is raised and the water with it. How many pounds of water can you raise if you press down 20 pounds on the outer end of the handle. The handle is 3 feet long and the pin is 4 inches from the shorter end.

6. What is the difference between saying that a force just balances a weight and saying that a force raises a weight? Will the rule which we have been using hold in a problem such as the one above?

7. A wheelbarrow is loaded with 45 bricks averaging 6 pounds apiece. What lifting force will be necessary at the handles if it is 4 1/2 feet from the center of the wheel to the end of the handles and the center of the load is 2 feet from the center of the wheel?

8. A stone slab weighing 2400 pounds rests with its edge on a crowbar 6 inches from the fulcrum. If the crowbar is 6 feet long, how many pounds must a man lift at the other end in order to raise the stone?

9. John and Roy weigh 100 and 110 pounds respectively. John places a stone on the board with him so that they balance when he is 6 feet from the fulcrum and Roy is 7 feet from the fulcrum. How much does the stone weigh?

10. Henry and James are 5 and 7 feet respectively from the fulcrum and the teeter balances. If Henry weighs 75 pounds, how much does James weigh?

11. Suppose you have a weight of 500 pounds to raise by means of a lever 6 feet long. Suggest some combinations of the length of the lever and the force you must apply to raise the weight. Will any combination suffice to raise the weight? Why? What condition must be satisfied by the combination in order that the weight of 500 pounds may be raised?

1. If sound travels 1080 feet per second, how far does it travel in 15 seconds?

In this problem two of the quantities mentioned are evidently, "1080 feet per second" and "15 seconds". Is a third implied in "how far"? The quantity represented by "1080 feet per second" is called velocity, speed, or rate. "15 seconds" is called the time and "how far" is called the distance. Does a relation exist between these quantities? If so, what is it? Express this relation in terms of symbols, explaining what each symbol represents.

Your symbolic statement will probably be

$$d = vt$$

What does this equation say? Upon what two quantities does the distance depend? Can you explain the nature of this dependence? Upon what does the time depend? The velocity? Illustrate.

2. If a transcontinental train averages 35 miles per hour, how far does it travel in 2 1/2 days?

3. Light travels at a speed of approximately 186,000 miles per second. How long will it take light to travel 240,000 miles? (This is approximately the distance from the earth to the moon.)

4. If sound travels 1080 feet per second, how many seconds will elapse between the time the flash of a gun is observed and the report is heard if it is a mile distant?

5. What is the speed of an automobile if it covers 78 miles in 5 hours? If it covers 108 miles in 9 hours? In 11 hours? In 7 1/2 hours? how does the speed change with the time required?

6. If you have a journey of 45 miles to make, how many hours will it take if you travel at a rate of 5 miles per hour? 7 miles per hour? 15 miles per hour? As you increase your speed how does the time required change?

7. In the following problems we shall consider the velocity of a bullet to be constant and the velocity of sound to be 1100 feet per second.

Two and one half seconds after a marksman fires his rifle he hears the bullet strike the target which is 550 yards distant. Find the velocity of the bullet.

8. One and three fourths seconds after a marksman fires his rifle he hears the bullet strike the target 50 rods distant. Find the velocity of the bullet.

9. A marksman fires at a target 1000 yards distant. The bullet passes over a boy who hears the sound of it striking the target and the report of the gun at the same instant. The velocity of the bullet is 1650 feet per second. Find the distance of the b boy from the target.

10. If you and another person whom we may call Mr. Smith, engage in business, you furnishing \$50 and Mr. Smith \$25, how will you divide a profit of \$30?

What quantities are there in this problem? What is their relation? State this relation in English and then in terms of symbols. See if your statement makes a rule which will work in the following problems.

11. Two partners have invested in an enterprise \$3000 and \$2000 respectively. How should a profit of \$360 be divided? A profit of \$850?

12. One partner has three times as much invested as the other. How should a profit of \$7000 be divided? Upon what does a partner's share of the profits depend?

13. A boy wishes to cut a border for a flower bed from a board 14 feet long so as to have no waste. If the length is to be twice the width, how must he cut it?

14. A farmer wishes to enclose a field with 80 rods of wire fencing. He wishes the field to be three times as long as it is wide. What must be its dimensions so that he may just use 80 rods of fencing?

15. The sum of the three angles of a triangle is 180°. If the three angles are equal, how many degrees in each? If one angle is 90° and the second is double the third, find the number of degrees in each.

16. A flag pole 60 feet high was broken so that the part broken off was two times the part left standing. Find the length of the two parts.

17. Divide a pole 20 feet long into two parts so that one part shall be four times as long as the other.

18. If a tennis ball rebounds to 2/3 of the hieght from which it was dropped, from what height was it dropped if the rebound was 5 feet? If the height of its second rebound was 4 1/2 feet?

Chapter II.

Supose you weigh 130 pounds and wish to raise a weight of 700 pounds by applying your weight at the end of a lever 7 feet long. How near the other end must you place the fulcrum so that your weight may just balance the 700 pounds?

You will recall having solved problems involving levers, but this problems you will find differs from those which we have previously solved, in that instead of one of the weights being unknown the distances of the weights from the turning point or fulcrum are not known.

Using our usual method of choosing abbreviations for the distances from the weights to the fulcrum, we represent the distance of the weight to be raised from the fulcrum by D, and your distance from the fulcrum by d. But since the problem states that the entire length of the lever is 7 feet, d must be equal to 7 - D. We usually write this thus:

If we represent the distance of the weight to be raised from the fulorum by D, then 7 - D must represent your distance from the fulcrum.

Proceeding as in former problems we get from the conditions of the problem,

$$700\ D = 130 \text{ times } 7 - D.$$

The marks ( ) are called parentheses and may be used to indicate that the same operation is to be performed on several numbers or symbols. We may then write

$$700\ D = 130(7 - D).$$

Performing the indicated process, the equation becomes

700 D = 910 - 130 D

In what respect does this equation differ from those which we have previously solved? You will notice that there is a D on each side of the equality sign. Some method must be found by means of which we can get both on the same side or we cannot use what we have already learned about solving equations. Suppose we add 130 D to both sides of the equation.

$$\begin{array}{l} 700\ D = 910 - 130\ D \\ \underline{130\ D = 130\ D \qquad\qquad} \\ 830\ D = 910 + 130\ D - 130\ D \end{array}$$

or

$$830\ D = 910$$

We are now able to finish the problem by our usual method. Doing so $D = \frac{910}{830} = 1.1$ feet the distance of the weight from the fulcrum. Your distance from the fulcrum is 7 - 1.1 or 5.9 feet.

To check our result we try it in the problem.

1.1 times 700 = 770

5.9 times 130 = 767

These products are not exactly equal but the result 1.1 feet was not exact, so we should not expect the products to be exactly equal. Therefore we may say that the result is sufficiently correct.

A moments consideration will show that the method we used for removing the D from the right hand side of the equation is a legitimate operation. For if the two expressions 700 D and 910 - 130 D are equal, the equality will certainly not be destroyed by increasing each of the expressions by the same amount.

Solve and check each of these equations.

1. $8x = 12(16 - 4x)$

2. $5x + 3 = 3(2 + 3x) + 25$

3. $57x = 9(24 - x)$

4. $41x = 23(14 - 3x)$

5. $7x - 12 = 3(9 - 2x)$

6. $5x + 21 = 72 - x$

7. $2(a - 7) = 11 + 5(24 - a)$

8. $3(8 - 5p) = 23 + 7(2 - 4p)$

9. $2(3y + 6) - 15 = 24 + 2(y + 3)$

10. $5 + 6x - 7 = 21 + 3(x - 5)$

11. $10x + 5 - 7x = 17 + 6(21 - 2x)$

12. $\frac{1}{2}x = \frac{3}{2}(10 - 3x)$

13. $.3x + 5 = .5(30 - 7x) + 12$

14. $\frac{1}{3}x = \frac{1}{2}(12 - x)$

15. $3x - .4 = .2(8 - .3x)$

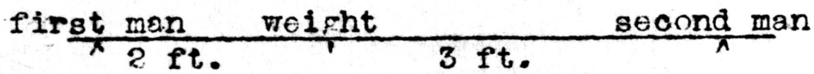
16. $\frac{1}{4}x + 3 = \frac{1}{2}(5 - 3x)$

17. Suppose you weigh 150 pounds and wish to raise a weight of 500 pounds by applying your weight to a lever 8 feet long. How near the weight must you place the fulcrum if you are to just balance the weight on the lever? At least how near the weight must the fulcrum be if you are to raise the weight?

18. Where would you have to place the fulcrum if you raise a weight of 200 pounds with the lever of the above problem? 400 pounds? 800 pounds? What would happen in the first case if you placed the fulcrum 1 foot from the 500 pound weight? 6 inches from the weight? 2 feet from the weight?

19. Two men are carrying a weight of 250 pounds which is suspended from a pole at a point 2 feet from one man. If the men are 5 feet apart, how many pounds is each man carrying?

In this problem, instead of wishing to find the weight or force which must be placed at one end of the lever in order to raise a given weight at the other end, we have the given weight placed at some point on the lever and supported by forces applied at the ends of the lever. It is our problem to find the manitude of these forces. The method of solving is very much the same as in previous problems. We may take one of the points of support as the turning point or fulcrum, and consider the force at the other point of support and the weight as the active forces.



Refering to the figure, we may take the point of support at the first man as the fulcrum. The distance from the fulcrum to the weight is two feet and the length of the pole is 5 feet. If we represent the force which the second man is supporting by F we have as our equation

$$5F = 250 \times 2$$

$$5F = 500$$

$$F = 100$$

If the second man supports 100 pounds, what does the other man support? What effect would it have on the number of pounds that each supports if the weight were moved to the left? To the right? If the pole were 10 feet long and the weight were 4 feet from the first man?

In solving such problems as the one above, it is often of great advantage to draw a rough diagram and mark upon it the distances and weights of the particular problem. Some care should be taken to have a unit of representation; thus if this line

represents 2 feet, then this line

will represent 3 feet. The line  is called the unit length and represents 1 foot. Such a diagram will always serve as a rough check upon your work and often is suggestive of the method of solution in a difficult problem. In solving problems in the future, draw a rough diagram whenever the problem will permit.

20. If two boys are carrying a weight of 90 pounds suspended from a pole 6 feet long and one boy is supporting 55 pounds, how many pounds is the other supporting? If one supports 48 pounds?

21. A basket weighing 50 pounds is carried by two boys who lift at the ends of a light stick. How much does each boy lift if the basket is at the middle point? Is the length of the pole concerned? If so, how?

22. A basket weighing 84 pounds, hangs on a stick 8 feet long at a point 2 feet from one end. It is carried by two boys, one at each end of the stick. How much does each boy lift?

23. A basket weighing 90 pounds hangs on a pole 6 feet long at a point 2 feet from the middle. If it is carried by two boys, one at each end of the pole, how much does each lift?

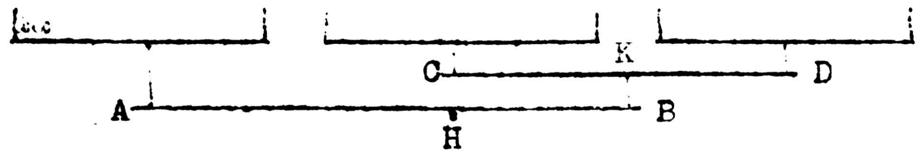
24. Two men lifting at the ends of a pole 8 feet long raise a certain weight. What is the weight, and at what point does it hang if one man lifts 25 pounds and the other 85 pounds?

25. A beam is 12 feet long. It carries a 40 pound weight at one end and a 60 pound weight at the other end. Where is the fulcrum if the beam balances?

26. Weights of 10 and 12 pounds are fastened to the ends of a 10 foot pole. At what point must the pole be supported in order that it may balance?

27. A beam is 16 feet long. At what point is it to be supported if it is to carry, when balanced, 460 pounds at one end and 690 pounds at the other?

28. When a farmer uses three horses, he often uses a three-horse double-tree of the type illustrated by the following diagram.

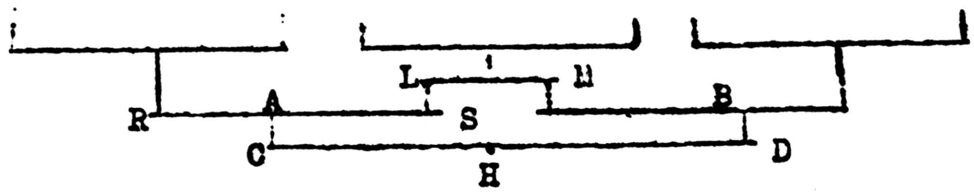


Where should the hole H be bored if it is 5 feet from A to B? We assume that each horse pulls an equal amount.

29. If in the above problem the total pull of the three horses is 4500 pounds, what effect would moving H 2 inches to the right have upon the individual horses? If H was moved 2 inches to the left? 4 inches to the left.

30. In planting corn with a lister it is necessary to have the horses walk 3 feet, 6 inches apart. Draw a diagram similar to the above showing a double-tree for three horses and determine the dimensions of the double-trees AB and CD and where the holes H and K should be bored if the holes A, B, C, and D are 3 inches from the ends of the double-tree.

31. Another style of a three-horse double-tree sometimes used is shown by the following diagram.



All and holes are 1 1/2 inches from the end. If LM is 10 inches long and RS is 2 feet, 3 1/2 inches, where should the hole A be bored? What will be the length of CD? Where should the hole H be bored? Assume that the pull per horse is 1200 pounds.

32. A beam is 12 feet long. It carries a 40 pound weight at one end, a 60 pound weight 3 feet from this end, and a 70 pound weight at the other end. Where is the filcrum if the beam balances?

In this problem we find three weights but the same rule applies. The sum of the products of the weights on one side of the fulcrum and their respective distances from the fulcrum is equal to the sums of the products of the weights on the other side of the fulcrum by their respective distances from the fulcrum.

33. A beam carries a weight of 240 pounds 7 1/2 feet from the fulcrum and a weight of 265 pounds at the other end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

34. Four boys wish to sit on a teeter. If two of the boys weighing 75 and 90 pounds sit on the same side of the fulcrum at distances of 3 and 5 feet respectively, and another boy weighing 82 pounds sits on the other side of the fulcrum at a distance of 3 feet from it, where should the fourth boy who weighs 100 pounds sit?

1. A beam of uniform thickness and weighing 100 pounds is used as a lever in raising a building. If the beam is 16 feet long and the fulcrum is placed 1 foot from the building and a force of 450 pounds applied at the other end, what weight is raided?

This problem introduces a new factor, the weight of the lever or beam. Consider a uniform bar 6 feet in length and weighing 20 pounds. If the fulcrum is applied at the middle point we would find that the beam balanced, which means that the fulcrum will support the lever without the application of an additional force. Would the lever still balance if it were 10 feet long and the support were kept at the middle point? If it were 2 feet long? It appears, then that the length of the lever, of uniform thickness, does not determine whether it will balance or not provided the fulcrum is placed at its middle point. We may then for the purpose of solving problems consider the weight of the lever to be concentrated at its middle point.

2. Two men carry an iron bar 12 feet long which weighs 7 pounds per foot. What weight does each man carry if they are at the ends of the bar? If a weight of 90 pounds is placed at the middle point of the bar, what weight does each carry?

3. Suppose a uniform bar weighing 30 pounds, and 10 feet long, is used by two men, one at each end, to carry a load of 170 pounds. How man pounds must each carry if the load is attached 2 feet from the left end?

4. A steel rail 30 feet long is supported at a point 14 feet from one end. A lifting force of 45 pounds at the other end holds the rail in place. What is the weight of the rail? The weight per yard of length?

5. Three boys desire to carry a log 12 feet long and weighing 240 pounds. Two of the boys lift at the ends of a hand spike placed cross wise underneath the log and the third boy carries the rear end of the log. Where must the hand spike be placed so that all may lift equally?

6. A bar weighing 7 pounds per lineal foot rests on a fulcrum 3 feet from one end. What must be its length, that a weight of 71 1/2 pounds suspended from that end may balance a 20 pound weight suspended from the other end? What is the pressure on the fulcrum?

7. A steel beam 24 feet long and weighing 120 pounds to the yard is being moved by placing an axle borne by a pair of wheels under it. How far from the end must the axle be placed in order that the weight supported at the other end may be 200 pounds? What will be the weight upon the axle?

8. How may a railroad rail 24 feet long and weighing more than a ton be weighed with a spring balance capable of weighing only 60 pounds? Refer to problem 4, page 11. Is the method which you devise here the same as the one you gave there? If not compare them.

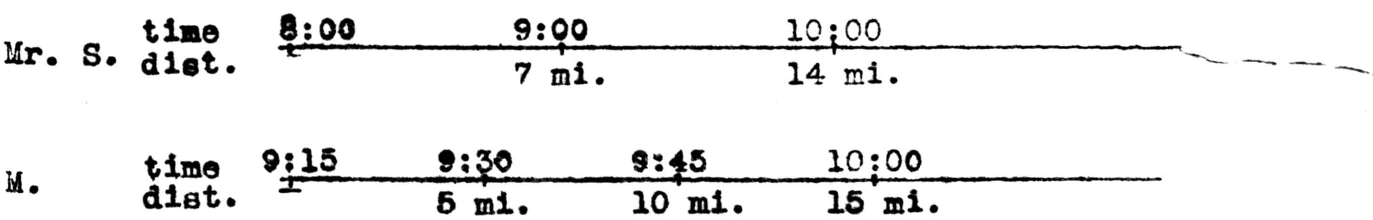
1. Mr. Smith leaves his home for the city at 8 o'clock driving a team which will make 7 miles an hour. At 9:15 important business arises which demands that he be reached and return home as soon as possible. A messenger in a motor car is sent after him traveling at a rate of 20 miles an hour. At what time and how far from his home will the messenger overtake Mr. Smith?

In the above problem, what are the quantities which determine the problem? What is the relation between the distance traveled (distance), and the number of hours (time) and the rate per hour (velocity)?

If we use abbreviations, this relation may be represented thus:

$$d = vt$$

We may draw a picture of this problem as follows:



Can you guess from this picture about where the messenger will overtake Mr. Smith?

2. A man with a loaded wagon leaves Columbia for Fulton and 1 hour and 20 minutes afterwards a second man departs. If the first travels 6 miles per hour and the second 15 miles per hour, how far from Columbia will the second overtake the first?

3. A freight train leaves Chicago for St. Paul at 11 a.m. At 3 and 5 p.m. of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 p.m. of the same day and the other which runs 10 miles per hour slower, at 3 a.m. of the next day. What is the speed of each trahn?

4. A camping party sends a messenger with mail to the nearest postoffice at 5 a.m. At 8 a.m. another messenger is sent out to overtake the first which he does in 2 1/2 hours. If the second messenger travels 5 miles an hour faster than the first, what is the rate of each?

5. Two trains start west over the same road at the same time, one from New York and the other from Philadelphia. If the New York train runs 65 miles per hour and the Philadelphia train 47 miles per hour, how long before they are 15 miles apart, the distance from New York to Philadelphia being 90 miles? How long before they are together?

6. An ocean liner making 21 knots an hour leaves a port when a freight boat, making 8 knots an hour, is already 1240 knots out. In how many hours will the liner overtake the freight?

7. A motor boat starts 7 2/3 miles behind a sail boat and runs 11 miles per hour, while the sail boat makes 6 1/2 miles per hour. How long will it require for the motor boat to overtake the sail boat?

8. A fleet making 11 knots per hour, 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

9. In an automobile race, one contestant drives his machine at an average rate of 53 miles per hour, while a second contestant, who starts 1/4 hour later, averages 57 miles per hour. How long before the second overtakes the first?

10. A fast freight leaves Chicago for New York at 8;30 a.m. averaging 32 miles per hour. At 2:30 p.m. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

11. An ocean liner making 21 knots an hour leaves port when a freight boat making 10 knots an hour is 960 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one answer? When will they be together?

12. A freight train traveling 20 miles per hour and a passenger train traveling 35 miles per hour are both going from A to B. If the freight starts 2 1/2 hours before the passenger, how far from A must the freight allow the passenger to pass?

13. A man walking travels from his home towards town at a rate of 5 miles per hour and a second man riding travels from town at the rate of 11 miles per hour. If the man's home is 19 miles from town and they both start at the same time, where will they meet?

14. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the two terminals are 500 miles apart, in how many hours will they meet? If they meet in 8 hours how long is the road?

15. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet?

16. A steamer leaves Liverpool for New York on Saturday at 9 a.m. averaging 18 knots per hour. Seven hours later another steamer leaves New York for Liverpool, making 20 1/2 knots per hour. What time (Liverpool time) will the steamers meet if the trans-Atlantic distance by their course is 2940 knots?

17. A beam of uniform thickness weighing 10 pounds is supported at the end by two props. A weight of 31 pounds hangs 4 feet from one end. Find the length of the beam and where a weight must be placed that the pressure on the two props may be 18 and 25 pounds respectively.

18. A sign board 4 feet long hangs in a horizontal position by hooks at its ends to an iron rod weighing 14 pounds. The rod is supported at its ends. If the sign board weighs 32 pounds how far from the ends of the rod must the hooks be placed that the pressure at the ends of the rod may be 12 and 34 pounds?

19. An iron bar of uniform thickness 10 feet long and weighing 1 1/2 cwt. is supported at its extremities in a horizontal position, and carries a weight of 400 pounds suspended from a point 3 feet from one end. Find the pressure on the points of support.

20. A loaded wagon standing on a bridge, has a load of 2500 pounds on its from axle and a load of 3000 pounds on its rear axle. The fron wheels are 4 feet and the rear wheels 10 feet from the left end of the bridge. If the bridge is 20 feet long, what is the pressure on its supports due to the wagon?

21. In the above problem what is the pressure on the supports when the wagon is in the middle of the bridge? When the rear wheels are 2 feet and the front wheels 8 feet from the right end of the bridge? When the rear wheels are not on the bridge and the front wheels are 5 feet from the end? Describe the change in pressure on the supports of the bridge as the wagon is driven over it from right to left.

22. A lumber wagon is coupled out to a distance of 9 feet between the axles and loaded with a pile of lumber 3 1/2x4x18. The load extends 3 feet in front of the front axle and the material averages 48 pounds per cubic foot. What is the pressure on the axles due to the load?

23. The water tank of a park sprinkler is a circular cylinder 51 inches in diameter and 9 feet in length. A driver weighing 180 pounds sits directly over the front axle. The center of the tank is 3 feet in front of the rear axle. The axles are 6 feet apart. If water weighs 62 1/2 pounds per cubic foot, what is the weight on each axle when the tank is full and the driver is on his seat? When the tank is half full?

24. The box of a three horse wagon is 12 feet long and is loaded with 6 tons of coal. If the box eztends three feet in front of the front axle and 4 feet back of the rear axle what are the weights upon the front and rear axles?

25. A foot bridge is 15 feet long and is supported at the ands. The bridge weighs 1000 pounds. Two men weighing 150 pounds each stand upon the bridge 5 feet from the end. Find the pressure upon the supports at the ends.

26. In the above problem frind the pressure upon the supports if one man is 5 feet from the left end and the other 7 feet from the right end. Describe the change in pressure upon the supports as they walk together across the bridge from right to left. Describe the change in pressure if the start from opposite ends and meet in the middle.

27. A wagon box ten feet long is loaded with 50 bushels of wheat weighing 60 pounds per bushel, It extends 1 1/2 feet in front of the front axle and 2 feet behind the rear axle. What is the load on each axle?

28. A bridge 20 feet long weighs 2400 pounds and supports two loads; one of 600 pounds 4 feet from the left end, and the other 800 pounds, 15 feet from the left end. What are the loads borne by the supports?

29. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how may minutes will they again be together, if they start at the same place and time and in the same direction?

30. The planet Mercury makes a circuit around the sun in 3 months and Venus in 7 1/2 months. Starting in conjunction, how long before they will again be in this position?

31. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together, in how many seconds will they be together again? Solve for the two cases, when they go in the same direction and in opposite directions.

32. Supose an officer sees a fugitive a block ahead of him and starts in pursuit. The fugitive goes around the block and it takes him five minutes to return to his starting place. If it takes the oficer 3 1/2 minutes to make the round trip, how many block must he travel before he catches his man?

33. At what time between 7 and 8 o'clock are the hour and minute hands of a clock together?

34. The earth and Mars were in conjunction July 12, 1907. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days?

35. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in 38 1/2 minutes. In what time will the faster machine gain one lap on the slower?

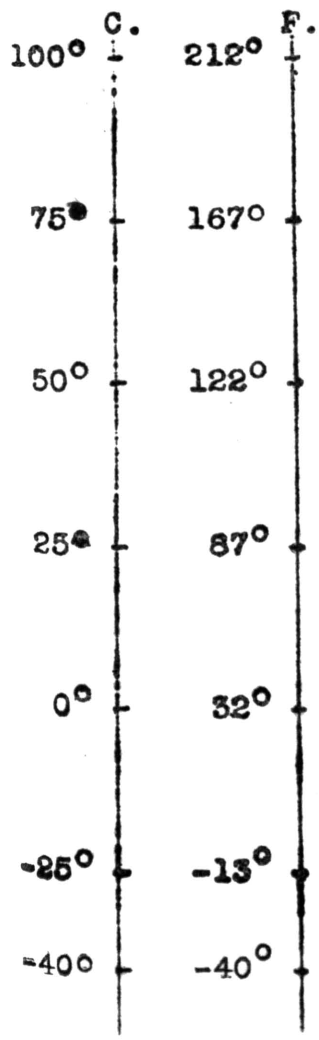
36. The length of a freight train is 1540 feet and the length of a passenger train 660 feet. When they run on parallel tracks in opposite directions they pass each other in in 20 seconds and when they run in the same direction they pass each other in 1 minute and 40 seconds. Find the rates of the trains.

37. Two bicyvlists travel in opposite directions around a quarter-mile track and meet every 22 seconds. When they travel in the same direction the faster passes the slower one every 3 minutes and 40 seconds. Find the rate of each rider.

38. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the avergae velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find the distance to the target and the length of time the bullet was in the air.

39. A gunner using one of the best modern rifles would hear the projectile strike the target 2640 yoards distant in 9 2/5 seconds after the report of his gun, provided the projectile maintained throughout its flight the same velocity as it had on leaving the gun. Find this velocity if the sound travels 1100 feet per second.

We measure temperature by means of an instrument called a thermometer, which is a vertical glass tube containing a liquid. The liquid rises or falls in the tube according as it becomes warmer or colder. Instead of measuring the height of the liquid in inches or metric units, arbitrary scales have been chosen, the unit of which is called a degree. The scale in common usage is called the Fahrenheit. It is characterized by having 32° as the temperature at which water freezes and 212° as the temperature of boiling water. In laboratories and in all scientific work a different scale called the Centigrade is used. On this scale the freezing temperature is marked 0° and the boiling temperature, 100°.



The accompanying diagram illustrates roughly the two scales.

Make a careful drawing on a somewhat larger scale, marking every 10° on the scale. As in this diagram, place the boiling and f freezing points on a level. Compaye your drawings with the scale on an actual thermometer.

How many degrees are there between the freezing and boiling points on the Fahrenheit scale? On the Centigrade scale? 1° Centigrade equals how many degrees Fahrenheit? What is 3° above freezing on the Fahrenheit scale? 15°? How many degrees above freezing is 60° F?

A temperature 95° F. is generally spoken of as quite warm. What would this temperature be in Centigrade degrees?

Can you state a rule for finding the temperature on the Centigrade scale when you have the temperature given on the Fahrenheit scale? After you have decided upon a rule, state it in terms of symbols. By means of this rule solve the following problems.

1. The normal temperature of a person is 98.4° F. What would this be on the Centigrade scale?

2. The usual temperature of a room is about 70° F. What is this temperature on the Centigrade scale?

3. Change to Fahrenheit the following temperatures given in Centigrade degrees; 40°, 23°, 65°.

4. The hourly readings of the thermometer at Kansas City on July 28, 1910 are given by the following table. Change these into Centigrade readings.

7 a.m.	82	5 p.m.	101
8 a.m.	85	6 p.m.	99
9 a.m.	88	7 p.m.	98
10 a.m.	91	8 p.m.	94
11 a.m.	94	9 p.m.	91
12 noon	95	10 p.m.	90
1 p.m.	98	11 p.m.	90
2 p.m.	100	12 midnight	88
3 p.m.	101	1. a.m.	87
4 p.m.	101	2 a.m.	86

5. Find the temperature Centigrade corresponding to 14° F.

Proceeding as in other problems, you have

$$C= \frac{5}{9}(14 - 32).$$

This presents a difficulty for we are accustomed to say that a larger number cannot be subtracted from a smaller, as in this case we have 32 to subtract from 14. Certainly there is a temperature 14° F. and there must be a temperature Centigrade which corresponds to it and hence it must be possible to solve this problem. To find this temperature we must devise some way to solve 14 - 32. We may do this by subtracting 14 from 32 and writing the result as -18 which we read minus 18. Taking $\frac{5}{9}$ of 18 and keeping the minus sign we have

$$C = -10^{\circ}.$$

What does this result mean? Locate this temperature on the diagram which you made. Does it correspond on the diagram to 14° F? When we speak of temperatures above zero, we say simply 10°, 65°, 91°, etc. The only other kinds of temperature which we have are temperatures below zero and we shall find it very convenient to refer to them as minus or negative temperatures. As you will observe from your diagram, the temperature which corresponds to 14° F. is below zero on the Centigrade scale.

Examine a weather map and note how temperatures below zero are represented on it. (See almost any Physical Geography.)

6. Write 20° below zero; 12° below; 25° above.

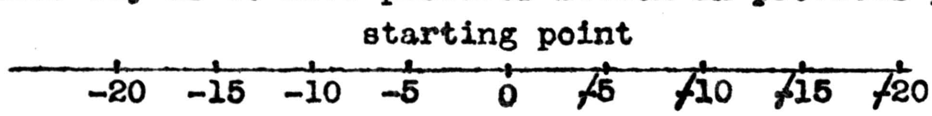
7. How could you write an overdraft of $62? A loss of $40?

8. How could you express the weight of a ballon? Can you think of other uses of negative numbers?

9. When the temperature rises 10° we say the change is 10°. How could we express the change when it falls 5°? 24°?

10. If we speak of a person who is walking due east as having gone +12 miles, how would we speak of a person who is traveling due west and who has gone 15 miles?

The conditions of the last problem may be pictured in much the same way as we have pictured motion in previous problems.



We call such numbers as these negative numbers. Thus we distinguish them from the numbers which we ordinarily use and which we call positive numbers. We see that negative numbers are necessary if we are to solve all equations and that they have a meaning. We have used them to represent temperatures below zero, an overdraft, a loss, etc. In other problems we may find that they have still other meanings.

Solve the following problems and check the results.

1. $5x + 8 = 4x + 5$
2. $7(5 - 2x) = 2(12x - 4)$
3. $3(7 - 6r) + 10 = 11r + 2(5r - 6)$
4. $4(5k + 1) + 3k = 21 + 5k$
5. $7 + 25k - 2(1 + 7k) = 19$
6. $2(a + 3) - 7a = 10 + 13a$
7. $4(2x - 20) = 2$
8. $5(3x - 1) - 6(2x + 5) = 21$
9. $3(2x + 3) - 2(3x + 2) = 7(1 - 2x)$
10. $5(2x - 1) + 2(2x + 1) = 15 - 3x$

If we may thus attach a real meaning to negative numbers and find them necessary for the solution of equations, we are then justified in learning how to use them. The operations of addition, subtraction, multiplication, and division have been defined only for positive numbers. We must now define these operations for negative numbers.

We have, in the preceding problems, used negative numbers very much as if they were positive. For instance when we multiplied -18 by $\frac{5}{9}$ we simply multiplied 18 by $\frac{5}{9}$ and gave the result a minus sign. So we have at once our rule for the multiplication of a negative by a positive or the reverse. Since division is the opposite of multiplication, we would expect a similar rule to hold for the division of a negative number by a positive or a positive number by a negative.

Multiply:

1.	-13 $+5$	-17 13	6 -8	$6a$ -5
2.	$-16ax$ $5b$	$-7ax$ 14	-23 45	-492 63

Divide:

3. -15 by 5; -72 by 8; 64 by -4.

4. -125 by 25; 216 by -6; 1728 by -24.

Simplify these fractions which are really problems in division.

5. $\frac{12}{-3}$; $\frac{-54}{9}$; $\frac{360}{-9}$; $\frac{-72}{24}$;

6. A man travels on successive days, 20 miles, 15 miles, 17 miles, 16 miles all in the same direction. How far is he from his starting point? Picture his movements. (See p. 30, prob. 10)

7. A man travels 25 miles west, 18 miles west, 15 miles west. How far is he from his starting point? Picture his movements.

8. If the temperature is -3° and it falls 11°, what is the temperature then? Express this problem symbolically?

Can you now state a rule for the addition of negative numbers?

9. What will be the total wealth of a man who has $1500 in the banl and owes $1200, if by a transfer of property he cancels a debt of $750?

10. A weight of 30 pounds is attached to three small balloons which pull up respectively 10 pounds, 15 pounds, and 20 pounds. What is the upward pull of the whole device? Consider the downward force to be positive.

11. A vessel capable of making 10 miles per hour is opposed by a current of six miles an hour. By hoisting sails use is made of the wind blowing in the same direction at 8 miles an hour. What is the speed at which the vessel moves? Express in terms of positive and negative numbers.

12. A vessel is making use of both sails and steam; the latter will alone propel the boat at 15 miles an hour, and the wind opposes the motion at the rate of 10 miles an hour, and an opposing current is flowing at the rate of 2 miles an hour. What is the speed of the vessel? What is the speed when the sails are lowered?

13. Add -5 and -8; -7 and -12.
-155 and -24; -3 and -5 and -7.

14. Add -7 and -6; -3 - 2 - 7 - 27.

If the problem contains both positive and negative numbers, they are added separately and the smaller sum subtracted from the larger and the result given the sign of the larger sum.

Thus 18 - 5 - 7 + 1 - 13 + 2 = 21 - 25 = -4

and 5 - 12 - 7 + 8 - 4 + 10 + 2 = 25 - 23 = +2 or simply 2.

15. Simplify 27 - 6 - 22 + 10 - 7

16. - 16 + 5 - 24 + 51

The subtraction of negative numbers is indicated in the same manner as positive numbers. Thus 7 - 4 means 4 subtracted from 7 and if we write 7 - (-4) we mean -4 subtracted from 7. When we subtracted four from 7, we decreased 7 by 4. Since -4 is the opposite of 4 we would naturally expect to do the opposite thing to 7, that is increase 7 by 4. Hence

$$7 - (-4) = 7 + 4 = 11.$$

We express this rule by saying that two minus signs give a positive, or that in subtraction we change sign and add. The above problem might be written

$$\begin{array}{r} 7 \\ \underline{-4} \\ 11 \end{array}$$

To this form of stating the problem, the last statement of the rule is more applicable.

It should be noted that in the expression 7 - (-4) the first minus sign is used to indicate the operation of subtraction while the second is used to indicate a negative number. This distinction in use is slight but should always be recognized.

17.	Subtract	17x -24x	23x -11x	7x -3x	29x -5x
18.		-17 6	-24 22	-14 29	-34 51
19.		-6 -5	-7 -9	-18 -24	-35a 23a
20.		-65 46	-3x -7x	-12x -5x	-34ax -18ax

The product or quotient of a negative and a positive number has been defined as negative. (See pages 30 and 31.) The product or quotient of two negative numbers is defined as <u>positive</u>. Thus $(-2)(-13) = +26$; $-32 \div -4 = +8$.

21.	Multiply:	-15 -3	-17 -6	24 -3	-48 -24

22. Divide: -6 by -2; -108 by -6; -196 by -28.

23. Simplify: $\frac{-12}{-4}$; $\frac{-2}{-6}$; $\frac{-21}{-7}$; $\frac{-4}{-24}$;

24. Multiply: (15 - 17x)(-7)

25. (2a - 13)(4)

26. (27 - 14ax)(-12ab)

We collect the rules of multiplication here for reference.

$$+5 \times +3 = +15$$
$$+5 \times -3 = -15$$
$$-5 \times +3 = -15$$
$$-5 \times -3 = +15$$

1. What is the weight of a body whose density is .75 and whose volume is 372 c.cm.? What is its weight in water?

By density we mean that an object is that many times as heavy as an equal volume of water. The weight of one cubic centimeter (c.cm.) of water is one gram.

2. What is the weight of an object whose density is 4.3 and whose volume is 250 c.cm.?

3. What is the density of an object whose weight is 23.5 grams and whose volume is 17 c.cm.?

4. What is the volume of an object whose weight is 24 grams and whose density is .65?

5. What is the weight in water of 750 c.cm. of cork whose density is .24?

6. If 500 c.cm. of alcohol, density .79 is mixed with 300 c.cm. of distilled water, density 1., what is the density of the mixture?

7. If 1200 c.cm. of cork, density .24, are combined with 64 c.cm. of steel, density 7.8, what is the average density of the mass? Will it float or sink in water?

8. How many cubic centimeters of cork, density .24, must be combined with 75 c.cm. of steel, density 7.8, in order that the average density shall be equal to that of water, i.e. so that the combination will just float?

9. What is the weight in water of a boat which displaces 25 cubic feet of water if the boat itself weighs 500 pounds? A cubic foot of water weighs 62 1/2 pounds.

10. A hollow sphere weighs 25 pounds and has a volume of 2.5 feet. What is its weight when it is submerged in water?

11. Coinage silver is an alloy of copper and silver. How many cubic centimeters of copper, density 8.83, must be added to 10 c.cm. of silver, density 10.57, to form coinage silver whose density is 10.38?

12. Brass is an alloy of copper and zinc. How many cubic centimeters of zinc, density 6.86, must be combined with 100 c.cm. of copper, density 8.83 to form brass whose density is 8.31?

13. What is the average density of 40 c.cm. of water, density 1, and 180 c.cm. of alcohol, density .79?

14. The density of pure gold is 19.36 and that of nickel is 8.57. How many cubic centimeters of nickel must be combined with 10 c.cm. of pure gold to form 14 carat gold whose density is 14.88?

15. When 960 cubic centimeters of iron, density 7.3, is fastened to 8400 c.cm. of white pine the combination just floats, i.e. has a density of 1. What is the density of white pine?

16. A wholesale house wishes to blend, by mixing two grades of tea worth 30 and 50 cents per pound, a mixture which they can sell for 42 cents per pound. What quantities of each must they use so as to obtain 25 pounds of the mixture?

17. How many pounds of spice worth 20 and 50 cents per pound must be taken to form 12 pounds of a mixture worth 30 cents per pound?

18. How can a merchant mix ten pounds of tea worth 31 cents a pound out of tea worth 30 and 35 cents a pound?

19. A merchant mixes two grades of vinegar which cost him 55 and 65 cents a gallon. How much of each must he take to make a 100 gallon mixture which he can sell at 75 cents and make a profit of 20%?

20. How much salt must be added to a 10% solution to make 5 pounds of a 20% solution?

21. Spirits of camphor is camphor gum disolved in alcohol. How many ounces of alcohol must be added to 20 ounces of 15% solution to make a 8% solution?

22. How many quarts of water must be mixed with 250 quarts of alcohol 80% pure to make a mixture 75% pure?

23. How much water must be added to a 20% solution of ammonia to make a 5% solution? Use one gallon as the amount of 20% solution.

24. A chemist wishes to obtain 1000 c.cm. of 25% acid solution. How much acid must he use?

Chapter III.

What must be the side of a tool shed in the shape of a square if it is to have as many square feet of floor surface as a shed 16 by 25 feet?

The area of a rectangle is the product of its length by its width: stated in terms of symbols, it is $A = lw$.

The area of a square is the product of a side mulriplied by itself: or stated in terms of symbols, if we represent the length of a side by s, $A = ss$, or as it is usually written $A = s^2$.

Since the product of a number by itself gives the area of a square, the length of whose side is represented by this number, the product ss or s^2 is called the square of s or s square. The 2 which is written a little above and to the right of the s means that the s is multiplied by itself. The 2 or any number which is used for a similar purpose is called an exponent.

Returning to our problem, if we represent the side of the square by s, we have as our equation

$$s^2 = (16)(25)$$

or

$$s^2 = 400.$$

We may think of this equation as asking the question, What number multiplied by itself gives 400? Since we know that 2 multiplied by itself gives 4, would we not expect the answer to be 20? Or $s = 20$

To check, we try 20 in the problem,

$$(20)(20) = 400.$$

Therefore we can say that the result checks and that the side of the tool-shed must be 20 feet.

This process of finding a number which when multiplied by itself will give a definite number, is called extracting the square root. With reference to the equation we say, we extract the square root of both sides of the equation.

It should be noted that, although we have found 20 to be the square root of 400, $(-20)(-20) = 400$ and therefore -20 might equally as well be said to be the square root of 400 as 20.

Thus if $$s^2 = 400$$

$$s = 20 \quad \text{or} \quad s = -20$$

or as it is sometimes written

$s = \pm 20$, the sign $\pm$ being read plus or minus. Since to say that the side of a square was -20 feet is absurd, we reject this result. However, as we shall see later, page 59, the negative result sometimes has a meaning

1. Find the side of a square which contains the same number of square feet as a rectangle 4 by 9 feet. 8 by 18 feet.

2. Find the area of a square whose side is 17 feet; 19 feet; 7 feet, 6 inches; 80 rods; 120 rods; 3wp rods;

3. The area of a circle is $\frac{22}{7}$ times the square of the radius. Find the area of a circle whose radius is 9 feet; 15 feet; 27 feet; 8 feet, 3 inches.

4. Suppose you wish to increase the area of a flower bed by 144 square feet by adding an equal strip on all sides. If the flower bed measures 14 by 18 feet, how wide should the strip be?

If we represent the width of the strip by x then the new length will be $18 + 2x$, since the width of the strip must be counted at both ends. Similarly, the new width will be $14 + 2x$.

Area = length times width

Old area = (14)(18)

New area = $(14 + 2x)(18 + 2x)$

From problem $(14 + 2x)(18 + 2x) = (14)(18) + 144$

We must now learn to reduce the expression $(14 + 2x)(18 + 2x)$ to an expression which contains no parentheses and which we can handle by our previous methods or some method which we can devise. As it is written, the expression means that the quantity $14 + 2x$ is to be multiplied by $18 + 2x$. Hence we must learn how to multiply $14 + 2x$ by $18 + 2x$.

Suppose we try to accomplish this multiplication as if it were a problem in multiplication in Arithmetic. Writing one quantity under the other as is our custom and following the same rule:

$$\begin{array}{rrr} & 14 + 2x & \\ & \underline{18 + 2x} & \\ & 28x & + 4x^2 \\ \underline{252 + } & \underline{36x} & \\ 252 + & 64x & + 4x^2 \end{array}$$

Multiply first by 2x and then by 18 and add the partial products, preserving the proper signs.

Solve:

1. $(2x + 3)(3x + 5)$
2. $(7 + x)(5 + 2x)$
3. $(4 + 3x)(4 - 3x)$
4. $(a + x)(a + x)$
5. $(a + b)^2$
6. $(x + 2)^2$
7. $(3x - 1)^2$
8. $(3a + x)^2$
9. $(2x + 3)(2x - 3)$

10. $(21 + 2x)(15 + 2x)$

11. $(2a - 3x)^2$

12. $(4x - 5)^2$

Refering to the result of our multiplication on the preceding page, our equation now is

$$252 + 64x + 4x^2 = 252 + 144$$

transposing $\quad 4x^2 + 64x = 252 + 144 - 252$

simplifying $\quad 4x^2 + 64x = 144.$

We may further simplify this equation by dividing through by 4, which gives us $\quad x^2 + 16x = 36.$

Now that we have the equation simplified, we find that it differs from the forms which we know how to solve in that there are both a term in x^2 and x. Previous to this we have not had terms of both these types occuring in the same equation. We must now devise a method for solving an equation of this type. We shall hereafter refer to this type of equation as a quadratic equation.

As in the simpler equation of the type $Ax^2 = B$, we shall solve this equation by extracting the square root of both sides of the equation. But first our equation $x^2 + 16x = 36$ must be changed into a suitable form as will be shown by the following discussion.

Consider the equality

$$(a + b)^2 = a^2 + 2ab + b^2$$

The expression, $a^2 + 2ab + b^2$, is spoken of as the square of $a + b$, or $a + b$ is called the square root of $a^2 + 2ab + b^2$. Note the relation which exists between $a + b$ and $a^2 + 2ab + b^2$. There are three terms in $a^2 + 2ab + b^2$ and only two in $a + b$.

The first term, a^2, of $a^2 + 2ab + b^2$ is the square of a, the first term of $a + b$; the last term, b^2, is the square of b the last term of $a + b$. The middle term, 2ab, is 2 times the product of a and b, or both terms of $a + b$.

We might then describe the expression, $a^2 + 2ab + b^2$, with respect to $a + b$ as being the square of the first term, a, plus twice the product of the first, a, by the second, b, plus the square of the second, b. An expression of this kind is called a perfect square, meaning by this that, if we take the square root of the first and last terms and connect them by a plus sign we have the square root of the whole expression.

Often the expression of which we wish to obtain the square root is not a perfect square. Hence we have first to determine whether the expression with which we are concerned is a perfect square, or not. If it is we can write down the square root at once.

Determine which of the following expressions are perfet squares and write out the square roots of those which are perfect squares.

1. $x^2 + 2x + 1$
2. $x^2 + 4x + 4$
3. $x^2 + 2x + 2$
4. $4x^2 + 4x + 1$
5. $9x^2 + 6x + 1$
6. $4x^2 + 12x + 9$
7. $4x^2 + 20x + 25$
8. $x^2 + 10x + 16$
9. $x^2 + 18x + 81$

10. $x^2 - 6x + 9$

In case the sign of the middle term is $-$ we write the difference of the square roots of the first and last term instead of the sum. Thus for the square root of $a^2 - 2ab + b^2$ we write $a - b$.

11. $x^2 - 16x + 64$

12. $4x^2 - 48x + 36$

13. $9x^2 - 6x + 1$

14. $9x^2 - 12x + 4$

15. $9x^2 + 8x + 1$

16. $x^2 - x + 1$

17. $4x^2 - 4ax + a^2$

18. $9a^2 - 30ab + 25b^2$

In our problem we have $x^2 + 16x = 36$ and the expression on the left hand side of the equality sign contains but two terms, $x^2 + 16x$. Before what we have just learned about finding square roots will be of use to us, we must have three terms in our expression. Compare this expression with problems 8 and 9 on the preceding page. It should be evident that the third term necessary to make $x^2 + 16x$ a perfect square is a number. Can you find what this number is? Refering to what we said on page 41, it must be a number such that when we take the square root of it and multiply it by 2 and x the product will be 16x. It is clear that the product of 8, 2 and x is 16x. If we multiply 8 by itself we have 64. Then

$$x^2 + 16 + 64$$

is a perfect square. To get this from our equation

$$x^2 + 16x = 36$$

we add to both sides $\underline{\quad 64 = 64}$

$$x^2 + 16x + 64 = 100$$

Extraction the square root of both sides of the equation

$$x + 8 = \pm 10$$

Solving for x $\quad x = -8 \pm 10$

that is $\quad x = -8 - 10 \quad$ or $\quad x = -8 + 10$

$$x = -18 \text{ feet} \quad \text{or} \quad x = 2 \text{ feet.}$$

Refering to our original problem, we find that x represented the width of a strip which was to increase the area of the flower bed 144 square feet. Since to add a border -18 feet wide is meaningless we may reject this result and check our problem by determining if a strip 2 feet wide will increase the area 144 square feet.

The new length is $18 + 2(2)$ or 22 feet.

The new width is $14 + 2(2)$ or 18 feet.

The new area is 18 times 22 or 396 square feet.

The old area was 252 square feet. $396 - 252 = 144$ square feet. Therefore we may say that the result checks.

Solve the following equations by the method just illustrated, which is usually spoken of as solving by completing the square.

1. $x^2 + 6x = 16$
2. $x^2 + 12x = 64$
3. $x^2 + 10x = 11$
4. $x^2 + 7x = \frac{15}{4}$
5. $x^2 + 14x = 51$
6. $x^2 + 5x + 6 = 0$

7. Suppose you wish to increase the area of a plot of ground by 87 square rods by adding an equal strip on one end and one side. If the plot of ground is now 10 by 16 rods, what is the width of the strip that should be added?

8. I wish to double the area of a two acre lot which is 8 by 40 rods by adding an equal strip on all sides. How wide should the strip be?

Forming the equation in the usual way we should obtain

$$4x^2 + 96x = 320$$

or

$$x^2 + 24x = 80$$

Completing the square $x^2 + 24x + 144 = 224$

whence $x + 12 = \pm\sqrt{224}$

The sign $\sqrt{}$ is used to indicate that we are to take the square root of the number written under it. The expression $\sqrt{224}$ is read, the square root of 224.

In previous problems we have been able to guess at the square roots of the numbers, but to find the square root of 224 it is necessary that we learn still another process. This process you may have studied in your Arithmetic.

		224.0000	14.96.....
		1	
20 + 4 =	24	124	
	4	96	
280 + 9 =	289	2800	
	9	2601	
2980 + 6 =	2986	19900	
	6	17916	

Beginning with the decimal point, divide the number into divisions of two digits each. Find the largest square which is less than the number in the first division to the left. In this case it is 1. Place the square root of this perfect square to the right of the number to be the first digit of the final square root. Place the square under the first division and subtract bringing down the next division. Multiply the partial square root by

2 times 10 and use this as a trial divisor to obtain the next digit of the square root. In this case it is 4. Add this to the trial diviscr and multiply the sum by it. Then subtract and repeat the process until as many digits are obtained as desired.

Our equation now is $x + 12 = \pm 14.96$

$x = -12 - 14.96$ or -26.96

or $x = -12 + 14.96$ or 2.96

Determine which result is to be used and check.

An equation of the above type is called a quadratic equation. It may be described as an equation which consists of a term which contains a square, a term which contains the first power, and a term which does not contain the unknown quantity. To state this briefly, a quadratic equation is an equation of the type $ax^2 + bx + c = 0$.

The new processes involved in the solution of problems which result in quadratic equations of this type are three, multiplication, completing the square, and extracting the square root. Be fore we can solve such problems efficiently, we will need to make ourselves familiar with these processes.

Extract the square root of the following:

1. 546 2. 729 3. 4045 4. 56147

5. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

A right triangle is a triangle having one right angle, or having two sides which are perpendicular. The side opposite the right angle is called the hypotenuse. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

6. The hypotenuse of a right triangle is nine inches and one of the sides is 6 inches. Find the length of the other side.

7. The hypotenuse of a right triangle is 12 inches and the other two sides are equal. Find their length.

8. The diagonal of a square is 8 feet. Find the area of the square. (The diagonal of the square divides it into two equal right triangles.)

9. The hypotenuse of a certain right triangle is 10 feet long; a piece of string which is exactly one fourth as long as one of the perpendicular sides is found to be one third as long as the other. How long is the string and what is the length of each of the sides?

10. An east bound train leaves Kansas City at 10 a.m. How far from Kansas City will the train be 10 miles from a town which is 30 miles east of Kansas City and 6 miles north of the railroad? Can you attach a meaning to both values of the unknown in this problem?

11. A bamboo reed 10 feet high is broken off so that the top reaches the ground just three feet from its base. How far from the ground is the reed broken off?

12. What is the length of the rafters of a house if the angle of the slope is 45° and the width of the house is 18 feet? (If one angle of a right triangle is 45° the perpendicular sides are equal.)

13. The angle of the slope of the rafters of a house is 30°. What is the length of the rafters is the house is 20 feet wide? (If one angle of a right triangle is 30°, the perpendicular side which is opposite it is one half the hypotenuse.)

14. The side of an equilateral triangle is 6 inches. Find its altitude. Find its area. (If the altitude of an equilateral triangle is drawn, it divides the triangle into two right triangles each of which have one angle equal to 30°.)

15. The side of an equilateral triangle is 10 inches. Find its altitude.

16. The altitude of an equilateral triangle is 6 inches. Find its side and its area.

17. If you live 7 miles east of a river which runs north and south, what point on the river is 15 miles from your home?

18. The radius of a circle is 16 feet. Find the area of the largest square which can be cut from it. Find the area of the parts which are cut off.

19. If the radius of a cylindrical jar is 7 inches, what is the area of the bottom of a square box which will just enclose the jar? What is the area of the open spaces in the corners?

20. In the middle of a square pond whose sides are 10 feet, there grows a reed which reaches 1 foot above the water. When the reed is bent over to the side, it just reaches the top of the water. How deep is the pond?

21. A rectangular pyramid on a square base is to be made by folding a figure like that shown. The altitude of the triangular sides is to be 6 inches. The surface area is not to exceed 100 square inches. What must be the side of the base to obtain precisely that surface area?

Multiply the following:

1. $(ax - 10)(4ax - 6)$
2. $(3x - 2y)(2x - 3y)$
3. $(6a + 1)^2$
4. $(5a - 2)^2$
5. $(3k - 2)^2$
6. $(3a + 4bc)^2$
7. $(8x - 3y)^2$
8. $(9x - 13)^2$
9. $(20s + 11)^2$
10. $(4x - 7)(4x + 7)$
11. $(x + 9)(x + 9)$
12. $(s - 21)(s + 21)$
13. $(2x - 3y)(2x + 3y)$
14. $(r - 12)^2$

15. $(2r - 12)^2$

16. $(4r + 15)^2$

17. $(x - 2ab)^2$

Can you state a rule for products such as problems 4 tp 9, 10 to 12 and 14 to 17 which will be of use to you in performing such multiplications? Apply this rule to the following:

18. $(2x + 7)^2$

19. $(5k - 16)^2$

20. $(12 - 15k)(12 + 15k)$

21. $(20 + 7x)(20 - 7x)$

22. $(7a + 2)^2$

23. $(3x + 4y)^2$

24. $(5a - 6)(5a + 6)$

25. $(x - 9)(x - 9)$

26. $(7a - 11b)^2$

Solve the following by completing the square:

27. $x^2 + 14x = -45$

28. $s^2 - 5s = 14$

29. $r^2 + 9r = 14$

30. $h^2 - 17h - 38 = 0$

31. $x^2 + 23x + 60 = 0$

32. $2x^2 + 3x - 2 = 0$

33. $x^2 + 17x + 36 = 0$

34. $x^2 + x - 1 = 0$

35. $3s^2 + 7s - 5 = 0$

1. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park by adding strips of equal width on one side and one end. How wide should the strip be?

2. A certain fancy quilt is 72 by 56 inches. It is decided to increase its area by 10 square feet by putting a border around it. Find the width of the border.

3. A rectangular piece of ground 60 by 14 feet is to be doubled in area by an equal increase of length and breadth. Determine the amount of increase.

4. A printed page is to have a margin of one inch and is to contain 35 square inches of printing. How large must the page be if the length is to exceed the width by 2 inches?

5. A farmer starts cutting grain going around a field 120 rods long by 70 rods wide. How wide a strip must he cut to make 12 acres?

6. A farmer has a wheat field 160 rods long by 80 rods wide. In cutting the wheat he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods? How wide is the strip when he has cut 10 acres?

7. A farmer in cutting his field of wheat, 160 rods long by 40 rods wide, cuts around the field. If he wishes to finish the field in three days, how wide a strip should he cut the first day? The second day? The third day?

8. How could you lay out a 5 acre field so that it is 20 rods longer than it is wide? What is its perimeter?

In solving quadratic equations after they have been simplified to the form $ax^2 + bx + c = 0$, the solution is often simpler if the equation is written as the product of two factors.

Take for instance the equation

$$x^2 - 5x + 6 = 0$$

This written as the product of two factors is

$$(x - 3)(x - 2) = 0$$

That this expression is equivalent to the original equation may be verified by performing the indicated multiplication.

The process of separating an expression, such as

$x^2 - 5x + 6$ into two factors is called factoring. As it will be seen, it is the reverse of multiplication.

To consider the equation $(x - 3)(x - 2) = 0$: can we find a number which will check in this equation? If we assign the value 3 to x and write 3 for x in $(x - 3)(x - 2) = 0$, we have $(3 - 3)(3 - 2) = 0(1) = 0$. We may therefore say that this value of x checks in our equation and hence must be one of the roots of the equation. In a like manner we may show that 2 is the other root of the equation.

As you will observe it is quite easy to determine what numbers to take for the roots when the equation is written in factored form. 3 and 2 are the numbers which appear in the factored form of the equation and are the same numbers that we should obtain if we solved the equations formed by placing each factor successively equal to zero. Thus in this problem we should have $x - 3 = 0$ and $x - 2 = 0$, whence $x = 3$ or $x = 2$. When you have any difficulty in determining the roots of an equation in factored form this process will be of value to you but it is not a necessary part of the solution.

This method of solving a quadratic equation is very convenient when the factoring can be easily performed. Since factoring is the reverse of multiplication, if the expression to be factored has resulted from some of the simpler types of multiplication (see pages 47 and 48), the factoring should be comparatively simple. You should first decide upon the type of multiplication from which the product has resulted or could result. If there is any doubt as to whether your factors are correct or not, you should check by multiplying the factors together. If their product is the same as the original product they are the correct factors. Thus $x - 3$ and $x - 2$ are the factors of $x^2 - 5x + 6$ because by actual multiplication $(x - 3)(x - 2) = x^2 - 5x + 6$.

Solve the following by factoring:

1. $x^2 - 4x + 4 = 0$
2. $x^2 + 6x + 9 = 0$
3. $x^2 + 7x + 12 = 0$
4. $x^2 - x - 12 = 0$
5. $x^2 - 25 = 0$
6. $x^2 - 9x + 20 = 0$
7. $x^2 + x - 20 = 0$
8. $x^2 + 2x - 8 = 0$
9. $x^2 - 8x - 20 = 0$
10. $x^2 - 64 = 0$
11. $x^2 - 24x + 144 = 0$
12. $x^2 + 11x + 30 = 0$
13. $x^2 + x - 42 = 0$
14. $x^2 - 3x - 40 = 0$
15. $x^2 - 13x + 40 = 0$
16. $x^2 - x - 56 = 0$
17. $x^2 + 11x - 42 = 0$
18. $x^2 - 13x - 30 = 0$
19. $x^2 + 17x + 72 = 0$
20. $x^2 - 25x + 100 = 0$
21. $x^2 - 6x + 5 = 0$

In solving quadratic equations in the future, use the method which appears to be the most convenient. Some expressions you cannot factor and hence you must expect to always obtain a solution by factoring. The method of solution by completing the square will always give a result.

1. How would you lay out a 20 acre field so that it will be twice as long as it is wide? What would be its perimeter?

2. If you wish to mark off a plot of ground which will contain 48 square yards, how can you do it so as to make it 13 yards longer than it is wide?

3. By what amount must the length and breadth of a rectangular plot of ground 300 feet long by 50 feet wide be equally increased in order that the area may be increased by 3600 square feet?

4. How wide a path may be laid out inside the margin of a park whose area is 100 by 250 feet, in order that the space taken up by the path may be 3400 square feet?
Such a problem as this may arise if we wish to know how wide we may make the path so that its area will not exceed a certain amount, say, 4000 squabe feet.

5. A city block is 400 by 480 feet measured to the outer edge of the sidewalk. If it costs 4 cents per square foot to lay a sidewalk around the bolck, what is the width of a walk which can be layed for $500?

6. A farm is 320 rods long and 280 rods wide. There is a road 2 rods wide running around the boundary of the farm and lying entirely within it. There is also a road 2 rods wide running across the farm parallel to the ends. What is the area of the farm exclusive of the roads?

7. A rectangular piece of ground 840 by 640 feet is divided into four equal city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?

8. A stone is dropped into a well of unknown depth and is heard to strike the bottom 2 seconds afterwards. What is the depth of the well if we neglect the velocity of sound?

When an object such as a stone falls freely, the distance it passes over is equal to 16 times the square of the number of seconds it falls, or in symbols $d = 16t^2$. This is very often written $d = \frac{1}{2}gt^2$, g being equal to 32.

9. A stone is dropped from a balloon which was passing over a river. How high was the balloon if the stone struck the water 15 seconds after it was dropped?

10. A man drops a stone over a cliff and hears it strike the ground below 6 1/2 seconds later. If sound travels 1152 feet persecond, find the height of the cliff.

11. A stone is dropped into a mine. How long will it take the stone to reach the bottom of the mine if it is 1200 feet deep?

12. How long will it take a stone to reach the bottom of a mine 1000 feet in depth, if it starts with an initial velocity of 20 feet per second?

If an object starts with an initial velocity and not from rest as in the preceeding problems, the distance is expressed by $d = vt + \frac{1}{2}gt^2$ where v represents the initial velocity.

13. When will a stone thrown downward at a speed of 74 feet per second from a height of one mile reach the earth?

14. A body is thrown downward with a speed of 50 feet per second from a distance of 2250 feet above the earth. How many seconds later will it strike the earth?

15. A stone is dropped into a well 60 feet deep. What is its velocity when it strikes the bottom?

The velocity of a falling body is given by $v = gt$, g being equal to 32. In order that this relation will be of use, we must first find t as in previous problems.

16. With what velocity would a ball have to be thrown upward to reach a window 50 feet above the ground?

This may be considered as a problem of a falling body reversed. All that we have said applies in an opposite sense in case the body is thrown upward.

17. With what initial velocity must a bullet be fired upward to reach a height of 6400 feet?

18. With what velocity must I throw a ball so that it may just reach a man on a scaffold 25 feet above me?

19. A body is thrown upward from a height of 1728 feet with a speed of 48 feet a second. When will it reach the earth? When will it reach the level from which it was thrown?

20. A body is thrown upward from a height of 500 feet with a speed of 30 feet per second. When will it reach the earth?

21. A stone is thrown horizontally from a cliff 400 feet high with a speed of 50 feet a second. When will the stone strike the earth? How far from the foot of the cliff will it strike?

If an object is thrown or is moving in a horizontal direction, it falls just the same as if it had been dropped, and it continues to move freely in the horizontal direction.

22. A mail bag is dropped from a train at a height of 4 feet above the platform while the train is going 30 miles per hour. How far in advance of its starting point does the bag strike the platform?

23. An object is dropped from an aeroplane which is going at a speed of 50 miles an hour. If the object is dropped at a height of 300 feet, how far in advance of its starting point will it strike the earth? What will be the result if the aeroplane is moving at a speed of 30 miles an hour?

24. One person starts north from a certain place going 4 miles an hour and a second person starts east from the same place at the same time going 3 miles an hour. In how many hours will they be 16 miles apart?

In solving such a problem as this, if you will draw a diagram of the motion which is at right angles, you will have a right triangle formed by the line joinging the two moving objects. To this right triangle you can apply your usual method of finding the third side when two are given.

25. Two electric cars are on tracks meeting at right angles. One starts from the crossing at the rate of 20 feet per second, the other starts 100 feet away from the crossing at the rate of 10 feet a second. When will they be 200 feet apart?

26. A train approaching Chicago from the south at the rate of 50 miles an hour is 75 miles away when a train starts west from Chicago at the rate of 25 miles per hour. How long after the second train starts will they be 50 miles apart measured diagonally across the country?

27. Two trains 100 miles apart on roads which cross at right angles are running towards the crossing. One train runs 10 miles an hour faster than the other. At what rates must they run if they both reach the crossing in 2 hours? How far were each from the crossing when they were 100 miles apart?

28. An automobile running northward at the rate of 15 miles per hour is 20 miles south of the intersection with an east and west road. At the same time another automobile running westward on the cross-road at the rate of 20 miles per hour is 15 miles east of the crossing. How far apart will they be 15 minutes later? One hour later? Two hours later?

29. Two men are on streets at right angles to each other, distant 7 and 8 feet respectively from the crossing. If they approach the corner at the same rate, how far must each walk so that their distance apart shall be 5 feet?

30. Two men stand on streets meeting at right angles, in positions 3 and 5 feet from the crossing. How far must each walk toward the crossing at the same rate in order that they may be 10 feet apart?

31. Two men standing 1 foot and 8 feet from the crossing of two streets at right angles to each other, walk away from the crossing at the same rate. When will they be 40 feet apart if they walk 250 feet per minute?

32. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

33. An athletic field is 800 feet long and 600 feet wide. The field is to be extended the same amount in length and width so that the longest possible straight course shall be increased 100 feet. What will be the dimensions of the new field?

34. A trunk 34 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally on the bottom. How much must the length be increased if it is to accomodate a gun 4 inches longer than the umbrella?

35. A picture is 15 by 20 inches. How wide a frame must be used if we wish to increase the diagonal 3 inches?

36. A rectangular lot is 12 by 16 rods. How wide a strip must be added to one end if you wish to increase the diangonal 1 rod?

37. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches? How much must be added to the width to increase the diagonal 4 inches?

38. A rectangular frame is so constructed that by means of a slide its width may be altered. When it is set at a width of 5 centimeters, a rod 13 centimeters long just fits as a diagonal. How much must the width be increased so that a rod 15 centimeters long may fit as a diagonal?

39. An open box is made from a square piece of tin by cutting out 5 inche squares from each of the corners and turning up the sides. How large is the original square if the box xontains 180 cubic inches?

In such a problem as this draw a diagram to represent the piece of tin and mark on this diagram the squares to be cut from the corners and the dimensions which are given and those to be found. Better still, take a piece of paper or cardboard in the shape of a square and actually cut from its corners the squares. Doing this together with what you know about areas and volumes should enable you to solve the problem.

40. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6 inche square from each corner and then turning up the sides. Find the dimensions of the original piece of tin.

41. Suppose you wish to make a tin box from a square piece of tin by cutting squares out of each of the corners and turning up the sides. If the box is to be 6 inches deep and is to contain 384 cubic inches, how large a piece of tin will you need?

42. How large a square should be cut from each corner of a piece of tin 18 inches square to form an open box whose total surface area is 260 square inches, by turning up the sides?

43. A piece of cardboard 5 inches longer than wide is used to make a box of capacity 108 cubic inches by cutting 3 inch squares from each of the corners. What must its dimensions be?

44. Suppose you wish to make a tin box whose length shall be 3 inches greater than its width and which shall be 4 inches deep. If it is to be made from a piece of tin as in the above problems and is to contain 352 cubic inches, what must be its dimensions?

45. Suppose you wish to construct a box which will be twice as long as it is wide by cutting out 5 inch squares from each of corners of a rectangular piece of tin and turning up the sides. If the box is to contain 840 cubic inches, what must be the dimensions of the piece of tin? How much tin is wasted?

46. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting 6 inch squares from each of the corners and turning up the sides. What are the dimensions of the box?

47. If you wish to measure your height so that the result will be correct to 1/4 an inch and to do so with a foot rule, what is the greatest error you can safely make in each application of the rule? For the purposes of the problem we will suppose your height to be between five and six feet.

We will assume that the division marked on the foot rule are correct and that the only error which enters into the result is due entirely to your inaccuracy in measuring. We will further suppose you have marked a height on the wall which is exactly equivalent to your height. When you apply the rule, beginning from the floor of thr room, you must mark its upper extremity on the wall for the next application of the rule. How near this extremity can you make this mark? Can you mark it to with 1/2 an inch? 1/4 an inch? 1/8 an inch? 1/16 an inch? 1/32 an inch? 1/64 an inch? 1/128 an inch? Is there any limit to the accuracy with which you can make this mark, using of course ordinary instruments?

Also in the next application of the rule, can you place it so that its lower extremity exactly coincides with the mark? Can you do this more accurately than you made the mark?

How many times will you apply the rule in measuring your height? Can you now answer the problem? Is it possible for you to measure your height with a foot rule so that you will know positively that the result is correct to 1/4 an inch?

About how accurately do you think that you can measure your height with a foot rule?

48. In measuring the length of a room, which is between 21 and 22 feet, with a foot rule to what fraction of a foot may you be sure your result is correct? Yow accurately will your result be if you use a yard stick?

49. Suppose a room is square and you measure one side finding it to be 15 feet, 3 inches. If your error in this measurement is 3 inches what will be the error in the area?

Magnitudes other than linear we do not measure directly, but compute them from liner measurements. That means that if we wisth to determine the area of this room, we would not measure the area directly by applying an unit square foot to the area of the room but we would measure the length and width and determine the area by taking their product. In the case of a square room, we should measure one side and take the square of it.

To consider the error in this particular problem, we shall assume that the error was made so that the length found was too large. Reducing the 15 feet to inches so that all our quantities will be in terms of the same units, the real area is 180 times 180 or 32400 square inches. But the length found by measurement is not 180 inches but $180 + 3$ inches. Hence, the computed area is $(180 + 3)^2$. This becomes $32400 + 1080 + 9$. The error in the area is the difference between this and the real area, 32400, which is 1089 square inches.

Thus we see that a comparatively small error (3 inches) in a linear measurement may make a very large error in our computed result. In general if we designate the error in the linear measurement by e, and the error in the area by E, and the real length of the side of the square by s, we have

$$E = (s + e)^2 - s^2$$

or

$$E = 2se + e^2$$

(s and e must be expressed in the same unit.)

The relation between E and e for the area of other figures and for volumes may be found in a similar manner.

50. Find the relation between E and e for a square whose side is 10 feet, e being given in inches.

51. Find relation between E and e for a square whose side is 25 feet, e being given in inches. For e being given in feet.

52. Find the relation between E and e for a rectangle 10 by 16 feet, e being given in inches.

53. If the error in inches made in measuring the side of a square 15 feet long is denoted by e, and the resulting error in square inches in the computed area by E, find E when e = 1; e = 3; 1/2; 0;

54. What would be meant by e = -5? Find E in the above problem for e = -5; e = -3; Compare with the second part of the preceding problem.

55. What is the error E (in square inches) in the computed area of a square 50 feet long, caused by an error e (in inches) in measuring the side? Find e if E = 2404; if E = _3591. If the measuring is done with a foot rule, how carefully must each mark be made if E is to be not more than 2404?

56. Find expression for E in terms of e for a rectangle which is 12 by 18 inches. Find e if E = 64. Find E if e = 1/4.

57. A lot is 150 by 300 feet. Find E (expressed in square feet) for this lot in terms of e (expressed in feet). Find e if E is 451. Find E if e is 1/2.

58. Find an expression for E in square rods in terms of e in rods if the side of a square is 80 rods. Find e if E = 159 square rods (note that this is nearly an acre). Find E if e = 1/2.

59. How carefully must one measure the radius of a circle whose radius is 21 feet in order that the computed area may be correct to with in 795 1/7 square inches.

Chapter IV.

1. How can you find the height of a flag pole if the length of its shadow is 40 feet? Suppose you know in addition that at the same time the shadow of a man who is 5 feet, 6 inches tall is 3 feet.

What are the quantities which occur in this problem? How many are there? Are there any besides the height of the flag pole and the height of the man and the lengths of their respective shadows? Does it seem reasonable to suppose that, if we know the relation existing between these quantities, we can determine the height of the flag pole when we know the other three? (It might be pointed out that all of the quantities except the height of the pole can be measured directly.)

Considering in particular the quantities involved in the above problem;

5.5 feet is the height of the man,

3 feet is the length of his shadow,

h will represent the height of the flag pole in feet,

40 feet is the length of its shadow.

Now it has been found in problems of this sort, that if we form two fractions using as numerators the heigh of the objects and for the denominators the length of the shadows of the corresponding objects, the two fractions are equal. Knowing this and applying it in the above problem, we have for our equation

$$\frac{h}{40} = \frac{5.5}{3}$$

This equation is not quite like the ones we have been meeting with, but a simple method by means of which we may simplify this equation is to multiply both sides by some number such that when the fractions are simplified, that is reduced to their lowest terms, their denominators reduce to unity and hence need not be written. The number used must be such that both of the denominators will exactly divide into it. It will of course be most convenient to use the smallest number which possesses this property since in that case we will avoid having to work with large numbers. This process is called clearing of fractions and will be refered to as such hereafter.

In this problem it is evident that both denominators will divide into 120. Hence we shall multiply both sides of the equation by 120 and simplify the fractions. Our equation formed on the other page is

$$\frac{h}{40} = \frac{5.5}{3}$$

Multiplying by 120, $\frac{120h}{40} = \frac{120 \times 5.5}{3}$

Simplifying $3h = 40 \times 5.5$

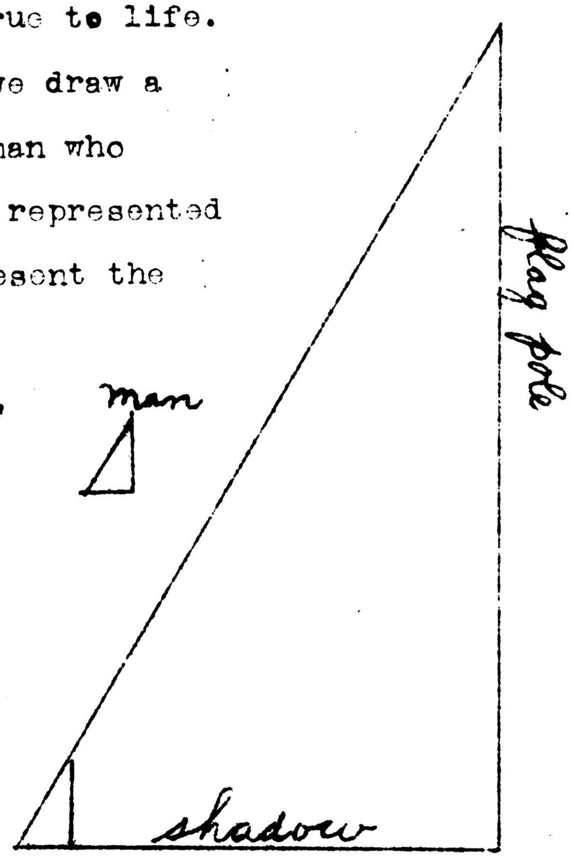
or $3h = 220$

$h = 73.3$ feet, the height of the flag pole

Checking $\frac{73.3}{40} = \frac{5.5}{3}$ reducing these fractions to decimal form $73.3 \div 40 = 1.84$ $5.5 \div 3 = 1.84$

Hence we may say that our result checks.

In solving problems of this type a figure will often be of considerable service to you. In drawing this figure, it is well to exercise some care in drawing it true to the conditions of the problem or as we might say, true to life. Thus in the problem just solved, if we draw a line to represent the height of the man who is 5.5 feet tall, the shadow will be represented by a somewhat shorter line. To represent the height of the pole and its shadow a considerable larger figure is needed.

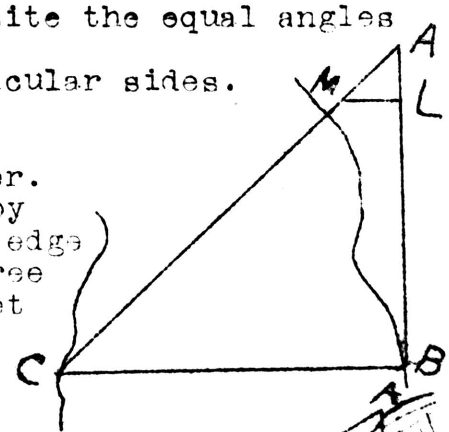


2. What is the height of a flag pole if the length of its shadow is 30 feet when the shadow of a man who is 5 feet 8 inches tall, is 3 feet 8 inches?

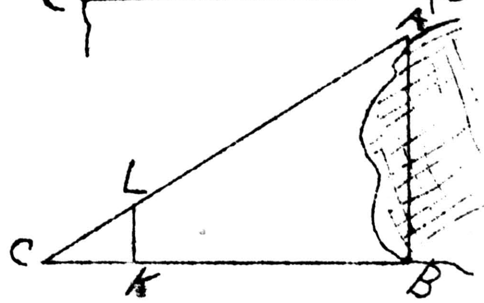
3. At noon the shadow of a building is 3 feet 2 inches. If at the same time the shadow of a post 4 1/2 feet high, is 10 inches, find the height of the building.

On page 60 we spoke of a rule which would apply to "problems of this sort". Can you distinguish problems of this sort from other problems? What are the characteristics of problems of this sort? Are there problems other than problems which involve the heights of objects and the length of their shadows which may come under this rule?

Consider the figure which we drew above to represent our first problem. This figure consists of two right triangles. These two triangles may be so placed that two sides of the one extend in the same direction as two sides of the other. This makes one angle of the one equal to an angle of the other. Hence we may describe this problem as one which may be represented by two right reiangles which have an angle of the one equal to a corresponding angle of the other. We may also describe these triangles as having the same shape. Now our rule is true for all problems which can be represented by triangles of the same shape. The only requirement which is placed upon the problem is that we must be able to represent it by two right triangles which have one acute angle of the one equal to an acute angle of the other. We may, then, form two equal fractions by taking the vertical lines as numerators and the horizontal lines as denominators. Or, if we wish to state the rule more accurately, we may take for the numerators the sides opposite the equal angles and for the denominators the other perpendicular sides.



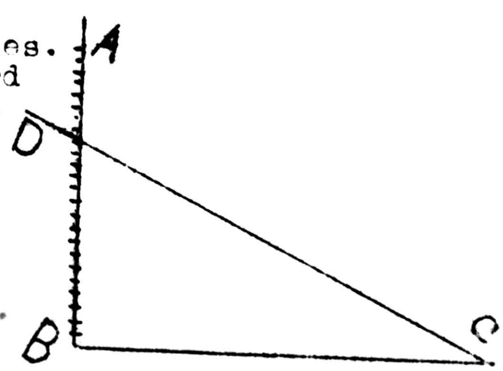
4. I wish to find the width of a river. To do so I measure the distances as shown by the diagram. C is a tree standing at the edge of the river. Directly across from the tree I fix a stake at B and then measure 100 feet upstream to A. I then mark off a small right triangle, ALM. AL is 6 feet, LM is 8 feet. Find the width of the river.



5. To find the distance between two points on the bank of a lake I measure CB 100 yards perpendicular to AB joining the two points. In the smaller triangle CKL, CK is 10 yards and LK is 12.5 yards. What is the distance between A and B?

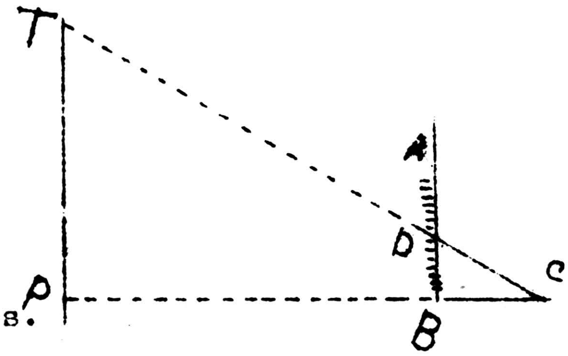
6. In the preceding problems we have been measuring off triangles which we have called right triangles and have been drawing lines perpendicular to other lines. In all this we have taken for granted that we could draw a line perpendicular to any other line. Can you suggest an easy method of drawing a line perpendicular to another or making a right angles? Show that your method is a correct one.

7. To measure the height of buildings, trees, etc., I construct the following instrument. I set two pieces, AB and BC at right angles. I make BC 20 inches and I fasten a third piece, CD, with a hinge at C so that it may be moved along AB. On AB I mark inches and fractions of an inch as on a ruler.



To measure the height of a building, I step a certain distance from it and then hold BC on a level, that is horizontal. I set the movable piece, CD, so that when I sight along CD it is just in line with the top of the building, as shown in the accompanying figure. P is point on the building on a level with my eye.

Remembering that BC is given as 20 inches, find TP when PC is 80 yards (steps) and BD is 12.5 inches.



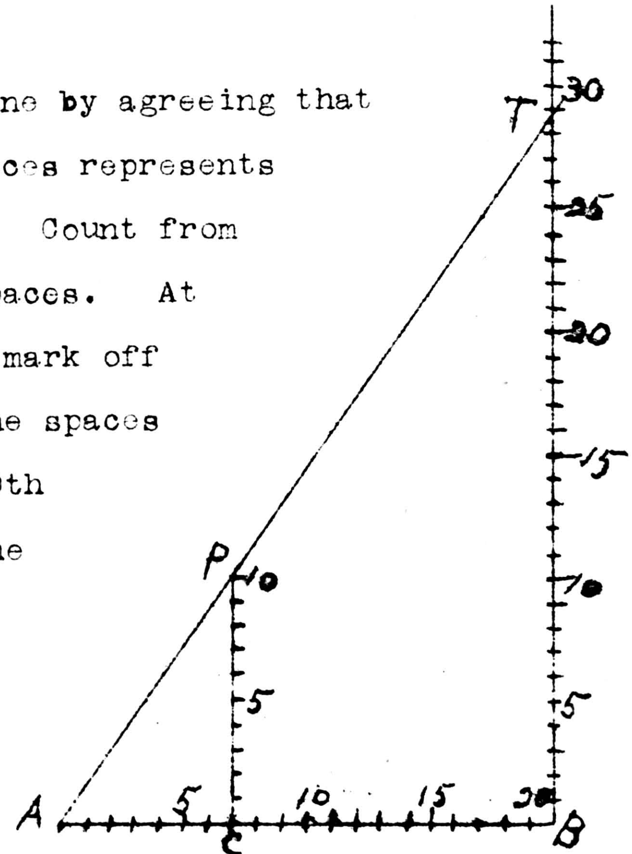
8. Find the height of a building if my eyes are 5 feet from the ground and PC is 72 yards and BD is 15 inches.

We have found that a figure drawn to represent a problem is of assistance in solving that problem. This has been especially true in the case of the problems of this chapter. In fact, in order to use our rule with any certainty a figure is almost a necessity. As has been suggested the figure will be of all the more assistance if we draw it so that it really represents the problem, that is, draw it accurately or, as it is sometimes called, draw it to scale.

9. Draw an accurate figure to represent this problem. What is the height of a tree which casts a shadow of 20 feet when a pole 10 feet high casts a shadow of 7 feet? Can you draw this figure accurately before you know the height of the tree?

(The student should at this stage of the course provide himself with a compass, a protractor and a ruler. The ruler should have divisions to at least 1/166 of an inch. A very satisfactory rule is one which has inches on one side and centimeters on the other. A triangle for drawing perpendicular lines would also be useful.)

To draw this figure accurately, or to scale, we may proceed as follows: Draw a horizontal line Ab (shown on the next page) and mark on it 20 or more equal spaces or divisions. (These divisions may be any convenient length. An eighth of an inch is used in the figure drawn.) We may now represent



the length of the shadows on this line by agreeing that one space represents one foot; 2 spaces represents 2 feet; 5 spaces, 5 feet; and so on. Count from one end, say the left, 7 of these spaces. At this point draw a vertical line and mark off on it a number of spaces equal to the spaces on the horizontal line. Mark the 10th space. Since we have agreed that one space represents one foot and since the pole is 10 feet high the length of this line represents the height of the pole, and the 7 spaces on the horizontal line the length of its shadow. Join the top of the pole with the extremity of the shadow and extend the line. Then at the 20th space draw a vertical line and mark on it spaces of the same size as we have marked on the other lines. Where it crosses the oblique line may be said to represent the top of the tree for the shadow of this vertical line is evidently 20 spaces or feet.

How many spaces on the vertical line Tb up to the point where it intersects the oblique line? Can you estimate the height of the tree from this number of spaces? There are about 28.5 spaces. Since we have agreed that one space represents one foot, the height of the tree acording to this estimate would be 28.5 feet.

Solving the problem by the usual method we have,

$$\frac{10}{7} = \frac{h}{20}$$

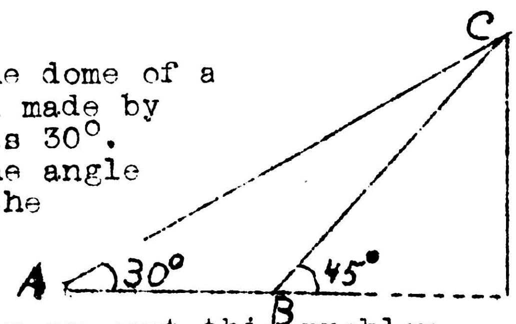
clearing of fractions $7h = 200$

$$h = 28\ 4/7$$

Thus we see that we can draw an accurate figure before we know the height of the tree and that when we have the figure drawn we can estimate the height of the tree from it.

10. To find the distance between two points H and K on opposite sides of a river, KL is measured perpendicular to HK and is 50 yards long. At L a small triangle, LOP, is drawn with OP perpendicular to LK. By measurement LO is found to be 20 feet and PO 28 feet. Draw an accurate figure of this problem and estimate the distance HK from your figure. Compare this result with the result xxxx xxx xxxxxx which you obtain from an equation.

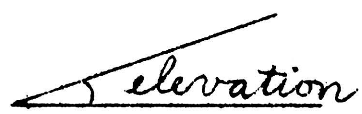
11. To determine the height of the dome of a building, I observe that the angle at A made by a line to the top with the horizontal is 30°. Going towards the building 150 feet, the angle at B is 45°. Determine the height of the dome from these measurements



In order to draw a figure to represent this problem you will need some convenient way of drawing an angle of 30° and 45°. For this purpose it is probably best to use a protractor which is a simple instrument for drawing angles of a definite size.

12. Tp determine the height of a tower the angle of elevation from one point is measured and found to be 25°, and from a second point 75 feet nearer it is 40°. Find the height of the tower.

(The angle of elevation is the angle measured above the horizontal.



The angle of depression is the angle measured below the horizontal.)

In this 12th problem, can you compute the result from an equation? Is it difficult to get from an accurate figure?

It might be pointed out that there are a large number of problems which may be solved by means of a figure but which are very difficult to compute from an equation or otherwise. Problems in which angles are measured as a part of the data usually cannot be solved without the use of the formulae of Trigonometry and trigonometric tables which at this stage of your work would be rather difficult to use and even when they are understood they are tedious.

13. At a horizontal distance of 120 feet from the foot of a building, the angle of elevation of the top was found to be 65°. Find the height of the building.

14. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 15°. Find the distance of the boat from the foot of the rock.

15. A distance AB is measured 96 feet along the bank of a river from a point A opposite a tree C on the other bank. The angle at B is 25°. Find the distance across the river.

16. Find the angle of elevation of the sun when a tower 120 feet high casts a horizontal shadow of 70 feet.

17. From the top of a hill the angles of depression of two objects on a straight level road are observed to be 5° and 15°. If the objects are one mile apart, how high is the hill?

18. The angle of elevation of an inacessable tower situated on a horizontal plain is 60°; at a point 500 feet farther from the base, the angle of elevation of its top is 25°. Find the height of the tower.

19. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60°, and froma point 40 feet more distant the angle of elevation is 50°. Find the width of the river.

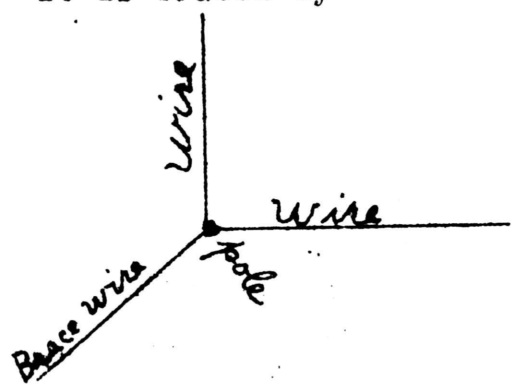
20. The length of a lake subtends, at a certain point, an angle of 45°, and the distances from this point to the two extremities of the lake are 346 and 290 feet. Find the length of the lake.

21. Along the bank of a river a base line is drawn 500 feet in length. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 55°; that of the second extremity from the same object, observed at the first, is 80°. Find the width of the river.

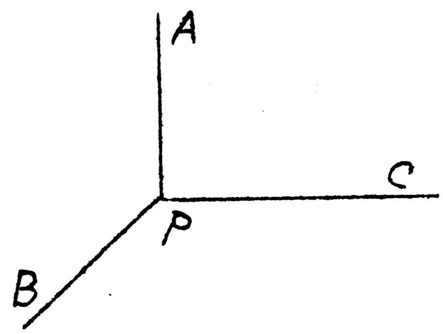
22. Two observers stationed on opposite sides of a cloud observe its angles of elevation to be 45° and 35°. Their distances from each other is 700 feet. Waht is the height of the cloud?

1. A telephone pole on a corner has wires attached to it from two directions at right angles. It is braced by a third wire as shown in the figure, which shows the pole and wires as you would see them from above the pole. What should be the angles between the brace wire and the telephone wires if the pull on each of the telephone wires is 500 pounds? What is the pull on the brace wire?

(It is assumed that the brace wire will entirely relieve the pole of any pull due to the telephone wires.)

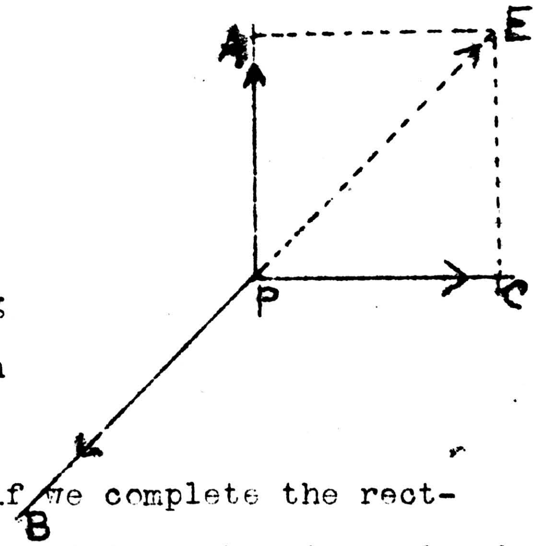


The quantities in this problem are evidently the pulls on the respective wires. If we draw another figure, using letters to designate the wires, we can talk about the problem much better. The telephone wires are represented by PA and PC respectively and the brace wire by PB. They are joined to the pole at P. Consider the pulls only along PA and PC. If the brace wire was not attached to the pole, the pulls along PA and PC would tend to pull the pole over in what direction? Could a single wire be attached which would pull the pole in this direction? What would be the pull upon it? If an equivalent pull can be produced by a single wire, our problem becomes quite simple. For our problem would then be, what pull is necessary and in what direction should it be applied to counteract a pull along a single wire. Suppose we have a pull of 220 pounds along PL how great a pull and in what direction is necessary to balance or counteract this single pull? It should be evident that the pull must be 220 pounds and in just the opposite direction, PK, But as has just been pointed out we have



pulls in two directions and we wish to balance them.

To proceed to find a single pull which is equivalent to the pulls along PA and PC, we may represent the forces on lines in very much the same way that we have been representing distances and thus draw a figure which really represents the problem. We may represent a pull of one pound by any convenient distance and the direction of the pull by the direction of the line. In this case it will be convenient to represent a pull of 100 pounds by 1/4 of an inch. Thus to represent the pulls on the telephone wires we draw two lines forming a right angle and measure off on each 5 spaces to represent the 500 pounds.



Now by a law of Mechanics, if we complete the rectangle of which these lines are a part and draw the diagonal, the length of the diagonal will represent the pull which is equivalent to the two pulls along PA and PC; the direction of the diagonal will represent the direction of that pull. In our figure, completing the rectangle and drawing its diagonal, we have PE whose length represents the magnitude of the single pull that is equivalent to the pulls due to both telephone wires. We can estimate the magnitude of that pull from the figure, or since EC is equal to PA and the triangle PCE is a right triangle we can calculate it.

$$(500)^2 + (500)^2 = d^2$$

$$d^2 = 500\ 000$$

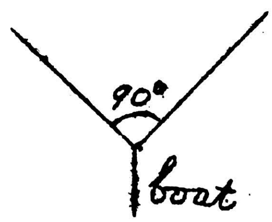
$d = 707.$ pounds, the pull along PE.

The angle between PE and PC is evidently 45° since the figure is really a square. PB should be drawn in the opposite direction to PE and equal to it in length. The angle between PB and PA is 135°. It is the same between PB and PC.

A problem of this kind is called "Composition of Forces." The force along PE is called the resultant of the two forces along PA and PC which are called the component forces. The force along PB is called the equilibrant. It is equal in magnitude but opposite in direction to the resultant.

2. Find the tension (pull) of a brace wire to a telephone pole which is arranged as in the above problem. The pulls on the telephone wires are 600 and 1000 pounds respectively. Estimate the direction of the brace wire with your protractor.

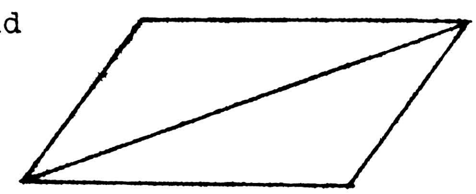
3. A boat is being towed upstream by means of two ropes as shown in the figure. If the angle between the ropes is a right angle and the pull on each is 100 pounds, what is the actual pull on the boat?



4. Under the conditions of the above problem, find the pull on the boat if the pull on the ropes is 75 and 100 pounds respectively.

5. Two ropes are attached to a stake and the angle between them is 60°. If the pulls on the ropes are 150 and 200 pounds respectively, what is the magnitude and direction of the resultant of these pulls.

In case the angle between the forces is not a right angle, you cannot draw a rectangle but you can construct a figure by drawing lines parallel to the lines representing the forces. Thus in the above problem we would have a figure such as that shown. This figure is called a parallelogram because the sides are formed by parallel lines. The diagonal of the parallelogram will represent the resultant of the two forces. However since the triangle is not a right triangle



it will be necessary for you to estimate the magnitude of the resultant from the figure.

6. The water in a river is flowing at the rate of 4 miles an hour. A man who can row 4 miles an hour starts directly across. How far will he be from his starting point after 15 minutes? What direction will he have gone?

7. A boat is headed due west and can move in still water at a rate of 8 miles an hour. It is in a current running due north at the rate of 3 miles per hour. How far will the boat go in 1 hour and in what direction?

8. A river 3/4 mile wide has a current of 3 miles per hour. A ferry boat can cross in 4.5 minutes. At what angle must the boat be headed upstream in order to go straight across?

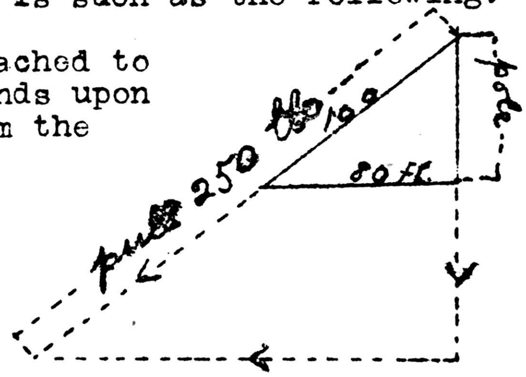
9. A man jumps from a train running at 10 miles per hour in a direction making an angle of 30° with the train and with a velocity of 9 feet per second. What is his velocity and direction relative to the ground?

10. Suppose under the conditions of the above problem, that the man had jumped at right angles to the train. What would be his velocity and direction relative to the ground in that case?

11. Suppose that you are on a train running at 30 miles per hour and can throw an object at a speed of 50 feet per second. If you wish to hit an object whose direction from you is at right angles to the train, at what angle should you throw?

12. If you are on a train running at 30 miles per hour and throw a ball with a speed of 50 feet per second at an angle of 60° with the direction of the train, what is the direction of the path of the ball with respect to the ground? What is its actual velocity?

An interesting type of problem which may be classed under this head for purposes of solution is such as the following.



1. I have a rope 100 feet long attached to the top of a pole. I am pulling 250 pounds upon the rope at its end which is 80 feet from the foot of the pole. How much am I pulling sideways upon the pole?

In this problem the pull on the rope is, of course, one force. What does it do to the pole? Does it tend to pull the pole in any other direction ~~other~~ than sideways? If you were 5 feet from the foot of the pole would the sideways pull be as great as when you were 50 feet away? Is this pull as great as that upon the rope? Can we say, then, that the pull on the rope is doing at least two things, pulling sideways and pulling downward? The direction of the forces are then, the direction of the rope, vertical, and horizontal. Therefore, we may represent them by lines whose directions are the same as that of the lines which represent the pole, the rope, and the ground. Doing this, extending the lines of our figure if necessary, we have the complete figure as shown above. You will recognize that this is the same sort of a figure as we had in the problems on page 60 and following. We may then apply the same rule as in those problems.

To form our equation, 250 is the pull along the hypotenuse of the right triangle, we will represent by p the pull sideways or along the horizontal line.

100 is the length of the hypotenuse.

80 is the length of the horizontal line.

Forming our equation we have,

$$\frac{p}{80} = \frac{250}{100}$$

Clearing of fractions $100p = 20000$

$p = 200$ pounds, the pull sideways.

This problem differs from the ones we classed as "Composition of Forces" in that here we start with a single force, the pull on the rope, and find that it was equivalent to two forces. This type of problem is called "Resolution of Forces", that is we start with the resultant and find the components. In the composition of forces it was the other way around.

2. In the above problem find the height of the pole and the downward pull on the pole.

3. What is the horizontal pull on a pole 40 feet high due to a pull of 500 pounds upon a rope attached to the top of the pole and meeting the ground 80 feet from the foot of the pole? What is the downward pull?

4. In the above problems is the sum of the horizontal and sideways pull equal to the pull on the rope? Can you explain?

5. In problem 3 what would be the horizontal pull if the rope met the ground 40 feet from the foot of the pole? 30 feet? 20 feet?

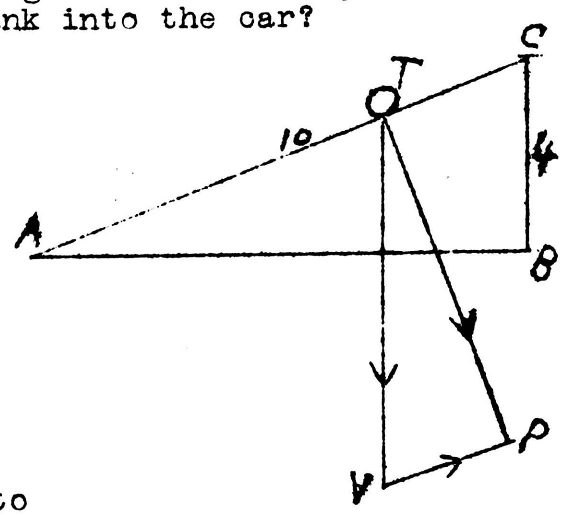
6. Is the horizontal pull doubled when the distance from the foot of the pole is doubled? In the case of the pole in problem 3, if the pull at 35 feet from the foot of the pole were 150 pounds, how far from the foot of the pole would it be necessary to go in order to double the horizontal pull?

7. I have a rope attached to the top of a pole 75 feet high. If the angle of elevation of the top of the pole is 35° at the point where I stand, how many pounds must I pull on the rope to produce a pull of 250 pounds horizontally on the pole?

8. A post is braced 3 feet from the ground by a plank which meets the ground10 feet away. A pull of 500 pounds is exerted on a wire which is fastened to the post at the point where the brace meets it. What is the vertical force on the post due to this pull?

9. What is the vertical force in the above problem if the foot of the brace is 15 feet away? 10 feet? 5 feet?

10. To load a 300 pound tank into a car which is 4 feet above the platform, a plank 10 feet long is used. How great a force must ge exerted to roll the tank into the car?



What are the forces acting upon the tank? How many are there? What is their direction? Can we say that their direction is the same as that of the plank, the vertical, and the horizontal line?

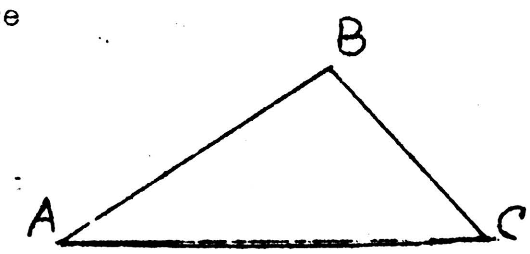
This problem will be found to be different from the ones we have been solving in that we cannot represent the forces acting upon the tank by lines whose direction is the same as the sides of the triangle formed by the car, the plank and the distance of the foot of the plank from the car.

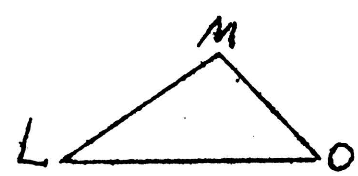
Consider the forces acting upon the tank when it is part way up the plank. The weight of the tank is pulling straight down, hence we could represent that force on a vertical line such as TV. The tank is pressing upon the plank and it is an established fact of Mechanics that when an object rests upon a plane, whether that plane is horizontal or not, the pressure of that object upon the plane is perpendicular to the plane. In this problem the plank is the plane, hence one force is at right angles to it and we may represent it on a line, TP, drawn at right angles to the plane. The other force is the one trying to move the tank along the plank and therefore it must be represented on a line whose direction is the same as the one representing the plank. In our figure this is marked VP.

Ncr that we have a figure drawn to represent the problem and understand its meaning, we find that it differs quite materially from the figures we have been meeting with. If we are able to form an equation from this figure, we must know just what relations exist between the sides of these two triangles. There is this likeness between this figure and those drawn in our earlier problems; there are in each case two right triangles, for there is a right angle at both B and P. Are any of the other angles equal? How about the angles at A and T? Can you show that they are equal? How about the angles at V and C? are they equal? In this figure they appear to be equal. This is really the case and this canbe shown by the methods of geometry.

The angles being equal by pairs really makes the same kind of a figure here as we have had in our earlier proklems. For this reason we may form the equation for this problem by the same rule.

It should be remarked that very much the same relation exists even when each of the triangles do not have a right angle. The only essential condition is that we shall be able to pair off each of the angles of one triangle with an equal angle of the other. Thus in the figure it is necessary that we know only that the angle at A is equal to the angle at L, the angle at B is equal to the angle at M, and the angle at C equal to the angle at O. If we can prove these pairs of angles equal, we may then write an equation using the length of the sides opposite any pair of equal angles for numerators and the lengths of the sides opposite any other pair of equal angles as





denominators, taking care that the sides are taken in the same order in each case.

Triangles which have their angles respectively equal as in the above case are called similar, and the relation between their sides is expressed by saying that their are proportional, meaning by that that you can form an equation as we have been doing. It might also be stated that in determining whether two triangles are similar or not, it is not necessary to show that all three pairs of angles are equal but that simply to show that two pairs are equal is sufficient.

To return to the problem on page 70 which started this discussion;

The force along TV is 300 pounds.

We will represent the force along PV by F.

The side corresponding to TV is AC (Why?) Its length is 10 feet.

The side corresponding to PV is CB. (Why?) Its length is 4 feet.

Thus we have for our equation

$$\frac{F}{4} = \frac{300}{10}$$

Clearing of fractions $\frac{20F}{4} = \frac{20 \times 300}{10}$

or $5F = 600$

$F = 120$ the number of pounds of force required to roll the tank into the car.

11. To get a barrel which weighs 200 pounds out of a cellar I use a plank and roll it. The plank is 12 feet long and the cellar is 3 feet deep. How many pounds must I push on the barrel to roll it up the plank?

12. A boy is able to exert a force of 75 pounds. How long an inclined plane must he have in order that he may be able to push a truck weighing 300 pounds up to a doorway 3 feet above the ground?

13. A boy pulls a loaded sled weighing 200 pounds up a hill which rises one foot in five. Neglecting friction how many pounds of force must he exert?

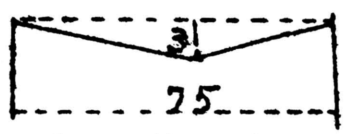
14. What force will be required to support a 50 pound ball on an inclined plane of which the length is 10 times the height?

15. The wind is blowing from due N. E. and its pressure upon a sail is 2000 pounds. What is its useful pressure if the boat wishes to go due south?

16. A drum weighing 1200 pounds is to be loaded into a car 4 feet above the platform by two men. If they are able to exert only about 350 pounds how long a plank should they have?

17. A hill rises 2 feet in 10. A team of horses pull a loaded wagon weighing 2 tons up this hill. How many more pounds are they required to pull than if the wagon were on level ground?

18. A team of horses is hitched to a harrow. The tugs are fastened to the collars at a point 3 1/3 feet above the ground. The distance from the collar to the front of the harrow is 8 feet. What is the horizontal pull on the harrow if the horses are pulling 2000 pounds?

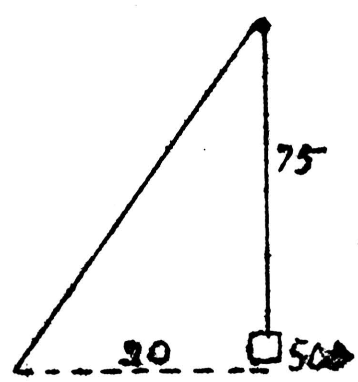


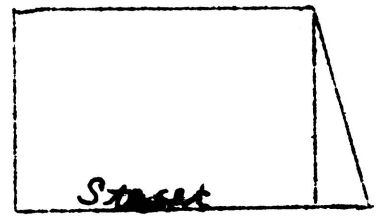
19. Two posts are set 75 feet apart, and a wire is stretched tightly between them. A man weighing 180 pounds is traveling along the wire from one post to the other. What is the horizontal pull on the pole when he is in the middle if the wire sags 3 feet?

20. In the above problem what will be the horizontal pull on the poles if the sag is 5 feet? What is the pull along the wire? (This pull along the wire is usually called the "tension".)

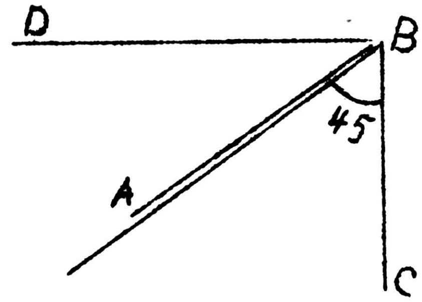
21. In hoisting material with a windlass, a derrick is arranged as shown in the diagram. If the horizontal distance of the windlass from the center of the derrick is 20 feet and the derrick is 75 feet high, what pull on the windlass is necessary to raise 500 pounds?

Can this problem be solved by the method which we have been using? If not, why?





22. In bracing a telephone pole on a corner it is necessary to carry the brace wire across the street to a second pole as is shown in the figure. The second pole is then braced with a second wire. If the pole is 20 feet high and the brace wire meets the ground 6 feet from the foot of the pole, what is the tension of the second brace wire if the tension of the horizontal wire is 700 pounds?



23. For lifting heavy weights a derrick of the type shown in the figure is used. There is a windlass at A and the rope passes from there to a pulley at B and thence to the weight at C. A brace wire BD is horizontal. What is the pressure on the beam AB in raising a weight of 1000 pounds, if the angle between AB and BC is 45°? What is the tension on BD?

24. I throw a ball upward at an angle of 45° with a velocity of 40 feet per second. What is its real upward velocity? What is its horizontal velocity?

25. If I throw a ball upward at an angle of 70° and with a velocity of 45 feet per second, what is its real upward velocity?

26. In the case of the above problem, how high will the ball be at the end of 2 seconds? How far will it have gone horizontally?

27. A weight of 5 pounds hangs by a string and it is pushed aside by a horizontal force until the string makes an angle of 45° with the vertical. Find the horizontal force and the tension of the string.

28. In the case of the above problem discuss the tension of the string as the ball is pushed out.

29. A boy who weighs 130 pounds is in a swing. Find the tension of the ropes of the swing when he is pushed out so that the popes makes an angle of 30° with the horizontal.

30. A man weighing 150 pounds is in a hammock stretched between two trees. If the trees are 15 feet apart and the sag in the hammock is 3 feet, what is the tension in the ropes of the hammock? (We will assume that the points at which the hammock is fastened to the trees are on a level.)

31. A man weighing 160 pounds is in a hammock stretched between two trees. What is the horizontal pull on the trees if the ropes of the hammock make an angle of 60° with each of the trees? What is the tension of the ropes?

32. A man weighing 150 pounds is in a hammock stretched between two trees. If the ropes make angles of 60° and 75° respectively with the trees, what is the horizontal pull on each of the trees? What is the tension of the ropes?

33. A ladder leans against a building, the foot of the ladder being 6 feet from the building. The ladder is 18 feet long and weighs 60 pounds. A man stands half way up the ladder. If we disregard the friction between the ladder and the building, what is the pressure against the building?

34. In the above problem, what is the vertical pressure on the ground due to the ladder and the man? What is the bending force upon the ladder?

35. A 20 foot ladder leans against a building with the foot of the ladder 5 feet from the building. A man weighing 180 pounds is 15 fett from the foot of the ladder. Disregarding the friction of the ladder against the building and the weight of the ladder, what is the pressure against the building What is the bending force against the ladder?

36. Solve the above problem if the weight of the ladder is given as 50 pounds.

37. A man wishes to load a barrel of oil weighing 250 pounds on to a wagon the bed of which is 3 1/2 feet above the ground. How long a plank should he have to roll it on if he wishes to push not over 75 pounds on the barrel?

38. Two rafters meet at an angle of 60° in a vertical plane and support a chandelier weighing 90 pounds. What is the force along each rafter?

39. A bar is supported by a chain as shown in the figure, the chain being attached to the ends of the bar. The chain is passed over a smoothe peg at the top. If the angle at the top is 90° and the bar weighs 100 pounds, find the tension on the chain.

40. A weight is suspended by means of two ropes. The angle between the ropes is 60° and the pull on them is 75 and 90 pounds respectively. What is the weight suspended by them?

1. A farmer has two plows. With one of them he can plow a certain field in 15 days and with the other he can plow it in 12 days. How long will it take to plow the field if he hires a hand and uses both plows?

Can you attach this problem by any of our previous rules? Is it similar to any problems we have solved? Can you attack this problem by considering the work done with each plow in one day? Thus if the farmer can plow the field with the first plow in 15 days, he can plow 1/15 of the field in one day. The hand with the other plow can plow 1/12 of the field in one day. If we represent by N the number of days it will take both to plow the field, 1/N is the part of the field they can plow in one day when working together. Since the part that they can plow together is equal to the part the farmer plows added to what the hand plows, we have for our equation

$$\frac{1}{N} = \frac{1}{15} + \frac{1}{12}$$

To clear this equation of fractions we do as in the other problems which we have been solving. Having three denominators does not change the conditions except that we must find a number which will contain all <u>three</u> denominators. Since N appears in the denominators, N must appear in that number. Try 60N. Multiplying each fraction by 60N we have

$$60N\frac{1}{N} = 60N\frac{1}{15} + 60N\frac{1}{12}$$

Simplifying $60 = 4N + 5N$

or $9N = 60$

Whence $N = 6\ 2/3$ days to plow the entire field when both are working.

2. A man can do a piece of work in 12 days and a second can do it in 9 days. How long will it take them when working together?

3. A farmer has two plows. With one of them he can plow a certain field in 25 days, with a second he can plow it in 18 days. How many days will it take to plow the field if he hires a hand to run the second plow?

4. One pipe can fill a cistern in 11 hours and a second pipe can fill it in 5 hours. How many hours will be required to fill the cistern when both pipes are running?

5. Two pipes empty a reservoir in 1 hour and 12 minutes. One of the pipes can empty the reservoir in 3 hours. How long would it take the other if running by itself?

6. A boy can fold 3000 advertising circulars in 3 hours less time than a second boy. The two working together can fold 7500 in 5 hours. How many can each fold in one hour?

7. It takes two hours longer for one pipe to empty a tank than for a second pipe to empty an equal tank. Both pipes together can fill either tank in 1 hour and 20 minutes. How long would it take each separately to do so?

8. Two boys together can fold 1000 circulars in an hour. It is observed that when each works separately at 1000 circulars one boy finishes 50 minutes before the other. How many circulars can each fold in one hour?

9. A and B working together can do a piece of work in 10 days. When A is working alone he can finish in 16 days. How many days would it take B to finish when he works alone?

10. $\frac{1}{2} + \frac{1}{3} + \frac{1}{x} = \frac{1}{36}$

11. $\frac{1}{4} + \frac{2}{3} = \frac{1}{x}$

12. $\frac{1}{2} + \frac{1}{2x} = \frac{1}{6}$

13. $5x + \frac{x - 2}{4} = 4x + 7$

14. $\frac{4}{x} + \frac{12 + x}{3x} = \frac{4}{3}$

15. $3x - \frac{x - 1}{4} - \frac{8x}{5} = 6$

16. $\frac{5}{6} - \frac{3x - 5}{4x} + \frac{x - 2}{3x} = 0$

17. $\frac{x + 5}{5x} - \frac{3(x + 1)}{x} = \frac{31}{5}$

18. $\frac{2x - 3}{3 + 2x} = 4$

19. $\frac{x - 2}{x - 3} = \frac{15}{16}$

20. $\frac{5}{7x + 5} - \frac{1}{8} = 0$

21. $\frac{1}{x - 2} = \frac{3}{x - 3}$

One of the main object of this year's work has been the study of the relations existing between the quantities arrearing in the various types of problems. Upon taking up a type of problem, these relations have been pointed out and it has been by means of these relations that we have been able to form and equation for the solving of problems. In every case the study of these relations has been the first part of our task; the second was to solve the equation which we formed from a knowledge of these relations.

We have spoken of one quantity depending upon another or others for its value; that is, an unknown quantity could be determined when certain others were known. Consider this problem: Sound travels 1100 feet per second. How far does it travel in 3 1/2 seconds? The quantities here are, as you already know, the distance, velocity, and time. Their relation is expressed by the equation d= vt, where d represents the distance, v the velocity and t the time. We may speak of this statement, d = vt, as meaning that d depends uppn v and t for its value, or since v is fixed for sound, it may be said to depend more directly upon t, the time. If we solve the equation for v, $v = \frac{d}{t}$. Thus we have v which does not change equal to quotient $\frac{d}{t}$. Hence this quotient must not change in value or as we usually say, it must be constant although d and t themselves do change in value. What we mean by saying that d and t change in value may be illustrated by this simple case. A man is walking 4 miles an hour. As the time increases, that is changes in value, the distance that the man has gone will increase, that is change its value.

This relation between d and t, when v is fixed, is often expressed by saying that the distance (d) varies as the time (t), or that the distance varies directly as the time. This form of expressing this sort of relation between quantities is sometimes very useful.

1. The distance which sound travels varies directly as the time. A man measures with a stop watch the time between the sight of the smoke of a hunters gun and the sound of its report. When the hunter was 1 mile distant, the time was 4 2/5 seconds. How far off was the hunter when the observed time was 2 seconds?

You can of course solve this problem by a method which you already know in case you happen to know the velocity of sound. We may, however, solve this problem by making use of the relation that the distance varies directly as the time which does not require that we remember the velocity of sound. In the problem we find these quantities:

Distance of 1 mile.

4 2/5 seconds which is the time for sound to travel 1 mile.

An unknown distance which we may call d.

2 seconds which is the time required for sound to travel the unknown distance.

If we now form two fractions with the distance as numerators and the respective time for denominators, the two fracions will be equal. Thus

$$\frac{1}{4\ 2/5} = \frac{d}{2}$$

Clearing of fractions $4\ 2/5\ d = 2$

$d = .45$ of a mile the distance of the hunter from the observer.

2. The cost of coal varies directly as the amount purchased. If 150 bushels cost $20. What will 400 bushels cost?

3. The weight of an object below the surface of the earth varies directly as its distance from the center of the earth. An object weighs 100 pounds at the surface of the earth, what would be its weight 1000 miles below the surface? (The distance from the surface of the earth to its center is 4000 miles.) What would be its distance 2000 miles below the surface? What would be its weight at the center?

In these problems which involve direct variation, we have formed an equation by making two fractions or quotients by using corresponding quantities for numerators and denominators. That this was a legitimate method can be shown by writing the equation d = vt in the form $v = \frac{d}{t}$. Since the quotient or fraction is always equal to the velocity which is always the same (for any given problem); any two quotients or fractions must be equal.

Any direct variation relation can always be expressed by either of these two forms of equations, that is d = vt or or $v = \frac{d}{t}$. On the other hand any relation between two changing quantities which can be expressed by means of either of these two forms of equations is a direct variation relation. Thus, the area of a square is equal to the square of a side, $A = s^2$: hence the area of a square varies directly as the square of a side. The area of a circle is equal to 22/7 times the square of the radius, $A = 22/7\ r^2$: hence the area of a circle varies directly as the square of the radius.

Thus we have characteristics by means of which we may recognize direct variation between two varying (changing) quantities. The use of this relation between two varying quantities will enable us to group together a large variety of problems.

4. How does the area of a square vary with the length of its side? If the side of a square is made 3 times as large how many times is the area increased?

5. If the side of one square is 1/2 the side of another square, how do their areas compare?

6. The area of one square is double that of a second square. How do the lengths of their side compare?

7. How does the area of a circle vary with the length of the radius? The area of a circle is 154 sq are inches and its radius is 7 inches. Find the radius of a circle whose area is 594.

8. The area of a circle is 10 square feet. What is the area of one whose radius is twice as great?

9. The radius of one circle is double that of another. How do their areas compare in size?

10. The radius of a circle is 16 feet. What is the radius of a circle one half as large?

11. On page 53 you have been told that the distance passed over by a falling body is found by multiplying the square of the time by 16 or, $d = 1/2\ gt^2$. How does the distance vary with the time? If a body falls 16 feet in one second, how far will it fall in 8 seconds?

12. The volume of a cube is equal to the cube of its dimension. How does the volume vary with the dimension? If you double the dimension of a cube , how is the volume increased?

13. The weight of a sphere of given material varies directly as the cube os its radius. Two spheres of the same material have radii 2 and 6 inches respectively. The first weighs 10 pounds. Find the weight of the second.

14. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters (39.37 inches) long vibrates once in one second, find the time of vibration of a pendulum 64 centimeters long.

15. Noting from the above problem that 100 centimeters is equivalent to 39.37 inches determine the time of vibration of a pendulum 12 inches in length.

16. Find the length of a pendulum which vibrates once in 2 seconds.

17. Find the length of a pendulum which vibrates once in 1/2 second; once in 1/4 second.

The problems dealing with the pendulum which you have just solved illustrate the value of the use of the variation relation. Without the use of this relation you could not have solved these problems unless you know more about pendulums.

18. I have 5 cubic feet of air in a cylinder under a pressure of 15 pounds to the square inch. I increase the pressure ~~of 15 pounds to~~ upon the air enough to reduce the volume to 2 1/2 cubic feet. I find that the pressure is then 30 pounds to the square inch. What is the variation relation between the pressure and volume?

The volume has changed when the pressure was changed, but is this change of the same sort of a change as we have been having in our problems? In our previous problems, the two quantities which have been connected with a variation relation have been so related that when one of them increased, the other increased; when one decreased the other did likewise. In some case one quantity increased or decreased much more rapidly than the other, but it always did the same thing as the other.

Now we have a case where the volume decreases when the pressure increases. This variation relation is of an entirely different sort and to distinguish it from the direct variation relation of our previous problems, we call this inverse or indirect variation. We may say then that the volume varies inversely as the pressure. We write this in equation form as $vp = k$ or $v = \frac{k}{p}$, where v represents the volume, p the pressure, and k is a fixed number.

Just how does this relation differ from the variation relation which we discussed on page 76 which we wrote as $d = vt$, where d and t were the varying or changing quantities and v was fixed? What is the difference in the position of the variables? Writing these two statements side by side will emphasize their difference in structure and meaning.

$v = \frac{k}{p}$, $d = vt$ or $vp = k$, $\frac{d}{t} = v$.

19. When the volume of air in a bicycle pump is 24 cubic inches, the pressure on the handle is 30 pounds. What is the volume when the pressure is 36 pounds?

20. In the case of the above conditions find the volume when the pressure is 48 pounds.

21. In the case of the above conditions find the pressure when the volume is 12 cubic inches.

22. Under ordinary conditions a gram of air occupies about 800 cubic centimeters. Find what volume a gram would occupy at the top of Mont Blanc (altitude 15,810 feet) where the pressure is only about 1/2 what it is under ordinary conditions.

23. The intensity (brightness) of light varies inversely as the square of the distance from the source of light. A reader holds his book 4 feet from a lamp, at what distance should he hold it if he wishes to double the brightness.

24. A book is held at 3 feet from a lamp and then 5 feet away. How many times has the intensity been increased?

25. A lamp shines on the page of a book 9 feet distant. Where should the book be held so that the page will receive four times as much light? Twice as much?

It would be interesting for you to try this out and see if it really is true that the intensity of light varies inversely as the square of the distance.

26. The weight of an object above the surface of the earth varies inversely as the square of the distance from the center of the earth. An object weighs 100 pounds at the surface of the earth. What will it weigh 1000 miles above the surface? 2000 miles? 4000 miles? (Compare this with problem 3, page 78.)

27. If a man in a balloon rises until the pressure is 1/4 what it was at the surface, how many inhalations would be be obliged to make in order to obtain the same amount of air as he could at the surface in one inhalation?

28. On December 26, 1910, Arch Hoxsey in a Wright bioplane attained a height of a trifle over 2 miles. Assuming that he weighs 150 pounds, how much did he weigh when at that height?

29. The force or pressure P of the wind against a surface in pounds per square foot, is computed from $P = 0.005\ V^2$ where V = velocity of the wind when it is blowing at 60 miles an hour. What would be the total force exerted by this wind against the side of a building 180x40 feet?

30. The pressure of the wind against a surface varies directly as the square of the velocity of the wind. If the pressure per square foot is 18 pounds when the velocity of the wind is 60 miles per hour, what is the pressure when the velocity is 40 miles per hour? What is the pressure when the velocity is 20 miles per hour? 10 miles per hour?

31. The pressure of wind on a flat (plane) surface varies jointly as the area of the surface and the square of the velocity of the wind. The pressure on one square foot is .9 pounds when the wind is blowing at thr rate of 15 miles per hour. Find the velocity of the wind when the pressure on one square yard is 18 pounds.

By saying that the pressure of the wind varies jointly as the area and the square of the velocity, we mean that the pressure is equal to a fixed number multiplied by the product of the area and square of the velocity, $P = k(v^2A)$. We can handle this form of variation in very much the same manner as the other forms.

32. A man carried a load of 150 pounds up a ladder to a point 16 feet above the ground. How much work has he done?

The quantities here are evidently the weight, the distance the weight was raised, and the work done. Hence we must be able to determine the work from the weight and the distance it was raised. The relation is a very simple one, the work, is equal to the product of the weight by the distance it was raised. We call the product foot-pounds.

For a given weight, how does the work vary with the distance?

33. A steam crane lifts a block of granite weighing 2 tons 80 feet. Find the work done . What is the work when it is raised 5feet? 10 feet? 30 feet?

34. A pump is raising 2000 gallons of water per hour from the bottom of a mine 400 feet deep. How much work is done in 3 hours, a gallon of water weighing 8.3 pounds?

35. In the above problem, how many pounds of work is being done per minute? If a horse power is 33,000 foot pounds per minute, how amny horse power of work are being done?
(It may be remarked that the horse power of work being done is approximately the horse power of the engine required to run the pump.)

36. A city uses 200 000 gallons of water per day. If this amount must be pumped from a well 800 feet deep, what is the horse power of the work done?

37. I have a 20 horse power engine. Neglecting friction, etc. what is the greatest number of gallons of water which I can raise from a well 40 feet deep in one hour?

38. A horse is attached to a capstan bar 12 feet long and does 75 000 foot pounds of work in going around the circle 100 times. Determine how many pounds the horse pulls.

39. A man using a windlass raises 500 pounds 12 feet. If the crank to the windlass is 18 inches and one turn of the windlass raises the weight 3 inches, what force must the man apply to the crank? What is the total amount of work which he does?

40. In a switch yard, a freight car weighing 40 tons and moving 8 miles an hour strikes a standing car weighing 24 tons. What is the velocity of the two after impact?

When an object is moving it is said to possess momentum. This momentum is the product of the weight and the velocity of the moving object. If an object is set in motion and is not acted on by any force afterwards tis momentum remains unchanged, that is constant. This is true if the moving object strikes one standing still and they move off together as in the above problem. Hence we may state that for a given momentum, the velocity varies inversely as the weight of the moving object.

41. A bowler uses a sixteen ounce ball to take down the last pin. The ball sends the pin off at a velocity of 6 feet per second, the weight of the pin being 48 ounces, while the velocity of the ball is reduced to 4 feet per second. With what velocity did the ball strike the pin?

42. Take a freight car weighing 60 tons. What is its momentum when it is moving 20 miles per hour? If the weight is fixed, how does the momentum vary with the velocity?

43. A billiard ball weighing 6 ounces and moving 16 feet per second meets another ball which it sends off at the rate of 10 feet per second. The rate of the first ball is reduced to 9 feet per second by the impact. What is the weight of the second ball?

44. A ball weighing 2 pounds is moving with a speed of 25 feet per second. What is its capacity for doing work?

It should be pointed out that momentum has no connection with work. but it is also true that a moving body possesses a capacity for doing work. This capacity for doing work is called <u>kinetic energy</u> and is equal to one half the product of the weight and the square of the velocity,

$$k.e. = 1/2\ wv^2.$$

Given a fixed weight, how does the capacity for doing work (kinetic energy) vary with the velocity? For a fixed weight, how does the momentum vary with the velocity?

45. A rock falls 64 feet. What is its momentum when it reaches the ground? What is its capacity for doing work?

46. In problem 40 on page 89, compute the capacity for doing work of the first car before the impact and the capacity for doing work of the two cars after the impact. Explain the difference.

47. A pile driver weighing 2000 pounds is raised 20 feet. How much work is done in raising it?

48. The pile driver in the above problem is allowed to fall freely. What is its capacity for doing work when it reaches the ground? Compare this with the amount of work which was done in raising it in the first place.

49. If the pile driver in the above two problems was used to drive a pile and drove it 6 inches, what was the pressure exerted?

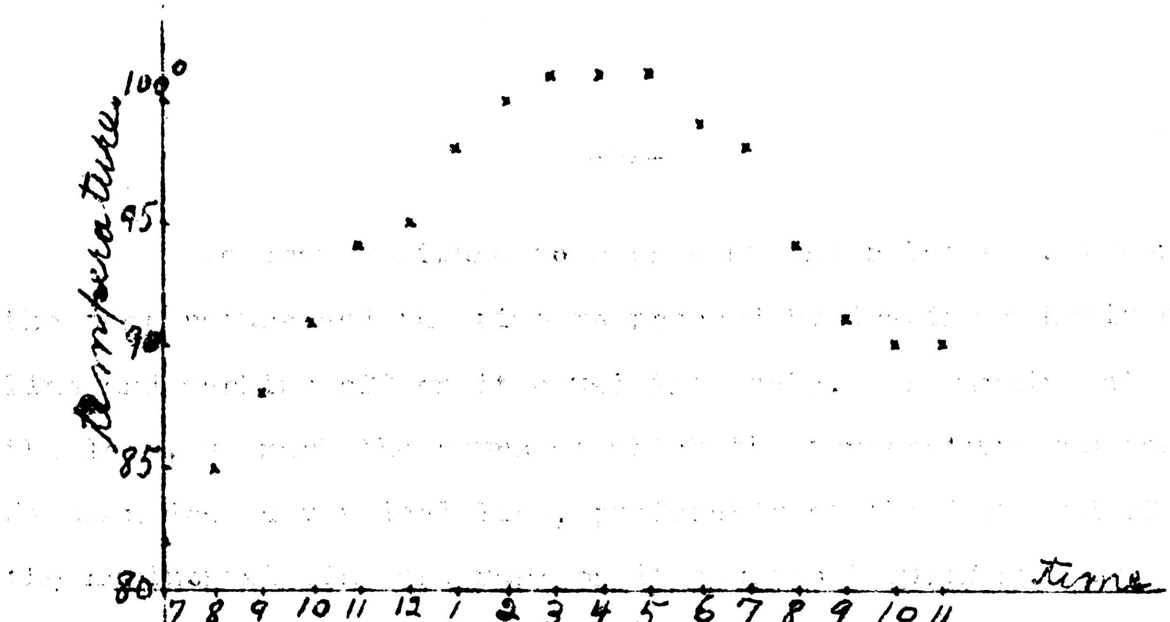
Chapter V.

1. The readings of a thermometer on a certain day were as follows: 7 a.m., 82; 8 a.m., 85; 9 a.m., 88; 10 a.m., 91; 11 a.m., 94; 12noon, 95; 1 p.m., 98; 2 p.m., 100; 3 p.m., 101; 4 p.m., 101; 5 p.m., 101; 6 p.m., 99; 7 p.m., 98; 8 p.m., 94; 9 p.m., 91; 10 p.m., 90; 11 p.m., 90; Express in some way the relation between the two quantities in this problem, the temperature and the time of day.

The two quantities in this problem, the temperature and the time of day, are both variable quantities such as we have been studying in our study of the variation relation between quantities. But before we attempt to express the variation relation between these, let us notice how they change or vary. The time increases always and at the same rate. The temperature increases a part of the day but decreases at other times as can be seen from noting the temperatures at 7 a.m., 11 a.m., and 9 p.m. Hence the variation relation connecting the temperature and the time of day is neither direct nor inverse. In fact, the relation between these quantities differs widely from either of these kinds of variation. It is a relation which we cannot express by means of an equation, but it is possible to express it by means of a diagram.

This diagram may be drawn in very much the same way as the diagrams you have been drawing to represent problems but you will find it of great assistance to use cross section paper of which you should purchase a supply. As the name implies, this paper is ruled with lines at right angles. They are ruled at convenient distances so that one does not need a ruler in marking off distances. In the figure on the next page, cross section paper is not used because it was not possible to do so since these copies were prepared on a mimeograph.

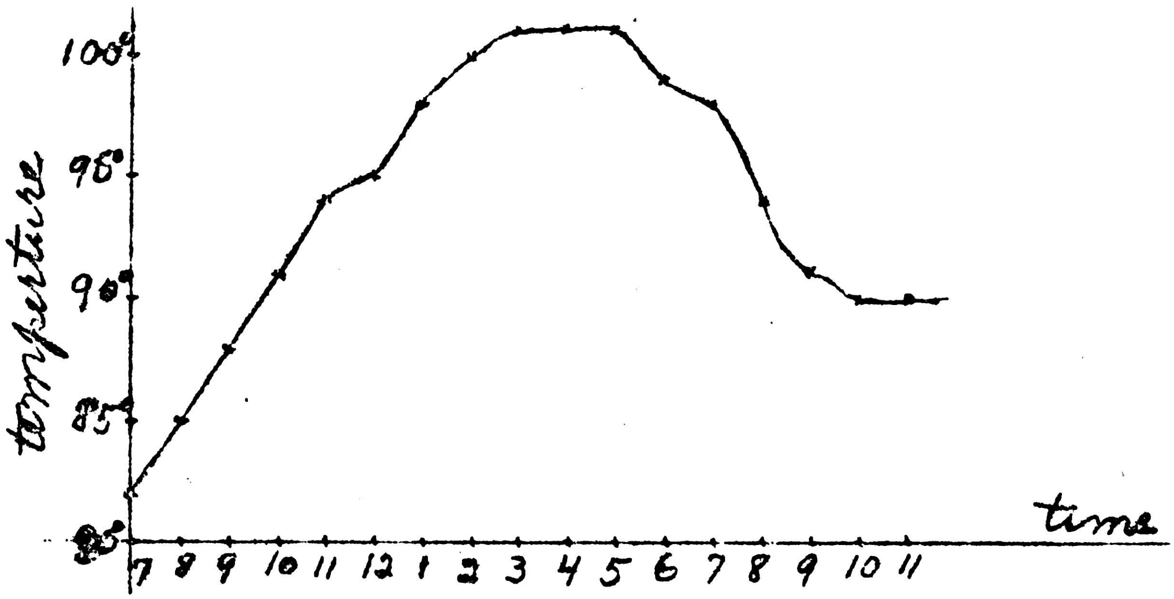
To draw a figure to represent the relation between the temperature and the time we proceed by drawing a horizontal line and marking off on it equal intervals. Beginning at the left, we mark the hours at which the temperature was taken. We then draw a vertical line, preferably at the left end of the horizontal line and mark on it degrees beginning, say with 80, at the horizontal line and then go up to 85, 90, etc. We now refer to our temperature readings and find that at 7 o'clock the temperature was 82. We go directly above 7 to a height which represents a temperature of 82; we mark this height by a samll cross. At 8 o'clock, the temperature has gone up



to 85; we mark a height directly above 8 which represents a temperature of 85. We continue to do this for each hour that is given, marking directly above the hour, a height which represents the temperature at that time.

Our figure does not represent just the true state of affairs because the temperature changes gradually. We have no temperature given between the hours, but we may improve our figure by joining the points which we have marked by a smooth curve, not by a straight line. Such a figure is shown on the next.

page. This figure is indentical with our first one except the points are joined.



This figure not only represents the relation between the temperature and the time of day, but represents this relation in such a way that one may tell at a glance several things about the change in temperature. For instance, it was hotest from 3 to 5 and was as hot at 5 as at 3. After 5 it cooled quite rapidly until 9 and then very slowly, indication a hot night. In the morning the rise of temperature was quite uniform until noon. Still other facts might be pointed out but these will serve to illustrate what amy be observed from this way of representing the variation relation between the temperature and time of day. Many of these facts would not be easily observed from the thermometer readings themselves.

Draw a diagram to represent the changes in temperature in the thermometer readings given below. After you have drawn the diagram, write a paragraph telling the facts which may be observed from the diagram about the changes or variation in temperature as was done in the paragraph above. The same facts will not be shown in all cases but you should be able to point out two or three in any case. Use cross section paper for drawing the diagram but write your description on plain paper.

2.
7 a.m.,,.......29
8 a.m...........29
9 a.m...........29
10 a.m..........30
11 a.m..........31
12 noon 33
1 p.m...........33
2 p.m...........34
3 p.m...........35
4 p.m...........36
5 p.m...........35
6 p.m...........34
7 p.m...........32
8 p.m...........32
9 p.m...........31
10 p.m..........29
11 p.m..........28
12 midnight 28
1 a.m...........27
2 a.m...........26

3.
7 a.m...........72
8 a.m...........65
9 a.m...........67
10 a.m..........70
11 a.m..........74
12 noon 76
1 p.m...........77
2 p.m...........78
3 p.m...........78
4 p.m...........79
5 p.m...........78
6 p.m...........77
7 p.m...........73
8 p.m...........72
9 p.m...........71
10 p.m..........70
11 p.m..........69
12 midnight 67

4.
7 a.m...........30
8 a.m...........33
9 a.m...........36
10 a.m..........39
11 a.m..........42
12 noon 45
1 p.m...........47
2 p.m...........49
3 p.m...........50
4 p.m...........49
5 p.m...........47
6 p.m...........45
7 p.m...........43
8 p.m...........42
9 p.m...........41
10 p.m..........40
11 p.m..........38
12 p.m..........38

5. The average height and weight for boys and girls at various ages are given in the following table which is taken from G. Standlet Hall's Adolescence.

	Boys		Girls	
Age	Height	Weight	Height	Weight
5.5	41.7	...	41.3	...
6.5	43.9	45.2	43.3	43.4
7.5	46.0	49.3	45.7	47.7
8.5	48.8	54.5	47.7	52.5
9.5	50.0	59.6	49.7	57.4
10.5	51.9	65.4	51.7	62.9
11.5	53.6	70.7	53.8	69.5
12.5	55.4	76.9	56.1	78.7
13.5	57.5	84.8	58.5	88.7
14.5	60.0	95.2	60.4	98.3
15.5	62.9	107.4	61.6	106.7
16.5	64.9	121.0	62.2	112.3

Draw on the same page a diagram to represent the height of boys and of girls. Do the same for the weight.

6. The following table gives the distance from Kansas City (in miles) and the elevation (in feet) above sea level of certain places along the line of the Santa Fe Railroad.

	Elevation	Distance
Kansas City	750	0
Newton	1440	200
Dodge City	2480	360
La Junta	4045	560
Ratin Pass	7608	680
Las Vegas	6383	780
Albuquerque	4935	920
Continental Divide	7258	1050
Winslow	4838	1200
Grand Canyon, rim	7000	1300
.. .. Bottom	2300	13 07
Needles	477	1500

7. The average temperatures for New York City are given in the table below.

Jan.	1-15	30	July	1-15	66
	16-31	28		16-31	68
Feb.	1-15	32	Aug.	1-15	68
	16-28	35		16-31	66
Mar.	1-15	38	Sept.	1-15	62
	16-31	40		16-30	60
April	1-15	45	Oct.	1-15	55
	16-30	48		16-31	50
May	1-15	50	Nov.	1-15	48
	16-31	56		16-30	45
June	1-15	60	Dec.	1-15	42
	16-31	65		16 -31	40

8. The population of the United States as given by the census reports from 1790 to 1900 is as follows:

1790....3.9 (million)	1830....12.9	1870...38.6
1800....4.3	1840....17.1	1880...50.2
1810....7.2	1850....23.2	1890...62.6
1820....9.6	1860....31.4	1900...76.3

It may be pointed out at this place that we have here another way of expressing the relation which exists between two quantities; in the problem which we discussed on page 91 and following, these quantities were the temperature and the time of day. We have from the beginning of the year be largely concerned with the relations which exist between quantities. In the first part of our course we were always able to express this relation by means of an equation; in the case of the lever, the first weight multiplied by its distance from the fulcrum is equal to the second weight multiplied by its distance from the fulcrum; in the case of falling bodies, the distance was equal to 16 times the square of the time; in the last chapter, a certain type of variation was designated by direct variation and another type, inverse variation.

But can you express the relation which exists between the temperature and the time of day by means of an equation? Can you conceive of an equation which will do this? It has already been pointed out that neither type of variation will apply. Thus it is important that we here recognize that we have another means of expressing the variation relation between varying quantities. Merely as a way of expressing this relation it is useful, but we shall find that this method of expressing the relation between variable quantities enables us to solve problems which we could not solve otherwise. For instance, in some problems we will find equations which we cannot solve by our usual methods but which yield readily to this method of treatment. This is particularly true if the equation has higher powers appearing in it that a square or if the equation contains two unknown quantities.

1. I have 80 yards of wire fencing and I wish to enclose a rectangular piece of ground with it. What dimensions should I make the rectangle if the area of the lot is as large as possible?

The two variable quantities in this problem are the area of the lot and the length of the fence around it. What is the relation between them? We may begin the consideration of this problem in our usual way.

The length of one side and one end is 40 yards (since that is half way around the lot.)

w represents the width of the lot.

Then 40 - w represents the length.

The area is the product of the dimensions, hence

$$\text{area} = w(40 - w)$$

or

$$\text{area} = 40w - w^2$$

But now that we have the relation between the area and the dimensions of the lot expressed by means of an equation, we find that it contains two unknowns. Hence, we cannot conclude our work in the usual way. Can you tell from the equation what w must be equal to in order that the area may have the largest value? Could you answer this question if we had this relation represented by means of a picture? We were able to tell from the diagram of the temperature and time relation the time at which it was hottest or coldest.

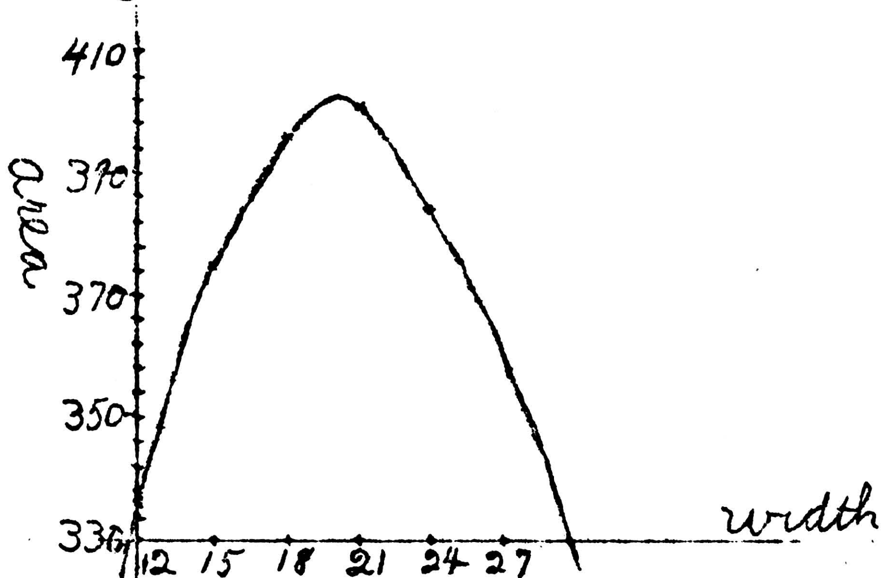
To draw a diagram of the relation between the width and the area, we must first find the area for a number of different widths.

When w is 9 yards, the area is 279 square yards

..	w	.. 12 ..	..	..	.. 336	..	..
..	w	.. 15 ..	..	..	.. 375	..	..
..	w	.. 18 ..	..	..	,, 396	..	..

When w is 21 yards, the area is 399 square yards
.. w .. 24 .. .. 284 .. ..

Using this data to draw a picture, we mark off the width on the horizontal line and the area on the vertical line. We find points whose heights above the horizontal line represents the area of the rectangle for the various widths. We then con-

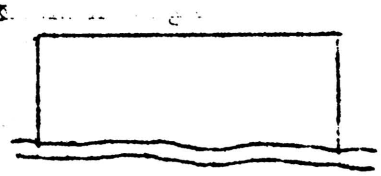


nect these points by a smooth curve. Since height represents area, the area will be the greatest at the highest point of the curve. What is the greatest area? Is it between 18 and 21? Is it nearer to 21 than to 18? What is the width which goes with this greatest area? Is it about 20? Find the area when the width is 20. If the width is 20, the length must be 20 also because the sum of one side and one end is 40. The area then is 400. How does this result agree with the greatest area shown in the diagram?

It is to be noted that, although we were able to express the relation between the width and area by means of an equation, we could not answer the question, what should be the width in order that the area would be greatest. However, when we expressed the relation by means of a diagram, we were able to answer this question at once. We may then conclude that it is of considerable advantage to be able to express the relations between variable quantities by means of a diagram even though it is also possible to express this same relation by means of an equation.

2. Determine the area and dimensions of the largest rectangle which you can fence with 150 rods of fencing.

3. What is the area of the largest rectangle and its dimensions which can be fenced with 100 rods of fencing if a river runs along one side as shown in the figure and no fence is required on that side? We will assume that the river is straight at the place where we are fencing the field.



4. I am fencing a garden in the shape of a rectangle and it is so arranged that a neighbor's fence will form one side. What should be the dimensions of the garden in order that the area may be as large as possible and to just use 60 yards of fencing?

5. We may vary the problem slightly by asking what should be the dimensions of a field in the shape of a rectangle which contains 30 acres in order that as little fence as possible may be needed?

Here the two quantities with which we are concerned are not the dimensions and the area for the area is known. The unknown quantities and hence the ones with which we are concernéd are the dimensions and the perimeter or the distance around the field. The area is 30 acres and if expressed in qsquare rods, is 30 x 160 or 4800 square rods. This area was obtained by multiplying the length by the width. Hence, if we represent the width by w, 4800 divided by w will represent the length. The perimeter as counted up from the figure is $w + \frac{4800}{w} + w + \frac{4800}{w}$ or

$$\text{perimeter} = 2w + \frac{9600}{w}$$

Determining some corresponding values of the width and the perimeter:

When w = 40, the perimeter = 320

.. w = 50, ... = 292

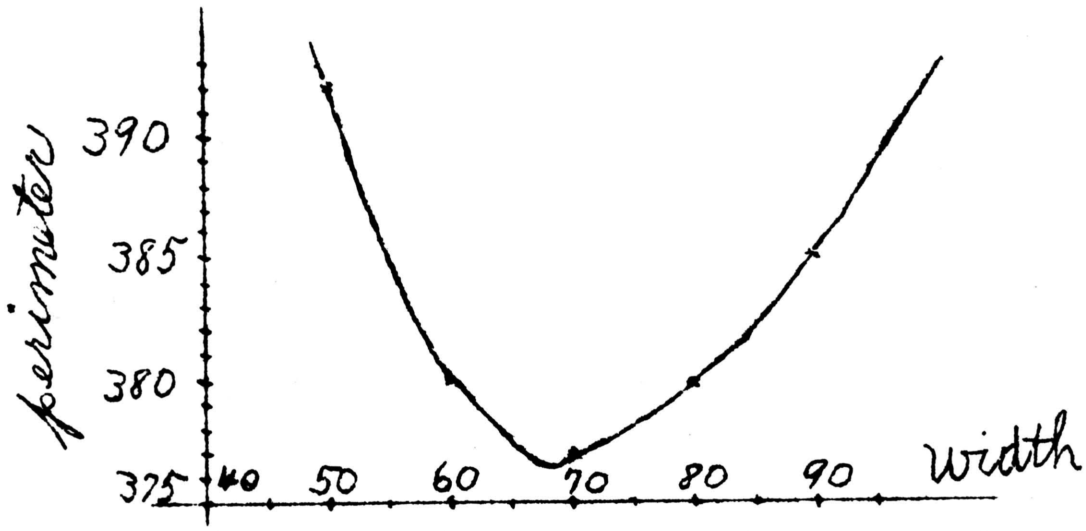
.. w = 60, ... = 280

.. w = 70, ... = 277

.. w = 80, ... = 280

.. w = 90, ... = 285

As in previous problems, we will represent the width on the horizontal lines and the corresponding perimeter by vertical distances as is shown in the figure below. Since



we are here concerned not with the greatest perimeter but with the least perimeter, we will look for the lowest point on the curve. What is it in the figure? What is the corresponding width? What is the length which goes with this width? Will these dimensions give an area of 30 acres?

6. What are the dimensions of a 20 acre field which will be cheapest to fence, that is which will have the least perimeter?

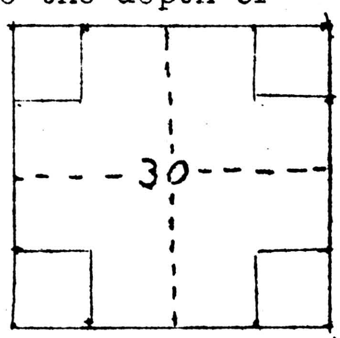
7. Suppose you wish to fence off a rectangular plot of ground which will contain 1/4 of an acre. What must be the dimensions of the rectangle in rods, if you use the smallest possible amount of fence and hence make the cheapest fence?

8. If you wish to fence off an acre lot along the side of a field so that the fence of the field will form one side of the lot, what must be the dimensions if the fence is made as cheaply as possible?

9. Suppose you have a ten-rod roll of wire fencing and wish to fence the largest chicken yard possible with it. If it is possible to utilize in making the fence, the wall of a building which is 25 feet long as a portion of one side of the fence, what should be the dimensions of the chicken yard? (Assume that the shape is rectangular.)

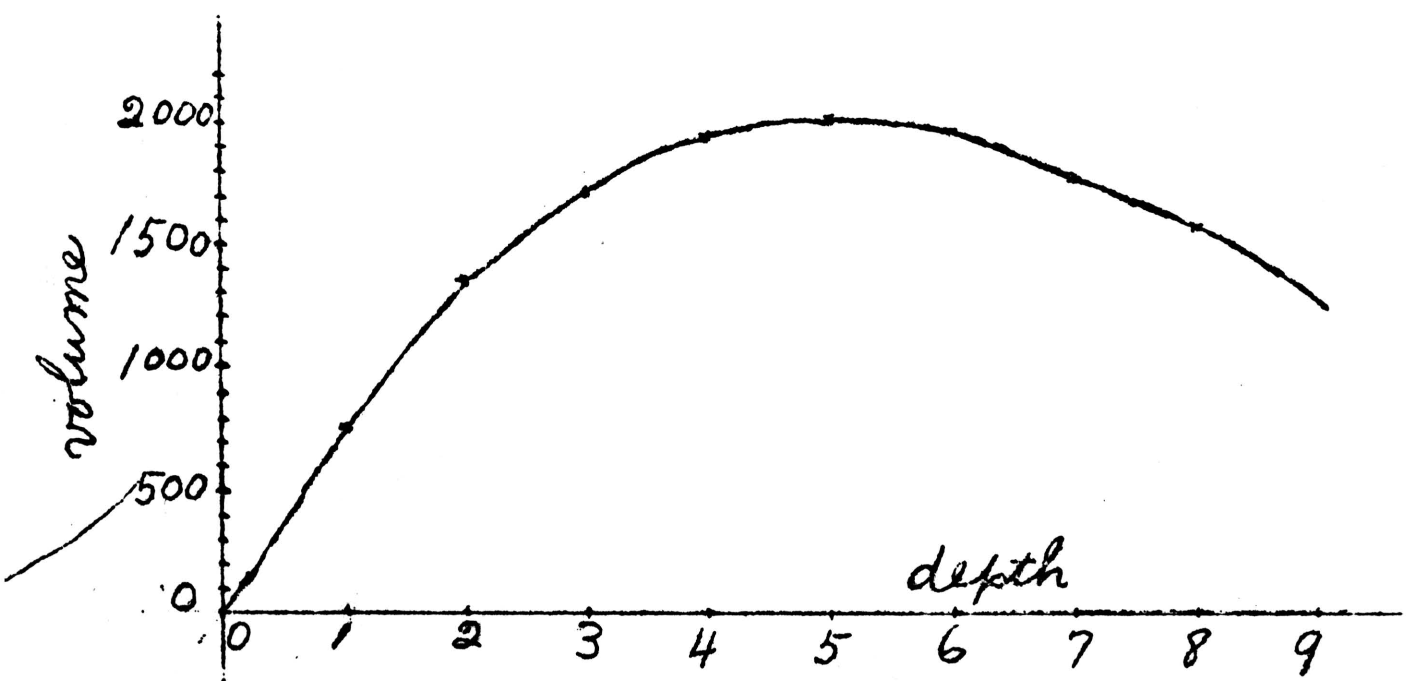
10. I have a piece of tin 30 inches square and wish to make a box with a square bottom by cutting out equal squares from each of the corners and turning up the sides. What should be the dimensions of the squares if I make the largest box?

The dimensions of the square will be the depth of the box. If we represent this depfh by d the dimensions of the bottom will be 30 -2d by 30 - 2d. Finding the volume by taking the product of the dimensions,



volume = d(30 - 2d) (30 - 2d)

By finding several pairs of corresponding values of the depth and the volume we may draw a curve to represent their relation as in other problems.



It should be noted that the relation between the volume and the depth is expressed not by an equation of the second degree but by one of the third degree, that is one which would contain a d^3 if it was multiplied out. The diagram is, however, very much the same as we have drawn for equations of the second degree. This just happened to be the case for in general they would not be anything alike.

11. A box is made from a piece of tin 24 inches square by cutting out squares from the corners. What should be the dimensions of the squares if the largest box is made?

12 I have a piece of card board 12 by 20 inches and I wish to make a box with no top by cutting equal squares from the corners and turning up the sides. What should be the dimensions of the squares to obtain the largest box?

13. I wish to build a box with a square bottom and no top which is to contain 50 cubic feet. What should be the dimensions in order that I may use the least possible amount of material in its construction?

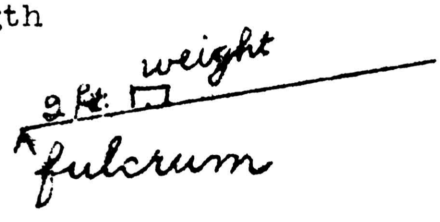
14. A 100 gallong tank is to be built with a square bottom and vertical sides and no top and is to be lined with copper. Find the most economical dimensions.

15. What are the best dimensions for a cylindrical half-bushel measure? (A half-bushel has no top and contains 1076 cubic inches.)

16. I am fencing a lot which I want to contain 1 acre. A building 75 feet long is to be used as a part of the fence on one side. What should be the dimensions of the lost to make the cheapest fence?

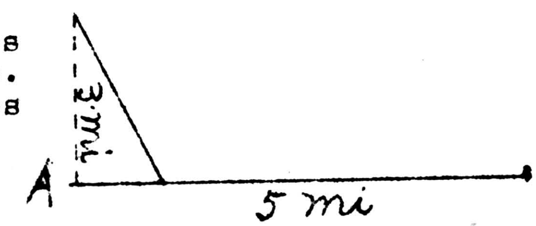
17. I have a piece of card board which is 10 inches by 20 inches. If I make a box by cutting out equal squares from each of the corners, what should be the dimensions of the squares if I make the largest box?

18. Find the most advantageous length for a lever by means of which to raise 100 pounds if the fulcrum and weight are placed as shown in the figure. The distance from the weight to the fulcrum is 2 feet and the lever weighs 3 pounds to the foot.



19. Under the conditions of the above problem, find the most advantageous length of a lever with which to raise a weight of 500 pounds if the lever weighs 9 pounds to the foot.

20. A man is in a row boat 3 miles from the nearest point A of a straight beach. He wishes to reach a point of the beach 5 miles from A in the shortest possible time. If he can walk at the rate of 4 miles an hour and can row only 3 miles per hour, what point of the beach ought he row for?



21. A man is traveling due east on a straight road which passes 1 mile south of his home. He is 5 miles west of his home and wishes to reach it in the shortest possible time. He can travel 5 miles per hour on the road but only 4 miles per hour if he goes across the fields. At what point should he leave the road?

22. What should be the dimensions of a tomato can which will contain a quart and in whose constructions the least possible amount of tin is used? (A quart is equivalent to 57.8 cubic inches.)

23. What should be the dimensions of a tin cup whose volume is one pint and in whose construction the least possible amount of tin is used?

24. What should be the dimensions of a 39 gallon cylindrical tank if the least possible amount of material is used in its construction?

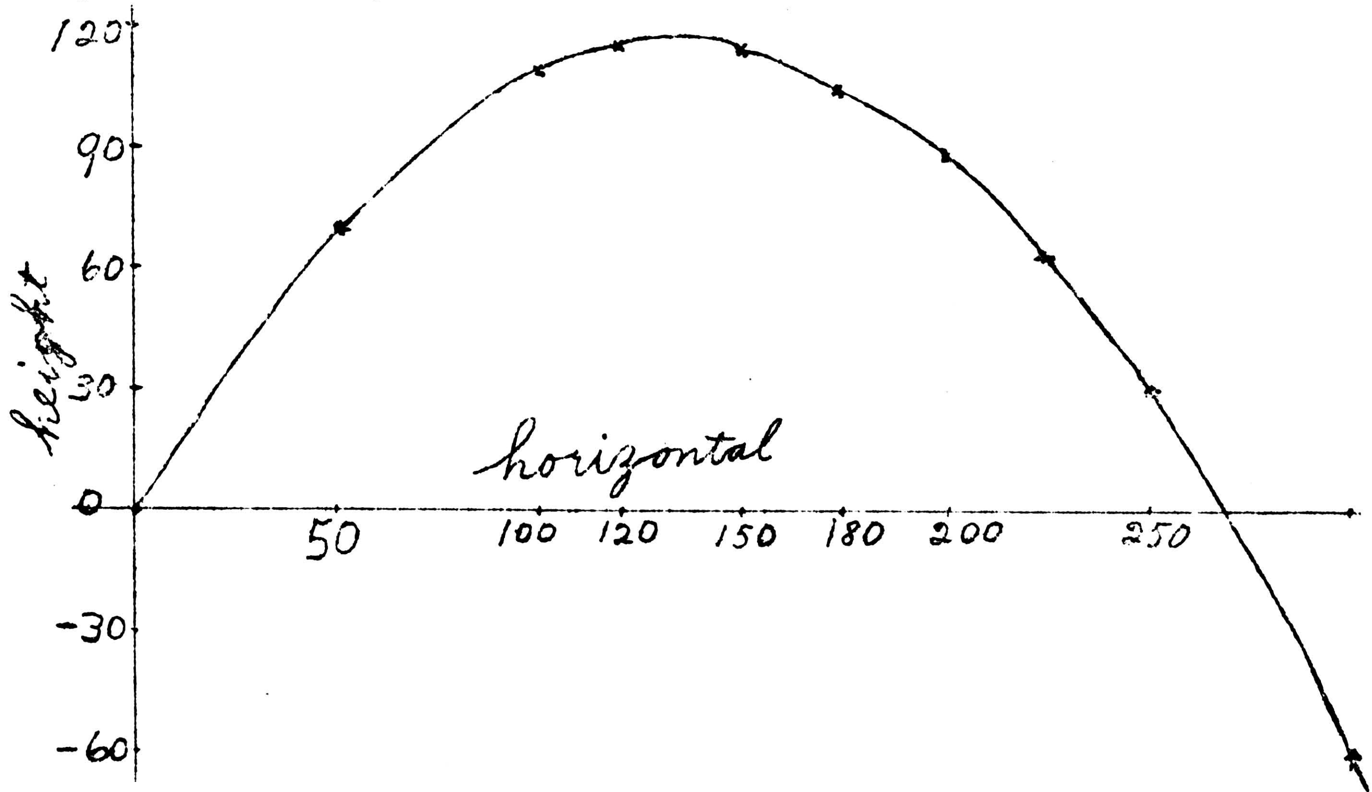
25. Draw a diagram to illustrate the part of a body thrown upward at an angle of 60° and with a velocity of 100 feet per second.

Refering to page 78, you should recall how to find the vertical and horizontal velocities. Do this and we find that the horizontal velocity is 50 feet per second and the vertical velocity 86 feet per second.

Consider the position of the body at the end of 1 sec., 2 sec., 2.5 sec., etc. as shown in the table on the next page. The horizontal distance passed over is easily determined since that motion is a simple uniform motion. But when we come to determine the vertical distance passed over, we must remember that the body begins to fall at once and the distance that it falls must be subtracted from the distance it would have gone if its motion were uniform. If hv represents horizontal velocity and vv vertical velocity, we may calculate the respective distances from these relations; horizontal distances = (hv) (t), virtical distance = (vv) (t) - $16t^2$.

No. of Sec.	Horizontal Distance	vertical distance
1	50 feet	70 feet
2	100 ..	108 ..
2.5	125 ..	115 ..
3	150 ..	114 ..
3.5	175 ..	105 ..
4	200 ..	88 ..
4.5	225 ..	63 ..
5	250 ..	30 ..
6	300 ..	-60 ..

Using these values we construct the figure as it is given below. The curve represents in minature the actual path of the body.



26. Determine from the figure the highest point that the reaches and how many seconds it takes it to reach it. If the body is thrown on level ground how far does it go horizontally? How man seconds before it reaches the ground?

27. An object is thrown with a velocity of 160 feet per second at an angle of 30^0. Draw a diagram to represent its path. From the diagram determine the highest point that it reached and how long after it was thrown that it reached this distance or point. How far did it go horizontally? (The horizontal distance is called the range.)

28. Take the above conditions and draw the paths of the object for various angles, say 30^0, 40^0, 45^0, 50^0, 60^0, on the same sheet. For what angle is the range the greatest?

29. Is it possible that an object may be thrown at two different angles and strike the ground at the same place? Give reasons for your answer.

30. How far can you throw a ball if it leaves your hand with a velocity of 64 feet per second?

31. You are standing on the ground 150 feet from a building. Can you hit a window 40 feet above the ground if you can throw with a velocity of 64 feet per second?

32. A revolver can give a bullet a muzzle velocity of 200 feet per second. Is it possible to hit the van on a church spire a quarter of a mile away, the height of the spire being 100 feet .

33. How far can you shoot with the revolver in the above problem?

34. With what velocity woulh it be necessary to throw a ball if it goes 300 feet horizontally? (It is assumed that it is thrown at the best angle.)

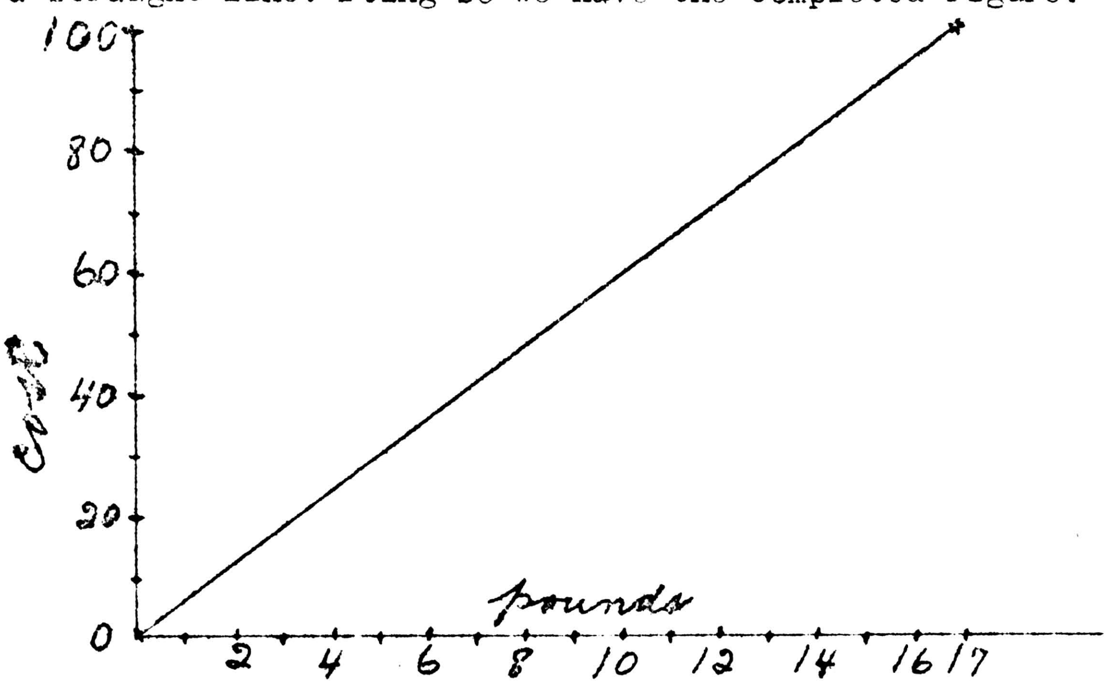
35. What would be the least velocity with which a bullet could be fired to hit a target a half mile away? At what angle would you aim?

36. If the initial velocity of a bullet is 1925 feet per second at what angle should you aim a rifle to hit an object 1 1/2 miles away? Determine the result approximately.

37. If a revolver gives a bullet a muzzle velocity of 200 feet per second, how far horizontally can you shoot if you aim the revolver at an angle of 20^0? 30^0? 40^0? 60^0? 80^0?

1. Draw a diagram which will represent the relation between the cost and amount of sugar, if sugar is selling 17 pounds for $1.00.

Suppose that we represent the number of pounds on a horizontal line and the cost on the vertical line. Mark the point which corresponds to $1.00 and 17 pounds. If we buy no pounds of sugar, it costs nothing. This gives us another point. Now since the cost increases just as fast as the number of pounds or as we may say, the cost varies directly as the number of pounds, we may complete the diagram by joining these two points by a straight line. Doing so we have the completed figure.



By means of this diagram we may answer a large number of questions, such as how manyn pounds of sugar should be given for 35¢. Determine from the figure the number of pounds which should be given for 25¢, 40¢, 80, 65¢. Also we can tell the cost of any number of pounds. Determine the cost of 5 pounds, 7 pounds, 12 1/2 pounds.

(This scheme is one which is really used and is called a "ready reckoner.")

2. Construct a diagram to show the relation between the cost and number of pounds of sugar when it sells for 19 pounds for $1.00. Answer for this diagram the questions which were asked above.

3. Construct a diagram to show the relation between inches and centimeters. 100 centimeters is equivalent to 19.37 inches. Determine from your figure the equivalent to 60 in., 2 ft., 40 cm., 75 cm., 1foot 8in., 40 in., 5 feet.

4. Construct a diagram to show the relation between the area of a square and the length of its side. Consider the side of the square measured in feet. Determine the area of a square whose side is 3.5 feet, 4.8 ft., 9.7 ft., Determine the side of a square whose area is 75 sq. ft., 40 sq. ft., 90 sq. ft.

In this problem you should not jump at the conclusion that it is just like the others. Remember that the area of a square varies directly as the square of its side. You should make a table of at least 5 values and locate the corresponding points before you draw the line joining them.

5. Show how the diagram in the above problem can be used to determine the square roots of numbers. Draw in a similar manner a diagram which will give the cube roots of numbers. from this diagram determine the cube roots of 40, 75, 80, 100.

(This same method may be applied to fourth, fifth and higher roots.)

It was pointed out on page 105 that the cost varied directly as the amount (number of pounds) purchased. In the problems 2 and 3 above what sort of figure did you draw to represent direct variation? Are we justified in concluding that the relation between quantities which can be represented by a straight line, is a direct variation relation?

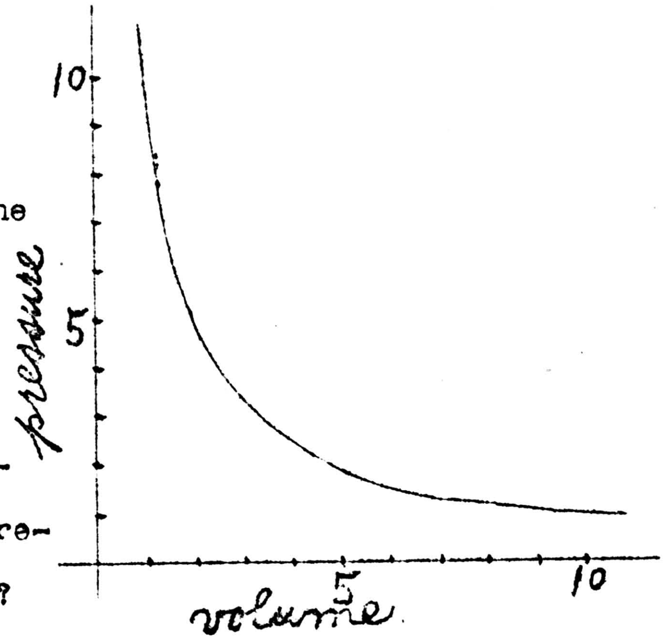
What is the variation relation between the area of a square and the length of its side? What sort of line or curve represents this relation? Is this the same sort of direct variation relation as we had in the case of the cost and amount? What is the distinguishing difference of their variation? What is the difference in the shape of the curve representing their relation?

In the first case the cost varies directly as the amount; in the second case the area of a square varies directly as the square of the length of its side. The distinguishing characteristic is that the cost varies directly as the first power or degree of the amount and this relation can be represented by means of an equation of the first degree, cost = price x amount: the area of a square varies directly as the second power of the length of its side and this relation is expressed by means of an equation of the second degree, area = $(side)^2$.

Thus although both are examples od direct variation, they possess this distinguishing characteristic and are represented by curves of quite different shape. And although we cannot say that direct variation is represented by means of a curve of definite shape, we can say that direct vvariation with respect to the first power of the variable is represented by a straight line; if the variation is with respect to the second power of the variable, the curve is of the general shape of the curve drawn to represent the relation between the area of a square and the length of its side. Variation with respect to higher powers of the variable are each represented by their own gharacteristic curves.

Now that we know what shapes of curves represent direcr variation, it will be interesting to learn what shapes of curves represent some cases of inverse variation.

6. Knowing that the pressure of a gas varies inversely as its volume, draw a curve to represent this relation. Take for your equation pv = 10.



The accompanying figure represents the relation between the quantities in the problem on the previous page.

What are the characteristics of this curve which distinguishes it from the curves which represent cases of direct variation?

If you had a diagram representing a variation relation, could you decide whether the variation was direct or inverse? It might be noted that a variation relation is not very definite when you merely know that it is direct or inverse. In the case of the volume pressure relation, pv = k, we need to know also the fixed value of k. (This value of k is fixed for only a definite mass (weight) of gas). Also there are several kinds of direct variation, variation with reppect to the first power, second power, etc. We should be able to determine these approximately.

7. A lath of yellow pine, 1 inch broad and .55 inch deep is supported at points 24 inches apart and weights placed upon it midway between the points of support. In the table below.

weight in pounds	0	8.6	18.6	28.6	38.6	48.6	58.6	63.6	68.6	69.6
bend in inches	0	0.15	0.36	0.57	0.78	1.00	1.23	1.36	1.70	1.78

Draw a diagram to represent the relation between the weight and the bend (deflection). Determine from your diagram a variation relation, which agrees very closely with the above data, and the fixed number or constant necessary in order to express the variation relation by means of an equation.

(For weights too heavy the variation relation will not hold and the lath will be permanently bent. Therefore disregard the largest weights if they do not seem to agree with the others.)

8. If a steel rod is clamped tightly at one end and a twisting force applied to the other end, the rod will be twisted a certain amount. In the table below are given the twisting force and the corresponding amount of twist. Determine by drawing a figure the variation relation between these quantities.

Twisting force	1.4	2.75	5.5	8.25	11.0	13.75	16.5
Twist	1.5	3.0	6.0	9.0	12.5	15.5	18.0

9. Four yellow pine laths of the same length, 24 inches, and of the same depth, 0.525 inches, but of different widths give for the same load the following deflections (bend). Draw a diagram and determine from it the variation relation between thw width of the lath and the deflection and also the constant of variation if possible.

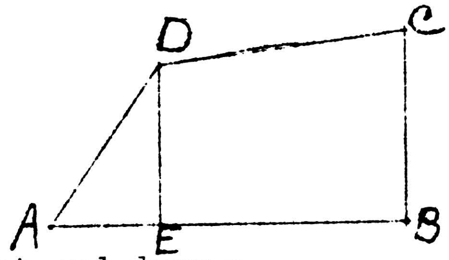
Width	0.54	0.79	1.02	1.26
Deflection	1.08	0.75	0.60	0.46

10. A cyclist sets out at 9 a. m. from a town A and rides two hours at a speed of 10 miles an hour; he rests half an hour and then returns at a speed of 8 miles an hour. A second cylist leaves A at 9:30 a.m. and rides at a speed of 7 miles an hour.

Draw upon the same sheet a diagram to represent the movements of each cyclist. Can you tell from the diagram when and where they will meet?

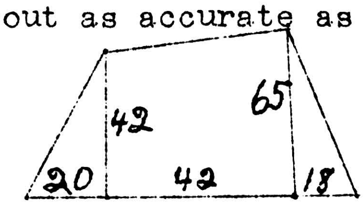
11. Two cyclists A and B set out at the same time. A rides for 2 hours at a speed of 9 miles per hour, rests 15 minutes and then continues at 6 miles per hour. B rides without stopping at a speed of 7 miles per hour. Determine as in the above problem when and where B will overtake A.

1. A farmer has an irregular shaped field as shown in the figure. He measures AE 10 rods, Cb 20 rods, DE 17 rods. Find the area.

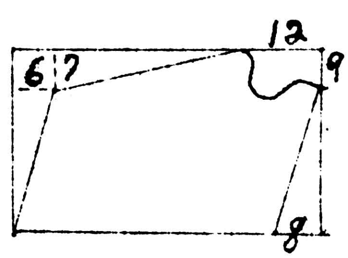


Take as large a scale as convenient and draw a careful diagram of the field on cross section paper. You should use the same scale for the vertical as for the horizontal distances. Then estimate the number of squares of the cross section paper inside the diagram of the field. Alog the edges if the lines cut the squares it will be necessary to actually count the squares. In counting, squares only partially included, count as a whole square those which are more than half inside, those less than half omit, those which are one half included count as one half. From your scale, you will know what area on square represents. Thus you can find the area of the field. The result obtained is only approzimate, it is true, but in the light of our discussion on page 57 and following it is doubtful if any computed area is more than a rough approximation. So really this method is about as accurate as any.

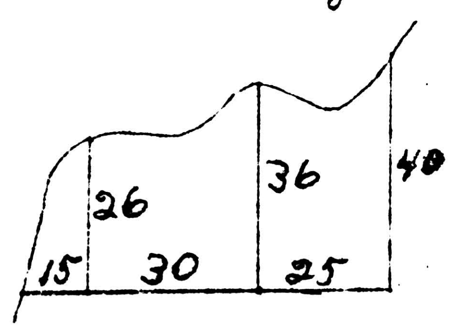
2. Find the area of the irregular tract of land which is measured as shown in the figure.



3. To measure an irregular tract of ground which is partially bounded by a creek, I lay off a rectangle 40 by 60 rods which just surrounds it as shown in the figure. I then measure the distances shown. Determine the area approximately.

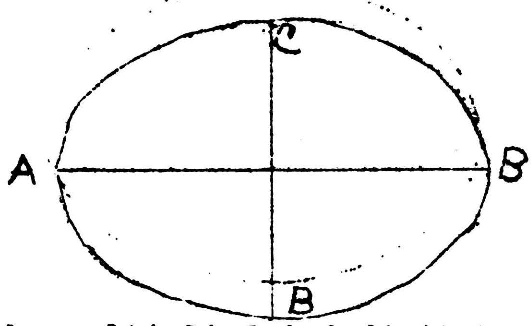


4. Find the area of an irregular tract formed by a creek running across a form in the manner shown in the figure. From the measurements given draw the creek in your figure and find the approximate area of the field.



5. By this same method find the area of a circle whose radius is 8 feet. Compare this result with your computed result. This will serve to indicate to you the accuracy of this method of determining areas.

6. Find the area of an oval figure of the shape and dimensions as given. AB is 18 inches, CD is 12 inches, and they bisect each other. Copy, upon a larger scale, this figure on cross section paper and estimate area from this.



Examples of this sort might be multiplied indefinitely but these will serve to indicate to you the value and efficiency of this method of determining areas. It should be clear that we can by this method determine an area approximately if we only know enough about it to draw a fairly accurate figure to scale.

7. In the table below are given the time table of certain trains on the Snata Fe Railroad between Chicago and Kansas City. On the next page a graphical picture of this time table is given.

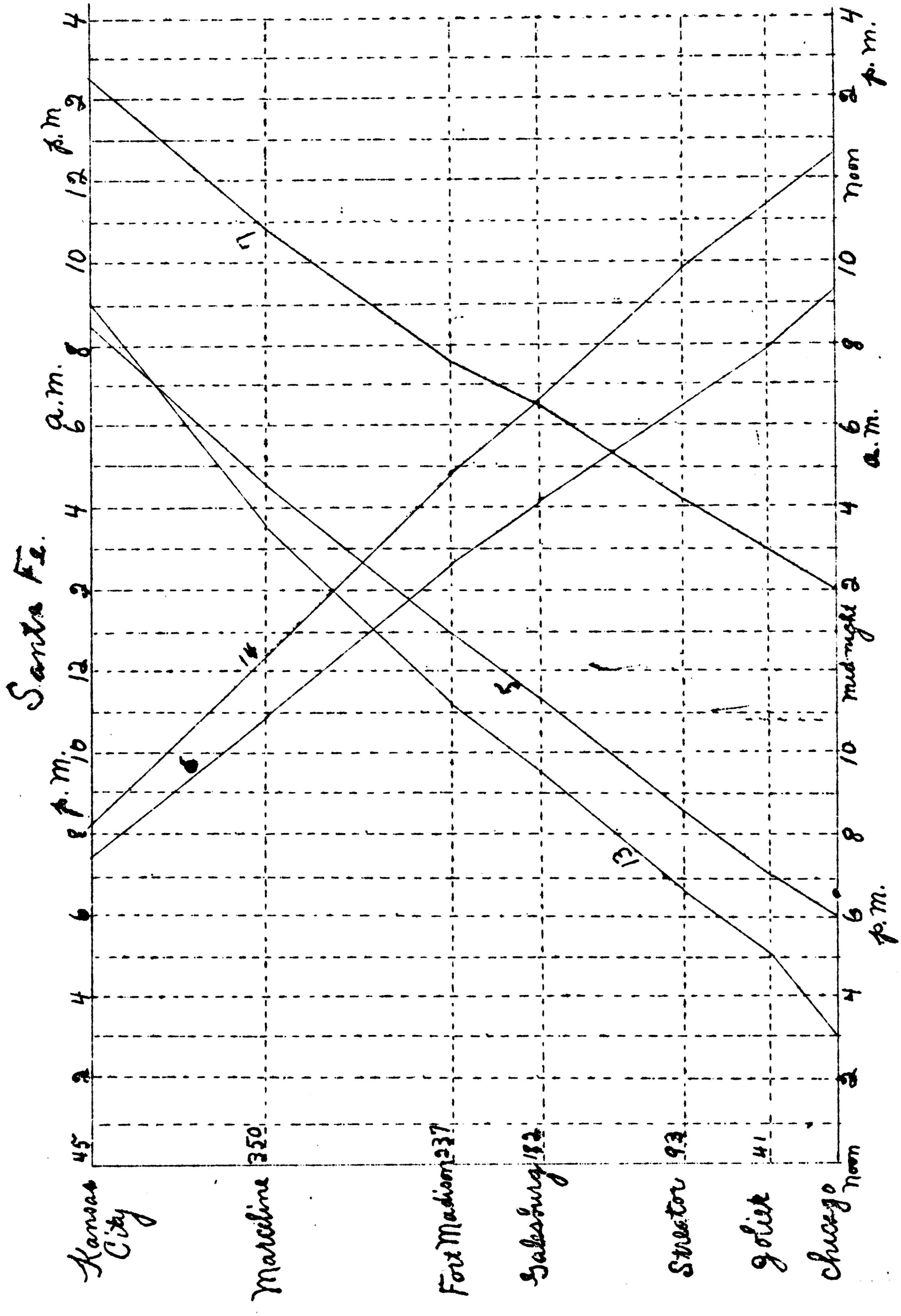
13	5	7		Dist.	6	14
p.m.	p.m.	a.m.	Station		a.m.	p.m.
3:43	6:00	2:00	Chicago	0	9:15	12:50
5:00	7:00	2:58	Joliet	41	7:55	11:30
6:30	8:30	4:12	Streator	93	6:25	9:37
9:26	11:20	6:30	Galesburg	182	4:07	6:30
11:05	12:55	7:45	Fort Madison	237	2:35	4:50
3:25	4:30	10:55	Marceline	351	10:55	12:10
8:50	8:30	2:15	Kansas City	458	7:30	8:10
a.m.	a.m.	p.m.			p.m.	p.m.

Write a paragraph telling what may be observed at a glance from a time table represented in this form.

It may be interesting to you to know that time tables are really used in this form in the railroad offices. I think that you may see why they are more usefull to a train dispatcher.

8. Represent graphically this time table of the Wabash trains betwee St. Louis and Kansas City.

3	9	1			2	12	4
a.m.	p.m.	p.m.	Stations	Dist.	p.m.	p.m.	a.m.
9:04	2:15	9:01	St. Louis	0	5:28	10:50	7:00
9:58	3:02	9;53	St. Charles	23	4:29	9:53	5:48
11:30	....	11:25	Montgomery	84	2:58	7:47	3:44
12:11	5:08	12:04	Mexico	110	2:20	6:58	2:52
12:35	5:32	12:27	Centralia	124	1:57	6:32	2:16
1:28	6:17	1:12	Moberly	148	1:20	5:44	1:20
2:35	7:14	3:30	Brunswick	187	12:08	4:23	11:42
3:19	7:49	4:20	Carrollton	211	11:29	3:28	11:00
5:34	9:40	7:00	Kansas City	277	9:30	1:20	9:00
p.m.	p.m.	a.m.			a.m.	p.m.	p.m.


Santa Fe.
Kansas City
Marceline
Fort Madison
Galesburg
Streator
Joliet
Chicago
45
350
237
182
93
41
noon
a.m.
p.m.
midnight
1
14
5
13
6

Chapter VI.

1. Two boys, James and Henry, have a 24 pound weight and a teeter board. To determine their weight they proceed as follows. The teeter board is placed so that it will just balance when neither of the boys are on it. James takes the weight on the board beside him at a distance of 6 feet from the fulcrum. In order to make the teeter balance Henry finds that he must be 9 feet from the fulcrum. Henry takes the weight beside him 5 feet from the fulcrum amd to make the teeter balance James must be 8 feet from the fulcrum. Find the weight of each boy.

You will noticethat this problem consists of two parts or two different sets of conditions. We may consider these one at a time in the same way as we have considered problems involving levers.

First part.

Represent by x the weight of James, then $x + 24$ will be the combined weight of James and the weight.

6 feet is his distance from the fulcrum.

Represent by y the weight of Henry.

9 feet is his distance from the fulcrum.

Following our usual rule, our equation is

$$9y = 6(x + 24)$$

Second part.

Using the same abbreviations and remembering that the 24 pound weight is now to be added to the weight of Henry our equation is $5(y + 24) = 8x$

Simplifying these equations we have

$$9y - 6x = 144$$

$$5y - 8x = -120$$

This problem now differs from all the others we have solved in that here we have two equations and two unknown quantities appearing in each equation. Our method of procedure will be to combine these two equations into one in such a way that there will remain only one of the unknown quantities. After finding the value of that quantity from this equation, it is a simpler matter to determine the value of the other unknown quantity.

To combine the two equations we may solve the equation

$$9y - 6x = 144$$

for y,

$$9y = 6x + 144$$

$$y = \frac{6x + 144}{9}$$

Writing this equivalent expression for y in the second equation

$$5y - 8x = -120$$

we have

$$\frac{5(6x + 144)}{9} - 8x = -120$$

This equation contains only x, the y has disappeared. This process is called elimination.

Simplifying

$$30x + 720 - 72x = -1080$$

$$42x = 1800$$

$x = 42.9$ pounds the weight of James.

To find the value of y we will write in the value we have just found for x in $y = \frac{6x + 144}{9}$

Doing this $y = \frac{6(42.9) + 144}{9}$

Simplifying $y = 44.6$ pounds the weight of Henry.

To check this problem we must go back to the problem as stated on page 114, and test these results in the problem itself.

The weight of James, we found to be 42.9 pounds.

The weight of Henry, we found to be 44.6 pounds.

In the first case James takes the 24 pound weight at a place 6 feet from the fulcrum, Henry is 9 feet from the fulcrum.

Hence $6\ (42.9 + 24) = 6\ (66.9) = 401.4$

$9\ (44.6) = 401.4$

Those products are the same, so we may say that our results check in the first part of the problem <u>but</u> it is necessary to check the second part also.

In the second case, Henry has the weight 5 feet from the fulcrum and James is 8 feet from the fulcrum. Hence

$5\ (44.6 + 24) = 5\ (68.6) = 343.0$

$8\ (42.9) = 343.2$

Our results check in the second part of the problem and we may now conclude that they are correct.

It might be pointed out that problems in which two equations occur, really have two separate sets of conditions. Our results must satisfy both sets of conditions. In the following problems, check your results in both parts of the problems.

2. Two boys, A and B, have a 30 pound weight and a teeter board. They proceed to determine their weight as follows. They find that they balance when A is 6 feet from the fulcrum. When B takes the weight on the board beside him they balance when B is 3 feet and A 7 feet from the fulcrum. Find the weight of each boy if in the first case B is 5 feet from the fulcrum.

3. An unknown weight 3 feet from the fulcrum of a lever is balanced by another unknown weight 8 feet from the fulcrum. An addition of 20 pounds to the first weight necessitates the removal of the second weight 3 feet farther from the fulcrum in order to preserve the balance. Find the two weights.

4. Two unknown weights balance when they are placed 7 and 9 feet from the fulcrum. If their positions are reversed, 11 pounds must be added to the lesser of the two weights to restore the balance. What are the weights?

5. E weighs 95 pounds and F 110 pounds. They balance at certain unknown distances from the fulcrum. E then takes a 30 pound weight on the board, which compels F to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

6. Two weights 35 and 40 pounds respectively balance when resting on a beam at unknown distances from the fulcrum. If 15 pounds is added to the 35 pound weight, the 40 pound weight must be moved 2 feet farther from the fulcrum in order to maintain the balance. What was the original distance of each of the weights from the fulcrum?

7. C is 6 1/2 feet from the point of support and balances D who is at an unknown distance from this point. C places a 33 pound weight beside himself on the board and when 4 2/3 feet from the fulcrum balances D who remains at the same point as before. C's weight is 84 pounds. What is D's weight and how far is he from the fulcrum?

8. A motor boat goes 10 miles per hour in still water. In 10 hours the boat goes 42 miles up a river and back again. What is the rate of the current?

9. A boatman trying to row up a river drifts back at the rate of 1 1/2 miles an hour. If he can row down the river at the rate of 12 miles an hour, what is the rate of the current?

10. A boat can be rowed down stream 4 miles in the same time as it can be rowed up stream 3 miles. A trip of 6 miles down stream and back requires 3 1/2 hours. Find the rate of the boat in still water and the rate of the current.

11. A boatman in 2 hours rows a certain distance up a stream where the rate of the current is known to be 2 miles an hu hour. Then he rows back to a place 1 mile beyond the starting point in one hour. Find the distance he rows each way and the rate of the boat in still water.

12. A crew rows upstream against a current of 3 miles an hour for a distance of 8 miles and back again. If the trip takes 5 hours, what rate could the crew make in still water?

13. A crew able to make 3 miles an hour in still water rows 8 miles upstream and back again ina total time of 6 hours. What is the rate of the current?

14. A crew able to make 5 miles per hour in still water rows 6 miles upstream and drifts back in a total time of 5 hours. How fast is the current flowing?

15. A stream flows at the rate of five miles an hour; a crew rows 6 miles with the stream and the same distance back in 3 1/2 hours. What is the rate of the boat in still water?

16. 200 cm. of white oak is fastened to 25 cm. of steel, making a combination whose average density is 1.56. If 250 cm. of oak is fastened to 20 cm. of steel, the average density of the combination is 1.3. Find the density of both white oak and steel.

17. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour the running time is decreased by 2 hours. Find the speed of the train.

18. The distance from New York to Buffalo is 442 miles. By decreasing the speed of a fast freight 8 miles per hour the running time is increased 4 hours. Find the speed of the freight.

19. If tin and lead lose respectively 5/37 and 2/23 of their weights when weighed in water, and a mass of 60 pounds loses 7 pounds when weighed in water, how much tin and lead are in the mass?

20. Archimedes suspected a crown of being made of a mixture of copper and gold and proved it by weighing it and then weighing the water which ran out of a full vessel into which the crown was placed. Suppose he found the weight of the crown to be 1140 grams and of the water to be 390 grams. If the gold is 19.3 and copper 8.9 times as heavy as water, find the number of grams of each metal in the crown.

21. Find the weight of a body which weighs 8 ounces on one pan of a balance and 9 ounces on the other. (Assume that the untrue weights are due to the arms of the balance being of unequal length.)

22. It was observed that a passenger train and a freight train going in the same direction completely pass each other in 3 minutes. The length of the passenger train is 600 feet and the freight is observed to pass a particular place in 50 seconds. Find the speed of each train.

23. The speed of a passenger train is 4p miles per hour. A man observes that a freight train going in the same direction on a parallel track passes it in 1 minute and 20 seconds. It is observed on another passenger train going at the rate of 25 miles per hour that the same freight passes it in 2 minutes What is the speed of the freight train?

24. A man on a passenger train observes that 12 seconds elapse while a freight train 1200 fett long going in the opposite direction on a parallel track passes his window. Later he observes that it takes 1 minute 15 seconds to pass a freight train 1440 feet long going in the same direction. Find the rate of each train, if both freight trains are traveling at the same speed.